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General Relativity: Examination

(1)

(1). This is a reasonable equation.

- Reasonable equation

- Nonsense since operating with the covariant derivative twice would not lead to a contravariant tensor.

(3,0) tensor.

- Reasonable equation.

- Reasonable equation.

- Reasonable equation.

(2) $ds^2 = xy dx^2 + x^2 dy^2$

$$\therefore g_{\mu\nu} = \begin{pmatrix} g_{xx} & 0 \\ 0 & g_{yy} \end{pmatrix}$$

∴ 5 Possible Christoffel symbols:

$$\Gamma_{xx}^{xx}, \Gamma_{xy}^{xx}, \Gamma_{xx}^{yy}, \Gamma_{xy}^{yy}$$

$$\Gamma_{yy}^{yy}, \Gamma_{yx}^{yy}, \Gamma_{xy}^{yy}$$

$$\Gamma_{xx}^{xx} = \frac{1}{2} \partial_y g^{xx} (\partial_x g_{xx} + \partial_y g_{xx} \partial_x g_{xx})$$

Diagonal metric $\Rightarrow \Gamma_{xx}^{yy} = -\frac{1}{2g_{yy}} \partial_y g_{xx}$

$$= -\frac{1}{2x^2} \partial_y (xy) = -\frac{1}{2x}$$

$$\Gamma_{xx}^{xx} = \partial_x \ln \sqrt{g_{xx}}$$

$$= \partial_x \left(\ln \sqrt{xy} \right) = \frac{\left(\frac{1}{2}y\right)}{\sqrt{xy}} = \frac{1}{2y}$$

(3)

$$(2) \text{ and } ds^2 = g_{xy} dx^2 + x^2 dy^2 \text{ mit } g_{xy} \text{ ist diagonal}$$

(2)

$$\therefore g_{\mu\nu} = \begin{pmatrix} xy & 0 \\ 0 & x^2 \end{pmatrix} \quad \text{ist diagonal}$$

\therefore Metric is diagonal. Christoffel symbols are:

$$\Gamma_{xx}^x, \Gamma_{xy}^x, \Gamma_{yx}^x, \Gamma_{yy}^x, \Gamma_{xx}^y, \Gamma_{xy}^y, \Gamma_{yx}^y, \Gamma_{yy}^y$$

$$\therefore \Gamma_{xx}^x = \partial_x \ln \sqrt{|g_{xx}|} = \partial_x \ln \sqrt{xy} = \frac{1}{2} - \frac{1}{2} x^{-1/2} y^{1/2}$$

$$\Gamma_{yy}^y = \partial_y \ln \sqrt{|g_{yy}|} = \partial_y \ln \sqrt{xy} = 0$$

$$\Gamma_{xy}^x = \partial_y \ln \sqrt{|g_{xx}|} = \partial_y \ln \sqrt{xy} = \frac{1}{2} y^{-1/2} \frac{1}{(xy)^{1/2}} = \frac{1}{2y}$$

$$\Gamma_{yx}^x = \frac{1}{2} \quad (\text{symmetry}), \quad \Gamma_{yy}^x = \frac{\partial_x (g_{yy})}{2g_{xx}} = \frac{-1}{2xy} = -\frac{1}{2x}$$

$$= -\frac{1}{2}$$

$$\Gamma_{xy}^y = \frac{-1}{2} \quad \partial_y (g_{xx}) = -\frac{1}{2x}, \quad \Gamma_{xy}^y = \frac{\partial_y (g_{yy})}{2g_{xx}} = \frac{1}{2x} = \partial_x (\ln \sqrt{x^2})$$

$$\therefore \Gamma_{yx}^y = \partial_x (\ln (x)) = \frac{1}{x}$$

\therefore To compute $R_{\mu\nu}^{\alpha\beta}$, we need $R_{\nu\mu\alpha}^{\beta}$ first:
 (Riemann tensor): $R_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R$

(2) Compute R^x_{yxy} and then R^y_{yyx} . We can then obtain other components by symmetry

$$\begin{aligned} R^x_{yxy} &= \partial_x \Gamma^x_{yy} - \partial_y \Gamma^x_{yx} - \Gamma^x_{xx} \Gamma^x_{yy} + \Gamma^x_{xy} \Gamma^y_{yy} - \Gamma^x_{yx} \Gamma^x_{yy} \\ &= \partial_x \left(\frac{1}{2y} \right) = 0 - \partial_y \Gamma^x_{yx} \left(\frac{1}{2y} \right) - \\ &= -\partial_y \left(\frac{1}{2y} \right) - \left(\frac{1}{2x} \right) \left(-\frac{1}{y} \right) = \frac{1}{2y} - \frac{1}{2x} \\ &= \frac{-1}{2y^2} - \frac{1}{2xy} - \frac{1}{2y} = -\frac{2}{2xy} \\ &= \frac{1}{2y^2} - \frac{1}{2y} = \frac{x+2y}{4xy} \end{aligned}$$

Since Riemann tensor is antisymmetric in last two indices we have

$$R^x_{yyx} = -R^x_{yxy} = -\frac{x+2y}{4xy}$$

$$\begin{aligned} \text{Compute } R^y_{xxy} &= -R^y_{xyx} \\ &= R^y_{yxy} = -\frac{x+2y}{4xy} \end{aligned}$$

is identical to the above, up to $-\partial_y \rightarrow -\partial_x \left(\frac{1}{2x} \right)$

$$R^y_{yxy} = -R^y_{xyx} = -\frac{x+2y}{4x^2y} \quad (\text{it is symmetric})$$

with $R^x_{yyx} \times$ (SEE END OF PAGE 7)

(4)

R^x_{xxx} and R^y_{yyy} vanish by definition of Riemann tensor.

$$R_{xx} \text{ s.t. } R^x_{yxx} = 0 \Rightarrow R^x_{xyx} = 0$$

$$\therefore R^x_{xx} = g^{xx} (R^x_{xxx} + R^y_{xyx}) = g^{yy} R^y_{yxy} = \frac{x+2y}{4x^2y}$$

$$\text{Similarly, } R_{yy} = g^{yy} R^y_{yxy} + R^y_{yyy} = \frac{x+2y}{4x^2y^2}$$

$$\therefore R_{xy} = \begin{pmatrix} \frac{x+2y}{4x^2y} & 0 \\ 0 & \frac{x+2y}{4x^2y^2} \end{pmatrix}$$

$$\therefore R^y_{xy} = g^{xy} R_{xy} = \frac{1}{xy} \cdot \frac{x+2y}{4x^2y} + \frac{1}{x^2} \cdot \frac{x+2y}{4xy^2} = \frac{x+2y}{4x^3y^2}$$

$$= \frac{x+2y}{4x^3y^2} + \frac{x+2y}{4x^3y^2} = \frac{2x+4y}{4x^3y^2} = \frac{x+2y}{2x^3y^2}$$

(3) Claim:

$$\nabla_u T^\mu_v = \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^\mu_v) - \Gamma^\lambda_{uv} T^\mu_\lambda$$

$$\nabla_u T^\mu_v = \partial_u T^\mu_v + \Gamma^\mu_{u\sigma} T^\sigma_v$$

(5)

$$\begin{aligned}
 &= \partial_u T^u_v + (\sqrt{-g})^{-1} \partial_\sigma (\sqrt{-g}) T^{\sigma u} - \Gamma_{\mu\nu}^\sigma T^{\mu u} \\
 \nabla_u T^u_v &= \partial_u \nabla_v T^u + \Gamma_{\mu\nu}^\sigma \nabla_v T^{\mu u} - \Gamma_{\mu\nu}^\sigma T^{\mu u} \\
 &= \partial_u \nabla_v T^u + (\sqrt{-g} \partial_\sigma) \sqrt{-g} \nabla_v T^{\mu u} - \Gamma_{\mu\nu}^\sigma T^{\mu u} \\
 &= \partial_u T^u_v + \frac{1}{\sqrt{-g}} \partial_\sigma (\sqrt{-g} T^u_v) - \Gamma_{\mu\nu}^\sigma T^{\mu u} \Rightarrow QED
 \end{aligned}$$

(where $\lambda = \sigma$)

Want to now show that:

$$\nabla_u T^u_v = \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^u_v) - \frac{1}{2} (\partial_v g_{\mu\nu}) T^{\mu u}$$

Case where $\sigma = \mu \neq v$

$$= \frac{1}{\sqrt{-g}} \partial_u T^u_v + \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^u_v) + \Gamma_{\mu\nu}^\mu T^{\mu u}$$

$$\therefore \Gamma_{\mu\nu}^\mu = \partial_\mu \ln(\sqrt{|g_{\mu\nu}|}) = \frac{1}{\sqrt{-g}} \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial \mu} = -\frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial \mu}$$

$$= \frac{1}{2 \sqrt{-g}} \frac{1}{2} g^{\mu\lambda} \frac{\partial g_{\mu\lambda}}{\partial \mu} T^{\mu u}$$

$$\therefore \nabla_u T^u_v = \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^u_v) + \partial_v (\ln(\sqrt{|g_{\mu\nu}|})) T^{\mu u}$$

$$= \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^u_v) + \frac{1}{2} \partial_v (g_{\mu\nu}) T^{\mu u}$$

$$T^u_\mu g^{\mu\lambda} = \frac{1}{\sqrt{-g}} \partial_u (\sqrt{-g} T^u_v) - \frac{1}{2} \partial_v (g_{\mu\nu}) T^{\mu u}$$

(6)

(4)a Schwarzschild Metric (General Form)

$$ds^2 = \left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$\text{Geodesic eqn general form: } \ddot{x}^\mu + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

$$\therefore \frac{d^2r}{d\tau^2} + \Gamma_{tt}^r \frac{dr}{d\tau} \frac{dt}{d\tau} + \Gamma_{rr}^r \frac{dr}{d\tau} \frac{dt}{d\tau} = 0 \quad (\text{set } \theta = \phi \text{ and } \Omega = 0)$$

$$= \frac{d^2r}{d\tau^2} + \Gamma_{tt}^r \frac{dr}{d\tau} \frac{dt}{d\tau} + \Gamma_{rr}^r \frac{dr}{d\tau} \frac{dt}{d\tau} = 0 \quad (\text{set } \theta = \phi \text{ and } \Omega = 0)$$

$$\therefore \frac{d^2t}{d\tau^2} + \frac{-2GM}{(2GM-r)r} \frac{dr}{d\tau} \frac{dt}{d\tau} = 0 \quad (1)$$

$$\frac{d^2r}{d\tau^2} + \Gamma_{tt}^r \left(\frac{dt}{d\tau}\right)^2 + \Gamma_{rr}^r \left(\frac{dr}{d\tau}\right)^2 = 0 + \Gamma_{00}^r \left(\frac{dt}{d\tau}\right)^2 + \Gamma_{\varphi\varphi}^r \left(\frac{d\varphi}{d\tau}\right)^2$$

$$\Rightarrow \frac{d^2r}{d\tau^2} + \frac{-GM(2GM-r)}{(2GM-r)r} \left(\frac{dt}{d\tau}\right)^2 + \frac{GM}{(2GM-r)r} \left(\frac{dr}{d\tau}\right)^2 + 2GM-r \left(\frac{d\varphi}{d\tau}\right)^2$$

$$+ (2GM-r)\sin^2(\theta) \left(\frac{d\varphi}{d\tau}\right)^2 = 0 \quad (2)$$

$$\frac{d^2\varphi}{d\tau^2} + 2\Gamma_{r\varphi}^0 \frac{dr}{d\tau} \frac{d\varphi}{d\tau} + \Gamma_{\varphi\varphi}^0 \left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

(7)

$$= \frac{d^2\theta}{dt^2} + \frac{2}{r} \frac{dr}{dt} \frac{d\theta}{dt} - \cos(\theta) \sin(\theta) \left(\frac{d\varphi}{dt} \right)^2 = 0$$

$$\frac{d^2\varphi}{dt^2} - 2r^2 \frac{d\varphi}{dt} \frac{dr}{dt} + 2r^2 \frac{d\theta}{dt} \frac{d\varphi}{dt} = 0$$

$$= \frac{d^2\varphi}{dt^2} + \frac{2}{r} \frac{d\varphi}{dt} \frac{dr}{dt} + 2 \cot(\theta) \frac{d\theta}{dt} \frac{d\varphi}{dt} = 0 \quad (4)$$

$$b. R = R_s \dots$$

(2) Recalculating Riemann and Ricci tensors for Question 2.

By definition, the Riemann tensor is given by:

$$R^{\mu}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\alpha\sigma} \Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\beta\sigma} \Gamma^{\sigma}_{\nu\alpha}$$

$$R^x_{xxxz} = \partial_x \Gamma^x_{xx} - \partial_x \Gamma^x_{xz} + \Gamma^x_{x\lambda} \Gamma^\lambda_{xx} - \Gamma^x_{x\lambda} \Gamma^\lambda_{xz}$$

$$= \partial_x \left(\frac{1}{2x} \right) - \partial_x \left(\frac{1}{2x} \right) + \Gamma^x_{xx} \Gamma^\lambda_{xx} - \Gamma^x_{xx} \Gamma^\lambda_{xz}$$

$\lambda = x \Rightarrow$ the above trivially vanishes:

$$\lambda = y \Rightarrow \Gamma^y_{xy} \Gamma^y_{xx} - \Gamma^y_{xy} \Gamma^y_{xx} = 0$$

Similarly for $R^y_{yyy} = 0$

$$R^x_{xxy} = \partial_x \Gamma^x_{xy} - \partial_y \Gamma^x_{xy} + \Gamma^x_{x\lambda} \Gamma^\lambda_{xy} - \Gamma^x_{y\lambda} \Gamma^\lambda_{xy}$$

$x = y = \lambda = x \Rightarrow = 0, \lambda = y \Rightarrow = 0$ (given Christoffel symbols).

By symmetry $R^x_{xyx} = 0$ (Similarly, any entry with more than two y 's is zero)

(8)

The non-vanishing components are R^x_{yxy} , R^x_{yyx} , R^y_{xxy} , R^y_{xyx} , R^y_{yyy} .

R^x_{yxy} and R^x_{yyx}

Hence $R^x_{yxy} = R^x_{yyx}$

$$= \partial_x \left(\Gamma^x_{yy} - \Gamma^y_{yx} \right) - \Gamma^x_{yy} + \Gamma^x_{xx} \Gamma^x_{yy} - \Gamma^x_{yy} \Gamma^x_{yx} = \frac{1}{4x^2} + \frac{1}{4x^2} = \frac{1}{2x^2}$$

$$= \frac{1}{2y} \left(\frac{1}{2y} \right) + \left(\frac{1}{2y} \right) \left(-\frac{1}{2y} \right) - \frac{1}{2y^2} = \frac{1}{4y^2}$$

$$= \frac{1}{2y^2} - \frac{1}{2y^2} = 0$$

R^y_{xxy} is computed with additional eq.

$$\text{By symmetry, } R^x_{yxy} = R^x_{yyx} = \frac{x+2y}{4x^2}$$

$$\text{Hence } R^y_{xxy} = -R^y_{yyx} = \frac{x+2y}{4x^2} \quad (\text{computation})$$

is identical, up to $(\partial_y \leftrightarrow \partial_x)$, and due to the symmetry in the form of the Christoffel symbols

Ricci tensor = $\text{contraction of Christoffel symbols}$

$$R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} = R^x_{yxy} + R^y_{xyx}$$

$$\therefore R_{xx} = R^\sigma_{x\sigma} = R^x_{xxy} + R^y_{xyx} = 0 + \frac{x+2y}{4x^2} = \frac{x+2y}{4x^2}$$

$$R_{xy} = R^\sigma_{x\sigma y} = R^x_{xyy} + R^y_{yyx} = 0$$

$$R_{yy} = -R^\sigma_{y\sigma y} = -R^x_{yyx} = 0$$

$$R_{yy} = R^\sigma_{y\sigma y} = R^x_{yyx} + R^y_{xyx}$$

(9)

$$\frac{\partial xy}{\partial x} = 0 \quad \frac{\partial xy}{\partial y} = 1$$

Value $\begin{pmatrix} \frac{\partial xy}{\partial x} & 0 \\ 0 & \frac{\partial xy}{\partial y} \end{pmatrix}$

Ricci scalar was computed earlier (trace of $R_{\mu\nu}$)

$$\frac{\partial xy}{\partial x} + \frac{\partial xy}{\partial x} = \frac{\partial^2 xy}{\partial x^2}$$

Want to write:

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

$$= \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad (\text{Carroll Ch. 4 p. 153})$$

2. b. γ^μ = parallel transported along $x^\mu(\tau)$, $u^\mu = \frac{dx^\mu}{d\tau}$

It must obey the eqs of parallel transport

$$(\text{lie } \nabla_u \gamma = u^\mu \nabla_\mu \gamma^\nu = 0)$$

Let $x^\mu(\tau)$ be an affine geodesic:

$$\Rightarrow x^\mu(\tau) = \alpha \tau + \beta, \quad \alpha, \beta \in \mathbb{R}$$

$$\therefore u^\mu = \dot{x}^\mu = g^{\mu\nu} g_{\nu\lambda} u^\lambda = g_{\mu\nu} u^\nu = g_{\mu\nu}(\alpha) =$$

(6)

$$= \frac{x+2y}{4xy^2} + 0 = \frac{x+2y}{4xy^2}$$

(9)

$$\therefore R_{\mu\nu} = \begin{pmatrix} \frac{x+2y}{4xy^2} & 0 \\ 0 & \frac{x+2y}{4y^2x} \end{pmatrix}$$

(Ricci scalar was computed earlier = Trace of $R_{\mu\nu}$)

$$= \frac{x+2y}{4y^2x} + \frac{x+2y}{4y^2x} = \frac{x+2y}{2x^3y^2}$$

(5) Q. Want to write:

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

$$= \nabla_\mu A_\nu - \nabla_\nu A_\mu \quad (\text{Carroll Ch. 4 p. 153})$$

b. y^μ = parallel transported along $x^\nu(\tau)$, $u^\mu = \frac{dx^\mu}{d\tau}$

It must obey the eqn of parallel transport

$$(\text{i.e. } \nabla_u^\mu y^\nu = u^\lambda \nabla_\lambda y^\mu = 0)$$

Let $x^\mu(\tau)$ be an affine geodesic:

$$\Rightarrow x^\mu(\tau) = \alpha \tau + \beta, \quad \alpha, \beta \in \mathbb{C}$$

$$u^\mu = \dot{x}^\mu = \frac{dx^\mu}{d\tau} = g_{\mu\nu} g^{\nu\lambda} \dot{x}^\lambda = g_{\mu\nu} \delta^\lambda_\nu (\alpha) =$$

(P)

10

$$\therefore u^\mu u_\mu = \alpha^2 = \text{constant.}$$

$$ds^2 = 0 \quad ds^2 + 0 =$$

prop

prop

$$Y^\mu u_\mu = Y^0 \alpha_0 + \alpha Y^1 + \alpha Y^2 + \alpha Y^3 \dots$$

$\approx \alpha Y^\mu$ (dot product is conserved since $u_\mu = \text{constant}$).

$$Y^\mu u_\mu =$$

$$\text{Know that } \nabla_\mu Y^\nu = 0 \Rightarrow Y^\nu \partial_\mu Y^\mu + \Gamma_{\mu\nu}^\mu = 0$$

(~~So we want to express $\partial_\mu Y^\mu$ in terms of $\Gamma_{\mu\nu}$~~)

$$\partial_\mu Y^\nu + \Gamma_{\mu\nu}^\mu * Y^\mu = 0$$

$$\therefore Y^\mu u_\mu = Y^\mu \frac{\partial x_\mu}{\partial \tau}$$

stays at tail.

modified variables are A, B, A_B, B_A

(Euler-Lagrange Equations)

$x^B = u$ (2) is path constraint following $= X \cdot d$

constraint follows from $dx^B/dt = 0$ from EL

$$(0 = P^B \dot{x}^B + L_B)$$

boundary conditions in (2) are fed

$$A \cdot d + B \cdot 0 = Q + L_B = (2)^{\text{initial}}$$

Hamiltonian $H = P_A \dot{A} + P_B \dot{B} + L_A A + L_B B$