Deformation theory

This is to attept to find a generalization of Kuranishi's theory of deformations to singular projective varieties, as suggested by David Mond.

Let Y be an irreducible scheme of finite type over $\mathbb{C}$. Suppose that the associated reduced scheme is a possibly singular projective variety. Let $\mathcal{P}$ be the radical ideal sheaf and K the field of rational (=meromorphic) functions on the reduced subscheme defined by $\mathcal{P}$. Let $Y_{\mathcal{P}}$ be the localization of Y at the radical. Note that $\Gamma(Y_{\mathcal{P}}, \mathcal{O}_{Y_{\mathcal{P}}})$ contains a field K_0 reducing isomorphically to K (what is called the 'coefficient field' in Cohen structure theory).

Definition. We'll say Y is a *deformation* if it satisfies these properties.

i) Y is flat over $\Gamma(Y, \mathcal{O}_Y)$.

ii) The natural map induced by the field isomorphism $K_0 \to K$ $\Gamma(Y, \mathcal{O}_Y) \otimes_{\mathbb{C}} K \to \Gamma(Y_{\mathcal{P}}, \mathcal{O}_{Y_{\mathcal{P}}})$ is an isomorphism.

Note that we are really referring to an *infinitesimal* deformation in this section, however we omit writing 'infinitesimal' only for typographical reasons. Also we have not said what Y is a deformation *of*.

Definition. If X and Y are two deformattions, then we will say that Y is a deformation *of* X if we have in mind a $\mathbb{C}$ algebra homomorhism $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ and an isomorphism between Y and the base extension of X along this map of finite-dimensional local $\mathbb{C}$ algebras.

Let us attempt to construct an initial object in the category of deformations of Y which happen to have that the kernel of $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is semisimple. Let π be the map sending Y to a point, and let Δ be the defining ideal sheaf of the diagonal in $Y \times Y$ viewed as a quasicoherent sheaf on Y (pushed forward) via the first projection $Y \times Y \to Y$. Let Y^{red} be the reduced subscheme defined by the radical. Consider the composite

$$\begin{array}{rcl} \mathbb{C}\otimes_{\mathbb{C}}\mathcal{O}_Y & \subset & \mathcal{O}Y\otimes_{\mathbb{C}}\mathcal{O}_Y \\ & \to & (\mathcal{O}_Y\otimes\mathcal{O}_Y)/\Delta^{i+1} \\ & = & (1\otimes\mathcal{O}_Y)\oplus\Delta/\Delta^{i+1} \\ & \to & 0\oplus\Delta/\Delta^{i+1} \end{array}$$

where the last map projects onto the second factor. This is not a map of coherent sheaves over $\mathcal{O}_Y$, only of complex vector spaces. Nevertheless, we can compose the map induced by this on global sections $H^0(Y,\mathcal{O}_Y)\to H^0(Y,\Delta/\Delta^{i+1}))$ with the Yoneda action

$$Ext^1_Y(\Delta/\Delta^{i+1},\mathcal{O}_Y^{red})\to Hom_Y(H^0(Y,\Delta/\Delta^{i+1}),H^1(Y,\mathcal{O}_Y^{red}))$$

to obtain a map

$$Ext^1_Y(\Delta/\Delta^{i+1},\mathcal{O}_Y^{red})\to Hom_Y(H^0(Y,\mathcal{O}_Y),H^1(Y,\mathcal{O}_Y^{red}))$$

for each $i=1,2,3,...$ let E_i be the kernel so that

$$0\to E_i\to Ext^1_Y(\Delta/\Delta^{i+1},\mathcal{O}_{Y^{red}})\to Hom_Y(H^0(Y,\mathcal{O}_Y),H^1(Y,\mathcal{O}_{Y^{red}}))$$

is exact.

The inclusion of E_i is an element of

$$Hom_{\mathbb{C}}(E_i,Ext^1_Y(\Delta/\Delta^{i+1},\mathcal{O}_{Y^{red}}))\cong Ext^1_Y(\Delta/\Delta^{i+1},\mathcal{O}_{Y^{red}}\otimes_{\mathbb{C}}\widehat{E_i})$$

corresponding to an extension of coherent sheaves on Y

$$0\to\mathcal{O}_{Y^{red}}\otimes_{\mathbb{C}}\widehat{E_i}\to\mathcal{M}\to\Delta/\Delta^{i+1}\to 0.$$

Let $\mathcal{A}$ be the inverse image of $1\otimes\mathcal{O}_Y$ under

$$\mathcal{O}_Y\oplus\mathcal{M}\to\mathcal{O}_Y\oplus\Delta/\Delta^{i+1}=\frac{\mathcal{O}_Y\otimes_{\mathbb{C}}\mathcal{O}_Y}{\Delta^{i+1}}$$

$\mathcal{A}$ is a sheaf of rings providing a scheme Z containing Y as a subscheme. Also $\mathcal{O}_Y\oplus\mathcal{M}$ is spanned by $\mathcal{A}$ as $\mathcal{O}_Y$ module so there is a surjective map

$$\mathcal{O}_Y\otimes_{\mathbb{C}}\mathcal{A}\xrightarrow{\epsilon}\mathcal{O}_Y\oplus\mathcal{M}.$$

If we let Δ' be the defining ideal sheaf of the diagonal Y in $Y \times Z$ then the kernel of ϵ is a complement of $1 \otimes N$ in $(\Delta')^{i+1} + \mathcal{O}_Y \otimes_{\mathbb{C}} N$ where N is the copy of $\mathcal{O}_{Y^{red}} \otimes \widehat{E_i}$ which now serves as the ideal sheaf on Z defining Y. It need not be a sheaf of ideals on $Y \times Z$. The image of $1 \otimes \widehat{E_i}$ in $\mathcal{M} \subset \mathcal{O}_Y \oplus \mathcal{M}$ is in embedded copy, and we now want to modify $\widehat{E_i}$ by choosing a complement for its intersection with the image of $(\Delta')^i$ in $\mathcal{O}_Y \oplus \mathcal{M}$. Reducing modulo this complement we obtain what we call F_i, a homomorphic image of $\widehat{E_i}$.

If we start again using F_i in place of $\widehat{E_i}$ what will happen is that the kernel of $\mathcal{O}_Y \otimes_{\mathbb{C}} \mathcal{O}_Z \to \mathcal{O}_Y \oplus \mathcal{M}$ will now be an ideal sheaf on $Y \times Z$ and it will in fact be $(\Delta')^i$.

The replacement of $\widehat{E}_i$ by F_i does not reduce the set of extensions that are possible; that is, all but finitely many F_i are zero, and writing $F = \oplus_{i=0}^{\infty} F_i$ there is now a universal scheme X and an exact sequence

$$0 \to F \otimes_{\mathbb{C}} \mathcal{O}_{Y^{red}} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

This induces an exact sequence of global section algebras

$$0 \to F \to \Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y) \to 0$$

and X is flat over $\Gamma(X, \mathcal{O}_X)$ and satisfies that the localization at the radical is the base extension of the 'coefficient subfield' copy of the rational function field of Y^{red} along the inclusion $\mathbb{C} \to \Gamma(X, \mathcal{O}_X)$.

Let's verify that the sheaf of algebras $\mathcal{A}$ really does define a deformation $Y \to X$ as we have defined it. First we will treat the case when F_i is all of $\widehat{E_i}$.

The definition of E_i as the kernel is chosen to ensure that if we choose basic functionals $\widehat{E_i} \to \mathbb{C}$ inducing $\widehat{E_i} \otimes \mathbb{O}_{Y^{red}} \to \mathcal{O}_{Y^{red}}$ the induced extension

$$0 \to \mathcal{O}_{Y^{red}} \to \mathcal{B} \to \mathcal{O}_Y \to 0$$

where $\mathcal{B}$ is the corresponding homomorphic image of $\mathcal{A}$, is split. This being so for every functional implies that $\mathcal{A} \to \mathcal{O}_Y$ is surjective as a map of sheaves of complex vector spaces, and therefore is onto on global sections. From this and the fact that Y satisfies our definition of a deformation will imply that also X does.

1. Question. Is X is the universal such extension? That is, it is an initial object in the category of deformations of Y whose global section map $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ has semisimple kernel?

If this is true, any 'deformation' of Y as we have defined it arises by repeatedly passing to this universal deformation finitely many times, each time reducing reducing modulo a vector subspace of F, and finally at the end performing a base extension.

There is a surjectivity theorem that ensures that no non-surjective base extension are needed until the last step.

2. Question. Is there a universal bound n depending only on Y^{red}, to the number of times it may be required to take the reducing subspace to be nonzero, and thereafter one need only iterate the universal extension procedure?

3. Question. Is the map $socle\Gamma(X, \mathcal{O}_X) \to socle\Gamma(Y, \mathcal{O}_Y)$ zero? If so there should also be a uniqueness assertion; if not, it means that one should modify F further to attempt to regain uniqueness.

4. Question. When this is continued past the number n of question one, for every initial choice of a sequence of n subspaces of the F, is the completion a finitely generated formal power series ring modulo an ideal?

5. Question. If so, can the generators be taken to have nonzero radius of convergence?

It would follow if the answer to questions 1,2,4 and 5 is 'yes' that the base of any infinitesimal deformation is a base extension given by a map from the base of the deformation to an analytic space which is a finite union of bundles where the base is an iterated fiber bundle with levels locally closed smooth submanifolds of Grassmannian varieties and fibers possibly singular Stein spaces.

Added note about deformation theory

Here is an explanation for why T^2 was ever needed:

The issue is that the most straightforward and natural universal extensions are the surjections $A \to B$ where the kernel N is semisimple and satisfies that $N \to \Omega_A \otimes B$ is injective.

It is possible to factorize any extension with semisimple kernel into ones like this unless one reaches an intermediate stage C where the actual deRham differential

$$d : C \to \Omega_C$$

has kernel larger than $\mathbb{C}$.

And that is where the issue lies, which one has to get around somehow.

If one presents a ring C as $\mathbb{C}[x_1, ..., x_n]$ modulo an ideal $I = (f_1, ..., f_m) \subset (x_1, ..., x_n)$, then the condition for dh to be zero in the differentials of $\mathbb{C}[x_1, ..., x_n]/I$ is

$$dh = \sum_i a_i df_i + \sum b_{ij} x_j df_i$$

with $a_i \in \mathbb{C}$. The issue is, does this really imply h is in I? Replacing h by $h - \sum_i a_i f_i$ as one may, one gets

$$d(h - \sum_i a_i f_i) = \sum_i b_{ij} x_j df_i$$

From just considering the left side one is led to ask if there is a polynomial $h \in (x_1, ..., x_n)$ with all its partial derivatives in I, but not itself in I.

If so, one can rename this h, and just take I to be the ideal generated by the $\partial h / \partial x_i$.

If it happens that $h(0) = 0$ and h is not contained in the ideal in the complete local ring at 0 generated by its own partial derivatives, then by reducing modulo a large power of the maximal ideal $(x_1, ..., x_n)$ one has found a finite dimensional local algebra C over the complex numbers for which the kernel of $d : C \to \Omega_C$ is larger than the scalars.

This is how a defining equation h of a hypersurface whose Milnor numbe is not the same as its Tjurina number, related to the difficulty that required the introduction of T^2 in deformation theory.