Simulataneous Dilation and Translation Tilings of $\mathbb{R}^n$

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March 23, 2024

ELEMENTARY WAVELETS AND THE INTO C(K) EXTENSION PROPERTY

A Dissertation

by

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Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 1997

Major Subject: Mathematics

Tilings

Definition

A collection of measurable sets $\{W_j \subset \mathbb{R}^n : j \in J\}$ is a *measurable tiling* of $\mathbb{R}^n$ if

- $m(W_j \cap W_{j'}) = 0$ whenever $j \neq j'$, and
- $\bigcup_{j \in J} W_j = \mathbb{R}^n$ up to a set of measure zero.

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Definition

Let A be an invertible, $n \times n$ matrix and let $\Gamma \subset \mathbb{R}^n$ be a full-rank lattice. An (A, Γ) *wavelet set* is a measurable set W such that

- $\{A^j(W) : j \in \mathbb{Z}\}$ is a measurable tiling of $\mathbb{R}^n$ and
- $\{W + k : k \in \Gamma\}$ is a measurable tiling of $\mathbb{R}^n$.

Examples

1. Let $A = 2$ and $\Gamma = \mathbb{Z}$. The set $W = [-1, -1/2] \cup [1/2, 1]$ is an $(A, \mathbb{Z})$ wavelet set.

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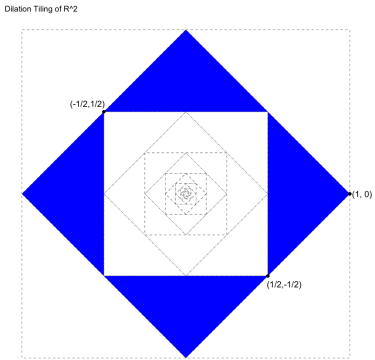
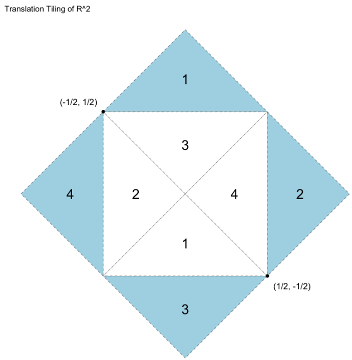
2. Let $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The shaded set below is an $(A, \mathbb{Z}^2)$ wavelet.





Relationship to wavelets

If W is an (A, Γ) wavelet set and ψ is the inverse Fourier transform of I_W, then ψ is a (B, Γ^*) orthogonal wavelet. That is,

$$\{|\det B|^{j/2}\psi(B^j \boldsymbol{x} + \boldsymbol{k}) : \boldsymbol{j} \in \mathbb{Z}, \boldsymbol{k} \in \Gamma^*\}$$

is an orthogonal basis for $L^2(\mathbb{R}^n)$. Here, $B = A^T$ and Γ^* is the dual lattice of Γ.

Question

Question For which pairs (A, Γ) does there exist a wavelet set?

Source of Question

CHAPTER VI

CONCLUSION

In Chapter II, it is shown that for any expansive matrix D, there exist wavelet sets with respect to D. Somewhat surprisingly, an example was recently constructed by the author which shows that being expansive is not a necessary condition on a matrix D in order for there to exist wavelet sets. This answers a question posed in [ILP]. A natural question, then, would be to characterize all matrices for which there are wavelet sets. Similar types of questions have been considered by Lagarias and Wang [LW], where they have attempted to characterize all matrices for which there exist Haar type wavelets.

Me in 1995

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Previous Results

- If $|\det A| = 1$, then no (A, Γ) wavelet set exists. Can and will assume $|\det A| > 1$.
- (Dai-Larson-S 1997) If all eigenvalues of A are greater than 1 in modulus, then (A, Γ) wavelets exist.
- (S 2004) If A is 2×2 and both eigenvalues are greater than or equal to one in modulus, then (A, Γ) wavelets exist.

Ionascu-Wang (2006)

Theorem

Let A be 2×2 matrix with $|\det A| > 1$ and let Γ be a full rank lattice in $\mathbb{R}^2$. Let λ_1 and λ_2 be the eigenvalues of A such that $|\lambda_1| \geq |\lambda_2|$. There exists an (A, Γ) wavelet set if and only if

- $|\lambda_2| \geq 1$, *or*
- $|\lambda_2| < 1$ *and* $\ker(A - \lambda_2 \mathbf{I}) \cap \Gamma = \{0\}$.

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Remark

The key to (IW) was showing that $\ker(A - \lambda_2 \mathbf{I}) \cap \Gamma = \{0\}$ *implies that* $\liminf \# \left(A^{-j}\left(B(0,1)\right) \cap \Gamma\right) = 1$. *It turns out these are equivalent in* $\mathbb{R}^2$

Key Object

Remark

The key object of study is the limiting behavior as $j \to \infty$ of

$$\# \left(A^{-j}(B(0,R) \cap \Gamma\right)$$

Higher Dimension Different

First noticed by Khintchine (1926), things are very different dimension 3 than dimension 2. The characterizing condition (ii) in the previous Theorem has several possible restatements in higher dimensions when the smallest eigenvalue of A is less than one in modulus. For example, these four statements are all equivalent in dimension $n = 2$ when $|\det A| > 1$.

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- for every sublattice $\Lambda \subset \Gamma$, if $V = \operatorname{span}(\Lambda)$ and $d = \dim V$, then

$$\liminf_{j\to\infty} m_d(A^{-j}(\mathbf{B}(0,1)) \cap V) < \infty,$$

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  where m_d denotes the Lebesgue measure on the subspace V,
- if V is the space spanned by the eigenvectors associated with eigenvalues less than one in modulus, then $V \cap \Gamma = \{0\}$.

Main Result

Theorem (Bownik-S, Am J Math, to appear)

Let A be an $n \times n$ matrix with $|\det A| > 1$. Let $\Gamma \subset \mathbb{R}^n$ be a full rank lattice. Then, there exists an (A, Γ) wavelet set if and only if

$$\sum_{j=1}^{\infty} \frac{1}{\# \left| A^{-j}(\mathbf{B}(0,1)) \cap \Gamma \right|} = \infty,$$

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Remark

The condition above is in between

for every $R > 0$, $\liminf_{j\to\infty} \#|A^{-j}(\mathbf{B}(0,R)) \cap \Gamma| < \infty$,

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Application

Theorem (Bownik-S)

Let A be an $n \times n$ matrix such that $|\det A| > 1$ and all eigenvalues of A are greater than or equal to one in modulus. Then, for every full rank lattice Γ, there exists an (A, Γ) wavelet set.

Application

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Remark

This is the largest class of matrices A with $|\det A| > 1$ for which (A, Γ) wavelet set exists regardless of the choice of the lattice Γ.

Key Ingredients

Theorem (Margulis 1971)

Let Γ be a full rank lattice in $\mathbb{R}^n$ and let U be a unipotent matrix (all eigenvalues 1). There exists $\delta > 0$ such that

$$\liminf_{j\to\infty} \#(U^{-j}B(0,\delta)\cap\Gamma) = 1.$$

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Theorem (Cheshmavar-Führ (2020))

Suppose that A is an $n \times n$ invertible matrix. Then, there exists a positive constant c and an $n \times n$ matrix $\tilde{A}$ such that all eigenvalues of $\tilde{A}$ are positive and

$$A^j(\mathbf{B}(0,r/c)) \subset \tilde{A}^j(\mathbf{B}(0,r)) \subset A^j(\mathbf{B}(0,cr)) \qquad \text{for all } j \in \mathbb{Z},\ r > 0.$$

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Remark

Cheshmavar-Führ allows us (essentially) to restrict to matrices with ***positive*** *eigenvalues.*

Proof

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3. Theorem [M] implies there exists $\epsilon > 0$ such that $\liminf \# \left(T^{-j}(B(0, \epsilon)) \cap \Gamma \right) = 1$.

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3. Theorem [M] implies there exists $\epsilon > 0$ such that $\liminf \# \left(T^{-j}(B(0,\epsilon)) \cap \Gamma\right) = 1$.
4. For large j, $A^{-j}(B(0,\epsilon)) \subset T^{-j}(B(0,\epsilon))$, so

$$\liminf \#(A^{-j}(B(0,\epsilon)) \cap \Gamma) = 1$$

and $\sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(\mathbf{B}(0,\epsilon)) \cap \Gamma|} = \infty$. Thus, (A, Γ) wavelet sets exist.

Summary

- Understanding the limiting behavior of $\#\left(A^{-j}B(0,1)\cap\Gamma\right)$ is crucial to understanding when (A,Γ) wavelet sets exist.

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Summary

- Understanding the limiting behavior of $\#\left(A^{-j}B(0,1)\cap\Gamma\right)$ is crucial to understanding when (A,Γ) wavelet sets exist.
- There are theorems and ideas developed over the last 100 years which are useful to understand this.
- Wavelet sets exist if and only if $\sum_{j=1}^{\infty}\frac{1}{\#|A^{-j}(\mathbf{B}(0,1))\cap\Gamma|}=\infty$

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- Understanding the limiting behavior of $\# \left(A^{-j}B(0,1) \cap \Gamma\right)$ is crucial to understanding when (A, Γ) wavelet sets exist.
- There are theorems and ideas developed over the last 100 years which are useful to understand this.
- Wavelet sets exist if and only if $\sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(\mathbf{B}(0,1))\cap\Gamma|} = \infty$
- Conjecture: (A^T, Γ^*) orthogonal wavelets exist if and only if $\sum_{j=1}^{\infty} \frac{1}{\#|A^{-j}(\mathbf{B}(0,1))\cap\Gamma|} = \infty$