Complete intersections

Let $X \subset \mathbb{P}^n$ be an r dimensional irreducible projective variety over a field k of characteristic zero, and let N be a number. We assume X is not contained in any hyperplane. Let G_N be the Grassmannian whose k rational points are the nonzero subspaces of the k vector space $H^0(X, \mathcal{O}_X(N))$. For each $V \in G_N$ denote by X_V the rational image of X in the projectification of the dual space of

$$H^0(X, \mathcal{O}_X(1)) \otimes V.$$

One knows that the set of $V \in G_N$ such that X_V is nonsingular is not an empty set for large N.

Let $I_N \subset G_N$ be the larger set consisting of those V such that X_V is the image of some nonsingular variety $\tilde{X}$ under an unramified morphism (ie a morphism which is an immersion of complex manifolds if $k = \mathbb{C}$). As N increases the sets I_N parametrize the resolutions $\tilde{X}$ via the map sending V to the Nash blowup of X_V. Although the larger set I_N is more manageable, I do not know of any construction of I_N, nor whether it is constructible. The theorem of this paper is that, under the assumption that X is a complete intersection of hypersurfaces there is a constructible subset of G_N contained in I_N

$$C_N \subset I_N,$$

which is large enough to parametrize all possible choices of $\tilde{X}$ as N increases.

Now we shall state things more precisely:

Theorem. Let $X \subset \mathbb{P}^n$ be an irreducible complete intersection defined over a field k of characteristic zero. Then there is a pair of finite filtrations of G_N by subvarieties

$$G_N = A_{N,0} \supset A_{N,1} \supset ...$$

and

$$G_N = B_{N,0} \supset B_{N,1} \supset ...$$

with the following properties:

1. For each N and i $A_{N,i} \subset B_{N,i}$.

2. Writing C_N for the (finite) union $\cup_i B_{N,i} \backslash A_{N,i}$, the variety C_N is nonempty for large N.

3. Each k-rational point of C_N is a subspace $V \subset H^0(X, \mathcal{O}_X(N))$ such that the rational image of X in the projectification of the dual of $H^0(X, \mathcal{O}_X(1)) \otimes V$ is the image of a nonsingular variety $\tilde{X}$ und er an unramified morphism, and $\tilde{X}$ is the Nash blowup of X_V.

4. Every resolution $\tilde{X}$ of X arises in this way from some point of a C_N.

We will prove the theorem a little later on.

Say the complete intersection X is defined by homogeneous polynomial equations

$$0 = f_1 = f_2 = ... = f_s$$

for $s = n-r$. Let $d_i = degree(f_i)$. The adjunction formula and induction upon s give an equivalence of divisors $K_X = cH$ for H a hyperplane and $d_1 + ... + d_s = c + n + 1$. Specifically, if $s = 0$ then $V = \mathbb{P}^n$ and $K_V = (-n-1)H$ for H a hyperplane. If $s = 1$ then let H be a hyperplane and apply the adjunction formula $K_X = (K_{\mathbb{P}^n} + X)X = ((-n-1)H + d_1H)X$ which shows K_X is $(-n-1+d_1)$ times the hyperplane section divisor $V \cap H$, so $c = -n-1+d_1$, and so-on.

There is however a more illuminating way of understanding the identity

$$d_1 + ... + d_s = c + n + 1.$$

Namely, the Atiyah sequence (displayed a little later on in this paper) of the sheaf $\mathcal{O}_X(N+1)$ gives a map from the the the tensor product of the highest exterior power of $\Omega_{X/k}(N+1)$ with $\mathcal{O}_X(N+1)$ to the $r+1$'st (=highest) exterior power of the principal parts sheaf $Pr(\mathcal{O}_X(N+1))$. This map is an isomorphism since $\mathcal{O}_X(N+1)$ is locally free, and the sequence is then locally split; and so the highest exterior power of $\mathcal{P}(\mathcal{O}_X(N+1))$ is just a copy of $\mathcal{O}_X(K_X + (N+1)(r+1)H)$ where K_X is the canonical divisor of X and H is a hyperplane section. The equivalence in the divisor class group

$$K_X + (N+1)(r+1)H \cong ((d_1 - 1) + ... + (d_s - 1) + (r+1)N)H$$

is realized by an actual isomorphism of line bundles

$$\begin{aligned} O_X((r+1)(N+1)H + K_X) &\cong \Lambda^{r+1}\mathcal{P}(O_X(N+1)) \\ &\to \mathcal{O}_X((r+1)N + (d_1-1) + ... + (d_s-1)) \end{aligned}$$

such that if $h_0, .., h_r$ are homogeneous polynomials of degree $N+1$

$$dh_0 \wedge ... \wedge dh_r$$

is sent to the homogeneous polynomial g of degree

$$(r+1)N + (d_1 - 1)... + (d_s - 1)$$

defined by the rule

$$gdx_0 \wedge ... \wedge dx_n = d(h_0) \wedge ... \wedge d(h_r) \wedge df_1 \wedge ... \wedge df_s \qquad (2)$$

For any number N the global sections of the principal parts sheaf $\mathcal{P}(O_{\mathbb{P}^n}(N+1))$ can be viewed as expressions $a_0dx_0 \oplus ... \oplus a_ndx_n$ where a_i are homogeneous polynomials of degree N (see [Stevens] for instance). Therefore there is map of finite dimensional k vector spaces

$$\Lambda^{r+1}(V \otimes \Gamma(\mathbb{P}^n, \mathcal{O}(1))) \to \Gamma(P^n, \Lambda^{r+1}\mathcal{P}(\mathcal{O}_{\mathbb{P}^n}(N+1))) \quad (1)$$

such that for any sequences

$$y_0, ..., y_r \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$$

$$v_0, ..., v_r \in V$$

one has

$$(y_0 \otimes v_0) \wedge ... \wedge (y_r \otimes v_r) \mapsto d(y_0 v_0) \wedge ... \wedge d(y_r v_r).$$

Restricting to X and composing with the isomorphism to $\mathcal{O}_X((N+1)r + (d_1 - 1) + ...d_s - 1))$ which we have just described, we obtain then a map which we'll call

$$E : A \to \Gamma(X, \mathcal{B})$$

for $A = \Lambda^{r+1}(V \otimes \Gamma(X, \mathcal{O}_X(1)))$ and $\mathcal{B} = \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1))) \cong \mathcal{O}_X((r+1)N + (d_1 - 1) + ... + (d_s - 1))$. The map E is defined such that

$$E((y_0 \otimes v_0) \wedge ... \wedge (y_r \otimes v_r)) = g$$

where g is defined by the rule

$$d(y_0 v_0) \wedge ... \wedge d(y_r v_r) \wedge df_1 \wedge ... \wedge df_s = g dx_0 \wedge ... \wedge dg_n.$$

The map E induces a second map

$$\begin{aligned}
G : \Lambda^{r+1}[\Gamma(X, \mathcal{O}_X(1)) \otimes V \otimes \Lambda^{r+1}[\Gamma(X, \mathcal{O}_X(1)) \otimes V]] &= \Lambda^{r+1}[\Gamma(X, \mathcal{O}(1)) \otimes V \otimes A] \\
&\to \Gamma(X, \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1) \otimes \mathcal{B}) \\
&= \Gamma(X, \Lambda^{r+1}(\mathcal{P}(\mathcal{O}_X(N+1) \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1)))) \\
&\cong \Gamma(X, \Lambda^{r+1}(\mathcal{P}(\mathcal{O}_X(N+1) \otimes \mathcal{O}_X((r+1)N + (d_1 - 1) + ... + (d_s - 1)))) \\
&\cong \Gamma(X, \mathcal{O}_X(K_X + (r+1)(N + 1 + (r+1)N + (d_1 - 1) + ... + d_s - 1)) \\
&= \Gamma(X, \mathcal{O}_X(d_1 + ... + d_s - n - 1 + (r+1)N + (r+1) + (r+1)^2 N + (r+1)(d_1 + ... + d_s - s)) \\
&= \Gamma(X, \mathcal{O}_X((r+2)(N(r+1) + (d_1 - 1) + ... + (d_s - 1)))
\end{aligned}$$

such that given sequences

$$y_0, ...y_r \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$$

$$v_0, ..., v_r \in V$$

$$a_0, ..., a_r \in A$$

we have

$$G(y_0 \otimes v_0 \otimes a_0) \wedge ... \wedge (y_r \otimes v_0 \otimes a_r) = g$$

where g is the homogeneous polynomial such that

$$d(y_0 v_0 E(a_0)) \wedge ... \wedge d(y_r v_r E(a_r)) \wedge df_1 \wedge ... \wedge df_s = g dx_0 \wedge ... \wedge dx_n.$$

The diagram below compares $r+2$'nd symmetric power of the of E, composed with the inclusion of the $r+2$'nd symmetric power of the global sections in the global sections of the $r+2$'nd symmetric power, with the map G.

$$\begin{array}{ccc} S^{r+2}\Lambda^{r+1}[\Gamma(X,\mathcal{O}_X(1))\otimes V] & \rightarrow & \Gamma(X, S^{r+2}\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1))/torsion) \\ & & \downarrow \\ \Lambda^{r+1}[\Gamma(X,\mathcal{O}_X(1))\otimes V \otimes \Lambda^{r+1}[\Gamma(X,\mathcal{O}_X(1))\otimes V]] & \rightarrow & \Gamma(X, \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1)\otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N+1)))/torsion) \end{array}$$

The upper arrow is the Wronskian or Wahl map in its action on a symmetric power. Because $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X) \cong \Lambda^r \Omega_X$, the vector spaces on the right are both isomorphic to the space of global sections of $O_X((r+2)K_X+(r+1)(r+2)(N+1))$ and the vertical map on the right side of the square is the isomorphism on global sections so induced.

Our first technical lemma will be that if V is a complete intersection then it is possible to lift this downward vertical map to a downward vertical map between the vector spaces on the left, completing the diagram to a commutative square; or, equivalently, that the image of the composite map from the upper left corner to the lower right, is contained in the image of the lower map.

1. Lemma. If V is a complete intersection then there is a vertical map completing the commutative square.

Proof. The definition of the map in question involves the rational number

$$\lambda = (N+1)/((r+2)(N+1)+r+c+1)$$

where r is the dimension of V, and c is the number such that

$$K_V = cH$$

for a hyperplane H. The map is to send the $r+2$'nd symmetric power

$$\gamma^{\circ(r+2)}$$

for

$$\gamma = (y_0 \otimes v_0) \otimes ... \otimes (y_r \otimes v_r)$$

to

$$\lambda \cdot (y_0 \otimes v_0 \otimes \gamma) \wedge ... \wedge (y_r \otimes v_r \otimes \gamma).$$

We have seen that $c = d_1 + ... + d_s - n - 1$.

Let us use calculate the difference

$$d(gy_0v_0) \wedge ... \wedge d(gy_rv_r) \wedge df_1 \wedge ... \wedge df_s - g^{r+2}dx_0 \wedge ... \wedge dx_n \\ = g^r dg \wedge \beta \wedge \delta$$

where

$$\beta = \sum_{u=0}^{r} (-1)^u h_u dh_0 \wedge ... \wedge \widehat{dh_u} \wedge ... \wedge dh_r$$

where $h_i = y_iv_i$ and

$$\delta = df_1 \wedge ... \wedge df_s.$$

In case $\lambda = 1$ it is this difference which we will need to show is equal to zero, and for other values of λ we will need to establish an equality involving both λ and the difference.

We may assume h_0 is not identically zero on X. Since $g^{degree(h_0)}$ and $h_0^{degree(g)}$ have the same degree, $g^{degree(h_0)}/h_0^{degree(g)}$ is a rational function on projective space, and because h_0 is assumed not to vanish identically on X this restricts to a rational function on X. By assumption, none of the x_i vanish on all of V. Writing $e = degree(h_0)g^{degree(h_0)-1}/h_0^{degree(g)}$ we may view the differential of our rational function as a differential form on affine space with the origin removed, and we have

$$d(g^{degree(h_0)}/h_0^{degree(g)}) = e \cdot (dg - (degree(g)/degree(h_0)(g/h_0)dh_0).$$

I claim

$$d(g^{degree(h_0)}/h_0^{degree(g)}) \wedge \beta \wedge \delta = 0.$$

To see this, let us change basis in the differentials of the rational function field of affine space over k , using as a new basis

$$dx_0, dx_1 - (x_1/x_0)dx_0, ..., dx_n - (x_n/x_0)dx_0$$

Then within the expression for β each dh_i can be written so that the coefficient of x_0 is the result of applying the Euler derivation to h_i

$$dh_i = (\sum_{u=0}^{n}(x_u/x_0)\partial h_i/\partial x_u)dx_0 + \sum_{v=1}^{n} \partial h_i/\partial x_v(dx_v - x_v/x_0dx_0) \\ = degree(h_i)(h_i/x_0)dx_0 + \sum_{v=1}^{n} \partial h_i/\partial x_v(dx_v - x_v/x_0dx_0).$$

Substituting for the dh_i in the definition of β, and expanding using the distributive law for exterior products, each product $dh_0 \wedge ... \wedge \widehat{dh_u} \wedge ... \wedge dh_r$ is a sum of $(n+1)^r$ exterior products of r factors, and each product which is not zero includes at most one exterior factor of the type $degree(h_i)(h_i/x_0)dx_0$. Moreover, each term occurs multiplied by the term h_u. Because of the alternating signs, applications of the identity

$$h_u h_i = h_i h_u$$

allow all the exterior monomials containing one of the exterior factors $degree(h_i)(h_i/x_0)dx_0$ to cancel out. It follows that β is contained in the r'th exterior power of the kernel of the map

$$\mathcal{P}(\mathcal{O}(N)) \to \mathcal{O}(N)$$

$$a_0 dx_0 + ... + a_n dx_n \mapsto a_0 x_0 + ... + a_n x_n.$$

The Atiyah sequence

$$0 \to \Omega_{\mathbb{P}^n/k}(N) \to \mathcal{P}(\mathcal{O}(N)) \to \mathcal{O}(N) \to 0$$

is exact, and our differential $d(g^{degree(h_0)}/h_0^{degree(g)})$ belongs then also to the the kernel of the same map. Or, directly, one sees that if q is a rational function on affine space, then the map sends dq to the euler derivation applied to q, and so if q is a rational function on projective space, having degree zero, it is sent to zero, and so belongs to the kernel.

The map which we have been discussing

$$\Lambda^{r+1}\mathcal{P}(\mathcal{O}_{\mathbb{P}^n}(N)) \to \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(N)) \cong \mathcal{O}((r+1)N + (d_1 - 1) + ... + (d_s - 1))$$

sends $d(g^{degree(h_0)}/h_0^{degree(g)}) \wedge \beta$ to the homogeneous polynomial p so that

$$p dx_0 \wedge ... \wedge dx_n = d(g^{degree(h_0)}/h_0^{degree(g)}) \wedge \beta \wedge \delta.$$

We have seen that the wedge product of the first two factors lies in the image of the $r+1$'st exterior power of the Kahler differentials of projective space (suitably twisted), and thus the triple wedge product can be lives in the $r+1$'st exterior power of a copy of the Kahler differentials of X. Since X is only r dimensional, the triple wedge product is zero, and so $p = 0$.

Then, from this, we have

$$dg \wedge \beta \wedge \delta = degree(g)/degree(h_0)(g/h_0)dh_0 \wedge \beta \wedge \delta.$$

Now, of the $r+1$ summands in the expression for $dh_0 \wedge \beta$ all are zero except

$$h_0 dh_0 \wedge dh_1 \wedge ... \wedge dh_r.$$

Therefore

$$degree(g)/degree(h_0)(g/h_0)dh_0 \wedge \beta \wedge \delta = degree(g)/degree(h_0)gdh_0 \wedge ... \wedge dh_r \wedge \delta$$
$$= (degree(g)/degree(h_0))g^2 dx_0 \wedge ... \wedge dx_n. \quad (3)$$

Now, we have not yet introduced the number λ into the proof of the lemma, but we have seen that

$$d(gh_0) \wedge d(gh_1) \wedge ... \wedge d(gh_r) \wedge df_1 \wedge ... \wedge df_r$$
$$= g^{r+2} dx_0 \wedge ... \wedge dx_r + g^r dg \wedge \beta \wedge \delta.$$

We also saw that the last term is the same as

$$(degree(g)/degree(h_0))g^{r+2} dx_0 \wedge ... \wedge dx_n.$$

Combining we see

$$d(gh_0) \wedge ... \wedge d(gh_r) \wedge df_1 \wedge ... \wedge df_s$$
$$= (1 + degree(g)/degree(h_0))g^{r+2} dx_0 \wedge ... \wedge dx_n.$$

Then the difference which we shall prove to be zero is

$$g^{r+2} dx_0 \wedge ... \wedge dx_n - \lambda d(gh_0) \wedge ... \wedge d(gh_r) \wedge df_1 \wedge ... \wedge df_s$$
$$= (g^{r+2} - \lambda(1 + degree(g)/degree(h_0)))g^{r+2})dx_0 \wedge ... \wedge dx_n$$

so it suffices to show λ is the reciprocal of $1 + degree(g)/degree(h0) = (N + 1 + degree(g))/(N + 1)$, and this follows from the fact that $degree(h_0) = N$ and $degree(g) = N(r + 1) + (d_1 - 1) + ... + (d_s - 1)$, and the lemma is proved.

The main consequence of this is that – since the right vertical map is injective – the image of the lower map will equal the image of the right vertical map if and only if the top map and lower map have the same rank. We define the A and B filtrations to be the rank filtrations for the image of the top and lower maps.

Proof of theorem: The constructibility of C_N follows from the fact that the appropriate inequality relating the ranks of two maps suffices to make their images equal. The fact that the Nash blowup of X_V is unramified for $V \in C_N$ is proven like this: let I be the subsheaf of $\mathcal{O}_X(N)$ generated by the vector space V of global sections. For any torsion free rank one coherent sheaf I on X let $F(I) = \Lambda^{r+1}\mathcal{P}(I)/torsion$. Now, X_V is the blowup of X along I and if we let $\tilde{X}$ be the Nash blowup of X_V then the pullback (mod torsion) of the map $F(I)^{r+2} \to F(IF(I))$ of [Finite generation] is the map of torsion free Kahler differentials whose whose surjectivity implies the Nash blowup is unramified. We have contrived that the inclusion of the image of the top map in the image of the lower map in our commutative square of vector spaces is in fact a map of generating global sections of one sheaf to the other, inducing the map of sheaves. Let us give now the proof that this is the case.

2. Lemma. Let M be an integer and let W be a nonzero vector space of global sections of $\mathcal{O}_X(M)$. Let $\mathcal{J} \subset \mathcal{O}_X(1)$ be the subsheaf generated by W. Then the image of the map

$$\Lambda^{r+1}(\Gamma(X, O_X(1)) \otimes W) \to \Gamma(X, \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{J}))$$

generates the sheaf $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{J})$.

Proof. Consider the open set U in X defined by the equation $x_0/ne0$. Let K be the rational function field of X. For any module S over the coordinate ring of U, the principal parts module $\mathcal{P}(S)$ over the same ring can be constructed as the submodule of $K \oplus \Omega_{K/k}$ generated by the $f \oplus df$ for $f \in S$. Taking S to be the module of sections of $\mathcal{O}_X(1) \otimes \mathcal{J}$ over U a global section y of $O(1)$ and an element $w \in W$ give rise to the section yw/x_0^{m+1} on U and to the principal part

$$(yw/x_0^{M+1}) \oplus d(yw/x_0^{M+1}).$$

On the other hand, the same pair of elements y, w determine the differential on affine space

$$d(yw) = \sum_{i=0}^{n} x_i \frac{\partial(yw)}{\partial x_i} dx_0 + \sum_{i=1}^{n} \frac{\partial(yw)}{\partial x_i}(dx_i - x_i/x_0 dx_0)$$

$$= \frac{y}{x_0} w \cdot (M+1)dx_0 + x_0 \cdot \sum_{i=1}^{n} \frac{\partial(\frac{y}{x_0}\frac{w}{x_0^{M+1}})}{\partial(\frac{x_i}{x_0})} d(\frac{x_i}{x_0}).$$

$$= \frac{y}{x_0}\frac{w}{x_0^M}(M+1)x_0^M dx_0 \oplus x_0 d(\frac{y}{x_0}\frac{w}{x_0^{M+1}}).$$

The map from the sheaf of differentials on affine space generated by the $d(yw)$ pushed forward to X and pulled back to U, on the one hand, to the principal parts sheaf of the module $(\mathcal{O}(1) \otimes \mathcal{J})(U)$ in $K \oplus \Omega_{K/k}$, on the other hand, is induced by the linear map of K vector spaces sending $(M+1)x_0^M dx_0$ to the basis vector $1 \oplus 0$ and sending $x_0 d(\frac{x_i}{x_0})$ to $0 \oplus d(\frac{x_i}{x_0})$. Under this map, the $d(yw)$ are sent to $f \oplus df$ for f running over a set of module generators times generators of the coordinate k algebra. By [On resolving] the images of the $d(yw)$ therefore generate the principal parts module and the lemma is proved.

Return now to our situation, where $\mathcal{I}$ is the subsheaf of $\mathcal{O}(N)$ generated by V. The most recent lemma with $W = V$ and $\mathcal{J} = \mathcal{I}$ implies that the image of the map $\Lambda^{r+1}(\Gamma(X, \mathcal{O}_X(1)) \otimes V) \to \Gamma(\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I}))$ is a space of global generating sections of the sheaf $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I})$. The same lemma , with $W = \Gamma(X, \mathcal{I} \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I}))$ and $\mathcal{J} = \Lambda^{r+1}\mathcal{P}(\mathcal{O}(1) \otimes \mathcal{I})$ shows that the image of $\Lambda^{r+1}(\Gamma(X, \mathcal{O}_X(1)) \otimes \Gamma(X, \mathcal{I} \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I})))$ generates the sheaf $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}(1) \otimes \mathcal{I}))$.

Moreover because tensor products of generating sections always generate a tensor product of sheaves (even while they need not span the global sections space of the tensor product of sheaves), $\Gamma(X, \mathcal{I} \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I})))$ is generated by global sections of I times global sections of $\mathcal{P}(\mathcal{O}_X(1) \otimes \mathcal{I})$.

Combining this information verifies that we have indeed contrived that whenever V belongs to C_N the map of sheaves is surjective, and so the Nash blowup is unramified, and therefore $\tilde{X}$ is nonsingular.

It remains to show every $\tilde{X}$ arises from some choice of $V \in C_N$. Also by [Finite generation and the Gauss process] if I is a sheaf of ideals so that $Bl_I(X) \to X$ is unramified then, after possibly replacing I by one of the terms of the sequence of torsion free rank one coherent sheaves

$$I, F(I), F(IF(I)), ...$$

where F is the functor so $F(I) = \Lambda^{r+1}\mathcal{P}(I)/torsion$, we obtain that the maps of sheaves inducing the lower map and the right vertical map do have the same image. We start with the sheaf of ideals I generated by the homogeneous polynomials belonging to V. We replace I by one of the terms in the sequence of sheaves such that the map of sheaves is surjective, then we twist by a suitable number N so that the induced map on global sections is also surjective. It is the case that the new sheaf can be obtained from an appropriate vector space of homogeneous polynomials, and also it is the case that our particular generators, the images of the maps in our commutative square, are global sections which generate the sheaves. However, it may not yet be true that the spaces we are interested in include all the global sections of the sheaves.

Construct then the doubly graded ring which consists in $i = 0$ of the sum of the global sections of the $\mathcal{O}_X(NH)$ and then in degree $i = 1$ put the forms of degree $(r+1)N + (d_1 - 1) + ... + (d_s - 1)$ generating $F(I(H))(NH)$ and products with forms of arbitrary degree. Then in degree $i = (r + 2)$ put the forms of degree $(r+1)((r+1)(N+1)+N+c)+c+r+1 = (r+2)((r+1)(N+1)+c)$ generating $F(I(H)F(I(H))(NH)$ and so-on. The ring thus created is closed under multiplication because of the lemma above. More precisely, let Q be the operator on graded ideals of the homogeneous coordinate ring, which sends an ideal I to the ideal generated by all g such that $g dx_0 \wedge ... \wedge dx_r = d(x_{v_0} i_0) \wedge ... \wedge d(x_{v_r} i_r) \wedge df_1 \wedge ... \wedge df_s$ where now the elements $i_0, ..., i_r$ are arbitrary homogeneous elements of the graded ideal. Because of Leibniz rule applied to each of the first $r + 1$ factors, the terms of level $i = (r + 2)^\alpha$ consist of the ideal L_α where the L_i are defined by setting

$$L_0 = Q(J)$$

$$L_{i+1} = Q(JL_0...L_I)$$

Since the maps from one i value to another correspond to maps of sheaves which are isomorphisms, the corresponding graded modules agree up to integral closure, and this means then as in an argument in the earlier paper that the ring (this time a doubly graded ring) is finitely generated and then therefore that eventually one of our actual maps defined on spanning sets is an isomorphism of finite dimensional vector spaces. In other words eventually $L_{i+1}^{r+2} = L_{i+2}$. Note that L_i is generated by its term of lowest N degree, and this vector space is the desired vector space V. The various replacements we have effected alter X_V by at most a Nash blowup and do not therefore alter $\tilde{X}$.