Primer on elliptic curves

I. Considerations of naturality...

An elliptic curve is a branched double cover of a Riemann sphere in more than one way. The underlying issue of naturality has its home in symmetry of torsors for the finite Klein four-group. The Jacobian of any elliptic curve contains a copy of the Klein four-group, but different elliptic curves have different Jacobians, and it is not right to say any elliptic curve is a representation of *the* Klein four-group.

This section can be subsumed into Galois theory of fields, or also into the theory of braid groups and mapping class groups, or into the theory of covering spaces of manifolds, or, really, into the theory of the finite group S_4 with its normal four-element subgroup. But it nice to start with elliptic curves without making any complicated definition of what they are, just assuming we know them as an axiomatic starting place like the plane in Euclid's theory.

I.1. ...for K_4 torsors.

When we consider the natural permutation representation on a four element set S, there is for each group element g the 'twisted' representation ${}^gS$, defined by the operation $\cdot^g$ so that

$$h \cdot^g s = g^{-1}hg \cdot s.$$

The restriction of the representation to the Klein four-group maps the 24 different twisted representations to six representations of the Klein four-group.

I.2. ... for covering spaces.

Now we can think of what this means on the level of covering spaces. Any four-sheeted cover of, let us say, a complex manifold M has a natural associated six sheeted cover $\widetilde{M} \to M$, which may be disconnected even if M is connected, and with Galois group S_4/K_4. Each fiber of $\widetilde{M} \to M$ is the set of six decompositions of the corresponding fiber of the original cover into an ordered pair of two-element subsets.

I.3. ... for sphere bundles

Now we can think in turn what this means for Riemann sphere bundles. The holomorphic fiber bundles with fiber a Riemann sphere with four points deleted are classified by four-sheeted covers of complex manifolds M together with a S_4/K_4-equivariant period map from the corresponding six-sheeted cover $\widetilde{M} \to \mathbb{P}^1 \setminus \{0,1,\infty\}$. Two such equivariant maps describe isomorphic bundles over M if and only if the period maps agree after composing with a translation of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ belonging to S_4/K_4, in other words a holomorphic automorphism of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The fundamental group of $P^1 \setminus \{0,1,\infty\}$ is the pure braid group on three strands on $\mathbb{P}^1$ and when M is connected equivariance under S_4/K_4 extends the induced map on fundamental groups to a map from the fundamental group of M to the full braid group.

For simplicity let's look at the case when M is connected. The connected components of $\widetilde{M}$ are isomorphic to each other, and each connected component is a copy of the lowest cover where the period map becomes well-defined. If M is a connected curve, this is the 'Riemann surface' of the multivalued period map on M, let us call it λ, and there is a field extension $\mathbb{C}(M) \subset \mathbb{C}(M,\lambda)$ describing the function field of one connected component of $\widetilde{M}$.

That is to say,

1. Remark. The period map underlying a fiber bundle with fibers Riemann spheres with four points deleted needn't be single-valued; it can be multivalued, defining a covering space of the base manifold M of degree at most six.

I.4. ...for bundles of pointed elliptic curves.

Now let's think of what this means for actual bundles of elliptic curves with basepoint (i.e., bundles with a section). Given an elliptic curve J with chosen point p, the subgroup $2H_1(J) \subset H_1(J) \cong \pi_1(J, p)$ defines a natural four sheeted regular cover $\widetilde{J} \to J$. The inverse image of p is a four-element subset of $\widetilde{J}$ which is a torsor for unique Klein four subgroup of any of the four group structures of $\widetilde{J}$ which correspond to a choice of lift of p.

There is a unique involution of $\widetilde{J}$ with this four element subset as its fixed point set, and the quotient modulo the involution is a Riemann sphere with a marked set of four (indistinguishable) points.

Applying what we've already said about Riemann sphere bundles, we can associate to any fiber bundle of elliptic curves with a (chosen) section a Galois cover (which is possibly disconnected even if M is connected) $\widetilde{M} \to M$ of degree six and a S_4/K_4 equivariant period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, uniquely determined up to the six translations (=automorphisms) of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The corresonding four-sheeted cover of $\widetilde{M}$ now has a globally-defined free K_4 action. The twists of the K_4 torsor are just translations and all have become isomorphic on $\widetilde{M}$. Under the covering maps $\widetilde{J} \to J$ the four points in the torsor become identified with one basepoint in each elliptic curve fiber J and we recover the section of the elliptic curve bundle.

I.5. ...for bundles of elliptic curves.

Finally for holomorphic bundles of elliptic curves which may not have a section, and for which we have chosen no section, we must specify a complex manifold M and a bundle J of pointed elliptic curves, and an element of $H^1(M, J)$ to choose a torsor. Thus

2. Theorem. A holomorphic bundle of elliptic curves (without chosen basepoint in each) is determined by choosing a complex manifold M, a four-sheeted cover of M, an S_4/K_4-equivariant period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$, where $\widetilde{M}$ is the corresponding six-sheeted cover, and finally an element of $H^1(M, J)$ where J is the corresponding bundle of pointed elliptic curves. Two S_4/K_4 equivariant period maps with cohomology class determine isomorphic elliptic curve bundles if and only if they agree after an automorphism of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

3. Corollary. In a bundle of elliptic curves which is connected, compact and projective all elliptic curve fibers are isomorphic to each other.

Here we speak literally; in its common usage the term ‘elliptic fibration’ is allowed to refer to a more general situation where not all fibers are elliptic curves. In fact, it is the truth of the corollary which has led to abandoning the use of the term ‘elliptic fibration’ in its literal sense to mean a bundle of elliptic curves, as such a thing in the compact projective case is merely a torsor of a trivial bundle.

To prove the corollary, just observe that once M and therefore $\widetilde{M}$ are compact and projective, the holomorphic period map $\widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ is proper and has compact image. The compact analytic subsets of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ however are discrete.

II. Isogenies

Once an elliptic curve $E \to S$ is a doubly branched cover of a Riemann sphere S, then for each choice of a pair of the four branch points there is an intermediate cover $S' \to S$ branched only at those two points, which is another Riemann sphere. The normalized pullback $E' \to S'$ is another elliptic curve branched over a Riemann sphere. The induced map $E' \to E$ is unbranched. Thus the isogeny $E' \to E$ covers the branched cover $S' \to S$.

The map $E' \to S'$ is now unbranched at the points of E' which map to the two critical points of $S' \to S$.

The isogeny underlies a reduction step in the classical theory of elliptic integrals. The one-form

$$\frac{dz}{\sqrt{z(z-1)(z-\lambda)}}$$

whose Riemann surface is an elliptic curve branched at $0, 1, \lambda, \infty$ is transformed by the substitution $z = t^2$ to the one-form

$$\frac{2dt}{\sqrt{(t^2-1)(t^2-\lambda)}}.$$

whose Riemann surface is again an elliptic curve, now branched at $1, -1, \pm\sqrt{\lambda}$ and unbranched at $0, \infty$.

III. Construction of elliptic curves

Let s be a global section of $T_{\mathbb{P}^1}^{\otimes 2}$. Then $s(\mathbb{P}^1)$ meets the zero section $\mathbb{P}^1$ at a divisor of degree four. The inverse image of $s(\mathbb{P}^1)$ under tensor square

$$T_{\mathbb{P}^1} \to T_{\mathbb{P}^1}^{\otimes 2}$$

is a 2-section of $T_{\mathbb{P}^1}$ which also meets the zero-section at four points. If the four points are distinct, the pullback of the 2-section of the tangent bundle of $\mathbb{P}^1$ to the 2-section viewed as a double cover, splits into a pair of mutually negative 1-sections without zeroes, showing that the 2-section itself has trivial tangent bundle.

While the operation of pulling back is undefined at the branching points, four undefined points in each of the two components amount to 'removable singularities.'

IV. Compactification.

IV.1 Introduction.

This chapter will be a special case of exercise 17 on page 25 of the Park City notes on surfaces by Miles Reid: the case $a = 1$ and $\alpha = 4$. We continue investigating elliptic curves, now knowing that we will need to include some singular fibers to projectively compactify a bundle of elliptic curves other than a torsor of a trivial fundle.

IV.2. Abstract theory.

Let $b \in \mathbb{P}^2$ be a point, and let $C \subset \mathbb{P}^2$ be a curve of degree four which does not pass through b. Consider the the pencil F_1 of projective lines through b considered disjoint from each other (a ruled surface), and its map to $\mathbb{P}^1$ viewed as the set of projective lines through b. Take as M the projective line with those points deleted which correspond to lines that fail to meet C transversely. Take as $\phi : \pi_1(M, m) \to S_4$ the monodromy action on the four points of intersection of m (viewed as a line) with C. Let $\widetilde{M}$ be the connected covering whose Galois group is the image $G = \phi(\pi_1(M, m)) \subset S_4/K_4$. The G-equivariant period map $\lambda : \widetilde{M} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ extends to a finite map from the completion of $\widetilde{M}$ to $\mathbb{P}^1$. Let L be the line bundle on F_1 whose section sheaf is the dual of the defining ideal of the curve C in F_1. Since this has class divisible by two, there is a line bundle N such that $N^{\otimes 2} = L$. Let s be a section of L whose zero variety defines C.

4. Corollary. The inverse image in N of the image $s(F_1)$ in L under the tensor square map is a complete projective variety which is a compactification of the bundle of elliptic curves defined by the period map λ.

IV.3. A double cover of $\mathbb{P}^2$.

Let's start slowly. We begin with the projective plane with given homogeneous coordinates $[x : y : z]$. We choose four general lines; up to automorphisms there is only one choice, and so we are free to choose the the lines defined by the equations

$$\begin{aligned} -2x + y - z &= 0 \\ x + y - z &= 0 \\ x - 2y - z &= 0 \\ -z &= 0 \end{aligned}.$$

Next, instead of considering the curve which is doubly branched over $\mathbb{P}^1$ at four *points*, we consider the surface which is doubly branched over $\mathbb{P}^2$ over these four *lines*. Uniqueness of the choice of lines up to automorphisms means the moduli is reduced to a single point; the inverse image of a general line is an elliptic curve in the surface and the moduli of elliptic curves is algebraically parametrized by the position of this one line.

Thus we consider the surface which is doubly branched over $\mathbb{P}^2$ along these lines. It has six nodes which can be resolved resulting in six pairs of Riemann spheres with each pair crossing at two points, all with normal degree -2.

The complement of a single point of $\mathbb{P}^2$ such as $[0 : 0 : 1]$ can be given the structure of a line bundle. Let M be a line bundle of degree one on $\mathbb{P}^1$. There is an open embedding

$$M \to \mathbb{P}^2$$

with image the complement of the point $[0 : 0 : 1]$, which can be defined like this: label two basic global sections of M with the names x, y, and for each pair of complex constants a, b map the global section $ax + by$ of M to the line in $\mathbb{P}^2$, which does not pass through $[0 : 0 : 1]$, which is defined by the equation $z = ax + by$. There is one point on this line for each value of the ratio $[x : y]$ and each section of M maps isomorphically to a line in $\mathbb{P}^2$ not passing through $[0 : 0 : 1]$.

In this way, we can view our four lines as really being sections of M.

We can construct the double cover of $\mathbb{P}^2$ simply like this. Call our four lines L_1, L_2, L_3, L_4 and consider a line bundle L on $\mathbb{P}^2$ with a section s such that the intersection of the image of s with the zero section equals the union of the four lines

$$s(\mathbb{P}^2) \cap \mathbb{P}^2 = L_1 \cup L_2 \cup L_3 \cup L_4$$

and is transverse except at the six points where two of the lines meet. The section sheaf of $\mathcal{L}$ of L is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(L_1 + L_2 + L_3 + L_4)$, the isomorphism given on local sections as

$$\mathcal{O}_{\mathbb{P}^2}(L_1 + L_2 + L_3 + L_4) \to \mathcal{L}$$
$$r \mapsto rs.$$

There is a line bundle N with $N^{\otimes 2} = L$ and the inverse image of $s(\mathbb{P}^2)$ under tensor square

$$N \to N^{\otimes 2} \cong L$$

is our surface with six nodes.

Let's name our four sections

$$\begin{aligned} e_1 &= -2x + y \\ e_2 &= x + y \\ e_3 &= x - 2y \\ e_4 &= 0 \end{aligned} \qquad (0)$$

Since e_4 and $e_1 + e_2 + e_3$ are zero, the tensor equation simplifies

$$v^{\otimes 2} = z^4 + (e_1e_2 + e_1e_3 + e_2e_3)z^2 - e_1e_2e_3z. \quad (1)$$

The extra factor of z on the right appears to be a slight correction of a mistake by Weierstrass; he had taken e_4 to be the exceptional section, and the surface could not be compactified because the divisor class of the sum of the four lines was not even.

Take our original vector bundle of degree 1 on $\mathbb{P}^1$, of which we've labelled two basic sections x and y, to be one which depends naturally on a choice of $0, 1, \infty$. Namely, we take the line bundle whose local sections are the local holomorphic one-forms with at worst simple (=logarithmic) poles of degree one at those three points. This line bundle extends the line bundle M_2 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose sections are modular forms of weight 2 for $\Gamma(2)$. Hence we may take

$$\begin{aligned} x &= \frac{\pi^2}{3}\theta(0,\tau)^4 d\tau \\ y &= \frac{\pi^2}{3}\theta(0,1+\tau)^4 d\tau \end{aligned}.$$

The map which converts a modular form to a meromorphic one-form with at worst simple poles at $\{0, 1, \infty\}$, both locally and globally, is the one which is represented symbolically by appending $d\tau$. As I've explained more carefully elsewhere, the multiplication occurs as a product of a zero form with a one-form on $\mathbb{H}$ whereas on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the multi-valued form $d\tau$ has a simple pole at all three points.

Let's use the letter M_2 to refer to this vector bundle of degree 1 on $\mathbb{P}^1$ whose restriction to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has as its global sections the weakly modular forms of weight two for $\Gamma(2)$, and as its global sections the actual modular forms of weight two for $\Gamma(2)$.

The projection

$$\mathbb{P}^2 \setminus \{[0:0:1]\} \to \mathbb{P}^1$$
$$[x:y:z] \mapsto [x:y]$$

is a line bundle projection, and the section sheaf of the vector bundle has isomorphism type $\mathbb{O}_{\mathbb{P}^1}(1)$, same as the section sheaf of M_2. For each pair of complex numbers a, b, the line with equation

$$0 = z - ax - by$$

not passing through $[0:0:1]$ is a section of the line bundle, and we may quite simply think that the equation asserts that the fiber coordinate z must equal the section $ax + by$ of $\mathcal{O}_{\mathbb{P}^1}(1)$.

At the same time, the right side of this *equation* of the line is a global section of $\mathcal{O}_{\mathbb{P}^2}(1)$ which defines the same line in $\mathbb{P}^2$ by its intersection with the zero section.

Our equation (1) describes a section v of $\mathbb{O}_{\mathbb{P}^1}(2)$ whose tensor square equals the product of the four sections

$$(z-e_1)(z-e_2)(z-e_3)(z-e_4)$$

a section of $\mathcal{O}_{\mathbb{P}^2}(4)$ defining the union of the four lines.

In the section after next, we'll return to looking at the double cover of $\mathbb{P}^2$. We'll start to consider the lines in the pencil of lines through $[0:0:1]$ to be disjoint, thus resolving the indeterminacy of the map to $\mathbb{P}^1$. This inserts two rational curves with normal bundle degree -1 into the double cover of $\mathbb{P}^2$. The new rational curves just map to points of $\mathbb{P}^2$ but each maps isomorphically to $\mathbb{P}^1$. The inverse image in the double cover of each line in the pencil which doesn't pass through an intersection point of the four lines is an elliptic curve. Before resolving the indeterminacy, all such elliptic curves meet at just two points. After resolving the indeterminacy, the elliptic curves become disjoint and the two points are resolved to two exceptional lines.

First let's include a discussion of Weierstrass' function.

IV.4. Weierstrass' $\wp$ function

Weierstrass' relation for the $\wp$ function

$$(\frac{\partial}{\partial w}\wp(w,\tau))^2 = 4(\wp(w,\tau)^3+(e_1e_2+e_1e_3+e_2e_3)\wp(w,\tau)-e_1e_2e_3) \quad (2)$$

does still hold.

Although (1) and (2) appear to be equations of different degrees, we will need to relate them.

Recall $T : \mathbb{H} \to \mathbb{H}$ is our transformation $T(\tau) = \tau + 1$. Whenever $f : \mathbb{H} \to \mathbb{C}$ is a function, write f^T to be the function defined by $f^T(\tau) = f(T^{-1}\tau)$. We also define the 'coboundary' $i(f) = f - f^T$. Starting with

$$A(w,\tau) = \frac{\theta(w,\tau)^2}{\theta(0,\tau)^2}$$

and

$$g(\tau) = \theta(0, 1+\tau)^4 - 2\theta(0,\tau)^4$$

5. Definition. The $\wp$ function can be defined to be $\pi^2/3$ times the eigenfunction for the action of multiplying by g on i, applied to A. That is,

$$\wp(w,\tau) = \frac{\pi^2}{3}\frac{i(gA)}{i(A)}$$

.

Here is where this definition comes from. For fixed τ, in the elliptic curve $\mathbb{C}$ modulo translation by 1 and τ, the rational functions which express the linear equivalence between the divisor of order two at the point $\tau/2$ and the divisor of order two at the point $(\tau-1)/2$ is a constant a multiple of

$$\frac{\theta(w,\tau)^2}{\theta(w+\frac{1}{2},\tau)^2}.$$

Choose the constant multiple (depending on τ) to be

$$\frac{\theta(w,\tau)^2\theta(0+\frac{1}{2},\tau)^2}{\theta(0,\tau)^2\theta(w+\frac{1}{2},\tau)^2}$$

Under the transformation $(w,\tau) \mapsto (w/(2\tau+1), \tau/(2\tau+1))$ the equation $w = \tau/2$ defining the pole and the equation $w = 1/2 + \tau/2$ defining the zero are both affected by adding an integer multiple of τ to w. The resulting function is therefore invariant under $\Gamma(2)$ modulo scalar multiplications. It is in fact invariant as can be checked on the two generators of $\Gamma(2)$.

Since $\theta(w+1/2,\tau) = \theta(w,\tau+1)$ the ratio

$$R = \frac{A^T}{A} = \frac{\theta(w,1+\tau)^2\theta(0,\tau)^2}{\theta(0,1+\tau)^2\theta(w,\tau)^2}$$

is therefore an invariant meromorphic function, and we write a pair of of $\mathbb{Z}^2 \rtimes \Gamma(2)$-invariant meromorphic coefficients to express $\wp(z,\tau)$ as a linear combination of basic one-forms x and y

$$\wp(w,\tau)d\tau = \frac{1}{R-1}((R+2)x - (2R+1)y).$$

We can write this

$$\wp(w,\tau)i(A) = \frac{\pi^2}{3}i(gA).$$

From the definition of i this is

$$\wp(w,\tau)(A - A^T) = gA - (gA)^T.$$

This expands out to be

$$\wp(w,\tau)(\frac{\theta(w,\tau)^2}{\theta(0,\tau)^2} - \frac{\theta(w,1+\tau)^2}{\theta(0,1+\tau)^2})$$

$$= \frac{\pi^2}{3}((\theta(0,1+\tau)^4 - 2\theta(0,\tau)^4)\frac{\theta(w,\tau)^2}{\theta(0,\tau)^2} - (\theta(0,\tau)^4 - 2\theta(0,1+\tau)^4)\frac{\theta(w,1+\tau)^2}{\theta(0,1+\tau)^2}).$$

The one-form

$$\wp(w,\tau)d\tau$$

on $\mathbb{C} \times \mathbb{H}$ is a $\mathbb{Z}^2 \rtimes \Gamma(2)$-invariant one-form double pole on $0 \times \mathbb{H}$ and various zeroes elsewhere.

By definition 5, the divisor of this meromorphic one-form compares the "translation" invariant subvariety of A with that of the gA. The restriction of gA to the divisor of zeroes of our one-form is invariant under T and the restriction of A to the divisor of poles of our one-form is invariant under T. The divisor of this one-form is also, hence, invariant under T.

The deRham differential of $\wp(w,\tau)d\tau$ is a differential two-form; we can calculate it in the coordinates (w,τ) on $\mathbb{C} \times \mathbb{H}$ as

$$d\wp \wedge d\tau = \frac{\partial}{\partial w}\wp(w,\tau)dw \wedge d\tau$$

and it too is $\mathbb{Z}^2 \rtimes \Gamma(2)$ invariant.

Weierstrass' relation concerns two line bundles. One is the second tensor power of the second exterior power of the cotangent bundle, let us say of $\mathbb{C} \times \mathbb{H}$, and concerns the global section

$$(d\wp \wedge d\tau)^{\otimes 2}$$

of that line bundle. The other is the pullback of of the line bundle $M_3 = M_1^{\otimes 3}$ viewed as a $\Gamma(2)$ equivariant line bundle on $\mathbb{H}$ along the second projection of $\mathbb{C} \times \mathbb{H}$, and concerns the section

$$(\wp(w,\tau)d\tau - e_1)(\wp(w,\tau)d\tau - e_2)(\wp(w,\tau)d\tau - e_3).$$

Weierstrass' relation implies (and follows from) the condition that if we write the first section as a meromorphic function times the basic tensor $(dw \wedge d\tau)^{\otimes 2}$ and if we write the second as a meromorphic function times the basic tensor $d\tau^{\otimes 3}$, the two coefficient functions will be identical.

From the equivariant isomorphisms

$$\Lambda^2\Omega_{\mathbb{C}\times\mathbb{H}} \cong \Omega_{(\mathbb{C}\times\mathbb{H})/\mathbb{H}} \otimes p_2^*\Omega_{\mathbb{H}}$$

and

$$\Omega^{\otimes 2}_{(\mathbb{C}\times\mathbb{H})/\mathbb{H}} \cong p_2^*\Omega_{\mathbb{H}}$$

we can produce an equivariant isomorphism ϕ

$$\phi : \Lambda^2 \Omega_{\mathbb{C}\times\mathbb{H}}^{\otimes 2} \cong p_2^* \Omega_{\mathbb{H}}^{\otimes 3}.$$

Weierstrass' relation thus identifies a pair of global sections which correspond with one another under the equivariant isomorphism ϕ between the second tensor power of one line bundle and the third tensor power of the other

$$\phi((d\wp \wedge d\tau)^{\otimes 2}) = (\wp d\tau - e_1) \otimes (\wp d\tau - e_2) \otimes (\wp d\tau - e_3).$$

IV.5. The elliptic surface

We may take our rational structural map to be $\lambda = \frac{(e_1-e_3)}{(e_1-e_2)}$. As a map whose domain is our double cover of $\mathbb{P}^2$ λ is only a rational map; once we resolve the indeterminacy of λ then we have our elliptic surface over $\mathbb{P}^1$ which still has six nodes.

The cross-ratio

$$\gamma = \frac{(e_3 - e_2)(e_1 - e_4)}{(e_1 - e_2)(e_3 - e_4)}$$

factorizes through the structural map defining a quadratic period map $\mathbb{P}^1 \to \mathbb{P}^1$.

Since we've taken the e_i to be functions of τ then λ is also a function of τ, it is precisely the classical lambda function.

The period map factors through the structural map, as we mentioned, and equals the quadratic map $\mathbb{P}^1 \to \mathbb{P}^1$ which is given

$$\gamma(\lambda) = \frac{1 - \lambda^2}{1 - 2\lambda}.$$

The degree-two map $\lambda \mapsto \gamma(\lambda)$ sends the values of λ which parametrize elements of the pencil which meet a crossing point of two of the same lines, which are

$$-1, 0, \frac{1}{2}, 1, 2, \infty,$$

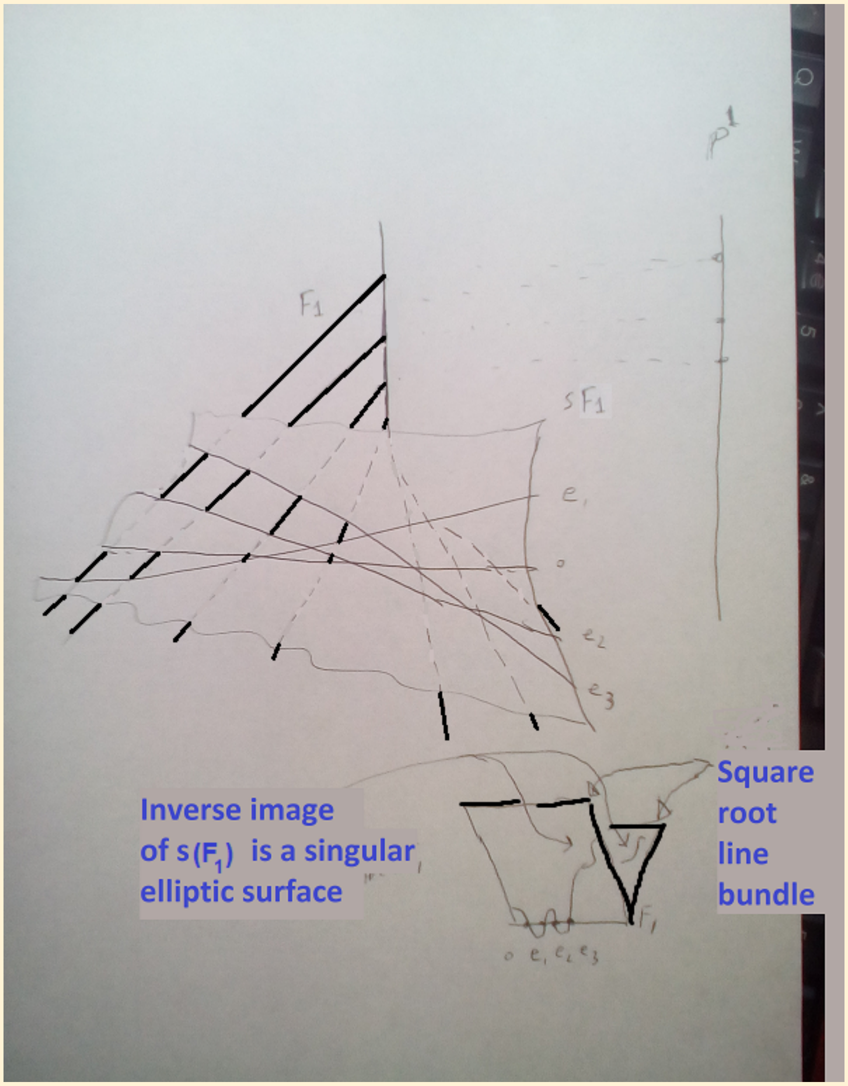
in order, to

$$0, 1, \infty, 0, 1, \infty.$$

The two branching points of the quadratic period map are when $\lambda(\tau) = e^{\pm 2i\pi/6}$, where $\tau = e^{i\pi/2 \mp i\pi/6}$.

The six singular fibers of the resulting compact projective surface have a node each, which is a node in the ambient surface. Once the six nodes are resolved, each singular fiber is a union of two rational curves intersecting at two points each, and each with normal bundle of degree -2.

The unresolved elliptic surface S is the inverse image under tensor square $N \to N^{\otimes 2} \cong L$ of the section image $s(F_1) \subset L$ with L being the line bundle whose section sheaf is dual to the defining ideal $\mathcal{I}$ of the union of four lines in F_1.



The nontrivial Galois automorphism of the map $\lambda \mapsto \frac{1-\lambda^2}{1-2\lambda}$ is easily calculated, if we say $\lambda \mapsto c$ then λ satisfies that $1 - \lambda^2 = c(1 - 2\lambda)$; the two solutions of this equation add to $2c$ so the other solution is $\frac{\lambda-2}{2\lambda-1}$.

Rather than preserving the image of the classical lambda function, this automorphism interchanges 0 and 2 and interchanges 1 and -1, and interchanges ∞ and $1/2$. The period map describes a Galois cover of $\mathbb{P}^1$ branched at two points, but it is not possible to lift the Galois automorphism to any automorphism of $\mathbb{H}$ because it sends three interior points to ideal points.

The cover $\widetilde{M} \to M$ is just trivial (six disjoint copies of M itself for $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}$), and a connected component is just a copy of M itself, so there is no need to consider equivariance.

The period map itself is a degree-two Galois cover; the nontrivial Galois automorphism of $\mathbb{P}^1$ induces a *non-algebraic* automorphism of our surface S once we view S as the pullback of a non-algebraic surface along the period map. Our surface S is *analytically* a branched cover of degree two, branched on two smooth fibers. While it is *algebraically* a branched cover of the scroll branched along four Riemann spheres.

The connected components of the Picard group of our surface are a free abelian group of rank 10. It is rationally the same as $H^1(S, \Omega_S)$ and since the higher derived functors of pushing forwards along the branched cover to the scroll are trivial, there is a basis conssting of the two classes which span Pic of the scroll, and a rational basis of 8 anti-invariant classes. Six of these merely had to do with resolving the six singular points. Two remaining anti-invariant classes remain to be understood.

IV.6 Analytic parametrization of the surface.

Let's choose as a fundamental domain for the action of $\Gamma(2)$, which is the same as for the action of $\langle T^2, ST^2S, \rangle$ since minus the identity acts trivially, the pair of ideal triangles in $\mathbb{H}$ with ideal vertex set $\{0, 1, i\infty\}$ and $\{1, 2, i\infty\}$. Let's call this fundamental domain D. The composite

$$Interior(D) \subset \mathbb{H} \to \Gamma(2) \setminus \mathbb{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\} \subset \mathbb{P}^1$$

extends to a map

$$D \to \mathbb{P}^1$$

by which we can interpret $\mathbb{P}^1$ as an identification space, made by gluings on the boundary of D.

We can lift the period map $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ to a map $\eta : D \to D$ such that the diagram commutes

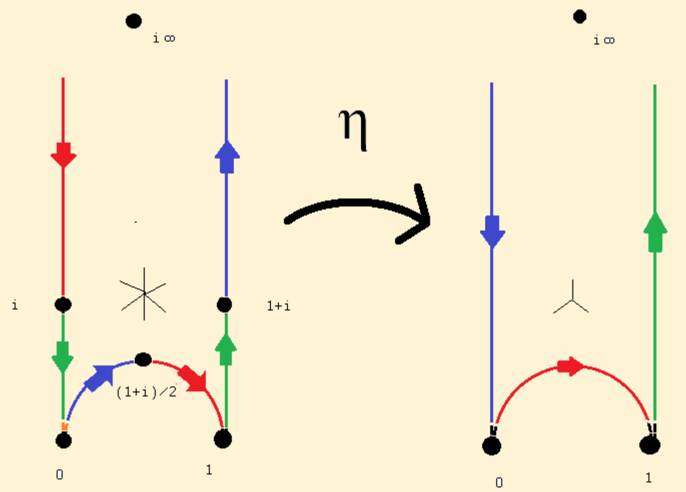
$$\begin{array}{ccc} D & \to & \mathbb{P}^1 \\ \downarrow \eta & & \downarrow \gamma \\ D & \to & \mathbb{P}^1 \end{array}.$$

On the ideal triangle in ideal vertices $0, 1, i\infty$, consider in order in the boundary of this triangle the six points $0, \frac{1+i}{2}, 1, 1+i, i\infty, i$. Consider the holomorphic map which sends the six geodesic segments in order between these points to the three edges of the same triangle, so our map on vertices is

$$\begin{array}{ccc} i\infty & \mapsto & 0 \\ i & \mapsto & 1 \\ 0 & \mapsto & \infty \\ \frac{1+i}{2} & \mapsto & 0 \\ 1 & \mapsto & 1 \\ 1+i & \mapsto & \infty \end{array}.$$

This describes a branched conformal map, let us call it η. It is a branched double cover of the ideal triangle, with the branch point of order two when $\tau = e^{2i\pi/6}$. It has the property that for our period map γ

$$\gamma(\lambda(\tau)) = \lambda(\eta(\tau)).$$



The map and the formula extend by symmetry to the second ideal triangle needed to cover $\mathbb{P}^1$ and the map has a second branch point in the second ideal triangle.

As the variable τ goes around the ideal triangle in $\mathbb{H}$ with vertices $i\infty, 0, 1$ the corresponding period ratio $\eta(\tau)$ goes twice around the same ideal triangle. We may write this as a ratio of trigonometric integrals using the isogeny we mentioend earlier. Write

$$\begin{aligned}\int_0^a \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} &= \int_0^{\sqrt{a}} \frac{2tdt}{\sqrt{t^2(t^2-1)(t^2-\lambda)}} \\ &= 2\int_0^{\sqrt{a}} \frac{dt}{\sqrt{(t^2-1)(t^2-\lambda)}} \\ &= \frac{2}{\sqrt{\lambda}}\int_0^{\sqrt{a}} \frac{dt}{\sqrt{(t^2-1)(\frac{t^2}{\lambda}-1)}} \\ &= \frac{2}{\sqrt{\lambda}}\int_0^{arcsin(\sqrt{a})} \frac{d\theta}{\sqrt{1-\frac{1}{\lambda}sin^2\theta}}\end{aligned}$$

This last would be known as

$$\frac{2}{\sqrt{\lambda}}F(arcsin(\sqrt{a}),\frac{1}{\lambda}).$$

Then we can recover a value of τ from λ by

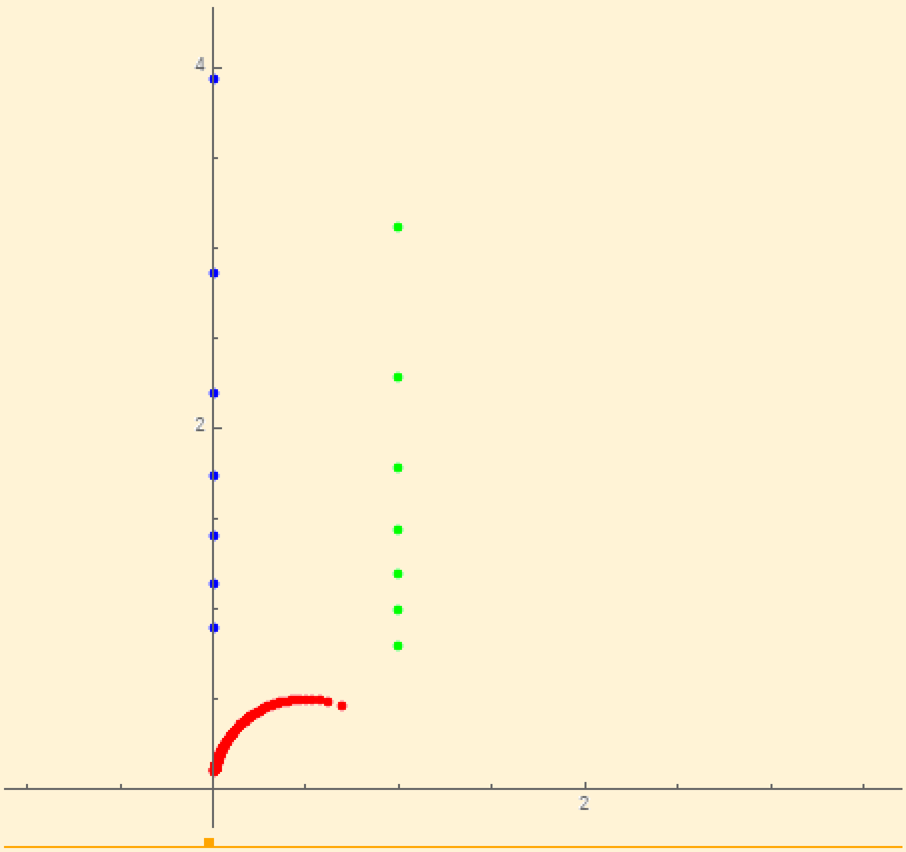
$$\tau = \frac{-F(arcsin(\sqrt{-\infty}),\frac{1}{\lambda})}{F(\text{arcsin}(\sqrt{\infty},\frac{1}{\lambda}) - F(arcsin(\sqrt{1},\frac{1}{\lambda})}$$

and

$$\eta(\tau) = \frac{-F(arcsin((\sqrt{-\infty},\frac{1}{\gamma(\lambda)})}{F(\text{arcsin}(\sqrt{\infty},\frac{1}{\gamma(\lambda)}) - F(arcsin(\sqrt{1},\frac{1}{\gamma(\lambda)})}$$

Graphing this using Wolfram Alpha, due to a typo I accidentally replaced $-\infty$ with ∞ and it worked even better, the corresponding point in $\mathbb{P}^1$ was unaffected but the image is in our originally chosen fundamental domain.

Here are $\eta(\tau)$ when τ is on the intervals $[i\infty, i], [i, 0]$, and the arc $\frac{1}{2} + \frac{1}{2}e^{2i\pi[\frac{1}{2},1]}$ coloured red, blue, green. These arcs cover half the boundary of the ideal triangle while their images cover the whole boundary.



Here is, I think, a transformation that establishes a correspondence line-by-line when we look at lines through $[0:0:1]$ in $\mathbb{P}^2$. Define

$$d(c) = \frac{1-c^2}{1-2c}.$$

This commutes with $c \mapsto 1-c$, that is

$$1 - d(c) = d(1-c).$$

The rational function

$$f(T,c) = (1-2c)\frac{1+(2c-1)T}{-1+7c-7c^2+(1-6c^2+4c^3)T} \quad (4)$$

satsifies that

$$f(\frac{1}{c-2}, c) = \frac{1}{d(c)-2}$$
$$f(\frac{1}{1+c}, c) = \frac{1}{1+d(c)}$$
$$f(\infty, c) = \frac{1}{1-2d(c)}$$
$$f(\frac{1}{1-2c}, c) = 0.$$

When we set $c = \frac{y}{x}$ so that $1-c = \lambda$ we see that $d(c) = 1 - \gamma(\lambda)$. When we consider a line through $[0:0:1]$ in $\mathbb{P}^2$ and fix the ratio $\frac{y}{x}$ to a value of c, then when we apply f to the x coordinate, the values where the point $[x:y:1]$ lies on one of L_1, L_2, L_4, L_3 are (in that order) sent to u coordinate of the points $[u:v:1]$ with ratio $\frac{v}{u} = \gamma(\lambda)$ which intersect L_1, L_2, L_3 and the single point $[0:0:1]$ which corresponds to the exceptional line in the scroll.

The rational map $\mathbb{P}^2 - \to \mathbb{P}^2$ sending $[x:y:z]$ to $[f(\frac{x}{z}, \frac{y}{x}) : f(\frac{x}{z:}, \frac{y}{x})\frac{x^2-y^2}{x^2-2xy} : 1]$ is written

$$[x:y:z] \mapsto$$

$$[(-z+x-2y)(x^2-2xy) : (-z+x-2y)(x^2-y^2) : x^3-x^2z+7xyz-7y^2z-6xy^2+4y^3].$$

This rational map is indeterminate at $[0:0:1]$. Also at the three points

$$[2:1:0],\ [-1:-1:1],\ [1:0:1].$$

It contracts the projective line $L_3 = V(-z+x-2y)$ on which they are located to the point $[0:0:1]$. And contracts the line from $[0:0:1]$ to each of the three points. When we resolve the indeterminacy at only the three points we obtain a singular Del Pezzo surface which can be resolved by un-contracting our copy of L_3.

If label the linear, quadratic, and cubic forms in x, y alone as

$$
\begin{aligned}
l &= -z + x - 2y \\
q_1 &= x^2 - 2xy \\
q_2 &= x^2 - y^2 \\
q_3 &= -x^2 + 7xy - 7y^2 \\
t &= x^3 - 6xy^2 + 4y^3
\end{aligned}
$$

then the map is

$$[x : y : z] \mapsto [lq_1 : lq_2 : zq_3 + t].$$

If a ratio $[x : y]$ is fixed then $[q_1 : q_2 : q_3]$. is determined and then in general there is a unique choice of $[l : z]$ realizing any linear relation among the three coordinates. This means if we fix a line through $[0 : 0 : 1]$ in the domain, the points on that line parametrize general hyperplane sections. Whereas special hyperplane sections consist of particular pairs of lines through $[0 : 0 : 1]$. An intersection of a special hyperplane section with a general hyperplane section is transverse, and consists of one point on each of the two lines.

The resulting correspondence between the two lines in each case is the one which induces an isomorphism between the elliptic curves which are double covers of the two projective lines over their intersection with L_1, L_2, L_3, L_4. For, the intersection points themselves do correspond, and there is a Galois automorphism between the two lines of each pair which shows that the correspondence is holomorphic.

Another way of constructing the isomorphism without using the Galois automorphism exists because the elliptic curve which is the double cover of a generally-chosen line in $\mathbb{P}^2$ through $[0 : 0 : 1]$ branched at the intersection of that line with L_3, L_1, L_2, L_4 is isomorphic with the elliptic curve over the image of that line under our rational map, branched over $[0 : 0 : 1]$ and over its intersection with the three lines L_1, L_2, L_3. And the two lines of each corresponding pair merely map to the same image line.

The rational map sends the intersection point of the general line through $[0:0:1]$ in the domain with L_1, L_2, L_4 to the intersection points of its image in the codomain with L_1, L_2, L_3 and sends the intersection point with L_3 to $[0:0:1]$.

By construction, if the four points and their images are deleted, the map of projective lines through $[0:0:1]$ induced by the rational map is a holomorphic isomorphism on each line, and therefore it induces a holomorphic isomorphism of the corresponding elliptic curves. The elliptic curve which is the branched cover over the line through $[0:0:1]$ in the domain is the general fiber in our elliptic surface S with six singular fibers.

Next let us complement our definition of η by finding a function ν such that the functions

$$\begin{aligned} x &= \frac{\pi^2}{3}\theta(0,\tau)^4 \\ y &= \frac{\pi^2}{3}\theta(0,1+\tau)^4 \\ z &= \wp(w,\tau) \\ u &= \frac{\pi^2}{3}\theta(0,\eta(\tau))^4 \\ v &= \frac{\pi^2}{3}\theta(0,1+\eta(\tau))^4 \\ r &= \wp(\nu(w,\tau),\eta(\tau)) \end{aligned}$$

satisfy the rule

$$[u:v:r] = [(-z+x-2y)(x^2-2xy) : (-z+x-2y)(x^2-y^2) : x^3-x^2z+7xyz-7y^2z-6xy^2+4y^3].$$

With $\eta(\tau)$ as we've explicitly calculated it, we determine r to make the ratio true; specifically,

$$r(w,\tau) = \frac{u}{f(\frac{x}{z},\frac{y}{x})}$$

for the rational function f as given in (4), and using this we could write

$$\nu(w,\tau) = \frac{1}{2}\int_{r(w,\tau)}^{\infty} \frac{dT}{\sqrt{(T-e_1(\eta(\tau)))(T-e_2(\eta(\tau)))(T-e_3(\eta(\tau)))}}$$

so that $(\nu(w,\tau),\eta(\tau))$ will correspond analytically with (w,τ).

V. Differential calculus on elliptic curves.

V.1. Fiberwise vector fields

Let $S \to M$ be a smooth holomorphic bundle of elliptic curves. Let L be the corresponding line bundle of Lie algebras on M. Let $\mathcal{S}$ be the coherent sheaf on M which consists of fiberwise vector fields on S which commute with addition by local sections of the corresponding Jacobian bundle $J \to M$.

7. Theorem. The sheaf $\mathcal{S}$ is naturally isomorphic with the sheaf of sections of L.

We want to consider more general surfaces mapping to $\mathbb{P}^1$ with some singular fibers.

It is useful now to speak of the line bundle M_k on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose sections correspond to modular forms of weight k for our group $\langle T^2,\ ST^2S\rangle$ for k a positive or negative integer.

A technicality is that there are no global modular forms of weight 1 for the subgroup of $Sl_2(\mathbb{Z})$ which is the inverse image of $\Gamma(2) \subset PSl_1(\mathbb{Z})$. Instead, we lift $\Gamma(2)$ isomorphically to the subgroup generated by T^2, ST^2S where $T(\tau) = \tau + 2$, $S(\tau) = -1/\tau$ and we define M_k for all integers k to be the vector bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose local sections are what are called 'weakly modular forms' of weight k for $\langle T^2, ST^2S\rangle$, that is, which locally satisfy the transformation law of weight k for that group. One way to see define what it means to be locally weakly modular is to directly construct the line bundle M_k for all k as $\mathbb{C} \times \mathbb{H}$ modulo the action of $\langle T^2, ST^2S\rangle$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (w, \tau) = ((c\tau + d)^k w, \frac{a\tau + b}{c\tau + d}),$$

and define local weakly modular forms of weight k to be local sections of M_k.

The fiberwise vector fields on M_k have divisor the pullback to M_k of the divisor defining that very line bundle, namely k times a point up to linear equivalence, where k may be positive or negative, and local sections of the sheaf of fiberwise vector fields once restricted to the complement of the singular fibers, pull back on $\mathbb{C} \times \mathbb{H}$ to the $\Gamma(2)$-invariant vector fields of the form $g(\tau)\frac{\partial}{\partial w}$ where g is locally weakly modular of weight $-k$.

With finer analysis one might be able to show more in more general situations that such vector fields fix only the singular subscheme of each fiber, and it is this which forces the restriction on the singular fibers in Kodaira's classification.

Let's try to construct a fiberwise meromorphic vector field on S with a simple pole on the fiber over one of the branching points of the period map.

The vector field $\frac{1}{2\pi\theta(0,\tau)^2}\frac{\partial}{\partial w}$ on $\mathbb{C} \times \mathbb{H}$ is invariant under $\mathbb{Z}^2 \rtimes \Gamma(2)$ for the action of weight -1. Starting at the point at infinity in each elliptic curve, the paralellogram of 'time' $\{2q\pi\theta(0,\tau)^2 + 2t\tau\pi\theta(0,\tau)^2 : 0 \leq q, r < 1\}$ is a fundamental domain, covering each elliptic curve precisely once when τ is fixed, and when τ ranges over a fundamental domain for $\Gamma(2)$ this set covers the complement of the singular fibers in the elliptic surface with three singular fibers. For each fixed value of τ, the arc of 'time' from 0 to 1 double covers the arc in the real projective plane where $\lambda \in (-\infty, 0)$ and the arc of 'times' from 0 to τ double covers the arc in the real projective where $\lambda \in (1, \infty)$. The two double covering arcs meet transversely at the point at infinity in each elliptic curve fiber.

For our surface with 6 singular fibers we have the basic period formulas

$$2\int_{-\infty}^{0} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda(\tau)))}} = 2\pi\tau\theta(0,\eta(\tau))^2$$

$$2\int_{1}^{\infty} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda(\tau)))}} = 2\pi\theta(0,\eta(\tau))^2.$$

Twice the integral between two branch points expresses the integral around each basic closed loop in each elliptic curve fiber.

That is to say, the vector field on each elliptic curve fiber such that the directional derivative of z is $\sqrt{z(z-1)(z-\gamma(\lambda))}$ integrates to zero when time ranges from 0 to $2\pi\theta(0,\eta(\tau))^2$ or from 0 to $2\pi\tau\theta(0,\eta(\tau))^2$.

Now let's use a different coefficient so that our vector field will have just one pole on one smooth fiber and no zeroes.

Let's choose as our smooth fiber, the fiber over the the point of $\mathbb{P}^1$ where $\lambda(\tau) = e^{2\pi i/6}$. The branching points of γ are fixed points of γ, so also $\gamma(\lambda(\tau)) = e^{2\pi i/6}$.

We choose a modular form of weight one whose square has a simple zero at the point $\gamma(e^{2\pi i/6})$. The *classical* lambda function takes the value $e^{2\pi i/6}$ when $\tau = e^{2i\pi/6}$.

Since it is a coincidence that $e^{2\pi i/6}$ is fixed by both maps, let's write this explicity:

$$\gamma(\lambda(e^{2\pi i/6})) = \gamma(e^{2\pi i/6}) = e^{2\pi i/6}.$$

We take as our modular form, because of the rule $1 + e^{2i\pi/3} = e^{i\pi/3}$, to be

$$f(\tau) = \sqrt{\theta(0, e^{2i\pi/3})^4\theta(0, 1+\tau)^4 - \theta(0, e^{i\pi/3})^4\theta(0,\tau)^4}.$$

There is a constant $c \cong 1.03...$ which seems to be a positive real number, such that

$$\theta(0, e^{2\pi i/3})^4 = ce^{-i\pi/6}$$

$$\theta(0, e^{i\pi/3})^4 = ce^{i\pi/6}.$$

Thus

$$f(\tau) = c^{1/2}\sqrt{e^{11i\pi/6}(\theta(0,1+\tau)^4 + e^{7i\pi/6}\theta(0,\tau)^4}.$$

The vector field

$$\frac{1}{f(\tau)}\frac{\partial}{\partial w}$$

on $\mathbb{C}\times\mathbb{H}$ is invariant under the action of $\Gamma(2)$ by which $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w,\tau) = \frac{w}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$, and defines a vector field on the bundle M_{-1} on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ which has a simple pole on the fiber over one of the two branching points of γ.

Since we denote the period map $\mathbb{P}^1 \to \mathbb{P}^1$ by γ, then the complement of the singular fibers in our surface S is the total space of $\gamma^* M_{-1}$, and our fiberwise vector field lifts to a vector field having a simple pole on one fiber.

Two basic periods for the elliptic curve over a point $\lambda \in \mathbb{C} \setminus \{0,1\} \subset \mathbb{P}^1 \setminus \{0,1,\infty\}$ are then obtained as follows: starting with λ we evaluate $\gamma(\lambda) = \frac{1-\lambda^2}{1-2\lambda}$. Then

$$\eta(\tau) = \frac{\int_{-\infty}^{0} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda(\tau)))}}}{\int_{1}^{\infty} \frac{dz}{\sqrt{z(z-1)(z-\gamma(\lambda(\tau)))}}}.$$

Then our two basic periods are

$$f(\tau), \eta(\tau)f(\tau).$$

V.3. General vector fields

It is easiest to start with just the double cover, let's call it $D,$ of $\mathbb{P}^2$ along $L_1 \cup L_2 \cup L_3 \cup L_4$. Calling the double covering map π we know that due to a nice and general property of logarithmic one-forms,

$$\Omega_D(log(L_1 \cup L_2 \cup L_3 \cup L_4) \cong \pi^*\Omega_{\mathbb{P}^2}(log(L_1 \cup L_2 \cup L_3 \cup L_4).$$

Here the L_i on the left side of the equation refer to the reduced divisors which support the pullback of the L_i as a divisor on $\mathbb{P}^2$.

As we may do with any branched cover, we can use the identification to explicitly construct one-forms on the cover. We construct the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \Omega_D & \to & \Omega_D(log(L_1+...+L_4)) & \to & \oplus_{i=1}^4 \mathcal{O}_{L_i} & \to & 0 \\ & & \uparrow & & \uparrow\cong & & \uparrow & & \\ 0 & \to & \pi^*\Omega_{\mathbb{P}^2} & \to & \pi^*\Omega_{\mathbb{P}^2}(log(L_1+...+L_4)) & \to & \oplus_{i=1}^4 \pi^*\mathcal{O}_{L_i} & \to & 0 \end{array}.$$

This shows that Ω_D is isomorphic to the kernel of the composite

$$\pi^*\Omega_{\mathbb{P}^2}(log(L_1+...+L_4) \to \oplus_{i=1}^4 \pi^*\mathcal{O}_{L_i} \to \oplus_{i=1}^4 \mathcal{O}_{L_i}.$$

We should now be able to analyze this very precisely. A crude first statement of the corresponding principle for vector fields is that the general holomorphic vector fields on $\mathbb{P}^2$ which preserve some subset of L_1, L_2, L_3, L_4 should lift to a meromorphic vector field on the double cover with simple pole on whichever of the L_i are not preserved in $\mathbb{P}^2$.

V.4. Transformation of a fiberwise vector-field

The vector field $z\frac{\partial}{\partial z}$ on affine space induces a well-defined vector field on $\mathbb{P}^2$ and is equal to that induced by $-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. The corresponding vector field on F_1 has a simple zero on the exceptional line and the line L_4 defined by the equation $z = 0$. It lifts to the double cover having a simple zero on each of the two disjoint exceptional lines E_1, E_2, and just a simple zero on the L_4, but acquires simple poles on the lines L_1, L_2, L_3. The divisor $E_1 + E_2 - L_1 - L_2 - L_3 + L_4$ restricts to a principal divisor on each elliptic curve fiber, but without restricting, it is equivalent to minus the class of a fiber.

We can adjust the divisor within its linear equivalence class by multiplying our vector field by a rational function on our elliptic surface so that the vector field will have just a simple pole on one smooth fiber. By the usual conventions, which involve a factor of two relating half-periods with periods, one defines

$$g_2 = -4(e_1e_2 + e_2e_3 + e_1e_3)$$

$$g_3 = 4e_1e_2e_3.$$

Once we multiply by $\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}} = \sqrt{\frac{(z-e_1)(z-e_2)(z-e_3)}{(z-e_4)^3}}$ the resulting vector field will have divisor $E_1 + E_2 - 2L_4$ and for numbers a, b if we multiply by $\frac{z}{ax+by}$ it will have divisor minus the inverse image of $[-b : a]$.

What we have done, then, is to start with the vector field on $\mathbb{P}^2$ which is induced by the vector field on affine space $-x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, and let it act on the affine part of $\mathbb{P}^2$ with coordinates $\frac{x}{z}, \frac{y}{z}$. It acts with eigenvalue -1 on each coordinate, so we would describe it in these coordinates as

$$-\frac{x}{z}\frac{\partial}{\partial \frac{x}{z}} - \frac{y}{z}\frac{\partial}{\partial \frac{y}{z}}.$$

When we multiply by our ratio $\frac{z}{ax+by}$ and then by $-\frac{1}{2}\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}}$ we obtain

$$\frac{1}{2}\sqrt{4 - \frac{g_2}{z^2} - \frac{g_3}{z^3}}\left(\frac{x}{ax+by}\frac{\partial}{\partial \frac{x}{z}} + \frac{y}{ax+by}\frac{\partial}{\partial \frac{y}{z}}\right).$$

If our choice of order of vanishing on the 'boundary' divisor is correct, this should lift to a vector field on our elliptic surface which has six nodes, having a simple pole on the fiber of one of the two connected irreducible components of the branching of the quadratic map induced by the period map.

Now let's construct a vector field on the copy of $\mathbb{P}^2$ which is the codomain of our rational map. Starting with the modular forms in the variable η now (which we'll later identify with $\eta(\tau)$ using the function with the same name)

$$u = \frac{\pi^2}{3}\theta(0,\eta)^4$$

$$v = \frac{\pi^2}{3}\theta(0,1+\eta)^4.$$

Take (ν, η) as our coordinate function pair on $\mathbb{C} \times \mathbb{H}$. For each pair of numbers α, β not both zero, the vector field

$$\sqrt{\frac{1}{\alpha u + \beta v}}\frac{\partial}{\partial w}$$

is invariant under $\Gamma(2)$ and descends to a vector field on M_{-1} which is a line bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$,

As we showed in a previous section, with the choice of values

$$\alpha = e^{7\pi i/6}$$

$$\beta = e^{11\pi i/6}$$

this vector field on the double cover has a simple pole on the fiber over one of the two branched (and fixed) points of $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$.

Let's lift this vector field, viewed as a two-valued vector-field on $\mathbb{P}^2$, through our birational map $\mathbb{P}^2 - \to \mathbb{P}^2$ and compare it to the one we constructed already on the domain (in coordinates x, y, z). Let's then apply both our fiberwise vector fields to the basic rational function field generators $\frac{x}{z}$ and $\frac{y}{z}$. Applying the vector field transformed to the domain to the generator $\frac{x}{z}$ we get

$$\frac{\partial}{\partial w}(\frac{x}{z}) = \frac{\partial}{\partial w}(\frac{\pi^2}{3}\frac{\theta(0,\tau)^4}{\wp(w,\tau)})$$

$$= \frac{-1}{\wp(w,\tau)}\sqrt{4(\wp(w,\tau)-e_1)(\wp(w,\tau)-e_2)(\wp(w,\tau)-e_3)}\frac{x}{z}$$

As for the occurrences of u, v in the coefficient, we look at how the rational map relates u, v with x, y :

$$\frac{1}{\wp(\nu,\eta)}u = \frac{(-\wp(w,\tau)+x-2y)(x^2-2xy)}{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}$$

$$\frac{1}{\wp(\nu,\eta)}v = \frac{(-\wp(w,\tau)+x-2y)(x^2-y^2)}{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}.$$

Adding α times the first equation with β times the second gives

$$\frac{1}{\wp(\nu,\eta}(\alpha u+\beta v) = \frac{(-\wp(w,\tau)+x-2y)(\alpha(x^2-2xy)+\beta(x^2-y^2))}{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}$$

Now the magic happens; the particular values we're using for α, β imply that the polynomial in the last factor of the numerator is a perfect square, so

$$\frac{1}{\wp(\nu,\eta)}(\alpha u+\beta v) = \frac{(-\wp(w,\tau)+x-2y)(e^{9\pi i/12}x+e^{17\pi i/12}y)^2}{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}$$

Solving for $\alpha u+\beta v$ and substituting the result as the denominator in the coefficient, we know now how to describe our derivation in the new coordinates. Applying it to $\frac{x}{z}$ gives

$$\frac{1}{\sqrt{\alpha u+\beta v}}\frac{\partial}{\partial w}(\frac{x}{z}) =$$

$$\sqrt{\frac{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}{\wp(\nu,\eta)(-\wp(w,\tau)+x-2y)}}\cdot\frac{1}{e^{9\pi i/12}x+e^{17\pi i/12}y}\cdot\frac{-1}{\wp(w,\tau)}\cdot\sqrt{4(\wp(w,\tau)-e_1)(\wp(w,\tau)-e_2)(\wp(w,\tau)-e_3)}\frac{x}{z}$$

Factors of $\wp(w,\tau)-e_3$ cancel to -1 and we have

$$\frac{1}{\sqrt{\alpha u+\beta v}}\frac{\partial}{\partial w}(\frac{x}{z}) =$$

$$\sqrt{-\frac{x^3-6x^2y+4y^3+\wp(z,\tau)(-x^2+7xy-7y^2)}{\wp(\nu,\eta)}}\cdot\frac{1}{e^{9\pi i/12}x+e^{17\pi i/12}y}\cdot\frac{-1}{\wp(w,\tau)}\cdot\sqrt{4(\wp(w,\tau)-e_1)(\wp(w,\tau)-e_2)}\frac{x}{z}.$$

Putting in z for $\wp(z,\tau)$ and r for $\wp(\nu,\eta)$ this becomes
$\frac{1}{\sqrt{\alpha u+\beta v}}\frac{\partial}{\partial w}(\frac{x}{z}) =$

$$-\sqrt{-\frac{x^3-6x^2y+4y^3+z(-x^2+7xy-7y^2)}{r}}\cdot\frac{1}{e^{9\pi i/12}x+e^{17\pi i/12}y}\cdot\sqrt{4(1-\frac{x}{z}-\frac{y}{z})(1+2\frac{x}{z}-\frac{y}{z})}\frac{x}{z}.$$

The same happens for $\frac{y}{z}$. This vector field, which we obtained by transforming the vector field on the other copy of $\mathbb{P}^2$, is $-\sqrt{\frac{-z}{z-x+2y}}$ times the vector field we constructed here. In other words, it has an additional zero on the line at infinity and an additional pole on L_3. This is very consistent with the fact that the birational transformation contracts L_3 to a point and sends the line at infinity to L_3.

And it does have a simple pole on the one line through $[0:0:1]$ in $\mathbb{P}^2$ defined by the equation $e^{9\pi/12}x+e^{17\pi/12}y=0$.

So that we have gone all the way from considering $\frac{\partial}{\partial w}$ on $\mathbb{C}\times\mathbb{H}$, and multiplied by a coefficient to make it invariant, so we could consder it to be a two-valued vector field on $\mathbb{P}^2$, then we transformed this partially-defined analytic vector-field through the birational transformation, and we see that after adjusting for the extra zero due to the line at infinity mapping to L_3, and the addditional pole due to L_3 being contracted, we arrive at a vector field one our elliptic surface which double covers of the F_1 where it has no zeroes or poles except a single pole on one smooth elliptic curve fiber.

It is a somewhat arbitrary series of transformations, visiting partway through the Del Pezzo surface where the line L_3 is blown up at three points before it is contracted; but it does go all the way from the analysis on $\mathbb{C}\times\mathbb{H}$ to the algebra of vector fields on complete projective varieties.

VI. Integral calculus on elliptic curves.

VI.1 Expression of $\wp(w,\tau)d\tau$ as a rational one-form

Consider M_{-1} as a line bundle on $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The map

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C}^2$$

$$(w,\tau) \mapsto (w\theta(0,\tau)^2, w\theta(0,1+\tau)^2)$$

contracts the zero section $0 \times H$ to a single point at the origin, and it is invariant for $\langle T^2, ST^2S\rangle$ so it descends to a map

$$M_{-1} \to \mathbb{C}^2$$

which contracts the zero section $\mathbb{P}^1 \setminus \{0,1,\infty\}$ to a single point.

There is a corresponding map which contracts the zero-section of the tensor square M_{-2}, induced by

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C}^2$$

$$(w,\tau) \mapsto (w^2\theta(0,\tau)^4, w^2\theta(0,1+\tau)^4).$$

This map

$$M_{-2} \to \mathbb{C}^2$$

extends to a map whose domain is the full line bundle M_{-2} on $\mathbb{P}^1$.

Up to isomorphism of (locally free rank one) coherent sheaves, nothing changes if we append $d\tau$ to both coordinates, and then each coordinate is w^2 times a one-form on $\mathbb{P}^1$ with at most simple (=logarithmic) poles at $0,1,\infty$. The factors $\theta(0,\tau)^4 d\tau$ and $\theta(0,1+\tau)^4 d\tau$ as sections of the dual line bundle to M_{-2} can be interpreted as functions $M_{-2} \to \mathbb{C}$.

We can compactify the image $\mathbb{C}^2$ by considering that $\mathbb{P}^2$ is nothing but M_2 and M_{-2} glued along the complement of their zero sections by the antipodal map on the fibers (to result in the scroll F_1), then with the zero-section of M_{-2} contracted to a point.

Another way of compactifying the image is to replace the factor w^2 by the periodic factor $\frac{\pi^2}{3\wp(w,\tau)}$. The meromorphic map

$$M_{-2} \to \mathbb{C}^2$$

$$(w,\tau) \mapsto (\frac{\pi^2\theta(0,\tau)^4}{3\wp(w,\tau)}, \frac{\pi^2\theta(0,\tau+1)^4}{3\wp(w,\tau)})$$

extends to

$$M_{-2} \to \mathbb{P}^2$$

$$(w,\tau) \mapsto [\frac{\pi^2}{3}\theta(0,\tau)^3 : \frac{\pi^2}{3}\theta(0,1+\tau)^4 : \wp(w,\tau)].$$

The restriction of M_{-2} to $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is a non-Galois cover of

$$(\mathbb{Z}^2 \rtimes \langle T^2, ST^2S\rangle)\backslash(\mathbb{C}\times\mathbb{H})$$

and once the zero section is contracted, this embeds in the double cover of $\mathbb{P}^2$ branched over L_1, L_2, L_3, L_4 as the complement of the lift of $L_4 = V(z)$ union $V(-2x+y) \cup V(x-2y) \cup V(x+y)$.

Thus one component of the critical locus is deleted, and also three lines are deleted which are asymptotic to the other three components of the critical locus, as $L_1 = V(z+2x-y), L_2 = V(z-x-y)$, and $L_3 = V(z-x+2y)$.

We obtain a different copy of M_{-2} on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ by pulling back M_{-1} along the period map

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1$$

$$[x:y] \mapsto [x^2 - 2xy : x^2 - y^2].$$

Although it did not matter in the previous section (since the coefficient was homogeneous of degree zero), here, let's change our definition of x and y not to be rational one-forms on $\mathbb{P}^1$ with at worst simple poles at $0, 1, \infty$, but, rather, we divide these by $d\tau$ obtaining instead

$$x = \frac{\pi^2}{3}\theta(0,\tau)^4$$

$$y = \frac{\pi^2}{3}\theta(0,1+\tau)^4.$$

We can still interpret x and y as sections of $\mathcal{O}(1)$ (the coherent sheaf structure is not affected by removing the symbol $d\tau$).

Now, as before, the classical lambda function is

$$\lambda = 1 - \frac{y}{x} = \frac{x-y}{x}.$$

From the formulas

$$d\ log\ \lambda = i\pi\theta(0, 1+\tau)^4 d\tau$$
$$= \frac{3}{\pi} iy d\tau$$

and

$$d\ log \lambda = \frac{x}{x-y}(-d\frac{y}{x})$$
$$= \frac{y}{y-x}(\frac{x}{y}d(\frac{y}{x}))$$
$$= \frac{y}{y-x}(\frac{dy}{y} - \frac{dx}{x}),$$

eliminating $d\ log\lambda$ gives

$$d\tau = \frac{\pi}{3i}\frac{1}{xy(y-x)}(xdy - ydx).$$
$$= \frac{i\pi}{3}\frac{1}{xy(x-y)}(xdy - ydx).$$

The expression $xdy - ydx$ which we might more rigorously have written $x\nabla(y) - y\nabla(x)$ is a global section in the kernel of

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{P}(\mathcal{O}_{\mathbb{P}^1}(1)) \to \mathcal{O}_{\mathbb{P}^1}(2)$$

and is therefore a global section of $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \Omega_{\mathbb{P}^1}$, a copy of the trivial line bundle and therefore this is the uniquely determined global section up to a scalar multiple.

Multiplying by the coefficient converts it to a rational section of $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}$. Therefore by multiplying by a linear form $ax + by$ we obtain a rational one-form.

Our equation is an identity that holds on $\mathbb{H}$ when x, y are interpreted as merely being modular forms.

In terms of our analytic parametrization, if we now instead write

$$x(w,\tau) = \frac{\pi^2}{3\wp(w,\tau)}\theta(0,\tau)^4$$

$$y(w,\tau) = \frac{\pi^2}{3\wp(w,\tau)}\theta(0,1+\tau)^4$$

then the same formula will instead describe $\wp(w,\tau)d\tau$. Let's record this

$$\wp(w,\tau)d\tau = \frac{i\pi}{3xy(x-y)}(xdy - ydx).$$

Let's state this

8. Theorem. The meromorphic one-form $\wp(w,\tau)d\tau$ on $\mathbb{C}\times\mathbb{H}$ is $\mathbb{Z}^2 \rtimes \langle T^2, ST^2S\rangle$-invariant for the action of degree -2. It is the pullback of the same rational one-form on $\mathbb{P}^2$ whose restriction to the affine (x,y) plane is

$$\frac{i\pi}{3xy(x-y)}(xdy - ydx).$$

Note that the one-form $xdy - ydx$ has the property that for any rational function f of x and y,

$$d(f(xdy - ydx)) = \delta(f)dx \wedge dy.$$

where δ is the derivation coming from the Euler derivation when we view x and y as sections of $\mathcal{O}_{\mathbb{P}^1}(1)$, namely

$$\delta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

In this case the coefficient is 'homogeneous of degree -3' so $\delta(f) = -3f$ we get rid of that denominator of 3. That is to say

$$\frac{\partial}{\partial w}\wp dw \wedge d\tau = d(\wp(w,\tau)d\tau)$$

$$= d\,\frac{i\pi}{3xy(x-y)}(xdy - ydx)$$

$$= \frac{-3i\pi}{3xy(x-y)} dx \wedge dy$$
$$= \frac{i\pi}{xy(x-y)} dx \wedge dy.$$

It makes sense to interject a particular calculation re-deriving from this the factor of $\pi\theta(0,\tau)^2$ in the periods of the classical elliptic integral of the first kind. Because we know a set of basic periods of the incomplete elliptic surface made with the $\wp$ function actually are $1, \tau$.

By our current conventions

$$x\wp = \frac{\pi^2}{3}\theta(0,\tau)^4, y\wp = \frac{\pi^2}{3}\theta(0,1+\tau)^4,$$

such that e_1, e_2, e_3 and also $x\wp, y\wp$ are functions of τ alone and

$$\int_{e_1}^{\infty} \frac{dT}{\sqrt{(T-e_1)(T-e_2)(T-e_3)}} = 1$$
$$\int_{e_2}^{\infty} \frac{dT}{\sqrt{(T-e_1)(T-e_2)(T-e_3)}} = \tau$$

setting $s = \frac{T-e_1}{3x\wp}$ and calculating

$$\frac{ds}{\sqrt{s(s-1)(s-\lambda)}} = \sqrt{3x\wp}\frac{dT}{\sqrt{(T-e_1)(T-e_1-3x\wp)(T-e_1-3x\wp\lambda)}}$$

$$= \sqrt{3x\wp}\frac{dT}{\sqrt{(T+2x\wp-y\wp)(T+2x\wp-y\wp-3x\wp)(T+2x\wp-y\wp-3x\wp(1-\frac{y}{x}))}}$$
$$= \sqrt{3x\wp}\frac{dT}{\sqrt{(T-e_1)(T-e_2)(T-e_3)}}.$$

which must then integrate to $\sqrt{3x\wp} = \pi\theta(0,\tau)^2$ if integrated from 1 to ∞ and to $\tau\pi\theta(0,\tau)^2$ if integrated from 0 to ∞.

VI.2 Geometric determination of the period lattice.

First consider only the ratio $[u : v]$. If we had written something simpler

$$u = x^2 - 2xy$$
$$v = x^2 - y^2$$
$$x(\tau) = \frac{\pi^2}{3}\theta(0,\tau)^4$$
$$y(\tau) = \frac{\pi^2}{3}\theta(0,1+\tau)^4$$

then we could have said

$$d\eta(\tau) = \frac{u(\tau)}{x(\eta(\tau))}\frac{2\pi i}{3}\frac{x^2 - xy + y^2}{xy(x-y)(x+y)(y-2x)(x-2y)}(xdy - ydx).$$

which also equals

$$\frac{v(\tau)}{y(\eta(\tau))}\frac{2\pi i}{3}\frac{x^2 - xy + y^2}{xy(x-y)(x+y)(y-2x)(x-2y)}(xdy - ydx).$$

In the first instance the numerator $u(\tau)$ would cancel the factors $x(x-2y)$ in the denominator, in the second instance the numerator $v(\tau)$ would cancel the factors $(x-y)(x+y)$.

Elsewhere, when we write x or y we mean $x(\tau)$ or $y(\tau)$.

If we choose not to make any cancellations, the last factor has zeroes on the two lines through $[0 : 0 : 1]$ which meet L_4 at the two branching points of the period map, and poles on the six lines through $[0 : 0 : 1]$ which meet a crossing among L_1, L_2, L_3, L_4.

Because of the homogeneity of the expression $xdy - ydx$, the product of all terms besides $\frac{1}{x(\eta(\tau))}$ is homogeneous of degree zero, and is unaffected when we change our definition of x, y by multiplying by an arbitrary function of w, τ; and in the second instance likewise for the product of all terms except $\frac{1}{y(\eta(\tau))}$. Therefore we have formulas for the invariant forms $x(\nu(w,\tau), \eta(\tau))d\eta(\tau)$

and $y(\nu(w,\tau),\eta(\tau))d\eta(\tau)$ as rational functions, even if we resume the the correct definitions

$$x(w,\tau) = \frac{\pi^2}{3}\frac{\theta(0,\tau)^4}{\wp(w,\tau)}$$

$$y(w,\tau) = \frac{\pi^2}{3}\frac{\theta(0,1+\tau)^4}{\wp(w,\tau)},$$

giving

$$x(\nu(w,\tau),\eta(\tau))\wp(\nu(w,\tau),\eta(\tau))d\eta(\tau)$$
$$= \frac{2\pi i(x(w,\tau)^2 - x(w,\tau)y(w,\tau) + y(w,\tau)^2)}{3y(w,\tau)(2x(w,\tau)-y(w,\tau))(x(w,\tau)-y(w,\tau))(x(w,\tau)+y(w,\tau))}(x(w,\tau)dy(w,\tau) - y(w,\tau)dx(w,\tau)),$$

$$y(\nu(w,\tau),\eta(\tau))\wp(\nu(w,\tau),\eta(\tau))d\eta(\tau)$$
$$= \frac{2\pi i(x(w,\tau)^2 - x(w,\tau)y(w,\tau) + y(w\,\tau)^2)}{3x(w,\tau)y(w,\tau)(x(w\,\tau)-2y(w,\tau))(y(w,\tau)-2x(w,\tau)))}(y(w,\tau)dx(w,\tau) - x(w,\tau)dy(w,\tau)).$$

The coefficients of $d\eta$ do not really depend on ν, dividing by these and integrating, the enticing formulas which result express a period ratio of the elliptic curve which double covers the line through the point (x,y) in the affine plane, branched on the lines L_1, L_2, L_3, L_4; there is a period lattice basis ω_1, ω_2 with $\omega_1/\omega_2 = \eta(\tau)$ and therefore

$$\frac{\omega_1}{\omega_2} = \int \frac{2i}{\pi\theta(0,\eta(\tau))^4} \;\; \frac{x^2 - xy + y^2}{y(2x-y)(x-y)(x+y)}(xdy - ydx)$$
$$= \int \frac{2i}{\pi\theta(0,1+\eta(\tau))^4} \;\; \frac{x^2 - xy + y^2}{xy(x-2y)(y-2x)}(xdy - ydx).$$

On each of the two parts of an open cover where $\theta(0,\eta(\tau))$ or $\theta(0,1+\eta(\tau))$ is not zero, the period ratio of the double cover of the line through a point (x,y) is determined now by the path integral of a rational one-form with poles on four of the six lines through $[0:0:1]$ which meet an intersection of two of L_1, L_2, L_3, L_4 and with zeroes on the two lines through $[0:0:1]$ which meet L_4 at a branching point of the period map.

While the period ratio only depends on the ratio $[x:y]$, we've put in a radial dependence using theta functions, and have multiplicatively separated out a rational one-form on each part of the open cover.

Let's test this formula approximately. In this code some definitions are inessentially different, so that what we call $\eta(\tau)$ needs to be spelled out as a different composite as $\eta(\gamma(\lambda(\tau)))$.

```
F[lambda_, a_] := 2*(1/Sqrt[lambda])*EllipticF[ArcSin[Sqrt[a]], 1/lambda]
eta[lambda_]:= (aa := Limit[F[lambda, T]/(F[lambda, T] - F[lambda, 1]), T → Infinity]; If[Im[aa] < 0, -aa, aa])
gamma[l_]:= (1 - l^2)/(1 - 2*l)
x[w_, tau_] := (1/3)*Pi^2*EllipticTheta[3, 0, Exp[I*Pi*tau]]^4/WeierstrassP[w, WeierstrassInvariants[{1/2, tau/2}]]
y[w_, tau_] := (1/3)*Pi^2*EllipticTheta[3, 0, Exp[I*Pi*(tau + 1)]]^4/WeierstrassP[w, WeierstrassInvariants[{1/2, tau/2}]]
d[f_, w_, t_] := 10000*(f[w, t + 0.0001] - f[w, t])
d0[f_, t_]:= 10000*(f[t + 0.0001] - f[t])
tau := 0.1 + .8*I
w := 0.3 + 0.5*I
N[d0[eta @* gamma @* ModularLambda, tau]]
2*I/EllipticTheta[3, 0, Exp[I*Pi*eta[gamma[ModularLambda[tau]]]]]^4 * (x[w, tau]^2 - x[w, tau]*y[w, tau] + y[w, tau]^2)* (x[w, tau]*d[y, w, tau] - y[w, tau]*d[x, w,
    tau])/(Pi*y[w, tau]*(x[w, tau] - y[w, tau])*(x[w, tau] + y[w, tau])*(y[w, tau] - 2*x[w, tau]))
2*I/EllipticTheta[3, 0, Exp[I*Pi*(1+eta[gamma[ModularLambda[tau]]])]]^4 *(x[w, tau]^2 - x[w, tau]*y[w, tau] + y[w, tau]^2)* (x[w, tau]*d[y, w, tau] - y[w, tau]*d[x, w,
    tau])/(Pi*x[w, tau]*y[w, tau]*(x[w, tau] - 2*y[w, tau])*(y[w, tau]-2*x[w, tau]))

Out[428]= -1.72811 - 0.0963306 i
Out[429]= -1.72741 - 0.0953794 i
Out[430]= -1.72741 - 0.0953794 i
```

VI.3 Variable of integration for $L(s,\chi,\tau)$.

In the first instance, we can use $\frac{y}{x}$ as a variable of integration. From

$$y = \frac{\pi^2\theta(0,1+\tau)^4}{3\wp(w,\tau)}$$

we can rewrite

$$\wp(w,\tau)d\tau = \frac{\pi^2\theta(0,1+\tau)^4}{3y}d\tau$$

and then

$$d\tau = \frac{3y}{\pi^2\theta(0,1+\tau)^4}\frac{1}{xy(y-x)}(xdy - ydx)$$

Then from $d\ log(\frac{\lambda}{q}) = i\pi(\theta(0,1+\tau)^4 - 1)$

$$L(s,\chi,\tau_0) = -\frac{1}{\Gamma(s)\pi^{1-s}}\int_0^{\tau_0}(\frac{\tau}{i})^{s-1}i\pi(\theta(0,\tau)^4 - 1)d\tau$$

$$= \frac{1}{\Gamma(s)(-i\pi)^{s-1}}\int_0^{\tau_0}\tau^{s-1}(1 - \frac{1}{\theta(0,1+\tau)^4})(\frac{1}{1-\frac{y}{x}})d\frac{y}{x}.$$

The upper limit of integration refers to the value of $\frac{y}{x}$ which is $1-\lambda(\tau_0)$.

A more direct way of deriving the same formula is from the rule

$$d\ log(\frac{\lambda}{q}) = (1 - \frac{1}{\theta(0,1+\tau)^4})d\ log(\lambda).$$

Let's rework this to remove the theta function of τ from the integrand.

Since when we constrain $w = \frac{\tau}{2}$ we have

$$0 = \frac{d}{d\tau}(-2x + y),$$

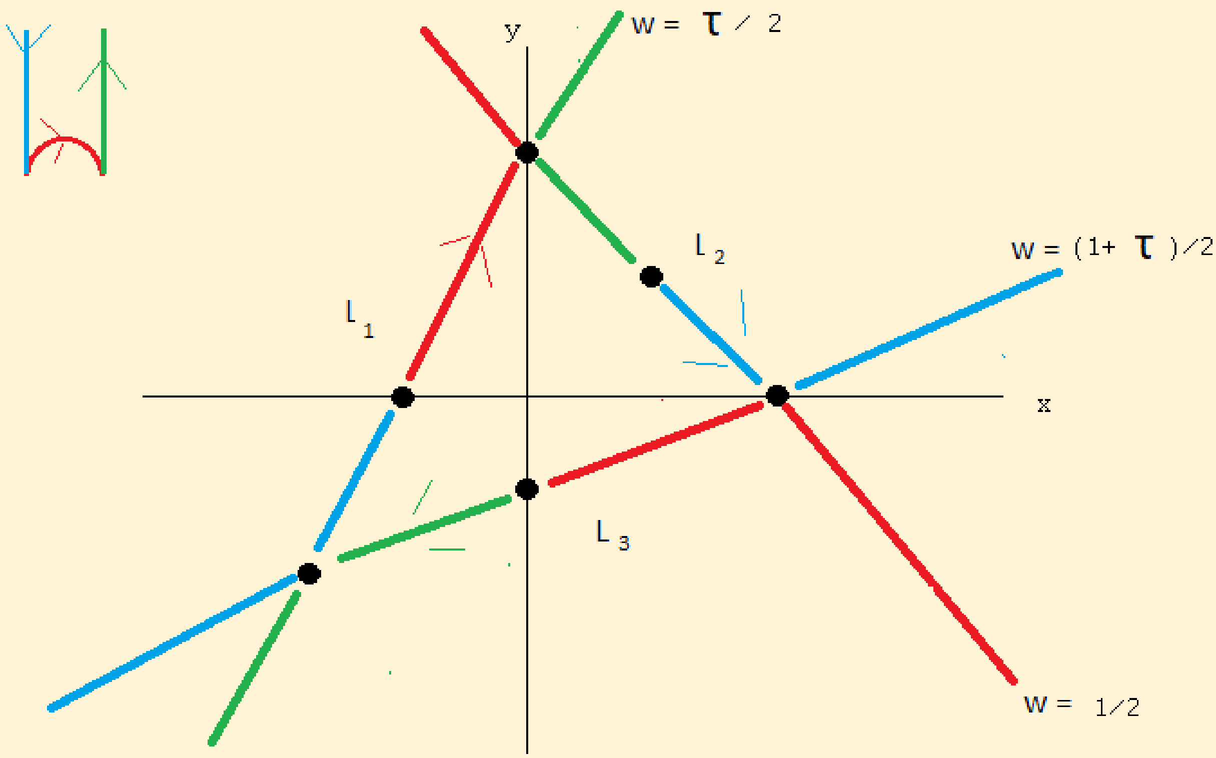
when we constrain $w = \frac{1}{2}$ we have

$$0 = \frac{d}{d\tau}(x + y)$$

and when we constrain $w = \frac{1+\tau}{2}$ we have

$$0 = \frac{d}{d\tau}(x - 2y)$$

then with each of these constraints, the path formed by $(x(w, \tau), y(w, \tau))$ as (w, τ) ranges along the corresponding real line in $\mathbb{C}$ is a straight real line in $\mathbb{C}^2$.



The three lifts of the boundary of our basic ideal triangle (half-fundamental domain) with ideal vertices $0, 1, i\infty$ map to the real points of the lines L_1, L_2, L_3.

If we write the separate variables of integration as dx and dy, on the non-compactifiable elliptic surface we can write

$$-L(s,\chi,\tau)(-i\pi)^{1-s}\Gamma(s) = \int_0^\tau t^{s-1} i\pi(\frac{3}{\pi^2}\frac{\wp(\frac{1}{2},t)-\wp(\frac{1+t}{2},t)}{3} - 1)dt$$

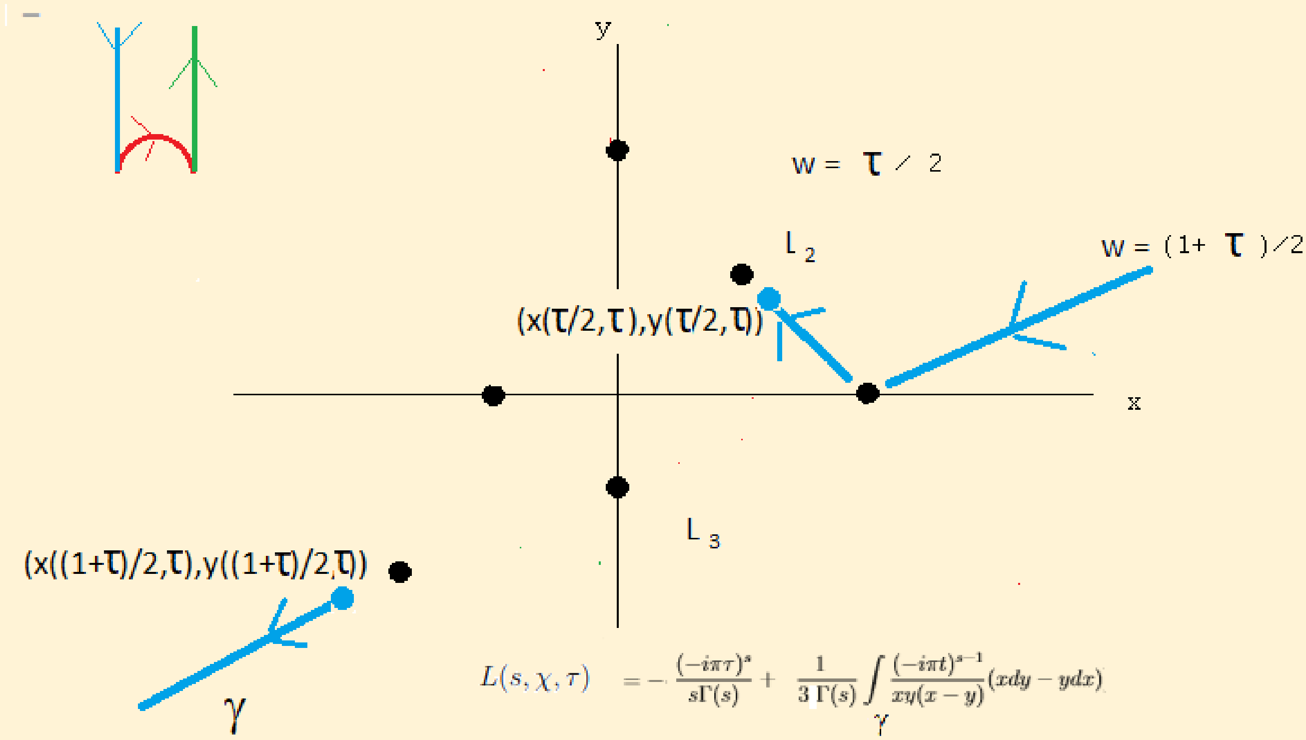
so

$$L(s,\chi,\tau) = \frac{(-i\pi)^{s-1}}{\Gamma(s)}(i\pi\frac{\tau^s}{s} - \frac{i}{\pi}\int_0^\tau t^{s-1}\mathcal{P}(\frac{1}{2},t)dt + \frac{i}{\pi}\int_0^\tau t^{s-1}\mathcal{P}(\frac{1+t}{2},t)dt)$$

$$= \frac{1}{\Gamma(s)}(-\frac{(-i\pi\tau)^s}{s} + \frac{1}{i\pi}\int_0^\tau (-i\pi t)^{s-1}\mathcal{P}(\frac{1}{2},t)dt - \frac{1}{i\pi}\int_0^\tau (-i\pi t)^{s-1}\mathcal{P}(\frac{1+t}{2},t)dt)$$

$$= -\frac{(-i\pi\tau)^s}{s\Gamma(s)} + \frac{1}{3\Gamma(s)}\int \frac{(-i\pi t)^{s-1}}{xy(x-y)}(xdy - ydx)$$

along two arcs of a corresponding path. If we use a purely imaginary τ near $i\infty$ the path in $\mathbb{C}\times\mathbb{H}$ goes along two of the irreducible components of the branching locus: first from from (w,t) ranging from $(\frac{1+\tau}{2},\tau)$ to $(\frac{1}{2},0)$ while $[x:y:z]$ goes along L_3 from a point near $[-1:-1:0]$ through the 'point at infinity ' at $[-2:1:0]$ on to $[1:0:1]$ and then on the second arc (w,t) ranges from $(\frac{1}{2},0)$ to $(\frac{1}{2},\tau)$ while $[x:y:z]$ goes in L_2 from $[1:0:1]$ to a point near $[\frac{1}{2}:\frac{1}{2}:1]$.



We should eventually perform the same analysis with u, v playing the role of x, y and with ν, η playing the role of w, τ. It is η which is the actual period ratio for the elliptic curves which double cover lines through $[0 : 0 : 1]$ branching on the intersections with L_1, L_2, L_3, L_4 – and these are merely inverse images of the same projective lines in a double cover of all of $\mathbb{P}^2$, and the relation between x, y, z and u, v, r is via a rational map which factors through a singular del Pezzo surface where L_3 becomes contracted to a point and L_4 maps to L_3.

In the meantime, as a reality check, we should spot-check that what we're saying makes sense. Let's evaulate $L(s, \chi)$ at $s =$

$.3 + .3i$, say. We have

$$L(s, \chi) = -8\zeta(s)\zeta(s-1)4^{-s}(2-2^s)(4-2^s) = -1.46167 - 0.519004i,$$

approximately. Whereas

$$\frac{-1}{\Gamma(s)}\int_0^{i\infty}(-i\pi\tau)^{s-1}(\frac{1}{i\pi}\wp(\frac{1+\tau}{2},\tau) - \frac{1}{i\pi}\wp(\frac{1}{2},\tau) - i\pi)d\tau$$

gives the same answer. We find $L(s, \chi, \tau)$ for, say $\tau = .9i$, by replacing the upper limit by $.9i$. It is $-1.43196 - 0.43822i$. If we also replace the lower limit by $0.1i$ we get $-0.641538 - 0.684334i$. We express this as the sum of three parts. Replacing $\wp(w, \tau)d\tau$ by $\frac{i\pi}{3}\frac{xdy-ydx}{xy(x-y)}$ the first part is the integral of $\frac{(-i\pi\tau)^{s-1}}{3\Gamma(s)}\frac{xdy-ydx}{xy(x-y)}$ along the path parametrized

$$(x, y) = (\frac{\pi^2\theta(0,\tau)^4}{3\wp(\frac{1}{2},\tau)}, \frac{\pi^2\theta(0, 1+\tau)^4}{3\wp(\frac{1}{2},\tau)})$$

for τ ranging from $.1i$ to $.9i$. Approximating this as a finite sum of 10000 parts, it is approximately $5.30538 + 4.02164I$. The next part is the is the path integral of $\frac{(-i\pi\tau)^{s-1}}{3\Gamma(s)}\frac{xdy-ydx}{xy(x-y)}$ along the path parametrized by

$$(x, y) = (\frac{\pi^2\theta(0,\tau)^4}{3\wp(\frac{1+\tau}{2},\tau)}, \frac{\pi^2\theta(0, 1+\tau)^4}{3\wp(\frac{1+\tau}{2},\tau)})$$

for τ ranging from $.1i$ to $.9i$. This is approximately $-5.23797 - 3.89999i$. The third part is just an evaluation of the minus remaining definite integral, it is $\frac{(-i\pi(.1i))^s}{s\Gamma(s)} - \frac{(-i\pi(.9i))^s}{s*\Gamma(s)}$ which is approximately $-0.708885 - 0.805857i$, the three parts add to $-0.641473 - 0.68421i$ which is correct to around three decimal places.

We have not yet looked at convergence of the path integrals when considered separately though this will be easy. The greater challenge is to understand the transform through our rational map which is related to the period mapping, so we may use $\eta(\tau)$ in place of τ. This may also finally allow thinking of a simpler function $L(s, \chi, \nu, \eta)$ which we could also parametrize as $L(s, \chi, u, v)$ under the relation between analytic and rational parametrizations.

The comparison between the two compactifications is likely to be confusing, so we should slow down a little. Going back to the diagram a few pages ago, which is displayed beneath the words "... with each of these constraints..." we see a map from the total space of the line bundle on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ which is a non-Galois cover of the non-compactified surface. The non-compactified surface is a bundle of smooth elliptic curves polarized with a $\mathbb{Z}/(2\mathbb{Z})$ homology basis such that each fiber represents each polarized isomorphism type exactly once.

Regarding the line bundle, its total space is $\mathbb{C} \times \mathbb{H}$ and the map to $\mathbb{P}^2$, in variables (ν, η), is given

$$(\nu, \eta) \mapsto [\pi^2\theta(0, \eta)^4 : \pi^2\theta(0, 1 + \eta)^4 : 3\wp(\nu, \eta)].$$

The map contracts the zero-section described by the equation $\nu = 0$ to the point $[x : y : z] = [0 : 0 : 1]$ and the map branches of order two at three lines in the domain $\nu = \frac{1}{2}$, $\nu = \frac{\tau}{2}$, $\nu = \frac{1+\tau}{2}$.

The way the total space of the line bundle is a non-Galois cover of the smooth elliptic surface is that the quotients of $\mathbb{C} \times \mathbb{H}$ by the action of $\mathbb{Z}^2 \rtimes \Gamma(2)$ describes the elliptic surface, while the quotient only by $0 \times \Gamma(2)$ describes the total space of the line bundle. However the group action of $\mathbb{Z}^2 \rtimes 1$ does not descend to any well-defined group action of $\mathbb{Z}^2$ on the line bundle.

The branching locus of the map from the line bundle to $\mathbb{P}^2$ maps locally isomorphically to the branching locus of the map from the elliptic surface to $\mathbb{P}^2$. Thus it is the union of the branching in each fiber where we fix τ to be a constant, and it occurs at the values $\nu = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$. This occurs at the three lines we've described, and the zero section which is contracted in the map to $\mathbb{P}^2$. That is to say, if we represent our smooth non-compact surface as a bundle of elliptic curves, the branching of the map to $\mathbb{P}^2$ occurs at the four sections where ν is the chosen zero section or one of the points of order two in its elliptic curve fiber.

We can make a naive partial compactification of $\mathbb{C} \times \mathbb{H}$ which the disjoint union $\mathbb{C} \times \ (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Z}))$. Note that $\mathbb{P}^1(\mathbb{Z}) = \mathbb{P}^1(\mathbb{Q})$. To topologize the disjoint union we first consider $\mathbb{P}^1(\mathbb{Q})$ to be the rational points of the real line boundary of $\mathbb{H}$ in $\mathbb{C}$ together with $i\infty$, and start with the topology generated by the induced topology from $\mathbb{C}$. Then we adjoin new open sets consisting of $\mathbb{H}$ union each single point of $\mathbb{P}^1(\mathbb{Z})$, and let these generate a finer topology. Now the induced topology on the subset $\mathbb{P}^1(\mathbb{Z})$ has been made discrete in a universal way, even while the points of that set are not isolated in the larger set.

We extend the map by continuity to the map

$$\mathbb{C} \times \mathbb{H} \to \mathbb{P}^2$$

such that the line of points $(\nu, 0)$ contracts to the point $[1 : 0 : 1]$, the line $(\nu, 1)$ contracts to the point $[0 : 1 : 1]$ and for each number ν the point of the line $(\nu, i\infty)$ maps to the point $[\pi^2 : \pi^2 : \wp(\nu, i\infty)]$.

Here $\wp(\nu, i\infty)$ refers to $lim_{\tau \to i\infty} \wp(\nu, \tau)$ and we will have to describe this limiting function. Our expression by theta functions describes an indeterminate limit requiring L'Hopital's rule, there will be a more direct way of describing the limiting function.

The group action of the subgrop $0 \rtimes \Gamma(2)$ on $\mathbb{C} \times \mathbb{H}$ is generated by the two transformations $(w, \tau) \mapsto (w, \tau + 2)$ and $(w, \tau) \mapsto (\frac{w}{2\tau+1}, \frac{\tau}{2\tau-1}$. The first of these extends continuously to an automorphism which pointwise fixes $\mathbb{C} \times \{i\infty\}$ acts freely on the union of all other boundary lines. The second extends continuously to an automorphism which acts freely on all boundary lines besides $\mathbb{C} \times \{0\}$ and fixes that one pointwise. In this way we have extended the action of $0 \rtimes \Gamma(2)$ to a partial compactification of the line bundle which also has a line fiber over each of $0, 1, \infty \in \mathbb{P}^1$. The set of boundary lines has three orbits. It is a question, for now, whether the quotient by the group action is the total space of a line bundle over $\mathbb{P}^1$ with lines corresponding to the three orbits lying over $0, 1, \infty$. The subset

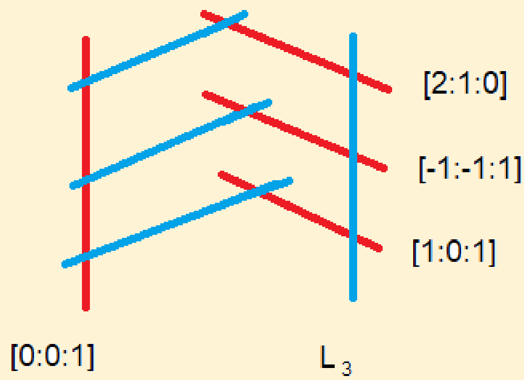
$$\{\frac{1}{2\pi\theta(0, \tau)^2}, \tau) : \tau \in \mathbb{H}\} \subset \mathbb{C} \times \mathbb{H}$$

is invariant under $0 \rtimes \Gamma(2)$ and descends to a section of our line bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. We need to consider whether our compactification of this line bundle is actually a holomorphic line bundle over $\mathbb{P}^1$ and, if so, whether this section extends to a *meromorphic* section of the line bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The tensor square of the line bundle over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has section sheaf isomorphic with vector fields on $\mathbb{P}^1$ which fix $0, 1, \infty$. This is isomorphic with the restriction of $\mathbb{O}(-1)$ to the complement of $\{0, 1, \infty\}$. However, $\mathbb{O}(-1)$ is not a tensor square.

Before we take the tensor square, we can think of our line bundle as being M_{-1}, and as we explained in section V.I the sheaf of fiberwise vector fields on M_{-1} once restricted to the zero section is isomorphic with the sheaf of sections of M_{-1}. If we wish to make the isomorphism explicit, we can think that a fiberwise tangent vector at the zero point of a fiber describes a point in that fiber, which would be the result of flowing along the vector-field for one unit of time. Two rough intuitive features of the subsequent analysis are that lifting through the period map doubles the degree, and the tensor square of fiberwise vector fields restricted to the zero section is isomorphic with vector fields along the zero section fixing $0, 1, \infty$.

Our algebriac compactification was done by a corresponding degree-two rational transformation of the projective plane. The plane which is shown in the diagram, coordinatized on the page by x and y, which we shall rename as u and v, rationally represents codomain of this transformation. We will use x, y for variables in the domain. The transformation maps the line at infinity to the line L_3 whose image in the (u, v) plane is defined by the equation $u - 2v = 1$, and it maps the line L_3 in the domain which is defined by the equation $x - 2y = 1$ to the single point $[0 : 0 : 1]$.

Here is a drawing of particular exceptional lines in the surface that results when we completely resolve the ambiguities of the rational map



Under the degree-two rational map, each of the red arcs comes from a single point in the domain (the point is given as a label of each red arc) while each blue arcs maps to a single point in the codomain. Thus, the blue arc connecting the red line labelled $[0:0:1]$ with the red line labelled $[2:1:0]$ corresponds to the actual line between the points with those $[u:v:w]$ coordiantes in the domain, but maps to a single point in the codomain. Three of the three blue lines are lines through $[0:0:1]$ in the domain and the fourth one is L_3 itself. The red line labelled $[0:0:1]$ in the diagram maps to the line L_3 in the codomain which is disjoint from the image of the blue line labelled L_3.

In terms of analytic functions, the fact that the line $\nu = \frac{1}{2}$ maps to the line L_2, corresponds to an analytical identity

$$\wp(\frac{1}{2}, \eta) = \frac{\pi^2}{3}(\theta(0,\eta)^4 + \theta(0, 1+\eta)^4)$$

and similarly for the other two branching lines. The relation between the naive compactification and the Kahler compactification will comprise a substantial extension of this type of analytical coincidence beyond just the branching lines.

VI.4 Integration by substitution.

If $p(t)$ gives a path in a manifold, starting at time 0 (and we may consider time to be a complex holomorphic entity), and if a flow is given on the manifold, whose corresponding operator on functions is the derivation δ, then as long as the path p is an integral curve of the flow, parametrized accordingly, we may explicitly calculate for each complex number c

$$\int_0^c \delta(f)(p(t))dt = \int_0^c \frac{df}{dt}(p(t))dt$$

$$= \int_0^c df(p(t)) = f(p(c)) - f(p(0)).$$

And we can continue the analysis

$$= (e^{c\delta}(f) - f))(p(0)).$$

The element $c\delta$ belongs to the Lie algebra, and knowing how its exponential acts on f allows the calculation of the original integral of the real valued function of t.

A variant of this analysis is the following:

$$\int_0^c \frac{df(p(t))}{\delta(f)(p(t))} = \int_0^c \frac{\delta(f)(p(t))}{\delta(f)(p(t))}dt$$

$$= \int_0^c dt = c.$$

It is this second variant which explains why elliptic integrals calculate periods in the lattice in the Lie algebra. Starting with a one-form of the type $\frac{dz}{g(z)}$ one considers the derivation δ such that $\delta(z) = g(z)$. In other words, so that the function $g(z)$ is the contraction $i_\delta(dz)$ of the one-form dz which is in the numerator. Then one allows the path $p(t)$ to form itself starting from a point $p(0)$, using the exponential formula if one wishes to be explicit; and the integral of $\frac{dz}{g(z)}$ along this particular path has the property that the integral from $p(0)$ to $p(t)$ is just t itself for all values of t.

If $p(c) = p(0)$, the value of c is a period of $\frac{dz}{g(z)}$, and $(e^{c\delta}(g) - g)(p(0)) = 0$.

We can write the one-form related to Riemann's hypothesis as $\frac{d\frac{\lambda}{q}}{\frac{\lambda\tau^{1-s}}{q}}$

or, the one which is slightly more directly related, writing $u = -i\tau$, is $\frac{d\frac{\lambda}{q}}{\frac{\lambda u^{1-s}}{q}}$ and so we can try to choose δ such that

$$\delta(\frac{\lambda}{q}) = \frac{\lambda}{q}u^{1-s}.$$

Given any path $p(t)$ in the upper half plane, say, for t ranging from 0 to 1, we determine values of u along the path by solving the ordinary differential equation

$$\frac{d}{dt}u(p(t)) = \frac{\frac{d}{dt}p(t)}{u(p(t))^{s-1}i\pi(\theta(0,1+iu(p(t)))^4-1)}.$$

Then Writing $a = p(0)$ and $b = p(1)$, in terms of the modular L series

$$\frac{-\pi^{s-1}}{\Gamma(s)}(L(s,\chi,iu(b)) - L(s,\chi,iu(a)))$$

$$= \int_0^1 u(p(t))^{s-1}i\pi(\theta(0,1+iu(p(t)))^4-1)\frac{d}{dt}u(p(t))dt$$

$$= \int_0^1 \frac{d}{dt}p(t)dt = b - a.$$

Now choose instead an arbitrary point ie in the upper half plane (so e has positive real part). Choose a smooth path p with endpoints $p(0) = 0$, $p(1) = \frac{\pi^{s-1}}{\Gamma(s)}L(s,\chi,ie)$. Solve the differential equation above with $u(p(0)) = e$, and let $b = u(p(1))$. We have

$$\frac{-\pi^{s-1}}{\Gamma(s)}(L(s,\chi,ib) - L(s,\chi,ie)) = \frac{\pi^{s-1}}{\Gamma(s)}L(s,\chi,ie)$$

and therefore

$$L(s,\chi,ib) = 0.$$

The point $\tau = iu(b)$ is then a zero of $L(s, \chi, \tau)$. Note we do not assume u to be real. Thus

9. Observation. For each point ie in the upper half plane and each smooth real path p connecting 0 to

$$\frac{\pi^{s-1}}{\Gamma(s)} L(s, \chi, ie),$$

a solution u of the differential equation above with $u(p(0)) = e$, if it exists, satisfies that the number $\tau = iu(p(1))$ is a zero of $L(s, \chi, \tau)$.

10. Example. Take $s = .4 + .2i$, $e = 1$, and $p(t) = (1 - t)e + t\frac{\pi^{s-1}}{\Gamma(s)} L(s, \chi, ie) = (1 - t) \cdot 1 + t \cdot (-.2057 + .1987i)$. Setting $u(p(0)) = u(0) = e$ and applying the differential equation gives $\tau = u(p(1)) = -.4142 + .1888i$, and for this value of τ we have $L(s, \chi, \tau) = 0$.

As we vary s we would see the position of τ changing.

11. Remark. The zeroes of $L(s, \chi, \tau)$ which tend to the ideal point $\tau = i\infty$ as s varies are those which correspond with zeroes of Riemann's zeta function.

Since we're restricting the value of s to have real part between zero and 1, the coefficient in the function

$$\mathbb{H} \to \mathbb{C}$$
$$\tau \mapsto \frac{-\pi^{1-s}}{\Gamma(s)} L(s, \chi, \tau)$$

is just a constant.

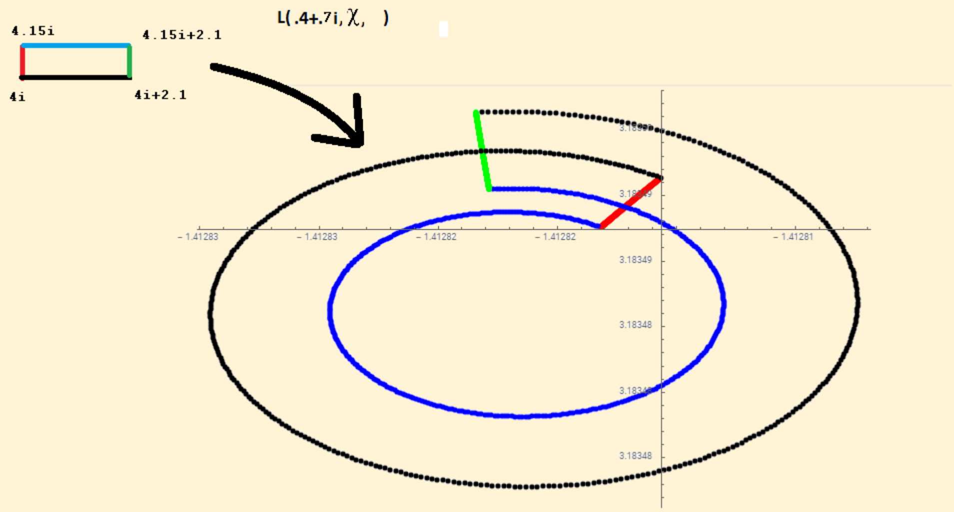
12. Observation. As a function of τ $L(s, \chi, \tau)$ is unbranched but not proper.

It is unbranched because $\theta(1/2, \tau) = \theta(0, 1 + \tau)$ does not take the value $0, 1, i, -i$ for any value of τ. Given this, then it is not proper because $\mathbb{H}$ is not conformally equivalent to $\mathbb{C}$.

VI.5. Deformation of T.

This opens the likelihood that it is almost certaintly not not single-valued; i.e., not an embedding. The choice of path p of 'complex times' is essential choice leading to all the possible solutions of $L(s, \chi, \tau) = 0$ starting with any one initial nonzero value.

An example shows it is not single valued. Fix $s = .4 + .7i$, chosen at random; this small rectangle in the τ plane overlaps itself under the mapping. Lines of constant imaginary value tend to a circle near the black arc as the imaginary coordinate tends to zero, and to a single point, which is the value $L(s, \chi) = L(s, \chi, i\infty)$, as the imaginary coordinate tends to infinity.



This raises the question whether there is a neighbourhood U of the ideal point $i\infty$ in the upper half-plane, which is invariant under the action of an automorphism T_s which is a deformation of the automorphism $T : \tau \mapsto \tau + 2$ in the case when $s = 1$, such that for all $\tau \in U$ we have

$$L(s, \chi, \tau) = L(s, \chi, T_s(\tau)).$$

The specific way an infinite set of nearby solutions might arise, if Riemann's $\zeta(s) \neq 0$ but is near zero, for $0 < Re(s) < 1$, is that $L(s, \chi, i\infty) = L(s, \chi)$ would be near zero so the open set U should contain a zero of $L(s, \chi, \quad)$, and therefore that the infinite cyclic orbit of this zero under the action of T_s would produce an infinite set of zeroes. That is to say, we might define a holomorphic isomorphism η_s in a neighbourhood of the ideal point by the rule

$$e^{i\pi\eta_s(\tau)} = L(s, \chi, \tau) - L(s, \chi),$$

From invariance of the exponential map we'd have $\eta_s(T_s(\tau)) = \eta_s(\tau) + 2$. Remembering the ordinary transformation T such that $T(\tau) = \tau + 2$ we would have a formula for T_s as the conjugate

$$T_s = \eta_s^{-1} T \eta_s.$$

where

$$\eta_s(\tau) = \frac{1}{i\pi} \left(log(L(s, \chi, \tau) - L(s, \chi)\right).$$

As s approaches 1, our deformation η_s approaches the identity, and T_s approaches T itself, which is the transformation $\tau \mapsto \tau + 2$ of $\mathbb{H}$.