HW 10 - Graph Theory

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March 12, 2023

1.

Proof. We know that $V(G) = V(H)$, so all vertices in G are the same vertices in H. Consider $V(H)$. We select each $v \in V(H)$ and write them in an unordered list. We then find the degree of each vertex, $d_H(v)$, and associate it with each vertex in our list. We use any sorting algorithm, i.e. bubble sort, to sort our list with respect to degree in non-descending order. We now have a list of degrees, D_H, of $V(H)$ in non-descending order, with each degree in our list having an associated vertex.

Consider $V(G)$. We write a list of each $v \in V(G)$ with each vertex in the same position as it is in D_H (which we can easily do, as each degree in D_H has an associated vertex), and call this list D_G. The degree of each vertex in D_G is unknown, but we do know that $d_G(v) \leq d_H(v)\ \forall v \in V(H)$. With the elements of $D_H =< h_1, h_2, ..., h_{\nu(H)} >$ and the elements of $D_G =< g_1, g_2, ..., g_{\nu(G)} >$, we know that $g_i < h_i$ for $0 \leq i \leq \nu(G)$, since these degrees correspond to the same vertex in $V(G)$ and $V(H)$. D_H is non-descending, but D_G might not be. We can apply a sorting algorithm like bubble sort to D_G now to ensure it is non-descending while preserving that $d_G(v) \leq d_H(v)\ \forall v \in V(H)$ (moving the smaller element towards the front of the list is fine as it swaps with the larger value which did not violate $d_G(v) \leq d_H(v)\ \forall v \in V(H)$. Moving the larger element towards the end of the list cannot violate $d_G(v) \leq d_H(v)\ \forall v \in V(H)$ since its value g_i is strictly less than or equal to its corresponding h_i, and all values in D_H after h_i are greater than or equal to h_i). So H degree-majorizes G.

□

2.

Proof. Let G be a graph on $r(k_1 - 1, k_2, k_3) + r(k_1, k_2 - 1, k_3) + r(k_1, k_2, k_3 - 1)$, and $v \in V$. We have 3 cases: 1) v is adjacent to a subset S of $r(k_1 - 1, k_2, k_3)$ vertices or 2) v is adjacent to a subset S of $r(k_1, k_2 - 1, k_3)$ vertices or 3) v is adjacent to a subset S of $r(k_1, k_2, k_3 - 1)$ vertices.

i: Either G[S] has a k_1 - 1 clique or a k_2 or k_3 independent set $S' \subset S$. Then $S + v$ is a k_1-clique in G.

ii: Either G[S] has a k_2 - 1 clique or a k_1 or k_3 independent set $S' \subset S$. Then $S + v$ is a k_2-clique in G.

iii: Either G[S] has a k_3 - 1 clique or a k_1 or k_2 independent set $S' \subset S$. Then $S + v$ is a k_3-clique in G.

The vertex induced subgraphs of these subsets + v necessitate G have a k_1 clique or a k_2 clique or a k_3 clique since 1 or 2 or 3 must hold.

□

3.

Proof. Each vertex in each tripartition of G connects to each other vertex in the other tripartitions. So letting $x + i|X_i|$, we find that the total degree of G is $x_1x_2 + x_2x_3 + x_1x_3$. We know that $T_{3,n}$ divides $n = 3k$ vertices as evenly as possible, so we have 3 subsets of k vertices, or our total degree is $3k^2$. Suppose the division of G is not as evenly as possible. Suppose one subset has 1 less vertices and one has one more. Then we get the total degree as $(k+1)(k-1) + k(k+1) + k(k-1) = k^2 - 1 + k^2 + k + k^2 - k = 3k^2 - 1$ which is less than $3k^2$. We showed this for the minimal difference, and any larger difference will clearly produce an equal or smaller number.

□

4.

i.

Proof. If G contains no K_{m+1}, then $\epsilon(G) \leq \epsilon(T_{m,\nu(G)})$. Suppose G contains no triangle. Then $\epsilon(G) \leq \epsilon(T_{2,\nu(G)})$. What is $\epsilon(T_{2,\nu(G)})$?

$\nu(G)$ Even: We know that $2|\nu(G)$. We let our bipartition be X, Y, with $|X| = |Y| = \nu/2$. So the degree of each $x \in X$ is $\nu/2$ and the degree of each $y \in Y$ is $\nu/2$. So the total degree of $T_{2,\nu(G)}$ is $\nu/2 \bullet \nu/2 + \nu/2 \bullet \nu/2 = 2\epsilon$, or $\epsilon(T_{2,\nu(G)}) = \frac{\nu^2}{4}$. But $\epsilon(G)$ is strictly less than or equal to this value, so any number of edges over it produces a K_3.

$\nu(G)$ Odd: We arbitrarily let $|X| = |Y| + 1$ (the difference can be no more than 1 by our statement and bipartitedness). Since $\nu(G)$ odd, we write $\nu = 2k + 1$, so $|X| = k + 1$ and $|Y| = k$. Finding the total degree again by a similar process as the even case, we get $2k(k+1) = 2\epsilon$, or $\epsilon(T_{2,\nu(G)}) = k^2 + k$. Since $\nu(G)$ odd, we get that $\frac{(2k+1)^2}{4} = k^2 + k + \frac{1}{4}$, so $\epsilon(T_{2,\nu(G)})$ is exactly $\frac{\nu(G)^2}{4} - \frac{1}{4} \geq \epsilon(G)$. So we have proven our statement for both cases (by contrapositive).

$\square$

ii.

Consider a complete bipartite graph upon n vertices divided evenly as possible. We showed in i) that this graph will always have the ceiling of $\frac{\nu^2}{4}$ edges. By simply writing our odd number of vertices as $2k-1$, we find that this graph will always have the floor of $\frac{\nu^2}{4}$ edges.

$\square$