E9 222
Signal Processing in Practice

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भारतीय विज्ञान संस्थान

Convolution

Convolution: The Standard View

1D Linear Convolution:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

1D Circular Convolution (length N):

$$y[n] = (x \circledast h)[n] = \sum_{k=0}^{N-1} x[k]h[(n-k)_N]$$

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- Convolution is linear: $y[n] = (x * h)[n] \quad \equiv \quad \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$
- $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ can only represent finite sequences $\rightsquigarrow$ How to define $\boldsymbol{A}$?

Handling Boundaries

- Signal $x[n]$ has length N
- Filter $h[n]$ has length M

What is the size of the output?

1. **Full Mode**
   - **Output:** $N + M - 1$
   - Assumes zero-padding
2. **Same Mode**
   - **Output:** N
   - Centre crop of 'Full'
3. **Valid Mode**
   - **Output:** $N - M + 1$
   - Filter *fully overlaps*

Full Mode Linear Convolution

Linear convolution represented as a **Toeplitz matrix**

- $h = [h_0, h_1, h_2]$, $M = 3$
- $x = [x_0, x_1, x_2, x_3, x_4]$, $N = 5$
- The output y has length $5 + 3 - 1 = 7$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \\ 0 & 0 & 0 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & h_2 \end{bmatrix}}_{\mathbf{T}\ \text{(Toeplitz)}} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Same Mode Linear Convolution

- $h = [h_0, h_1, h_2]$, $M = 3$
- $x = [x_0, x_1, x_2, x_3, x_4]$, $N = 5$
- The output y has length 5

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \\ 0 & 0 & 0 & h_2 & h_1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Valid Mode Linear Convolution

- $h = [h_0, h_1, h_2]$, $M = 3$
- $x = [x_0, x_1, x_2, x_3, x_4]$, $N = 5$
- The output y has length $5 - 3 + 1$

$$\begin{bmatrix} y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Circular Convolution $\Leftrightarrow$ Linear Convolution + Periodic Extension

What does the modulo operator $(n-k)_N$ actually imply?

$$\underbrace{\begin{bmatrix} h_0 & 0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \\ \hline 0 & 0 & 0 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & h_2 \end{bmatrix}}_{\text{Linear (Toeplitz)}} \xrightarrow{\text{Wrap Tails}} \underbrace{\begin{bmatrix} h_0 & 0 & 0 & \mathbf{h_2} & \mathbf{h_1} \\ h_1 & h_0 & 0 & 0 & \mathbf{h_2} \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix}}_{\text{Circular (Circulant)}}$$

Circular Convolution

Circular convolution maps $\mathbb{R}^N \to \mathbb{R}^N$
The operator is a **circulant matrix**

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & 0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & h_2 & h_1 & h_0 \end{bmatrix}}_{\mathbf{C}\ \text{(Circulant)}} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Each row is a cyclic shift of the previous one (periodic wrap-around).

Eigenstructure of Circulant Matrices

Theorem

Every circulant matrix $\mathbf{C}$ *is diagonalized by the DFT matrix* $\mathbf{F}$*, i.e.*

$$\mathbf{C} = \mathbf{F}^{-1}\mathbf{\Lambda}\mathbf{F}$$

where $\mathbf{\Lambda} = \textit{diag}(\lambda_0, \ldots, \lambda_{N-1})$

The eigenvalues λ_k are exactly the DFT coefficients of the filter h.

$$\lambda_k = H[k] = \sum_{n=0}^{N-1} h[n] e^{-\mathrm{j}2\pi nk/N}$$

Zeros in the DFT and Rank of the Convolution Matrix

Since the eigenvalues are the DFT coefficients $H[k]$

$$\det(\mathbf{C}) = \prod_{k=0}^{N-1} \lambda_k = \prod_{k=0}^{N-1} H[k]$$

- If $H[k] \neq 0$ for all k, then $\det(\mathbf{C}) \neq 0$
  The matrix is **full rank**
  The convolution is invertible
- If $H[k] = 0$ for some k, the matrix is **singular**

The Cocktail Party Problem

Reverberation/Echo
The model:

$$y[n] = x[n] * h[n] + \eta[n]$$

The goal: Recover $x[n]$ given $y[n]$ and $h[n]$
This is known as **deconvolution** or **channel equalization**

The Inverse Filter

Naive solution in Fourier domain

$$Y(\omega) = X(\omega)H(\omega) + N(\omega)$$

$$\hat{X}(\omega) = \frac{Y(\omega)}{H(\omega)} = X(\omega) + \frac{N(\omega)}{H(\omega)}$$

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The Noise Amplification Problem

Typically, $H(\omega) \to 0$ as $\omega \to \pi$

Therefore, $\dfrac{N(\omega)}{H(\omega)} \to +\infty$

The Wiener Filter

Instead of perfect inversion, we minimize the mean-square error

$$\hat{X}(\omega) = G(\omega)Y(\omega)$$

The optimal filter $G(\omega)$ is:

$$G(\omega) = \frac{H^*(\omega)}{|H(\omega)|^2 + \text{NSR}(\omega)}$$

where $\text{NSR} = \frac{S_{nn}(\omega)}{S_{xx}(\omega)}$ is the noise-to-signal ratio

Intuition:

- Where SNR is high ($|H(\omega)|^2 \gg \text{NSR}$), $G \approx 1/H$ (Inverse Filter)
- Where SNR is low ($|H(\omega)|^2 \ll \text{NSR}$), $G \approx 0$ (Attenuate Noise)

Discrete Cosine Transform (DCT)

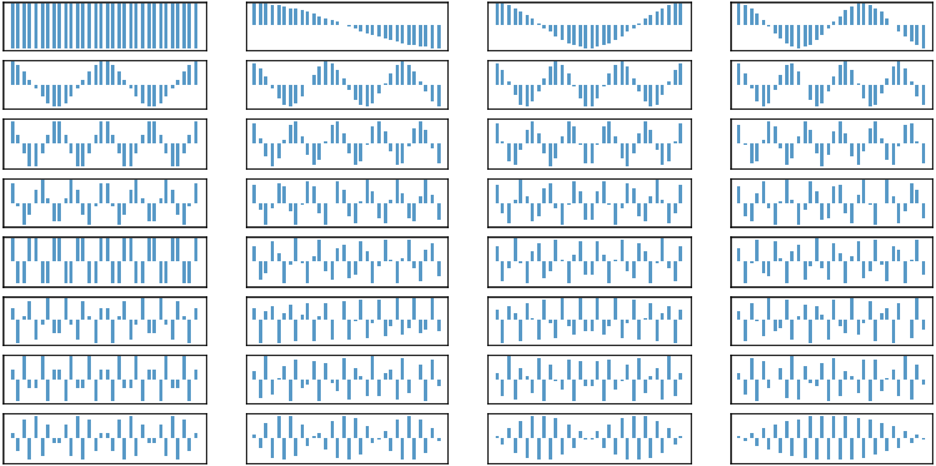
What does the DCT do?

Key idea

Project a signal onto a set of **cosine basis functions**

- Real-valued
- Ordered from smooth to oscillatory
- Energy concentrates in low indices

1D DCT basis functions



DCT-II definition

For a signal $x[n]$, $n = 0, \ldots, N-1$:

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n] \cos\left(\frac{\pi}{N}(n + \tfrac{1}{2})k\right)$$

- Linear transform
- Orthogonal basis
- Perfect reconstruction/Unitary transform

Matrix form: $\hat{\boldsymbol{x}} = \boldsymbol{D}\boldsymbol{x}$

From 1D to 2D

For a signal $x[m,n]$

$$\hat{x}[k,\ell] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x[m,n] \cos\left(\frac{\pi}{M}(m+\tfrac{1}{2})k\right) \cos\left(\frac{\pi}{N}(n+\tfrac{1}{2})\ell\right)$$

Separability

2D DCT $=$ DCT on rows, then columns

$$\hat{\boldsymbol{x}} = \boldsymbol{D}\boldsymbol{x}\boldsymbol{D}^\top$$

2D DCT basis functions