HW 5 - Graph Theory

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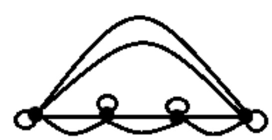
1.

a.



G has $\delta > 4$ as each vertex has degree = 6. G has $\kappa = 2$ as no one vertex can be removed to disconnect the graph since G is cyclical. Consider vertex 1 and 3: deleting vertex 1 and 3 clearly disconnects the graph. G has $\kappa' = 4$ since each vertex has 2 edges between them, and G is cyclical, so we must remove the connection between two vertices twice to disconnect the graph.

b.

Proof: G is simple, so any vertex in $V(G)$ may only be adjacent to $\nu - 1$ vertices (since G is simple, each vertex pair is joined by at most one edge), so each $v \in V(G)$ is incident to at most $\nu - 1$ edges.

$$\text{So } \delta \geq \nu - 2 \Rightarrow \delta = \nu - 2 \text{ or } \delta = \nu - 1.$$

Case 1: $\delta = \nu - 1$: We know that the vertex with minimal degree, $v_0 \in V(G)$, is connected to each other vertex in $V(G)$, so all other $v \in V(G)$ have degree $=$ $\nu - 1$ since G is simple and $\delta = \nu - 1$, the maximal degree in a simple graph.

$\therefore$ G is complete. By convention, $\kappa(K_n) = n - 1 = \nu - 1 = \delta$. So $\kappa(G) = \delta(G)$.

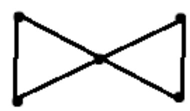
Case 2: $\delta = \nu - 2$: We know there exists some vertex, $a \in V(G)$ with degree $=$ $\nu - 2$. Since G is simple, it is clear that a is connected by $\nu - 2$ edges to $\nu - 2$ distinct vertices. Furthermore, there is exactly one other vertex with degree $=$ $\nu - 2$- the vertex that a is not adjacent to. It is also clear that removing this set of edges incident to a, which we will call E', disconnects the graph. E' is minimal, also, since the rest of the graph is complete (bar 1 vertex- the vertex not connected to a).

$$\text{So } d(a) = \delta = \nu - 2 = \kappa'.$$

Let $A =$ all vertices adjacent to a. We see that $G - E'$ produces exactly the same effect as $G - A$— the graph is disconnected by a minimal edge/vertex cut (since a has minimal degree and only one edge in G is not connecting each vertex). Clearly, $|A| = \nu - 2 = \kappa(G) = \delta(G)$.

$\square$

c.



2

a.

Proof: Suppose there exists some $v_0 \in V(S)$ not neighboring H_1 or H_2. Without loss of generality, we consider H_1. So v_0 shares no edge with any $v \in H_1$. It is clear, then, that the edge cut $S - \{v_0\}$ still disconnects H_1 from H_2 since v_0 is connected to no $v \in \ V(H_1) \cup V(H_2)$ and cannot connect H_1, H_2 if it is not adjacent to them. But this means $S - \{v_0\}$ is a smaller vertex cut of G. This is a contradiction, as S is the minimal vertex cut in G.

$$\therefore \text{Every } v \in V(S) \text{ must have a neighbor in } H_1 \text{ and a neighbor in } H_2.$$

$\square$

b.

Proof: G is 3-regular, so every $v \in \ V(G)$ has $d(v) = 3$, and $\delta = 3$. We know from part A that each $v \in V(S)$, where S is some minimal vertex cut, neighbors two components H_1 and H_2. By definition, a minimal vertex cut is a set of vertices such that $G - S$ is disconnected. We apply our argument from part A on the components of G, calling them G_1, G_2, G_3. We find that every $v \in V(S)$ has a neighbor in G_1 and G_2, every $v \in V(S)$ has a neighbor in G_1 and G_3, and every $v \in V(S)$ has a neighbor in G_2 and G_3.

Suppose $G - S$ had > 3 components. A minimal vertex cut, S, that has $G - S$ produce n components necessitates that G remain connected while deleting $v \in V(S)$ from G except for the last (if it did not remain connected, then there would exist a smaller vertex cut, which is impossible). So there exists some $q \in V(S)$ that is the last vertex where G remains connected before removed.

Deleting this q produces > 3 components. But to produce > 3 components, > 3 edges must connect to q (if ≤ 3 edges, then q is only connected to 3 vertices, and it would be impossible to produce > 3 components since these vertices cannot belong to multiple components).

$\therefore$ This is impossible since G is 3-regular. q may have at most 3 edges, so $G - S$ has at most 3 components.

$\square$

c.

Proof: We know that $\kappa \leq \kappa' \leq \delta = 3$. We consider all cases of κ.

Case 1: $\kappa = 1$: We know there exists some $v \in V(G)$ such that G-v is disconnected. Since G is 3-regular, we know v is incident to 3 edges and since G is simple, v is adjacent to 3 vertices (proven earlier in this homework), with at least one neighboring vertex being disconnected from the others in $G - v$ called u (if this was not the case, then v would not disconnect G and would not be a vertex cut). We know that there exists an edge $e \in E(G)$ between u and v. G is simple, so only 1 edge joins u and v. We may delete e to produce the same effect of $G - v$, since cutting a vertex deletes both the vertex and all incident edges, causing the graph to be disconnected. So only 1 edge was removed $= \kappa'$, and this is precisely κ.

Case 2: $\kappa = 2$: We know there exists some $u, v \in V(G)$ such that $G - \{u, v\}$ is disconnected. Since G is 3-regular, we know u, v are incident to 3 edges each and since G is simple, u, v are adjacent to 3 vertices, with at least two neighboring vertices being disconnected from the others in $G - \{u, v\}$ called a, t (if this was not the case, then u, v would not disconnect G and would not be a vertex cut). We know that there exist edges $e_1, e_2 \in E(G)$ between that join u, a and v, t. G is simple, so only one edge joins each vertex pair. We may delete e_1, e_2 to produce the same effect of $G - \{u, v\}$, since cutting a vertex deletes both the vertex and all incident edges, causing the graph to be disconnected. So only 2 edges were removed $= \kappa'$, and this is precisely κ.

Case 3: $\kappa = 3$: We know there exists some $u, v, w \in V(G)$ such that $G - \{u, v, w\}$ is disconnected. Since G is 3-regular, we know u, v, w are incident to 3 edges each and since G is simple, u, v, w are adjacent to 3 vertices, with all neighboring vertices being disconnected from the others in $G - \{u, v, w\}$ called a, t, s (if this was not the case, then u, v, w would not disconnect G and would not be a vertex cut). We know that there exist edges $e_1, e_2, e_3 \in E(G)$ between that join u, a, v, t, and w, s. G is simple, so only one edge joins each vertex pair. We may delete e_1, e_2, e_3 to produce the same effect of $G - \{u, v, w\}$, since cutting a vertex deletes both the vertex and all incident edges, causing the graph to be disconnected. So only 3 edges were removed $= \kappa'$, and this is precisely κ.

$\therefore$ We have shown that in each possible case of κ in a 3-regular graph, there is an edge cut with the same size as vertex cut S, or κ.

□

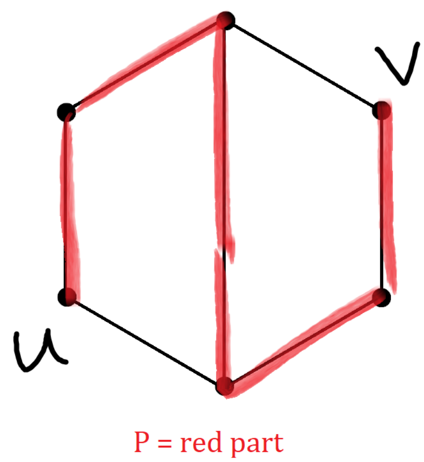
d.

In part C, we showed that $\kappa = \kappa'$ for all possible values of κ in a 3-regular graph. So $\kappa = \kappa'$.

□

3.

a.



b.

This does not contradict theorem 3.2 as each vertex in G is still connected by 2 internally disjoint paths (G is essentially a cycle which we know is 2-connected). Theorem 3.2 still holds as I have ≥ 3 vertices and each vertex in G has at least 2 internally disjoint paths to every other vertex. It is only because I specifically chose this convoluted path that another internally disjoint path may not exist.

c.

Proof: We prove by induction.

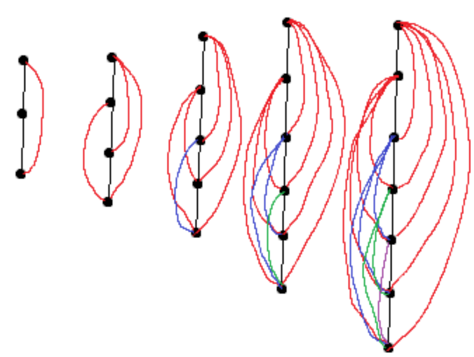
Base Case: $\nu = 2$. We call the two vertices in $V(G)$ u_0 and v. We know that u_0, v are connected by exactly two internally disjoint paths. It is evident, then, that a maximum of 2 edges may exist between u_0 and v (if this was not true, then there would be 3 ways to traverse a edge/path between u_0 and v which is impossible). So 2 edges, e_0, e_1 connect u_0, v, forming cycle $C = u_0e_0ve_1u_0$. C is one internally disjoint path; C^{-1} is the other.

Inductive Step: We assume our statement holds for $\nu = k$. We now show that the statement holds for $\nu = k + 1$. It still remains that a maximum of 2 edges may exist between any $u, w \in V(G)$ (else there would be >2 ways to traverse a path if at least one edge in a path is joined by 2 others connecting the same u, w). We know k vertices form a cycle. We add 1 vertex to G, u_{k-1}. So $\nu = k + 1$. We take the edge immediately after u_0 in our k-cycle C, e_0, and connect it to u_{k-1}. We connect u_{k-1} to the vertex that was immediately after u_0 with a new edge, e_k. Our cycle $C = u_0e_0u_1...u_{k-1}e_kv$ is once again connected. We connected only 2 edges to u_{k-1}, so we do not have more than 2 internally disjoint paths joining u_0, v, and G is still a cycle.

□

4.

a.



b.

Proof: We prove by induction.

Base Case: G is connected, so there exists a path within G. We know in a path with length ≥ 3 that some $d(u, v \in V(G))$=2 (if it did not, then there is no path in G with length 3 which implies G is not connected- a contradiction). So, in a path of length 3, there is at least one pair of vertices with $d(u, v) = 2$. By our algorithm, we may add an edge e to our path such that it has one cycle. So e connects 2 vertices in G. We need not worry about our 3rd vertex, as it it either belongs in the cycle C and has 2 internal disjoint paths or does not belong in C (and if it does not belong in C, it will only belong to paths starting or ending with itself). Inductive Step: We assume our statements holds for k.

We now show it holds for k+1. Clearly k+1 will produce at least 1 more cycle than k. But we add only one vertex in k+1. So we connect another cycle and our graph is even more traversable. So each vertex still has at least 2 internally disjoint paths and $\kappa = 2$.

□