First introduction to Projective Toric Varieties

Chapter 1

Projective toric varieties are a type of possibly singular complex manifolds indexed by easy combinatorial data having to do with poles and zeroes of meromorphic functions.

It is maybe easiest to introduce the combinatorial data first. We start with an abstract lattice of some rank n which we may take to be just the set Z^n, whose elements we could denote on paper by columns of length n with integer entries.

Then any finite set $S \subset Z^n$ describes a projective toric variety, if it is not empty.

The easiest way of describing the singular manifold is to say that S has an "affine structure." Although it is not always possible to add elements of S to obtain other elements of S, it is certainly possible to determine the truth or falsehood of any additive equation involving elements of S. Because we won't be caring about the choice of origin $0 \subset Z^n$ we will restrict attention to homogeneous equations, such as the equation $a + b + c = d + e + f$ among six elements of $S \subset Z^n$. This particular equation is homogenous of degree three, but any degree is allowed.

We say that an element $x \in Z^n$ is in the *convex hull* of S if an equation holds such as

$$x + x + x = a + b + c$$

for elements $a, b, c \in S$ not necessarily distinct. And we say that S is *convex* if it is equal to its convex hull.

Ordinarily one only considers convex subsets $S \subset Z^n$ and if one uses a non-convex subset, the resulting complex manifold is not called 'toric' by convention, though there is a well-defined process called 'normalization' that can be applied to the singular manifold itself to get back to the one corresponding to the convex hull.

A function $h : S \to C$ to the complex numbers is called, let us say, an *affine map* if it preserves the affine structures, where we use multiplication in C. This means that whenever we have an equation in S like

$$a + b + c = d + e + f$$

we should also have

$$h(a)h(b)h(c) = h(d)h(e)h(f).$$

The projective toric variety corresponding to S is defined (usually only when S is convex) to be the equivalence classes of affine maps $S \to C$ which are not identically zero, under the relation that two maps h, h' are considered equivalent if there is a nonzero complex number λ such that

$$h = \lambda h' .$$

Example. If the elements of S are affinely linearly independent (meaning that if one is considered to be the origin the others are linearly independent), then there are no homogeneous equations among elements of S except tautologies like a+b=b+a etcetera. This means that the affine maps $h : S \to C$ are merely all functions. Labelling the elements of S as

$$x_0, ..., x_m$$

where m is the dimension of the affine span of $S \subset Z^n$ then we see that the projective toric variety is just all ratios $[x_0 : ... : x_m]$ which is the set of points in the projective space P^m of dimension m .

Example. The previous example is a little restrictive if we require S to be convex, because there are not many convex affinely independent sets. They are, I think, just the affine bases of (cosets of) summands of Z^n.

However, if we ever expand a convex set S, simply taking the multiples ks, for $s \in S$ where k is a fixed integer, and take the convex hull, we are in a situation where any affine map from the vertex set kS to C extends uniquely to an affine map from the convex hull.

So the process of expanding S by an integer and taking the convex hull does not affect the projective toric variety.

And it follows

Example. If S is an affine basis of a summand of the lattice and k is a nonzero integer, then the convex hull of kS defines the projective toric variety which is a projective space.

In a certain sense these are the universal examples, because when there are relations among elements of S it just means that some of the the functions $S \to C$ are not allowed, and we get a closed subset of projective space insteadof the whole projective space.

For simplicity say S spans Z^n affinely.

The affine maps $S \to C$ which send *no* element of S to zero extend uniquely to affine maps

$$Z^n \to C^\times$$

where $C^\times$ is the group of nonzero complex numbers under multiplication.

Instead of modding out by the action of nonzero scalars, we can equivalently here just pass to the subset of affine maps sending 0 to 1. These are just the ordinary group homomorphisms $Z^n \to C^\times$.

This set is bijective with $(C^\times)^n$ and it is a group under multiplication. It is called a "torus" because each factor of $C^\times$ contains the unit circle, and a cartesian product of n copies of a circle is sometimes called an n dimensional torus.

But here we call the larger group of real dimension $2n$ a torus also.

And thus we see that any projective toric variety contains a torus.

Example.

Let's look at the four points in Z^2 which have entries of 0 and 1. These are the four columns of the matrix

$$\binom{0 \quad 0 \quad 1 \quad 1}{0 \quad 1 \quad 0 \quad 1}.$$

We call these a, b, c, d and the only nontrivial relation

is

$$a + d = b + c.$$

We visualize the letters a, b, c, d as the four corner of a square (with coordinates as given by the columns w chose),

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

The various afine functions h merely assign numbers to a, b, c, d and the affine requirment together with the nonzero requirement just means be the entries of a matrix of rank one.

Thus we can see the elements $a, b, c, d \in S$ as ordinary variables, and the arrangement as a square, the positions we have writen them on the page, require us to substitute only such numbers as yield a rank one matrix.

So the variety is rank one matrices modulo multiplication by nonzero scalars. And we see that such a thing is uniqly determined by just the ratios $[a : b]$ and $[a : c]$ so that our variety is just the cartesian product of two copies of one dimensional projective space.

We could have seen this more easily by working component-by-component. Our set S was after all only the cartesian product of two copies of $0, 1 \subset Z$.

However, here we have constructed the same surface as the solution set of the homogeneous equation $ad = bc$ and it is a subset of projective three-space.

Note that the square diagram above is uniquely a linear degeneration of a tetrahedron; the linear degeneration is the defining constraint for the surface within projective space.

Chapter 2 – Divisors

In this section I'll use define some words which I won't end up using, so if something doesn't make sense, don't worry and just keep reading.

Now let's talk about divisors. An irreducible divisor on a complex manifold means a possibly singular codimension one submanifold which is not a union of smaller ones.

A divisor means either a finite union of irreducible divisors, or, more generally, a finite sequence of irreducible divisors to which are attached integers.

A meromorphic function on a normal complex projective variety has a divisor of poles and zeroes, and by convention we count poles as being negative zeroes, so that if a meromorphic function could have only zeroes, its divisor would have only positive numbers attached to its irreducible components. But in fact this never happens. A meromorphic function without any poles would be defined everywhere on the complex projective variety, but there is no such function except constants.

One way around this difficulty is to consider instead of only functions, rather to consider the more general things which are sections of line bundles. For any locally principal divisor, one can make a holomorphic line bundle such that your divisor is exactly equal to the divisor of zeroes (and poles) of a meromorphic section.

Because you can always multiply a meromorphic section of a line bundle by a meromorphic function, then once you can obtain your divisor from a line bundle, you can equally well obtain any divisor which differs from your divisor by the divisor of a meromorphic function.

Divisors coming from meromorphic functions (meromorphic sections of the trivial bundle) are called "principal divisors" and two divisors which differ only by a principal divisor are called "linearly equivalent."

One can easily show by this correspondence that isomorphism types of holomorphic line bundles are naturally bijective with the a group of divisors (the locally principal divisors) modulo principal divisors. If the variety is smooth, all divisors are locally principal.

Smoothness exercise. Show that the manifold is smooth if and only if S is convex and each vertex of the convex hull together with the vertices connected by an edge form an affine lattice basis.

Although it is actually easier, because it is abstract, we won't here talk about line bundles, but only about divisors.

If we choose any codimension-one face of our convex set S, it will of course be a convex subset on its own, and will define a codimnsion one subvariety.

The way this works is that a face $S_0 \subset S$ corresponds to the classes mod scalars of affine maps $S \to C$ which send *all but* the elements of S_0 to zero.

Exercise. Show that the set of elements *not* sent to zero under any affine map $S \to C$ is either empty or a face of some dimension.

If the codimension one faces of S are $F_1, ..., F_u$ then the divisor in which each F_i is given coefficient -1 is denoted

$$-F_1 - F_2 - ... - F_u$$

and it is called a "canonical divisor" because it is the divisor of (zeros and) poles of a meromorphic differential n forms where n is the dimension of the variety (we call it n consistently with the superscript in Z^n as if S spans affinely our variety has dimension n).

Actually, I've used the same letter F_i for the codimension one face of S as I have for the corresponding codimension one subset of our variety – the latter of course is all the affine maps which are zero except on F_i.

Anyway, with this abuse of notation accepted, the only divisors we ever have to worry about are going to be integer linear combinations of the F_i.

Again, this is something you don't yet need to know, but the torus acts on the variety and there are no torus invariant divisors except linear combinations of the F_i. And every divisor is linearly equivalent to a torus invariant divisor.

And this means we won't go wrong if we just pretend that there are no divisors except which we make by assigning integers to the faces of S.

If we view S or perhaps its real convex hull as a cell complex, then what we are talking about is exactly a cellular $n-1$ chain.

Let me give you the next thing to visualize without explaining it. What I want you to visualize is that instead of assinging integers to the faces of S, we are going to move the faces in and out.

If you think of the faces as infinite hyperplanes in Z^n, then as these hyperplanes are translated from one position to another in R^n , they collide at certain times with points of Z^n. Or, if we think of the hyperplanes as just being subsets of Z^n, the translates actually are cosets of Z^n modulo whichever translate contains 0, and the cosets are elements of an infinite cyclic quotient group.

Whenever I speak of 'how far to move' a face of S I am going to mean using counting in the infinite cyclic quotient group that has to do with that face.

Now, it is best to start by choosing an element of S to be the origin of the lattice, and having our movable hyperplanes $F_1, ..., F_u$ chosen as being the ones meeting at the origin. To construct S, we will move each one in the originally chosen direction (outward) some number of steps. This gives a positive integer to assign to each F_i , and if we call these integers $e_1, ..., e_u$ then we will write down the divisor $e_1F_1 + ... + e_uF_u$

This is an *effective* divisor because all e_i are positive. The hyperplanes F_i each moved out e_i steps provide the faces of the convex set S. This choice of the e_i, or any other which defines a finite subset of the lattice defining the same projective variety, makes $e_1F_1 + ... + e_uF_u$ be a *very ample* divisor.

I should have said this earlier, but if we choose any two points of Z^n, say x, y, then for each point h of our variety, the ratio $h(x)/h(y)$ is well defined as long as $h(y) \neq 0$.

It is convenient to take y to be the origin in Z^n and then if you recall when we restrict h to the torus as no element is sent to zero, instead of reducing modulo scalar multiplication it is better to assume $h(0) = 1$. Then our chosen element $x \in Z^n$ corresponds to the function

$$h \mapsto h(x)$$

and this is something very familiar. For, just as the torus is group homomorphisms $Z^n \to C^\times$ we can recover the group Z^n as analytic group homomorphisms $(C^\times)^n \to C^\times$.

Each lattice element $x \in Z^n$ already corresponds to an analytic function on the torus, and one which is a group homomorphism (a "character" of the torus).

And when we take y to be the origin of Z^n the function sending h to the well defined ratio $h(x)/h(y)$, when we restrict to the torus in our variety, is nothing but x itself, viewed as a character of the torus.

Now I am going to just state something, without getting us lost in any details. I said already that the convex polyhedron we started with, because we needed to move the faces out in order that it should have any volume at all, has positive integers assigned to its faces. The particular integers depended on a choice of origin of the lattice.

With respect to the same origin, all the lattice points contained in the polyhedron correspond to particular meromorphic functions, and being inside the polyhedron (or on the boundary) means that all of these functions have poles *no worse* than the prescribed divisor

$$e_1F_1 + ... + e_uF_u$$

This means that we can reinterpret our chosen convex set S as being the torus characters which have divisor of poles no worse than as prescribed by that divisor.

Now, each lattice point q also has a principal divisor. The origin of the lattice has the divisor zero. Starting with all the hyperplanes in $\mathbb{Z}^n$ intersecting at the origin, if we choose any other lattice point q, the number of steps (with outwards counting positively) each face has to move so that all meet at the point q, counting steps in the infinite cyclic quotient group modulo that hyperplane, defines a divisor $a_1F_1 + ... + a_uF_u$, and this is the divisor of zeroes and poles of q.

The fact that not all of the a_i can be positive corresponds to the fact that the point q has no volume, so not all F_i can be moved outwards if they are all to meet at one point.
Changing the origin, or equivalently translating our convex polyhedron, then modifies our effective divisor

$$e_1F_1 + ... + e_uF_u$$

by adding to it all possible principal divisors.

To say that we do not care about translating our polyhedron is the same as saying that we don't care about the divisor, only the divisor class.

And we see that the divisor class group is spanned by the F_i as an abelian group, with n linearly independent relations coming from translation. So it is an abelian group of rank $u - n$, the number of faces of S minus the rank of the lattice which S spans.

Now, I have cheated a bit because in defining linearly equivalent divisors one is suposed to use principal divisors coming from arbitrary meromorphic functions, here we only used torus invariant principal divisors. But, that is OK because if a difference of torus invariant divisors is principal it is certainly a principal torus invariant divisor.

Example.

Consider the four elements of Z^2 which are the columns of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

The slanted face connecting the last two points has coefficient 2, the ones containing 0 have coefficient 0, the remaining one has coefficient one. This is because the slanted face mets latice points if it is to be translated to the origin.

An example of a principal divisor is the difference of the two vertical edges minus the slanted edge. This is the divisor of the lattice point $(1, 0)$.

Exercise. For convex sets spaning Z^n show that a torus equivariant map from one projective toric variety to another amounts to saying that after expanding the first convex set by an integer multiple and taking the convex hull again, it is a union of tanslates of the other.

Since the convex set in the previous example already is a union of three vertical line segments, then there is a map from this complex projective surface to the projective line (Rieman sphere).

That map is in fact a fiber bundle map, and our surface is a fiber bundle of the Riemann shpere over the Rieman sphere. It is the Hirzebruch surface, also called a scroll.

There is a ring called the Chow ring which is a historical precursor to the cohomology (and for smooth projective toric varieties is the commutative ring is the even dimensional part of integer cohomology).

Here we could define it by saying that it is a graded ring which is Z in degree 0. In degree 1 it is spanned by the F_i modulo the n linear relations.

Then we say that the ring multiplication comes from intersection, and the ring in degree i is spanned by codimension i faces. More precisely, if two faces f, g intersect transversely the product fg is the intersection, and if they do not, relations in the divisor class group are used to arrange that they do.

Ampleness, Local Principality

We started with a finite set S containing an origin of the lattice, and interpreted the linear combination $e_1F_1 + ... + e_uF_u$ to mean that we have moved each F_i outwards by e_i steps so that the F_i define the convex hull of S. Note that if we replace each e_i by the *same* multiple λe_i where λ is a positive integer, it will define the convex hull of the equivalent set λS.

There is a definition that is not very relevant for toric varieties, a divisor D is called *ample* if some positive integer multiple λD is very ample. In view of the statement above,

Observation A divisor on a toric variety is ample if and only if it is very ample.

Let's give an explicit criterion of ampleness. Assuming that two finite sets S, T both span the same lattice Z^n, the projective varieties defined by S and T are the same if and only if some expansion λS of S by a positive integer λ is a union of translates of T and vice-versa. Thus when S is convex, a divisor $a_1F_1 + ... + a_uF_u$ is ample (=very ample) if and only if for some positive integer λ the set of points enclosed in what we call we call $\lambda a_1F_1 + ... + \lambda a_uF_u$ is a union of translates of S. Thus

Fact. A divisor D is (very) ample if and only if the convex hull of the set of torus characters with poles no worse than D is combinatorially equivalent to the convex hull of S, with corresponding faces of all dimensions parallel.

Let's now give a description of the local defining equations of each divisor F_i as a codimension one subset of our variety.

If we choose a vertex of S then a corresponding open subset of the variety (it is covered by such open sets) consists of the affine maps $S \to C$ sending this vertex to $1 \in C$. Those points belonging to F_i are as we said just those affine maps sending the points of S which are enclosed in the polyhedron which we visualize as $e_1F_1 + ... + (e_i - 1)F_i + ... + F_u$ to zero.

Since we have taken the e_i to be large, if the chosen vertex does not belong to F_i this will be all the points of the open set (= all affine maps sending the chosen vertex to 1). If the vertex does belong to F_i it will be those affine maps which not only send the chosen vertex to 1 but also send the points of S which do not belong to the face which we F_i. to zero. That is, if we view our chosen vertex as the origin of the lattice, we are talking about monoid homomorphisms from the lattice span of S to $C^\times$ which send the monomial ideal spanned by the elements of S which are not in F_i to zero.

To say that the divisor F_i is locally principal is to say that for each choice of vertex (or equivalently for choice of an element of S) this monomial ideal is locally principal.

For each vertex, the ideal is generated by a single element if the divisor F_i restricts to an actual principal divisor on the open subset corresponding to each vertex. If we assume that S is convex, this is the same as saying that the combinatorial type of the convex hull of S – that is, the combinatorial type of the polyhedron defined by the expression $e_1F_1 + ... + e_uF_u$ – is the same as the combinatorial type of the polyhedron defined by the expression $e_1F_1 + ... + (e_i - 1)F_i + ... + F_u$.

And that is equivalent to ampleness (and also then to very ampleness) of that divisor.

Let's write that down. Suppose that S is convex. Then

Fact: For each i, the following are equivalent

i) The defining ideal of the subvariety where F_i meets the affine open subset corresponding to each vertex of S is not only locally principal, but actually principal.

ii) for some (equivalently all) ample divisors H there is a positive integer λ such that $\lambda H - F_i$ is ample.

If the manifold is smooth then the monoid spanned by S when each vertex is viewed as the origin is just a free commutative monoid on n generators, and every locally free ideal is free.

This shows that when the manifold is smooth with initially chosen very ample divisor $e_1F_1 + ... + e_uF_u$, after we multiply all e_i by the same number λ to make them large enough then for each i the divisor $e_1F_1 + ... + (e_i - 1)F_i + ... + F_u$ remains very ample.

Arguing analagously for a divisor which is not just a single face,

Proposition. Let H be a an ample divisor on a smooth projective toric variety, and D any divisor. Then there is a (positive) number λ such that $\lambda H + D$ is ample.

For varieties which are not toric (even if they are still assumed to be smooth) this is false.

In the special case when we take $a_1, ..., a_u$ to be positive integers, and $e_1F_1 + ... + e_uF_u$ our original very ample divisor, as long as λ is large enough, what we see is that the combinatorial type of the convex hull of the set of of torus characters with poles no worse than $(\lambda e_1 - a_1)F_1 + ... + (\lambda e_u - a_u)F_u$ is the same as the the combinatorial type of the convex hull of the original set S of torus characters with poles no worse than $e_1F_1 + ... + e_uF_u$. And the former is a polyhedron contained in the expansion of the latter by the number λ, resulting from expanding by λ, then moving the face which we call F_i inwards by a_i steps in the cyclic quotient group Z^u/F_i.

When the variety is not smooth (when S contains vertices not part of an affine lattice basis in the boundary) there can be a change of of combinatorial type between one of the one of the polyhedra and the other. It corresponds to a blowing down if an integer expansion of the first polyhedron happens to be a union of lattice translates of the second, and a blowing up if an integer expansion of the second polyhedron happens to be a union of lattice translates of the first.

Note that the smaller polyhedron is not degenerate since we chose the e_i to be large compared to the a_i.

The edges span the $n-1$ degree term of the Chow ring, and the multiplication of the divisor $a_1F_1 + ... + a_uF_u$ with an edge Y is an element of the n degree term, which is naturally isommorphic to the ordinary integers, spanned by the class of any point of the variety.

The integer which corresponds to the product $a_1F_1Y + ... + a_uF_uY$ is the the number of lattice points (=torus characters) in Y as an edge of the original polyhedron, minus the number of lattice points in the corresponding edge of the smaller polyhedron.

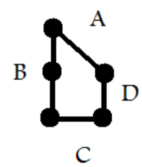
That number can be positive or negative.

Example.

The root system of type A_2 (using the centerless model of the group so that the torus is not artificially extended) is made starting with an affine basis of the two-dimensional lattice of characters, and making a hexagon from six such triangles using reflections. Once we double the size of the hexagon each edge contains three lattice points, while if moved inwards one step contains four. Although the product of two consecutive edges is 1, because they meet at one point (either thinking about them as edges of the hexagon or as projective lines in a variety they meet at the same point), the self-intersection of the projective line corresponding to each edge is $3 - 4 = -1$. We also see from how we can move in three disjoint edges two different ways to make two different triangles that we can blow down three non-intersecting projective lines two different ways to obtain the projective plane. The resulting birational transformation of the projective plane is called the quadratic transform.

Example. Let's return to the example where we start with S the set of points of $\mathbb{Z}^2$ which are the columns of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$



The convex hull includes also the point $(0, 1)$. The four codimension-one faces are labelled A, B, C, D. Calling the complex manifold M, it is nonsingular since the triangle where A and B meet is a lattice basis. It follows that the Chow ring is $\mathbb{Z} \oplus H^2(M, \mathbb{Z}) \oplus H^4(M, \mathbb{Z})$, and let us calculate it.

The commutative group $H^2(M, \mathbb{Z})$ is spanned by A, B, C, D with two relations coming from translation. If the face B is moved outward (to the left) while the face D is moved inward (to the right) the diagram is congruent to the same diagram if the face A had been moved outward (upward) to the next coset in the lattice. Thus leftward translation in the lattice gives the relation

$$B - D = A$$

in the Cohomlogy

group. Upward translation gives

$$A = C.$$

The pairwise intersections of adjacent faces are four actual points of the manifold which we might label AB, BC, CD, DA. Because of the two relations we can replace A by C. Since A and C are disjoint this implies that in the Chow ring $0 = A^2$. And we can replace B by $C + D$. Then the four points become $C(C + D), (C + D)C, CD, DC$. All of these are equal since $C^2 = 0$. We knew anyway that $H^4(M, \mathbb{Z}) = \mathbb{Z}$ and any point is Poincare dual to the fundamental class.

The elements C, D map to a basis of H^2 and the product CD is a basis of H^4.

We see from the diagram that $D^2 = -1$ because when the face D is moved inwards it becomes one bit longer. Here the number -1 denotes minus the fundamental class of M. This also can be derived from the relation $D = B - A$ as then $D^2 = DB - DA$ and the first term is zero while the second term is -1.