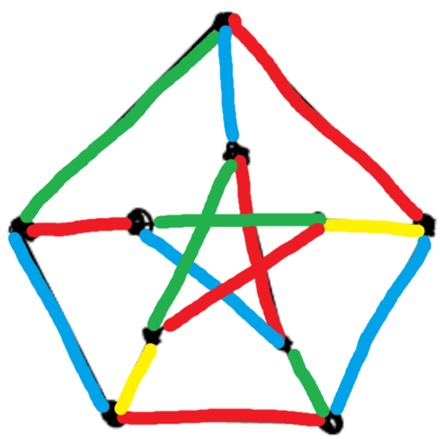
HW 9 - Graph Theory

Ethan Beaird

March 12, 2023

1.



So $\chi(G) = 4$.

G has no proper 3-edge-coloring. Suppose G had a proper 3-edge-coloring. G's outer edges are a 5-cycle C. Suppose C is 2-edge-colorable. Then no two edges of the same color may be adjacent, and each vertex in C must be incident to exactly 2 distinct colors H_1, H_2 (otherwise, two colors would be adjacent). Denote the edges in C, in order, as $\{e_0, e_1, e_2, e_3, e_4\}$. We arbitrarily let $e_0 \in H_1$. So $e_1 \in H_2$, $e_2 \in H_1$, $e_3 \in H_2$, and $e_4 \in H_1$. Since C is a 5-cycle, each e_i is adjacent to e_{i-1} and e_{i+1}, with e_0 adjacent to e_4. So e_0 and e_4 are adjacent but share the same color, and C has no proper 2-edge-coloring. We attempt to color C with 3 colors, H_1, H_2, H_3. We arbitrarily let $e_0 \in H_1$. So $e_1 \in H_2$, $e_2 \in H_3$, $e_3 \in H_1$, and $e_4 \in H_2$. No like colors are adjacent, so this edge-coloring is proper.

Each vertex in C, v_0, v_1, v_2, v_3, v_4 is incident to 2 distinct colors, but each vertex is also incident to one uncolored edge. We color this edge with the third color that each vertex is not incident to. With each $v \in C$ incident to e_i, e_{i+1} and

v_4 incident to e_0, we denote the vertices of C, $v_0 ... v_4$, by their incident edge colors, $H_1H_2, H_2H_3, H_3H_1, H_1H_2, H_2H_1$. So the final edge incident to each $v \in C$ must be H_3, H_1, H_2, H_3, H_3, respectively. As such, the inner vertices of G, $S = u_0, u_1, u_2, u_3, u_4$, must also be incident to H_3, H_1, H_2, H_3, H_3, respectively. Consider u_1. We know that u_1 is adjacent to both u_3 and u_4 (by the nature of the graph). But u_1 is incident to H_1 already, so we know that we must color the edges u_1u_3 and u_1u_4 as H_2 and H_3, distinctly. But u_3, u_4 are both colored by H_3, so we may not properly color one of u_1u_3 or u_1u_4. $\therefore$ G is not 3-edge-colorable.

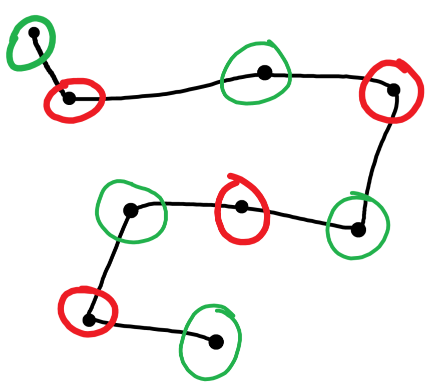
$\square$

2.

Here, $\epsilon(G) = 83$ and $\Delta(G) = 24$. By theorem 6.1, if a graph is bipartite, then $\chi'(G) = \Delta(G)$. Our graph is bipartite, so the required number of periods, p, is $\Delta(G) = 24$. We know from theorem 6.3 that $\lfloor \frac{\epsilon}{p} \rfloor \leq M_i \leq \lceil \frac{\epsilon}{p} \rceil$, or $\lfloor \frac{83}{p} \rfloor \leq 3 \leq \lceil \frac{83}{p} \rceil$. If we may use a maximum of 3 rooms in any given period, by letting $p = 28$, we satisfy our equation, as $\lfloor \frac{83}{28} = 2.964 \rfloor \leq 3 \leq \lceil \frac{83}{28} = 2.964 \rceil$, and we have a maximum of 3 rooms being used in any period.

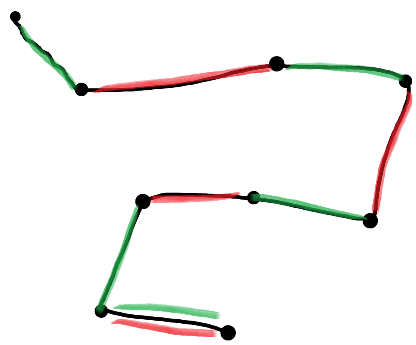
3.

a.



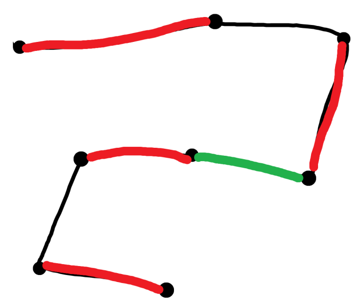
Consider the graph above. Our green vertex covering is clearly minimal (as removing any vertex is no longer a vertex covering) and has size 5. Our red covering has size 4 and is minimum, but is clearly not contained in our green covering. So this statement is false.

b.



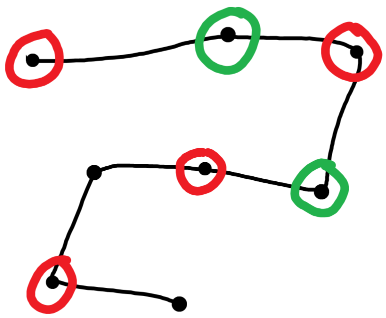
Consider the graph above. Our green edge covering is clearly minimal (as removing any edge is no longer an edge covering) and has size 5. Our red covering has size 4 and is minimum, but is clearly not contained in our green covering. So this statement is false.

c.



Consider the graph above. Our red matching is clearly maximum as it is perfect. Our green matching is another matching that is not contained in our red matching. So this statement is false.

d.



Consider the graph above. Our red independent set is clearly maximum. Our green independent set is another independent set that is not contained in our red independent set. So this statement is false.

4.

i.

If $\alpha(G) = n$, then no independent subset $S \subset V(G)$ may have size $> n$. So this graph has independent subset(s) with maximum size n. (I am unsure of how general of an answer this question wants. If it wants a specific example, then consider the graph with n components with each component complete.)

ii.

If $\alpha(G) = 1$, then every independent subset of $V(G)$ may contain at most 1 vertex. So all vertices are connected to eachother, or G is complete.

iii.

If $\beta'(G) = 1$, then the minimum size of an edge covering in G is 1. So all vertices of G must be adjacent to a common edge. So the graph must be two vertices joined by an edge.