Graph Theory Notes

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October 3, 2022

0.1 Euler Tours

A walk is a *trail* if it crosses every edge at most once. Recall that an Euler trail is a trail that crosses every edge (necessarily once).

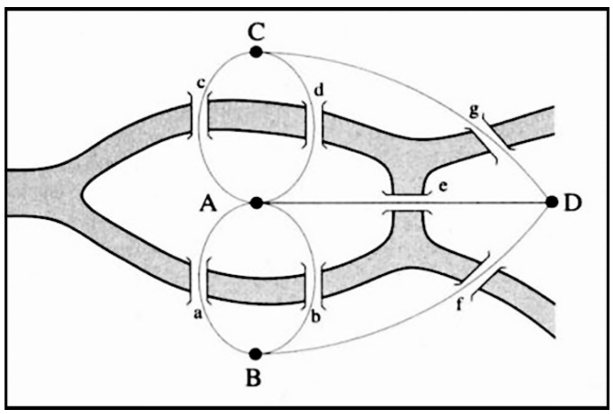

Figure 1: 7 Bridges of Konigsberg Problem

An *Euler tour* is a closed Euler trail. A graph G is *Eulerian* if it contains an Euler tour.
Theorem 4.1: A Graph G is Eulerian iff it is connected and every vertex has even degree.
Lemma: If H is a nontrivial connected graph where each vertex has even degree, then for all vertices in H, H has a closed trail (of positive length) starting and ending at v.
Proof: Construct this trail W as follows: Start at v. Since H is nontrivial and connected, there is some edge e_1 incident to v and some vertex v_1. Now, $W = ve_1v_1$. Having constructed $W = ve_1v_1...e_nv_n$, add an edge e_{n+1} incident to v_n not in W and its other end to W if you can. Stop when you can no longer do this.
Claim : We stop when W's last vertex is v.
Proof : Suppose we've stopped constructing W. Let $u = v_n$. Every time $v_i = u$

$(i \neq n)$, W crossed an even number of half edges incident to u. If $u \neq v$, then we also crossed exactly 1 more half edge at the end of W. Thus W crossed an odd number of half edges at u. But $deg(u)$ is even, so there's another edge we can add to W. So we didn't stop here. This is a contradiction. $\therefore u = v$

$\square$

Proof of Theorem 4.1: ($\Rightarrow$) G is Eulerian if it has an Euler tour (closed walk with each edge once). Left as exercise.
($\Leftarrow$) Let W be a maximal closed trail in G. Suppose W is not an Euler tour. Let $G' = G - E(W)$. This has some edges in it. As discussed before $\forall v \in V(G)$, E(W) contains an even number of half edges incident to v. Thus, every vertex in G' has even degree. Let G'_0 be some component in G'. Since G is connected, some vertex of G'_0 is a vertex v_i in W (Why? Connect v_0, a vertex in G'_0, to a vertex in G by some path and take the last vertex in that path which is in W.) By Lemma, there exists a closed trail W' in G'_0 starting and ending at v. Then, paste W' into W = W''. W' used no edges in W. SO W'' is a longer closed trail. This is a contradiction as W is maximal. $\therefore$ W has every edge.

$\square$

Corollary 4.1: A graph contains an Euler Trail iff it has 0 or 2 vertices of odd degree in G.

0.2 Hamilton Cycles

A *Hamilton Path* is a path which contains all vertices of G. A *Hamilton Cycle* is a cycle which contains all vertices of G. Unfortunately, checking whether or not a graph is Hamiltonian is Hell for computers to check.
We will see some necessary conditions for being Hamiltonian (i.e. it holds if it is Hamiltonian) and sufficient conditions (i.e. if it holds then G is Hamiltonian).

Theorem 4.2: If G is Hamiltonian, then for each nonempty proper subset $S \subset V(G)$, $\omega(G - S) \leq |S|$.
Proof: Let C be a Hamiltonian cycle in G. Then for each $S \in V(G)$, $\omega(C - S \leq |S|$. Since C contains all V(G), $V(C - S) = V(G - S)$. Thus, if components of $C - S$ are $C_1, C_2, ..., C_n$, $G - S = C - S +$ some edges, so we many connected some C_i together but no new components appear. $\therefore \omega(C - S) \leq n = \omega(C - S) \leq |S|$.

$\square$

Theorem 4.3: If G is simple with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, then G is Hamiltonian.
Proof: Suppose that there is some graph G with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$ that is not Hamiltonian. Whatever G is, it can't be complete since complete graphs are Hamiltonian. So let G be a graph with ν vertices and is maximal with respect to edges, with $\delta \geq \frac{\nu}{2}$ and is not Hamiltonian. I.e. G+e does satisfy the theorem. Pick 2 non-adjacent vertices $u, v \in V(G)$. Then the graph G+uv

must be Hamiltonian by maximality of G. But since G is not Hamiltonian, every Hamilton cycle in G+uv must cross uv. Deleting uv from the cycle gives us a Hamilton path P from u to v.
Define $S = \{v_i | v_{i+1}$ is adjacent to $u\}$ in G. Define $T = \{v_i | v_i$ is adjacent to $v\}$ in G.
Notice that v is in neither S nor T. So $S \cap T$ is empty. So W is a Hamilton cycle in G. But G has no Hamilton cycle! This is a contradiction. But d(u) + d(v) = $|S| + |T| = |S \cup T| - |S \cap T| = |S \cup T| \leq \nu(G) - 1$. This contradicts $\delta(u) + d(v) = \delta(G) + \delta(G) = \nu(G)$. Contradiction!

□

Lemma: Suppose G is simple and u and v are non-adjacent vertices such that $d(u) + d(v) \geq \nu(G)$. Then G is Hamiltonian iff G + uv is Hamiltonian.
Proof: Same argument as previous proof.

□

0.3 Closure

The *closure* of G is the graph obtained from G by recursively joining pairs of nonadjacent vertices u,v, satisfying $d(u) + d(v) \geq \nu(G)$ until all are gone. The graph created we call $C(G)$.
Lemma 4.4.2: $C(G)$ is well defined- no matter what choices you make, you get the same result.
Proof: G is a graph. Make graphs H, H' by applying the procedure but in different ways.
$H = G + \{u_1v_1, u_2v_2, ..., u_kv_k\}$. $H' = G + \{u'_1v'_1, u'_2v'_2, ..., u'_kv'_k\}$ (u_iv_i are two nonadjacent vertices in $G + \{u_1v_1, u_2v_2, ..., u_kv_k\}$ with $d(G + \{...\}(u_i) + d(G + \{...\}(v_i) \geq \nu(G)$. This also true for H'.
Observe that if $d_H(u) + d_H(v) \geq \nu(G)$ then u and v are adjacent in H. If not, the process would have added the edge. This is also true for H'. By induction, $H \subset H'$ (symmetric shows $H' \subset H$).
Base Case: u_1v_1 is in H'. $d_{H'}(u_1) + d_{H'}(v_1) \geq d_G(u_1) + d_G(v_1) \geq \nu(G)$ (since it was added in process for H). By observation, $u_1v_1 \in E(H')$.
Inductive Step: $\{u_1v_1, u_2v_2, ..., u_{i-1}v_{i-1}\} \in E(H') \Rightarrow u_iv_i \in H'$. So $d_{H'}(u_i) + d_{H'}(v_i) \geq d_{G+\{...\}(u_i} + d_{G+\{...\}(v_i)} \geq \nu(G)$ since we added it to the graph in the other process. By observation, $u_iv_i \in H'$. Induction tells us $\{u_1v_1 ... u_kv_k\} \leq H'$. So $H \subset H'$. Symmetric argument shows $H' \subset H$. So H = H'.

□

0.3.1 Chinese Postman Problem

Edges = Streets (edges have positive weight). Vertices = intersections. Find a shortest closed walk that crosses all edges If G is Eulerian (every degree is even) then ever tour is optimal.

0.3.2 Traveling Salesman Problem

Given graph G with positive edge weights, find a Hamilton cycle with the least weight.
Easier Problem: Find a minimum weight Hamilton cycle in a weighted complete graph. There exists an algorithm which approximates an optimal solution.

A simplified version of an algorithm: Start with some Hamilton cycle in a complete graph

$$C = v_1 v_2 ... v_n$$

Look at all pairs of edge that don't share common vertices. Compare the weights and see if it is connected.