Euler's equations for fluid flow

NOTE: There is a mistake in the proof of one of the main results here (Corollary 5). Where I speak of a sum of $(n + n^2)^{1+4i}$ terms, these are terms of 'degree $i+1$' in the x_i, and have operators $\partial/\partial u_j$ applied. The mistake is that I write these as sums of operators involving the $\partial/\partial u_i$ times the x_j without taking account of the increasing number of terms due to Leibniz rule. This can be significant, for instance $(\partial/\partial u)^d x^d$ becomes the integer factorial d. Here since the x_i are polynomials this specific counterexample does not apply, but it is not clear whether the series really always does have infinite radius of convergence even for polynomial initial conditions.

The purpose of this note is to write down a formal series solution to Euler's equations of fluid flow, to look for generalizations of the theorem relating radius of converence of meromorphic functions to positions of poles, in an attempt to begin to understand examples by Xinyu He.

One observation will be that for $\mathbb{R}^n$ every vector field at time zero, whose coefficients in the usual basis are polynomials, extends to a solution for all time and space, which is allowed to have 'rip-tides' extending to infinity. The existence of entire harmonic functions implies that such solutions are far from unique.

Let E be Euclidean space with coordinates u_i, and let $x_1, ..., x_n$ belong to a space V of smooth functions on E so that the divergence of the vector field

$$\delta = \sum x_i \frac{\partial}{\partial u_i} \qquad (1)$$

is zero. Suppose that V is preserved by the Laplacian and the restriction to V is surjective, so there is at least one τ a linear differential operator such that

$$\Delta \circ \tau = identity$$

where Δ is the Laplacian restricted to V.

1. Lemma. Suppose it is possible to solve the differential equation making x_i functions of time (with the $x_i(0)$ being the originally given functions in (1))

$$\frac{\partial}{\partial t} x_k = -\sum_i x_i \frac{\partial}{\partial u_i} x_k + \frac{\partial}{\partial u_k}\ \tau \sum_{i,j} \frac{\partial}{\partial u_j}(x_i \frac{\partial}{\partial u_i} x_j) \qquad (2)$$

Then the vector field given by (1) for all time solves the Euler equations of fluid flow.

Proof. The Euler equations require that the extended vector field δ has divergence zero for all time and the curl of the total derivative (the time derivative of δ plus the directional derivative along δ of δ) is zero.

Since the divergence is zero when $t = 0$ (the given value of δ) it suffices to show that the time derivative of the divergence is zero. This is the same as the divergence of the time derivative. Thus we sum $\frac{\partial}{\partial x_k}$ applied to the right side of (2). The sum of the partial derivatives of the second terms is the Laplacian of the part after and including τ and therefore equals the part of (2) after τ. The sum of the partial derivatives of the first terms is the negative of the same expression, and the sum is zero.

Regarding the curl, the second term on the right side of (2) gives a gradient which contributes zero. The first terms contribute minus the curl of the directional derivative, and when the directional derivative is added the result is zero. QED

The next lemma may be used later but is not used in this paper.

3. Lemma. Let ω be any nowhere vanishing n form on a Riemannian n manifold M and for any smooth function f on M let $\Delta(f)$ be the Laplacian, that is, the unique function so that

$$d\ i_{\nabla f}(\omega) = \Delta(f)\omega$$

where $i_{\nabla f}$ is the contracting operator associated to the vector field ∇f. Then for any two smooth functions f, g

$$df \wedge d\ i_{\nabla g}(\omega) = \langle df, dg\rangle\omega = dg \wedge d\ i_{\nabla f}(\omega).$$

Hence two terms cancel in the calculation of the de Rham differential

$$d(f\ i_{\nabla g}(\omega) - g\ i_{\nabla f}(\omega))$$

leaving

$$d(f\ i_{\nabla g}(\omega) - g\ i_{\nabla f}(\omega)) = (f\Delta(g) - g\Delta(f))\omega.$$

If in addition $\Delta(g) = 0$ then $g\Delta(f)$ is an integrating factor such that $g\Delta(f)\omega$ is exact.

Proof of Lemma 3: Each part follows from the previous one. QED

4. Lemma. Suppose the space of smooth functions V contains the x_i as elements, and is closed under multiplication by the x_i and under the $\frac{\partial}{\partial u_i}$ and also under Δ. Suppose that the restriction of Δ to V is surjective so there is a linear τ such that $\Delta \circ \tau = identity$. If the radius of convergence of the series below for each k is infinite for all values of $u_1, ..., u_n$ then the x_i extend to functions which are analytic in time, whose Taylor coefficient functions with respect to time belong to V and which satisfy the Euler equations of fluid flow.

$$x_k(t) = \{x_k\} + \frac{t}{1!}\{-\sum_{q=1}^{n} x_q \frac{\partial}{\partial u_q} x_k + \frac{\partial}{\partial u_k} \tau(\sum_{q,j=1}^{n} \frac{\partial}{\partial u_j}(x_q(\frac{\partial}{\partial u_q}(x_j))))\}$$

$$+\frac{t^2}{2!}\{-\sum_{q=1}^{n} \quad [-\sum_{\alpha_0=1}^{n} x_{\alpha_0} \frac{\partial}{\partial u_{\alpha_0}} x_q + \frac{\partial}{\partial u_q} \tau(\sum_{\alpha_0,\beta_0=1}^{n} (\frac{\partial}{\partial u_{\beta_0}}(x_{\alpha_0}(\frac{\partial}{\partial u_{\alpha_0}}(x_{\beta_0}))] \quad \frac{\partial}{\partial u_q} x_k$$

$$+\frac{\partial}{\partial u_k} \tau(\sum_{q,j=1}^{n} (\frac{\partial}{\partial u_j}(\quad [-\sum_{\alpha_1=1}^{n} x_{\alpha_1} \frac{\partial}{\partial u_{\alpha_1}} x_q + \frac{\partial}{\partial u_q} \tau(\sum_{\alpha_1,\beta_1=1}^{n} (\frac{\partial}{\partial u_{\beta_1}}(x_{\alpha_1}(\frac{\partial}{\partial u_{\alpha_1}}(x_{\beta_1})))))] \quad (\frac{\partial}{\partial u_q}(x_j)))))$$

$$+(-\sum_{q=1}^{n} x_q \frac{\partial}{\partial u_q} \quad [-\sum_{\alpha_2=1}^{n} x_{\alpha_2} \frac{\partial}{\partial u_{\alpha_2}} x_k + \frac{\partial}{\partial u_k} \tau(\sum_{\alpha_2,\beta_2=1}^{n} (\frac{\partial}{\partial u_{\beta_2}}(x_{\alpha_2}(\frac{\partial}{\partial u_{\alpha_2}}(x_{\beta_2}))] \quad)$$

$$+\frac{\partial}{\partial u_k} \tau(\sum_{q,j=1}^{n} (\frac{\partial}{\partial u_j}(x_q(\frac{\partial}{\partial u_q}(\quad [-\sum_{\alpha_3=1}^{n} x_{\alpha_3} \frac{\partial}{\partial u_{\alpha_3}} x_j + \frac{\partial}{\partial u_j} \tau(\sum_{\alpha_3,\beta_3=1}^{n} (\frac{\partial}{\partial u_{\beta_3}}(x_{\alpha_3}(\frac{\partial}{\partial u_{\alpha_3}}(x_{\beta_3}))] \quad))\}$$

$$+\frac{t^3}{3!}(\ldots \qquad (4)$$

Proof. First let's explain how the series is made. From the coefficient of $\frac{t}{1!}$ the occurrences of x_q and x_k in the first term are one or the other replaced by the whole coefficient with subscripts re-indexed (shown in square brackets), and the two terms added; and likewise for x_q and x_j in the second term. This gives the coefficent of $\frac{t^2}{2!}$ consistent with the product rule for a derivation. A rigorous way of describing the series is to apply the rule $[\partial/\partial t,\ \tau] = 0$ along with (2) to deduce a recursive formula for all $\frac{\partial}{\partial t}^{\alpha} x_k$ as

$$\frac{\partial}{\partial t}^{\alpha} x_k = \sum_{\beta=0}^{\alpha-1} \binom{\alpha-1}{\beta} (-\sum_i (\frac{\partial}{\partial t}^{\beta} x_i) \frac{\partial}{\partial u_i} (\frac{\partial}{\partial t}^{\alpha-1-\beta} x_k)$$

$$+\frac{\partial}{\partial u_k} \tau \, (\sum_{i,j} \frac{\partial}{\partial u_j} ((\frac{\partial}{\partial t}^{\beta} x_i) \frac{\partial}{\partial u_i} (\frac{\partial}{\partial t}^{\alpha-1-\beta} x_j))))$$

Assume that for all values if $(u_1, .., u_n)$ the radius of convergence of the series as a formal power series in t is infinite. Then the series does represent some analytic function of t. The coefficients can be expressed in terms of the functions $x_1, ..., x_n$ through finite expressions involving τ and the $\frac{\partial}{\partial u_i}$ and are therefore smooth functions. It follows by uniform convergence of power series on compact sets that the resulting functions $x_i(u_1, ..., u_n, t)$ are smooth and everywhere well defined.

The Lemma 1 gave a sufficient condition upon these functions for the vector field $\sum x_i(u_1, ..., u_n, t) \frac{\partial}{\partial u_i}$ to solve the Euler equations. It suffices to verify the rule in the lemma within the ring of formal power series in t with coefficients in the smooth functions in $u_1, ..., u_n$. The left side of the formula in Lemma 1, $\frac{\partial}{\partial t} x_k$, is represented by the power series above but where we have removed the constant term and shifted each coefficient one place. The right side of the formula in Lemma 1 is represented by the expression where we have replaced x_k with the full series above which *represents* x_k, and likewise replaced the single occurrence of x_i and x_j on the right side of the formula in Lemma 1 with analagous infinite series. It remains to verify that the two sides are the same.

An entire solution for polynomial vector fields

It is shown in course notes on Alex Freire's website [1] that for homogeneous polynomials, the Laplacian is merely adjoint to multiplication by r^2. then $\Delta \circ r^2$ is invertible on polynomials, where r^2 means the operation of multiplying by $r^2 = \sum_{i=1}^n u_i^2$. If we take for τ the operation $r^2 \circ (\Delta \circ r^2)^{-1}$ then the hypotheses of the lemma is true, that $\Delta \circ \tau = \textit{identity}$ and the radius of convergence is infinite for all values of $(u_1, ..., u_n)$.

This operator τ is very easy to understand. For example, for polynomials x and y in variables u, v we have

$$\tau(a_0 + a_1u + a_2v + a_3u^2 + a_4uv + a_5v^2 + ...)$$

$$= r^2(\frac{1}{4}a_0u+\frac{1}{8}a_1u+\frac{1}{8}a_2v+(\frac{7}{96}a_3-\frac{1}{96}a_5)u^2+\frac{1}{12}a_4uv+(\frac{-1}{96}a_3+\frac{7}{96}a_5)v^2+...$$

The coefficient of $\frac{t^i}{i!}$ is an expression which is a sum of $(n + n^2)^{1+4i}$ terms. Each term includes at most i occurrences of τ and $i + 1$ occurrences of an iterated partial derivative of one of the x_i with respect to the u_j. Writing $\tau = r^2 \circ (\Delta \circ r^2)^{-1}$ we may multiply this out using $r^2 = \sum_{i=1}^n u_i^2$ and we then obtain $n^i(n + n^2)^{4i+1}$ terms in total, each of which contains at most $2i + 1$ occurrences of one of the iterated partial derivatives of a u_i. The operator $(\Delta \circ r^2)^{-1}$ has its largest eigenvalue the eigenvalue of 1 which is $\frac{1}{2n}$. We only need that this is less than or equal to one.

Therefore on a neigbourhood in space and time where all the iterated partial derivatives of the x_i, even when all terms in each polynomial are taken positive, are less than a number M in magnitude, the absolute value of the i'th term of the series is less than $\frac{n^i(n+n^2)^iM^{2i+1}}{i!}t^i$ and the partial sums of the absolute value of the Taylor series terms, for each fixed value of $(u_1, ..., u_n)$ in that neighbourhood, are all bounded above. Thus

[1] Alex Friere, course notes on Math 435, University of Kansas

5. Corollary. If $x_1, .., x_n$ are polynomials in $u_1, ..., u_n$ then the series (4) for $\tau = r^2 \circ (\Delta \circ r^2)^{-1}$ converges uniformly on bounded subsets of $x_1, ..., x_n, t$ space and therefore converges everywhere to an analytic solution of the Euler differential equations.

Remark. In contrast with the series where τ is chosen to be the integral (3), this entire solution is in some sense only spurious.[2]. It does not belong to any nice space of entire functions that I know of, it is not unique in any particular sense I'm aware of, and does not respect any particular boundary conditions.

Remark In cases where V contains no nonzer harmonic functions, it is also true that τ commutes with the $\frac{\partial}{\partial u_i}$. For, we know $0 = [\Delta, \partial/\partial u_i]$ for all i, and then from *identity* $= \Delta \circ \tau$ we have

$$\Delta[\frac{\partial}{\partial u_i}, \tau] = [\frac{\partial}{\partial u_i},\ \Delta \circ \tau]$$

and this is zero since the identity commutes with $\frac{\partial}{\partial u_i}$. So that the commutator $[\frac{\partial}{\partial u_i}, \tau]$ is always a map to the space of harmonic functions.

[2]On the claymath website Fefferman's article would deem such solutions as not 'physically reasonable'

Example. Let's take as standard variables on the plane the familiar letters x, y instead of u_1, u_2. beginning with the divergence free vector field

$$x^2\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}$$

the time-dependent vector field generated by the previous corollary (using $(0,0)$ as an origin) is

$$(x^2+0.333tx^3+0.999txy^2+0.104t^2x^4+0.875t^2x^2y^2-0.062t^2y^4+0.014t^3x^5$$

$$+0.437t^3x^3y^2 - 0.177t^3xy^4 + 0.002t^4x^6 + 0.117t^4x^4y^2 - 0.333t^4x^2y^4$$

$$+0.043t^4y^6 + 0.001t^5x^7 - 0.010t^5x^5y^2 - 0.426t^5x^3y^4 + 0.169t^5xy^6$$

$$+0.001t^6x^8-0.041t^6x^6y^2-0.387t^6x^4y^4+0.380t^6x^2y^6-0.012t^6y^8...)\frac{\partial}{\partial x}$$

$$+(-2xy-tx^2y-0.333ty^3-0.416t^2x^3y-0.583t^2xy^3-0.072t^3x^4y-0.437t^3x^2y^3$$

$$+0.035t^3y^5 - 0.014t^4x^5y - 0.156t^4x^3y^3 + 0.133t^4xy^5 - 0.012t^5x^6y$$

$$+0.018t^5x^4y^3 + 0.255t^5x^2y^5 - 0.024t^5y^7 - 0.013t^6x^7y$$

$$+0.083t^6x^5y^3 + 0.310t^6x^3y^5 - 0.108t^6xy^7...)\frac{\partial}{\partial y}$$

Generalization, simplification

One thing that would be nice to do is to remove the dependence on the actual metric, but for now let's just assuming a metric, and also not working holomorphically, just hodge theory on Riemannian manifolds.

Identifying vector fields with $n-1$ forms, we get to call the divergence d and the curl $*d*$ or let's call it d'.

Let's say we're on a compact manifold so Hodge decomposition works.

Any solution I can think of comes from choosing a decomposition of the identity map on $n-1$ forms,

$$1 = E + F$$

where the only rule needed is that

$$0 = d \circ F$$

$$0 = d' \circ E$$

The amount of choices in doing this is for each of E and F you get to choose a linear map from all n-1 forms to harmonic ones. So if there are no harmonic n-1 forms this is unique.

Then there is a flow on the tangent bundle of your manifold, zero on the zero section, it only operates on vector fields, and for any vector field δ it is given

$$d/dt\delta = -F \circ s(\delta)$$

Here s is self directional derivative. If you view the vector field as a map $\Omega \to O_M$ for M your manifold, then

$$\Omega \to \Omega \otimes \Omega \stackrel{i_\delta \otimes i_\delta}{\to} \mathcal{O}_M$$

where i_δ is the contracting map from one forms to zero forms. This composite is what we mean by $s(\delta)$. Or someone might write it as $\nabla_\delta(\delta)$ in differential geometry notation.

Then you see that from the rule $0 = d \circ F$, if you apply d to $d/dt\delta$ you get zero.

Whereas if you take the total derivative and apply d' you get

$$d'(d/dt + s)\delta = d'E \circ s\delta = 0.$$

So that being able to integrate the equation

$$d/dt\delta = -F \circ s\delta$$

solves the Euler equation.

So the non-unqueness comes down to the possible existence of harmonic functions, which shows up as non uniqueness in E and F. But note the non-unique parts add to zero since $E + F = 1$.

If you write $n-1$ forms as coexact + harmonic + exact, the matrix of F is

$$\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

In the example of $\mathbb{R}^3$, when we just now made an operator on functions $r^2 \circ (\Delta \circ r^2)^{-1}$, this fits into the picture in that we can take

$$E = grad \circ r^2 \circ (\Delta \circ r^2)^{-1} \circ div.$$

That is, in the solution this operator on functions was only useful because implicitly it was making an operator on vector fields.

The operator which we exponentiated was

$$\begin{aligned} -F \circ s &= -(1 - grad \circ r^2 \circ (\Delta \circ r^2)^{-1} \circ div) \circ s \\ &= (grad \circ r^2 \circ (div \circ grad \circ r^2)^{-1} \circ div - 1) os \end{aligned}$$

So that complicated series was taking i powers of this difference and dividing by $i!$.

Note that the operator

$$F = (1 - grad \circ r^2 \circ (div \circ grad \circ r^2)^{-1} \circ div)$$

really has the right shape matrix. Or better, just to note that if I take div of F I get $(div - div \circ grad \circ r^2 \circ (div \circ grad \circ r^2)^{-1} \circ div) = 0$ and if I take curl of E I get zero.

So all that complication really comes down to solving

$$0 = doE$$

$$0 = d'oF$$

$$1 = E + F$$

in such a way that letting t be a variable for time, the operator

$$-tF \circ s$$

can be exponentiated

$$\sum_{i=0}^{\infty} (-tF \circ s)^i / i!$$

Now, Fefferman's condition (for the Millennium prize, but he requires nonzero viscosity) does allow one to restrict to the case of x_i so that all iterated partials are bounded by a particular continuous function of position.

Then when you look at the action of $F \circ s$, all s does is apply such partials and multiply, so that is not a probem at all.

When I took a particular version of F

$$\begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}$$

(and we can write down the *'s but let us not) then it converged when the initial vector field is polynomial, but to something with bad behaviour at infinity.

If one takes for instead an initial vector field whose coordinates x_i and all iterated partial derivatives with respect to the u_i are bounded by a continuous function of positin, and take the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for F – which is likely what the integral in formula (2) is doing, then the issue is, how badly does applying this projection to a vector field, affect LATER partial derivatives?

If they only increase exponentially, that is if the L^∞ size of the coefficients of the $(F \circ s)^i$ only increases exponentially with i, with constant of exponentiation varying continuously with position, then the exponential series will again converge.

So, it has been possible to arrange convergence for the polynomial case by using nonzero entries in the middle row of F, but which had bad behaviour at infinity and is far from unique (changing the origin changes the definition of r and changes the solution).

If those three matrix entries are set to zero, one would sort of expect the convergence to be even better. But the issue is, after doing a projection, the coordinate functions of the vector field have been affected such that they no longer fall under the imposed hypothesis of the Millennium prize, to the effect that all iterated partials remain bounded by a continuous function of position.

One would sort of expect that projection would be nice, and would not affect further partials. But, if one thinks that the class of functions with partials bounded like that is an artificial class, not preserved on the coefficients when a contracting operator is applied to the space of vector fields, then one would not at all expect the Euler problem to have a convergent good solution for every choice of initial vector field.

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