HW 11 - Graph Theory

Ethan Beaird

March 12, 2023

1.

G has $\chi(G) = k$, so G contains a k-critical graph K. It follows from theorem 8.1 that $\delta(K) \geq k - 1$, or $\delta(K) + 1 \geq k$. Considering the vertex-induced subgraph $G[V(K)]$, we certainly have the same or more edges than in K. So $\delta(K) \leq \delta(G[V(K)]) \leq \max\delta(H)$. It follows that $k - 1 \leq \delta(K) \leq \max\delta(H)$, or $k \leq \max\delta(H) + 1 \implies \chi \leq \max\delta(H) + 1$.

$\square$

2.

If every S-component of G has a k-1 coloring where all vertices of S are colored the same, then as in the proof of 8.2 in class, we can produce a k-1 coloring of G. But this is impossible since $\chi(G) = k$. Therefore G is not uniquely k-colorable as we have multiple colorings.

$\square$

3.

a.

We need a graph with $\chi(G) = 1$, i.e. G can be colored by just one color and every proper subgraph of G can be colored by 0 colors. To be colored by just 1 color, there may be no edges in G, so G is the empty graph. But for every proper subgraph of G to be colored by 0 colors, each proper subgraph of G must result in no vertices, so we may only have one vertex in our graph, or G is a K_1.

b.

We need a graph with $\chi(G) = 2$, i.e. G can be colored by 2 colors and every proper subgraph of G can be colored by 1 color. To be colored by 2 colors, G must be bipartite with at least one edge. But for every proper subgraph of G to be colored by 1 color, we need each proper subgraph to yield the empty graph. So each vertex in G must be incident to each edge in G, i.e. G is a K_2.

c.

Our graph has $\chi(G) = 3$, and we know that $\chi - 1 \leq \Delta$, so $\Delta \geq 2$. Similarly, $\delta \geq k - 1 = 2$. But Δ is certainly no more than 2 since our graph is 3-colorable and it is impossible to 3-color G if any $v \in V(G)$ is adjacent to ≥ 3 vertices (as that would necessitate like colors adjacent). It follows that $\Delta = \delta = 2$, or each $v \in V(G)$ has degree 2. From a theorem from class, G contains a cycle. Even cycles are 2-colorable, so our cycle cannot be even. So G must be an odd cycle with $\nu(G) \geq 3$ (since $\nu = 1$ and $\nu = 2$ are both 1-colorable and 2-colorable, respectively).

$\square$

4.

a.

The minimum length of a cycle in G is 3, so each cycle encloses a "face". Each face has at least k edges, and each edge bounds 2 faces (either both interior or one interior and the other exterior). So $2\epsilon \geq k\phi$. It follows from Euler's Formula that $2\epsilon \geq k(2 + \epsilon - \nu) \to 2\epsilon \geq 2k + k\epsilon - k\nu \to (2 - k)\epsilon \geq 2k - k\nu \to (k - 2)\epsilon \leq k(\nu - 2) \to \epsilon \leq \frac{k(\nu-2)}{k-2}$.

$\square$

b.

If $\epsilon > \frac{k(\nu-2)}{k-2}$, then G is not connected or G is not planar or G does not have girth ≥ 3. By inspection, the Petersen graph has $\epsilon = 15$, $\nu = 10$, and $k = 5$. So $\frac{k(\nu-2)}{k-2} = \frac{5(8)}{3} = \frac{40}{3} = 13.333$, so $\epsilon = 15 > 13.333$, satisfying our condition. G is clearly connected and has girth 5, so G is not planar.

$\square$

5.

We consider our points in the plane as vertices of a graph G. Clearly, then, G is planar. By corollary 9.5.2, we know that if G is a simple planar graph with $\nu > 3$, then $\epsilon \leq 3\nu - 6$. Considering two vertices connected by an edge as a "pair", we have at most $3n - 6$ pairs of points in the plane, and thus at most $3n - 6$ pairs of points with distance 1 in the plane.

$\square$