Intuitive explanation of the Riemann hypothesis II

In the previous paper, we mentioned that for real numbers ω_0 and c, if once we let ω be the unitary character of the connected multiplicative group which is given $\omega(g) = e^{i\omega_0 g}$, there is an associated real two-form on $T \times (0, i\infty)$ given as

$$g^{2c-2}\omega(g)\mu_1^*(\alpha - d\tau) \wedge \mu_2^*(\alpha - d\tau).$$

Theorem 1 said that this form is exact in the sense of Schwartz forms if and only if Riemann's function ζ satisfies $\zeta(c + i\omega_0) = 0$.

Now let us call this A_s to remind us of 'area,' for $s = c+i\omega_0$. Note, by the way, that if δ is any vector field on our domain, the 'divergence,' which means $\frac{\delta(A_s)}{A_s}$ where the numerator denotes the Lie derivative action, is equal to $\frac{d\ i_\delta(A_s)}{A_s}$ and so the divergence function is always an 'integrating factor,' when we multiply by it we end up with the exact form. If there is any form B so that $A_s = \delta(B)$ then again A_s is exact and again $\zeta(s) = 0$.

The integral of A_s is just the squared magnitude of the integral of $g^{c-1}\omega(g)(\alpha - d\tau)$ itself, and a holomorphic indefinite integral of $\alpha - d\tau$ is $\frac{1}{i\pi} log(\lambda/q)$.

I'm not sure if I explained this in detail, but if we let $\tau = ie^t$, we can use 'integration by parts' to write

$$\frac{1}{i\pi}\int e^{(c-1)t+i\omega_0 t} d\, log(\lambda/q) = \frac{c-1+i\omega_0}{i\pi}\int e^{(c-1)t+i\omega_0 t} log(\lambda/q(ie^t))dt.$$

Therefore

$$\int\int A_s = (\frac{|(s-1)|}{\pi})^2 |\int_{-\infty}^{\infty} e^{i\omega_0 t}e^{(c-1)t} log(\lambda/q(ie^t))dt|^2.$$

The second term on the right is just the ordinary magnitude of the Fourier tranform value, at frequency ω_0, of the real function

$$e^{(c-1)t} log(\lambda/q(ie^t)),$$

and a disjoint part of the previous paper was an attempt to understand intuitively why it cannot be zero unless $c = 1/2$. This function approximates $log(16)$ times the unit step function if $c = 0$, and it approximates π times the complementary downard unit step function if $c = 1$. Although the peak of the function isn't quite at zero (it seems to be at the negative number $log(log(16)/\pi)$ when $c = 1/2$), our heuristic argument should say that when $0 < c < 1/2$ we should have

$$|\int_{-\infty}^{0} e^{(c-1)t+i\omega_0 t} log(\lambda/q(ie^t))| < |\int_{0}^{\infty} e^{(c-1)t+i\omega_0 t} log(\lambda/q(ie^t))|$$

for all values of ω_0. Then the triangle inequality would show that the absolute value of the sum cannot be zero.

Our model of the dynamical system, viewed from a rotating reference frame, can be simplified.

Imagine a lathe where a spinning disk is held at one point. In order to center this point, there is a mounting; one might imagine two ball-bearings or disks, one on each side, holding the spinning disk at its point of rotation. To center the disk one can roll the contact point between the two bearings. We mark the initial contact point, say, with a red dot. Now later after we have rolled the two bearings in one direction, instead of a point, we will see the trace of a circle made by the red point orbiting the new pivot.

At the end of the motion, the area of the circle is π times the squared magnitude of that same absolute value we are talking about.

That is, we are integrating the form A_s by choosing the frequency of spinning to be ω and the rolling distance of the pivot sideways as $e^{(c-1)t} log(\lambda/q(ie^t))$.

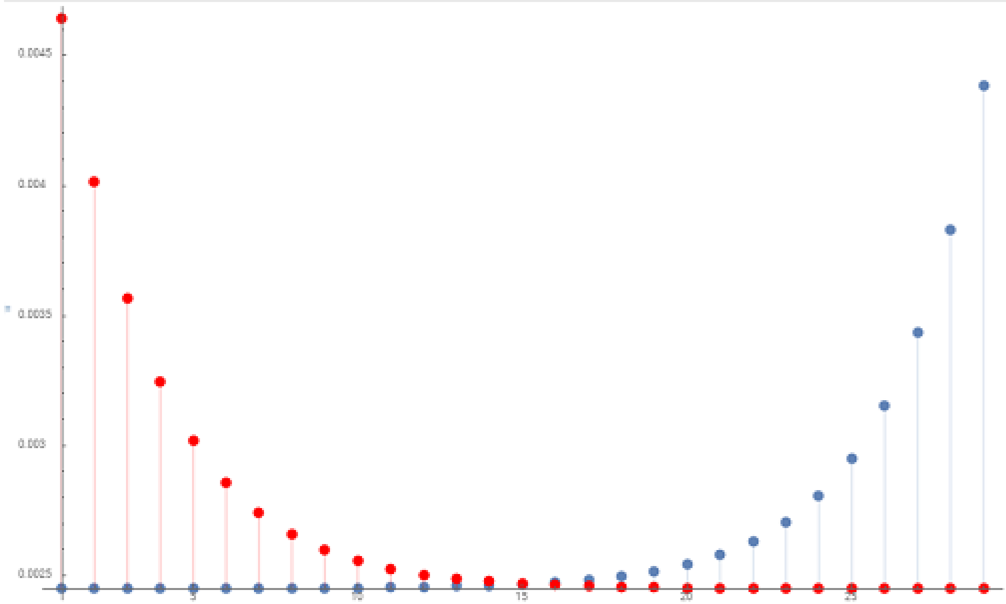
In fact, the area inside the red circle is exactly

$$\frac{\pi^3}{|s-1|^2} \int A_s$$

. Our heuristic argument says, if we did this twice, once for time from $-\infty$ to 0 and again for time from 0 to ∞, unless the two circles have the same area, they will not have the same radius. If they do not have the same radius, the total displacement cannot bring the original pivot point back to its position.

To justify our heuristic argument, we need only make a graph of the absolute value of the two integrals, as a function of c.

Here is that graph.



We have chosen $\omega_0 = 1000$.

We let $c = n/30$ for $n = 1, 2, 3, ..., 29$ and the red dots show the absolute value of the integral of $e^{(c-1)t} log(\lambda/q(ie^t))dt$ over the interval $(-10, 0)$ while the blue dots show the value over the interval $(0, 10)$.

The value $c = 2/3$ corresponds to the number 20 on the horizontal axis. Above this, the blue dot is much higher than the red dot. If this difference persists (as it obviously will) when we extend our domain of integration all the way to infinity, then we see that $\zeta(2/3 + 1000i) \neq 0$.

My ‘qualitative’ explanation is that when $c = 2/3$ so $(1-c) = 1/3$ the exponential rate is more severe on the interval $(0, \infty)$ than it is on the interval $(-\infty, 0)$ and a phenomenon reminiscent of ‘falling out of a boat’ occurs. The blue dot can be moved a distance no more than the height of the red dot, such that the area of the blue circle becomes $\frac{\pi^3}{|s-1|^2} \int A_s$. When $s = 2/3 + 1000i$ the height of the red dot is too low to allow the blue dot to move to the origin.