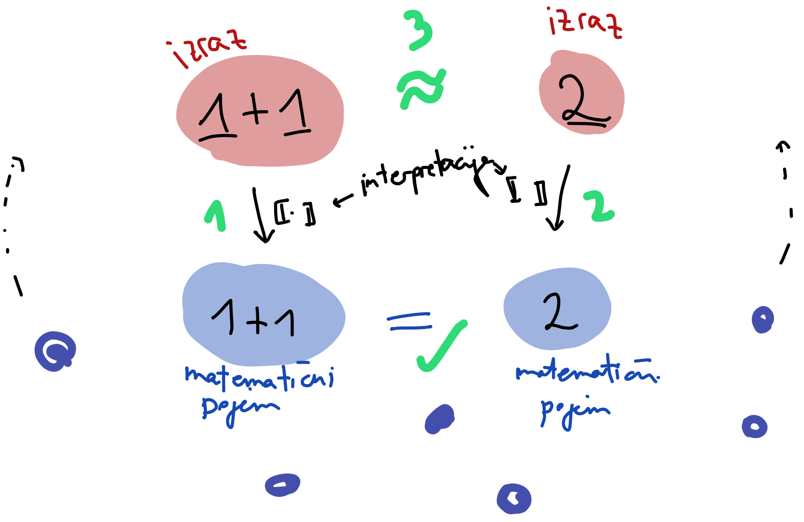


$$M, N ::= x \mid \lambda x.M \mid MN \mid \underline{n} \mid M+N \mid M*N \mid M<N$$
$$\mid \text{true} \mid \text{false} \mid \text{if } M \text{ then } N_1 \text{ else } N_2$$
$$\mid \text{rec } f\,x.M$$

izrazi

$$A, B ::= \text{int} \mid \text{bool} \mid A \to B$$

tipi

Za vsak tip A definirajmo množico $[\![A]\!]$

$$[\![\text{int}]\!] = \mathbb{Z} \qquad [\![\text{bool}]\!] = \mathbb{B} = \{ff, tt\} \qquad [\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$$

Izrazom brez tipov ne bomo pripisovali pomena

Izrazom s tipi priredimo preslikave

$$y:\text{int}, x:\text{int} \vdash x > 2+y : \text{bool}$$

$$[\![\underbrace{\Gamma \vdash M : A}_{\text{drevo izpeljave}}]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

$$[\![x_1:A_1, x_2:A_2, \dots, x_n:A_n]\!] = [\![A_1]\!] \times [\![A_2]\!] \times \dots \times [\![A_n]\!]$$

$$\boxed{\Gamma \vdash M : A}$$

tipi

kontekst – seznam, ki vsaki izmed spr. $x_1, \dots, x_n$ priredi natanko en tip. $x_1:A_1, x_2:A_2, \dots, x_n:A_n$

$$\frac{x:A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\,N : B}$$

$$\Gamma \vdash \underline{n} : \text{int} \qquad \frac{\Gamma \vdash M:\text{int} \quad \Gamma \vdash N:\text{int}}{\Gamma \vdash M+N : \text{int}} \qquad \frac{\Gamma \vdash M:\text{int} \quad \Gamma \vdash N:\text{int}}{\Gamma \vdash M*N : \text{int}} \qquad \frac{\Gamma \vdash M:\text{int} \quad \Gamma \vdash N:\text{int}}{\Gamma \vdash M<N : \text{bool}}$$

$$\frac{}{\Gamma \vdash \text{true} : \text{bool}} \qquad \frac{}{\Gamma \vdash \text{false} : \text{bool}} \qquad \frac{\Gamma \vdash M:\text{bool} \quad \Gamma \vdash N_1 : A \quad \Gamma \vdash N_2 : A}{\Gamma \vdash \text{if } M \text{ then } N_1 \text{ else } N_2 : A}$$

$$\frac{\Gamma, f:A \to B, x:A \vdash M : B}{\Gamma \vdash \text{rec } f\,x.M : A \to B}$$

$$[\![x_1:A_1, \dots, x_n:A_n \vdash x_i : A_i]\!](a_1, \dots, a_n) = a_i$$

$$[\![\Gamma \vdash \lambda x.M : A \to B]\!](\eta) = a \in [\![A]\!] \mapsto [\![\Gamma, x:A \vdash M : B]\!](\eta, a)$$

$$[\![\Gamma \vdash M\,N : B]\!](\eta) = \Big(\underbrace{[\![M]\!](\eta)}_{\in [\![B]\!]^{[\![A]\!]}}\Big)\Big(\underbrace{[\![N]\!](\eta)}_{\in [\![A]\!]}\Big)$$

$$[\![\Gamma \vdash \underline{m} : \text{int}]\!](\eta) = m$$

$$[\![\Gamma \vdash M+N : \text{int}]\!](\eta) = [\![\Gamma \vdash M : \text{int}]\!](\eta) + [\![\Gamma \vdash N : \text{int}]\!](\eta)$$

$$[\![\Gamma \vdash M*N : \text{int}]\!](\eta) = [\![M]\!](\eta) \cdot [\![N]\!](\eta)$$

$$\vdots$$

$$[\![\Gamma \vdash \text{if } M \text{ then } N_1 \text{ else } N_2]\!](\eta) = \begin{cases} [\![N_1]\!](\eta) & [\![M]\!](\eta) = tt \\ [\![N_2]\!](\eta) & [\![M]\!](\eta) = ff \end{cases}$$

<u>Trditev</u> Če velja $\emptyset \vdash M : A$ in $M \leadsto M'$, tedaj velja $[\![M]\!](\cdot) = [\![M']\!](\cdot)$.

Dokaz Doma.

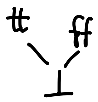
$$(\mathrm{rec}\, f\, x.\, M)\, V \rightsquigarrow M[V/x, (\mathrm{rec}\, f\, x.\, M)/f]$$

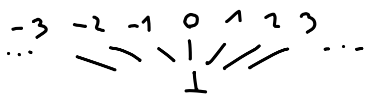
$$[\![\mathrm{rec}\, f\, x.\, M]\!](\cancel{[\![V]\!]}) = [\![M]\!](\cancel{[\![V]\!]}, [\![\mathrm{rec}\, f\, x.\, M]\!])$$

Definicija Domena $(D, \leq)$ je delna urejenost $\leq$ na množici D, za katero velja:

- obstaja najmanjši element $\bot_D$, ki mu pravimo dno,
  $\forall x \in D.\ \bot \leq x$
- obstaja supremum vseh števnih naraščajočih verig
  $x_0 \leq x_1 \leq x_2 \leq \dots \qquad \bigvee_i x_i$
  $(x_i)_{i \in \mathbb{N}} \in D, \quad \forall i \in \mathbb{N}.\ x_i \leq x_{i+1} \Rightarrow \bigvee_i x_i \in D$

Primer

- $([0,1], \leq)$ ✓
- $((0,1], \leq)$ ✗ nima dna
- $([0,1), \leq)$ ✗ $\bigvee_i (1-\frac{1}{i}) \notin D$
- $([0,1) \cup \{2\}, \leq)$ ✓
- $\mathbb{B}_\bot$ 

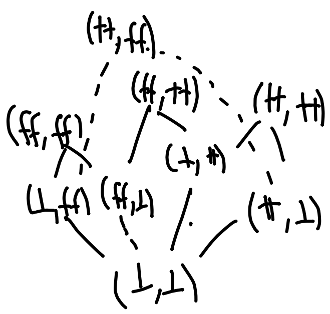
- $\mathbb{Z}_\bot$ 

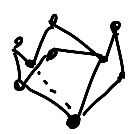
- $A_\bot = (A + \{\bot\}, \leq) \qquad x \leq y \Leftrightarrow x = \iota_1(\bot) \vee x = y$
  dvig A

Definicija Za domeni $(D_1, \leq_1)$ in $(D_2, \leq_2)$ definiramo njun produkt kot $(D_1 \times D_2, \leq)$, kjer je
$$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2$$

Doma: to je res domena.

Primer $\mathbb{B}_\bot \times \mathbb{B}_\bot$





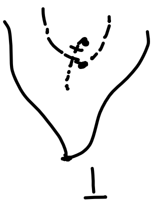
Definicija Vzemimo domeni $(D_1, \leq_1)$ in $(D_2, \leq_2)$. Preslikava $f: D_1 \to D_2$ je <u>zvezna</u>, če:

- je monotona $\quad x \leq_1 x' \Rightarrow f(x) \leq_2 f(x')$
- ohranja supremume verig

$$f(\bigvee_i x_i) = \bigvee_i f(x_i)$$

obstaja, ker je D_1 domena

obstaja, ker je D_2 domena in f monotona



Trditev Vzemimo domene $(D_1, \leq_1)$, $(D_2, \leq_2)$ in $(D_3, \leq_3)$ ter zvezni preslikavi $f: D_1 \to D_2$ in $g: D_2 \to D_3$. Tedaj je $g \circ f: D_1 \to D_3$ zvezna.

Dokaz
- monotonost

$$x_1 \leq_1 x_1' \Rightarrow f(x_1) \leq_2 f(x_1') \Rightarrow g(f(x_1)) \leq_3 g(f(x_1'))$$

- ohranjanje supremumov

$$(g \circ f)(\bigvee_i x_i) = g(f(\bigvee_i x_i)) = g(\bigvee_i f(x_i))$$
$$= \bigvee_i (g(f(x_i))) = \bigvee_i (g \circ f)(x_i)$$

Izrek (Tarski) Naj bo $(D, \leq)$ domena in $f: D \to D$ zvezna preslikava. Tedaj ima f najmanjšo fiksno točko.

Dokaz Definirajmo $x_0 = \bot$, $x_{i+1} = f(x_i)$.
Ker je $x_0 = \bot \leq f(\bot) = x_1$ je $x_1 = f(x_0) \leq f(x_1) = x_2 \ldots$
Zato dobimo verigo $x_0 \leq x_1 \leq x_2 \leq \ldots$
Definirajmo $x = \bigvee_i x_i$. Tedaj velja

- $f(x) = f(\bigvee_i x_i) = \bigvee_i f(x_i) = \bigvee_i x_{i+1} = x$
- Naj velja $y = f(y)$. Tedaj velja $x_0 = \bot \leq y$. Po indukciji velja tudi $x_{i+1} = f(x_i) \leq f(y) = y$. Torej je $x_i \leq y$ za vse i, zato je tudi $x = \bigvee_i x_i \leq y$.

Definicija Vzemimo domeni $(D_1, \leq_1)$ in $(D_2, \leq_2)$ ter definirajmo <u>domeno zveznih funkcij</u> $[D_1 \to D_2]$ kot

$$(\{f: D_1 \to D_2 \mid f \text{ je zvezna}\}, \leq), \text{ kjer je}$$

$$f \leq g \iff \forall x \in D_1.\ f(x) \leq_2 g(x).$$

Pokažimo, da res dobimo domeno.

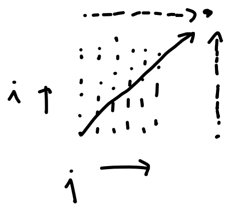
$$\bot_{[D_1 \to D_2]} = x \mapsto \bot_{D_2} \quad ✓$$

je zvezna ✓

Vzemimo verigo $f_1 \le f_2 \le \dots$ in pokažimo, da je $\bigvee_i f_i = f$, kjer je $f(x) := \bigvee_i f_i(x)$. Najprej pokažimo, da je f zvezna.

$$f\left(\bigvee_j x_j\right) = \bigvee_i f_i\left(\bigvee_j x_j\right) = \bigvee_i \bigvee_j f_i(x_j)$$

$$\bigvee_j f(x_j) = \bigvee_j \bigvee_i f_i(x_j) \quad \text{D.N.}$$



Pokažimo, da je f res supremum. Vzemimo $g \ge f_i$ in pokažimo, da je $g(x) \ge f(x)$ za vse $x \in D_1$.

$$f(x) = \bigvee_i f_i(x) \le \bigvee_i g(x) = g(x) \ \checkmark$$

Lema Naj bo $x_{i,j} : \mathbb{N}^2 \to D$, da je $x_{i,j} \le x_{i',j'}$ za vse $i \le i'$, $j \le j'$. Tedaj je $\bigvee_i \bigvee_j x_{ij} = \bigvee_k x_{kk}$.

Dokaz $(\le)$ $x_{ij} \le x_{\max(i,j),\max(i,j)} \le \bigvee_k x_{kk}$

$\bigvee_j x_{ij} \le \bigvee_k x_{kk}$

$\bigvee_i \bigvee_j x_{ij} \le \bigvee_k x_{kk}$

$(\ge)$ $x_{kk} \le \bigvee_j x_{kj} \le \bigvee_i \bigvee_j x_{ij}$

Trditev Preslikava $ev : [D_1 \to D_2] \times D_1 \to D_2$ je zvezna.

$ev : (f, x) \longmapsto f(x)$

Dokaz

- monotonost ✓
- $ev\left(\bigvee_i (f_i, x_i)\right)$

$$= ev\left(\bigvee_i f_i, \bigvee_j x_j\right)$$
$$= \left(\bigvee_i f_i\right)\left(\bigvee_j x_j\right)$$
$$= \bigvee_i \left(f_i\left(\bigvee_j x_j\right)\right)$$
$$= \bigvee_i \bigvee_j f_i x_j$$
$$= \bigvee_k f_k x_k$$
$$= \bigvee_k ev(f_k, x_k) \ \checkmark$$

Trditev Naj bodo D, D_1, D_2 domene. Tedaj je $f : D \to D_1 \times D_2$ zvezna natanko tedaj, kadar sta zvezni $\pi_1 \circ f : D \to D_1$ ter $\pi_2 \circ f : D \to D_2$.

Trditev Naj bodo D, D_1, D_2 domene. Tedaj je $f : D_1 \times D_2 \to D$ zvezna natanko tedaj, kadar so zvezne

$f^1_y : D_1 \to D \qquad f^1_y(x) = f(x,y)$ za vse $y \in D_2$

$f^2_x : D_2 \to D \qquad f^2_x(y) = f(x,y)$ za vse $x \in D_1$

Vsak tip A bomo interpretirali z domeno $[\![A]\!]$

$$[\![\mathrm{int}]\!] = \mathbb{Z}_\perp \qquad [\![\mathrm{bool}]\!] = \mathbb{B}_\perp \qquad [\![A \to B]\!] = [[\![A]\!] \to [\![B]\!]]$$

Kontekst $\Gamma = x_1 : A_1, \dots, x_n : A_n$ bomo interpretirali s produktom $[\![A_1]\!] \times \dots \times [\![A_n]\!]$

Izraz $\Gamma \vdash M : A$ bomo interpretirali z zvezno preslikavo

$$[\![\Gamma \vdash M : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

- $\underbrace{[\![x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i]\!]}_{f}(a_1, \dots, a_n) = a_i$

$(i \neq j) \quad f(x_1, \dots, \bigvee_j x_j, \dots, x_n) = x_i = \bigvee_j f(x_1, \dots, x_n)$

$(i = j) \quad f(x_1, \dots, \bigvee_i x_i, \dots, x_n) = \bigvee_i x_i = \bigvee_i f(x_1, \dots, x_n)$

- $[\![\Gamma \vdash MN : A]\!](\eta) = ([\![M]\!](\eta))([\![N]\!](\eta))$

$$= ev([\![M]\!](\eta), [\![N]\!](\eta))$$

- $[\![\Gamma \vdash \lambda x. M : A \to B]\!](\eta) = \underbrace{a \mapsto [\![\Gamma \vdash M : B]\!](\eta, a)}_{\text{zv. funkcija, ker } [\![M]\!] \text{ ohranja sup. v vseh komp.}}$

- $[\![\lambda x. M]\!](\bigvee_i \eta_i)(a) = [\![M]\!](\bigvee_i \eta_i, a)$

$$= \bigvee_i [\![M]\!](\eta_i, a)$$

$$= \bigvee_i ([\![\lambda x. M]\!](\eta_i)(a))$$

$$= (\bigvee_i [\![\lambda x. M]\!](\eta_i))(a)$$

torej je $[\![\lambda x. M]\!](\bigvee_i \eta_i) = \bigvee_i [\![\lambda x. M]\!](\eta_i)$ po ekstenzionalnosti.

- Ostale doma / na vajah / na izpitu.

<u>Trditev</u> Preslikava $\mathrm{fix} : [D \to D] \to D$, ki vsaki zvezni preslikavi priredi njeno najmanjšo fiksno točko, je zvezna.

$\perp \leq f(\perp) \leq f(f(\perp)) \leq \dots$

$\mathrm{fix}\, f = \bigvee_j f^j(\perp)$

<u>Dokaz</u>

- monotona ✓
- $\mathrm{fix}(\bigvee_i f_i) = \bigvee_j (\bigvee_i f_i)^j(\perp)$

$$\stackrel{?}{=} \bigvee_j \bigvee_i f_i^j(\perp)$$

$$= \bigvee_i \bigvee_j f_i^j(\perp)$$

$$= \bigvee_i \mathrm{fix}\, f_i$$

$$\left(\bigvee_i f_i\right)^0 (x) = x = \bigvee_i f_i^0(x)$$

$$\begin{aligned}\left(\bigvee_i f_i\right)^{n+1} (x) &= \left(\bigvee_i f_i\right)\left(\bigvee_j f_j\right)^n (x)\\ &= \bigvee_i f_i\left(\bigvee_j f_j^n(x)\right)\\ &= \bigvee_i \bigvee_j f_i\left(f_j^n(x)\right)\\ &= \bigvee_k f_k^{n+1}(x)\end{aligned}$$

$$[[[A]] \to [[B]]] \to [[[A]] \to [[B]]]$$

$$[[\Gamma \vdash \mathrm{rec}\, f\, x.\, M : A \to B]](\eta) = a \longmapsto \mathrm{fix}\left(f \mapsto [[M]](\eta, f, a)\right)(a)$$