Graph Theory Notes

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Definition: A vertex v is incident to an edge e if v is an end of e. 2 vertices are adjacent if they're connected by an edge. 2 edges are adjacent if they are incident to a common vertex. A graph is planar if it can be drawn in the plane such that no edges cross and an edge only meets its ends.
Definition: An isomorphism between graphs G and H is a pair of bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_g(e) = uv$ if and only if $\psi_h(\phi(e)) = \theta(u)\theta(v)$.
Definition: A graph is *simple* if it has no loops and at most one edge joining a pair of vertices. A graph is *empty* if it has no edges. A simple graph where all vertices are adjacent is a *complete* graph. A graph is *bipartite* if it is simple and there is a partition of the vertices into 2 sets such that every edge in G has one end in X and Y.
Definition: A graph is *complete bipartite* if every vertex in X is adjacent to every vertex in Y. $K_{m,n}$ is the notation.
Definition: The *incidence matrix* is an n x m matrix $M(G)$ such that $M_{i,j}$ is the number of times edge e_j is incident to vertex v_i
Definition: The *adjacency matrix* is the n x n matrix $A(G)$ such that $A_{i,j}$ is the number of edges between v_i and v_j.
Definition: A graph H is a *subgraph* of G ($H \subset G$) if all vertices, edges, and ψ_h are the same.
Definition: The subgraph of G induced by $V' \subset V$, denoted by $G[V']$, is the subgraph with vertices V' and includes every edge with ends in V'.
Definition: The subgraph of G induced by $E' \subset E$, denoted by $G[E']$, is the subgraph with edges E' and includes every vertex that is the end of one of the edges.
Definition: The *Ramsey number*, denoted $r(k, l)$, is the number of vertices of G required such that G contains either a clique or an empty graph on l vertices.
Definition: The *degree* of a vertex is the number of edges incident to v where loops counts twice (number of incident half-edges).
Theorem : $svscode \sum_{v \in V} d(v) = 2\epsilon$
Corollary : In every G, the number of vertices with odd degree is even.
Definition: A graph is $k - regular$ if the degree is always k.
Definition: A *walk* is an alternative sequences of vertices and edges, starting and ending with a vertex. A *trail* is a walk where all edges are distinct. A *path*

is a walk where all vertices are distinct.
Definition: 2 vertices u and v are *connected* if there is a path from u to v in G. G is *connected* if all pairs of vertices are connected.
Fact: u and v have a path between iff there exists a walk between them.
Definition: If $u, v \in G$ are in the same component, then the distance from u to v is $d_G(u, v) = min\{l(P)|P$ is a path from u to v$\}$
Definition: A walk/trail is *closed* if the initial and final vertex are the same.
Definition: An *Euler trail* is a trail that crosses all edges (necessarily once).
Definition: A walk is a *cycle* if it is closed and all vertices are distinct (bar the initial and final).
Theorem : A graph G is bipartite iff every cycle has even length.
Shortest Path Algorithm: Pick two vertices, u and v in a weighted graph G. Consider H a subgraph of G. For each edge e with one end in H and 1 end not in H, we compute $d(u) + w(e)$. For some e with minimal value of $d(u) + w(e)$, add it and its incident vertex and direct it towards u. If this incident vertex is v, stop.
Definition: A connected graph with no cycles is a *tree*. A graph (possible disconnected) without cycles is a $forest$ (acyclic graph).
Theorem : In a tree, any pair of vertices is connected by a unique path.
Theorem : If G is a tree, then $\epsilon = \nu - 1$.
Definition: A $leaf$ is a vertex with degree = 1.
Theorem : An edge e in a graph G is a cut edge iff e is not contained in any cycle of G.
Corollary : A graph G is a forest iff every edge is a cut edge.
Definition: A *spanning tree* $T \subset G$ is a subgraph which is both spanning and a tree.
Theorem : Every connected graph contains a spanning tree.
Corollary : If G is connected and $\epsilon(G) = \nu(G) - 1$, then G is a tree.
Theorem : Suppose T is a spanning tree in G and $e \in \epsilon(G) - \epsilon(T)$. Then $T + e$ has a unique cycle.
Definition: An *edge cut* of a graph is a subset of edges such that $\omega(G - E') > \omega(G)$. We call a minimal edge cut a *bond*.
Theorem : Let T be a spanning tree of a connected graph G. Let $e \in E(T)$. Then $E(G) - E(T)$ contains no edge cut and $E(G) - E(T) + e$ contains a unique minimal edge cut of G.
Corollary : G is connected $\Rightarrow \epsilon(G) \geq \nu(G) - 1$.
Definition: A vertex is a *cut vertex* if $E(G)$ can be partitioned into the subsets E_1, E_2 such that $G(E_1)$ and $G(E_2)$ intersect only at v. If G is loopless and $\nu(G) > 1$, then v is a cut vertex iff $w(G - v) > w(G)$.
Theorem - Cayley's Algorithm: Let $\tau(G)$ be the number of spanning trees in a graph G. Then, if e is not a loop, $\tau(G) = \tau(G \bullet e) + \tau(G - e)$.
Theorem - Cayley's Theorem: $\tau(K_n) = n^{n-2}$.
Theorem - Krustal's Algorithm: We want to find a spanning tree of minimal weight. Let T be all vertices will no edges. Find a non-loop edge of minimal weight and add it to T. If T is a spanning tree, stop. Else, repeat this process such that no cycles form.

Theorem : Krustal's Algorithm produces an optimal tree.
Definition: A *vertex cut* of G is a subset of vertices such that $G - v'$ is disconnected.
Fact: Complete graphs have no vertex cuts.
Definition: The connectivity $\kappa(G)$ is the minimal size of a vertex cut for graphs with ≥ 2 non-adjacent vertices. Otherwise $\kappa(G) = \nu(G) - 1$. Note that if $\nu(G) = 1$ or G is disconnected, $\kappa(G) = 0$. If $\kappa(G) \geq k$, G is k-connected.
Definition: The edge-connectivity $\kappa'(G)$ is 0 if $\nu(G) = 1$ or G is disconnected. Otherwise, $\kappa'(G)$ is the minimal size of an edge cut. If $\kappa'(G) \geq k$, G is k-edge-connected.
Theorem 3.1: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$
Theorem 3.2: A graph G is 2-connected if it has ≥ 3 vertices and any 2 vertices are connected by 2 internally disjoint paths.
Corollary 3.2.1: Suppose $\nu(G) \geq 3$. G is 2-connected iff any 2 vertices of G lie on a common cycle.
Corollary 3.2.2: Suppose $\nu(G) \geq 3$ and G is a block. Any 2 edges of G lie on a common cycle.

0.1 Euler Tours

A walk is a *trail* if it crosses every edge at most once. Recall that an Euler trail is a trail that crosses every edge (necessarily once).

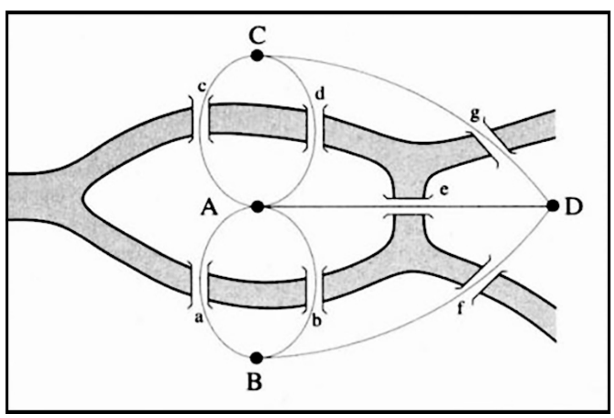

Figure 1: 7 Bridges of Konigsberg Problem

An *Euler tour* is a closed Euler trail. A graph G is *Eulerian* if it contains an Euler tour.
Theorem 4.1: A Graph G is Eulerian iff it is connected and every vertex has even degree.
Lemma: If H is a nontrivial connected graph where each vertex has even degree, then for all vertices in H, H has a closed trail (of positive length) starting and

ending at v.
Proof: Construct this trail W as follows: Start at v. Since H is nontrivial and connected, there is some edge e_1 incident to v and some vertex v_1. Now, $W = ve_1v_1$. Having constructed $W = ve_1v_1...e_nv_n$, add an edge e_{n+1} incident to v_n not in W and its other end to W if you can. Stop when you can no longer do this.
Claim : We stop when W's last vertex is v.
Proof : Suppose we've stopped constructing W. Let $u = v_n$. Every time $v_i = u$ $(i \neq n)$, W crossed an even number of half edges incident to u. If $u \neq v$, then we also crossed exactly 1 more half edge at the end of W. Thus W crossed an odd number of half edges at u. But $deg(u)$ is even, so there's another edge we can add to W. So we didn't stop here. This is a contradiction. $\therefore u = v$

□

Proof of Theorem 4.1: $(\Rightarrow)$ G is Eulerian if it has an Euler tour (closed walk with each edge once). Left as exercise.
$(\Leftarrow)$ Let W be a maximal closed trail in G. Suppose W is not an Euler tour. Let $G' = G - E(W)$. This has some edges in it. As discussed before $\forall v \in V(G)$, E(W) contains an even number of half edges incident to v. Thus, every vertex in G' has even degree. Let G'_0 be some component in G'. Since G is connected, some vertex of G'_0 is a vertex v_i in W (Why? Connect v_0, a vertex in G'_0, to a vertex in G by some path and take the last vertex in that path which is in W.) By Lemma, there exists a closed trail W' in G'_0 starting and ending at v. Then, paste W' into W = W''. W' used no edges in W. SO W'' is a longer closed trail. This is a contradiction as W is maximal. $\therefore$ W has every edge.

□

Corollary 4.1: A graph contains an Euler Trail iff it has 0 or 2 vertices of odd degree in G.

0.2 Hamilton Cycles

A *Hamilton Path* is a path which contains all vertices of G. A *Hamilton Cycle* is a cycle which contains all vertices of G. Unfortunately, checking whether or not a graph is Hamiltonian is Hell for computers to check.
We will see some necessary conditions for being Hamiltonian (i.e. it holds if it is Hamiltonian) and sufficient conditions (i.e. if it holds then G is Hamiltonian).

Theorem 4.2: If G is Hamiltonian, then for each nonempty proper subset $S \subset V(G)$, $\omega(G - S) \leq |S|$.
Proof: Let C be a Hamiltonian cycle in G. Then for each $S \in V(G)$, $\omega(C - S \leq |S|$. Since C contains all V(G), $V(C - S) = V(G - S)$. Thus, if components of $C - S$ are $C_1, C_2, ..., C_n$, $G - S = C - S +$ some edges, so we many connected some C_i together but no new components appear. $\therefore \omega(C - S) \leq n = \omega(C - S) \leq |S|$.

$\square$

Theorem 4.3: If G is simple with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, then G is Hamiltonian.
Proof: Suppose that there is some graph G with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$ that is not Hamiltonian. Whatever G is, it can't be complete since complete graphs are Hamiltonian. So let G be a graph with ν vertices and is maximal with respect to edges, with $\delta \geq \frac{\nu}{2}$ and is not Hamiltonian. I.e. G+e does satisfy the theorem. Pick 2 non-adjacent vertices $u, v \in V(G)$. Then the graph G+uv must be Hamiltonian by maximality of G. But since G is not Hamiltonian, every Hamilton cycle in G+uv must cross uv. Deleting uv from the cycle gives us a Hamilton path P from u to v.
Define $S = \{v_i | v_{i+1}$ is adjacent to $u\}$ in G. Define $T = \{v_i | v_i$ is adjacent to $v\}$ in G.
Notice that v is in neither S nor T. So $S \cap T$ is empty. So W is a Hamilton cycle in G. But G has no Hamilton cycle! This is a contradiction. But d(u) + d(v) $= |S| + |T| = |S \cup T| - |S \cap T| = |S \cup T| \leq \nu(G) - 1$. This contradicts $\delta(u) + d(v) = \delta(G) + \delta(G) = \nu(G)$. Contradiction!

$\square$

Lemma: Suppose G is simple and u and v are non-adjacent vertices such that $d(u) + d(v) \geq \nu(G)$. Then G is Hamiltonian iff G + uv is Hamiltonian.
Proof: Same argument as previous proof.

$\square$

0.3 Closure

The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices u,v, satisfying $d(u) + d(v) \geq \nu(G)$ until all are gone. The graph created we call $C(G)$.
Lemma 4.4.2: $C(G)$ is well defined- no matter what choices you make, you get the same result.
Proof: G is a graph. Make graphs H, H' by applying the procedure but in different ways.
$H = G + \{u_1v_1, u_2v_2, ..., u_kv_k\}$. $H' = G + \{u_1'v_1', u_2'v_2', ..., u_k'v_k'\}$ (u_iv_i are two nonadjacent vertices in $G + \{u_1v_1, u_2v_2, ..., u_kv_k\}$ with $d(G + \{...\}(u_i) + d(G + \{...\}(v_i) \geq \nu(G)$. This also true for H'.
Observe that if $d_H(u) + d_H(v) \geq \nu(G)$ then u and v are adjacent in H. If not, the process would have added the edge. This is also true for H'. By induction, $H \subset H'$ (symmetric shows $H' \subset H$).
Base Case: u_1v_1 is in H'. $d_{H'}(u_1) + d_{H'}(v_1) \geq d_G(u_1) + d_G(v_1) \geq \nu(G)$ (since it was added in process for H). By observation, $u_1v_1 \in E(H')$.
Inductive Step: $\{u_1v_1, u_2v_2, ..., u_{i-1}v_{i-1}\} \in E(H') \Rightarrow u_iv_i \in H'$. So $d_{H'}(u_i) + d_{H'}(v_i) \geq d_{G+\{...\}(u_i} + d_{G+\{...\}(v_i)} \geq \nu(G)$ since we added it to the graph in the other process. By observation, $u_iv_i \in H'$. Induction tells us $\{u_1v_1...u_kv_k\} \leq H'$. So $H \subset H'$. Symmetric argument shows $H' \subset H$. So H = H'.

$\square$

0.3.1 Chinese Postman Problem

Edges = Streets (edges have positive weight). Vertices = intersections. Find a shortest closed walk that crosses all edges If G is Eulerian (every degree is even) then ever tour is optimal.

0.3.2 Traveling Salesman Problem

Given graph G with positive edge weights, find a Hamilton cycle with the least weight.
Easier Problem: Find a minimum weight Hamilton cycle in a weighted complete graph. There exists an algorithm which approximates an optimal solution. A simplified version of an algorithm: Start with some Hamilton cycle in a complete graph

$$C = v_1 v_2 ... v_n v_1$$

Look at all i, j such that $1 < i + 1 < j < \nu$ and compare $w(v_i v_{i+1}) + w(v_j v_{j+1})$ to $w(v_i v_j) + w(v_{i+1} v_{j+1})$. If the latter is smaller for some i, j, then do the following: C becomes $C_2 = v_i ... v_j v_{j-1} ... v_{i+1} v_{j+1} v_\nu v_i$ and the new weight of C_2 is $w(C) + w(v_i v_j) + w(v_{i+1} v_{j+1}) - w(v_i v_{i+1}) - w(v_j v_{j+1})$, which is strictly less than $w(C)$. Repeat this process until you can no longer do so.

0.4 Matchings

Definition: A subset $M \subset E(G)$ is a matching if $E(G)$ has no loops and no 2 edges in M are adjacent. *(When we consider matchings, assume G is always simple!)* A matching M *saturates* v or v is *M-saturated* if v is an end of some edge in M. M is a *perfect matching* if every vertex is M-saturated. M is a *maximum matching* if its a matching of biggest size.
An M-alternating path is a path where edges alternate between M and $E - M$. An M-alternating path is an M-augmenting path if the starting and ending vertices are M-unsaturated.
Theorem 5.1: A matching M in G is a maximum matching iff G contains no M-augmenting path.
Proof: Suppose there exists some M-augmenting path in G called

$$P = v_0 v_1 v_2 ... v_{2m} v_{2m+1}$$

The edges $\{v_0 v_1, v_2 v_3, ..., v_{2m} v_{2m+1}\}$ are clearly not in M.

$$\{v_1 v_2, v_3 v_4, ..., v_{2m-1} v_{2m}\} \subset M$$

So $M' = M - \{v_1, v_2, ..., v_{2m-1} v_{2m}\} \cup \{v_0 v_1, ... v_{2m} v_{2m+1}\}$ is a new matching. $\{v_0 v_1, ... v_{2m} v_{2m+1}\}$ are unsaturated in $M - \{v_1 v_2, ..., v_{2m-1} v_{2m}\}$. So throwing in the other edges is a matching for $v_0, v_1, v_2, ..., v_{2m+1}$. But $|M'| = |M| - m + m + 1 = |M| + 1$. So M is not maximal. ($\Leftarrow$) $\Box$
($\Rightarrow$) Suppose M is not maximal, i.e. there exists a matching M' with $|M'| > |M|$. Let $H = G[M \triangle M']$ where $M \triangle M' = M \cup M' - (M \cap M')$. Each vertex in H

has degree 1 or 2 since every vertex is an end of at most 1 edge in M and 1 edge in M'. The components of H are cycles and paths. In fact, each component of H is one of the following:

- An alternating cycle with edges alternating between M and M'
- A path with edges alternating between M and M'

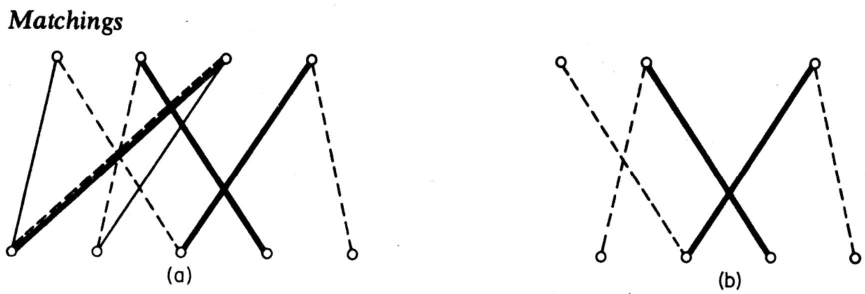

Figure 5.2. *(a) G, with M heavy and M′ broken; (b) G[M ΔM′]*

In the first possibility, the component has just as many M edges as M' edges. Since $|M'| > |M|$, $|M' - (M \cap M')| > |M - (M \cap M')|$. Therefore H has some component C which is a path with more M' than M edges.
$\Rightarrow$ C must both start and end with M' edges. So C's ends are M' saturated in H. So C's are M saturated in G. So C is an augmenting path in G.

$\square$

Definition: For $S \subset V(G)$, the neighbor set of S, $N_G(S)$, is the set of vertices adjacent to at least 1 element of S.
Theorem 5.2: Let G be a bipartite graph. Then G contains a matching that saturates X iff $|N_G(S)| \geq |S|$.
Proof: ($\Rightarrow$) Let M be a matching that saturates X. Let $S \subset X$. Then every $v \in S = \{v_1, v_2, ..., v_n\}$ is an end of some edge e_i in M. Since G is bipartite, the other end u_i of e_i is in Y. Because M is a matching, all u_i are distinct. But $\{u_1, u_2, ..., u_n\} \subset N(S)$. So $|N(S)| \geq n = |S|$.
($\Leftarrow$) Suppose no matching saturates X. Let M be a maximum matching. Let $u \in X$ that is unsaturated. There is no M-augmenting path since every M-alternating path that starts at u must end in an M-saturated vertex. Let Z be all vertices connected to u by an M-alternating path. Then u is the only M-unsaturated vertex in Z. Let $S = X \cap Z$ and $T = Y \cap Z$. Each vertex in T is matched by M to some vertex in S. Thus, $T \subset N(S)$. We also know that $N(S) \subset T$ since if u has an alternating path to $v \in S$, then every $v' \in N(v)$ is either in P (so $v' \in T$) or the last edge in P taken to v is in M, so the edge from v to v' is not in M, and we can extend P to a longer alternating path ending at v'. Thus $T = N(S)$. T is matched to 1 vertex in S, but u is not M-saturated. So $|N(S) = |T| \leq |S| - 1 < |S|$. Contradiction!

$\square$

Corollary 5.2: If G is k-regular and bipartite with $k > 0$, then G has a perfect matching.
Proof: Let G be a graph with bipartition X, Y. Since G is bipartite and k-regular, $\sum_{v \in X} d(v) = |E(G)| = k|X| = k|Y|$. So $|X| = |Y|$. Let $S \subset X$ and let E_1 be all edges incident to S and E_2 be all edges incident to the neighbors of S, $N(S)$. $E_1 \subset E_2$ so $|E_2| = k|N(S)| \geq |E_1| = k|S|$. $|N(S)| \geq |S|$. This works for all S. So by theorem 5.2, there exists a matching M saturating X. Every edge in M has exactly 1 $v \in X, Y$. So M saturates the same number of X vertices and Y vertices. But $|Y| = |X|$, so M saturates all vertices and is a perfect matching.

$\square$

0.5 Coverings

Definition: A *cover* of a graph G is a set $K \subset V(G)$ such that every edge is incident to some vertex in K.
Observation: By the nature of a cover K and a matching M, $|M| \leq |K|$. Why? Every $e \in M$ is incident to at least one vertex $v \in K$, but no other $e' \in M$ is incident to this same vertex. For maximum matching M^* and minimum cover K', $M^* \leq K'$.
Theorem 5.3: In a bipartite graph, $|M^*| = |K'|$.
Lemma 5.3: If M is a matching, K is a cover, and $|M| = |K|$, then M is a maximum matching and K is a minimum cover.
Proof: Let M^* be a maximum matching and K' a minimum cover. Then, $|M| < |M*| \leq |K'| \leq |K| = |M|$ so $|M| \leq |M^*| \leq |M| \to |M|$ and $|K| \leq |K'| \leq |K| \to |K'| = |K|$.

$\square$

Proof of Theorem 5.3: Let G be bipartite with bipartition X,Y and maximum matching M^*. Let $U \subset X$ be all M^*-unsaturated vertices in X. Suppose $U \neq \emptyset$. Let Z be all vertices in G reachable by an M^*-alternating path which starts at some vertex in U). Let $S = U \cap X$ and $T = U \cap Y$. Observations: In every M^*-alternating path with P starting at some $u \in U$,

1. P starts with a non-$M*$ edge
2. Every $x \in X - U$ in P is preceded by an M^* edge.
3. Every $y \in Y$ in P is preceded by a non-M^* edge.
4. Every $y \in Y$ in P is $M*$-saturated (if this were not true, then we would have an M^*-augmenting path, contradicting maximality of M^*.

Consequently,

1. All T vertices are M^*-saturated

2. $N(S) \subset T$: if $u \in U \subset S$, then any edge from u is an M^*-alternating path so $N(U) \subset T$. If $v \in S - U$ then it has one neighbor along an M^* edge and no neighbor along that edge is in T. We can reach all other neighbors of v by extending the alternating path to v by connecting edge because of (2).

3. $T \subset N(S)$: We start alternating path in $U \subset X$ so every T vertex is preceded in alternating path by some $v \in V$ which must be in S.

Thus $N(S) = T$. Let $K' = (X - S) \cup T$. K' is a cover because every edge has an end in X and if e has an end in S, it also has an end in T.
Now, we just need to show every M^* edge is incident to exactly one K' vertex. Notice all $X - S$ vertices are M^*-saturated since $U \subset S$. The only way for $e \in M^*$ to be incident to two K' vertices is if one end is in $X - S$ and one end is in T. This is impossible using observation (3). An alternating path from U to T can be extended to v, but then $v \in S$. Contradiction! Proves the claim. By lemma 5.3, we are done.

□

Definition: A component of a graph is called *odd* if it has an odd number of vertices. Otherwise it is called *even*.
Theorem 5.4 - Tutte's Theorem: Let $o(G)$ be the number of odd components in G. G has a perfect matching iff $o(G - S) \leq |S| \; \forall S \subset V(G)$.
Corollary 5.3: Every 3-regular graph without a cut vertex has a perfect matching.
Proof: It suffices to prove this for simple graphs.
($\Rightarrow$) Suppose G has a perfect matching M. Let $S \subset V(G)$. Take a component H of $G - S$. If every vertex of H is matched in with another vertex of H by M, then H is even. If $G_1, G_2, ..., G_n$ are odd components of $G - S$, then for each $e \in M$, some vertex $v_i \in V(G_i)$ is matched by M with some vertex $u_i \in S$. Since M is a matching, u_i are distinct, so $o(G - S) = |\{u_1, ...u + n\}| \leq |S|$.

($\Leftarrow$) Conversely, suppose that G satisfies $o(G - S) \leq |S| \; \forall S \subset V(G)$. Suppose G does not have a perfect matching.
Step 1: Add edges to G to get some simple H, where H satisfied our condition, and H is maximal with respect to not having a perfect matching (i.e. H plus any edge gives us a perfect matching). Note that we are assuming G is simple, and in this proof, we are assuming all graphs are simple. Notice that if $G' = G + e$ $\forall$ new edge e, then $o(G' - S) = o(G - S) \leq |S|$ because $G' - S = G - S$ (i.e. e is in S) or:

Case 1) e connects 2 odd components means that we get 1 even component and $o(G' - S) < o(G - S) \leq |S|$.

Case 2) e connects 2 even components which means $o(G' - S) = o(G - S) \leq |S|$.

Case 3) e connects an even and an odd component which means $o(G' - S) = o(G - S) \leq |S|$.

Observe that if $S = \emptyset$, then $o(G - S) = o(G) \leq |S| = 0$, so G has no odd components and an even number of vertices. Keep adding edges to G, which doesn't make a perfect matching, until you can't. Call this graph H. Our goal is to show that H can't exist, i.e. Show H can't have a perfect matching, but we assume H has a perfect matching. Let $U \subset V(H)$ be the set of vertices adjacent to all other vertices, $U = \{u \in V(H) | d_H(u) = \nu(H) - 1$.

Step 2: Show that the components of H-U are complete. Suppose they are not, i.e. there exists a component of H-U with vertices x,y,z such that $xy, yz \in E(H)$ but xz is not in $E(H)$ (H is simple, so a component of H-U is simple but also not complete, so we need at least 3 vertices which are not connect but not all adjacent). Since y is not in U, there exists some $w \in V(H)$ such that yw is not in $E(H)$. Remember that H has a property that $H + e$ has a perfect matching. By the maximality of H, $H + xz$ and $H + yw$ have perfect matchings, M_1 and M_2. Let $H' \subset H + xz + yw$ be a graph produced by $M_1 \triangle M_2$. Then every vertex in H' has degree 2. Every vertex $v \in V(H + xz + yw)$ is incident to exactly $m_1 \in M_1$ and $m_2 \in M_2$; if $m_1 = m_2$, then v is not in $V(H')$. If $m_1 \neq m_2$, then $d_{H'}(v) = 2$.

Notice that $xz \in M_1 - M_2$, $yw \in M_2 - M_1$ (or else M_1 and M_2 would not be perfect matchings). Since $d_{H'}(v) = 2$ for all vertices in H', H' is a disjoint union of cycles. Also, $xy, yw \in E(H')$.

Case 1: xz and yw are in different cycles/components of H'. Say $xz \in C_1$, $yw \in C_2$, of cycle C_i, are of even length and alternate between M_1 and M_2. Take M' to be M_1 edges outside C_1 and M_2 edges inside C_2.

Case 2: xyz, wy are in the same component of H'. We pick edges and yz to get a perfect matching.

Claim: H has a perfect matching. Let $B_1, ..., B_n$ be the odd components of $H - U$ and $C_1, ..., C_m$ be the even components. For C_i, C_i is complete with an even number of vertices, so we can easily find a perfect matching in each C_i, M_i. For B_i, B_i is complete with an odd number of vertices, so we can find a matching N_i in B_i saturated all but one vertex b_i. Since $|U| \leq o(H - U) =$ size of $\{b_1, ..., b_n\}$, there exists at least one vertex in U for every v, we can find a subset $\{u_1, ..., u_n\} \in U$ of size n. It is obvious that $U - \{u_1, ..., u_n\}$ is even number. Since U is complete, there exists a matching M_{max}. Set $U - \{u_1, ..., u_n\}$. So $M =$Union of the M_i Union of the UN_i Union of the M_{max} Union of the $b_i u_i$ is a perfect matching of H. This contradicts that H has no perfect matching.

□

Proof of Corollary 5.4 Let G be such a graph. Let $S \subset V(G) be a proper subset.$ Let $G_i ... G_n$ be odd components of $G - S$. Let m_i be the number of edges with one end in G_i and one end in S. By 3-regularity, the sum of the degrees of the vertices in each components is 3 times the number of vertices in each component (*), and the sum of the degrees of S is the size of S. By (*), $2\epsilon(G_i)$ is the sum of the degrees of the component minus m_i, so $m_i = \sum_{v \in V(G_i)} d(v_i) - 2\epsilon(G_i)$ is odd. Since all edges going out of G_i must go to S and there are no cut edges, m_i must be ≥ 3. So we get

$$o(G - S) = n = \frac{1}{3}(3n) \leq \sum_{i=1}^{n} m_i \leq \frac{1}{3} \sum_{v \in S} d(v) = \frac{1}{3}(3|S| = |S|.$$

By theorem 5.4, G has a perfect matching.

$\square$

Algorithm to find a perfect matching in a bipartite graph: With n employees and n tasks, each employee can do some subset of tasks. The *Hungarian Algorithm* presents the following idea: Given some non-perfect matching M, there is some M-unsaturated $u \in X$, and we construct an M-alternating tree rooted at u to find an M-augmenting path or $|S| \subset X > |N(S)|$.
Definition: For a bipartite G, $H \subset G$ is an M-alternating tree rooted at u if its a tree and every (u-v)-path is M-alternating.
Observe that in an M-alternating tree rooted at u, every path from u to $x \in X$ has last edge in M.
M-alternating tree subroutine: Given M and $u \in X$ unsaturated:

1. Start with H = $\{u\}$.
2. Look for $y \in Y - V(H)$ adjacent to some $x_0 \in V(H) \cap X$ via an edge e. If no such y exists, then $S = X \cap V(H)$ has the property that $|S| > |N(S)|$, so no perfect matching. Else, add y and e to H.
3. If y is M-unsaturated, then $u - y$ path in H is augmenting. Output the path.
4. If y is M-saturated and is paired by some $e' \in M$ with some $x \in X - V(H)$. Add x, e' to H ans return to step 2.

Things to Check:

- H is always M-alternating tree rooted at u. We just want if H starts as such then it remains so as we do steps 2-4. In step 2, we add an edge from $x_0 \in V(H) \cap X$ to some $y \in Y - V(H)$ and add y to H.
- In the original H, the $(u - x_0)$-path has last edge in M and previous vertex was in H, so the new y is not paired to x by an M-edge. So e is not in M, and our $(u - y)$-path is alternating. So H has $\epsilon = \nu - 1$ and we increment ϵ and ν by one each time, so it holds, H is still a tree.

- Notice that before step 2, for every x in V(H), x is M-saturated and the edge it is incident to in M is an edge in H. As we go through step 4, the new y vertex is incident to a new x vertex via an M-edge not in H. So still a tree, and since the (u-y) path ended in a non-M-edge, the (u-x) path will be M-alternating.
- H starts with 1 x-vertex. After running through 2-4, we add 1 x and 1 y vertex to H. When we get to the beginning of step 2, the number of X vertices in H = the number of Y + 1. If we can't find an $y \in Y - V(H)$ adjacent to a vertex to a vertex in $X \cap V(H)$, then the neighbors of $X \cap V(H)$ is already in $H \subset V(H) \cap Y$.
- We found an $S \subset X$ with $N(S) \subset V(H) \cap Y$ and $|N(S)| \leq V(H) - Y| < |S|$.

Full algorithm:

- A. Start with a matching.
- B. If M is perfect, stop.
- C. Else, run subroutine for some M-unsaturated vertex u in X.
- D. If subroutine stops at step 2, no perfect matching. STOP.
- E. Else the subroutine outputs an M-augmenting path P. Replace M with $M \triangle E(P)$ which is a larger matching.
- F. Go to (B).

0.6 Edge Colorings

Definition: A $k-$*edge coloring* of a loopless graph G is an assignment of "colors" $1, 2, 3, ..., k$ to the edges of G. I.e. label each edge with $\{1, 2, 3, ..., k\}$. The coloring is *proper* if no two adjacent edges have the same color.
Definition: G is $k-$*edge colorable* if G has a proper k-coloring. The *edge chromatic number* $\chi'(G)$ is the smallest k for which G is k-edge-colorable. Also, we could say a coloring C as a part. of $E(G)$ by $E_i = \{e \in E(G) | e \text{ color is i}\}$.

$$E(G) = E_1 \cup E_2 \cup ... \cup E_k$$

We denote C as $\{E_1, E_2, ..., E_k\}$. Observe that for C a proper coloring, $\forall e_1, e_2 \in E_i$ with $e_1 \neq e_2 \Rightarrow e_1, e_2$ are not adjacent. So E_i is a matching. Recall that $\Delta(G)$ is the maximum degree of any vertex in G.
Observe that $\chi'(G) \geq \Delta(G)$ as there exist $|\Delta(G)|$-many edges incident to the same vertex.
Theorem 6.1: If G is bipartite, $\chi'(G) = \Delta(G)$.
Proof: If G is k-regular, $k > 0$, then we have k perfect matchings. So we

construct a supergraph H of G where H is k-regular, $k = \chi'(G)$. We construct H.
Start with G. Add vertices to X if $|X| < |Y|$ or add vertices to Y if $|Y| < |X|$ until $|X| = |Y|$. We have $H_0 = G+$ new vertices. G is bipartite, so the total degree of X in H_0 = total degree of Y in H_0 = number of edges in H_0.

Since $d_{H_0}(v) \leq k$, either $\sum_{v \in V(X)} d_{H_0}(v) = \epsilon(H_0) = \sum_{v \in V(Y)} d_{H_0}(v)$ and $d_{H_0}(v) = k$, or $d_{H_0}(v) < k$ for some $v \in V(X)$ or $d_{H_0}(v) < k$ for some $v \in V(Y)$. If H_0 is k-regular, we are done. If H_0 is not k-regular, then there exists some $x \in X, y \in Y$, with $d_{H_0}(x) < k$, and $d_{H_0}(y) < k$. Add an edge from x to y and call the graph H_1. If H_1 is k-regular, then stop. Otherwise, repeat this process until you get a k regular graph H. By a homework problem, H has k-disjoint perfect matchings, $E_1, E_2, ..., E_k$. Since E_i is perfect, $|E_i| = \frac{1}{2}|V(H_0)| = |X|$. So $\sum_{i=1}^{k} |E_i| = k|X| = |E(H)|$, so C is a proper coloring of H and is thus a proper coloring of G.

$\square$

Theorem - Vizing's Theorem: If G is simple, then its edge-chromatic-number is either $\Delta(G)$ or $\Delta(G) + 1$.
Time Tabling Problem: m teachers need to teach n classes P_{ij} times a week. Our goal is to teach all classes or sections in as few of periods as possible. Each period is a matching, so a time table is a coloring of a bipartite graph. One more consideration is the number of rooms available. If ϵ is the total number of classes, can you find a time table where in each period, at most the ceiling of $\frac{\epsilon}{p}$ classes are being taught? Yes!
Lemma 6.3: If M,N are disjoint matchings of a graph G with $|M| > |N|$, then there are disjoint matchings M', N' such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and $M \cup N = M' \cup N'$.
Proof: Let H = $G[M \cup N]$. Every $v \in V(H)$ has degree 1 or 2. Every component of H is a path or a cycle, and in any component, the edges alternate between M and N. Since $|M| > |N|$, some component is a path

$$P = v_0 e_1 v_1 e_2 v_2 ... e_{2k+1} v_{2k+1}$$

where $\{e_i | i \text{ odd}\} \in M$ and $\{e_i | i \text{ even}\} \in N$. Let $M' = M - \{e_i | i \text{ odd}\} \cup \{e_i | i \text{ even}\}$ and $N' = N - \{e_i | i \text{ even}\} \cup \{e_i | i \text{ odd}\}$. In a sense, we swap the M edges in P with the N edges in P. Observe that $|M'| = |M| - 1$ and $|N'| = |N| + 1$. M', N' are matching because no M,N edges in P are incident to vertices in P.

$\square$

Theorem 6.3: Let G be bipartite and $p \leq \Delta(G)$. Then there exist p disjoint matchings $M_1, M_2, ..., M_p$ such that $E(G)$ =Union of the $M_i s$ and

$\lfloor \frac{\epsilon}{p} \rfloor \leq M_i \leq \lceil \frac{\epsilon}{p} \rceil$.

Proof: By theorem 6.1, G is Δ-colorable, so we can find matchings $M_1, M_2, ..., M_p$ where $E(G)$ =Union of the M_is. If the second part of our statement is false, then one M_i is too large and one M_j is too small. Use lemma 6.3 to find M_i', M_j' with M_i' one smaller and M_j' one larger. Repeat until the statement holds.

$\square$

Independent Sets and Cliques

Recall that $S \subset V(G)$ is an *independent set* if no 2 vertices in S are adjacent in G. It is maximum if it is of maximum size.
Theorem 7.1: $S \subset V(G)$ is an independent set iff $V(G) - S$ is a covering.
Proof: S is an independent set $\Rightarrow$ Any edge incident to an S vertex is incident to a non-S vertex $\Rightarrow$ $V - S$ is every edge, a covering.

$\square$

Equivalently, $V - S$ a covering $\Rightarrow$ every edge is incident to some $V - S$ vertices $\Rightarrow$ No edge is incident to only S vertices $\Rightarrow$S is independent.

$\square$

$S \subset V(G)$ independent set means that all v are non-adjacent in S. $S \subset V(G)$ clique means that all v adjacent in S.
Let $\alpha(G)$ be the *independence number*, the maximum size of an independent set.
Let $\beta(G)$ be the *vertex covering number*, the minimum size of a vertex covering.
Corollary 7.1: $\alpha(G) + \beta(G) = \nu(G)$
Proof: Let S be a maximum independent set and K be a minimum covering. Then $\nu(G) = |S| + |\nu - S| \geq |S| + |K| = \alpha + \beta$. But $\nu(G) = |K| + |\nu - K| \leq |K| + |S| = \alpha + \beta$. So $\nu(G) = \alpha + \beta$.

$\square$

$L \subset E(G)$ is an *edge covering* if every $v \in V(G)$ is incident to some edge in L. An edge covering exists iff $\delta(G) > 0$.

$\alpha'(G)$ is the *edge independence number*, the maximum size of a matching in G.
$\beta(G)$ is the *edge covering number*, the minimum size of an edge covering.

M matching does not imply M^C a covering.
L covering does not imply L^C a matching.

Theorem 7.2: If $\delta(G) > 0$, then $\alpha' + \beta' = \nu$.
Proof: Let M be a maximum matching in G. Let U be all M-unsaturated vertices. Since $\delta > 0$ and M is maximum, each $u \in U$ is only incident to edges

whose other end is M saturated, else M is not maximum. Thus there exists a set $E' \subset E(G)$ of $|U|$ edges where each $u \in U$ is incident to an edge in E' and $E' \cap M$ is empty. This is an edge covering, so $\beta' \leq |M \cup E'| = |M| + |E'| = \alpha' + \nu - 2\alpha' = \nu - 2\alpha'$. So $\alpha' + \beta' \leq \nu(G)$.

Now, let L be a minimum edge covering, $H = G[L]$, M a matching on H. Let $U \subset V(H)$ be M-unsaturated vertices. Since M is maximum, $H[U]$ has only loops and no other edges, so each $u \in U$ is incident to some edge $e \in L - M$ that no other vertex in U is incident to. Therefore $|L| - |M| = |L - M|$ since $M < L$. We get $|L - M| \geq |U| = \nu - 2|M| \Rightarrow |L| + |M| \geq \nu \Rightarrow \alpha' + \beta' \geq \nu$, so $\alpha' + \beta' = \nu$.

□

Ramsey Theory

For this section, assume all graphs to be simple.
Basic Idea: If a graph doesn't have a "large" independent set, then it has many edges and should have a "large" clique.
Definition: $r(k,l)$ is the smallest positive integer such that if G is simple and has $\nu \geq r(k,l)$, then either: G has a k clique or an l independent set.

Observe that $r(k,l) = r(l,k)$. If G has a k-clique, then G^C has a k-independent set. $r(1,l)$ is $r(k,1) = 1$. $r(2,l) = l$, $r(k,2) = k$.
Theorem 7.4: For all $k, l \geq 2$, $r(k,l) \leq r(k-1,l) + r(k,l-1)$. Moreover, if $r(k-1,l)$ and $r(k,l-1)$ are even, then $r(k,l) < r(k-1,l) + r(k,l-1)$.
Proof: Let G be a graph with $\nu(G) = r(k-1,l) + r(k,l-1)$. Let $v \in V(G)$. Then, either

1. v is non-adjacent to all vertices in a set S of at least $r(k,l-1)$ vertices
2. or there exists a set S with $\nu(S) \geq r(k-1,l)$ and v is adjacent to all vertices.

So $d(v)+$ the number of vertices non-adjacent to v $= r(k-1,l) + r(k,l-1) - 1$.
Case 1: Either $G[S]$ has a k-clique or an $l-1$ independent set $S' \subset S$. Then $S'v$ is an l-independent set in G.
Case 2: Either $G[S]$ has a $k-1$ clique or an l independent set $S' \subset S$. Then $S'v$ is a k-clique in G. In either case, G has a k-clique or an l-independent set.

Even Part: Let G be a graph with $\nu(G) = r(k-1,l) + r(k,l-1) - 1$. G has an odd number of vertices, so some vertex has even degree (since $2\epsilon =$ sum of $d(v)$). So either v is adjacent to $\geq r(k-1,l)$ vertices or v is adjacent to $\leq r(k-1,l) - 2$ vertices. This implies that v is non-adjacent to $\geq r(k,l-1)$ vertices. Then run the same argument.

□

Theorem 7.5: $r(k,l) \leq \binom{k+l-2}{k-1} = \frac{(k+l-2)!}{(k-1)!(k+l-2-k+1)!} = \frac{(k+l-2)!}{(k-1)!(l-1)!}$.
Proof:
Base Case:
$r(k,1) = 1$, $\binom{k+1-2}{k-1} = \frac{(k-1)!}{(k-1)!(0)!} = 1 \Rightarrow$ our statement holds.
$r(1,l) = 1$, $\binom{1+l-2}{0} = \frac{(l-1)!}{(0)!(l-1)!} = 1 \Rightarrow$ our statement holds. (Also check for $r(k,2)$ and $r(2,l)$).

Inductive Step: Assume $r(k,l) \leq \binom{k+l-2}{k-1}$. If $k+l = n$, show it for $k+l = n+1$. Take $k+l = n+1$ ($k,l \geq 3$). Then, by theorem 7.4 and induction, $r(k,l) \leq r(k-1,l) + r(k,l-1) \leq \binom{k-1+l-2}{k-1} + \binom{k+l-1-2}{k-2} \leq \binom{k+l-3}{k-1} + \binom{k+l-3}{k-1} \leq \binom{k+l-2}{k-1}$.

$\square$

Theorem 7.6: $r(k,k) \geq 2^{k/2}$
Proof: We see from theorem 7.5 that $r(k,k) \leq \binom{2k-2}{k-1} = \frac{(2k-2)!}{(k-1)!(k-1)!} = \frac{(2k-2)(2k-3)...(k+1)(k)}{(k-1)(k-2)...(3)(2)(1)} \geq 2^k$. We see that $r(1,1) = 1$ and $r(2,2) = 2$. We assume $k \geq 3$. Let $\mathbb{G}_\ltimes$ be the set of all simple graphs on the vertex set $\{v_1, ..., v_n\}$ (Alternatively, all subgraphs of complete graphs on $\{v_1, ..., v_n\}$). It is clear that $|\mathbb{G}_\ltimes| = 2^{\binom{n}{2}}$. Let $\mathbb{G}_\ltimes^\urcorner \subset \mathbb{G}_\ltimes$ all subgraphs of K_n with a k-clique. The number of graphs with a k-clique on $\{v_1, v_2, ..., v_k\}$ is $2^{\binom{n}{2}-\binom{k}{2}}$. There are $\binom{n}{k}$ subsets of $V(K_n)$, so $|\mathbb{G}_\ltimes^\urcorner| \leq \binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}}$, and $\frac{|\mathbb{G}_\ltimes^\urcorner|}{|\mathbb{G}_\ltimes|} \leq \frac{\binom{n}{k}2^{\binom{n}{2}-\binom{k}{2}}}{2^{\binom{n}{2}} < \frac{n^k 2^{-\binom{k}{2}}}{k!}}$. Suppose $n < 2^{k/2}$, then $\frac{\mathbb{G}_\ltimes^\urcorner}{\mathbb{G}_\ltimes} < \frac{2^{k^2/2}2^{-\binom{k}{2}}}{k!} = \frac{2^{k/2}}{k!} < 1/2$.
The number of graphs with k-independent sets is $\mathbb{G}_\ltimes^\urcorner$ since there exists a bijection between graphs with a k-clique and graphs with k-independent sets via completeness. I.e the number of graphs with a k-clique or k-independent set divided by $\mathbb{G}_\ltimes$ is $2\frac{\mathbb{G}_\ltimes^\urcorner}{\mathbb{G}_\ltimes < 2 \bullet \frac{1}{2} = 1.}$ So not all graphs in $\mathbb{G}_\ltimes$ have a k-clique or k-independent set. So our statement holds.

$\square$

Corollary I: f $m = min(k,l)$, then $r(k,l) \geq 2^{m/2}$ and $r(k,l) \geq r(m,m)$.
Another interpretation of $r(k,l)$: We can thing of a simple $G \subset K_n$. We can edge color (improperly) the edges of K_n as $C = \{E(G), E(K_n) - E(G)\}$. Then $r(k,l)$ is the smallest n such that for any 2 edge colorings of $K_n = \{E_1, E_2\}$, either E_1 has a k clique or E_2 has an l clique. We can generalize this to m colors, i.e. $r(k_1, k_2, ..., k_m$ is the smallest number such that if $n \geq r(k_1, ..., k_m)$, then for each m-edge-coloring of K_n, one of the E_i has a k_i clique. One can prove via recursion that $r(k_1+1, k_2+1, ..., k_m+1) \leq \frac{(k_1+k2+...+k_m)!}{k_1!k_2!...k_m!}$.
Turan's Theorem Introduced: The general idea is that if G has no M clique, then $\epsilon(G) \leq ?$. We need new concepts! The *degree sequence* of G is the vector $< d_1, d_2, ..., d_{\nu(G)} >$ of degrees of vertices of G in non-decreasing order. A graph G is degree majorized by H if $\nu(G) = \nu(H)$ and $d_i \leq d_i' \forall 1 \leq \nu(G)$ and d_i, d_i' are

degree sequences of G,H.

G is an M-partite graph if there exists a partition $V(G) = X_1 \sqcup X_2 \sqcup ... \sqcup X_m$ such that for all $1 \leq i \leq m$, no vertex in X_i is adjacent to a vertex in X_i. G is a complete m-partite graph if, additionally, for every $i \neq j$, every vertex in X_i is adjacent to every vertex in X_j.

Theorem Turan's Theorem - 7.8: If a simple graph G contains no K_{m+1}, then G is degree majorized by some complete m-partite graph H. If G and H have the same degree sequence, then G is isomorphic to H.
Proof: We prove by induction.
Base Case: $m = 1$. G has no k_2, ie no edges. A 1-partite graph is an empty graph. H is isomorphic to G.
Inductive Step; Assume theorem 7.8 holds for $m < n$. We prove for $m = n$. Suppose G has no K_{m+1}. Let's pick a vertex with degree maximum $= \Delta(G)$. Let G_1 be $G[N(u)]$. Notice G has no K_m. By inductive hypothesis, G_1 is degree-majorized by some complete (m-1) partite graph, H_1. Let V_1 be $N(u)$ and $V_2 = V(G) - V_1$. Let H_2 be the empty graph with vertex set V_2. Let $G' = G_1 \vee H_2$, the *join* of G_1 and H_2, i.e. take the union and add all possible edges between $V(G_1)$ and $V(H_2)$.
Observations: $\Delta(G) = d_G(u) = |V_1|$, so for $v \in V_2, d_{G'}(v) = |V_1| = \Delta(G) \geq d_G(v)$. For $v \in V_1$, $N_{G'}(v) \supseteq N_G(v)$, so $d_{G'}(v) \geq d_G(v)$. Thus G' degree majorizes G, but we still need to show a matching 1-by-1. We did this in the homework.

But $H_1 \vee H_2$ is m-partite. $V(H) = X_1 \sqcup X_2 \sqcup ... \sqcup X_{m-1}$ where there are no edges in each partition. Set $X_m = V(H_2)$, same property. $V(H_1 \vee H_2) = X_1 \sqcup X_2 \sqcup ... \sqcup X_m$ and as H_2 had no edges, it is still true that for each i, no vertices in X_i are adjacent. Notice that $H_1 \vee H_2$ is complete.

Let $T_{m,n}$ be the complete m-partite graph on n vertices, where $V(T_{m,n}) = X_1 \sqcup ... \sqcup X_m$ satisfies $\lfloor \frac{n}{m} \rfloor \leq X_i \leq \lceil \frac{n}{m} \rceil$.
Theorem 7.9 - Turan's Theorem: If G contains no K_{m+1}, then $\epsilon(G) \leq \epsilon(T_{m,\nu(G)})$. If $\epsilon(G) = \epsilon(T_{m,\nu(G)})$, then G is isomorphic to $T_{m,\nu(G)}$.
Proof: By theorem 7.8, G is degree majorized by some complete m-partite graph H. One can show that $T_{m,\nu(G)}$ has the most edges among all m-partite graphs with $\nu(G)$ edges, i.e. $\epsilon(H) \leq \epsilon(T_{m,\nu(G)})$. So $2\epsilon(G) = \sum_{v \in V(G)} d(v) \leq \sum_{v \in V(H)} d(v) = 2\epsilon(H) \implies \epsilon(G) \leq \epsilon(H) \leq \epsilon(T_{m,\nu(G)})$. If $\epsilon(G) = \epsilon(T_{m,\nu(G)})$, then $\epsilon(G)\epsilon(H)\ \epsilon(T_{m,\nu(G)})$, so G and H have the same degree sequence, i.e. G is isomorphic to H. If H is not isomorphic to $T_{m,\nu(G)}$, then by exercise, one can show that $\epsilon(H) \leq \epsilon(T_{m,\nu(G)})$. But they are equal. So H is also isomorphic.

$\square$

Geometric Application of Turan's Theorem: The *diameter* of a finite set S in $\mathbb{R}^2$ is the maximum distance between two points in S.

Vertex Colorings

A $k-vertex-coloring$ is an assignment of a "color" in $1, 2, 3, ..., k$ to each vertex of a graph. It's a *proper coloring* if no to adjacent vertices have the same color. (Alternatively, one can think of vertex colors as partitions of $V(G)$. A proper coloring is a partition into independent set). G is $k - vertex - colorable$ if it has a proper k-vertex-coloring.
$\chi(G)$ is the *chromatic number*, or the smallest number such that G is k-vertex-colorable. We use the shorthand *coloring* to indicate a proper vertex coloring. K-colorable means k-vertex-colorable. What is $\chi(G)$? We know that $\chi(G) \leq \Delta(G)+1$. G is $k-critical$ if $\chi(H) < \chi(G)$ for any proper subgraph $H \subset G$ and if G is $k-chromatic$. Note that any graph with $\chi(G)$ has a k-critical subgraph.
Theorem 8.1: If G is k-critical, then $\delta \geq k - 1$.
Proof: Suppose not, i.e. k-critical G with $\delta < k-1$. Let $v \in V(G)$ be a vertex of degree δ. G being k-critical $\implies$ $G-v$ is $k-1$ colorable. Let $V_1, V_2, ..., V_{k-1}$ be a k-1 coloring of $G-v$. By definition, v is adjacent in G to $\delta < k-1$ vertices, and therefore v must be nonadjacent in G to every vertex in some V_j. But then $V_1, V_2, ..., V_j \cup \{v\}$ is a k-1 coloring of G, a contradiction. Thus $\delta \geq k - 1$.

$\square$

Corollary 8.1.1: Every $k - chromatic$ graph G has at least k-vertices of degree at least k-1.
Proof: Let G be such that $\chi(G) = k$. Then G contains a k-critical graph H. By theorem 7.1, $\delta(H) \geq k-1$, and moreover $|V(H)| \geq k$ since $\chi(H) = k$. Thus H has k-vertices of degree at least $k - 1$ and G does too.

$\square$

Corollary 8.1.2: $\chi(G) \leq \Delta(G) + 1$
Proof: By corollary 8.1.1, $\Delta(G) \geq \chi(G) - 1$. This is obvious.

$\square$

Theorem 8.4: If G is not a complete graph nor an odd cycle, then $\chi \leq \Delta$.

Question: Given a graph G, what is $\chi(G)$? It's hard.
Hadwiger Conjecture: If G is loopless and has no K_n minor, then $\chi(G) < n$. (Contrapositive: G loopless $\chi(G) \geq n \implies$ G has no K_n minor.)

H is a *minor* of G if H can be obtained from G by deleting an edge, deleting a vertex, and/or edge contraction. Subgraphs are obviously minors. Note that the converse is false.

Hajos Conjecture: $\chi(G) = n \rightarrow$ G has a subdivided K_n.
Since the book came out, the conjecture has been proven to be false (for at least n=7 or something).
G has a K_n minor $\leftarrow$ G has subdivided K_n, but not the other way around.

Theorem 8.5 - Hajos' Theorem: If $\chi(G) = 4$, then G contains a subdivided K_4.
Definition: Let S be a vertex cut of G and let the components of $G - S$ have vertex sets $V_1, V_2, ..., V_n$. Then, the vertex induces subgraphs $G[V_i \cup s$ are called S components.

Theorem 8.2: In a critical graph, no vertex cut is in a clique.
Proof: Suppose G is k-critical and S is a vertex cut and a clique. Let $G_1, ..., G_n$ be S-components. Then each G_i has a k-1 coloring, C_i. Since $V(G_i) \geq S$ and S is a clique, C_i colors each vertex of S by a different color. Consequently, we can "swap" colors in $C_1, C_2, ..., C_n$ so that $C_1, ..., C_n$ "agrees" on S. If $V_i = V(G) - S$, then there are no edges between V_i, V_j in G. So G = Union of the $G_i s$ and so $C_1, ..., C_n$ gives a k-1 coloring of G, but $\chi(G) = k$. Contradiction!

$\square$

Corollary I: f G is k-1 critical, then G has no vertex cut of size 1.
$\rightarrow$ We begin doing stuff to build up the proof of 8.5. Pick some k. If $\{u, v\}$ is a cut of G, we call a $\{u, v\}$-component G_i of *type* 1 if every k-1 coloring of G_i colors u,v the same and *type* 2 if every k-1 coloring of G_i colors u,v differently.

Theorem 8.3: If G is k-critical with vertex cut $\{u, v\}$, then i) $G = G_1 \cup G_2$ is a $\{u, v\}$-component of type 1 and ii) $G_1 + uv$ and $G_2 \bullet uv$ are both k-critical, where $G_2 \bullet uv$ denotes the graph obtained from G_2 by identifying u and v.
Proof: i) If every $\{u, v\}$-component of G has a k-1 coloring where u,v are the same, then as in the proof of 8.2, we can produce a k-1 coloring of G. The same is true if every $\{u, v\}$ component colors u, v differently. But this is impossible since $\chi(G) = k$. Therefore there exists some $\{u, v\}$-component G_1 of type 1 and $\{u, v\}$-component G_2 of type 2.