Sai Pemmaraju   Quantum Mechanics Final Exam   (1)

(1) a. $\hat{x} = i\hbar \partial_p$, $\hat{p} = p$

∴ Want to show that $[\hat{x}, \hat{p}_x] = i\hbar$

∴ Let $\Psi(x)$ be a wavefunction on which operators can act:

$$\cancel{(\hat{x}\hat{p})\Psi - \hat{p}\hat{x}} = (\hat{x}\hat{p}\Psi) - (\hat{p}\hat{x}\Psi)$$

$$= i\hbar \frac{\partial}{\partial p}(p\Psi) - \left(p\, i\hbar \frac{\partial}{\partial p}\Psi\right)$$

$$= i\hbar\left(\Psi + p\frac{\partial \Psi}{\partial p}\right) - p\left(i\hbar\frac{\partial \Psi}{\partial p}\right) = i\hbar\Psi$$

∴ $[\hat{x}, \hat{p}_x] = i\hbar \Rightarrow$ QED

b. $a|0\rangle = 0$ = Ground state.

$C_f =$

, Let $0 = \Psi_0$

$$\therefore a|0\rangle = \frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\hbar\omega}} = 0$$

$$\therefore \hat{x} = i\hbar\frac{\partial}{\partial p}$$

$$\therefore \frac{m\omega i\hbar\frac{\partial \Psi_0}{\partial p} + ip\Psi}{\sqrt{2m\hbar\omega}} = 0$$

$$\therefore m\omega i\hbar\frac{\partial \Psi_0}{\partial p} + ip\Psi = 0$$

(2)

$$\hat{a}|0\rangle = \frac{m\omega i\hbar \frac{\partial \psi_0}{\partial p} + ip\psi}{\sqrt{2m\hbar\omega}} = 0$$

$$\therefore m\omega i\hbar \frac{\partial \psi_0}{\partial p} + ip\psi = 0$$

$$\therefore \frac{\partial \psi_0}{\partial p} + \frac{ip\psi}{m\omega i\hbar} = 0 \Rightarrow \frac{\partial \psi_0}{\partial p} + \frac{p\psi}{m\omega\hbar} = 0$$

$$\therefore \psi_0' + \alpha\psi = 0 \quad \left(\alpha = \frac{p}{m\omega\hbar}\right)$$

$$\therefore \psi_0' = -\alpha\psi$$

$$\therefore \frac{\psi_0'}{\psi} = -\alpha$$

$$\therefore \int \frac{\psi_0}{\psi} dp = \int -\alpha \, dp$$

$$= \ln|\psi(p)| = \frac{-p^2}{2m\omega\hbar} + c$$

$$\therefore \psi_0(p) = Ae^{-p^2/2m\omega\hbar} \Rightarrow$$

$$\therefore |\psi_0(p)|^2 = A^2 e^{-p^2/2m\omega\hbar} = \int \frac{A^2}{2\pi\hbar} e^{2p^2/2m\omega\hbar} dp = 1$$

$$\cancel{A^2\left[e^{-p^2/2m\omega\hbar}\right]^2}$$

$$\int \frac{A^2}{2\pi\hbar} e^{-p^2/m\omega\hbar} dp = 1 \Rightarrow \frac{A^2 e^{1/m\omega\hbar}}{2\pi\hbar} \sqrt{\pi} = 1$$

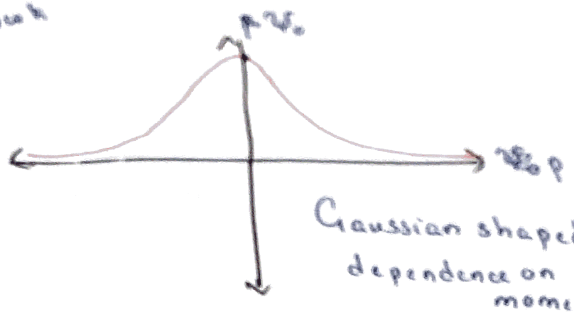
$$\pi - \pi^{1/2}$$

$$\therefore \frac{2\pi\hbar}{\sqrt{\pi}} = A^2 e^{1/m\omega\hbar} \Rightarrow \frac{\cancel{2\pi\hbar}}{} \frac{2\sqrt{\pi}\hbar}{e^{1/m\omega\hbar}} = A^2$$

$$\therefore A = \left(\frac{2^{1/2}\sqrt{\pi}^{1/4}\hbar^{1/2}}{e^{1/m\omega\hbar}}\right)^{1/2}$$

$$\therefore \psi_0(p) = \left(2\frac{\sqrt{\pi}\,\hbar}{e^{1/m\omega\hbar}}\right)^{1/2}\sqrt{\pi}\; e^{-p^2/2m\omega\hbar}$$

(2)



Gaussian shaped dependence on momentum

(3) a. $\{\hat{a}_\alpha, \hat{a}^\dagger_{\alpha'}\} = \delta_{\alpha\alpha'}$, $\{\hat{a}_\alpha\, \hat{a}_{\alpha'}\} = 0$, $\{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_{\alpha'}\} = 0$

$$\therefore \quad \hat{a}_+ = \frac{(\hat{a}_1 + \hat{a}_2)}{\sqrt{2}} \quad \text{and} \quad \hat{a}_- = \frac{(\hat{a}_1 - \hat{a}_2)}{\sqrt{2}}$$

$$\therefore \{\hat{a}_+, \hat{a}^\dagger_+\} = \frac{1}{\sqrt{2}\,2}\{\hat{a}_1 + \hat{a}_2, \hat{a}^\dagger_1 + \hat{a}^\dagger_2\}$$

$$= \frac{1}{2}\{\hat{a}_1, \hat{a}^\dagger_1 + \hat{a}^\dagger_2\} + \frac{1}{2}\{\hat{a}_2, \hat{a}^\dagger_1 + \hat{a}^\dagger_2\}$$

$$= \frac{1}{2}\{\hat{a}_1, \hat{a}^\dagger_1\} + \frac{1}{2}\{\hat{a}_1, \hat{a}^\dagger_2\}$$

$$= \delta$$

$$\left\{\frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}, \frac{\hat{a}^\dagger_1 + \hat{a}^\dagger_2}{\sqrt{2}}\right\}$$

$$= \frac{1}{2}\{\hat{a}_1 + \hat{a}_2, \hat{a}^\dagger_1 + \hat{a}^\dagger_2\}$$

$$= \frac{1}{2}\{\cancel{\hat{a}_1 + \hat{a}^\dagger_1 +} \hat{a}_1\{\hat{a}_1 + \hat{a}_2, \hat{a}^\dagger_1\} + \{\hat{a}_1 + \hat{a}_2, \hat{a}^\dagger_2\}$$

$$= \frac{1}{2}\left(\{\hat{a}_1, \hat{a}^\dagger_1\} + \{\hat{a}_2, \hat{a}^\dagger_1\} + \{\hat{a}_1, \hat{a}^\dagger_2\} + \{\hat{a}_2, \hat{a}^\dagger_2\}\right)$$

$= \frac{1}{2}(2\delta)_{\alpha\alpha'} = \delta_{12} \Rightarrow$ As required for fermionic ~~op~~ operators

Question 2

(4)

(3)

$$= \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} - \hat{x}F(t) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) - \frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)F(t)$$

$$= \quad C_f = \delta_{fi} - \frac{i}{\hbar}\int_0^t dt'\, e^{i(E_f - E_i)t'/\hbar} \langle f|\hat{H}_1(t')|i\rangle$$

(4)

a.

$$C_f = \delta_{fi} - \frac{i}{\hbar}\int_0^t dt'\, e^{i(E_f - E_i)t'/\hbar} \langle f| \frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)F(t)|0\rangle$$

$$\langle f|\frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)|0\rangle = \frac{l}{\sqrt{2}}\langle f|\cancel{\tfrac{l}{\sqrt{2}}}\,\hat{a} + \hat{a}|0\rangle$$

$$= \frac{l}{\sqrt{2}}\left[\langle f|\hat{a}|0\rangle + \langle f|\hat{a}^\dagger|0\rangle\right]$$

$$= \frac{l}{\sqrt{2}}\left[\langle f|\hat{a}|0\rangle + \cancel{\langle f|\hat{a}|}\langle f|1\rangle\right]$$

Question 3

b. $$C_{01} = \delta_{10} - \frac{i}{\hbar}\int_0^t dt'\, e^{i(E_1 - E_0)t'/\hbar} \langle 1| \frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)|0\rangle \qquad \Theta(t)\,\Theta(T_0 - t)F_0$$

$$\therefore C_1 = \delta_{10} - \frac{i}{\hbar}\int_0^t dt'\, e^{i\omega t'/\hbar} \langle 1|\frac{l}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger)|0\rangle$$

$$\therefore C_1 = \delta_{10} - \frac{i}{\hbar}\int_0^t dt'\, e^{i\omega t/\hbar} \langle 1|(\hat{a} + \hat{a}^\dagger)|0\rangle \frac{l}{\sqrt{2}} \quad F(t)$$

$$\Rightarrow C_1 = \delta_{10} - \frac{i}{\hbar}\int_0^t dt'\, e^{i\omega t/\hbar}\, \cancel{F(t)(\Theta(t)\Theta}\, F(t')\frac{l}{\sqrt{2}}$$

$$\therefore C_1 = \delta_{10} - \frac{i}{\hbar}\int_0^t dt'\, e^{i\omega t/\hbar}\left(\Theta(t')\,\Theta(T_0 - t')F_0\right)\frac{l}{\sqrt{2}}$$

$$\cancel{\therefore C_1 = \delta_{10} - \frac{i}{\hbar}\int_0^t dt}\; \frac{i}{\hbar}\cdot\frac{\hbar}{i\omega}\left[\int^0 e^{i\omega t/\hbar}\,\delta(t)\,\delta(T_0 - t)\right]\frac{l}{\sqrt{2}}$$

(4)

$$= \frac{-1}{\omega}\left[e^{i\omega t/\hbar}\,\delta(t)\,\delta(T_0 - t) - \left| \cancel{e^{i\omega(T_0 - t)}}\, \delta(1)\,\delta(T_0 - 1)\right]\right] \frac{\ell}{\sqrt{2}}$$

$$= \frac{-1}{\omega}\left[e^{i\omega t/\hbar}\,\delta(t)\,\delta(T_0 - t) - \delta(1)\,\delta(T_0 - 1)\right] \frac{\ell}{\sqrt{2}} \quad ?$$

(4) a. $\varepsilon_p = \dfrac{p^2}{2m}$.

~~N~~ Density of states:

$$n = \frac{(2m\varepsilon_F)^{1/2}}{3\pi^2\hbar} \quad \text{(no. of states per unit volume)}.$$

Since for 1D $p_F = (3\pi^2\hbar n)$

(by analogy with 2D case covered in lectures)

$\therefore$

$$\therefore \nu(\varepsilon) = \frac{dn}{d\varepsilon} = \frac{\frac{1}{2}\cdot(2m)^{1/2}\varepsilon_F^{-1/2}}{3\pi^2\hbar} \Rightarrow \frac{1}{2}\,\frac{(2m)^{1/2}}{\varepsilon_F^{1/2}\cdot 3\pi^2\hbar}$$

At $\varepsilon = 0$, $\dfrac{dn}{d\varepsilon}$ is undefined (i.e. singular)

b. $\varepsilon_p = \dfrac{(p - p_0)^2}{2m}$

$$\therefore \varepsilon_p = \cancel{\frac{p^2 - 2pp_0 + p_0^2}{2m}}$$

$= \cancel{\varepsilon\, p^2}$

Consider $\varepsilon_0 = \dfrac{p_0^2}{2m}$

$$\therefore n = \frac{(2m\varepsilon_F)^2}{\quad}.$$

$$\therefore \cancel{A} = N\frac{\frac{2V}{2}\pi p_F^2}{(2\pi\hbar)^2} = \Rightarrow n = \frac{2\pi p_F^2}{(2\pi\hbar)^2} \quad \text{(~~electron degeneracy factor included~~)}$$

$$\hat{a}_- = \frac{(\hat{a}_1 - \hat{a}_2)}{\sqrt{2}}$$

$$\therefore \{\hat{a}_-, \hat{a}_-^+\} = \frac{1}{2}\{\hat{a}_1 - \hat{a}_2, \hat{a}_1^+ - \hat{a}_2^+\}$$

$$= \cancel{\frac{1}{2}\{a_1, \hat{a}_1^+} -$$

$$\frac{1}{2}\left[\{\hat{a}_1 - \hat{a}_2\}, \hat{a}_1^+\} - \{\hat{a}_1 - \hat{a}_2, -\hat{a}_2^+\}\right]$$

$$= \frac{1}{2}\left[\{\hat{a}_1, \hat{a}_1^+\}, - \{\hat{a}_2, \hat{a}_1^+\} - \{\hat{a}_1, -\hat{a}_2^+\} + \{\hat{a}_2, \hat{a}_2^+\}\right] = 0$$

$$\hat{a}_- = \frac{(\hat{a}_1 - \hat{a}_2)}{\sqrt{2}} \Rightarrow \cancel{\hat{a}_- = \frac{1}{2}\{\hat{a}_1, -a}$$

$$\therefore \{\hat{a}_-, \hat{a}_-^+\}$$

$$= \{\hat{a}_1 - \hat{a}_2, \hat{a}_1^+ - \hat{a}_2^+\}\frac{1}{2}$$

$$= \left[\{\hat{a}_1 - \hat{a}_2\}, \hat{a}_1^+\} - \{\hat{a}_1 - \cancel{\hat{a}_2}, \hat{a}_2^+\}\right]$$

$$= \cancel{\{\hat{a}_1, -\hat{a}_1}$$

$$\left[\{\hat{a}_1 - \hat{a}_2\}, \hat{a}_1^+\} - \{\hat{a}_1 - \hat{a}_2, -\hat{a}_2^+\}\right]\frac{1}{2}$$

$$= \left[\{\hat{a}_1, \hat{a}_1^+\} - \{\hat{a}_2, \hat{a}_1^+\} - \{\hat{a}_1, \hat{a}_2^+\} + \{-\hat{a}_2, -\hat{a}_2^+\}\right]\frac{1}{2}$$

$= \underline{\underline{\delta_{12}}}$ = As required for fermionic operators.

b. $\cancel{\frac{(\hat{q}_1 + \hat{q}_2)}{\sqrt{2}}} = \hat{q}_+$ Want to show that:

(3) b. $\frac{(\hat{a}_1 + \hat{a}_2)}{\sqrt{2}} = \hat{a}_+$ Can write Hamiltonian in matrix form and diagonalise

$$\therefore H = \begin{pmatrix} \varepsilon & \varepsilon \\ -t & -t \end{pmatrix}, \quad \begin{pmatrix} \hat{a}_+ \\ \hat{a}_+ \end{pmatrix} =$$

PDP^{-1}

$$\varepsilon \quad \begin{pmatrix} \cancel{\varepsilon} & \varepsilon \\ -t & \cancel{-t} \end{pmatrix} \quad \cancel{a}\, \rho^{-1}$$

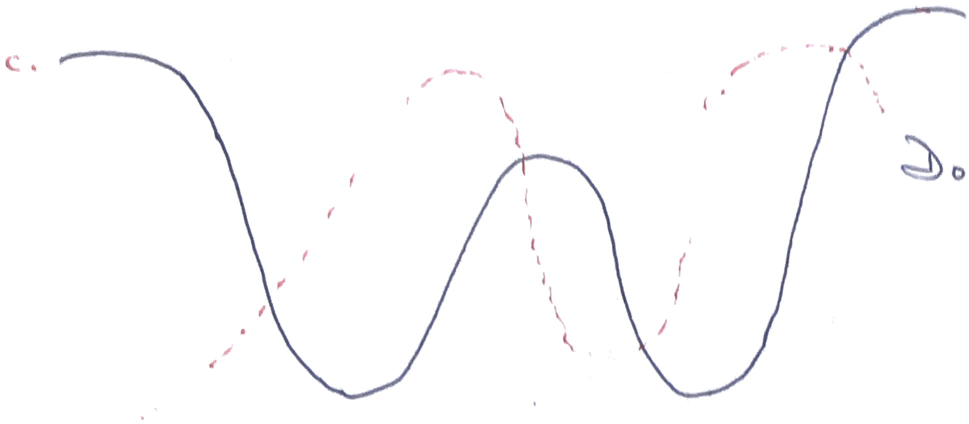
$$\text{P's Matrix } P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} a_+ \\ a_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Write Hamiltonian as: $H = \begin{pmatrix} \varepsilon & \varepsilon \\ -t & -t \end{pmatrix}$

$$\therefore \cancel{\tfrac{\varepsilon}{J}} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}$$

= Have to write ~~PD~~ $H = PDP^{-1}$ where D is a diagonal matrix and whose ~~entri~~ ~~dia~~ entries ~~will~~ ~~demonst~~. represent the eigenvalues of $\hat{H}$ $\Rightarrow$ ~~th~~ this will show that the ~~symmet~~ a_+ and a_- are eigenstates of $\hat{H}$



Double well potential