Date :

Assignment 4

3) $A = m \times n$ matrix $(m \leq n)$.

$P = A^T A$, $Q = AA^T$ (all values real).

$(n \times n)$ $\quad$ $(m \times m)$

a) Let y be an $n \times 1$ vector.

$\therefore$ To prove, $y^T P y \geq 0$

We have $y^T P y$

$= y^T (A^T A) y = y^T A^T A y$

$= (Ay)^T (Ay)$

Now, Ay is an $m \times 1$ vector.

$\therefore (Ay)^T (Ay) = \|Ay\|_2^2$

$\|Ay\|_2^2 \geq 0$ (sum of squares of real numbers)

$\therefore y^T P y \geq 0$, Hence proved

Let z be an $m \times 1$ vector.

We have $z^T Q z$ (a scalar)

$= z^T A A^T z$

$= (A^T z)^T (A^T z)$

Now, $A^T z$ is an $n \times 1$ vector

$\therefore (A^T z)^T (A^T z) = \|A^T z\|_2^2$

$\|A^T z\|_2^2 \geq 0$ (sum of squares of real numbers)

$\therefore z^T Q z \geq 0$

P and Q are symmetric matrices

$P^T = (A^TA)^T = A^TA = P$, $\;$ \& $\; Q^T = (AA^T)^T = AA^T = Q$

The eigen values of a symmetric matrix are real.

Proof: Let $\quad Mx = \lambda x$

Let $\bar{x}, \bar{\lambda}$ be the complex conjugates of x and λ

$\because$ x is real valued, so is M

$\therefore \quad \bar{x} = x$

$\therefore \quad \overline{Mx} = \overline{\lambda x} \quad \Rightarrow \quad M\bar{x} = \bar{\lambda}\bar{x} \quad \therefore \quad Mx = \bar{\lambda} x$

$\therefore \quad \lambda = \bar{\lambda}$.

As the eigen values of P and Q are real, we can use them in inequalities.

Let u be an eigenvector of P, (u is $n \times 1$)

We know $\; y^T P y \geq 0 \quad \forall \vec{y}$

$\therefore \quad u^T P u \geq 0$

Let $\quad Pu = \lambda u \quad$ (eigenvector u, eigen value λ

$\therefore \quad u^T \lambda u \geq 0$

$\therefore \quad \lambda u^T u \geq 0 \quad$ (λ is a scalar, can be taken out)

$\lambda \|u\|^2 \geq 0$

$\forall u \neq \vec{0}$, $\quad \|u\|^2 > 0$

$\therefore \quad \lambda \geq 0 \quad \forall \vec{u} \neq 0$

$\therefore$ For any eigenvalue λ, we see that it is always non-negative when it corresponds to a non-zero eigenvector.

we know: $z^T Q z \geq 0 \quad \forall \vec{z}$

$\therefore$ let $\vec{v}$ be an eigenvector of Q with eigenvalue μ $\quad (Qv = \mu v)$

$\therefore z^T Q z \geq 0 \Rightarrow$ ~~$z^T\mu$~~ $\quad v^T Q v \geq 0$

$\therefore v^T \mu v \geq 0$

$\therefore \mu v^T v \geq 0 \Rightarrow \mu \|v\|^2 \geq 0$

$\therefore$ if $\vec{v} \neq \vec{0}$, $\|v\|^2 > 0$

$\therefore \mu \geq 0$

$\therefore$ All the eigenvalues of Q are also non-negative.

b) i) $Pu = \lambda u$ $\quad (P: n\times n,\ u = n\times 1)$

$\therefore$ $A^T A u = \lambda u$

Premultiply by A (valid, as we have $n\times 1$ vectors on both sides)

$\therefore$ $AA^T Au = A(\lambda u)$

$AA^T Au = \lambda Au$ (as λ is a scalar.

$\because$ $AA^T = Q$, Au is an $m\times 1$ vector, say u'

$\therefore$ $Qu' = \lambda u'$

$\therefore$ u' is an eigenvector of Q with value λ

$\therefore$ Au is an eigenvector of Q, eigenvalue λ.

ii) $Qv = \mu v$ $\quad (Q = m\times m,\ v = m\times 1)$

$\therefore$ $AA^T v = \mu v$

Premultiply by A^T (valid as we have $m\times 1$ vectors on both sides and A^T is $(n\times m)$)

$\therefore$ $A^T AA^T v = A^T \mu v$

$\therefore$ $(A^T A)(A^T v) = \mu (A^T v)$ (μ is scalar)

$A^T A = P$ and $A^T v$ is an $n\times 1$ vector

$\therefore$ $P(A^T v) = \mu (A^T v)$

$\therefore$ $A^T v$ is an eigenvector of P with eigenvalue μ.

$\vec{u}$ has 'n' elements

$\vec{v}$ has 'm' elements.

c) Given: v_i is an eigenvector of Q

$\therefore$ Let $Q\vec{v}_i = \mu_i \vec{v}_i$ (μ_i is a scalar)

Given $u_i \doteq \dfrac{A^T v_i}{\| A^T v_i \|_2}$ $\Rightarrow$ u_i is an $n \times 1$ vector ($n \times m \times m \times 1$)

Premultiply by $m \times n$ matrix A:

$$Au_i = \frac{AA^T v_i}{\| A^T v_i \|_2} = \frac{Q v_i}{\| A^T v_i \|_2}$$

$$\therefore Au_i = \frac{\mu_i}{\| A^T v_i \|_2} v_i$$

From part ⓐ, $\mu_i \geq 0$, as eigenvalues of P & Q are non-negative.

$\| A^T v_i \|_2 \geq 0$, as it is the sum of squares of real numbers.

$\therefore$ Let $\gamma_i \doteq \dfrac{\mu_i}{\| A^T v_i \|_2}$

$\therefore$ $Au_i = \gamma_i v_i$, where $\gamma_i \geq 0$

Hence Proved

d) Given: $U = \left[\vec{v}_1 \,|\, \vec{v}_2 \,|\, \dots \,|\, \vec{v}_m\right]$, $V = \left[\vec{u}_1 \,|\, \vec{u}_2 \,|\, \dots \,|\, \vec{u}_m\right]$

here $\vec{v}_i$ is an eigenvector of Q (AA^T)
and since $\vec{u}_i = \alpha A^T \vec{v}_i$, from (b)
$\vec{u}_i$ is an eigenvector of P (A^TA)
($m \leq n$, so at max, only 'm' rank of A,
hence rank(P) and rank(Q) $\leq m$ also, so
a maximum of 'm' eigenvectors for either of them.

Given: U and V are orthonormal matrices $(u_i^T u_j = 0, v_i^T v_j = 0)$
A is $m \times n$ $(m \leq n)$
Γ is a diagonal matrix with non-negative $\gamma_1, \gamma_2, \gamma_3, \gamma_m$

Lemma 1: A, A^TA, AA^T (P, Q) have the same rank.

Proof:

say $Ax = 0$
$\Rightarrow A^TAx = 0$
$\therefore x \in N(A^TA)$
$\therefore N(A) \subseteq N(A^TA)$

say $A^TAx = 0$
$\therefore x^TA^TAx = 0$
$(Ax)^T Ax = 0$
$\Rightarrow Ax = 0$
$\therefore x \in N(A)$
$\therefore N(A^TA) \subseteq N(A)$

$N(A) \subseteq N(A^TA)$ and $N(A^TA) \subseteq N(A) \Rightarrow N(A) = N(A^TA)$
$\therefore$ Dimensionality of null space are equal
$\therefore$ rank(A) = rank(P) = rank(Q)

We know that rank(P), rank(Q) = m, because there exist 'm' distinct, orthonormal eigenvectors.

$\therefore$ We have, rank(A) = m also (full row rank).

Hence, we have: $u_i = \frac{A^T v_i}{\|A^T v_i\|_2}$ $\forall i \in 1 \ldots m$

(non of the values are 0).

v_i is $m \times 1$ and u_i is $n \times 1$.

$\therefore$ We have the system of 'm' equations:

$$u_i = \frac{A^T v_i}{\|A^T v_i\|_2} \quad i \in 1 \ldots m.$$

$\therefore$ ~~Ir~~ $\therefore$ $A^T v_i = u_i \|A^T v_i\|_2$

Writing these in matrix form

$$A^T U = V \Gamma$$

$$(n \times m) \times (m \times m) = (n \times m) \times (m \times m)$$

$v_1, v_2 \ldots v_m$ are columns of U

$u_1, u_2 \ldots u_m$ are columns of V

Γ is a diagonal matrix with non-negative values $\|A^T v_i\|_2$ (norms are non-negative)

(let these be the γ_i's required).

v_i's and u_i's are unit vectors.

$\therefore$ U, V are orthonormal.

$$U^T U = \left[\langle v_i, v_j \rangle \right] = I$$

$\therefore$ $U^T = U^{-1}$

$$\therefore \quad A^T U = V\Gamma$$

Taking transpose: $\quad (A^T U)^T = (V\Gamma)^T$

$$\therefore \quad U^T A^{TT} = \Gamma^T V^T$$

$$\therefore \quad U^T A = \Gamma V^T \qquad (\text{as } \Gamma \text{ is symmetric}, \ \Gamma = \Gamma^T)$$

$\therefore$ Premultiply by U

$$U U^T A = U \Gamma V^T$$

But $\quad U^T = U^{-1}$

$$\therefore \quad U U^{-1} A = U \Gamma V^T$$

$$\therefore \quad \boxed{A = U \Gamma V^T}$$

Hence, there exists a singular value decomposition for the matrix A, with the required matrices U, Γ and V.