Assignment 1: Question 2

Saturday, 12 March 2022   11:37 am

b) Electric and magnetic field operators in Heisenberg picture:

$$\hat{E}(\underset{\sim}{r},t) = i\sum_{ks}\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\left(\hat{a}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)$$

$$\hat{B}(\underset{\sim}{r},t) = i\sum_{\underset{\sim}{k}s}\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}\,[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\left(\hat{a}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)$$

- Required to compute average values for $\hat{E}(\underset{\sim}{r},t)$ and $\hat{B}(\underset{\sim}{r},t)$ in the classical single mode state (occupations of all photonic modes = 0 except for the single mode)
  For $\hat{B}(\underset{\sim}{r},t)$ $\underset{\sim}{k} = \underset{\sim}{e}_x k_0$, polarization $\underset{\sim}{e}_0 = \underset{\sim}{e}_y$ and the occupation is as the coherent state. α.

Solution:

$$\langle\hat{\vec{E}}(\underset{\sim}{r},t)\rangle = \langle\alpha|\hat{E}|\alpha\rangle$$

$$= \langle\alpha|\hat{E}|\alpha\rangle = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\langle\alpha|\left(\hat{a}_{ks}\underset{\sim}{e}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}\underset{\sim}{e}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)|\alpha\rangle$$

$$= i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\langle\alpha|\hat{a}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}|\alpha\rangle$$

$$= i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\left(\alpha e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \alpha^* e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)$$

$$2i\times i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\cdot\frac{1}{2i}\left(\alpha e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \alpha^* e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)$$

$$= -2\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\frac{1}{2i}\left(|\alpha|e^{i\delta}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - |\alpha|e^{-i\delta}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)$$

$$= -2\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\frac{1}{2i}\left(|\alpha|e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)} - |\alpha|e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)}\right)$$

$$= -2\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\underset{\sim}{e}_{ks}\cdot|\alpha|\sin(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)$$

Similarly for $\langle\hat{\vec{B}}(\underset{\sim}{r},t)\rangle$

$$= \langle\alpha|\vec{B}(\underset{\sim}{r},t)|\alpha\rangle$$

$$= i\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}\langle\alpha|[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\left(\hat{a}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right)|\alpha\rangle$$

$$= i\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\langle\alpha|\hat{a}_{ks}e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \hat{a}^{+}_{ks}e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}|\alpha\rangle$$

$$= i\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\left[\alpha e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)} - \alpha^* e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t)}\right]$$

- Let $\alpha = |\alpha|e^{i\delta}$

$$\Rightarrow\; i\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\left[|\alpha|e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)} - |\alpha|e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)}\right]$$

$$= 2i\times i\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\frac{1}{2i}\left[|\alpha|e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)} - |\alpha|e^{-i(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)}\right]$$

$$= -2\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\left[|\alpha|\sin(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)\right]$$

$$= -2|\alpha|\sqrt{\frac{2\pi\hbar c^2}{\omega_k V}}[\underset{\sim}{k}\times\underset{\sim}{e}_{ks}]\sin(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)$$

- In both of the above scenarios, the average values of $\hat{E}$ and $\hat{B}$ for the coherent state $|\alpha\rangle$ are of the form:

$$\vec{E} = E_0\sin(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)\underbrace{\hat{\underset{\sim}{e}}}_{\text{electric field vector}} \quad\text{and}\quad E_0 = -2|\alpha|\sqrt{\frac{2\pi\hbar\omega_k}{V}}$$

and

$$\vec{B} = B_0\sin(\underset{\sim}{k}\cdot\underset{\sim}{r}-\omega t+\delta)\underbrace{[\hat{\underset{\sim}{k}}\times\hat{\underset{\sim}{e}}]}_{\text{direction of magnetic field vector}} \quad\text{and}\quad B_0 = -2|\alpha|\sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}$$

- Take $\underset{\sim}{k} = k_0\underset{\sim}{e}_x$ and take $\underset{\sim}{e} = \underset{\sim}{e}_0 = \underset{\sim}{e}_y$. Hence:

$$\cdot\;\vec{E}(\underset{\sim}{r},t) = E_0\sin(k_0\underset{\sim}{e}_x\cdot(\hat{x},\hat{y},\hat{z}) - \omega t+\delta)\,\underset{\sim}{e}_y$$

$$\Rightarrow\;\vec{E}(x,t) = E_0\sin(k_0 x - \omega t+\delta)\,\underset{\sim}{e}_y$$

$$\vec{B}(\underset{\sim}{r},t) = B_0[k_0\underset{\sim}{e}_x\times\underset{\sim}{e}_y]\sin(k_0x-\omega t+\delta)$$
$$= B_0k_0\underset{\sim}{e}_z\sin(k_0x-\omega t+\delta)$$

(for $\vec{E}$ and $\vec{B}$)

- Note that both of the above expressions are in phase with one another, and are also orthogonal to one another, thereby mimicking the spatial and the temporal behaviour of the classical EM wave.