HW 8 - Graph Theory

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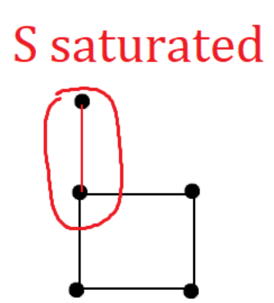
1.

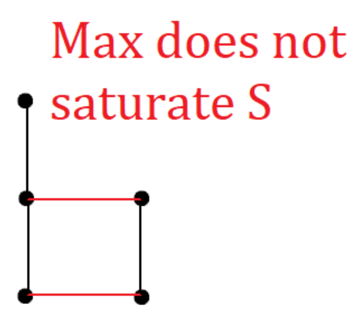
i.

Proof. We know that $S \subset V(G)$ is saturated by some matching, M. If M is maximum in G, then S is saturated by a maximum matching. Suppose M is not maximum, i.e. there exists some matching M' such that $|M'| > |M|$. Since M is not maximum, we know from Theorem 5.1 that G must contain some M-augmenting path, $P = v_0v_1...v_{2m}v_{2m+1}$ with $V(S) = \{v_1, v_2, ..., v_{2m}\}$. It is evident that the edges $\{v_0v_1, v_2v_3, ..., v_{2m}v_{2m+1}\}$ are clearly not in M, and the edges $\{v_1v_2, v_3v_4, ..., v_{2m-1}v_{2m}\}$ are clearly in M. We let $M' = M - \{v_1, v_2, ..., v_{2m-1}v_{2m}\} \cup \{v_0v_1, ...v_{2m}v_{2m+1}\}$. $\{v_0v_1, ...v_{2m}v_{2m+1}\}$ are unsaturated in $M - \{v_1v_2, ..., v_{2m-1}v_{2m}\}$. So adding the other edges is a matching for $v_0, v_1, v_2, ..., v_{2m+1}$. All vertices of S are saturated by this matching, so we have a larger matching and P is now alternating (since both the beginning and vertices of P are saturated). Repeat this process of converting augmenting paths to alternating paths for all M-augmenting paths in G. When no augmenting paths remain, we have a maximum matching that saturates S.

□

ii.



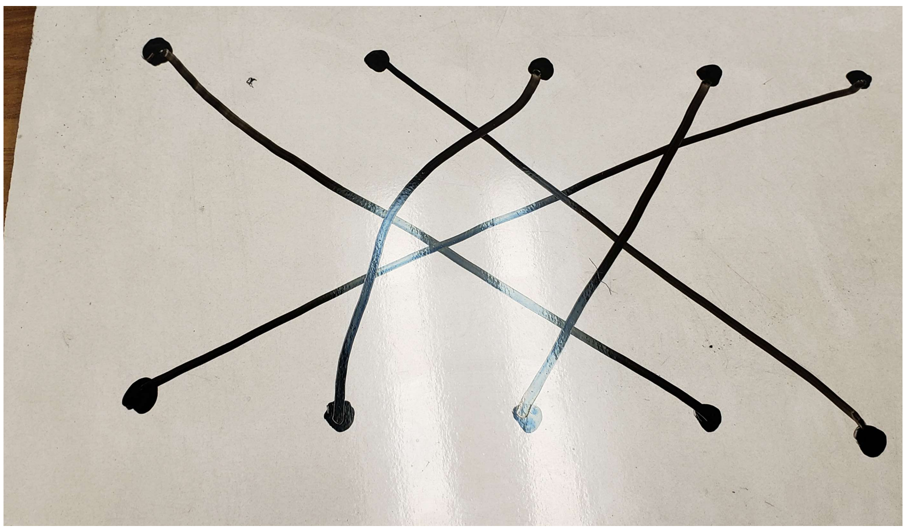


2.

i.

Each element in X must match with an element in Y in a bipartite graph (by nature of bipartite graphs; no vertex in X is adjacent to another vertex X and vice versa). But we have 5 elements in X and only 4 elements in Y. So we can at most match 4 pairs of vertices, but we will always have one vertex unmatched (since X > Y). So there is no perfect matching. (Need even number of vertices for a perfect matching).

ii.



iii.

For all $S \subset V(G)$, $o(G-S) \leq |S| \Leftrightarrow$ G has a perfect matching. Let S be the two innermost vertices in the pictured graph. Deleting these two vertices produces 4 components, 3 of which are odd. So we have $o(G-S) = 3$. But 3 is not less than or equal to $|S| = 2$. So there is no perfect matching.

iv.

Let S be the two rightmost vertices in the bottom partition of G. We delete them and produce 4 odd components. But 4 is not less than or equal to $|S| = 2$. So there is no perfect matching.

3.

i.

A connected simple graph G with size of maximum matching M = 1 necessitates that all vertices of G are adjacent to u or v (with u and v saturated by M) and no other vertices.

ii.

A connected simple graph G with size of minimum cover K = 1 necessitates that all edges are incident to a shared vertex. As such, all vertices in G are adjacent to this vertex as well (indeed, this single vertex is our cover).

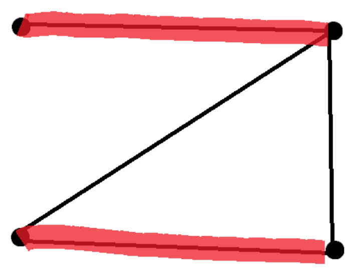
4.

i.

Proof. Suppose there exist ≥ 2 perfect matchings in a tree, say M and M'. Consider $G' = G[M \cup M']$. Clearly $V(G) = V(G')$ as both M and M' are perfect matchings. Each component of G', denoted G'_i, must have at least one edge (if this were not the case, then we would have an isolated vertex not part of M or M' in G which is impossible as M and M' are perfect). If a component G'_i has exactly 1 edge, then that edge must belong to both M and M' (if this were not the case, then only one matching would be saturating these 2 vertices in G as G'_i is a component and is disconnected from the rest of G'). If a component G'_i has more than 1 edge, then some $u, v \in V(G'_i)$ are connected by only an M edge (WOLOG). It is clear to see that the edge uv may not belong to both M and M', as $\epsilon(G'_i) > 1$, so we have at least one more edge connecting two vertices in G'_i that is saturated by at least M or M'. Suppose $\epsilon(G'_i) = 2$ with edge uv and edge vw. Our component is connected, so we must connect our edge saturated by M or M' to the rest of our edges/vertices. But this causes two edges from the same matching to be adjacent, which is impossible. So edge $uv \in M$ and edge $vw \in M'$. But since G'_i is a component and is disconnected from the rest of G', vertices u and w are not saturated by either M or M'. If we repeat this process for any $\epsilon > 1$, it is clear that each leaf of G'_i are at least M or M' unsaturated. Only by completing a loop can we guarantee each leaf in G'_i is both M and M' saturated. But our graph is acyclic, so this is impossible. $\therefore$ A tree has ≤ 1 perfect matching.

$\square$

ii.



iii.

Proof. We know that if G has a perfect matching, then for every $S \subset V(G)$, $o(G-S) \leq |S|$. We are always removing only one single vertex, so $|S| = 1$ for any $v \in V(G)$, and we get $o(G-v) \leq 1$. So the number of odd components in $G-v$ is either 0 or 1. Suppose $o(G-v) = 0$. Then each component, G_i of $G-v$ is even. So $\sum_i \nu(G_i)$ is also even. But $\sum_i \nu(G_i) = \nu(G) - |S| = \nu(G) - 1$, and $\nu(G)$ is even (G has a perfect matching). So we have an even number = odd number, which is impossible. $\therefore o(G-v) = 1$.

□