An explicit Chern character

Let M be a normal singular complex manifold, and let D be an effective irreducible Cartier divisor on M.

1. Theorem. The characteristic cohomology class in $H^1(M, \Omega_M)$ of the divisor D is the image in $Ext^1(\mathcal{O}_M, \Omega_M)$ of the extension class of the Poincare residue exact sequence itself

$$0 \to \Omega_M \to \Omega_M(logD) \to i_*\mathcal{O}_D \to 0$$

under the map induced by $\mathcal{O}_M \to i_*\mathcal{O}_D$ in the contravariant argument of Ext.

We'll give the proof after giving a lemma which explains the situation. First some preliminaries. For $\mathcal{L}$ locally free coherent of rank one on M let L be the line bundle with section sheaf $\mathcal{L}^{-1}$ dual to $\mathcal{L}$. Also then $\mathcal{L}$ is isomorphic to the push-forward to M of the sheaf of holomorphic functions on L which are linear transformations when restricted to each line fiber.

The sheaf$\mathcal{P}_M(\mathcal{L})$ of first principal parts of $\mathcal{L}$ is isomorphic to the pullback $i^*\Omega_L(log\ M)(-M)$ where $i : M \to L$ is the inclusion of M as the zero section of L.

Since $\Omega_L(log\ M)(-M)$ is the kernel of $\Omega_M \to i_*\Omega_M$ there is the exact sequence

$$0 \to \Omega(log\ M)(-M) \to \Omega_L \to i_*\Omega_M \to 0.$$

Upon pulling back to M the inclusion of the kernel term is no longer an inclusion; rather, there results a four term exact sequence which can be interpreted as the result of splicing together the two sequences below

$$\begin{array}{ll} 0 \to \mathcal{L} \otimes \Omega_M \to i^*\Omega_L(log\ M)(-M) \to \mathcal{L} \to 0 & \\ & 0 \to \mathcal{L} \to i^*\Omega_L \to \Omega_M \to 0. \end{array}$$

This implies, as I've mentioned elsewhere, that $\mathcal{L}\otimes\Omega_M \cong \mathcal{T}or^1_L(\mathcal{O}_M, \Omega_M)$.

Note that $i^*\mathcal{O}_L(-M) \cong \mathcal{L}$. This restriction is also the same as $\mathcal{I}_M \otimes \mathcal{O}_M \cong \mathcal{I}_M/\mathcal{I}_M^2$ if $\mathcal{I}_M$ is the defining ideal sheaf of the zero section M.

When L has a nonzero global section, we can interpret this as a submanifold of L which intersects M along a divisor D, and the defining ideal sheaf $\mathcal{I}_{M'}$ of the section M' is abstractly isomorphic to $\mathcal{I}_M$ as an invertible sheaf on L. And now $\mathcal{L}$ is not only isomorphic to $\mathcal{I}_M/\mathcal{I}_M^2$, it is also isomoprhic to $i^*\mathcal{I}'_M$ which we can interpret as the defining ideal sheaf of the effective divisor which is the restriction of the divisor $M' \subset L$ along the inclusion $M \to L$.

If we call this divisor D then we have $\mathcal{O}_M(-D) \cong \mathcal{L}$.[1]

Now let's use the hypothesis that D is irreducible. The theorem will follow from this lemma relating the residue sequence of D with the principal parts sequence.

2. Lemma. When D is irreducible there is a doubly exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \to & \mathcal{L} & \to & \mathcal{L} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_M & \to & i^*\Omega_L(log\ M) & \to & \mathcal{O}_M & \to & 0 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
0 & \to & \Omega_M & \to & \Omega_M(log\ D) & \to & \mathcal{O}_D & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the middle row is isomorphic to the principal parts sequence twisted by $\mathcal{L}^{-1}$, that is,

$$0 \to \Omega_M \to \mathcal{L}^{-1} \otimes \mathcal{P}(\mathcal{L}) \to \mathcal{L}^{-1} \otimes \mathcal{L} \to 0$$

and the lower row is the residue sequence for the effective Cartier divisor D on M.

[1] If D is not only effective, but also ample, then L has a one point compactification which has an analytic structure.

3. Remark. If D is assumed only to be a normal crossing divisor rather than irreducible, the same is true with $\mathcal{O}_D$ replaced by the direct sum $\mathcal{O}_{D_1} \oplus ... \oplus \mathcal{O}_{D_m}$ where $D_1, ..., D_m$ are the irreducible components of D.

We will not give the full proof of the lemma here here, in this setting where everything is Cartier, it can be easily verified in coordinates.

Note that this whole diagram can be reconstructed starting only with an effective Cartier divisor D on M. That is we can take $\mathcal{L} = \mathcal{O}_M(-D)$ and we can take L to be the line bundle whose section sheaf is $\mathcal{O}_M(D)$.

The diagram shows that $i^*\Omega_L(log\ M)$ can be constructed as the pullback[2] of $\mathcal{O}_M$ and $\Omega_M(log\ D)$ along the maps of coherent sheaves to $\mathcal{O}_D$.

Proof of theorem. In the diagram of the lemma, since the middle row is just a pullback of the lower row, the relevant cohomology class which all the other descriptions give actually is the image under the map

$$Ext^1_M(i_*\mathcal{O}_D, \Omega_M) \to Ext^1_M(\mathcal{O}_M, \Omega_M) = H^1(M, \Omega_M)$$

That is, the desired cohomology class is the image of the very extension class represented by the residue sequence for the divisor D on M.

[2]Incidentally, twisting the whole diagram by $\mathcal{L}$ also shows that the sheaf $\mathcal{P}_M(\mathcal{L})$ of first principal parts of M can be constructed as a pullback of $\Omega_M(log\ D)(-D)$ and $\mathcal{L} = \mathcal{O}_M(-D)$ along maps to $i^{\mathcal{L}} \cong \mathcal{I}_D/\mathcal{I}_D^2 \cong \mathcal{N}_{D/M}$, the latter being the pushforward to M of the conormal sheaf of D.

4. Remarks(relation with existing definitions). The diagram in the lemma relates any of the four or five equivalent definitions of characteristic classes to an image of something lower dimensional.

i) The extension class of the middle row, in $Ext^1(\mathcal{O}_M, \Omega_M) \cong H^1(M, \Omega_M)$ is the type $(1,1)$ cohomology class associated to the divisor D for example by Atiyah's definitions.

ii) At the same time, the Ω_M torsor consisting of the inverse image of $1 \in \Gamma(D, \mathcal{O}_D)$ in $\Omega_M(log\ D)$ is the additive Ω_M torsor on M which consists of one forms on M with a simple pole of residue 1 on D.

iii) Thirdly, in this same construction, if we choose local defining equations f_i for D on a trivial open cover of M, a Cech cocycle defining the torsor consists of the differences $df_i/f_i - df_j/f_j = d\ log(f_i/f_j)$. Here f_i/f_j is invertible on a pairwise intersection $U_i \cap U_j$ in the open cover, so that $d\ log(f_i/f_j) \in \Gamma(U_i \cap U_j, \Omega_M)$.

iv) Griffiths and Harris' text explains (though this needs some minor corrections) that we can trivialize the cocycle in the smooth category, choosing smooth functions h_i on U_i so that $dh_i - dh_j = df_i/f_i - df_j/f_j$. The differences being meromorphic are sent to zero under d' with d' the conjugate deRham differential and so $d'd(h_i - h_j)$ is a global $(1,1)$ form representing the same cohomology class in real deRham cohomology.

v) the cocycle assigning to a local function r and a section ω of $\mathcal{L}$ the tensor product $\omega \otimes dr$ is a Hochschild cocycle of $\mathcal{L}$ with coefficients in $\mathcal{L} \otimes \Omega_M$ and it can be interpreted as defining the principal parts exact sequence as a nontrivial bimodule extension of two trivial bimodules (trivial meaning the left and right actions agree), which happens to be determined unqely by one of the two-sided extensions.

Higher terms of the Chern character

It is convenient now to define the classes $i!ch_i(D)$ which in the case when M is compact Kahler are just the cup product powers D^i.

The first theorem above gave a description of the first Chern class of the divisor D on the manifold M, which equals $ch_1(D)$. However the same proof more generally shows the following:

5. Theorem. Let D be an irreducible divisor on a complex manifold M and let $\mathcal{L} = \mathcal{O}_M(-D)$. Then for each number i the i-extension

$$0 \to \Lambda^i\Omega_M \to \mathcal{L}^{-i} \otimes \Lambda^i\mathcal{P}(\mathcal{L}) \to \mathcal{L}^{-i+1} \otimes \Lambda^{i-1}\mathcal{P}(\mathcal{L}) \to ...$$
$$... \to \mathcal{L}^{-1} \otimes \mathcal{P}(\mathcal{L}) \to \mathcal{O}_M \to 0$$

representing $i!ch_i(D)$ can be built from the $i-1$'st by splicing the exact sequences

$$0 \to \Lambda^i\Omega_M \to \mathcal{L}^{-i} \otimes \Lambda^i\mathcal{P}(\mathcal{L}) \to \Lambda^{i-1}\Omega_M \to 0$$
$$0 \to \Lambda^{i-1}\Omega_M \to \mathcal{L}^{-i+1} \otimes \Lambda^{-i+1}\mathcal{P}(\mathcal{L}) \to ... \to \mathcal{O}_M \to 0.$$

Furthermore the first of these is isomorphic to the pullback of

$$0 \to \Lambda^i\Omega_M \to \Lambda^i\Omega(log\ D) \to i_*\Lambda^{i-1}\Omega_D \to 0$$

along $\Lambda^{i-1}\Omega_M \to i_*\Lambda^{i-1}\Omega_D$ where as before $i : D \to M$ is the inclusion.

Using properties of i extensions we can then replace the defining i extension of the Chern character with a sequence in which all intermediate terms are sheaves of logarithmic differentials on successively smaller varieties, under suitable hypotheses

6. Corollary. Let D be an irreducible effective Cartier divisor and let i be a number. Suppose that D is sufficiently movable that there are linearly equivalent effective irreducible divisors $D_1, D_2, ..., D_i$ such that the filtration $D_1 \supset D_1 \cap D_2 \supset ... \supset D_1 \cap D_2 \cap ... \cap D_i$ consists of distinct irreducible subvarieties. Then the element $i!ch_i(D) \in H^i(M, \Lambda^i\Omega_M)$ is the image of the extension class of the sequence

$$0 \to \Lambda^i\Omega_M \to \Lambda^i\Omega_M(log\ D) \to \Lambda^{i-1}\Omega_D(log\ D^2) \to \Lambda^{i-2}\Omega_{D^2}(log(\ D^3)) \to ...$$
$$... \to \Omega_{D^i}(log\ D^i) \to \mathcal{O}_{D^i} \to 0.$$

in $Ext^i(\mathcal{O}_{D^i}, \Lambda^i\Omega_M)$ under the map induced by $\mathcal{O}_M \to \mathcal{O}_{D^i}$.

Note that more rigorously if we label the inclusion $D^i \to M$ by a letter j we should be writing $j_*\mathcal{O}_{D^i}$ however we suppress writing the push-forwards explicitly, and this is not going to cause confusion as push-forwards is essentially a forgetful functor in this setting.

Note also that we're using an imprecise notation by referring to an intersection $D_1 \cap ... \cap D_i$ as D^i; making the statement precise would involve unending complexity whereas what is important is only the proof of the corollary.

Note finally that one can state and prove a similar result for normal crossing divisors rather than irreducible divisors; for example, for the highest term of the Chern character of a normal crossing divisor on a manifold M of dimension n there will be an extension for each n fold transverse intersection which consists of a single point of M, and extension classes can be added in $Ext^n(\mathcal{O}_M, \Lambda^n\Omega_M) \cong \mathbb{C}$ once pulled back to $\mathcal{O}_M$ along the separate maps $\mathcal{O}_M \to \mathbb{C}$ to the residue fields of the individual points.

The example of a line in the projective plane will illustrate how the proof goes in general.

7. Example. For $\mathbb{P}^1 \subset \mathbb{P}^2$ we have the two exact sequences

$$0 \to \Lambda^2\Omega_{\mathbb{P}^2} \to \Lambda^2\Omega_{\mathbb{P}^2}(log\ \mathbb{P}^1) \to \Lambda^1\Omega_{\mathbb{P}^1} \to 0$$

and

$$0 \to \Lambda^1\Omega_{\mathbb{P}^2} \to \Lambda^1\Omega_{\mathbb{P}^2}(log\ \mathbb{P}^1) \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

which we can pull back along

$$\Lambda^1\Omega_{\mathbb{P}^2} \to \Lambda^1\Omega_{\mathbb{P}^1}$$

and

$$\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^1}$$

respectively to obtain the two principal parts sequences

$$0 \to \Lambda^2\Omega_{\mathbb{P}^2} \to \mathcal{O}(2\mathbb{P}^2)\Lambda^2\mathcal{P}(\mathcal{O}(-\mathbb{P}^1) \to \Lambda^1\Omega_{\mathbb{P}^2} \to 0$$

and

$$0 \to \Lambda^1\Omega_{\mathbb{P}^2} \to \mathcal{O}(\mathbb{P}^1) \otimes \Lambda^1\mathcal{P}(\mathcal{O}(-\mathbb{P}^1) \to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

which splice to give a sequence representing the fundamental class of $\mathbb{P}^2$.

On the other hand, instead of pulling back the second sequence along $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^1}$ we can push it forwards by the map of kernels

$$\Omega_{\mathbb{P}^2} \to \Omega_{\mathbb{P}^1}$$

to the exact sequence

$$0 \to \Lambda^1\Omega_{\mathbb{P}^1} \to \mathcal{O}(\mathbb{P}^0) \otimes \mathcal{P}(\mathcal{O}(-\mathbb{P}^0)) \to \mathcal{O}^1_{\mathbb{P}} \to 0.$$

This in turn is the pullback along

$$\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^0}$$

of the exact sequence

$$0 \to \Lambda^1\Omega_{\mathbb{P}^1} \to \Lambda^1\Omega_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^0} \to 0.$$

Instead of pulling back both, we could have pushed forwards the second sequence along the map of kernels and spliced to get an element of $Ext^1(\mathbb{C}, \Lambda^2\Omega_{\mathbb{P}^2})$. The image of this under $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^0}$ where $\mathbb{P}^0$ is the intersection of two lines is the fudnamental class of $\mathbb{P}^2$.

To see that this is so, one can look at this diagram in which the given second sequence occurs as the second row

$$\begin{array}{ccccccccc}
0 & \to & \Lambda^1\Omega_{\mathbb{P}^2} & \to & \mathcal{O}(\mathbb{P}^1)\otimes\Lambda^1\mathcal{O}(-\mathbb{P}^1) & \to & \mathcal{O}_{\mathbb{P}^2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^1\Omega_{\mathbb{P}^2} & \to & \Lambda^1\Omega_{\mathbb{P}^2}(log\ \mathbb{P}^1) & \to & \mathcal{O}_{\mathbb{P}^1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^1\Omega_{\mathbb{P}^1} & \to & \mathcal{O}(\mathbb{P}^0)\otimes\Lambda^1\mathcal{P}(\mathcal{O}(-\mathbb{P}^0)) & \to & \mathcal{O}_{\mathbb{P}^1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Lambda^1\Omega_{\mathbb{P}^1} & \to & \Lambda^1\Omega_{\mathbb{P}^1}(log\ \mathbb{P}^0) & \to & \mathcal{O}_{\mathbb{P}^0} & \to & 0
\end{array}.$$

The top right and bottom right squares are pullbacks while the left middle square is a push-forward. Note that the sequence of superscripts on the left column is $2,\ 2,\ 1,\ 1$ while the sequence of superscripts on the right column is $2,\ 1,\ 1,\ 0$.

There is a diagram similar to the top two rows here where the exterior powers are increased by one. Each of the two rows of the two-row diagram splices with two of the rows in the diagram shown above to make four possible two-extensions. The two-extensions involving the middle two rows of the diagram shown above then represent just one element of $Ext^2(\mathcal{O}_{\mathbb{P}^1}, \Lambda^2\Omega_{\mathbb{P}^2})$. This element is the image of our element of $Ext^2(\mathbb{C}, \Lambda^2\Omega_{\mathbb{P}^2})$ under the map to the residue field $\mathcal{O}_{\mathbb{P}^1} \to \mathbb{C}$ and in turn it maps to the fundamental class of $\mathbb{P}^2$ as we've constructed it, under the map induced by $\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^1}$ as needed.

Although the example only deals with intersecting two projective lines in the projective plane to arrive at a single point, it is clear that the proof works generally.

i-extensions coming from filtrations

It is possible to view i-extensions as generalizations of filtered objects. For example, if there is a coherent sheaf $\mathcal{F}$ on a complex manifold (or scheme) M with with a filtration

$$0 = F_0 \subset F_1 \subset ... \subset F_i = \mathcal{F}$$

such that

$$F_1(\mathcal{F}) = \Lambda^i\Omega_M$$

then there is the long exact sequence

$$0 \to \Lambda^i\Omega_M \to F_2/F_0 \to F_3/F_1 \to ... \to F_i/F_{i-2} \to F_i/F_{i-1} \to 0$$

which describes an i-extension of F_i/F_{i-1} by $\Lambda^i\Omega_M$.

In the case of a point in the projective plane, the 2-extension is

$$0 \to \Lambda^2\Omega_{\mathbb{P}^2} \to \Lambda^2\Omega_{\mathbb{P}^2}(log\ \mathbb{P}^1) \to \Lambda^1\Omega_{\mathbb{P}^1}(log\ \mathbb{P}^0) \to (\mathcal{O}_{\mathbb{P}^0} \to 0.$$

Writing $\omega_{\mathbb{P}^2}$ as the canonical sheaf of $\mathbb{P}^2$ and $\omega_{\mathbb{P}^1}$ for the caonical sheaf of $\mathbb{P}^1$, This 2-extension can be interpreted as the extension caused by the inclusion

$$\omega_{\mathbb{P}^2} \subset \omega_{\mathbb{P}^2}(\mathbb{P}^1)$$

followed by the extension of the pullback of $\omega_{\mathbb{P}^2}(\mathbb{P}^1)$ to $\mathbb{P}^1$,

$$\omega_{\mathbb{P}^1} \subset \omega_{\mathbb{P}^1}(\mathbb{P}^0).$$

this is not the 2 extension underyling any filtered coherent sheaf; if we were to interpret it as underyling a filtered object, this would have to be something a little more general than a coherent sheaf, where we consider sections over subsets which are not open sets.

That is, we can imagine a single object which we describe as differential two-forms defined on open subwsets of $\mathbb{P}^2$ and which are allowed simple poles on $\mathbb{P}^1$, together with differential one-forms defined on open subsets of $\mathbb{P}^1$ which are allowed simple poles on $\mathbb{P}^0$. We might start with a pair of transverse lines, and the copy of $\mathbb{P}^1$ which we are referring to here is one of the two projective lines, while the copy of $\mathbb{P}^0$ is the intersection point of the two projective lines.

If the lines are labelled L_1, L_2 then the object which corresponds to the two-extension can be considered to consist of sections of $\omega_{\mathbb{P}^2}(L_1+L_2)$ on open subsets of $\mathbb{P}^2$ and also sections of the same sheaf, but on open subsets of $L_1 \subset \mathbb{P}^2$.

The combined object would not be called a sheaf because the definition of a sheaf only specifies sections on open sets of a single space.

Note that the choice of the second line does not affect the object, and it is determined by the filtration

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2.$$

This latter object is a point of the variety of complete flags in a three dimensional vector space. The extra information needed to build the cohomology class (the fundamental class of $\mathbb{P}^2$) in this case is just the choice of defining ideal sheaf of each subspace in the next.

Relation with duality

It is tempting to consider our example further, of the fundamental class of a point in the projective plane. There we considered that the characteristic cohomology class comes from a 2-extension which underlies a filtration, but not a filtration of coherent sheaves.