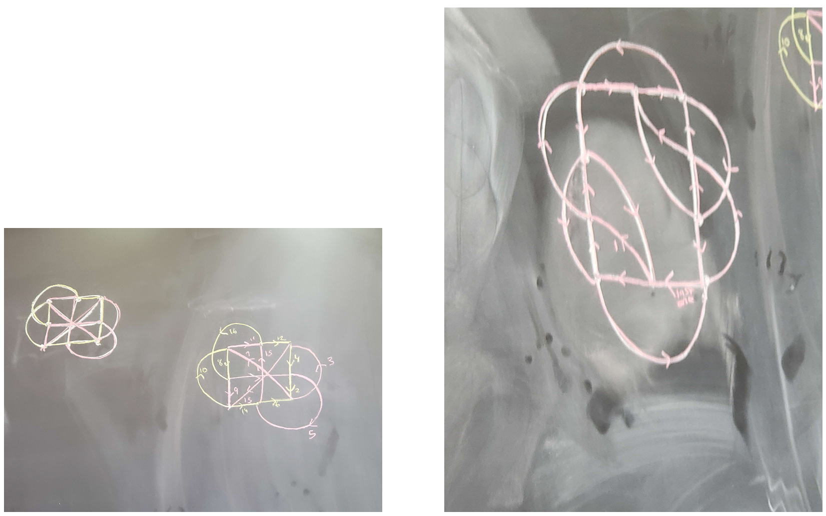
HW 6 - Graph Theory

Ethan Beaird

March 12, 2023

1.

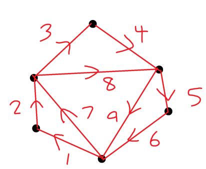


(a) (b)

Figure 1

2.

a.



b.

Proof: Suppose some bipartite graph G contains an Euler cycle. So there exists some cycle in G that crosses each edge necessarily once without repeating vertices. Since G is bipartite $V(G) = X \sqcup Y$ with no vertex in X adjacent to another vertex in X and no vertex in Y adjacent to another vertex in Y. WOLOG, we consider X. Each degree in X is the number of incident edges with the other end in Y (since G is simple and bipartite, there are no loops, multiple edges, or edges between two elements in the same subset). So $\epsilon(G)$ is exactly the degree of X. G is Eulerian, so all vertices in X must have even degree. $\therefore$ The sum of all of these even degrees is still even, and the total degree of X is the number of edges. So G has an even number of edges.

□

c.

Proof: We know any Eulerian bipartite graph has an even number of edges. Since G is complete, in $K_{m,n}$, vertices in $X \subset V(G)$ must have degree n and vertices in $Y \subset V(G)$ must have degree m. Since all vertices have even degree, and by part b, any even value of m and n holds.

□

3.

a.

Proof: Suppose the restriction $\nu \geq 3$ is no longer necessary. A graph may not have negative degrees, so we consider the following cases:
Case 1: $\nu = 0$: There are no vertices in G. Clearly, G contains no cycles. ν cannot be 0.
Case 2: $\nu = 1$: We know that δ must be $\geq \nu/2$. So δ must be at least 1. But our graph is simple and we have only one vertex, so its degree is 0. So ν cannot be 1.
Case 3: $\nu = 2$: Consider G with $\nu = 2$ and $\delta = \frac{2}{2} = 1$. If G is disconnected, then G has two components of 1 vertex with no edges (since G is simple). We showed in Case 2 that these components both violate $\delta = \frac{\nu}{2}$. Assume G is connected. Since G is simple, there may only be $\binom{2}{2}$= 1 edge in G. So $u, v \in V(G)$ have maximum degree 1, and $\delta = 1 = \frac{\nu}{2}$.
However, despite not violating our delta condition, starting at either u or v and traversing to the other leaves us "stranded" and unable to return to the starting vertex to complete the cycle (since there is only 1 edge joining u and v and it is already crossed).

□

b.

Proof: For $K_{m,n}$ to be Hamiltonian, it must contain a Hamilton cycle, i.e. a cycle that visits each vertex in G necessarily once. Starting from partitioned subset X in $K_{m,n}$, it is clear that one must perform an even number of steps to return to X (if this were not true, then there would exist some edge between 2 vertices in the same subset which is impossible).

So odd # of steps, starting from X, means you are in Y.
and even # of steps, starting from X, means you are in X.

So in order to return to the initial vertex in a Hamilton cycle, the cycle's length must be even. Additionally, $\nu(Y)$ must equal $\nu(X)$. If this were not the case, then one may either start in the smaller set or larger set.
1) WOLOG, consider X to be the smaller set, so $m < n$. We start in X. With m vertices in X, we have $m-1$ remaining vertices to touch in X and n remaining vertices to touch in Y. Every 2 steps, we touch one more vertex in X and one more vertex in Y. After we perform 2-steps k times (as our cycle must be even), we have $m-k-1$ vertices in X left and $n-k$ vertices in Y left. It follows that when 0 vertices in X remain after k_f 2-steps, $n-k_f$ vertices remain in Y, and we are currently on the last vertex in X. But since $m < n$, we know that some number of vertices remain in Y, so we step to Y. But we can no longer reach our initial X vertex to close the cycle since G is bipartite.
2) WOLOG, consider X to be the larger set, so $m > n$. We start in X. With m vertices in X, we have $m-1$ remaining vertices to touch in X and n remaining vertices to touch in Y. Every 2 steps, we touch one more vertex in X and one more vertex in Y. After we perform 2-steps k times (as our cycle must be even), we have $m-k-1$ vertices in X left and $n-k$ vertices in Y left. Since $m > n$, we know that Y will reach 0 remaining vertices before X will. It follows that when 0 vertices remain in Y after k_f steps, $m-k_f-1$ vertices remain in X, and we are in X. The vertex we are on may be the last untouched vertex in X, but it is still not a cycle (since G is bipartite and it is necessary to traverse a vertex in Y to return to the initial vertex in X, but no vertices remain in Y).
$\therefore \nu(X) = \nu(Y)$. However, we know that if $\nu(X) = \nu(Y) = 1$, then there would be only edge joining the two vertices. As shown earlier, there is no Hamilton cycle. So $k_{m,n} is Hamiltonian$ if $\nu(X) = \nu(Y)$ so long as ν is at least 2.

□

c.

Proof: Suppose G is Hamiltonian. Then each vertex in G belongs to one cycle. We know cycles are 2-connected (since removing only 1 vertex cut leaves a path connected all $v \in V(G)$ but removing 2 vertex cuts leaves two paths that are disconnected). So G is 2-connected.

□

4.

a.

Proof: G is connected an an edge-disjoint union of cycles. So these cycles may share a number of vertices (since G is connected, we know they must share at least 1), but they may share no edges. We construct an Euler cycle as follows: Consider cycles $C_1, C_2, C_3, ..., C_n$ in G. Consider two arbitrary cycles, C_a and C_b. Suppose C_a and C_b shared more than one vertex, say $u, v \in V(G)$. Then there exists a cycle starting at u, following the path from u to v in C_a, then following the path from v to u in C_b (if there were no cycle, then C_a and C_b are not edge-disjoint which is impossible). But clearly this newly created cycle is not edge disjoint with C_a and C_b. So no two cycles in G may share more than one vertex.

We can now construct an Euler cycle in G. Pick an arbitrary point v_0 in C_a and begin traversing it in any direction. If you encounter another cycle, C_b at $u_0 \in V(G)$, follow C_b until you either return to u_0 or encounter another cycle C_k. If you return to u_0, continue in the same direction as previously in C_a and return to v_0, completing an Euler cycle. If you encounter another cycle C_k, repeat this process until you return to v_0, completing an Euler cycle. So G always contains an Euler cycle and is thus Eulerian.

□

b.

Proof: G is Eulerian and connected, so there exists some cycle in G that uses each edge in G necessarily once. We prove by induction.
Base Case: Each $v \in V(G)$ has maximal degree = 2. We know that if G is Eulerian, each vertex in G has even degree. So each vertex may have degree 0 or 2. But since G is connected, only two cases exist: G has one vertex with degree 0 or all vertices with degree 2 (if this were not the case, then at least one vertex with degree 0 would be disconnected from the rest of G since it is connected to no other vertices). But if $d(v_0) = 0$, and v_0 is the only vertex in G, this contradicts our maximal degree of 2. So this is impossible. Then, $d(v) = 2$ for all $v \in V(G)$. Start at any arbitrary vertex in G, say v_0. All vertices have degree = 2, so moving to an adjacent vertex burns 1 degree of v_0 and 1 degree of v_1. We repeat this process for $v_2, ..., v_n$, burning 1 degree from v_i and 1 degree from v_{i+1} each time. G is finite, so this path must eventually end. Only one vertex remains with an unburnt edge: v_0 (since we only burned one degree initially). On v_n, we move to v_0, burning 1 degree from both v_n and v_0, completing our Euler cycle.

Inductive Step: We know that an Eulerian graph has all even degrees vertices. Suppose the maximal degree in $G = 2k$ holds and G is Eulerian. We must show this holds for maximal degree in $G = 2(k+1)$. Some $v \in V(G)$ has degree

$= 2(k+1)$, which we know is even since it is divisible by 2. Start at any arbitrary vertex in G, say v_0. If $d(v_0) = 2k$, our assumption holds. If $d(v_0) = 2(k+1)$, we know just 2 more degree is added. We follow one of these incident edges, burning 1 degree, and continue with our construction. We repeat this process (each time following new edges on vertices with degree $2(k+1)$ if they exist) until we are forced to eventually return to v_0, burning our extra degree in each recursive return. We then complete our cycle like normal or repeat this step if any more vertices with degree $= 2(k+1)$ exist.

□

1 5.

1.1 a.

Consider $K_{m,n}$ partitioned into two subsets, X and Y, with $\nu(X) = m$ and $\nu(Y) = n$. Suppose $m < n-1$ (or equivalently $n < m-1$. WOLOG, we will only consider our first scenario). We start in X. With m vertices in X, we have $m-1$ remaining vertices to touch in X and $n-1$ remaining vertices to touch in Y. Every step, we touch one more vertex in Y then one more vertex in X. But we know that m is at least 2 vertices smaller than n, so suppose we have stepped through such that $m = 0$. After k steps, $m = 0$. But we know there are at least 2 other vertices in Y that must be touched, so we step to one of them. However, now we have nowhere to move to in X to touch the final vertex in Y. So we know that $n-1 \leq m \leq n+1$ (since WOLOG we may swap X and Y and reach a similar conclusion to find both bounds).

□

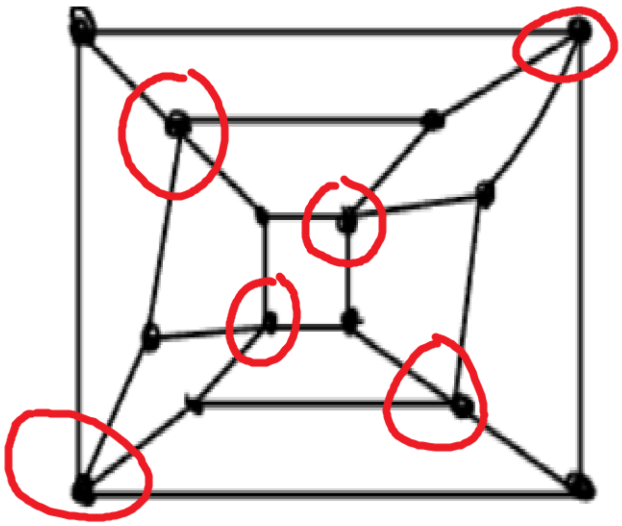
1.2 b.

Proof: Suppose $|S| = 1$, so S is just one vertex. Then, deleting S from G breaks G into at most 2 components, G_1 and G_2, since V is on a Hamilton path. If $|S| > 1$, say k, each vertex removed from partitions S_i breaks each G_i into at most $S_i + 1$ components (by assumption).
So $|S_1| + 1 + |S_2| + 1 = (|S| - 1) + 2 = |S| + 1 = k + 1$ ($|S| - 1$ since $|S_1| + |S_2|$ is missing just one vertex- the initially removed vertex). So our statement holds.

□

1.3 c.

Proof: If G has a Hamilton path, then $\omega(G-S) \leq |S|+1$. Let S be the vertices indicates in the graphic below. So $|S| = 6$. Clearly, $\omega(G-S) = 8$.



This violates our inequality, so G has no Hamilton path.

$\square$