Functional equations and symmetry

In 'Intuitive explanation' we mentioned that three integrals add to $-i\pi$ plus a horocycle integral tending to zero as $a \to \infty$. Writing $\theta(\tau)$ for $\theta(0,\tau)$, Jacobi's third theta function with the first argument zero, the three integrals were equal to

$$-\int_0^{i\infty} i\pi\tau^{s-1}(\theta(1+\tau)^4-1)d\tau,$$

$$\int_0^{ia} i\pi(1+\tau)^{s-1}(\theta(\tau)^4-1)d\tau,$$

$$-\int_{i/a}^{i\infty} i\pi(\frac{1}{1-\tau})^{s-1}(\theta(1+\frac{1}{1-\tau})^4-1)d(\frac{1}{1-\tau}).$$

In the limit as $a \to \infty$ the sum describes an integral over the real points of the Riemann sphere evaluating to $-i\pi$.

In 'Primer on Elliptic curves' we made various observations such as that we can view $\wp(w,\tau)d\tau$ as a rational differential form on the projective plane. The aim was to clarify the geometric setting in which we might begin to understand relations among values of the zeta function.

Currently, 'Intuitive explanation' remains a huge mess, and we have not even started talking about things like Riemann's functional equation.

In this note we are only going to start to straighten out that mess just a little.

We want to write these eventually as our path integrals on our compact elliptic surface.

For now, let's just try to collect and de-clutter what we know about them already.

The first integral, we have mentioned, has the number-theoretic interpretation, it equals

$$i^{s-1}L(s,\chi)\Gamma(s)(\pi)^{1-s}$$

with

$$L(s,\chi) = -8\zeta(s)\zeta(s-1)4^{-s}(2-2^s)(4-2^s).$$

The related $L(s,0)$ satisfies

$$L(s,0) = 8\zeta(s)\zeta(s-1)(1-4^{1-s}).$$

The product

$$L(s,0)\Gamma(s)\pi^{1-s}$$

is symmetric under $s \mapsto 2-s$, by an argument similar to Riemann's proof of the functional equation.

It is possible to deduce the functional equation for ζ from this symmetry. The point is that the product of the first three factors in $\zeta(s)\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s-1}{2})$ is invariant under $s \mapsto (1-s)$ if and only if the product is invariant under $s \mapsto 2-s$. By Legendere Duplication, $2^{1-s}\Gamma(s)\Gamma(1-\frac{s}{2})\Gamma(\frac{1}{2}-\frac{s}{2})$ is an odd function of s and therefore the invariance under $s \mapsto 2-s$ is equivalent to the one which $L(s,0)\Gamma(s)\pi^{1-s}$ satisfies.

Here's a test verification that it really is an odd function (to be sure we are not making typos)

```
f[s_] := 2^(1 - s) Gamma[s] Gamma[1 - s / 2] Gamma[1 / 2 - s / 2]
f[2.3]
f[-2.3]
```

```
13.7657

-13.7657
```

This symmetric product $L(s,0)\Gamma(s)\pi^{1-s}$ equals i^{1-s} times the integral

$$\int_0^{i\infty} i\pi\tau^{s-1}(\theta(\tau)^4 - 1)d\tau$$

which is not one of the integrals above, rather it is the second integral above plus the correction term

$$\int_0^{\infty} i\pi((1+\tau)^{s-1} - \tau^{s-1})(\theta(\tau)^4 - 1)d\tau.$$

The third integral might be interesting to transform by writing

$$\theta(1 + \frac{1}{1-\tau})^4 = \theta(\frac{1}{1-\tau} - 1)^4$$

$$= \theta(\frac{\tau}{1-\tau})^4 = -(\frac{1-\tau}{\tau})^2\theta(\frac{\tau-1}{\tau})^4$$

When we combine this with

$$d(\frac{1}{1-\tau}) = \frac{d\tau}{(1-\tau)^2}$$

so that the integral becomes

$$\int_0^{i\infty} i\pi(\frac{1}{1-\tau})^{s-1}[\tau^{-2}\theta(1-\frac{1}{\tau})^4 + (1-\tau)^{-2}]d\tau.$$

Letting $y = -\tau^{-1}$ and also interchanging the limits to remove the negative sign, so y ranges from 0 to $i\infty$ again, our integral becomes

$$-\int_0^{i\infty} i\pi(\frac{y}{1+y})^{s-1}[\theta(1+y)^4 + \frac{1}{(1+y)^2}]dy$$

Replacing y with τ we get

$$-\int_0^{i\infty} i\pi(\frac{\tau}{1+\tau})^{s-1}[\theta(1+\tau)^4 + \frac{1}{(1+\tau)^2}]d\tau$$

Here is a test verification that the three relevant integrals really do add to 1, although note that these are improper in the sense that the sum only converges when they are combined

```
s := .4 + I
T[tau_] := EllipticTheta[3, 0, Exp[I * Pi * tau]] ^ 4
NIntegrate[
 tau ^ (s - 1) * (T[1 + tau] - 1)
 - (1 + tau) ^ (s - 1) * (T[tau] - 1)
 + (tau / (1 + tau)) ^ (s - 1) * (T[1 + tau] + 1 / (1 + tau) ^ 2),
 {tau, I / 10000, I * 10000}]
0.9999 - 0.0000603416 i
```

We could combine the third integral with the first and the coefficient of $\theta(1+\tau)$ would become

$$i\pi(\tau^{s-1} + (\frac{\tau}{1+\tau})^{s-1})$$

though it would be better to combine the third integral with the second somehow.

We can relate the fact that $L(s,0)\Gamma(s)\pi^{1-s}$ is symmetric under $s \mapsto 2-s$ with a symmetry of our integrals. That is because

$$L(s,0) = -L(s,\chi)\frac{1-4^{1-s}}{4^{-s}(2-2^s)(4-2^s)}$$

which is the product of

$$\frac{(-i\pi)^{s-1}}{\Gamma(s)}\frac{1-4^{1-s}}{4^{-s}(2-2^s)(4-2^s)}$$

with the first integral.

This is

$$-\frac{(-i\pi)^{s-1}}{\Gamma(s)}\frac{1-4^{1-s}}{4^{-s}(2-2^s)(4-2^s)}\int_0^{i\infty} i\pi\tau^{s-1}(\theta(1+\tau)^4-1)d\tau.$$

Let us verify this

```
L[s_] := 8 * Zeta[s] * Zeta[s - 1] * (1 - 4 ^ (1 - s))
Ltest[s_] := (-I * Pi) ^ (s - 1) / Gamma[s] * (1 - 4 ^ (1 - s)) / (4 ^ (-s) * (2 - 2 ^ s) * (4 - 2 ^ s)) * NIntegrate[I * Pi * (tau) ^ (s -
    1) * (EllipticTheta[3, 0, Exp[I * Pi * (1 + tau)]] ^ 4 - 1), {tau, I / 1000, I * 1000}]
L[1.4 + 3.2 * I]
Ltest[1.4 + 3.2 * I]
3.12557 - 2.27067 i

3.12603 - 2.27118 i
```

And let's verify that if we omit the factor of $\pi^{s-1}/\Gamma(s)$ it is symmetric under $s \mapsto 2-s$

```
W[s_] := (-I) ^ (s - 1) * (1 - 4 ^ (1 - s)) / (4 ^ (-s) * (2 - 2 ^ s) * (4 - 2 ^ s)) * NIntegrate[I * Pi * (tau) ^ (s -
    1) * (EllipticTheta[3, 0, Exp[I * Pi * (1 + tau)]] ^ 4 - 1), {tau, I / 1000, I * 1000}]
W[1.4 + 3.2 * I]
W[2 - 1.4 - 3.2 * I]
-0.0902778 - 0.0719809 i

-0.0947466 - 0.0739693 i
```

Using the relation that the integrals add to 1, we see that the following expression is invariant under $s \mapsto 2-s$

$$(-i)^{s-1}\frac{1-4^{1-s}}{4^{-s}(2-2^s)(4-2^s)}(1+\int_0^{i\infty}(1+\tau)^{s-1}(\theta(\tau)^4-1)-(\frac{\tau}{1+\tau})^{s-1}(\theta(1+\tau)^4+\frac{1}{(1+\tau)^2})d\tau).$$

The effect of this substitution under the integral sign is the same as reciprocating both coefficients $1+\tau$ and $\frac{\tau}{1+\tau}$.

Since the integral from 0 to $i\infty$ of $\frac{\tau^{s-1}}{(1+\tau)^{s+1}}$ is $1/s$ we have

Theorem. The expression below is convegent for $0 < Re(s) < 2$ except for a finite number of poles, and it is invariant under $s \mapsto 2-s$

$$(-i)^{s-1}\frac{1-4^{1-s}}{4^{-s}(2-2^s)(4-2^s)}\Big(\frac{s-1}{s}+lim_{a\to\infty}\int_{i/a}^{ia}(1+\tau)^{s-1}(\theta(\tau)^4-1)-(\frac{\tau}{1+\tau})^{s-1}\theta(1+\tau)^4 d\tau\Big).$$

The integral is given explicitly as a limit rather than just an improper integral $\int_0^\infty$ because we haven't checked whether our having transformed various terms by $\tau \mapsto -1/\tau$ means the upper and lower limits must correspond under that same transformation before we take the limit.

In fact, we've merely described a particular convergent expression for the function

$$\frac{1}{i\pi}L(s,0)\Gamma(s)\pi^{1-s} = \frac{8}{i\pi}\zeta(s)\zeta(s-1)(1-4^{1-s})\Gamma(s)\pi^{1-s}.$$

which we already knew is invariant under $s \mapsto 2-s$. It might be worth unwinding this a bit, writing

$$\frac{8i^{s-1}}{i\pi}4^{-s}(2-2^s)(4-2^s)\Gamma(s)\pi^{1-s}\zeta(s)\zeta(s-1)$$

$$= \frac{s-1}{s} + lim_{a\to\infty}\int_{i/a}^{ia}(1+\tau)^{s-1}(\theta(\tau)^4-1)-(\frac{\tau}{1+\tau})^{s-1}\theta(1+\tau)^4 d\tau.$$

where it is now clear that the Riemann hypothesis is equivalent to the notion that for $0 < Re(s) < 1$ the condition that the right side of the equation is zero forces the integral to have a value which lies on the unit circle.