Arc lifting for the Nash manifold

Abstract. Let M be a singular normal irreducible complex manifold of dimension n. Let $f : U \to M$ be the smooth Nash manifold. Let $\gamma : R \to M$ be an analytic map from a connected Riemann surface whose image contains a non singular point of M. Then there is a unique $\gamma' : R \to U$ such that $\gamma = f \circ \gamma'$.

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1. Theorem. Let M be a singular normal irreducible complex manifold of dimension n. Let $f : U \to M$ be the smooth Nash manifold. Let $\gamma : R \to M$ be an analytic map from a connected Riemann surface whose image contains a nonsingular point of M. Then there is a unique $\gamma' : R \to U$ such that $\gamma = f \circ \gamma'$.

This follows from

2. Theorem. Let M be an irreducible normal complex singular manifold of dimension n. Let $V \subset M$ be an irreducible singular manifold including a nonsingular point of M. Then V has a proper transform in the Nash manifold $U \to M$ if and only if the union of the stable locus of proper transforms of V in finite sequences of Nash blowups of M is proper over V.

Proof. Earlier we mentioned a necessary and sufficient condition to factorize γ through $f : U \to M$.[1] Namely, there are relatively base point free divisors D_i on U such that

$$-\frac{1}{n+1}K_U = D_1 + D_{(n+2)} + D_{(n+2)^2} + \ldots.$$

Somewhat strangely, this formula only makes sense formally or p-adically for some prime divisor p of $n+2$, in the sense that the right side describes in the coefficients of each prime divisor what is eventually a geometric series to the base of $n+2$. On unbounded subsets the tails may intersect trivially.

A complete curve in U where the right side restricts to a finite sum merely maps to a singular point of M (always sharing its fiber with a complete Riemann surface of the other type).

If one point of R is singular in M, the necessary and sufficient condition to lift the arc γ is that the series of intersection numbers

$$\gamma \cdot D_{(n+2)} + \gamma \cdot D_{(n+2)^2} + \gamma \cdot D_{(n+2)^3} + \ldots$$

is eventually (necessarily nonzero) geometric to the base of $(n+2)$.

[1]Theorem 12, Functorial affinization of Nash's manifold, 22 April 2010

Consider first the case when the anticanonical divisor $-K_U$ is effective. Write the right side as

$$\frac{1}{n+1}(-D_1 + ((n+2)D_1 - D_{(n+2)}) + ((n+2)D_{(n+2)} - D_{(n+2)^2}) + ...)$$

Multipying both sides by $n+1$ gives

$$-K_U = -D_1 + ((n+2)D_1 - D_{(n+2)}) + ((n+2)D_{(n+2)} - D_{(n+2)^2}) + ...$$

and each term in round parentheses is the negative of an effective divisor.

It does not matter whether we view these as divisors on the Nash manifold itself or, term-by-term, each on a sufficiently high Nash blowup that they become Cartier. This expresses an effective Cartier divisor $-K_U$ as another Cartier divisor $-D_1$ (which is necessarily effective as a consequence of our assumption that $-K_U$ is effective and the fact that the $((n+2)D_{(n+2)^{i+1}} - D_{(n+2)^i})$ are) minus effective divisors. D_1 is the extension of K_M across U, starting with the smooth locus of M, obtained by pulling back $\Lambda^n \Omega_M$ along $U \to M$.

The hypothetical limiting natural number $-K_U \cdot \gamma$ is the Lefschetz product $-D_1 \cdot \gamma$ plus a sum of negative numbers, which must be finite. (We allow some of the finitely many components P of $|-D_1|$ containing our smooth point of $\gamma(R)$ to contain the whole of it under all linear equivalences which keep $|-K_U|$ effective; the same integer multiple of each $P \cdot \gamma$ occurs on both sides of the equation and it does not matter what value we should assign to it.)

In the case when we are considering V with a stable formal proper transform, the divisors restrict to that formal stable proper transform, where we have a decreasing limit of effective divisors. Any such limit converges.

In either case, what we have then is a directed system of line bundles which converges to a limiting line bundle.

Let's represent the limiting bundle as the restriction of the canonical bundle of U along an actual morphism. A rough argument is that on any bounded open set, the effective Cartier discrepancy

$$K_{i+1} - K_i = D_{(n+2)^{i+1}} - (n+2)D_{(n+2)^i}$$

must restrict to a principal divisor for sufficiently large i, which can be moved away from the stable proper transform of V.

This is true, but a more careful consideration shows that when the divisor restricts to a principal divisor, it actually only implies that the Nash blowups are finite maps, at points of the tansform of V within that bounded neighbourhood. Consideration of the analytic local rings of Nash blowups of M at such points of the stable proper transform, and the theorem of finitness of normalization, implies then that each bounded open subset of the stable proper transform is eventually contained in the smooth locus of a finite Nash blowup, and therefore contained in an open subset of U, which is the union of these.

This finishes the proof of theorems 1 and 2 in the case $-K_U$ is effective. Next, one can argue as follows: it is likely that if we consider any point $p \in M$, there is a neighbourhood of p in M whose smooth Nash manifold has an effective anticanonical divisor. This can be seen various ways. One way to think of it is that however singular the point p may be, we can find n flows which are everywhere defined in a neighbourhood W of p and transverse where they cross some other point q in that neighbourhood.

For a moment use the letter U to denote the Nash manifold of W. Since $U \to W$ is natural the flows lift to flows on U. Evaluating on the wedge product of the corresponding global vector fields on U gives a map $\mathcal{O}_U(K_U) \to \mathcal{O}_U$ showing that $-K_U$ is effective.

The manifold U is admittedly slightly strange, as the critical locus of the map to W can, we must allow, consist of infinitely many irreducible components. Thus it may make sense to argue a bit more carefully, also for generalization to the relative or number theoretic case, and this can be done using the actual $\mathcal{F}_i$ which pull back to $\mathcal{O}(D_i)$. The tuple of vector fields on W gives a map

$$\Omega_W/torsion \to B,$$

where B is nothing but the section sheaf of a trivial bundle of rank n. The deRham differential gives a connection

$$d : \mathcal{O}_W \to \mathcal{O}_W \otimes B$$

of course.

Starting from $\mathcal{F} = \mathcal{O}_W$, the sequence of sheaves

$$\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, ...$$

now each has a corresponding holomorphic connection

$$\nabla : \mathcal{F}_i \to \mathcal{F}_i \otimes B$$

and the sheaf of n forms mod torsion in the i'th Nash blowup $\pi : W' \to W$ is

$$(\pi^*\mathcal{F}^{n+1}\pi^*\mathcal{F}_{(n+2)^i-1}/torsion)^{-1}\pi^*\mathcal{F}_{(n+2)^i}/torsion.$$

In more detail, let

$$I = \mathcal{F}_1...\mathcal{F}_{(n+2)^{i-1}}$$

then

$$\mathcal{F}_{(n+2)^i-1} = I^{n+1}$$

From the inductive definition of $\mathcal{F}_{(n+2)^i}$ the n forms are expressed as

$$(\pi^*\mathcal{F}\pi^*I/torsion)^{-n-1}\pi^*\Lambda^{n+1}\mathcal{P}(\mathcal{F}I)/torsion \qquad (1)$$

Once t_i are local sections which span I and x_i are local coordinates generating $\mathcal{O}_W$ then letting the s_i be the products $t_j x_k$ we have that $\Lambda^{n+1}\mathcal{P}(I)/torsion$ is spanned by the

$$(s_0 \oplus \nabla s_0) \wedge ... \wedge (s_n \oplus \nabla s_n).$$

The connection $1 \oplus \nabla$ is a connection with values in principal parts rather than differentials, which is essentially a formal object whose existence is assured because of properties of the pullback of a coherent sheaf to its own Fibré Vectoriel.

This can be expanded in terms of a tangible connection ∇ though as

$$\sum_{i=0}^{n}(-1)^i s_i \nabla(s_0) \wedge ... \wedge \widehat{\nabla\ s_i} \wedge ... \wedge \nabla s_n \qquad (2)$$

$$= s_0 ... s_n \sum_{i=0}^{n}(-1)^i \nabla log(s_0) \wedge ... \wedge \widehat{\nabla\ log s_i} \wedge ... \wedge \nabla log s_n$$

where the hat denotes a deleted term.

For any i, j we have

$$\nabla(s_j) = \nabla(s_i \frac{s_j}{s_i})$$

$$= \frac{s_j}{s_i}\nabla(s_i) + s_i \otimes d(\frac{s_j}{s_i})$$

and so

$$\nabla log\ s_j - \nabla log\ s_i = dlog(\frac{s_j}{s_i}).$$

Using this, eliminate ∇ from the answer to obtain an expression which is just one term, not a sum and not involving ∇

$$s_0 ... s_n \cdot dlog(s_1/s_0) \wedge ... \wedge dlog(s_n/s_0).$$

This shows that the answer does not depend on ∇; we are allowed to choose it.

When i is not an $(n+2)$ power but has an expansion

$$\sum a_j (n+2)^j$$

with $0 \le a_j < (n+2)$ $\mathcal{F}_i$ is defined to be $\prod \mathcal{F}_{(n+2)^j}^{a_j}$. Thus all $\mathcal{F}_i$ contain $\mathcal{F}_1^i$ and any connection ∇ on $\mathcal{F}_1$ extends to a meromorphic connection

$$\mathcal{F}_i \xrightarrow{\nabla} \mathcal{F}_i \otimes \Omega_W.$$

However, the connection which is constructed explicitly using a basis of local derivations (which we can even take to be commuting) is a holomorphic connection

$$\mathcal{F}_i \to \mathcal{F}_i \otimes B$$

for all i. The formula (2) is linear of degree $n+1$ with respect to multiplying the s_i by a meromorphic function, as the later calculation explains. By the product rule the connection gives

$$\mathcal{F}I \to \mathcal{F}I \otimes B$$

and the coefficients in $\mathcal{F}I$ which is locally principal once pulled back to W' pass through (2) and cancel in (1) leaving

$$\Lambda^n \Omega_{W'}/torsion \to \Lambda^n(\mathcal{O}_{W'} \otimes B) \cong \mathcal{O}_{W'}.$$

The transition map

$$(\tau^*(\mathcal{F}\mathcal{F}_1 ... \mathcal{F}_{(n+2)^{i-1}})/torsion)^{-n-1}\tau^*\mathcal{F}_{(n+2)^i}/torsion$$
$$\to (\tau^*(\mathcal{F}\mathcal{F}_1 ... \mathcal{F}_{(n+2)^i})/torsion)^{-n-1}\tau^*\mathcal{F}_{(n+2)^{i+1}}/torsion.$$

for the Nash blowup $\tau : W'' \to M$ of W' is the evident rearrangement of terms in

$$(\mathcal{F}\mathcal{F}_1 ... \mathcal{F}_{(n+2)^i})^{n+1}\mathcal{F}_{(n+2)^i} \to (\mathcal{F}\mathcal{F}_1 ... \mathcal{F}_{(n+2)^{i-1}})^{n+1}\mathcal{F}_{(n+2)^{i+1}}.$$

induced by carrying $\mathcal{F}_{(n+2)^i}^{n+2} \to \mathcal{F}_{(n+2)^{i+1}}$.

The inclusion commutes with this transition map; and in the limit on U it embeds $\mathcal{O}_U(K_U) \to \mathcal{O}_U$ as the coherent sheaf of ideals defining an effective anticanonical divsor. Thus if one doesn't wish to trust an argument based on symmetry alone, it is possible to prove that the anticanonical divisor of the Nash manifold is linearly equivalent to a particular effective divisor when restricted to the inverse image of sufficiently small open neighbourhoods of a point. QED

Remark. It is probably nicer to show that there is always a locally principal sheaf J on M conducting n forms into an ideal sheaf on U, analogous to how $dlog(s_1/s_0)$ has no zeroes on the projective line being a section of $\mathcal{O}(-[s_1]-[s_0])$. Maybe $\mathcal{F} \cong \mathcal{O}_W$ on open sets W are restrictions $\mathcal{F}(W)$ of some $\mathcal{F}$ with connection on M. J might be $\Lambda^{-n}B$ for B containing the one forms of M, with the meromorphic extension $\mathcal{F}_i \to \mathcal{F}_i \otimes B$ holomorphic everywhere.