Midterm 2 - Graph Theory Corrections

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Proof. Each induced graph $G[V_i \cup V_j]$, $i \neq j$, is a k-regular bipartite graph with bipartition (V_i, V_j), so each $G[V_i \cup V_j]$ has a perfect matching (corollary 5.2). Consider any $G[V_i \cup V_j]$, $i \neq j$ in G and let $E_1 \subset E(G)$ be the edges joining V_i and V_j. By corollary 5.2, E_1 contains a perfect matching between these two subsets. Two other vertex subsets remain in G, say V_k, V_l; similarly, let the edges joining them in $G[V_k \cup V_l]$ be $E_2 \subset E(G)$; by corollary 5.2, E_2 contains a perfect matching between these two subsets. No $e \in E_1$ is adjacent to any $e \in E_2$ since the the ends of each $e \in E_1$ are in V_i, V_j and the ends of each $e \in E_2$ are in V_k, V_l; they are disjoint. It follows that no edges in the perfect matching contained in E_1, denoted M_1, are adjacent to edges in the perfect matching contained in E_2, denoted M_2. Therefore $M_1 \cup M_2$ is a matching in G that saturates all $v \in V(G)$, or G has a perfect matching.

For any vertex subset V_i, each vertex is incident to exactly k edges in $G[V_i \cup V_j]$, with the degree of each $v \in G[V_i \cup V_j] = k$ (by the problem statement). Repeat this process twice more (for the other 2 possible vertex-induced subgraphs including V_i that our construction allows), each time adding exactly k edges to each $v \in V_i$. It follows that each $v \in G$ has exactly degree 3k. By theorem 6.1, $G[V_i \cup V_j]$ has $\chi' = \Delta = k$ and is colored by k colors; coloring the edges between V_i and V_j leaves 2k uncolored incident edges to each vertex. Each vertex is already incident to k colors, then, so to color the edges in $G[V_i \cup V_k]$ and $G[V_i \cup V_l]$, the other two vertex-induced subgraphs including V_i, we must use k more colors each. So we need exactly 3k colors to color the edges incident to vertices in V_i.

V_j now has 2k uncolored incident edges to each vertex; we color the edges in $G[V_j \cup V_k]$ with the k colors we used to color V_i's incident edges that are not already incident to vertices in V_j or V_k (there are k colors touching V_j and k colors touching V_k, so we use the remaining k colors; otherwise, two of the same colored edges would be incident). We repeat this process similarly to the edges in $G[V_j \cup V_l]$. We have colored V_j's incident edges with the same $3k$ colors as V_i. Now, both V_k and V_l are not colored by the k colors between V_i and V_j, so we color the edges between V_k and V_l by those k-colors. Again, we have colored V_k and V_l's incident edges with the same 3k colors as the other subsets. Therefore our graph is now properly colored with exactly $\chi'(G) = 3k$ colors. ■