8. The residue calculation

Earlier I was mentioning that it is not difficult to write down hypotheses which strengthen Mordell's conjecture, with the conclusion that particular modular curves have no noncuspidal rational points.

The hypotheses where this was written down most carefully included a type of Galois condition, that if a fiber contains any rational point, all must be rational.

The Fermat curves, also, as I mentioned, arise with homogeneous coordinate rings the global sections of symmetric powers of one-forms on the upper half-plane $\mathbb{H}$ holomorphic at cusps on $\mathbb{H}$, which are fixed by the p commutator subgroup of $\Gamma(2)$. The structure map to $X(2)$ is a Belyi map as always, and a corresponding integer structure on the Fermat curve coming from specifying the branch points to be $0, -1, \infty$ gives the usual integer structure of the p'th Fermat curve.

Recent news stories remind us of the depth of history which underlies the Fermat theorem, the friendship between Shimura and Taniyama in Japan in the 1950's. Wiles' proven theorem is an instance of the same strenghtened conclusion, that the Fermat curve has no noncusidal rational points except when its genus is less than two. The case of genus one also has no noncusipdal ratoinal points, leaving the pythagorean triples and the triples with an entry of zero being the only integer solutions of the Fermat equation.

In earlier cases, I was relating the different element to duality and trace forms. That would be the aim here too, and to relate it to the intersection matrix among the components of F, if a suitable intersection theory can be found.

Let's take p to be a prime, although this is not necessary.

The way to calculate integer points in a fiber over a noncuspidal point $[x^p : y^p] = [a : b]$, I've claimed, is to choose integers

$$\begin{aligned} A &= -b \\ B &= a \\ C & \\ D & \end{aligned}$$

with C, D chosen so that $1 = AD - BC$.

The matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

belongs to $SL_2(\mathbb{Z})$.

The residue of the form

$$d\ log \frac{Ax^p + By^p}{Cx^p + Dy^p}$$

on the fiber F defined by the equation $Ax^p + By^p = 0$ among the set of all residues, defines a 'different element' in the restricted canonical sheaf. Let F be divisor defined by the numerator, and F' the divisor defined by the denominator. Since we've used $SL_2(\mathbb{Z})$ they are disjoint even scheme-theoretically.

The way we will calculate the residue is to use principal parts. Although there are nice ways of describing this, in terms of meromorphic connections, let's just consider it as some notation to write $\nabla(x), \nabla(y), \nabla(z)$ for the deRham differentials of x, y, z in *affine* space which can be viewed as global sections of first principal parts in various ways which are a basis rationally (in the sense of rational functions).

In the first instance, let's work over $\mathbb{Z}[1/p]$ so that we can identify, on the projective plane, the first principal parts of $\mathcal{O}(p)$ with $\mathcal{O}(p-1)$ tensor the first principal parts of $\mathcal{O}(1)$.

Then we may write

$$\mathcal{P}(\mathcal{O}(p)) = \mathcal{O}(p-1)\nabla(x) \oplus \mathcal{O}(p-1)\nabla(y) \oplus \mathcal{O}(p-1)\nabla(z).$$

This direct sum of three line bundles contains a trivial line bundle, spanned by

$$\frac{1}{p}(\nabla(x^p) + \nabla(y^p) - \nabla(z^p)).$$

Here we are allowed to use the rule of a derivation, and this is the same as

$$x^{p-1}\nabla(x) \oplus y^{p-1}\nabla(y) \oplus z^{p-1}\nabla(z).$$

Because the basic sections have no common zero anywhere on the projective plane, this is a line bundle inclusion (locally split).

Therefore the third exterior power of $\mathcal{P}(\mathcal{O}(p))$ is the same as the second exterior power of the quotient vector bundle. The quotient vector bundle restricts on the Fermat curve X to the principal parts mod torsion of $\mathcal{O}_X(D)$ where now D is the restriction to X of any hypersurface of degree p. We may take in particular our fiber F.

Putting things together a bit, the third exterior power of our rank three vector bundle corresponds to a hypersurface of degree $3p-3$ on the projective plane; and so the second exterior power of the torsion free princpal parts of the locally free sheaf $\mathcal{L} = \mathcal{O}_X(F)$ on the Fermat curve corresponds to the intersection of the Fermat curve with a degree $3p-3$ hypersurface, and we may think of this as a divisor of degree $3p^2-3p$ on the Fermat curve.

It is always true, because of the sequence

$$0 \to \Omega_X \otimes \mathcal{L} \to \mathcal{P}(\mathcal{L}) \to \mathcal{L} \to 0$$

which exists even if neither Ω_X nor $\mathcal{L}$ is locally free, that there is a map $\phi : \Omega \otimes \mathcal{L}^{\otimes 2} \to \Lambda^2\mathcal{P}(\mathcal{L})$, and in cases like the case at hand, when $\mathcal{L}$ is locally free, so the exact sequence is locally split, this is an isomorphism. Here the letter $\mathcal{P}$ should be taken to be torsion free first principal parts (the reduction moduo torsion) and the kernel term should refer to the torsion-free Kahler differentials, which is the reduction of the one-forms modulo torsion.

We have said that this corresponds to a degree $3p^2 - 3p$ divisor, the degree of the canonical divisor of the Fermat curve is $p^2 - 3p$ and we see that this agrees with the degree of $\mathcal{L} \otimes \mathcal{L} \otimes \Omega$, that is, it is larger than the canonical degree by $2p^2$ which is twice the degree of F.

The inverse isomorphism, on local sections f, g of $\mathcal{L}$, is given

$$\nabla(f) \wedge \nabla(g) \mapsto fg\ d\ log(f/g).$$

1. Remark. There is a slight subtlety in that we are not allowed to factorize the expression $fg\ d\ log(f/g) = g^2 d(f/g) = f^2 d(g/f)$ as a tensor such as

$$g^{\otimes 2} \otimes d(f/g)$$

because the right factor is not a section of one forms except when $g \neq 0$. However working locally the way to see that the expression $fdg - gdf$ corresponds to a local section of $\mathcal{L} \otimes \mathcal{L} \otimes \Omega$ in a neighbournood of a point of the Fermat curve is to choose a local section s of $\mathcal{L}$ not zero at that point. Then (and the analagous thing is true for higher exterior powers in cases of higher dimension, it is a property of contracting under the Euler derivation) there is a homogeneity property of the expression so that

$$fdg - gdf = s^2(\frac{f}{s}d\frac{g}{s} - \frac{g}{s}d\frac{f}{s}).$$

The right side is an expression involving rational functions which are well-defined at our point, giving a local section of Ω, and the factor s^2 is a local section of $\mathcal{L}^{\otimes 2}$. So this can be interpreted as a tensor product then, as we've worked locally

$$s^{\otimes 2} \otimes (\frac{f}{s}d\frac{g}{s} - \frac{g}{s}d\frac{f}{s}).$$

Returning to our exposition, to obtain our ‘different’ element, under the isomorphism between global sections of $\mathcal{L}$ and polynomials of degree p in x, y, z modulo the Fermat relation, we shall multiply our logarithmic derivative by two terms

$$(Ax^p + By^p)(Cx^p + DY^p)d\ log\ \frac{Ax^p + By^p}{Cx^p + Dy^p}$$

and pass to the image under the isomorphism ϕ, which is

$$(A\nabla(x^p) + B\nabla(y^p)) \wedge (C\nabla(x^p) + D\nabla(y^p)).$$

To interpret this as an element of the third exterior power of principal parts of $O(p)$ on the projective plane (later restricted to F) we will wedge with the basis element of the trivial sub bunde, that is we will wedge this expression with

$$\nabla(x^p) + \nabla(y^p) - \nabla(z^p).$$

The result is

$$(xyz)^{p-1}\ determinant \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 1 & 1 & -1 \end{pmatrix} \nabla(x) \wedge \nabla(y) \wedge \nabla(z)$$

$$= -(xyz)^{p-1}\nabla(x) \wedge \nabla(y) \wedge \nabla(z).$$

The coefficient monomial is now interpreted as a section of the line bundle $\mathcal{O}(3p-3)$ on the projective plane, and the intersection of F with the locus where this is zero is the subscheme defined by the ‘different’ element.

It follows that when working over $\mathbb{Z}[1/p]$, if F has a rational point then the different ideal of $\mathcal{O}_F$ is generated by $(xyz)^{p-1}$ which on each compopnent is a root of unity times an integer, and the integer does not depend on which component we are looking at.

2. Remark. In the way of explaining why we have obtained our different element as a restriction of a section of $\Omega(2F)$ when one should expect to use $\Omega(F)$, if we interpret $\mathcal{L}$ as the sheaf of rational functions with at worst simple poles on F, then the correspondence between homogeneous polynomials of degree p modulo the Fermat relations, and global sections of $\mathcal{L}$, is that a polynomial f corresponds to the rational function $\frac{f}{Ax^p+By^p}$. Thus when we multiplied our logarithmic derivative by the product of polynomials $(Ax^p + By^p)(Cx^p + Dy^p)$ the corresponding sections which we multiplied by were $(1)(\frac{Cx^p+Dy^p}{Ax^p+By^p})$. The numerator of the coefficient has divisor of zeroes F' disjoint from F and acts to multiply the residue by 1 since it restricts to 1 on F. The denominator has a simple zero on the Cartier divisor F; the product represents the same residue as the logarithmic derivative, though now on a pole of order two.

We can now state what we've proved.

3. Theorem. Let $X \subset \mathbb{P}^2$ be the Fermat curve defined by $x^p+y^p = z^p$ for p a prime number. For each $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sl_2(\mathbb{Z})$, let F be the fiber of $X \to \mathbb{P}^2 = \{[x^p : y^p]\}$ over the point defined by $Ax^p + By^p = 0$. Then

i) the scheme-theoretic zero locus of the residue $Res\ d\ log\ \frac{Ax^p+By^p}{Cx^p+Dy^p}$ on F agrees on the complement of the scheme $p = 0$ with the restriction to F of the scheme-theoretic zero locus of the section $(xyz)^{p-1}$ of the line bundle $\mathcal{O}_{\mathbb{P}^2}(3p-3)$ (which restricts to $\mathcal{L}^{\otimes 2} \otimes \omega_X$).

ii) If F has a rational point $[a : b : c]$ then each of x, y, z restricts on each irreducible component of F to an integer times a root of unity.

iii) Still assuming a rational point $[a : b : c]$, taking a, b, c pairwise coprime, the different ideal of $\mathcal{O}_F[1/p]$ is principal generated by the element $(abc)^{p-1}$ of the integers $\mathbb{Z}$ viewed as the characteristic subring.

iv) Still assuming a rational point $[a : b : c]$ with a, b, c coprime, if $s \in \mathbb{Z}$ is such that the scheme $Spec(\mathcal{O}_F[1/s])$ is not connected then abc is a divisor of a power of s.

Proof. We've proved all except iv) which will follow from first considerations of an intersection theory for the components of F.

It is interesting to consider the case of $p = 2$. Then the different element is precisely compatible with the intersections of the four components of F in the Pythagorean case. F is comprised of four copies of $Spec(\mathbb{Z})$ which each intersect the other three, one each, with intersection number a, b, c.

In general, when there is a rational point and $[a^p : b^p]$ is not one of the cusps, F has $(p+2)$ irreducible components corresponding to equivalence classes of solutions $[a\omega^i : b\omega^j : c\omega^k]$, and note that adding the same constant modulo p to (i, j, k) does not affect the solution point, nor does multiplying (i, j, k) by the same nonzero number modulo p, which acts as an automorphism on an irreducible component. Thus the $p+2$ irreducible components of F, when there is a rational point, correspond bijectively with orbits of the one dimensional affine group acting on F_p^3.

One naive type of intersection theory would associate to a pair of minimal prime ideals $P, Q \subset \mathcal{O}_F$ the number of elements in the cokernel of

$$\mathcal{O}_F \to \mathcal{O}_F/P \times \mathcal{O}_F/Q.$$

Our calculation of the different element (up to p torsion) would be no different if we had started with an equation which does have a solution such as

$$-8x^3 + 7y^3 = z^3.$$

This has the solution $[x : y : z] = [2 : 3 : 5]$, it still has different element $(2.3.5)^2 \in \mathcal{O}_F[1/3]$. Because it does have a rational solution, we can look at the naive intersection numbers.

We can consider any such cubic equation, obtained by modifying the Fermat equation by including fixed rational integer coeffficients of x^p and y^p so that it does have a solution, Writing $C_{i,j,k}$ for the irreducible component corresponding to the orbit including $[a\omega^i : b\omega^j : c\omega^k]$ we can calculate intersection numbers in a few examples. In terms of representatives for our subscript sequences, the five irreducible components of $\mathcal{O}_F$ are $C_{0,0,0}, C_{0,0,1}, C_{0,1,0}, C_{1,0,0}, C_{0,1,2}$. The first is a copy of $Spec(\mathbb{Z})$, and the others are copies of $Spec\mathbb{Z}[\omega]$ for ω a p'th root of unity.

From a few examples it seems that the naive intersection numbers in thise sense are

$$C_{0,0,0} \cdot C_{0,0,1} = 3c^3$$
$$C_{0,0,0} \cdot C_{0,1,0} = 3b^3$$
$$C_{0,0,0} \cdot C_{1,0,0} = 3a^3$$
$$C_{0,0,0} \cdot C_{0,1,2} = 3$$

$$C_{0,1,2} \cdot C_{0,0,1} = 3a^2b^2c$$
$$C_{0,1,2} \cdot C_{0,1,0} = 3a^2bc^2$$
$$C_{0,1,2} \cdot C_{1,0,0} = 3ab^2c^2$$

$$C_{0,0,1} \cdot C_{0,1,0} = 3a^2bc$$
$$C_{1,0,0} \cdot C_{0,1,0} = 3abc^2$$
$$C_{0,0,1} \cdot C_{1,0,0} = 3ab^2c.$$

To reiterate, we're considering a cubic equation of the type $qx^3 + ry^3 = z^3$ for q, r integers such that there is a solution with integers $[a : b : c]$

The fact that the different element belongs to the characteristic subring requires that for any prime s besides 3 which divides into any of a, b, c the intersection graph must remain connected when we replace all the intersection numbers by their s-adic valuation (or s primary part). The equations above verify that this is true in examples we've considered. The intersection numbers between fiber components and in fact the isomorphism type of $\mathcal{O}_F$ appear to be independent of the choice of q, r. The intersection numbers in examples have fixed expressions as monomials in a, b, c, p.

Let's restrict to the case p odd and write the equation $x^p + y^p + z^p = 0$, to make the Fermat curve more symmetrical. Also, let's use a simplified assignment of coordinates, taking $\lambda(\tau) = \frac{y^p}{z^p}$ without any factor of 16.

That is, any finite group which has a two dimensional representation has a corresponding action on the Riemann sphere, and here we consider $\mathbb{P}^1$ to be the projectivication of the two dimensional irreducible representation of the symmetric group $\mathbb{S}_3$. The group we take as acting by permuting the coordinate variables $[x : y : z]$.

The quotient of the Fermat curve X/S_3 is then 'modular' with group Γ so that

$$\Gamma(2)^{(p)} \subset \Gamma \subset \Gamma(1)$$

and it is index p^2 in $\Gamma(1)$, but not a normal subgroup. That is, the quotient $\Gamma(1)/\Gamma(2)^{(p)}$ is isomorphic to a semidirect product $F_p^2 \rtimes S_3$ and Γ is the inverse image of a subgroup copy of S_3 obtained by choosing a trivialization of the extension cocycle.

The map $X \to X/S_3$ covers the map $\mathbb{P}^1 \to \mathbb{P}^1$ which sends the λ invariant to the j invariant, at least if we parametrize things so that

$$j = \frac{(1+\lambda+\lambda^2)^3}{\lambda^2(1+\lambda)^2} = \frac{(x^{2p} + (xy)^p + y^{2p})^3}{(xyz)^{2p}}$$

To consider $j_0 = \frac{q}{s}$ with $q, s \in \mathbb{Z}$ we again choose A, B, C, D with $A = s, B = -q, AD - BC = 1$, as before the residue over the fiber at j_0 agrees in a neighbourhood of that fiber with

$$Res\ d\ log\ \frac{A(x^{2p} + (xy)^p + y^{2p})^2 + B(xyz)^{3p}}{C(x^{2p} + (xy)^p + y^{2p}) + D(xyz)^{3p}}.$$

In some sense, we know the answer already. If we invert 6 the fiber in X/S_3 over j_0 is the isomorphic image of the fiber in X of any one of the λ values mapping to j_0, and its different ideal must again be generated by $(abc)^{p-1}$ viewed as an element of the characteristic subring with p inverted, if there is a rational point.

But whereas this fiber has $(p+2)$ irreducible components in this case, its inverse image, the full fiber in X over j_0 now has $6(p+2)$ irreducible components, and so we expect the different element of the full fiber to have an additional factor which corresponds to intersections between fibers over different λ values.

The residue calculation is similar to what we have done already; one interesting thing is that the formula does not explicitly involve j_0 at all, meaning, there is again a polynomial expression representing a section of a line bundle, now $\mathcal{O}(13p-3)$, which defines in its restriction to the fiber over each value j_0 to the different subscheme of that fiber.

An issue is that the there is a line sub bundle of $\mathcal{P}(\mathcal{O}(6p))$ once we invert $6p$ which is 'spanned' by $\nabla(x^p+y^p+z^p)$

$$\mathcal{N} = \mathcal{O}(5p)\nabla(x^p+y^p+z^p).$$

Exactly the determinant which we already have calculated, in this case a Jacobian matrix made of the numerator and denominator of the fraction shown above along with the Fermat equation, is a generating global section of the line bundle made by wedging our global section of $\Lambda^2\mathcal{P}(\mathcal{O}(6p))$ with the whole line bundle $\mathcal{N}$, but then tensoring with the dual $\widehat{\mathcal{N}}$.

The generating section is $AD-BC$ which is 1, times

$$2p(xyz)^{2p-1}3(x^{2p}+(xy)^p+y^{2p})^2 det\begin{pmatrix} yz & xz & xy \\ 2px^{2p-1}+px^{p-1}y & 2py^{2p-1}+py^{p-1}x & 0 \\ x^{p-1} & y^{p-1} & z^{p-1} \end{pmatrix}$$

$$= 6p^2(xyz)^{2p-1}(x^{2p}+(xy)^p+y^{2p})^2(x^p-y^p)(x^p-z^p)(y^p-z^p).$$

Ignoring the factor of $6p^2$ we see that the different element for the six fibers viewed as disjoint, which was $(xyz)^{p-1}$ has now been multiplied by factors $x^py^pz^p(x^p-y^p)(x^p-z^p)(y^p-z^p)(x^{2p}+(xy)+y^{2p})^2$.

Let's show that all eight of the new factors, even with identical multiplicity (!) describe points where the six fibers intersect coming solely from congruences among values of the λ function.

4. Lemma Let a, b be coprime integers, and let $j \in \mathbb{P}^1/S_3$ be the image of $[a : b]$ in $\mathbb{P}^1$. Write $c = -a - b$. The fiber in $\mathbb{P}^1$ over j consists of the copies of $Spec(\mathbb{Z})$ indexed by $[a : b : c], [a : c : b], [b : a : c], [b : c : a], [c : a : b], [c : b : a]$, and the different element is represented on every irreducible component (after inverting 6) by the integer $6(abc)(a-b)(b-c)(c-a)(a^2+ab+b^2)^2 \in \mathbb{Z}[1/6]$.

Proof. This time the two by two determinant corresponds to the elements $(a^2 + (ab) + b^2)^3$ and $(abc)^2$. The coefficient of 6 may be removed since we are working over $\mathbb{Z}[1/6]$.

Next we will consider the diagram for X the Fermat curve

$$\begin{array}{ccc} X & \rightarrow & X/S_3 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \rightarrow & \mathbb{P}^1/S_3 \end{array}$$

The fiber over a rational point of $\mathbb{P}^1/S_3$ is an image of a tensor product of the fibers $\mathbb{P}^1$ and of X/S_3 over that point; this describes a map from a tensor product of an algebra of rank p^2 as an abelian group, and one of rank 6 as an abelian group. The map is injective and finite. Even locally, if the discriminant of the fiber is equal to the product of the discriminants, then the coordinate ring of the fiber decomposes as the tensor product precisely with no further partial normalization.

5. Remark. Another way of thinking about the explanation in Remark 2. for why $\mathcal{L}^{\otimes 2}$ occurred, is that is that we are constructing a section of the *relative* canonical sheaf, and the factor of $\mathcal{L}^{\otimes 2}$ represents the pullback of the inverse of the canonical sheaf of $\mathbb{P}^1$. That is, when we wrote $\mathcal{L}^{\otimes 2} \otimes \Omega_X$ we might as well have written $f^*\Omega_{\mathcal{P}^1}^{-1} \otimes \Omega_X$.

This analysis will continue in 'the meaning of positive and negative.'