Graph Theory Exam 1 Study Guide

Ethan Beaird

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Definition: A vertex v is incident to an edge e if v is an end of e. 2 vertices are adjacent if they're connected by an edge. 2 edges are adjacent if they are incident to a common vertex. A graph is planar if it can be drawn in the plane such that no edges cross and an edge only meets its ends.

Definition: An isomorphism between graphs G and H is a pair of bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_g(e) = uv$ if and only if $\psi_h(\phi(e)) = \theta(u)\theta(v)$.

Definition: A graph is *simple* if it has no loops and at most one edge joining a pair of vertices. A graph is *empty* if it has no edges. A simple graph where all vertices are adjacent is a *complete* graph. A graph is *bipartite* if it is simple and there is a partition of the vertices into 2 sets such that every edge in G has one end in X and Y.

Definition: A graph is *complete bipartite* if every vertex in X is adjacent to every vertex in Y. $K_{m,n}$ is the notation.

Definition: The *incidence matrix* is an n x m matrix $M(G)$ such that $M_{i,j}$ is the number of times edge e_j is incident to vertex v_i

Definition: The *adjacency matrix* is the n x n matrix $A(G)$ such that $A_{i,j}$ is the number of edges between v_i and v_j.

Definition: A graph H is a *subgraph* of G ($H \subset G$) if all vertices, edges, and ψ_h are the same.

Definition: The subgraph of G induced by $V' \subset V$, denoted by $G[V']$, is the subgraph with vertices V' and includes every edge with ends in V'.

Definition: The subgraph of G induced by $E' \subset E$, denoted by $G[E']$, is the subgraph with edges E' and includes every vertex that is the end of one of the edges.

Definition: The *Ramsey number*, denoted $r(k,l)$, is the number of vertices of G required such that G contains either a clique or an empty graph on l vertices.

Definition: The *degree* of a vertex is the number of edges incident to v where loops counts twice (number of incident half-edges).

Theorem : $\sum_{v \in V} d(v) = 2\epsilon$

Corollary : In every G, the number of vertices with odd degree is even.

Definition: A graph is $k - regular$ if the degree is always k.

Definition: A *walk* is an alternative sequences of vertices and edges, starting and ending with a vertex. A *trail* is a walk where all edges are distinct. A *path* is a walk where all vertices are distinct.

Definition: 2 vertices u and v are *connected* if there is a path from u to v in G. G is *connected* if all pairs of vertices are connected.
Fact: u and v have a path between iff there exists a walk between them.

Definition: If $u, v \in G$ are in the same component, then the distance from u to v is $d_G(u,v) = min\{l(P)|P$ is a path from u to v$\}$
Definition: A walk/trail is *closed* if the initial and final vertex are the same.

Definition: An *Euler trail* is a trail that crosses all edges (necessarily once).

Definition: A walk is a *cycle* if it is closed and all vertices are distinct (bar the initial and final).

Theorem : A graph G is bipartite iff every cycle has even length.

Shortest Path Algorithm: Pick two vertices, u and v in a weighted graph G. Consider H a subgraph of G. For each edge e with one end in H and 1 end not in H, we compute $d(u)+w(e)$. For some e with minimal value of $d(u)+w(e)$, add it and its incident vertex and direct it towards u. If this incident vertex is v, stop.

Definition: A connected graph with no cycles is a *tree*. A graph (possible disconnected) without cycles is a $forest$ (acyclic graph).

Theorem : In a tree, any pair of vertices is connected by a unique path.

Theorem : If G is a tree, then $\epsilon = \nu - 1$.

Definition: A *leaf* is a vertex with degree $= 1$.

Theorem : An edge e in a graph G is a cut edge iff e is not contained in any cycle of G.

Corollary : A graph G is a forest iff every edge is a cut edge.

Definition: A *spanning tree* $T \subset G$ is a subgraph which is both spanning and a tree.

Theorem : Every connected graph contains a spanning tree.

Corollary : If G is connected and $\epsilon(G) = \nu(G) - 1$, then G is a tree.

Theorem : Suppose T is a spanning tree in G and $e \in \epsilon(G) - \epsilon(T)$. Then $T + e$ has a unique cycle.

Definition: An *edge cut* of a graph is a subset of edges such that $\omega(G - E') > \omega(G)$. We call a minimal edge cut a *bond*.

Theorem : Let T be a spanning tree of a connected graph G. Let $e \in E(T)$. Then $E(G) - E(T)$ contains no edge cut and $E(G) - E(T) + e$ contains a unique minimal edge cut of G.

Corollary : G is connected $\Rightarrow \epsilon(G) \geq \nu(G) - 1$.

Definition: A vertex is a *cut vertex* if $E(G)$ can be partitioned into the subsets E_1, E_2 such that $G(E_1)$ and $G(E_2)$ intersect only at v. If G is loopless and $\nu(G) > 1$, then v is a cut vertex iff $w(G - v) > w(G)$.

Theorem - Cayley's Algorithm: Let $\tau(G)$ be the number of spanning trees in a graph G. Then, if e is not a loop, $\tau(G) = \tau(G \bullet e) + \tau(G - e)$.

Theorem - Cayley's Theorem: $\tau(K_n) = n^{n-2}$.

Theorem - Krustal's Algorithm: We want to find a spanning tree of minimal weight. Let T be all vertices will no edges. Find a non-loop edge of minimal weight and add it to T. If T is a spanning tree, stop. Else, repeat this process such that no cycles form.

Theorem : Krustal's Algorithm produces an optimal tree.

Definition: A *vertex cut* of G is a subset of vertices such that $G - v'$ is disconnected.

Fact: Complete graphs have no vertex cuts.

Definition: The connectivity $\kappa(G)$ is the minimal size of a vertex cut for graphs with ≥ 2 non-adjacent vertices. Otherwise $\kappa(G) = \nu(G) - 1$. Note that if $\nu(G) = 1$ or G is disconnected, $\kappa(G) = 0$. If $\kappa(G) \geq k$, G is k-connected.

Definition: The edge-connectivity $\kappa'(G)$ is 0 if $\nu(G) = 1$ or G is disconnected. Otherwise, $\kappa'(G)$ is the minimal size of an edge cut. If $\kappa'(G) \geq k$, G is k-edge-connected.

Theorem 3.1: $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Theorem 3.2: A graph G is 2-connected if it has ≥ 3 vertices and any 2 vertices are connected by 2 internally disjoint paths.

Corollary 3.2.1: Suppose $\nu(G) \geq 3$. G is 2-connected iff any 2 vertices of G lie on a common cycle.

Corollary 3.2.2: Suppose $\nu(G) \geq 3$ and G is a block. Any 2 edges of G lie on a common cycle.