5. Conclusion about modular forms

The conclusion of this sequence of notes about modular forms has to really only be an acknowledgement. Maybe it is now or never to write such a thing; there is no cable to charge this laptop, and the wi-fi signal is far away and weak, miles away for some reason. It is never any shame anyway if the actual content of a paper is just wrong, while also including an acknowledgement; but one should understand that this particular paper was written in five minutes, to say something that now seems easy about the Maths; but about how Maths is actually formulated, something deeper, relying also on an idea not many years ago by an undergraduate student who was here.

Three people from Queen Mary College have never had any acknowledgement from me, one is Charles Leedham-Green, we had conversations about logic, which went something like my claiming to have found the fastest-growing class of functions, and him eventually replying with something which in an obvious way could be increased, a detail left undone as idiotic as he could find.

Another is Peter Kropholler, I remember him sitting with a pint of beer with the chairman Roxburgh, when my time there was coming to an end, saying 'No one proves a great theorem, and then nothing for four or five years, and then proves another.' At the time I was privately angry, not recognizing the subtle way he had thus given me credit for his work, and a challenge to face the future.

Another is Robert Wilson, whom I hadn't met before, but had been there at Queen Mary, coming to Warwick last year to give a talk about his ideas about physics, the standard model, and recent theories in cosmology. This was intentionally incompetent (intentionally not making any connection with his work in Lie algebras), pretending to be superstitious, finding exact coincidences to many decimal places between ratios between constants in particle physics, and the number of days of the year, the effects of the moon on the earth, saying things along the lines that these relations between fundamental constants are caused by the tides. I haven't yet completely taken on board all the things which he has been saying; nor know who will.

My comments about modular forms were in answer to Cremona, Loeffler, Zerbes, Bruin, and Bartel at Warwick. The origin of an idea is a look, an expression during a conversation or passing in the hallway, in a very specific context among ambient conversations and ideas. When people say that there is a natural transformation between such and such, this is not to say that there is a pre-ordained transformation. But one can get enough confidence to assert something. The point is supposed to be that *anyone* can do that, people do it all the time, and there was once something like that, which was called the algebraic deRham theorem.

An integer form X of a modular curve has associated to it a topological space $X(\mathbb{C})$. The algebraic deRham theorem might come from Deligne wanting to solve a basic question of his advisor, or might be a common property as it is explained in Griffiths and Harris' book. Let's suppose that X is absolutely irreducible too, which I think means that $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}$.

The 'ordinary' or Eilenberg-Steenrod cohomology of $X \backslash cusps$, where $cusps$ is the finite set of cusps of X, is the cohomology of the sheaf of locally constant functions from $X(\mathbb{C}) \setminus cusps$ to the integers, also called the 'simplicial' or 'singular' cohomology, denoted $H^i(X(\mathbb{C}) \setminus cusps,\ \mathbb{Z})$. But it is known that this has an algebraic definition too.

If we revert to the older numbering of modular forms, this finitely-generated abelian group when i is 1 is the same as what is known as

$$M_2(X) \oplus \frac{M_4(X)}{M_2(2)M_2(X) + M_4(2)M_0(X)}$$

where $M_k(X)$ is modular forms of weight k and level X, as long as the group or 'level' of X is a subgroup of the congruence subgroup $\Gamma(2)$.

There is also another finitely-generated abelian group associated to X, this one depends on a choice of a rational (=integral) non-cusp point $p \in X(2) = X(\Gamma(2))$, and it is the underyling abelian group of the structure sheaf of the inverse image of p under the branched covering map $f_X : X \to \mathbb{P}^1$.

And this is a free abelian group of rank one smaller. Moreover we can just say that there is a natural map of abelian groups

$$H^1(X(\mathbb{C}) \setminus cusps, \mathbb{Z}) \to \mathcal{O}_{f_X^{-1}p},$$

which has a kernel free abelian of rank one.

In the case when f_X is the identity, so $X = X(2)$, this kernel is a rank one subgroup of a rank two lattice, as $H^1(X(2) \setminus cusps,\ \mathbb{Z})$ is a free abelian group of rank two.

And the identification of the rational point p with the rank one sublattice, or perhaps with the map itself which has that rank one sublattice as its kernel, is there, and a generator of the free abelian group of rank one is a modular form of weight two and level $X(2)$, which also then has any level above 2. Under one choice of the identification, if $p = [-b : a]$ with a, b relatively prime, then this modular form is

$$\gamma(p) = a\omega_0 + b\omega_1.$$

Here we can think of ω_0 and ω_1 as basic global sections of the sheaf of one-forms on $X(2) = \mathbb{P}^1$ allowed simple poles (=logarithmic poles since this is the one dimensional situation) on the three cusps.

Somehow, in terms of homogeneous coordinates u_0, u_1 these also can be written

$$\omega_0 = \frac{u_1}{u_1 - u_0} d(\frac{u_0}{u_1})$$

$$\omega_1 = \frac{u_0}{u_0 - u_1} d(\frac{u_1}{u_0}).$$

If we think of these as analytic functions $\mathbb{H} \to \mathbb{C}$ satisfying modularity of weight two they are

$$\omega_0 = \theta(0, \tau)^4$$

$$\omega_1 = \theta(1/2, \tau)^4,$$

and we append $d\tau$ if we want to think of the lifted one-forms on $\mathbb{H}$ itself.

So that a rational point of the projective line determines such an analytic function.

The class of a point in the projective line, in the cohomology of the compact complex manifold, can be identified with the isomorphism type of the torsor for the sheaf of one-forms, which consists of one-forms with a pole of residue exactly one at that point. This torsor is sometimes directly related to the $(1,1)$ form, in the notions of Kodaira, in this case describing the projective embedding which is the identity map.

That residue can be thought of as being a generator of the infinite cyclic quotient group that arises when we reduce $H^1(\mathbb{P}^1 \setminus cusps,\ \mathbb{Z})$ modulo the sublattice of rank one through $\gamma(p)$.

That is, we can think of the sheaf $\mathcal{O}_p$ as the free module of rank one consisting of all such residues, it is the image of the residue map from one forms with (logarithmic) poles at p and arbitrary poles and zeroes elsewhere, with kernel the one forms which do not have a pole at p.

When we pass to thinking about X instead of just $X(2)$, we can think of the coincidence

$$\text{covering degree} = rank\ H^1(X \setminus cusps,\ \mathbb{Z}) - 1$$

as coming from the fact it is still true on X that we have this residue map. The point here is that $M_2(X)$ modulo $\gamma(p)$ in the modular forms is the same as

$$M_2(X)\tau(p)\ modulo\ M_0(X)\gamma(p)\tau(p).$$

Here

$$\tau(p) = c\omega_0 + d\omega_1$$

with c, d chosen such that $ad - bc = 1$.

The choices of c, d are parametrized by $\mathbb{Z}$ analagous to the integers and the point at infinity on the boundary of the Poincare disk.

Now, we know that

$$M_4(X) = \frac{M_2(X)\gamma(p) \oplus M_2(X)\tau(p)}{M_0(X)\gamma(p)\tau(p)} \oplus H^1(X, \mathcal{O}_X)$$

or rather that it contains the first factor naturally with quotient being the second factor, so a splitting of the filtration describes such a direct sum decomposition.

The correspondence between modules over a graded ring and coherent sheaves can be made nice by thinking about graded modules as equivariant coherent sheaves on an affine variety, and here we are seeing that the even degree part of the ring $M(X)$ modulo the element $\gamma(p)$ is the coordiate ring of the fiber over p, this is the reduced fiber when p is not one of the three cusps.

If we think of $\gamma(p) = a\omega_0 + b\omega_1$ as a section of the sheaf of one forms on X with at most simple (=logarithmic) poles at the points of the fiber over p, and think of this as sections in the sense of sections of a vector bundle, then this section has a first principal part $\nabla(\gamma(p))$, a global section of first principal parts, and by reducing modulo $\gamma(p)$ and pulling back to the fiber we obtain a global section of first principal parts on the fiber. This happens to belong to the kernel of the map to the sheaf itself and can be interpreted as the restriction to the fiber of one forms with logarithmic poles.

Now, this is a little complicated but we can see through it! The line bundle is the one whose sections sheaf is isomrphic to one forms with at most simple (=logarithmic) poles on the *cusps*. But now we are taking sections which have simple (=logarithmic) poles on the *fiber*.

Remember that I said that the residues at the fiber, which comprise the quotient of $H^1(X \setminus cusps,\ \mathbb{Z})$ modulo that line in the lattice, are allowed to be residues of functions which can have arbitrary poles and zeroes away from the fiber. Here then we are allowed to have those poles at the cusps, because the fiber is disjoint from the cusps in $X(\mathbb{C})$ so contains no cusp in X.

The principal parts bundle is a rank two vector bundle, and $\nabla(\gamma(p))$ is not contained in the sub bundle which is one forms with poles allowed on the fiber. But when we restrict to the fiber it does end up belonging to that sub bundle for principal parts on the fiber. This is talking about how we have a principal different element for that ring of (perhaps not normal) integers.

From these considerations there is now an action, in principle determined by the structure of the ring $M(X)$, by which $\nabla(\gamma(p))$ can act on the quotient abelian group $H^1(X(\mathbb{C}) \setminus cusps,\ \mathbb{Z})/(\mathbb{Z}\gamma(p))$.

I have said other things, like about how X itself in the complex sense is a leaf of a foliated vector bundle, and so-on, and these are not really connected to what we have here. But what we have here is that there is an action L by which $L(\nabla\gamma(p))$ is a matrix acting on this quotient group.

I am purposely being adventurous now and imagining somehow, in analogy with the Weil conjectures, that we should lift of L to an endomorphism L_1 of $H^1(X(\mathbb{C}) \setminus cusps,\ \mathbb{Z})$; let L_0 be the induced action on $\mathbb{Z}\gamma(p)$ which I write as $M_0(X)\gamma(p) = H^0(X(\mathbb{C})\setminus cusps,\ \mathbb{Z})$, we have that L_i assigns an integer matrix once a cohomology basis is chosen, to each cohomology group, and we are contriving that the determinant $det(L)$ is a ratio of two determinants. Then

Theorem.

$$\prod_{i=0}^{1} det(L_i(\nabla\gamma p))^{(-1)^i} = \frac{1}{\prod_{j=1}^{m} disc(\mathcal{O}_j/\mathbb{Z})}$$

where *disc* means discriminant and $\mathcal{O}_j$ is the structure ring of the j'th connected component of the scheme theoretic fiber of X over p, and m is the number of connected components.

The left side we've written looking like a multiplicative Lefshetz character; when the theorem implies that the points p where its absolute value is 1 are exactly those points above which the points of X are scheme-theoretically disjoint rational (=integer) points.

And

$$| \ [R : \mathcal{O}_{f_X^{-1}p}]^2 \ \prod_{i=0}^{1} det(L_i(\nabla \gamma p))^{(-1)^i} \ |= 1$$

if and only if all $\mathbb{C}$ points above p are rational, where $\mathcal{O}_{f^{-1}p} \subset R = \prod_{k=1}^{s} \mathcal{O}_{f_X^{-1}p}/P_k$, with $P_1, ..., P_s$ the minimal prime ideals and s the number of irreducible components.

In the case when X is suituably Galois, all the $\mathcal{O}_i$ are isomorphic. Then the left side as a function of p precisely determines the set of rational non-cusps of X.

To give an algebraic formula for the left side would amount to considering the graph of the map from X to $\mathbb{P}^1$, and residues on the graph. This can obviously be done if the ring structure of $M(X)$ is known. We've given a Lefshetz formula for the second factor on the left side. For the first factor, under the relations $\gamma(p) = 0, \tau(p) = 1$ the whole of $M(X)$ retracts isomorphically to $\frac{M_4(X)}{M_2(X)\gamma(p)}$ which becomes the affine coordinate ring of the fiber over p and determines this factor too.

The dependence on the level Γ may not be a Σ_0 formula; any algebraic or topological formalism is meant to be only an allegory. For example, the fact that modular forms are complex analytic entities means that the search for rational points can take place in the domain of analytic number theory. Though that was clear at the outset too.