Graph Theory Final Exam Study Guide

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Definition: A vertex v is incident to an edge e if v is an end of e. 2 vertices are adjacent if they're connected by an edge. 2 edges are adjacent if they are incident to a common vertex. A graph is planar if it can be drawn in the plane such that no edges cross and an edge only meets its ends.

Definition: An isomorphism between graphs G and H is a pair of bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_g(e) = uv$ if and only if $\psi_h(\phi(e)) = \theta(u)\theta(v)$.

Definition: A graph is *simple* if it has no loops and at most one edge joining a pair of vertices. A graph is *empty* if it has no edges. A simple graph where all vertices are adjacent is a *complete* graph. A graph is *bipartite* if it is simple and there is a partition of the vertices into 2 sets such that every edge in G has one end in X and Y.

Definition: A graph is *complete bipartite* if every vertex in X is adjacent to every vertex in Y. $K_{m,n}$ is the notation.

Definition: The *incidence matrix* is an n x m matrix $M(G)$ such that $M_{i,j}$ is the number of times edge e_j is incident to vertex v_i

Definition: The *adjacency matrix* is the n x n matrix $A(G)$ such that $A_{i,j}$ is the number of edges between v_i and v_j.

Definition: A graph H is a *subgraph* of G ($H \subset G$) if all vertices, edges, and ψ_h are the same.
Definition: The subgraph of G induced by $V' \subset V$, denoted by $G[V']$, is the subgraph with vertices V' and includes every edge with ends in V'.

Definition: The subgraph of G induced by $E' \subset E$, denoted by $G[E']$, is the subgraph with edges E' and includes every vertex that is the end of one of the edges.

Definition: The *Ramsey number*, denoted $r(k, l)$, is the number of vertices of G required such that G contains either a clique or an empty graph on l vertices.

Definition: The *degree* of a vertex is the number of edges incident to v where loops counts twice (number of incident half-edges).

Theorem : $svscode \sum_{v \in V} d(v) = 2\epsilon$

Corollary : In every G, the number of vertices with odd degree is even.

Definition: A graph is $k - regular$ if the degree is always k.

Definition: A *walk* is an alternative sequences of vertices and edges, starting and ending with a vertex. A *trail* is a walk where all edges are distinct. A *path* is a walk where all vertices are distinct.
Definition: 2 vertices u and v are *connected* if there is a path from u to v in G. G is *connected* if all pairs of vertices are connected.

Fact: u and v have a path between iff there exists a walk between them.

Definition: If $u, v \in G$ are in the same component, then the distance from u to v is $d_G(u, v) = min\{l(P)|P$ is a path from u to v$\}$

Definition: A walk/trail is *closed* if the initial and final vertex are the same.

Definition: An *Euler trail* is a trail that crosses all edges (necessarily once).

Definition: A walk is a *cycle* if it is closed and all vertices are distinct (bar the initial and final).

Theorem : A graph G is bipartite iff every cycle has even length.

Shortest Path Algorithm: Pick two vertices, u and v in a weighted graph G. Consider H a subgraph of G. For each edge e with one end in H and 1 end not in H, we compute $d(u) + w(e)$. For some e with minimal value of $d(u) + w(e)$, add it and its incident vertex and direct it towards u. If this incident vertex is v, stop.

Definition: A connected graph with no cycles is a *tree*. A graph (possible disconnected) without cycles is a $forest$ (acyclic graph).

Theorem : In a tree, any pair of vertices is connected by a unique path.

Theorem : If G is a tree, then $\epsilon = \nu - 1$.

Definition: A $leaf$ is a vertex with degree = 1.

Theorem : An edge e in a graph G is a cut edge iff e is not contained in

any cycle of G.
Corollary : A graph G is a forest iff every edge is a cut edge.

Definition: A *spanning tree* $T \subset G$ is a subgraph which is both spanning and a tree.

Theorem : Every connected graph contains a spanning tree.

Corollary : If G is connected and $\epsilon(G) = \nu(G) - 1$, then G is a tree.

Theorem : Suppose T is a spanning tree in G and $e \in \epsilon(G) - \epsilon(T)$. Then $T + e$ has a unique cycle.

Definition: An *edge cut* of a graph is a subset of edges such that $\omega(G - E') > \omega(G)$. We call a minimal edge cut a *bond*.

Theorem : Let T be a spanning tree of a connected graph G. Let $e \in E(T)$. Then $E(G) - E(T)$ contains no edge cut and $E(G) - E(T) + e$ contains a unique minimal edge cut of G.

Corollary : G is connected $\Rightarrow \epsilon(G) \geq \nu(G) - 1$.

Definition: A vertex is a *cut vertex* if $E(G)$ can be partitioned into the subsets E_1, E_2 such that $G(E_1)$ and $G(E_2)$ intersect only at v. If G is loopless and $\nu(G) > 1$, then v is a cut vertex iff $w(G - v) > w(G)$.

Theorem - Cayley's Algorithm: Let $\tau(G)$ be the number of spanning trees in a graph G. Then, if e is not a loop, $\tau(G) = \tau(G \bullet e) + \tau(G - e)$.

Theorem - Cayley's Theorem: $\tau(K_n) = n^{n-2}$.

Theorem - Krustal's Algorithm: We want to find a spanning tree of minimal weight. Let T be all vertices will no edges. Find a non-loop edge of minimal weight and add it to T. If T is a spanning tree, stop. Else, repeat this process such that no cycles form.

Theorem : Krustal's Algorithm produces an optimal tree.

Definition: A *vertex cut* of G is a subset of vertices such that $G - v'$ is disconnected.

Fact: Complete graphs have no vertex cuts.

Definition: The connectivity $\kappa(G)$ is the minimal size of a vertex cut for graphs with ≥ 2 non-adjacent vertices. Otherwise $\kappa(G) = \nu(G) - 1$. Note that if $\nu(G) = 1$ or G is disconnected, $\kappa(G) = 0$. If $\kappa(G) \geq k$, G is k-connected.

Definition: The edge-connectivity $\kappa'(G)$ is 0 if $\nu(G) = 1$ or G is disconnected. Otherwise, $\kappa'(G)$ is the minimal size of an edge cut. If $\kappa'(G) \geq k$, G is k-edge-connected.

Theorem 3.1: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Theorem 3.2: A graph G is 2-connected if it has ≥ 3 vertices and any 2 vertices are connected by 2 internally disjoint paths.

Corollary 3.2.1: Suppose $\nu(G) \geq 3$. G is 2-connected iff any 2 vertices of G lie on a common cycle.

Corollary 3.2.2: Suppose $\nu(G) \geq 3$ and G is a block. Any 2 edges of G lie on a common cycle.

0.1 Euler Tours

A walk is a *trail* if it crosses every edge at most once. Recall that an Euler trail is a trail that crosses every edge (necessarily once).

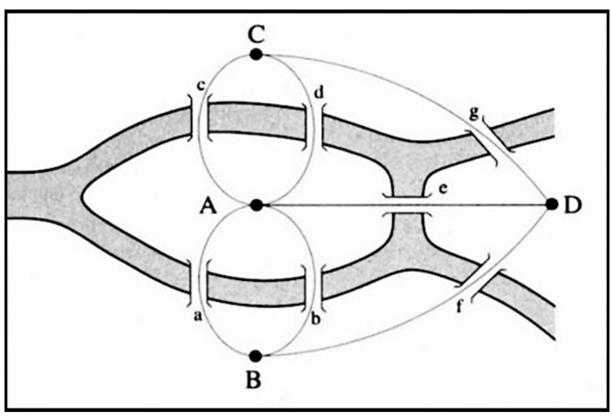

Figure 1: 7 Bridges of Konigsberg Problem

An *Euler tour* is a closed Euler trail. A graph G is *Eulerian* if it contains an Euler tour.

Theorem 4.1: A Graph G is Eulerian iff it is connected and every vertex has even degree.

Lemma: If H is a nontrivial connected graph where each vertex has even degree, then for all vertices in H, H has a closed trail (of positive length) starting and ending at v.

Corollary 4.1: A graph contains an Euler Trail iff it has 0 or 2 vertices of odd degree in G.

0.2 Hamilton Cycles

A *Hamilton Path* is a path which contains all vertices of G.
A *Hamilton Cycle* is a cycle which contains all vertices of G.

Unfortunately, checking whether or not a graph is Hamiltonian is Hell for computers to check.

We will see some necessary conditions for being Hamiltonian (i.e. it holds if it is Hamiltonian) and sufficient conditions (i.e. if it holds then G is Hamiltonian).

Theorem 4.2: If G is Hamiltonian, then for each nonempty proper subset $S \subset V(G)$, $\omega(G-S) \leq |S|$.

Theorem 4.3: If G is simple with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, then G is Hamiltonian.

Lemma: Suppose G is simple and u and v are non-adjacent vertices such that $d(u)+d(v) \geq \nu(G)$. Then G is Hamiltonian iff G + uv is Hamiltonian.

0.3 Closure

The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices u,v, satisfying $d(u)+d(v) \geq \nu(G)$ until all are gone. The graph created we call $C(G)$.

Lemma 4.4.2: $C(G)$ is well defined- no matter what choices you make, you get the same result.

0.3.1 Traveling Salesman Problem

Given graph G with positive edge weights, find a Hamilton cycle with the least weight.

Easier Problem: Find a minimum weight Hamilton cycle in a weighted complete graph. There exists an algorithm which approximates an optimal solution.

A simplified version of an algorithm: Start with some Hamilton cycle in a complete graph

$$C = v_1 v_2 ... v_n v_1$$

Look at all i, j such that $1 < i+1 < j < \nu$ and compare $w(v_i v_{i+1}) + w(v_j v_{j+1})$ to $w(v_i v_j) + w(v_{i+1} v_{j+1})$. If the latter is smaller for some i, j, then do the following: C becomes $C_2 = v_i ... v_j v_{j-1} ... v_{i+1} v_{j+1} v_\nu v_i$ and the new weight of C_2 is $w(C) + w(v_i v_j) + w(v_{i+1} v_{j+1}) - w(v_i v_{i+1}) - w(v_j v_{j+1})$, which is strictly less than $w(C)$. Repeat this process until you can no longer do so.

0.4 Matchings

Definition: A subset $M \subset E(G)$ is a matching if $E(G)$ has no loops and no 2 edges in M are adjacent. *(When we consider matchings, assume G is always simple!)*

A matching M *saturates* v or v is *M-saturated* if v is an end of some edge in M.

M is a *perfect matching* if every vertex is M-saturated. M is a *maximum matching* if its a matching of biggest size.

An $M-$ *alternating path* is a path where edges alternate between M and $E - M$. An M-alternating path is an $M-$ *augmenting path* if the starting and ending vertices are M-unsaturated.

Theorem 5.1: A matching M in G is a maximum matching iff G contains no M-augmenting path.

Definition: For $S \subset V(G)$, the neighbor set of S, $N_G(S)$, is the set of vertices adjacent to at least 1 element of S.

Theorem 5.2: Let G be a bipartite graph. Then G contains a matching that saturates X iff $|N_G(S)| \geq |S|$.

Corollary 5.2: If G is k-regular and bipartite with $k > 0$, then G has a perfect matching.

0.5 Coverings

Definition: A *cover* of a graph G is a set $K \subset V(G)$ such that every edge is incident to some vertex in K.

Observation: By the nature of a cover K and a matching M, $|M| \leq |K|$. Why? Every $e \in M$ is incident to at least one vertex $v \in K$, but no other $e' \in M$ is incident to this same vertex. For maximum matching M^* and minimum cover K', $M^* \leq K'$.

Theorem 5.3: In a bipartite graph, $|M^*| = |K'|$.

Lemma 5.3: If M is a matching, K is a cover, and $|M| = |K|$, then M is a maximum matching and K is a minimum cover.

Definition: A component of a graph is called *odd* if it has an odd number of vertices. Otherwise it is called *even*.

Theorem 5.4 - Tutte's Theorem: Let $o(G)$ be the number of odd components in G. G has a perfect matching iff $o(G - S) \leq |S| \; \forall S \subset V(G)$.

Corollary 5.3: Every 3-regular graph without a cut vertex has a perfect matching.

0.6 Edge Colorings

Definition: A $k-$ *edge coloring* of a loopless graph G is an assignment of "colors" $1, 2, 3, ..., k$ to the edges of G. I.e. label each edge with $\{1, 2, 3, ..., k\}$. The coloring is *proper* if no two adjacent edges have the same color.

Definition: G is $k-$ *edge colorable* if G has a proper k-coloring. The *edge chromatic number* $\chi'(G)$ is the smallest k for which G is k-edge-colorable. Also, we could say a coloring C as a part. of $E(G)$ by $E_i = \{e \in E(G)|e$ color is i$\}$.

$$E(G) = E_1 \cup E_2 \cup ... \cup E_k$$

We denote C as $\{E_1, E_2, ..., E_k\}$. Observe that for C a proper coloring, $\forall e_1, e_2 \in E_i$ with $e_1 \neq e_2 \Rightarrow e_1, e_2$ are not adjacent. So E_i is a matching. Recall that $\Delta(G)$ is the maximum degree of any vertex in G.

Observe that $\chi'(G) \geq \Delta(G)$ as there exist $|\Delta(G)|$-many edges incident to the same vertex.

Theorem 6.1: If G is bipartite, $\chi'(G) = \Delta(G)$.

Theorem - Vizing's Theorem: If G is simple, then its edge-chromatic-number is either $\Delta(G)$ or $\Delta(G) + 1$.

Time Tabling Problem: m teachers need to teach n classes P_{ij} times a week. Our goal is to teach all classes or sections in as few of periods as possible. Each period is a matching, so a time table is a coloring of a bipartite graph. One more consideration is the number of rooms available. If ϵ is the total number of classes, can you find a time table where in each period, at most the ceiling of $\frac{\epsilon}{p}$ classes are being taught? Yes!

Lemma 6.3: If M,N are disjoint matchings of a graph G with $|M| > |N|$, then there are disjoint matchings M', N' such that $|M'| = |M| - 1$, $|N'| = |N| + 1$ and $M \cup N = M' \cup N'$.

Theorem 6.3: Let G be bipartite and $p \leq \Delta(G)$. Then there exist p disjoint matchings $M_1, M_2, ..., M_p$ such that $E(G) =$Union of the $M_i s$ and $\lfloor \frac{\epsilon}{p} \rfloor \leq M_i \leq \lceil \frac{\epsilon}{p} \rceil$.

Independent Sets and Cliques

Recall that $S \subset V(G)$ is an *independent set* if no 2 vertices in S are adjacent in G. It is maximum if it is of maximum size.

Theorem 7.1: $S \subset V(G)$ is an independent set iff $V(G) - S$ is a covering.

$S \subset V(G)$ independent set means that all v are non-adjacent in S. $S \subset V(G)$ clique means that all v adjacent in S.

Let $\alpha(G)$ be the *independence number*, the maximum size of an independent set.
Let $\beta(G)$ be the *vertex covering number*, the minimum size of a vertex covering.

Corollary 7.1: $\alpha(G) + \beta(G) = \nu(G)$
$L \subset E(G)$ is an *edge covering* if every $v \in V(G)$ is incident to some edge in L. An edge covering exists iff $\delta(G) > 0$.

$\alpha'(G)$ is the *edge independence number*, the maximum size of a matching in G.
$\beta(G)$ is the *edge covering number*, the minimum size of an edge covering.

M matching does not imply M^C a covering.
L covering does not imply L^C a matching.

Theorem 7.2: If $\delta(G) > 0$, then $\alpha' + \beta' = \nu$.

Ramsey Theory

For this section, assume all graphs to be simple.

Definition: $r(k, l)$ is the smallest positive integer such that if G is simple and has $\nu \geq r(k, l)$, then either: G has a k clique or an l independent set.

Observe that $r(k, l) = r(l, k)$. If G has a k-clique, then G^C has a k-independent set. $r(1, l)$ is $r(k, 1) = 1$. $r(2, l) = l$, $r(k, 2) = k$.

Theorem 7.4: For all $k, l \geq 2$, $r(k, l) \leq r(k-1, l) + r(k, l-1)$. Moreover, if $r(k-1, l)$ and $r(k, l-1)$ are even, then $r(k, l) < r(k-1, l) + r(k, l-1)$.

Theorem 7.9 - Turan's Theorem: If a simple graph G contains no K_{m+1}, then G is degree majorized by some complete m-partite graph H. If G and H have the same degree sequence, then G is isomorphic to H.

Let $T_{m,n}$ be the complete m-partite graph on n vertices, where $V(T_{m,n}) = X_1 \sqcup ... \sqcup X_m$ satisfies $\lfloor \frac{n}{m} \rfloor \leq X_i \leq \lceil \frac{n}{m} \rceil$.

Theorem 7.9 - Turan's Theorem Cont.: If G contains no K_{m+1}, then $\epsilon(G) \leq \epsilon(T_{m,\nu(G)})$. If $\epsilon(G) = \epsilon(T_{m,\nu(G)})$, then G is isomorphic to $T_{m,\nu(G)}$.

Vertex Colorings

A $k-vertex-coloring$ is an assignment of a "color" in $1, 2, 3, ..., k$ to each vertex of a graph. It's a *proper coloring* if no to adjacent vertices have the same color. (Alternatively, one can think of vertex colors as partitions of $V(G)$. A proper coloring is a partition into independent set). G is $k-vertex-colorable$ if it has a proper k-vertex-coloring.

$\chi(G)$ is the *chromatic number*, or the smallest number such that G is k-vertex-colorable. We use the shorthand *coloring* to indicate a proper vertex coloring. K-colorable means k-vertex-colorable. What is $\chi(G)$? We know that $\chi(G) \leq \Delta(G) + 1$. G is $k-critical$ if $\chi(H) < \chi(G)$ for any proper subgraph $H \subset G$ and if G is $k-chromatic$. Note that any graph with $\chi(G)$ has a k-critical subgraph.
Theorem 8.1: If G is k-critical, then $\delta \geq k - 1$.

Corollary 8.1.1: Every $k-chromatic$ graph G has at least k-vertices of degree at least k-1.

Corollary 8.1.2: $\chi(G) \leq \Delta(G) + 1$

Theorem 8.5 - Hajos' Theorem: If $\chi(G) = 4$, then G contains a subdivided K_4.
Definition: Let S be a vertex cut of G and let the components of $G - S$ have vertex sets $V_1, V_2, ..., V_n$. Then, the vertex induces subgraphs $G[V_i \cup s$ are called S components.

Theorem 8.2: In a critical graph, no vertex cut is in a clique.

Corollary I: f G is k-1 critical, then G has no vertex cut of size 1.

Theorem 8.3: If G is k-critical with vertex cut $\{u, v\}$, then i) $G = G_1 \cup G_2$ is a $\{u, v\}$-component of type 1 and ii) $G_1 + uv$ and $G_2 \bullet uv$ are both k-critical, where $G_2 \bullet uv$ denotes the graph obtained from G_2 by identifying u and v.

0.7 Planar Graphs

Theorem 7.5: If G is a connected graph in the plane, then $\nu - \epsilon + \phi = 2$.

Corollary 9.5.2: If G is a simple planar graph with $\nu \geq 3$, then $\epsilon \leq 3\nu - 6$.

Corollary 9.5.3: If G is a simple planr graph, then $\delta \leq 5$.

Theorem 5 Color: : Every planar graph is 5 (and 4)-colorable.

0.8 Directed Graphs

Definition: A vertex $v \in V(D)$ is reachable from u to v if there is a directed path from u to v. u and v are diconnected if each is reachable from the other.

Definition: The *dicomponent* of u is the graph induced by all vertices that are diconnected to u. A digraph is *diconnected* if there is only one dicomponent.
Theorem 10.1: A digraph contains a directed path of length $\chi(G) - 1$.

0.9 Networks

Definition: A $flow$ in a network N is an integer-valued function f defined on E such that $0 \leq f(e) \leq c(e) \forall e \in E$ and $f^-(v) = f^+(v) \forall v \in I$.

Definition: The value of the flow $= val f = f^+(X) - f^-(X)$.

Definition: A cut in N is a set of edges in the form of (S, S^C) where $x \in S$

and $y \in S^C$. The capacity of a cut K is the sum of the capacities of its edges.

Lemma 11.1: For any flow f and any cut (S, S^C) in N, $valf = f^+(S) - f^-(S)$.

Theorem 11.1: For any flow f and any cut K in K, $valf \leq capK$.

Theorem 11.2: A flow f in N is a maximum flow if and only if N contains no f-incrementing path.

Theorem 11.3- Max Flow Min Cut: In any network, the value of a maximum flow is equal to the capacity of a minimum cut.

Theorem Menger's Theorem: In a graph G, the size of a minimum cut K is equal to the maximum number of internally disjoint paths between any two pairs of vertices in $V(G)$.