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Metric space change point detection

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Abstract Let $(x_t)_{t=1}^N$ be a time series with values in a metric space X, which is locally isometric to Euclidean space. A transformation of the data is proposed which produces a multi-dimensional time series of real numbers. If the original sequence of data points has a single change point in mean at time t_0, then with high probability the transformed data will also have a single change point in mean at time t_0. Applications to time series of persistence diagrams are considered.

Keywords change point detection · metric space · persistence · tda ·

Mathematics Subject Classification (2000) MSC code 1 · MSC code 2 ·

1 Introduction

A simple change point detection problem is the following: suppose that $(x_t)_{t=1}^{t_0}$ are iid Gaussian random variables with mean μ_0 and variance σ, and $(x_t)_{t=t_0+1}^N$ are iid Gaussian with mean μ_1 and variance σ. The goal is to determine whether $\mu_0 = \mu_1$, and if not, then what the value of t_0 is.

Grants or other notes about the article that should go on the front page should be placed here. General acknowledgments should be placed at the end of the article.

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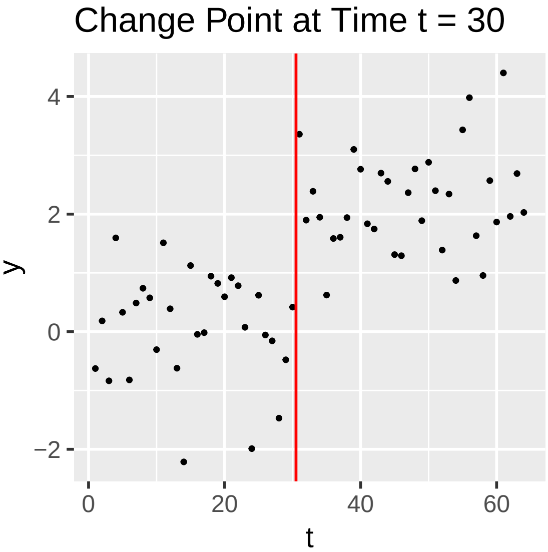


More general problems include having multiple change points, noise that is not Gaussian, underlying signals that are not constant, and multi-dimensional signals.

In this paper, we consider the following set-up. Let X be a metric space. Supose that $f : [0, 1] \to X$ is a piecewise continuous function with at most one discontinuity. Let $(a_t)_{t=1}^{N}$ be an increasing sequence of numbers in $(0, 1)$ and $\delta > 0$ such that for each $y \in [0, 1]$, $B_\delta(f(y))$ is isometric to Euclidean space. Let $x_t = f(a_t) + \epsilon_t$, where ϵ_t is a uniformly distributed random variable on $B_\delta(f(a_t))$. Our problem is to determine whether there exists a point of discontinuity y_0 of f and a $k < N$ such that the discontinuity is contained between a_k and a_{k+1}. If there does exist such a k, then we should also estimate it.

Our motivation is that we wish to find change points in time series of **persistence diagrams**. Persistence diagrams are a way of measuring the topological structure of a point cloud in Euclidean space. More here.

2 Metric space change point detection algorithm

With the notation as set up in Section 1, we proceed to describe the algorithm for change point detection for time series data in metric spaces. The main step is transforming the data into a multi-dimensional time series of real numbers in such a way as to preserve the change point, see Algorithm 1.

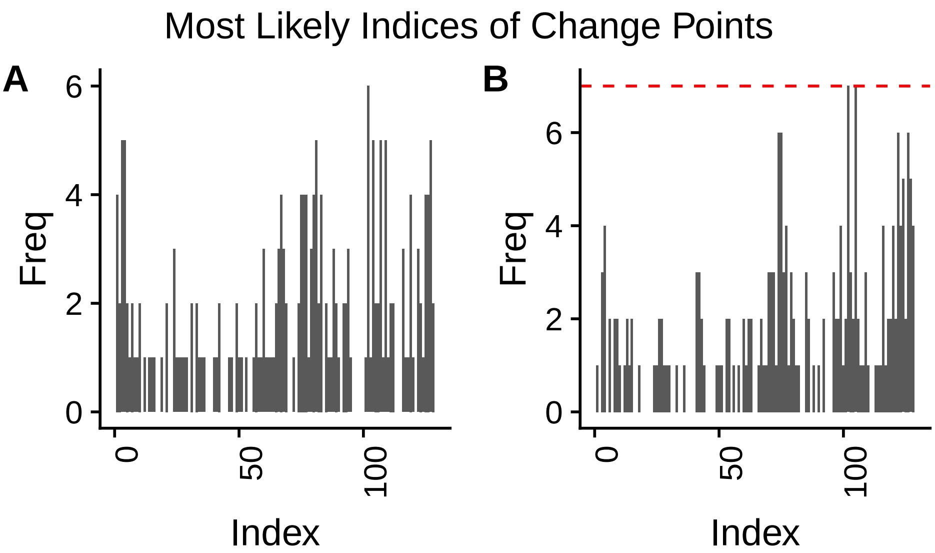
Once the metric space valued time series is transformed into a multi-dimensional real valued time series, standard techniques can be used to determine if and where a change point occurs. In this paper, we use the `cpbaywave` package, as it finds change points in high dimensional, smooth data with a single discontinuity. We illustrate the usage with some examples.

Algorithm 1 Transform to Real

```
procedure MyProcedure
    $time\_length$ ← length of $time\_series$
    $i \leftarrow 1$
    while $i$ ¡ $time\_length$ do
        $A,\ B \leftarrow$ sample(N, 2, replace = FALSE)
        for $j \leftarrow 1 : time\_length$ do
            $dists[j,i] \leftarrow (d(A, x_j)^2 - d(B, x_j)^2)/d(A, B)$
        $i{+}{+}$
```

\begin{example} Suppose that the time series lives in $\mathbb{R}^M$ for some M. For example, we could have a time series of length 128 in 100 dimensional data, with a change point at time $t = 80$. We consider two examples in this case; in both examples, the data has mean zero in all dimensions until time 80. In the first case, it has mean 0.1 in all dimensions from dimension 81 through 128 and in the second case it has mean 1 and all dimensions from dimension 81 through 128.

In the case that the mean increases to 0.1, the algorithm is not able to locate the change point, and incorrectly determines that there is no change. In the case that the mean increases to 1, the algorithm strongly indicates that the change point is at time $t = 80.



The following plot indicates that a change point is detected in both the original and the transformed data at time $t = 80$.

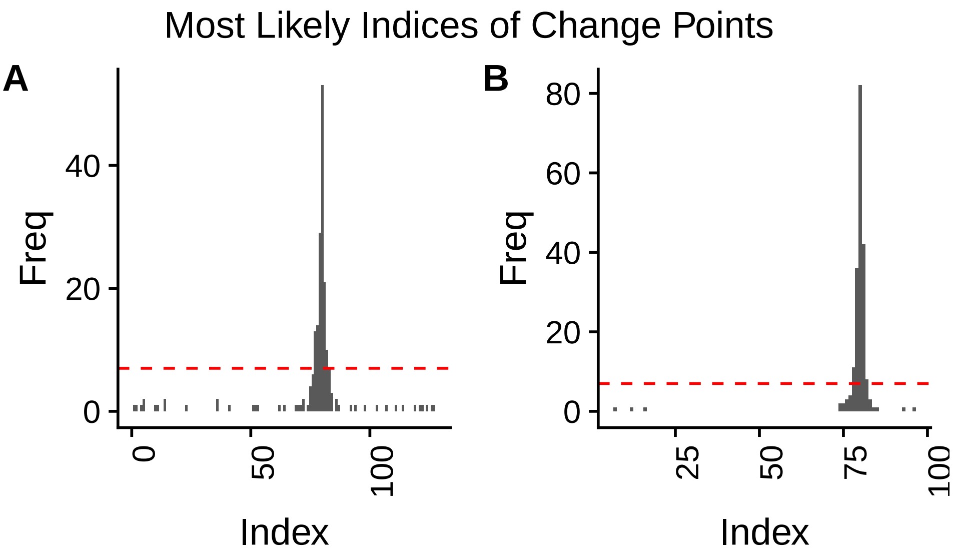


Table 1: Indices which are significant, together with their frequency. True change point at t = 80.

Raw Data		Transformed Data	
Index	Freq	Index	Freq
77	13	78	11
78	14	79	36
79	29	80	82
80	53	81	42
81	21	82	8
82	10		

This picture gives a typical strong indication that there is a change point at or near time $t = 80$, which we know is the correct value.

We note here that the `cpbaywave` algorithm also fails to detect a change point in the first example when using the untransformed data. \end{example}

Next, we consider the same change point scenario as above, but we imagine the time series living in ℓ^4_{100} rather than in ℓ^2_{100}.

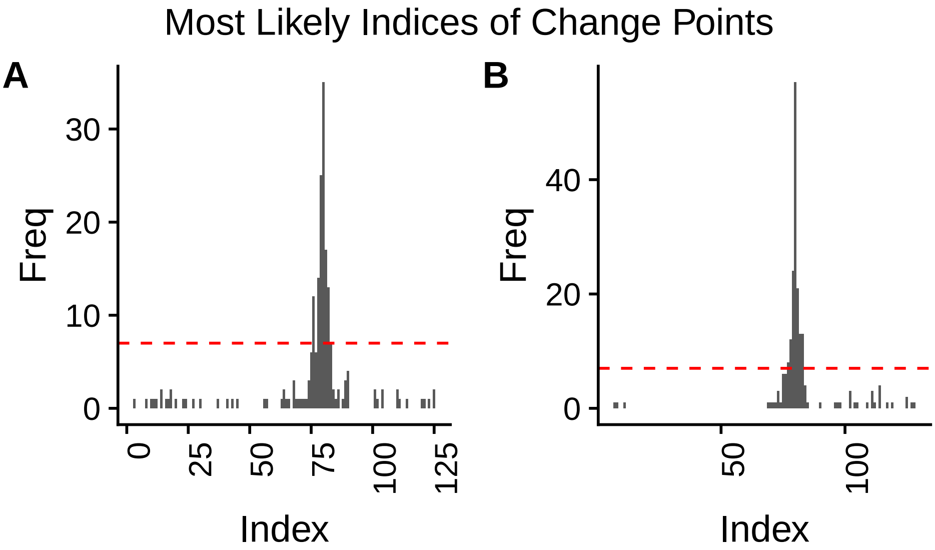


Table 2: Indices which are significant, together with their frequency.

Raw Data		Transformed Data	
Index	Freq	Index	Freq
76	12	77	8
78	14	78	12
79	25	79	24
80	35	80	57
81	17	81	21
82	13	82	13
		83	13

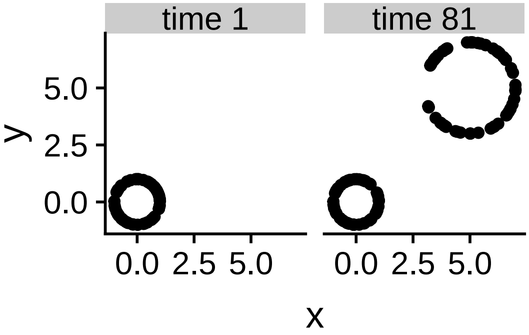
The algorithm is still easily able to detect the change point at time $t = 80$, and in fact, does so better than the algorithm applied directly to the unstranformed data.

3 Change points in persistence diagrams

The space of persistence diagrams is not locally isometric to Euclidean space. However, it does seem to be closer to being isometric to Euclidean space than ℓ^4 is. We apply our algorithm now for time series of persistence diagrams. First, we apply it to simulated data, then to data in the wild of various types.

Our first example is a proof of concept. We start with a single circle, sampled randomly, and then at time $t = 80$, we add a second disjoint circle.

Any good change point detection algorithm based on persistence diagrams should be able to find such a change. Here are two plots of typical point clouds in $\mathbb{R}^2$.



The change point detection algorithm easily detects the change:

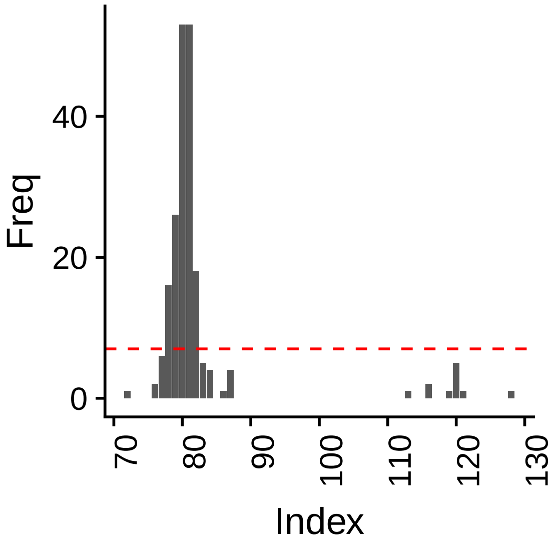
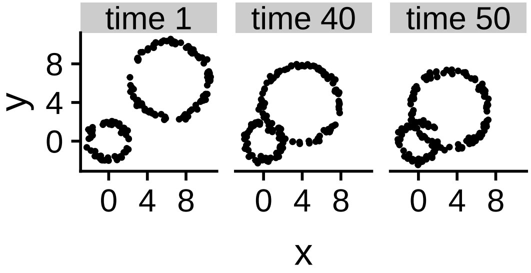


Table 3: Indices which are significant, together with their frequency. True change point at t = 80.

Index	Freq
78	16
79	26
80	53
81	53
82	18

Next, we consider two circles which start out being disjoint, but one of the two circles passes through the other circle. From a topological point of view, there are two potential change points. The first change point is when there are no longer two connected components, but rather one. The second change is when there are three loops rather than two. The algorithm currently under discussion only finds a single change point; how many and which dimensions are included in the persistence diagram and distance calculation will determine which change is detected.

Here are plots from various times in the time series.



We see that the change point detection algorithm detects approximately when there are three distinct loops. Note that in this case, we only used 1-dimensional homology, so it would not be able to detect changes in clusters.

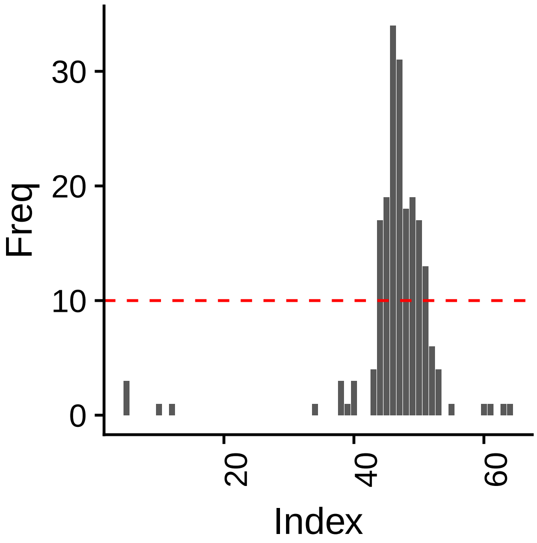


Table 4: Indices which are significant, together with their frequency. True change point ambiguous.

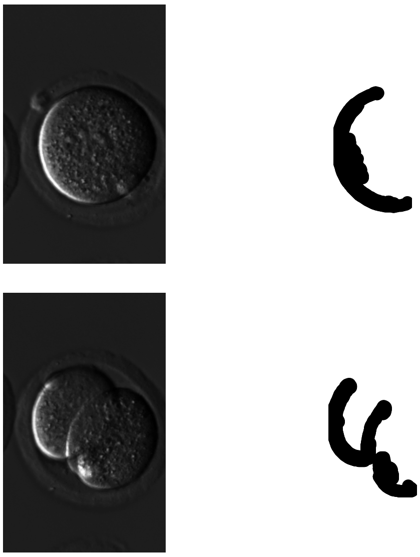
Index	Freq
44	17

45	19
46	34
47	31
48	18
49	19
50	17
51	13

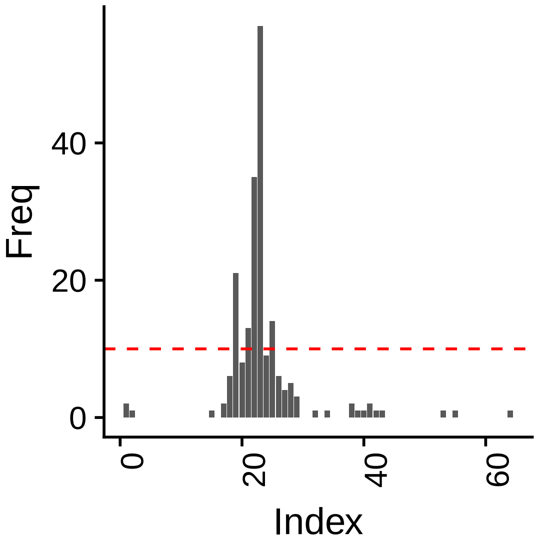
Next, we have two examples of time series of images. In the first example, we examine the Archive of Many Outdoor Scenes. We picked a camera that was taking pictures of Table Mountain from Bloubergstrand in South Africa. For each day, we averaged all of the pictures that were taken on that day. We then used edge detection to extract edges from the pictures, and we removed the time stamp. Finally, we randomly sampled points from the detected edges of the pictures to use as our point cloud. So, the time series consists of randomly sampled points from edges of the average of all pictures on 128 consecutive days.

In the next, we have a series of images of a cell, which divides at time $t = 23$. We use the `cannyEdges` function in the `imager` R package to detect the edges of the cells. We then randomly sample 200 points from the detected edges at each time stamp to form our time series of point clouds. We then applied the change point detection algorithm to obtain the results below.

Here are sample images from before and after the splitting of the cell, together with samples from the detected edges.



When running the algorithm, we obtain the following.



The significant indices from the bootstrap are:

Table 5: Indices which are significant, together with their frequency. True change point at time t = 23

Index	Freq
19	21
21	13
22	35
23	57
25	14

4 References