HW 7 - Graph Theory

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1.

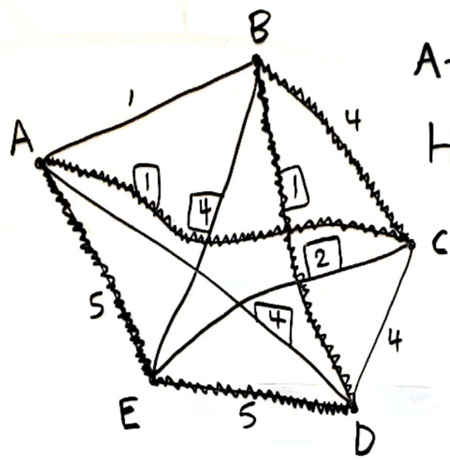
1)

$H_1 = ABCDE$

A-C , B-D

$H_2 = ACBDE$

$\ell = 16$

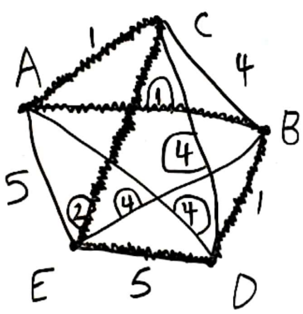


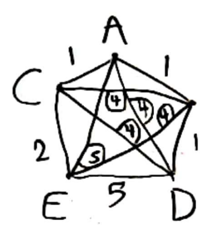
2)

E-C, A-B

$H_3 = ABDEC$

$\ell = 10$





We stop.
No pair of edges "inside" (the diagonals) have smaller weight than any pair of edges "outside" (our cycle).

2.

Let G be a graph with bipartition X,Y. Since G is bipartite and k-regular, $\sum_{v \in X} d(v) = |E(G)| = k|X| = k|Y|$. So $|X| = |Y|$. Let $S \subset X$ and let E_1 be all edges incident to S and E_2 be all edges incident to the neighbors of S, $N(S)$. $E_1 \subset E_2$ so $|E_2| = k|N(S)| \geq |E_1| = k|S|$. $|N(S)| \geq |S|$. This works for all S. So by theorem 5.2, there exists a matching M saturating X. Every edge in M has exactly 1 $v \in X, Y$. So M saturates the same number of X vertices and Y vertices. But $|Y| = |X|$, so M saturates all vertices and is a perfect matching. But we know $\sum_{v \in X} d(v) = k|X| = k|Y|$, and G is simple. WOLOG, each vertex in X is incident to exactly k vertices in Y. It is clear that $k \leq |X| = |Y|$ (if this were not true, then we would have more edges that vertices available for these edges to connect). Delete all edges in our found perfect matching, leaving us with a $k-1$-regular bipartite graph. We repeat this process $k-1$ times, and we find k disjoint perfect matchings.

$\square$

3.

a.

Consider $\mathbb{Q}_n$. $\mathbb{Q}_n$ has 2^n vertices. Each vertex has a unique binary sequence with length n. If two vertices are adjacent, then their binary strings may only differ by exactly one bit (per our construction). So one of n bits may vary between a vertex and its neighbors, i.e. we have n ways to vary the binary string and reach an adjacent vertex. So each vertex is connected to n other vertices in $V(\mathbb{Q}_n)$, or $\mathbb{Q}_n$ is n-regular. We know that $\sum_{v \in V(\mathbb{Q}_n)} d(v) = 2\epsilon(\mathbb{Q}_n)$. But each vertex has degree n and we have 2^n vertices, so $n2^n = 2\epsilon(\mathbb{Q}_n)$, or $\epsilon(\mathbb{Q}_n) = n2^{n-1}$.

$\square$

b.

Let

$$X = \{v \in V(\mathbb{Q}_n) : v\text{'s binary string has an even number of 1s}\}$$
$$Y = \{v \in V(\mathbb{Q}_n) : v\text{'s binary string has an odd number of 1s}\}$$

A number is either odd or even, so clearly $V(\mathbb{Q}_n) = X \sqcup Y$. Flipping exactly 1 bit in a binary string changes either a $0 \to 1$ (and the number of 1s increases by one) or a $1 \to 0$ (and the number of 1s decreases by one). WOLOG, the neighbors of $v \in X$, $u_0, u_1, ..., u_k$, differ from v by one bit in their binary sequences. So each neighbor's binary string has 1 less or 1 more number of 1s than v's binary string. But v's binary string has an even number of 1s, so each neighbor has an even number of 1s $\pm$ 1 $\Rightarrow$ each neighbor of v has an odd number of 1s in its binary string, or each neighbor of v is in Y. So $\mathbb{Q}_n$ is bipartite.

$\square$

c.

We have shown that $\mathbb{Q}_n$ is both n-regular and bipartite. By corollary 5.2, $\mathbb{Q}_n$ has a perfect matching for $n > 0$.

$\square$

4.

Consider a game board that is a 6 by 6 grid of squares. Let each square in this grid be represented by a vertex labeled by the sum of its column and row position (e.g. the square at 5, 4 would be called labeled v_9). Each vertex is connected by an edge to each vertex that its square shares an edge with. Let $X = \{$vertices with even label$\}$ and $Y = \{$vertices with odd label$\}$. For all $v_i \in G$, $v_{i\pm1}$ differs by either one row position or one column position. WOLOG, $v_i \in X$ means that i must be even, so $i \pm 1$ must be odd, and $v_{i\pm1} \in Y$. But differing by only one row position or column position means that all $v_{i\pm1}$ must be neighbors of v_i. $\therefore$ Every neighbor of each vertex in X must be in Y. Removing the corner vertices of the game board does not connect any $v \in X$ to any $v \in Y$ (since we are removing a vertex and not connecting anything), so our game board is bipartite. We removed the top left and bottom right square, so we removed v_2 and v_{12}. By counting each $v \in X$ and each $v \in Y$, we have 18 vertices in Y and 16 vertices in X. We know that each domino must occupy 2 adjacent squares. Furthermore, we know that all adjacent squares are of opposite color since our game board is bipartite. So a domino must occupy a black square and a white square, or a vertex in X and a vertex in Y, to be placed on our board. We can match at most 16 vertices in X to 16 vertices in Y, but we still have 2 vertices in Y that remain unsaturated. So our game board cannot be tiled by dominoes.
$\square$