Assignment 2: Question 5

Monday, 28 March 2022 7:16 pm

(5) Consider two point charges of charge q at a fixed distance d apart that orbit around each other in the xy plane at frequency ω

a. Calculate electric dipole moment of the system
b. Calculate the magnetic dipole moment of the system
c. Calculate all Cartesian components of the electric quadrupole moment
d. Compute the total power radiated.

Solution:

a. $\underset{\sim}{p} = \int \rho(\underset{\sim}{x}')\, \underset{\sim}{x}'\, d^3x'$: Electric dipole moment

$$\rho = q\left[\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + \delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right]\delta(z)$$

$$\therefore\ \underset{\sim}{p} = \int q\left[\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + \delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)(x\hat{\underset{\sim}{x}} + y\hat{\underset{\sim}{y}} + z\hat{\underset{\sim}{z}})\right]$$

$$= \left[\frac{qd}{2}\cos(\omega t) - \frac{qd}{2}\cos(\omega t)\right] + \left[\frac{qd}{2}\sin(\omega t) - \frac{qd}{2}\sin(\omega t)\right] = \underline{\underline{0}}$$

b. Magnetic dipole moment:

$$\underset{\sim}{m} = \frac{1}{2}\int \underset{\sim}{x} \times \vec{J}\, d^3\underset{\sim}{x}$$

$$\vec{J} = q\underset{\sim}{v}$$

$$= \left[q\,\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + q\,\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right]\left[\frac{d\omega}{2}\sin(\omega t)\,\underset{\sim}{i} - \frac{d\omega}{2}\cos(\omega t)\,\underset{\sim}{j}\right]$$

$$= -\frac{q\omega d}{2}\sin(\omega t)\,\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\underset{\sim}{i} + \frac{q\omega d}{2}\cos(\omega t)\,\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\underset{\sim}{j}$$

$$\frac{q\omega d}{2}\sin(\omega t)\,\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\underset{\sim}{i} - \frac{q\omega d}{2}\cos(\omega t)\,\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\underset{\sim}{j}$$

$$= \frac{q\omega d}{2}\sin(\omega t)\left[\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right) - \delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\right]\underset{\sim}{i} + \frac{q\omega d}{2}\cos(\omega t)\left[\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) - \delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right]\underset{\sim}{j} = \vec{J}$$

$$\therefore\ \underset{\sim}{m} = \frac{1}{2}\int \underset{\sim}{x} \times \vec{J}$$

$$\therefore\ \underset{\sim}{m} = \frac{1}{2}\int (x\hat{\underset{\sim}{x}} + y\hat{\underset{\sim}{y}} + z\hat{\underset{\sim}{z}}) \times \vec{J}\, d^3\underset{\sim}{x}$$

Evaluating the above by computer yields:

$$\boxed{\underset{\sim}{m} = \underline{\underline{\frac{q\omega d}{4}\hat{\underset{\sim}{z}}}}}$$

c. $Q_{ij}(t) = \int (3x_i x_j - r^2\delta_{ij})\rho\, d^3x$

$$= 2q\int [3x_i x_j - r^2\delta_{ij}]\left(\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + \delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right]\delta(z)\, d^3x$$

Q_{11}:

$$= 2q\iiint 3x^2 - (x^2 + y^2 + z^2)\left(\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + \delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right]\delta(z)\, dx\,dy\,dz$$

$$= 2q\iint (2x^2 - y^2)\quad\left(\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right) + \delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\right)dx\,dy$$

- Evaluating this by computer, one can obtain a value of $Q_{11} = \frac{d^2 q}{4}(1 + 3\cos(2\omega t))$

Q_{22}:

$$= \iint 3y^2 - (x^2 + y^2 + z^2)\,\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)^{\delta(z)} dx\,dy\,dz$$

$$\iint (2y^2 - x^2)\,\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)\Big)\, dx\,dy$$

Evaluating this expression computationally yields:

$$Q_{22} = \frac{d^2 q}{4}(1 - 3\cos(2\omega t))$$

$$Q_{33} = \int \left(3z^2 - (x^2 + y^2 + z^2)\right)\delta\left(x - \frac{d}{2}\cos(\omega t)\right)\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)^{\delta(z)} dx\,dy\,dz$$

$\Rightarrow\ Q_{33} = -\frac{qd^2}{2}$ (by computer)

$Q_{12} = Q_{21}$:

$$\iiint 3xy - (x^2 + y^2 + z^2)\,\delta\left(y - \frac{d}{2}\sin(\omega t)\right)\delta\left(x + \frac{d}{2}\cos(\omega t)\right)\delta\left(y + \frac{d}{2}\sin(\omega t)\right)^{\delta(z)} dx\,dy\,dz$$

- By computer, $Q_{12} = \frac{3d^2 q}{4}\sin(2\omega t)$

All of the remaining entries are equal to 0. Hence, the quadrupole moment is given by:

$$\boxed{Q = \begin{pmatrix} \frac{d^2q}{4}(1 + 3\cos(2\omega t)) & 3d^2q/4\sin(2\omega t) & 0 \\ \frac{3d^2q}{4}\sin(2\omega t) & \frac{d^2q}{4}(1 - 3\cos(2\omega t)) & 0 \\ 0 & 0 & -\frac{qd^2}{2} \end{pmatrix}}$$

d. The total power radiated by a quadrupole source:

$$P = \frac{c^2 Z_0 k^6}{1440\pi} Q_{ij}^* Q_{ij}$$

- Compute $Q_{ij}^* Q_{ij}$:

$$Q = \frac{qd^2}{2}\begin{pmatrix} \frac{1}{2} + \frac{3}{2}\cos(2\omega t) & \frac{3}{2}\sin(2\omega t) & 0 \\ \frac{3}{2}\sin(2\omega t) & \frac{1}{2} - \frac{3}{2}\cos(2\omega t) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- One can neglect the non-oscillatory terms since these do not contribute to the total power radiated.

$$Q = \frac{qd^2}{2}\begin{pmatrix} \mathrm{Re}\left(\frac{3}{2}e^{2i\omega t}\right) & \mathrm{Re}\left(\frac{3}{2}e^{(2i\omega t - \pi/2)}\right) \\ \mathrm{Re}\left(\frac{3}{2}e^{(2i\omega t - \pi/2)}\right) & \mathrm{Re}\left(-\frac{3}{2}e^{2i\omega t}\right) \end{pmatrix}$$

$$\therefore\ Q_{ij}^* Q_{ij} = \left(\frac{3}{2}\right)^2 \times \left(\frac{qd^2}{2}\right)^2 = \frac{9q^2d^4}{4}$$

$\therefore$ The total power radiated is given by:

$$P = \frac{c^2 Z_0 \left(\frac{2\omega}{c}\right)^6}{1440\pi}\cdot\frac{9q^2d^4}{4}$$

$$= \frac{c^2 Z_0\, 64\omega^6/c^6}{1440\pi}\cdot\frac{9q^2d^4}{4}$$

$$= \frac{Z_0\, 64\omega^6}{1440\pi c^4 \times 4} \times 9q^2d^4$$

$$= \frac{64 Z_0 \omega^6 q^2 d^4}{640} = \boxed{\underline{\underline{\frac{\omega^6 q^2 d^4}{10}}}}$$