First introduction to Projective Toric Varieties

Chapter 1

Projective toric varieties are a type of possibly
singular complex manifolds indexed by easy
combinatorial data having to do with poles and
zeroes of meromorphic functions.

It is maybe easiest to introduce the combinatorial
data first. We start with an abstract lattice of
some rank n which we may take to be just
the set Z^n, whose elements we could denote
on paper by columns of length n with integer
entries.

Then any finite set $S \subset Z^n$ describes a projective
toric variety, if it is not empty.

The easiest way of describing the singular manifold
is to say that S has an "affine structure." Although
it is not always possible to add elements of S to
obtain other elements of S, it is certainly possible
to determine the truth or falsehood of any additive
equation involving elements of S. Because we won't
be caring about the choice of origin $0 \subset Z^n$ we
will restrict attention to homogeneous equations, such
as the equation $a + b + c = d + e + f$ among six elements
of $S \subset Z^n$. This particular equation is homogenous
of degree three, but any degree is allowed.

We say that an element $x \in Z^n$ is in the *convex hull* of S if an equation holds such as

$$x + x + x = a + b + c$$

for elements $a, b, c \in S$ not necessarily distinct. And we say that S is *convex* if it is equal to its convex hull.

Ordinarily one only considers convex subsets $S \subset Z^n$ and if one uses a non-convex subset, the resulting complex manifold is not called 'toric' by convention, though there is a well-defined process called 'normalization' that can be applied to the singular manifold itself to get back to the one corresponding to the convex hull.

A function $h : S \to C$ to the complex numbers is called, let us say, an *affine map* if it preserves the affine structures, where we use multiplication in C. This means that whenever we have an equation in S like

$$a + b + c = d + e + f$$

we should also have

$$h(a)h(b)h(c) = h(d)h(e)h(f).$$

The projective toric variety corresponding to S is defined (usually only when S is convex) to be the equivalence classes of affine maps $S \to C$ which are not identically zero, under the relation that two maps h, h' are considered equivalent if there is a nonzero complex number λ such that

$$h = \lambda h'.$$

Example. If the elements of S are affinely linearly independent (meaning that if one is considered to be the origin the others are linearly independent), then there are no homogeneous equations among elements of S except tautologies like a+b=b+a etcetera. This means that the affine maps $h : S \to C$ are merely all functions. Labelling the elements of S as

$$x_0, ..., x_m$$

where m is the dimension of the affine span of $S \subset Z^n$ then we see that the projective toric variety is just all ratios $[x_0 : ... : x_m]$ which is the set of points in the projective space P^m of dimension m.

Example. The previous example is a little
restrictive if we require S to be convex, because
there are not many convex affinely independent
sets. They are, I think, just the affine bases of
(cosets of) summands of Z^n.

However, if we ever expand a convex set S, simply
taking the multiples ks, for $s \in S$ where k is a fixed
integer, and take the convex hull, we
are in a situation where any
affine map from the vertex set kS to C extends
uniquely to an affine map from the convex hull.

So the process of expanding S by an integer and taking
the convex hull does not affect the projective toric variety.

And it follows

Example. If S is an affine basis of a summand
of the lattice and k is a nonzero integer, then
the convex hull of kS defines the projective toric
variety which is a projective space.

In a certain sense these are the universal examples,
because when there are relations among elements
of S it just means that some of the
the functions $S \to C$ are not allowed, and
we get a closed subset of projective space
insteadof the whole projective space.

For simplicity say S spans Z^n affinely.

The affine maps $S \to C$ which send
no element of S to zero extend uniquely
to affine maps

$$Z^n \to C^\times$$

where $C^\times$ is the group of nonzero
complex numbers under multiplication.

Instead of modding out by the action of nonzero
scalars, we can equivalently here just
pass to the subset of affine maps sending 0 to 1.
These are just the ordinary group homomorphisms
$Z^n \to C^\times$.

This set is bijective with $(C^\times)^n$ and it is
a group under multiplication. It is called a
"torus" because each factor of $C^\times$
contains the unit circle, and a cartesian product of
n copies of a circle is sometimes called an n
dimensional torus.

But here we call the larger group of real
dimension $2n$ a torus also.

And thus we see that any projective toric
variety contains a torus.

Example.

Let's look at the four points in Z^2 which have entries of 0 and 1. These are the four columns of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

We call these a, b, c, d and the only nontrivial relation

is

$$a + d = b + c.$$

We visualize the letters a, b, c, d as the four corner of a square (with coordinates as given by the columns w chose),

$$\begin{array}{cc} a & b \\ c & d \end{array}$$

The various afine functions h merely assign numbers to a, b, c, d and the affine requirment together with the nonzero requirement just means be the entries of a matrix of rank one.

Thus we can see the elements $a, b, c, d \in S$ as ordinary variables, and the arrangement as a square, the positions we have writen them on the page, require us to substitute only such numbers as yield a rank one matrix.

So the variety is rank one matrices modulo
multiplication by nonzero scalars. And we
see that such a thing is uniqully determined
by just the ratios $[a : b]$ and $[a : c]$ so that
our variety is just the cartesian product of
two copies of one dimensional projective space.

We could have seen this more easily by working
component-by-component. Our set S was after
all only the cartesian product of two copies of
$0, 1 \subset Z$.

However, here we have constructed the same surface
as the solution set of the homogeneous equation
$ad = bc$ and it is a subset of projective three-space.

Note that the square diagram above is uniquely a linear
degeneration of a tetrahedron; the linear degeneration is
the defining constraint for the surface within projective
space.

Chapter 2 – Divisors

In this section I'll use define some words
which I won't end up using, so if something
doesn't make sense, don't worry and just keep
reading.

Now let's talk about divisors. An irreducible
divisor on a complex manifold means a possibly
singular codimension one submanifold
which is not a union of finitely many smaller ones.

A divisor means either a finite union of
irreducible divisors, or, more generally,
a finite sequence of irreducible divisors
to which are attached integers.

A meromorphic function on a normal complex
projective variety has a divisor of poles and
zeroes, and by convention we count poles
as being negative zeroes, so that if a meromorphic
function could have only zeroes, its divisor
would have only positive numbers attached
to its irreducible components. But in fact
this never happens. A meromorphic function
without any poles would be defined everywhere
on the complex projective variety, but there
is no such function except constants.

One way around this difficulty is to consider
instead of only functions, rather to consider

the more general things which are sections of
line bundles. For any locally principal divisor, one
can make a holomorphic line bundle such that
your divisor is exactly equal to the divisor of
zeroes (and poles) of a meromorphic section.

Because you can always multiply a meromorphic
section of a line bundle by a meromorphic function,
then once you can obtain your divisor from a line
bundle, you can equally well obtain any divisor which
differs from your divisor by the divisor of a meromorphic
function.

Divisors coming from meromorphic functions
(meromorphic sections of the trivial bundle) are
called "principal divisors" and two divisors which
differ only by a principal divisor are called
"linearly equivalent."

One can easily show by this correspondence
that isomorphism types of holomorphic line
bundles are naturally bijective with the a group
of divisors (the locally principal divisors) modulo
principal divisors. If the variety is smooth, all
divisors are locally principal.

Smoothness exercise. Show that the manifold is smooth
if and only if S is convex and each vertex of
the convex hull together with the vertices
connected by an edge form an affine lattice basis.

Although it is actually easier, because it is
abstract, we won't here talk about line bundles,
but only about divisors.

If we choose any codimension-one face of our
convex set S, it will of course be a convex
subset on its own, and will define a codimnsion
one subvariety.

The way this works is that a face $S_0 \subset S$
corresponds to the classes mod scalars of
affine maps $S \to C$ which send *all but* the
elements of S_0 to zero.

Exercise. Show that the set of elements *not* sent to
zero under any affine map $S \to C$ is either empty
or a face of some dimension.

If the codimension one faces of S are $F_1, ..., F_u$ then
the divisor in which each F_i is given coefficient -1
is denoted

$$-F_1 - F_2 - ... - F_u$$

and it is called a "canonical divisor" because
it is the divisor of (zeros and) poles of a meromorphic
differential n forms where n is the
dimension of the variety (we call it n consistently
with the superscript in Z^n as if S spans affinely
our variety has dimension n).

Actually, I've used the same letter F_i for the codimension one face of S as I have for the corresponding codimension one subset of our variety – the latter of course is all the affine maps which are zero except on F_i.

Anyway, with this abuse of notation accepted, the only divisors we ever have to worry about are going to be integer linear combinations of the F_i.

Again, this is something you don't yet need to know, but the torus acts on the variety and there are no torus invariant divisors except linear combinations of the F_i. And every divisor is linearly equivalent to a torus invariant divisor.

And this means we won't go wrong if we just pretend that there are no divisors except which we make by assigning integers to the faces of S.

If we view S or perhaps its real convex hull as a cell complex, then what we are talking about is exactly a cellular $n-1$ chain.

Let me give you the next thing to visualize without explaining it. What I want you to visualize is that instead of assinging integers to the faces of S, we are going to move the faces in and out.

If you think of the faces as infinite hyperplanes in Z^n, then as these hyperplanes are translated from one position to another in R^n , they collide at certain times with points of Z^n. Or, if we think of the hyperplanes as just being subsets of Z^n, the translates actually are cosets of Z^n modulo whichever translate contains 0, and the cosets are elements of an infinite cyclic quotient group.

Whenever I speak of 'how far to move' a face of S I am going to mean using counting in the infinite cyclic quotient group that has to do with that face.

Now, it is best to start by choosing an element of S to be the origin of the lattice, and having our movable hyperplanes $F_1, ..., F_u$ chosen as being the ones meeting at the origin. To construct S, we will move each one in the originally chosen direction (outward) some number of steps. This gives a positive integer to assign to each F_i , and if we call these integers $e_1, ..., e_u$ then we will write down the divisor $e_1 F_1 + ... + e_u F_u$

This is an *effective* divisor because all e_i are positive. The hyperplanes F_i each moved out e_i steps provide the faces of the convex set S. This choice of the e_i, or any other which defines a finite subset of the lattice defining the same projective variety, makes $e_1 F_1 + ... + e_u F_u$ be a *very ample* divisor.

I should have said this earlier, but if we choose any
two points of Z^n, say x, y, then for each point h
of our variety, the ratio $h(x)/h(y)$ is well defined
as long as $h(y) \neq 0$.

It is convenient to take y to be the origin in Z^n and
then if you recall when we restrict h to the torus
as no element is sent to zero, instead of reducing modulo
scalar multiplication it is better to assume $h(0) = 1$.
Then our chosen element $x \in Z^n$ corresponds to the
function

$$h \mapsto h(x)$$

and this is something very familiar. For, just
as the torus is group homomorphisms $Z^n \to C^\times$
we can recover the group Z^n as analytic group
homomorphisms $(C^\times)^n \to C^\times$.

Each lattice element $x \in Z^n$ already corresponds to
an analytic function on the torus, and one which is
a group homomorphism (a "character" of the torus).

And when we take y to be the origin of Z^n the function
sending h to the well defined ratio $h(x)/h(y)$, when
we restrict to the torus in our variety, is nothing but
x itself, viewed as a character of the torus.

Now I am going to just state something, without
getting us lost in any details. I said already that the
convex polyhedron we started with, because we needed
to move the faces out in order that it should have
any volume at all, has positive integers assigned to its
faces. The particular integers depended on a choice
of origin of the lattice.

With respect to the same origin, all the lattice
points contained in the polyhedron correspond to particular
meromorphic functions, and being inside the polyhedron
(or on the boundary) means that all of these functions
have poles *no worse* than the prescribed divisor

$$e_1 F_1 + \ldots + e_u F_u$$

This means that we can reinterpret our chosen convex
set S as being the torus characters which have divisor
of poles no worse than as prescribed by that divisor.

Now, each lattice point q also has a principal divisor.
The origin of the lattice has the divisor zero. Starting
with all the hyperplanes in $\mathbb{Z}^n$ intersecting
at the origin, if we choose any other lattice point q, the
number of steps (with outwards counting positively)
each face has to move so that all meet at
the point q, counting steps in the infinite
cyclic quotient group modulo that hyperplane,
defines a divisor $a_1 F_1 + \ldots + a_u F_u$, and this is
the divisor of zeroes and poles of q.

The fact that not all of the a_i can be positive
corresponds to the fact that the point q
has no volume, so not all F_i can be moved outwards
if they are all to meet at one point.
Changing the origin, or equivalently translating our
convex polyhedron, then modifies our effective divisor

$$e_1 F_1 + ... + e_u F_u$$

by adding to it all possible principal divisors.

To say that we do not care about translating
our polyhedron is the same as saying that
we don't care about the divisor, only the
divisor class.

And we see that the divisor class group
is spanned by the F_i as an abelian
group, with n linearly independent
relations coming from translation. So
it is an abelian group of rank $u - n$,
the number of faces of S minus
the rank of the lattice which S spans.

Now, I have cheated a bit because in defining
linearly equivalent divisors one is suposed to
use principal divisors coming from arbitrary
meromorphic functions, here we only used
torus invariant principal divisors. But, that
is OK because if a difference of torus invariant
divisors is principal it is certainly a principal
torus invariant divisor.

Example.

Consider the four elements of Z^2 which are the
columns of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$

The slanted face connecting the last
two points has coefficient 2, the ones containing 0
have coefficient 0, the remaining one has coefficient
one. This is because the slanted
face mets latice points if it is to be
translated to the origin.

An example of a principal divisor is the difference
of the two vertical edges minus the slanted edge.
This is the divisor of the lattice point $(1,0)$.

Exercise. For convex sets spaning Z^n show
that a torus equivariant map from one projective toric
variety to another amounts to saying that after expand-
ing the first convex set by an integer multiple and taking
the convex hull again, it is a union of tanslates
of the other.

Since the convex set in the previous example already is
a union of three vertical line segments, then there is
a map from this complex projective surface to the
projective line (Rieman sphere).

That map is in fact a fiber bundle map,
and our surface is a fiber bundle of the Riemann
shpere over the Rieman sphere. It is the
Hirzebruch surface, also called a scroll.

There is a ring called the Chow ring which is
a historical precursor to the cohomology
(and for smooth projective toric varieties is
the commutative ring is the even dimensional part of
integer cohomology).

Here we could define it by saying that
it is a graded ring which is Z in degree 0.
In degree 1 it is spanned by the F_i modulo
the n linear relations.

Then we say that the ring multiplication
comes from intersection, and the ring in
degree i is spanned by codimension i faces.
More precisely, if two faces f, g intersect
transversely the product fg is the intersection,
and if they do not, relations in the divisor class
group are used to arrange that they do.

Ampleness, Local Principality

We started with a finite set S containing an origin of
the lattice, and interpreted the linear combination
$e_1 F_1 + ... + e_u F_u$ to mean that we have moved
each F_i outwards by e_i steps so that the F_i
define the convex hull of S. Note that if we replace
each e_i by the *same* multiple λe_i
where λ is a positive integer, it will define
the convex hull of the equivalent set λS.

There is a definition that is not very relevant for
toric varieties, a divisor D is called *ample*
if some positive integer multiple λD is
very ample. In view of the statement above,

Observation A divisor on a toric variety is
ample if and only if it is very ample.

Let's give an explicit criterion of ampleness. Assuming
that two finite sets S, T both span the same lattice
Z^n, the projective varieties defined by S and T
are the same if and only if some expansion λS
of S by a positive integer λ is a union of
translates of T and vice-versa. Thus when S is convex,
a divisor $a_1 F_1 + ... + a_u F_u$ is ample (=very ample)
if and only if for some positive integer λ
the set of points enclosed in what we call
we call $\lambda a_1 F_1 + ... + \lambda a_u F_u$ is a union
of translates of S. Thus

Fact. A divisor D is (very) ample if and
only if the convex hull of the set of torus characters
with poles no worse than D is combinatorially equivalent
to the convex hull of S, with corresponding faces of all
dimensions parallel.

Let's now give a description of the local defining
equations of each divisor F_i as a codimension one
subset of our variety.

If we choose a vertex of S then a corresponding
open subset of the variety (it is covered by such
open sets) consists of the affine maps $S \to C$
sending this vertex to $1 \in C$. Those points
belonging to F_i are as we said just those
affine maps sending the points of S which
are enclosed in the polyhedron which we visualize
as $e_1 F_1 + ... + (e_i - 1) F_i + ... + F_u$ to zero.

If the chosen vertex does not belong to F_i this
will be empty; while if the the chosen vertex does
belong to F_i it will be those affine maps which
not only send the chosen vertex to 1 but also send
those points of S which do not belong to the face
F_i. to zero. That is, if we view our chosen vertex
as the origin of the lattice, we are talking about monoid
homomorphisms from the lattice span of S to $C^\times$
which send the monomial ideal spanned by the elements
of S which are not in F_i to zero.

To say that the divisor F_i is locally principal is to
say that for each choice of vertex (or equivalently for
choice of an element of S) this monomial ideal is locally
principal.

For each vertex, the ideal is generated by
a single element if the divisor F_i
restricts to an actual principal divisor on the open
subset corresponding to each vertex. If S is convex
and e_i sufficiently large, this is the same as
saying that the combinatorial type of S – that is,
the combinatorial type of the polyhedron defined
by the expression $e_1 F_1 + ... + e_u F_u$ – is the
same as the combinatorial type of the polyhedron defined
by the expression $e_1 F_1 + ... + (e_i - 1) F_i + ... + F_u$.

And that is equivalent to ampleness (and also then to very ampleness) of that divisor.

Let's write that down. Suppose that S is convex. Then

Fact: For each i, the following are equivalent

i) The defining ideal of the subvariety where F_i meets the affine open subset corresponding to each vertex of S is not only locally principal, but actually principal.

ii) for some (equivalently all) ample divisors H there is a positive integer λ such that $\lambda H - F_i$ is ample.

If the manifold is smooth then the monoid spanned by S when each vertex is viewed as the origin is just a free commutative monoid on n generators, and every locally free ideal is free.

This shows that when the manifold is smooth with initially chosen very ample divisor $e_1 F_1 + ... + e_u F_u$, after we multiply all e_i by the same number λ to make them large enough then for each i the divisor $e_1 F_1 + ... + (e_i - 1)F_i + ... + F_u$ remains very ample.

Arguing analagously for a divisor which is not just
a single face,

Proposition. Let H be a an ample divisor
on a smooth projective toric variety, and D any
divisor. Then there is a (positive) number
λ such that $\lambda H + D$ is ample.

For varieties which are not toric (even if they are
still assumed to be smooth) this is false.

In the special case when we take $a_1, ..., a_u$ to
be positive integers, and $e_1 F_1 + ... + e_u F_u$
our original very ample divisor, as long as
λ is large enough, what we see is
that the combinatorial type of the
convex hull of the set of of torus charac-
ters with poles no worse than
$(\lambda e_1 - a_1) F_1 + ... + (\lambda e_u - a_u) F_u$ is the same as
the the combinatorial type of
the convex hull of the original set S
of torus characters with poles no worse than
$e_1 F_1 + ... + e_u F_u$. And the former is a
polyhedron contained in the expansion of
the latter by the number λ, resulting
from expanding by λ, then moving
the face which we call F_i inwards by
a_i steps in the cyclic quotient group Z^u / F_i.

When the variety is not smooth (when S
contains vertices not part of an affine lattice
basis in the boundary) there can be a change of
of combinatorial type between one of the
one of the polyhedra and the other. It corresponds
to a blowing down if an integer expansion of the
first polyhedron happens to be a union of lattice
translates of the second, and a blowing up if an integer
expansion of the second polyhedron happens to be
a union of lattice translates of the first.

Note that the smaller polyhedron is not degenerate
since we chose the e_i to be large compared to the a_i.

The edges span the $n - 1$ degree term
of the Chow ring, and the multiplication of the
divisor $a_1 F_1 + ... + a_u F_u$ with an edge Y is an
element of the n degree term, which is naturally
isommorphic to the ordinary integers, spanned
by the class of any point of the variety.

The integer which corresponds to the product
$a_1 F_1 Y + ... + a_u F_u Y$ is the the number of lattice
points (=torus characters) in Y as an edge
of the original polyhedron, minus the number of
lattice points in the corresponding edge of the
smaller polyhedron.

That number can be positive or negative.

Example.

The root system of type A_2 (using the centerless
model of the group so that the torus is not artificially
extended) is made starting with an affine basis of
the two-dimensional lattice of characters, and
making a hexagon from six such triangles using
reflections. Once we double the size of the hexagon
each edge contains three lattice points, while
if moved inwards one step contains four.
Although the product of two consecutive
edges is 1, because they meet at one point
(either thinking about them as edges of the hexagon
or as projective lines in a variety they meet at the
same point), the self-intersection of the
projective line corresponding to each edge is
$3 - 4 = -1$. We also see from how we can move in
three disjoint edges two different ways to make
two different triangles that we can blow down three
non-intersecting projective lines two different ways
to obtain the projective plane. The resulting
birational transformation of the projective plane
is called the quadratic transform.

Example. Let's return to the example where we start with S the set of points of $\mathbb{Z}^2$ which are the columns of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}.$$
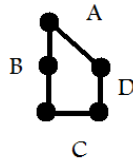
The convex hull includes also the point $(0, 1)$. The four codimension-one faces are labelled A, B, C, D. Calling the complex manifold M, it is nonsingular since the triangle where A and B meet is a lattice basis. It follows that the Chow ring is $\mathbb{Z} \oplus H^2(M, \mathbb{Z}) \oplus H^4(M, \mathbb{Z})$, and let us calculate it.

The commutative group $H^2(M, \mathbb{Z})$ is spanned by A, B, C, D with two relations coming from translation. If the face B is moved outward (to the left) while the face D is moved inward (to the right) the diagram is congruent to the same diagram if the face A had been moved outward (upward) to the next coset in the lattice. Thus leftward translation in the lattice gives the relation

$$B - D = A$$

in the Cohomlogy

group. Upward translation gives

$$A = C.$$

The pairwise intersections of adjacent faces
are four actual points of the manifold which
we might label AB, BC, CD, DA. Because
of the two relations we can replace A by
C. Since A and C are disjoint this
implies that in the Chow ring $0 = A^2$. And
we can replace B by $C + D$. Then the four
points become $C(C + D), (C + D)C, CD, DC$.
All of these are equal since $C^2 = 0$. We
knew anyway that $H^4(M, \mathbb{Z}) = \mathbb{Z}$
and any point is Poincare dual to the
fundamental class.

The elements C, D map to a basis of H^2
and the product CD is a basis of H^4.

We see from the diagram that $D^2 = -1$
because when the face D is moved inwards
it becomes one bit longer. Here the number -1
denotes minus the fundamental class of M.
This also can be derived from the relation $D = B - A$
as then $D^2 = DB - DA$ and the first term is
zero while the second term is -1.

Chapter 3 – Fourier series

Recall that I mentioned that if $F_1, ..., F_m$ are
the faces of a convex lattice polyhedron then any
divisor is equivalent to one of the type $a_1 F_1 + ... + a_m F_m$
for $a_1, ..., a_m$ integers. If the a_i are positive
so one may speak of the the polyhedron described-
described by the symbol $a_1 F_1 + ... + a_m F_m$, then
the set of lattice points in the interior are the
torus characters with poles no worse than
$a_1 F_1 + ... + a_m F_m$

In fact, the vector space of *all* meromorphic
functions on the projective variety with poles
no worse than $a_1 F_1 + ... + a_m F_m$ is described by
this set of characters. That is, it is the finite
dimensional vector space with torus action which is
the direct sum of a one-dimensional representation
(with multiplicity one) with character each interior
lattice point.

A way of understanding this is to first note that
just because the variety is projective, the vector
space must be finite-dimensional (the finite-
dimensionality can be taken to be a definition
of what it means for a variety to be projective,
by the way, and the condition once verified for
the powers of any ample divisor holds for every
divisor).

And then because the torus action preserves the
finite-dimensional space it decomposes
into one dimensional representations; and these
can have no multiplicity larger than one as the whole
function field has no component of multiplicity larger
than one.

The relation between the one dimensional
representations and their characters is exactly the
one which occurs in Fourier theory, and one could
create a projective toric variety by starting with an
infinite Fourier series in several variables, and
putting bounds on the characters to describe a finite
subset. The convex hull of this set arises naturally
then, and is associated to a projective toric variety
with an ample divisor (polarization), and the space of
corresponding Fourier series is the space of meromorphic
functions with poles no worse than the chosen ample di-
visor.

Such a bounding process is done in the classical way that
one obtains number-theoretic approximations of spectral
frequencies of the atoms, and there one also considers
differential forms (as one can do of course for toric
projective varieties in general).

In that case the projective variety is not toric and
so the restriction is not as simple as bounding an area
in a lattice. Its geometry in any case is not related
to the physical world; but is caused by choices which

one makes in order to simplify the physical world; for example choices of what are called 'electron configurations.'

Yet, it is needed to know the geometry of one's own choices before one can use coordinates to describe the natural world. For, it is a projection of infinite series onto finite series which leads to being able to write calculations in a finite amount of time; and typically the projection requires an assumption of a Hilbert space structure or bilinear form. In general one can consider projective geometry to be the 'finitization' of infinite series in analysis, and the issue is that one does not usually only want to know the exact value of a quantity, for instance at one time and one purported 'point' of space. Rather one wants to understand things more like ideas; as one contemplates a picture of subtle complexity it is not only that one wants to know, where exactly is this ant orthat grain of dust, but more meaningful things.

The passage from an infinite calculation
to a finite calculation must it seems be done based on intuition only, and the geometry of the choice which one has made does reflect something which is internal to the observer, only. In that sense, projective geometry constitutes introspection rather than observation; and yet without introspection, observation of nature and then action amount to recklessness.