Preface

This paper will perhaps be combined with two others on this website, one is called ‘Traces’ and follows up on the comment here about the Hodge conjecture, and the other is called ‘An explicit Chern character’ and derives the extension class here from a sequence of lower-dimensional classes, with the aim to resolve i-extensions by geometric flags. Both are also related to the exposition of the Riemann Roch theorem in ‘Riemann-Roch after Fulton’ and the further comments.

In the current paper here, there are global generation theorems proved about various sheaves, considering, but not resolving, the Takao Fujita conjecture about these ideas.

Ampleness

Let E be an r-dimensional projective variety.

1. Proposition.

> Ampleness of a Cartier divisor D on E is equivalent to the existence of an analytic (=algebraic) one-point compactification of the line bundle with setion sheaf $\mathcal{O}_E(D)$,
>
> It is also is equivalent to geometric contractibility of the zero section of the line bundle V with section sheaf $\mathcal{O}_E(-D)$.

Precise scheme-theoretic contractibility holds not for the reduced scheme structure of the zero section $E \subset V$ but it holds for an m'th order infinitesimal neighbourhood of E which we call the divisor mE.

More precisely, define a variety W and an ideal sheaf $\mathcal{I}$ by

$$W = \mathcal{S}pec \oplus_{i=0}^{\infty} \Gamma(E, \mathcal{O}_E(iD),$$

$$\mathcal{I} = \oplus_{i=m}^{\infty} \Gamma(E, \mathcal{O}_E(iD)).$$

Suppose that D is ample and m chosen so that mD is very ample.

2. Proposition.

> The blowup $V = Bl_{\mathcal{I}} W$ has the structure of a line bundle over E with a rational section without zeroes whose divisor of poles is D, in other words
>
> $$Bl_{\mathcal{I}} W = V = \mathbb{V}(\mathcal{O}_E(D)),$$
>
> the Fibré vectoriel of $\mathcal{O}_E(D)$, the line bundle whose section sheaf is $\mathcal{O}_E(-D)$.

A natural extension class

The $\mathcal{O}_V$ linear contracting map which underlies the Euler derivation ϵ is an exact differential on $\Lambda^{\cdot}\Omega_V(log\ E)$ resulting an exact sequence of coherent sheaves on V, locally free if E is smooth, which is

$$0 \to \Lambda^{r+1}(\Omega_V(log\ E)) \xrightarrow{\epsilon} ... \to \Lambda^1(\Omega_V(log\ E)) \to \mathcal{O}_V \to 0. \qquad (1)$$

This corresponds to a natural class in $H^r(V, \mathcal{O}_V(K_V + E))$.

Let $i : E \subset V$ be the inclusion of the zero section. The sequence (1) remains exact if restrict to E, and gives a natural extension class in the one dimensional vector-space

$$H^r(E, i^*\mathcal{O}_V(K_V + E)) = H^r(E, \mathcal{O}_E(K_E)).$$

Here K_V and K_E are canonical divisors for V and E respectively.

The restriction of the extension class to E is likely to be the fundamental class of type (r, r) in $H^{2r}(E, \mathbb{C})$ times the Chern number D^r of the divisor D on E.

If D is replaced by a positive integer multiple the sequence (1) for mD pulls back isomorphically, under the natural cyclic covering induced by tensor power, to the sequence for D, with the leftmost term pulling back by the integer multiple m^r.

The extension class in the singular case

If $\mathcal{F}$ is torsion free and rank one, not necessarily invertible, and if E is not necessarily smooth, there is still the exact sequence $S^{\cdot}\mathcal{F} \otimes \Lambda^{\cdot}\mathcal{P}(\mathcal{F})$ with differential of degree $(-1, 1)$ which was mentioned in 'Functorial affinization of Nash's manifold.' For any locally projective birational $f : E'' \to E$ for which $f^*\mathcal{F}/torsion$ is an invertible sheaf $\mathcal{O}'_E(D)$, once the this natural sequence is pulled back to E' reduced modulo torsion and tensored with $\mathcal{O}_{E'}(-(r+1)D)$ it equals the pullback of the sequence (1) on on $E' = Bl_{\mathcal{F}}(E)$ pulled back along the induced map $g : E'' \to E'$. Therefore it gives a class in $H^r(E'', g^*\Lambda^r\Omega'_E/torsion)$. Unless $E'' \to E$ factorizes through the Nash blowup of E' the coefficient sheaf is not invertible.

If $\mathcal{F}$ is chosen to resolve the singularities of E, so that E' is nonsingular, then the Nash blowup of E' is the identity. Taking $g : E'' \to E'$ to be the identity, The cohomology class of (1) on E' encodes the Chern number of $\mathcal{O}'_E(D)$ then; this is a defining ideal sheaf of the 'exceptional Cartier divisor,' and thus

Remark.

> In this case the pullback mod torsion under f, explicitly which is $\mathcal{O}_{E'}(-(r+1)D) \otimes f^*(S^{\cdot}\mathcal{F} \otimes \Lambda^{\cdot}\mathcal{P}(\mathcal{F})/torsion$, yields the sequence (1) on E' whose the cohomology class is the Chern number (=integer degree of the r'th power) of the negation D of the exceptional Cartier of the resolution $E' \to E$ multiplied by the fundamental cohomology class of E' in $H^r(E', \omega_{E'})$.

A functorial way of representing the Chern character.

A different approach than the one above is to consider the sequence (1) only for the trivial divisor $D = 0$. The restriction to E is $\Lambda^{\cdot}\mathcal{P}(\mathcal{O}_E)$ which is a split long exact sequence, and when we tensor with $\mathcal{F}$ we have a split exact sequence with differential

$$\nabla h_1 \wedge ... \wedge \nabla h_i \otimes f$$

$$\mapsto \sum_{j=0}^{i} (-1)^j h_j \nabla(h_1) \wedge ... \wedge \widehat{\nabla(h_j)} \wedge ... \wedge \nabla(h_i) \otimes f.$$

However, we now give this a different $\mathcal{O}_E$ module structure, one where for a local section r of $\mathcal{O}_E$

$$r \cdot \nabla(h_i) = \nabla(rh_i).$$

Thus

$$r \cdot \nabla(h_1) \wedge ... \wedge \nabla(h_i) \otimes f = \nabla(rh_1) \wedge ... \wedge \nabla(h_i) \otimes f.$$

The augmentation kernels are $\mathcal{F} \otimes \Lambda^i \Omega_E$. We see that this is exact even if E is singular, because it comes from a split exact sequence by changing the action. It follows that from $\mathcal{F}$ we obtain a sequence of elements in $Ext^i(\mathcal{F}, \mathcal{F} \otimes \Omega_i)$ and I believe that these are the terms of the Chern character of $\mathcal{F}$. As far as I know, it is not impossible that every element of $Ext^i(\mathcal{O}_E, \Lambda^i \Omega_E)$ is the difference of traces of two such a Chern character components, coming from two coherent sheaves $\mathcal{F}_0$, $\mathcal{F}_1$. This would imply the Hodge conjecture if true.

The adjunction isomorphism

The leftmost term of the sequence (1) is the canonical sheaf of V twisted by E. For any canonical divisor K_V on V there is distinguished isomorphism between the canonical sheaf and $\mathcal{O}_V(K_V)$.

The leftmost term of (1) restricts on the zero section E to the canonical sheaf of E. It follows that for any canonical divisors K_V and K_E on V and E respecively there is a distinguished isomorphism

$$i^*\mathcal{O}_V(K_V + E) \cong \mathcal{O}_E(K_E).$$

In other words if ω_V and ω_E are the canonical sheaves of V and E then there is a natural isomorphism

$$i^*\omega_V(E) \cong \omega_E.$$

Either the distinguised isomorphism depending on K_V and K_E, or the natural isomorphism here, may be called the adjunction isomorphism.

Adjunction for the Chern character

Let $S \subset E$ be a Cartier divisor on a possibly singular manifold E. Let $\mathcal{L}$ be a locally free sheaf such that S is the zero set of a global section f of $\mathcal{L}$.

Let $i : S \to E$ be the inclusion. Then there is the exact sequence

$$0 \to \mathcal{O}_S \nabla(f) \to i^*\mathcal{P}(\mathcal{L}) \to \mathcal{P}(i^*\mathcal{L}) \to 0. \qquad (2)$$

(I have explained this in more detail in 'easy things which number theorists know.') It follows that the Chern characters are related by

$$ch\ \mathcal{P}(i^*\mathcal{L}) = i^* ch \mathcal{P}(\mathcal{L}) - 1.$$

We have

$$\mathcal{L} \cong \mathcal{O}_E(S)$$

If there is a section of $\mathcal{L}$ which is not zero on any component of S then we may let S^2 denote the effective divisor of zeroes of this second section on S itself. Then also

$$i^*\mathcal{L} = \mathcal{O}_S(S^2).$$

Even if there is not such a section we define a linear equivalence class of divisors, not necessarily effective, any one of which we could call S^2, just defining $\mathcal{O}_S(S^2)$ to be $i^*\mathcal{L}$.

From the exact sequence (2) we have

$$ch\mathcal{P}(\mathcal{O}_S(S^2)) = i^* ch\mathcal{P}(\mathcal{O}_E(S)) - 1.$$

Effective adjunction (first notions)

If $\mathcal{L}$ is locally free of rank one on a possibly singular variety E and $i : S \subset E$ is the inclusion of an effective Cartier divisor, we have an exact sequences

$$0 \to \mathcal{N}_{S/E} \otimes i^*\mathcal{L} \to i^*\mathcal{P}(\mathcal{L}) \to \mathcal{P}(i^*\mathcal{L}) \to 0,$$

$$0 \to \mathcal{O}_E(-S) \otimes \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L}) \to i_*i^*\mathcal{P}(\mathcal{L}) \to 0$$

where $\mathcal{N}_{S/E} = i^*\mathcal{O}_E(-S)$ is the conormal sheaf.

Remark.

> Now suppose that E is a smooth projective variety of dimension r, and $\mathcal{L}$ an ample locally free sheaf of rank one. Existing Kodaira vanishing theorems give partial results: Choose any number $m \geq 2$ and let $i : S \to E$ be the inclusion of a smooth effective divisor of dimension ≥ 1 which is precisely defined by the vanishing one global section of $\mathcal{L}^{\otimes m}$. Then for any number $j \leq r - 2$ the natural map on global sections
>
> $$\Gamma(E, \Lambda^j\mathcal{P}(\mathcal{L})) \to \Gamma(E, i_*\Lambda^j\mathcal{P}(i^*\mathcal{L})) = \Gamma(S, \Lambda^j\mathcal{P}(i^*\mathcal{L}))$$
>
> is surjective.

Proof. The two exact sequences above induce exact sequences of exterior powers. The map on global sections is induced by the composite the rightmost map in the second sequence with the push-forward of the rightmost map in the first sequence The two kernels are made of extensions of $i - 1$ forms by $i - 2$ forms on E and S twisted by various minus ample divisors, all of which have trivial H^1 by Kodaira vanishing.

Relative $\Leftrightarrow$ absolute

There are no line bundles on V except pullbacks from E. For example the Cartier divisor E itself is equivalent on V to the pullback of $-D$. To see this, choose a rational section of $\mathcal{O}(D)$ on E. For instance we may choose the constant function 1. This induces a rational function on $V = \mathbb{V}(\mathcal{O}(D))$ with divisor $\phi^{-1}D + E$, showing E is linearly equivalent to $-\phi^{-1}D$.

3. Proposition.

> For $\mathcal{L}$ the section sheaf of a line bundle on V, the base locus of $i^*\mathcal{L}$ is the intersection of E with the support of the cokernel of $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$.

Proof. The relative base locus of a line bundle on V with section sheaf $\mathcal{L}$ is just the support of the cokernel B in

$$\pi^*\pi_*\mathcal{L} \to \mathcal{L} \to B \to 0.$$

Since $\mathcal{L}$ must be of the form $\phi^*\mathcal{L}_0$ for $\mathcal{L}_0$ the section sheaf of a line bundle on E, then when we apply i^* to the sequence above, the first two terms are

$$i^*\pi^*\pi_*\phi^*\mathcal{L}_0 \to i^*\phi^*\mathcal{L}_0.$$

From the definitions, left term is the section sheaf of the trivial vector bundle based on the global sections of $\mathcal{L}_0$ while the second term is $\mathcal{L}_0$. Thus the support of the cokernel is the base locus of $\mathcal{L}_0 = i^*\mathcal{L}$.

4. Corollary.

> A line bundle $\mathcal{L}$ on V is baspoint free relative to π if and only if its restriction to E is basepoint free.

Proof. In view of the previous proposition we need to show that the base locus of a line bundle $\mathcal{L}$ on V can't be a nonemtpy subset disjoint from E. But it is a closed subset invariant under the scalar action on the line bundle $V \to E$. Once it contains a point of V it contains the nonzeero elements of the line fiber through that point, and its closure which meets E.

Relative basepoint-free criteria

In view of Corollary 4, we can use relative criteria on V to decide basepoint freeness on E. Here are some; note that iii) and iv) have no absolute analogue.

5. Proposition. Let $\mathcal{I}$ be an ideal sheaf on a reduced irreducible scheme W of dimension r. Let $V = Bl_{\mathcal{I}}(W)$ and $\pi : V \to W$ the natural map. Let $\mathcal{L}$ be a locally free rank one coherent sheaf on V. Suppose that V is normal. The following are equivalent:

i) $\mathcal{L}$ is basepoint free relative to π.

ii) The natural embedding $\pi^*\pi_*\mathcal{L}/torsion \to \mathcal{L}$ is an isomorphism.

iii) The blowup map $\pi : V \to W$ factorizes through $Bl_{\pi_*\mathcal{L}}W \to W$,

iv) The inclusion $(\pi_*\mathcal{L})^{\otimes(r+1)} \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{I})/torsion \subset \Lambda^{r+1}\mathcal{P}(\mathcal{I}\otimes\pi_*\mathcal{L})/torsion$ of torsion-free rank one sheaves on W becomes an equality upon passing to integral closures of of coherent sheaves,

v) There is a torsion-free coherent rank one $\mathcal{G}$ such that both $\mathcal{G}$ and $\mathcal{L} \otimes \mathcal{G}$ are relatively spanned, and the map $\pi_*G \otimes \pi_*\mathcal{L} \to \pi_*(\mathcal{G} \otimes \mathcal{L})$ is surjective.

vi) There is a torsion-free coherent rank one $\mathcal{G}$ such that both $\mathcal{G}$ and $\mathcal{G} \otimes \mathcal{L}$ are relatively spanned, and a surjective map $\mathcal{V} \to \pi_*\mathcal{G}$ for $\mathcal{V}$ locally free of some rank, such that the sequence $\pi^*\mathcal{V} \otimes \mathcal{L} \to \mathcal{G} \otimes \mathcal{L} \to 0$ remains exact upon applying π_*.

Relative global generation of principal parts

Let $\pi : X \to Y$ be a locally projective birational morphism of reduced irreducible schemes. Let $\mathcal{F}$ be a torsion-free coherent sheaf of rank one on Y. Let $\mathcal{P}$ be the first principal parts functor.

6. Lemma.

For sufficiently large numbers m the support of the cokernel of

$$\pi^*\mathcal{F}^{\otimes m} \otimes \pi^*\mathcal{P}(\mathcal{F}) \to \pi^*\mathcal{F}^{\otimes m} \otimes \mathcal{P}(\pi^*\mathcal{F})/torsion$$

is equal to the transform on X of the indeterminacy locus of the rational map

$$Bl_{\mathcal{F}}Y - \to X.$$

Proof. You can reduce $\pi^*\mathcal{F}$ and $\mathcal{P}(\mathcal{F})$ mod torsion without affecting the statement. Then we may use that modulo torsion in this sense, principal parts commutes with pulling back to $Bl_{\mathcal{F}}(Y)$; the intersection over m of the support of the cokernel is the image in X of the support of the cokernel of the pulled-back map which is he ramification of the projection $X \times_Y Bl_{\mathcal{F}}Y \to Bl_{\mathcal{F}}Y$. The complementary set is the maximal open subset of X where the pullback map is an isomorphism; that is where $Bl_{\mathcal{F}}Y$ dominates X.

7. Corollary.

If $\mathcal{L}$ is an invertible sheaf on X relatively very ample for $X \to Y$ then $\mathcal{P}(\mathcal{L})$ is relatively baspoint free.

Proof. Take $\mathcal{F} = \pi_*\mathcal{L}$; the hypothesis that $\mathcal{L}$ is relatively very ample means that $Bl_{\mathcal{F}} - \to X$ is an isomorphism. The previous lemma gives for large m

$$\mathcal{L}^m \otimes \pi^*\mathcal{P}(\mathcal{F}) \to \mathcal{L}^m \otimes \mathcal{P}(\mathcal{L})$$

is surjective. Since $\mathcal{L}$ is invertible the same is true when $m = 0$. Then $\pi^*\pi_*\mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is surjective with kernel the torsion subsheaf, and $\mathcal{P}(\mathcal{L})$ is relatively basepoint free.

Absolute global generation of principal parts

As before let E be an irreducible projective variety (allowed to be singular) and D an ample divisor on E. Let m be such that mD is very ample. Also as before let $\phi : V = \mathbb{V}(\mathcal{O}_E(D)) \to E$ be the line bundle projection and $\pi : V \to W$ the morphism contracting mE. Let $i : E \to V$ be the inclusion of E as the zero-section. Let $\mathcal{L} = \mathcal{O}_V(-mE)$. Then $\mathcal{L}$ is very ample relative to π so Corollary 7 gives that $\mathcal{P}(\mathcal{O}_V(-mE))$ is relatively basepoint free. Then so is $i^*\mathcal{P}(\mathcal{O}_V(-mE))$. The exact sequence (2) which we used to describe the adjunction for the Chern character directly shows that there is a surjective map $i^*\mathcal{P}(\mathcal{O}_V(-mE)) \to \mathcal{P}(i^*\mathcal{O}_V(-mE)) = \mathcal{P}(\mathcal{O}_E(mD))$. Therefore

8. Corollary.

Suppose that D is an ample divisor on E and mD is very ample; then $\mathcal{P}(\mathcal{O}_E(mD))$ is generated by global sections.

Homogeneity principle

Let E be a (possibly singular) irreducible complex manifold, or a reduced irreducible scheme, and let $\Lambda^{r+1}\mathcal{P}(\,-\,)/torsion$ be the functor of $r+1$ exterior power of first principal parts modulo torsion, acting on torsion-free coherent sheaves of rank one.

9. Lemma.

Let U be an open subset of E and $s \in \Gamma(U, \mathcal{O}_E)$. Then the functor $\Lambda^{r+1}\mathcal{P}(\,-\,)/torsion$ applied to the homothety

$$s : \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F})$$

gives the homothety

$$s^{r+1} : \Gamma(U, \Lambda^{r+1}\mathcal{P}(\mathcal{F}))/torsion \to \Gamma(U, \Lambda^{r+1}\mathcal{P}(\mathcal{F}))/torsion.$$

Proof. This follows for example from part iv) of Proposition 5 in the case π is the identity $U \to U$ and $\mathcal{L}$ is the principal subsheaf $s\mathcal{O}_U \subset \mathcal{O}_U$. Or see Lemma 11 for something more explicit. Note that the functor $\mathcal{P}(\,-\,)$ itself is not linear because the definition involves differentiation, it is only upon taking the exterior power and reducing modulo torsion that it becomes homogeneous of degree $r+1$.

Remark. The homogeneity principle implies that if $\mathcal{L}$ is an invertible sheaf and $\mathcal{F}$ is a coherent sheaf, both on a scheme or complex manifold E of dimension r, then there is a natural isomorphism[1]

$$\mathcal{L}^{\otimes r+1} \otimes \Lambda^{r+1}\mathcal{P}(\mathcal{F}) \to \Lambda^{r+1}(\mathcal{P}(\mathcal{L} \otimes \mathcal{F}))/torsion.$$

[1] More generally, we've also mentioned elsewhere that there is such a map even if $\mathcal{L}$ is only assumed to be a rank one torsion-free coherent sheaf. Once pulled back modulo torsion to the blowup of $\mathcal{F} \otimes \mathcal{L}/torsion$ the cokernel support is the ramification locus of the map to the blowup of just $\mathcal{F}$.

Serre's criterion for global generation of principal parts

We can combine the Serre ampleness criterion with the homogeneity principle (as it occurs in the previous remark):

10. Theorem.

> Let E be a possibly singular projective variety of dimension r and $\mathcal{F}$ any coherent sheaf on E, say torsion free of rank one. Let D be an ample divisor on E. Then there is a number j so that $\Lambda^{r+1}\mathcal{P}(\mathcal{F}(jD))/torsion$ is generated by global sections.

Proof. By Serre's criterion $\mathcal{O}_E(jD) \otimes \mathcal{P}(\mathcal{F})$ is generated by global sections for all $j >> 0$. By the homogeneity lemma 9

$$\Lambda^{r+1}\mathcal{P}(\mathcal{F}\otimes\mathcal{O}_E(jD))/torsion \cong \mathcal{O}_E((r+1)jD)\otimes\Lambda^{r+1}\mathcal{P}(\mathcal{F})/torsion.$$

Remark. The earlier Corollary 8 (with $m = 1$, D itself very ample and $\mathcal{F} = \mathcal{O}_E(D)$) showed that when $\mathcal{F}$ is itself very ample invertible on E then $\mathcal{P}(\mathcal{F})$ is generated by global sections without requiring any tensor product; it is not necessary to enlarge m. The later Theorem 14 will provide particular generators in the case when E is smooth.

Remark. Reducing modulo torsion is not necessary here and is done just for compatibility with other statements.

Principle of recollement

Again start with E a possibly singular projective variety of dimension r and D ample on E. Choose a natural number n.

Choose an open subset $U \subset E$ so that $\mathcal{O}_U(D)$ is generated by a single global section. Let $g \in \Gamma(U, \mathcal{O}_E(D))$ be such a section so that $\mathcal{O}_U(D) = g \cdot \mathcal{O}_U$. Also use the same letter g to denote the induced holomorphic function $V \to \mathbb{C}$ of degree one on the line fibers of $V \to E$ and whose divisor of zeroes where it is defined agrees with E. Let $h_0, ..., h_r \in \Gamma(U, \mathcal{O}_E(nD)) = g^n \Gamma(U, \mathcal{O}_U)$ viewed as rational functions $V - \to \mathbb{C}$ which are constant along fibers of $\phi : V \to E$. If we factorize each h_i as

$$h_i = h_i/g^n \cdot g^n$$

this is simply a product of two rational functions on E, or, if we choose to interpret the second factor as a rational function $V - \to \mathbb{C}$ which is well-defined on the inverse image $\phi^{-1}U \subset V$, then the same formula instead describes h_i locally factorized into a product of two holomorphic functions defined on the open subset $\phi^{-1}U \subset V$; one the function h_i/g^n constant on fibers of $V \to E$, the other g^n of degree n on fibers of $V \to E$.

Let $\omega = dh_0 \wedge ... \wedge dh_r$ now viewed as a locally defined $r+1$ form on V in which the h_i are considered to be of degree n on the fibers of $V \to E$. Let ϵ be the Euler contraction $\Lambda^{r+1}\Omega_V(log\ E) \to \Lambda^r \Omega_V(log\ E)$ (the leftmost map in the exact sequence (1)).

11. Lemma.

Under these conditions if s is any rational function on E viewed as a rational function on V constant along fibers of $V \to E$

$$0 = \epsilon(\omega) \wedge ds$$

Note.

As the h_i when viewed as being degree n on the fibers of $V \to E$ are sections of $\Omega_V(log\ E)(-nE)$ we could have used in place of ϵ the restriction of ϵ to a map on the negatively twisted subsheaves

$$\Lambda^{r+1}(\Omega_V(log\ E)(-nE)) \to \mathcal{O}_V(-nE) \otimes \Lambda^r \Omega(\Omega_V(log\ E)(-nE)).$$

In each term in the formula for $\epsilon(\omega)$

$$(-1)^i h_i d(h_0) \wedge \ldots \wedge \widehat{dh_i} \wedge \ldots \wedge dh_r$$

the coefficient is a section of $\mathcal{O}_V(-nE)$ while the second term is a section of $\Lambda^r(\Omega_V(log\ E)(-nE))$ and so the expression fits nicely with the tensor product decomposition of the second term.

Proof. Using the formula $h_i = h_i/g^n \cdot g^n$, in which the first factor is interpreted as being constant along fibers of $V \to E$, we have

$$\epsilon(\omega) = \sum_{i=0}^{r} (-1)^i (h_i) d(h_0) \wedge \ldots \wedge \widehat{d(h_i)} \wedge \ldots \wedge d(h_r)$$

where the hat denotes a deleted factor. This expression is homogeneous of degree $r+1$ in the h_i and can be rewritten as

$$g^{n(r+1)} \sum_{i=0}^{r} \frac{h_i}{g^n} d\frac{h_0}{g^n} \wedge \ldots \wedge \widehat{d\frac{h_i}{g^n}} \wedge \ldots \wedge d\frac{h_r}{g^n}.$$

Then the entire expression $\epsilon(\omega) \wedge dlog\ s$ is the rational function $g^{n(r+1)}$ times the pullback under ϕ of an $r+1$ form from E. This is zero since E has dimension r only.

12. Corollary.

For n as above, let $h_0, ..., h_r \in \Gamma(E, \mathcal{O}_E(nD)$ be global sections, viewed as global functions $V \to \mathbb{C}$ constant along fibers of $\phi : V \to E$. Cover E by open sets U_i such that $\mathcal{O}_E(D)$ is principal on each U_i, and let g_i be a local generator on U_i viewed as a locally defined function $V \to \mathbb{C}$ of degree one on each fiber of $V \to E$. Let ω_i be the local $r+1$ form $d(h_0) \wedge ... \wedge d(h_r)$ on V. Then there is a unique global $r+1$ form on V which is the recollement of the $\epsilon(\omega_i) \wedge dlog\ g_i$ on U_i. That is, the restriction of $\epsilon(\omega_i) \wedge dlog\ g_i$ and $\epsilon(\omega_j) \wedge dlog\ g_j$ on an intersection $U_i \cap U_j$ are equal.

Proof. On U_i we can let $\tau = \sum_{t=0}^{r} \frac{h_0}{g_i^n} \wedge ... \wedge \widehat{\frac{h_t}{g_i^n}} \wedge ... \wedge \frac{h_r}{g_i^n}$. This is a section of $\Lambda^r \Omega_E$ and so $\epsilon\omega = g^{(r+1)n} \phi^*(\tau)$ is also section of $\phi^* \Lambda^r \Omega_E$. When we restrict to $U_i \cap U_j$ and take the difference of the restrictions of the two forms, the first factor in the expression

$$\epsilon(\omega) \wedge (dlog\ g_i - dlog\ g_j)$$

is then a pullback from E. This can be rewritten

$$\phi^*(\tau) \wedge dlog\ r$$

where $r = g_i/g_j$ is constant along fibers of $E \to V$, and now $dlog\ r$ is also a pullback from E, so the wedge product is as well. The wedge product is in the $r+1$ exterior power of the pullback of one forms on E, and this is the same as the pullback of $r+1$ forms on E, of which there are none except zero.

Remark. Continuing to interpret h_i as degree n on the fibers of $V \to E$, the recollement recovers nothing but the integer multiple $(-1)^r\omega$ itself from its Euler contraction $\epsilon(\omega)$. This is because locally (or as a rational form)

$$\omega = (-1)^r \epsilon(\omega) \wedge dlog\ g + g^{(r+1)n} d\frac{h_0}{g^n} \wedge ... \wedge d\frac{h_r}{g^n}.$$

In the calculation of the first term a factor of $1/n$ cancels a factor of n. The second factor is zero since it is a wedge product of $r+1$ forms which are pullbacks from E.

Remark. The foregoing shows that although there is not actually any splitting of the Euler contraction, it is because the $dlog\ g_i$ patch together only up to an error coboundary, not precisely; yet this error coboundary is inessential and becomes zero once wedged with the Euler contraction $\epsilon(\omega)$ of ω.

Remark. Intead of choosing g_i what we will do later on is focus on the case when $n = 1$, and let u be the rational function $u : V - \to \mathbb{C}$ induced by the element 1 as a rational section of $\mathcal{O}_E(D)$. The divisor of u is $E + \phi^{-1}(D)$ where $\phi^{-1}(D)$ is the divisorial inverse image of D which is not assumed to be effective. When $h_0, ..., h_r$ are local sections of $\mathcal{O}_E(D)$ we view the h_i as functions $V \to \mathbb{C}$ constant along fibers of $V \to E$ so as local sections of $\mathcal{O}_V(\phi^{-1}(D))$. Then the products $h_i u$ are local holomorphic functions $V \to \mathbb{C}$ of degree one on the fibers of $V \to E$ If Z_i is the divisor of zeroes of k_i on E then the divisor of zeroes of uh_i is on V is

$$(h_i u) = E + \phi^{-1}(Z_i).$$

What we'll do is to choose a particular open subset $U \subset E$ which will be the complement of a divisor S linearly equivalent to mD, where mD is very ample. We'll choose a number j and take h_i to be global sections of $\mathcal{O}_E((mj+1)D)$ of $\mathcal{O}_E((mj+1)D)$. Choosing f with divisor $jS - mD$ the rational functions h_i/f^j will be sections of $\mathcal{O}_E(D)$ on the open set U and the $h_i u/f^j$ will be functions degree one on the fibers of $V \to E$ and holomorphic on $\phi^{-1}U$. Since multiplying u by a local holomorphic function constant on fibers of $V \to E$ does not affect the answer, we have that if we let $\omega = d(h_0 u) \wedge ... \wedge d(h_r u)$ then $\omega = (-1)^r \epsilon(\omega) \wedge dlog\ u$. Although u is not holomorphic even on $\phi^{-1}(U)$ still the wedge product is, and is equal to ω.

We'll approach things from a rather different direction, interpreting V as the blowup of W along the ideal $I = \oplus_{i \geq m} \Gamma(E, \mathcal{O}_E(iD))$. The open set U is affine and we will very concretely construct the relevant coordinate rings and modules.

Geometric meaning of the cokernel

In ordinary local calculus, a familiar fact is that a map on an r dimensional manifold defined by a sequence $f_1, ..., f_r$ of functions is ramified at the zero locus of the form $df_1 \wedge ... \wedge df_r$. This corresponds to non-transverse intersections of the level sets $f_i = c_i$ for constants c; that is, points which can have a nonzero velocity vector while the point $(c_1, ..., c_r)$ has velocity zero.

When working projectively, with functions $h_0, ..., h_r$, if a map is given by the ratio $[h_0 : ... : h_r]$ it is no longer the level sets $h_i = c$ which are relevant. The condition which determines whether a point p can have nonzero velocity vector while either $[h_0(p) : ... : h_r(p)]$ is undefined or has velocity zero is exactly the linear depenence of the principal parts $\nabla(h_0), ..., \nabla(h_r)$. Therefore

13. Theorem.

Let E be a complex manifold of dimension r and let $\mathcal{L}$ be a locally free sheaf on E. Let $h_0, ..., h_r \in \Gamma(E, \mathcal{L})$ and suppose it is of finite dimension $n+1$ Then the support of the cokernel of the map $\mathcal{O}_E \nabla h_1 \oplus ... \oplus \mathcal{O}_E \nabla\ h_r \to \mathcal{P}(\mathcal{L})$ is equal to the union of the indeterminacy locus and the ramification locus of the rational map

$$E - \to \mathbb{P}^n - \ \to \mathbb{P}^r$$

$$e \mapsto [h_0(e) : ... : h_r(e)]$$

which is the composite of the projective rational morphism associated to $\mathcal{L}$ and the projection to r dimensional projective space.

Remark. When u is the rational function $V - \to \mathbb{C}$ induced the rational section 1 of $\mathcal{O}_E(D)$, and h is a global section of $\mathcal{O}_E(nD)$ for some n, then the relation between $h, u, \nabla h$ is that $\nabla h \in \mathcal{P}(\mathcal{O}_E(nD)$ is the restriction to E of $d(u^n h)$. Let's record this:

$$\nabla(h) = i^* d(u^n h). \qquad (3)$$

Local description of generators of $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(nD))$

We can combine the foregoing to give a local description of generating sections of the principal parts sheaf $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(nD))$ when E is smooth and nD is very ample on E. Note that global generation of this sheaf was proved abstractly in Theorem 10 without the assumption that E is smooth. When E is smooth Theorem 13 will provide global generators, and Corollary 12 will provide the local description.

14. Theorem.

> If E is smooth and n is a number such that nD is very ample then $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(nD))$ is spanned by global sections $\nabla h_0 \wedge ... \wedge \nabla h_r$ for $h_i \in \Gamma(E, \mathcal{O}_E(nD))$. Each ∇h_i is defined to be the restriction to E of the form dh_i where h_i is now viewed as a function $V \to \mathbb{C}$ of degree n on each line fiber of $\phi : V \to E$. Moreover the form $\omega = dh_0 \wedge ... \wedge dh_r$ is equal to the recollement of the wedge products $\frac{1}{n}\epsilon(\omega) \wedge dlog\ g$ where g ranges over local generators of $\mathcal{O}_E(D)$ viewed as locally defined functions $V \to \mathbb{C}$ (in other words so that each equation $g = 0$ is a local defining equation of E), and where ϵ is the Euler contraction.

Proof. For a very ample linear system on a smooth r dimensional variety E there is always a projection to $\mathbb{P}^r$ which is an isomorphism in a neighbourhood of any chosen point.

Then by Theorem 13 for every point $p \in E$ there are $h_0, ..., h_r \in \Gamma(E, \mathcal{O}_E(nD))$ so that $\nabla(h_0) \wedge ... \wedge \nabla(h_r) \in \Gamma(E, \mathcal{P}(\mathcal{O}_E(nD))$ is a local generator at p. Letting $u : V - \to \mathbb{C}$ be the rational map induced by 1 as a rational section of $\mathcal{O}_E(D)$, formula (3) gives

$$\nabla(h_i) = i^*(u^n h_i)$$

Finally, when when g is a local generator of $\mathcal{O}_E(D)$ we have by the remark following Corollary 12 a local identity $\nabla h_0 \wedge ... \wedge \nabla h_r = \epsilon(\omega) \wedge dlog\ g$.

Geometric interpretation of Theorem 14

Taking E smooth and $\mathcal{L} = \mathcal{O}_E(nD)$ (an arbitrary very ample locally free coherent sheaf of rank one), Theorem 14 now has a straightforward geometric interpretation. Just because we're assuming that $\mathcal{L}$ is locally free, the sequence

$$0 \to \Omega_E \otimes \mathcal{L} \to \mathcal{P}(\mathcal{L}) \to \mathcal{L} \to 0$$

is locally split, so that the map

$$\mathcal{L}^{r+1} \otimes \Lambda^r \Omega_E \to \Lambda^{r+1}\mathcal{P}(\mathcal{L})$$

is an isomorphism. Given $h_0, ..., h_r$ global sections of $\mathcal{L}$, the theorem gives a way of understanding the expression

$$\sum_{i=0}^{r} (-1)^i h_i \nabla h_0 \wedge ... \wedge \widehat{\nabla h_i} \wedge ... \wedge \nabla h_r$$

as representing a global section of $\Lambda^r \Omega_E \otimes \mathcal{L}^{r+1}$. The geometric interpretation in terms of logarithmic forms on V shows that the vanishing locus of the section is the locus where the map $[h_0 : ... : h_r] : E - \to \mathbb{P}^r$ which is the composite of the projective morphism of $\mathcal{L}$ with the projection on the linear projective space corresponding to $h_0, ..., h_r$, is either indeterminate or ramified.

Let us call the section above an 'adjoint section.' Then the fact that $\mathcal{L}$ is very ample, because it implies that for every point of E there are $h_0, ..., h_r$ global sections such that the rational map $[h_0 : ... : h_r]$ is well defined and not ramified at p it therefore implies that at each point the h_i chosen in this way define an adjoint section which is not zero at p.

For $\mathcal{L}$ which is merely ample, but not very ample, there may not be any 'adjoint sections' at all.

Choose $f \in \Gamma(E, \mathcal{O}_E(mD))$ and view f as a function $V \to \mathbb{C}$ which is degree m on each fiber of $V \to E$. Letting $i : E \to V$ be the inclusion, there is the doubly exact diagram which does not commute, as for a j form ω we have

$$\epsilon(\omega \wedge dlog\ f) = \epsilon(\omega) \wedge dlog\ f + (-1)^j m\omega$$

since $\epsilon(dlog\ f) = m$.

$$
\begin{array}{cccccccc}
 & & & & & & 0 & \\
 & & & & & & \uparrow & \\
 & & & & 0 & \to & \mathcal{O}_E(2mD) \otimes \Lambda^{r+1}(\Omega_V(log\ E)(-mE)) & \dots \\
 & & & & \uparrow & & \uparrow\ - \wedge dlog\ f & \\
 & & 0 & \to & \mathcal{O}_E(mD) \otimes i^*\Lambda^{r+1}(\Omega_V(log\ E)(-mE)) & \xrightarrow{\epsilon} & \mathcal{O}_E(2mD) \otimes \Lambda^{r}\Omega_V(log\ E)(-mE)) & \dots \\
 & & \uparrow & & \uparrow\ - \wedge dlog\ f & & \uparrow\ - \wedge dlog\ f & \\
0 & \to & i^*\Lambda^{r+1}\Omega_V(log\ E)(-mE)) & \xrightarrow{\epsilon} & \mathcal{O}_E(mD) \otimes i^*\Lambda^{r}(\Omega_V(log\ E)(-mE)) & \xrightarrow{\epsilon} & \mathcal{O}_E(2mD) \otimes i^*\Lambda^{r-1}(\Omega_V(log\ E)(-mE)) & \dots \\
 & & \uparrow\ - \wedge dlog\ f & & \uparrow\ - \wedge dlog\ f & & \uparrow\ - \wedge dlog\ f & \\
 & \to & i^*\Lambda^{r}(\Omega_V(log\ E)(-mE)) & \xrightarrow{\epsilon} & \mathcal{O}_E(mD) \otimes i^*\Lambda^{r-1}(\Omega_V(log\ E)(-mE)) & \xrightarrow{\epsilon} & \mathcal{O}_E(2mD) \otimes i^*\Lambda^{r-2}(\Omega_V(log\ E)(-mE)) & \dots \\
 & & \uparrow\ - \wedge dlog\ f & & \uparrow\ - \wedge dlog\ f & & \uparrow\ - \wedge dlog\ f & \\
 & & \vdots & & \vdots & & \vdots &
\end{array}
$$

The augmentation kernels of the left column are the $\Lambda^i\Omega_E$. The failure of commutativity explains why the map

$$(- \wedge dlog\ f) \circ \epsilon$$

in the upper left corner is multiplication by $(-1)^r m$ rather than zero. The occurrence of m here is explained by the fact that if locally we take $f = g^m$ then locally $dlog\ f = m\ dlog\ g$.

The maps labelled $(- \wedge dlog\ f)$ actually refer to the maps induced by $(- \wedge dlog\ f)$ acting on $\Lambda^{\cdot}\Omega_V(log\ E)(-mE)$ after applying the functor i^*.

Linear systems of divisors on V

Continue with the situation where E is a nonsingular projective variety with ample divisor D and $\phi : V = \mathbb{V}(\mathcal{O}_E(D)) \to E$ the bundle projection. Choose a number m so that mD is ample, and let $f \in \Gamma(E, \mathcal{O}_E(mD))$ be such that f has divisor of poles exactly mD and divisor of zeroes a smooth irreducible hypersurface S. Let $Y = \phi^{-1}S \subset V$. Let u be the rational function $V - \to \mathbb{C}$ of degree one on the line fibers which is induced by the rational section 1 of $\mathcal{O}_E(D)$.

15. Lemma

For every natural number j, the subspace of the (infinite dimensional) linear system $|jY|$ on V which consists of the fixed component E together with a fiberwise divisor is equal to the set of divisors $(uh/f^j)+\phi^{-1}(jm+1)D$ for $h \in \Gamma(E, \mathcal{O}_E((jm+1)D)$.

Proof. Write

$$Z = (h) + (jm + 1)D$$

effective on E. We have precisely

$$(u) = \phi^{-1}D + E,$$

$$(f) = S - mD.$$

Then

$$(uh/f^j) = \phi^{-1}D + E + \phi^{-1}(Z - (jm + 1)D) - jY - mj\phi^{-1}D$$

$$= E + \phi^{-1}Z - jY.$$

Then

$$(uh/f^j) + jY = E + \phi^{-1}Z.$$

Generators of $\Gamma(V \setminus Y, \Omega_V(log\ E)(-E))$.

Recall that E is smooth and projective of dimension r and D is an ample (not necessarily effective) divisor on E. Let $V = \mathbb{V}(\mathcal{O}_E(D)) \xrightarrow{\phi} E$ be the Fibré Vectoriel. For any number s and any global section $h \in \Gamma(E, \mathcal{O}_E(sD))$ we will use the same letter h to denote the composite rational function

$$V \xrightarrow{\phi} E - \to \mathbb{C}.$$

Thus rational functions on E will be viewed as rational functions on V which are constant along the line fibers $V \to E$.

The constant function 1 will not belong to $\Gamma(E, \mathcal{O}_E(D))$ if D is not effective, nevertheless it defines a rational map

$$u : V - \to \mathbb{C}$$

which is degree one on every line fiber where it is defined; and so that the divisor (u) satisfies

$$(u) = E + \phi^{-1}(D)$$

with ϕ^{-1} the divisorial inverse image.

Points of the very ample linear system $|mD|$ can be thought of as ideal sheaves in $\mathcal{O}_E$ which are isomorphic to $\mathcal{O}_E(-mD)$. A nonzero element $f \in \Gamma(E, \mathcal{O}_E(mD))$ gives by multiplication with f an embedding

$$\mathcal{O}_E(-mD) \to \mathcal{O}_E$$

and the cokernel is the coordinate sheaf of a subscheme of $S \subset E$ which depends precisely on the class of f modulo scalars. In other words, the ideal of definition $\mathcal{I}_{S/E}$ of S in E is $f \cdot \mathcal{O}_E(-mD)$.

Bertini's theorem provides a choice of f so that S is a smooth (and connected) reduced variety however we will not require this choice. Let $Y = \phi^{-1}(S)$ be the inverse image of S depending on f. The divisor

$$E + Y$$

is a simple normal crossing divisor on V with crossing locus S. Let $j : Y \to V$ be the inclusion.

We will now calculate spanning elements of the restriction of $\Omega_V(log\ E)(-E)$ the open subset $U = V\backslash Y$ which is the complement of Y. As S moves such open sets provide an open cover of V.

Recall that $W = Spec\ \oplus_{i=0}^{\infty}\Gamma(E, \mathcal{O}_E(iD))$ the result of contracting mE to a point p.

The coordinate ring $\mathcal{O}_W$ is the familiar homogeneous coordinate ring of E, and W itself is the familiar affine cone of E polarized by D. If we pull back a graded (meaning scalar equivariant) finitely generated $\mathcal{O}_W$ module to $W\backslash p$ and push forward again we obtain the saturated module – an intersection of powers of primes times localizations, not including the inessential localization. That is, the isomorphism types of saturated graded $\mathcal{O}_W$ modules is naturally bijective with isomorphism types of scalar equivariant coherent sheaves on $V\setminus E = W\setminus p$

The category of coherent sheaves on E itself is naturally equivalent to scalar equivariant coherent sheaves on $V\setminus E = W\setminus p$. In this way we see that saturated graded finitely generated $\mathcal{O}_W$ modules is an additive category naturally equivalent to the category of coherent sheaves on the exceptional divisor E which results when we blow up p with the appropriate scheme structure, the one defined by $I = \oplus_{i=m}^{\infty}\mathcal{O}_E(iD)$.

16. Lemma.

For any natural number s let $h \in \Gamma(E, \mathcal{O}_E(sD))$. View h as a rational function on V which is constant along fibers of $V \to E$. Let $u : V - \ \to \mathbb{C}$ be the rational function induced by 1 as a rational section of $\mathcal{O}_E(D)$, which is linear of degree one on the line fibers of $V \to E$ where it is defined. Then the product $u^s h$ is an entire function $V \to \mathbb{C}$ and as a result it is induced by an entire function $W \to \mathbb{C}$.

Proof. Viewing h as a rational function on V constant along the fibers of $V \to E$. The divisor of h is $\phi^{-1}(T - sD)$ where $T \subset E$ is the effective intersection of h viewed as a section of a vector bundle with its zero section. The divisor of u is $E + \phi^{-1}(D)$. Then the divisor of $u^s h$ is

$$sE + s\phi^{-1}(D) - s\phi^{-1}(D) + \phi^{-1}T$$

$$= sE + s\phi^{-1}(T)$$

which is effective as claimed.

17. Corollary.

The coordinate ring $\mathcal{O}_W$ embeds in the global sections of $\mathcal{O}_V$ as

$$\mathcal{O}_W = \oplus_{i=0}^{\infty}\Gamma(E, \mathcal{O}_E(iD)) \cdot u^i.$$

The elements of $\mathcal{O}_W$ can be considered to be polynomials,

$$a_0 + a_1 u + a_2 u^2 + ...$$

with $a_i \in \Gamma(E, \mathcal{O}_E(iD))$; note however that no a_i is allowed to equal 1 for $i \geq 1$ if D is not effective.

We can view f as a global section of $\mathcal{O}_W$ Assembling together the inclusions for each i

$$fu^m \cdot u^i\Gamma(E, \mathcal{O}_E(iD)) \subset u^{i+m}\Gamma(E, \mathcal{O}_E((i+m)D))$$

gives the principal ideal of $\mathcal{O}_W$ generated by fu^m,

$$\oplus_{i=0}^{\infty} f^m \cdot u^i\Gamma(E, \mathcal{O}_E(iD)) \subset \oplus_{i=0}^{\infty} u^{i+m}\Gamma(E, \mathcal{O}_E((i+m)D)).$$

The open subset $U \subset V = Bl_I W$ of the blow-up corresponding to the global section fu^m is the spectrum of the ring

$$\bigcup_{j=0}^{\infty} I^j/(fu^m)^j,$$

and this decomposes as

$$\bigcup_{j=0}^{\infty} \Gamma(E, \mathcal{O}_E(jmD))/(fu^m)^j \oplus \bigcup_{j=0}^{\infty} u\Gamma(E, \mathcal{O}_E((jm+1)D))/(f^m)^j \oplus \bigcup_{j=0}^{\infty} u^2\Gamma(E, \mathcal{O}_E((jm+2)D))/(f^{m}$$

Multiplication by fu^m now induces an isomorphism which decreases the level of the filtration by one while increasing the degree in the grading by m. The ideal of terms of degree at least m is the principal ideal generated by f.

$$
\begin{array}{rccccccccrc}
 & \mathcal{O}_{E\setminus S} & \oplus & \mathcal{O}_{E\setminus S}(D)\cdot u & \oplus & \mathcal{O}_{E\setminus S}(2D)\cdot u^2 & \oplus & \dots & \oplus & & f\mathcal{O}_{E\setminus S}\cdot u^m \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \uparrow \\
 & \vdots & & \vdots & & \vdots & & & & & \vdots \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \\
 & \Gamma(E,\mathcal{O}_E(4mD))/f^4 & \oplus & \Gamma(E,\mathcal{O}_E((4m+1)D))u/f^4 & \oplus & \Gamma(E,\mathcal{O}_E((4m+2)D)u^2/f^4 & \oplus & \dots & & & \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \\
 & \Gamma(E,\mathcal{O}_E(3mD))/f^3 & \oplus & \Gamma(E,\mathcal{O}_E((3m+1)D))u/f^3 & \oplus & \Gamma(E,\mathcal{O}_E((3m+2)D))u^2/f^3 & \oplus & \dots & & & \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \vdots \\
 & \Gamma(E,\mathcal{O}_E(2mD))/f^2 & \oplus & \Gamma(E,\mathcal{O}_E((2m+1)D))u/f^2 & \oplus & \Gamma(E,\mathcal{O}_E((2m+2)D))u^2/f^2 & \oplus & \dots & & & \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \uparrow \\
1\in & \Gamma(E,\mathcal{O}_E(mD))/f & \oplus & \Gamma(E,\mathcal{O}_E((m+1)D))u/f & \oplus & \Gamma(E,\mathcal{O}_E((m+2)D))u^2/f & \oplus & \dots & \oplus & & \Gamma(E,\mathcal{O}_E(2mD))u^m/f \\
 & \uparrow & & \uparrow & & \uparrow & & & & & \uparrow \\
 & \Gamma(E,\mathcal{O}_E) & \oplus & \Gamma(E,\mathcal{O}_E(D))u & \oplus & \Gamma(E,\mathcal{O}_E(2D))u^2 & \oplus & \dots & \oplus & fu^m\in & \Gamma(E,\mathcal{O}_E(mD))u^m
\end{array}
$$

The lower row is the whole of $\mathcal{O}_W$. Since $\Gamma(E,\mathcal{O}_E((jm+i)D))/f^j = \Gamma(E,\mathcal{O}_E(jS+iD))$ the j'th row is just the rational functions on V with pole at worst jY. The top row, the union of all the rows, is $\mathcal{O}_{V\setminus Y}$ where $Y=\phi^{-1}S$.

The leftmost term, allowing the union over all j, is the ring of rational functions on E which are well defined on the complement of S.

For any number j the finite dimensional vector space

$$\Gamma(E,\mathcal{O}_E(jm+1))u/f^j \subset \mathcal{O}_{V\setminus Y}$$

consists of elements integral over the polynomial sub-algebra $\mathcal{O}_{E\setminus S}[fu^m]$. Since the (invertible) module $\mathcal{O}_{E\setminus S}(D)\cdot u$ is finitely generated over the $\mathcal{O}_{E\setminus S}$ and also generates the ring $\mathcal{O}_{V\setminus Y}$ over $\mathcal{O}_{E\setminus S}$, we can choose j large enough that all module generators are contained in this vector space. Therefore

18. Lemma.

For sufficiently large j the finite-dimensional vector space $\Gamma(E,\mathcal{O}_E((jm+1)D))u/f^j$ generates $\mathcal{O}_{V\setminus Y}$ over the polynomial algebra $\mathcal{O}_{E\setminus S}[fu^m]$.

For every $x\in\mathcal{O}_{V\setminus Y}$, there is a unique natural number e so that

$$(fu^m)^e x \in \mathcal{O}_W \setminus 0.$$

That is, if we multiply x by the largest power of fu^m possible so that the answer is not actually zero, the answer will belong to $\mathcal{O}_W$.

The module $\Omega_{V\setminus Y}(log\ E)(-E)$ is the submodule of $\Omega_{V\setminus Y}$ generated by differentials of elements whose degree zero part is zero, in other words elements of the ideal

$$0 \oplus \mathcal{O}_{E\setminus S}(D)u \oplus \mathcal{O}_{E\setminus S}(2D)u^2 \oplus ...$$

Therefore, for any number j sufficiently large, it is spanned by $d(u^m f)$ together dh for

$$h \in \Gamma(E, \mathcal{O}_E((jm+1)D))u/f^j.$$

Equivalently, for any sufficiently large number j, any section of $\Omega_V(log\ E)(-E)$ over $V \setminus Y$ is an $\mathcal{O}_{V\setminus Y}$ linear combination of $d(u^m f)$ and $d(hu/f^j)$ for $h \in \Gamma(E, \mathcal{O}_E((jm+1)D))$.

We can detect a generating set by seeing whether wedge products generate the highest exterior power modulo torsion. By Corollary 4 in turn this can be detected only looking at the restriction to E. And yet $d(u^m f)$ is zero to high order on E, therefore it is a redundant generator and the $d(hu/f^j)$ alone must suffice as generators. Finally,

19. Theorem.

Let D on E be ample. Choose m so that mD is very ample. Let $f \in \Gamma(E, \mathcal{O}_E(mD))$. Let $S = mD + (f)$ so that S is an effective divisor in the linear system $|mD|$, and let $Y = \phi^{-1}S \subset V = \mathbb{V}(\mathcal{O}_E(D))$. We view f as a rational function on V via the line bundle projection map $V \to E$. Let u be the rational section 1 of $\mathcal{O}_E(D)$ viewed as a rational function on V of degree 1 on the fibers where it is defined. Then for any sufficiently large number j, $\Gamma(V \setminus Y, \Omega_V(log\ E)(-E))$ is spanned as an $\mathcal{O}_{V\setminus Y}$ module by $d(ku)$ for $k \in \Gamma(E, \mathcal{O}_E(jS + D))$.

Proof. Take $k = h/f^j$. We have seen that it is spanned by $d(hu/f^j)$ for $h \in \Gamma(E, \mathcal{O}_E((jm+1)D))$ and so $k \in \Gamma(E, \mathcal{O}_E(jS+D))$.

This corollary is not surprising,

20. Corollary.

For any ample divisor D on E and m such that mD is very ample, and for any $S \in |mD|$ there is a number j such that $\Gamma(E \setminus S, \mathcal{P}(\mathcal{O}_E(D))$ is spanned as a $\mathcal{O}_{E\setminus S}$ module by ∇h for $h \in \Gamma(E, \mathcal{O}_E(jS + D))$.

Discussion

Working locally now on the complement of S choose a local function g so that gu is a local defining equation of E. The divisor of u is actually $E+\phi^{-1}(D)$ where ϕ^{-1} here refers to divisorial inverse image, and so we will use a local function g which has D as its divisor of poles (and no zeroes).

Now $\Gamma(E \setminus S, \Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(D))$ is spanned as $\mathcal{O}_{E\setminus S}$ module by the coherent sheaf restriction to E of elements

$$\epsilon(\omega) \wedge dlog\ (gu)$$

for $\omega = (dh_0u/f^j) \wedge ... \wedge (dh_ru/f^j)$ and ϵ the Euler contraction; and that the expression is homogeneous of degree $r + 1$ in the h_i and unaffected by replacing $dlog\ (gu)$ by $dlog\ u$.

Once $j - 1$ is large enough that $D + m(j - 1)D$ is basepoint free on E then $(mj + 1)D$ is very ample and Theorem 13 implies that $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E((mj+1)D)$ is generated by the particular global sections $\nabla h_0 \wedge ... \wedge \nabla h_r$ for $h_0, ..., h_r \in \Gamma(E, \mathcal{O}_E(mj + 1)D))$.

Returning to look at the left portion of the diagram

$$\begin{array}{ccccccccccc}
\uparrow & & \uparrow & & \uparrow & & & & & & \\
\Gamma(E,\mathcal{O}_E(3S)) & \oplus & \Gamma(E,\mathcal{O}_E(3S+D))u & \oplus & \Gamma(E,\mathcal{O}_E((3S+2D))u^2 & \oplus & ... & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & \vdots & & \\
\Gamma(E,\mathcal{O}_E(2S)) & \oplus & \Gamma(E,\mathcal{O}_E(2S+D))u & \oplus & \Gamma(E,\mathcal{O}_E(2S+2D))u^2 & \oplus & ... & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\Gamma(E,\mathcal{O}_E(S)) & \oplus & \Gamma(E,\mathcal{O}_E(S+D))u & \oplus & \Gamma(E,\mathcal{O}_E(S+2D)u^2 & \oplus & ... & \oplus & \Gamma(E,\mathcal{O}_E(S+(r+1)D))u^{r+1} & \oplus & ... \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \\
\Gamma(E,\mathcal{O}_E) & \oplus & \Gamma(E,\mathcal{O}_E(D))u & \oplus & \Gamma(E,\mathcal{O}_E(2D))u^2 & \oplus & ... & \oplus & \Gamma(E,\mathcal{O}_E((r+1)D))u^{r+1} & \oplus & ...
\end{array}$$

Takao Fujita's conjecture is true if we can find such a sequence of elements h_i/f^j in $\Gamma(E, \mathcal{O}_E(jS + D))$ for each point $p \in E \setminus S$ that the meromorphic $r + 1$ form

$$d(h_0 u/f^j) \wedge ... \wedge d(h_r u/f^j)$$

is not identically zero on the fiber line of $V \to E$ through p, and has its pole on Y of order equal to zero or less. Then the form extends to a global holomorphic $r + 1$ form on V. Since S is movable the restriction $p \notin S$ is inessential.

Discussion

Let E be a projective variety with D an ample divisor. Associated to any coherent sheaf $\mathcal{F}$ on E is the finitely-generated graded $\mathcal{O}_W$ module $M = \pi_*\phi^*\mathcal{F}$, which has no cotorsion for the ideal of positive degree elements of the ring $\mathcal{O}_W$. The coherent sheaf $\mathcal{F}$ is characterised by the rule $v^*\mathcal{F} = j^*M$, with j, v as in the diagram below.

$$
\begin{array}{ccccc}
 & & E & & \\
 & & \uparrow \phi & & \\
V \setminus E & \subset & V & \stackrel{i}{\leftarrow} & E \\
\downarrow \cong & & \downarrow \pi & & \\
W \setminus p & \stackrel{j}{\subset} & W & & \\
\downarrow v & & & & \\
E & & & &
\end{array}
$$

When $\mathcal{F}$ is torsion free coherent on E there is a direct way to recover $\mathcal{F}$ from its graded $\mathcal{O}_W$ module. Since $-E$ is ample relative to π choose a number s so that $(\phi^*\mathcal{F})(-sE)$ is basepoint free relative to π. From the linear equivalence $\phi^{-1}D \equiv -E$ we have

$$(\phi^*\mathcal{F})(-sE) \cong \phi^*(\mathcal{F}(sD)).$$

The isomorphic sheaf being relatively basepoint free means that the natural map

$$\pi^*\pi_*\phi^*\mathcal{F}(sD)/torsion \to \phi^*\mathcal{F}(sD)$$

is an isomorphism. Then

Remark.

For $\mathcal{F}$ a torsion-free coherent sheaf on E for sufficiently large s, letting

$$N = \pi_*\phi^*\mathcal{F}(sD)$$

we recover $\mathcal{F}$ as

$$\mathcal{F} \cong \mathcal{O}_E(-sD) \otimes i^*(\pi^*N/torsion).$$

Appendix: homogeneous coordinate rings

Lemma. Let R be a graded integral domain. Then there is an irreducible projective variety E and an ample Cartier divisor D so that $R_i = \Gamma(E, \mathcal{O}_E(iD))$ if and only if

i) $R_0 = \mathbb{C}$ and R_i is finite-dimensional over $\mathbb{C}$ for all i,

ii) R is finitely generated over $\mathbb{C}$; in other words there is an m with $R_{mi} = R_m^i$ for $i = 1, 2, 3, ...$

iii) With m as in ii), there is a number t so that for all sufficiently large s $R_m^s(R_{mt+1})^m = R_m^{s+m+tm^2}$,

iv) R has no cotorsion for the ideal of positive degree elements.

Proof. Supposing R has these properties let $R^{(m)}$ be the subring of elements whose degree is a multiple of m and let $E = Proj\ R = Proj\ R^{(m)}$. The terms of degree congruent to one modulo m are a graded module over $R^{(m)}$ and condition iii) implies that the corresponding invertible sheaf $\mathcal{L}$ on E satisfies $\mathcal{L}^{\otimes m} \cong \mathcal{O}_E(H)$ for H the appropriate very ample divisor. Then there is a Cartier divisor D with $mD = H$. Then condition iv) gives $\Gamma(E, \mathcal{O}_E(iD)) = R_i$.

Appendix: Fujita's conjecture in terms of coordinate rings.

Lemma.

> The Fujita conjecture is equivalent to the assertion that if R is the graded ring of an ample divisor on a smooth r-dimensional projective variety, then for $t >> 0$ the submodule of $\Lambda^{r+1}\Omega_R/torsion$ generated by $R_{tm}\Lambda^{r+1}(RdR_m)$ is divisible by the ideal generated by $R_{(r+1)(m-1)+tm}$.

Proof. Writing $R = \oplus_{i=0}^{\infty}\Gamma(E, \mathcal{O}_E(iD))$ choose m so that mD is very ample. As usual let $I = R_m + R_{m+1} + ...$ and $\pi : V = Bl_I W \to W$, so that $V = \mathbb{V}(\mathcal{O}_E(D))$. Then $\mathcal{O}_V(-mE)$ is relatively very ample and by Theorem 14, $\mathcal{P}(\mathcal{O}_E(mD))$ is generated by global sections. This shows that the restriction of $\Lambda^{r+1}\Omega_V(log\ E)(-mE)$ to E is basepoint free. By Corollary 4, $\Lambda^{r+1}\Omega_V(log\ E)(-mE)$ is basepoint free relative to π. More precisely the $\nabla h_0 \wedge ... \wedge \nabla h_r$ for $h_i \in \Gamma(E, \mathcal{O}_E(mD))$ span $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(mD))$ and therefore for u the rational section 1 of $\mathcal{O}_E(D)$ the restriction of the $d(h_0u^m) \wedge ... \wedge d(h_ru^m) \in \Lambda^{r+1}\pi^*\mathcal{O}_W dI/torsion$ span the restriction of $\Lambda^{r+1}\Omega_V(log\ E)(-mE)$ to E; and then as Lemma 4 says the sheaf is spanned by global sections, these are in fact sections of $\Lambda^{r+1}\pi^*\mathcal{O}_W dI/torsion$; hence

$$\Lambda^{r+1}\Omega_V(log\ E)(-mE) = \Lambda^{r+1}\pi^*\mathcal{O}_W dI/torsion.$$

Choose t using relative very ampleness of I so that

$$\pi^*(I^t\Lambda^{r+1}\Omega_W dI)/torsion \cong \mathcal{O}_E(-mt) \otimes \Lambda^{r+1}\Omega_V(log\ E)(-mE).$$

The hypothesis is equivalent to the existence of a choice of t so that also this is the tensor product of a pullback modulo torsion under π^* with $\mathcal{O}_V(-(r+1)(m-1)E - tmE)$, equivalently then that

$$\Lambda^{r+1}\Omega_V(log\ E)(-E)$$

is a pullback modulo torsion under π. This is equivalent to relative basepoint freeness which is equivalent to absolute basepoint freeness of its pullback $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(D))$ to E by Corollary 4. And of course $\Lambda^{r+1}\mathcal{P}(\mathcal{O}_E(D)) \cong \mathcal{O}_E(K_E + (r+1)D)$.

In the lemma above, the divisibility is not required to respect the grading.

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