Assignment 1: Question 1

Friday, 4 March 2022 10:16 pm

(1) Hamiltonian for the 1D Harmonic Oscillator with frequency ω:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

- Ladder operators:

$$\hat{a} = \frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\hbar\omega}} \qquad \hat{a}^\dagger = \frac{m\omega\hat{x} - i\hat{p}}{\sqrt{2m\hbar\omega}}$$

a. Assume that a ket eigenstate of the creation operator exists such that:
$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle$

- β can be decomposed in the basis $|n\rangle$ as follows:

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$$

- Applying the raising operator yields:

$$\hat{a}^\dagger|\beta\rangle = \hat{a}^\dagger\sum_{n=0}^{\infty}c_n|n\rangle \quad , c_n \in \mathbb{C}$$
$$= \sum_{n=0}^{\infty}c_n\sqrt{n+1}\,|n+1\rangle$$
$$= \sum_{n=1}^{\infty}c_{n-1}\sqrt{n}\,|n\rangle$$

If $|\beta\rangle$ is an eigenstate of $\hat{a}^\dagger$ with eigenvalue β, then:

$$\sum_{n=0}^{\infty}c_n|n\rangle = \sum_{n=1}^{\infty}c_{n-1}\sqrt{n}\,|n\rangle$$

- RHS does not have an $n=0$ term $\Rightarrow$ the only possible solution to the above equality is that $c_n = 0$ $\forall n$. Hence, there is no eigenstate of the creation operator $\hat{a}^\dagger$

b. Seek to calculate the evolution of dispersion for $\hat{x}(t)$ and $\hat{p}(t)$ in the Heisenberg picture for the coherent state $|n\rangle$

$$D_n^x = \sqrt{\langle n|\hat{x}_H(t)^2|n\rangle - \langle n|\hat{x}_H(t)|n\rangle^2}$$

- In the Heisenberg picture, the position operator $\hat{x}_H$ is given by:

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}_H^\dagger(t) + \hat{a}_H(t)\right)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the ladder operators in the Schrödinger picture

$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
$\therefore \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$

$$\therefore \hat{x}_H^2(t) = \frac{\hbar}{2m\omega}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right)$$
$$= \frac{\hbar}{2m\omega}\left(\hat{a}\hat{a}e^{-2i\omega t} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger e^{2i\omega t}\right)$$

$\therefore \langle n|\hat{x}_H(t)^2|n\rangle$

$$= \frac{\hbar}{2m\omega}\langle n|\hat{a}\hat{a}e^{-2i\omega t} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger e^{2i\omega t}|n\rangle$$
$$= \frac{\hbar}{2m\omega}\left(e^{-2i\omega t}\langle n|\hat{a}\hat{a}|n\rangle + \langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle + \langle n|\hat{a}^\dagger\hat{a}^\dagger|n\rangle e^{2i\omega t}\right)$$
$$= \frac{\hbar}{2m\omega}\left(\langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle\right)$$
$$= \frac{\hbar}{2m\omega}\left((n+1)\langle n+1|n+1\rangle + n\langle n-1|n-1\rangle\right) = \frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) = l^2\left(n+\frac{1}{2}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle n|\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}|n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\langle n|\hat{a}e^{-i\omega t}|n\rangle + \langle n|\hat{a}^\dagger e^{i\omega t}|n\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|\hat{a}|n\rangle e^{-i\omega t} + \langle n|\hat{a}^\dagger|n\rangle e^{i\omega t}\right)]$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle n|n-1\rangle e^{-i\omega t} + \sqrt{n+1}\langle n|n+1\rangle e^{i\omega t}\right)] = \underline{\underline{0}} \Rightarrow \langle n|\hat{x}_H(t)|n\rangle^2 = \underline{\underline{0}}$$

$$\therefore D_n^x = \left(l^2(n+1/2)\right)^{1/2} = l(n+1/2) = \sqrt{\frac{\hbar}{m\omega}}\left(n+\frac{1}{2}\right)^{1/2}$$

- $D_n^p = \sqrt{\langle n|\hat{p}_H(t)^2|n\rangle - \langle n|\hat{p}_H(t)|n\rangle}$

$\therefore \langle n|\hat{p}_H(t)^2|n\rangle$

- In the Heisenberg picture, $\hat{p}$ is given by:

$$\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}_H(t) - \hat{a}_H^\dagger(t)\right) = \sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right)$$

$$\therefore \hat{p}_H^2 = -\frac{m\omega\hbar}{2}\left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right)^2$$
$$= -\frac{m\omega\hbar}{2}\left(\hat{a}\hat{a}e^{-2i\omega t} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger e^{2i\omega t}\right)$$

$$\therefore \langle n|\hat{p}_H(t)^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n|\hat{a}\hat{a}e^{-2i\omega t} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger e^{2i\omega t}|n\rangle$$
$$= -\frac{m\omega\hbar}{2}\left[\langle n|\hat{a}\hat{a}e^{-2i\omega t}|n\rangle - \langle n|\hat{a}\hat{a}^\dagger|n\rangle - \langle n|\hat{a}^\dagger\hat{a}|n\rangle - \langle n|\hat{a}^\dagger\hat{a}^\dagger e^{2i\omega t}|n\rangle\right]$$
$$= -\frac{m\omega\hbar}{2}\left[-\langle n|\hat{a}\hat{a}^\dagger|n\rangle - \langle n|\hat{a}^\dagger\hat{a}|n\rangle\right]$$
$$= -\frac{m\omega\hbar}{2}\left[-(n+1) - n\right] = \frac{m\omega\hbar}{2}\left[2n+1\right] = m\omega\hbar\left(n+\frac{1}{2}\right), \quad l = \sqrt{\frac{\hbar}{m\omega}} \Rightarrow l^2 = \frac{\hbar}{m\omega} \Rightarrow \frac{\hbar^2}{l^2} = \frac{m\omega}{\hbar}\cdot\hbar^2 = \hbar m\omega$$
$$\Rightarrow \langle n|\hat{p}_H^2(t)|n\rangle = \frac{\hbar^2}{l^2}\left(n+\frac{1}{2}\right)$$

- $\langle n|\hat{p}_H(t)|n\rangle = 0$ by similar arguments to $\langle n|\hat{x}_H|n\rangle = 0$

$$\therefore D_n^p = \sqrt{\langle n|\hat{p}_H^2|n\rangle - \langle n|\hat{p}_H|n\rangle^2} = \sqrt{\frac{\hbar^2}{l^2}\left(n+\frac{1}{2}\right)} = \frac{\hbar}{l}(n+1/2)^{1/2}$$

- Want to show that $D_n^p = m\omega D_n^x$

$$D_n^p = \frac{\hbar}{l}(n+1/2)^{1/2}, \quad D_n^x = \sqrt{\frac{\hbar}{m\omega}}(n+1/2)^{1/2}$$
$$\therefore \sqrt{\hbar m\omega}\,(n+1/2)^{1/2} = D_n^p$$
$$m\omega D_n^x = \sqrt{\frac{\hbar}{m\omega}}\cdot\sqrt{m^2\omega^2}\,(n+1/2)^{1/2} = \sqrt{\hbar m\omega}\,(n+1/2)^{1/2} = D_n^p$$

$\Rightarrow$ $\underline{\underline{\text{QED}}}$. This relation resembles the momentum relation of a classical harmonic oscillator

c. Seek to calculate the evolution of dispersion for the coordinate $\hat{x}(t)$ and momentum operators in the Heisenberg picture for the coherent state $|\alpha\rangle$

$$D_\alpha^x = \sqrt{\langle\alpha|\hat{x}_H(t)^2|\alpha\rangle - \langle n|\hat{x}_H(t)|\alpha\rangle^2}$$

- Compute $\langle\alpha(t)|\hat{x}_H^2(t)|\alpha(t)\rangle$

$$\therefore \frac{\hbar}{2m\omega}\langle\alpha(t)|\left(\hat{a} + \hat{a}^\dagger\right)^2|\alpha(t)\rangle$$
$$= \frac{\hbar}{2m\omega}\langle\alpha(t)|\hat{a}^2 + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}|\alpha(t)\rangle$$
$$= \frac{\hbar}{2m\omega}\langle\alpha(t)|\hat{a}^2 + (\hat{a}^\dagger)^2 + 2\hat{a}\hat{a}^\dagger + 1|\alpha(t)\rangle \qquad \text{(using the relation } [\hat{a},\hat{a}^\dagger] = 1\text{)}$$
$$= \frac{\hbar}{2m\omega}\left[\alpha(t)^2 + \alpha^*(t)^2 + 2\alpha(t)\alpha^*(t) + 1\right]$$
$$= \frac{\hbar}{2m\omega}\left[\alpha(t)^2 + \alpha^*(t)^2 + 2|\alpha(t)|^2 + 1\right] = \frac{\hbar}{2m\omega}\left[\left(\alpha e^{2i\omega t} + \alpha^{*2}e^{-2i\omega t}\right) + 2|\alpha| + 1\right]$$

- Similarly, $\langle\alpha(t)|\hat{x}_H(t)|\alpha(t)\rangle$:

$$\langle\alpha(t)|\hat{x}_H(t)|\alpha(t)\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha(t)|\hat{a} + \hat{a}^\dagger|\alpha(t)\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha(t) + \alpha^*(t)\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha e^{i\omega t} + \alpha^* e^{-i\omega t}\right)$$

$$\therefore \langle\alpha(t)|\hat{x}_H(t)|\alpha(t)\rangle^2 = \frac{\hbar}{2m\omega}\left(\alpha e^{i\omega t} + \alpha^* e^{-i\omega t}\right)^2$$
$$\Rightarrow \frac{\hbar}{2m\omega}\left(\alpha^2 e^{2i\omega t} + 2|\alpha|^2 + \alpha^{*2}e^{-2i\omega t}\right)$$

$$\therefore \langle\alpha(t)|\hat{x}_H(t)^2|\alpha(t)\rangle - \langle\alpha(t)|\hat{x}_H(t)|\alpha(t)\rangle^2$$
$$= \frac{\hbar}{2m\omega}\left[\left(\alpha e^{2i\omega t} + \alpha^{*2}e^{-2i\omega t}\right) + 2|\alpha| + 1\right] - \frac{\hbar}{2m\omega}\left[\alpha^2 e^{2i\omega t} + \alpha^{*2}e^{-2i\omega t} + 2|\alpha|\right]$$
$$= \frac{\hbar}{2m\omega}\left(\left[\alpha^2 e^{2i\omega t} + \alpha^{*2}e^{-2i\omega t} + 2|\alpha| + 1\right] - \left[\alpha^2 e^{2i\omega t} + \alpha^{*2}e^{-2i\omega t} + 2|\alpha|\right]\right)$$
$$= \frac{\hbar}{2m\omega}$$

- Hence, $D_\alpha^x = \sqrt{\langle\alpha(t)|\hat{x}_H^2|\alpha(t)\rangle - \langle\alpha(t)|\hat{x}_H(t)|\alpha(t)\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$

- Similarly one can compute D_α^p

$$D_\alpha^p = \sqrt{\langle\alpha(t)|\hat{p}_H(t)^2|\alpha(t)\rangle - \langle\alpha(t)|\hat{p}_H(t)|\alpha(t)\rangle^2}$$

$\therefore$ Compute $\langle\alpha(t)|\hat{p}_H(t)^2|\alpha(t)\rangle$:

$$= -\frac{m\omega\hbar}{2}\langle\alpha(t)|(\hat{a} - \hat{a}^\dagger)^2|\alpha(t)\rangle$$
$$= -\frac{m\omega\hbar}{2}\langle\alpha(t)|\hat{a}^2 - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2}|\alpha(t)\rangle$$
$$= -\frac{m\omega\hbar}{2}\langle\alpha(t)|\hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{a} - 1|\alpha(t)\rangle$$
$$= \frac{m\omega\hbar}{2}\left[2|\alpha(t)|^2 + 1 - \alpha(t)^2 - \alpha^*(t)^2\right]$$

$$\therefore \langle p_x\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}\langle\alpha(t)|\hat{a} - \hat{a}^\dagger|\alpha(t)\rangle$$
$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\alpha(t) - \alpha^*(t)\right]$$
$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left[\alpha e^{i\omega t} - \alpha^* e^{-i\omega t}\right]$$
$$\therefore \langle p_x\rangle^2 = -\frac{m\omega\hbar}{2}\left[\alpha^2 e^{2i\omega t} - 2|\alpha|^2 + \alpha^{*2}e^{-2i\omega t}\right]$$

$$\therefore \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = \left(\frac{m\omega\hbar}{2}\left[2|\alpha|^2 + 1 - \alpha^2 e^{2i\omega t} - \alpha^{*2}e^{-2i\omega t}\right] + \frac{m\omega\hbar}{2}\left[\alpha^2 e^{2i\omega t} - 2|\alpha|^2 + \alpha^{*2}e^{-2i\omega t}\right]\right)$$
$$= \frac{m\omega\hbar}{2}$$

$$\therefore D_\alpha^p = \sqrt{\frac{m\omega\hbar}{2}}$$

- Taking the product of the two dispersion relations yields: $D_\alpha^p D_\alpha^x = \sqrt{\frac{m\omega\hbar}{2}}\sqrt{\frac{\hbar}{2m\omega}} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2}$ $\Rightarrow$ this is the minimum allowed by the Heisenberg Uncertainty principle. This means that the spread of the wavefunction remains at a minimum as it oscillates back and forth

d. Seek to determine the coherent state wavefunction in position space

- Note that the raising operator is given by:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}_x\right)$$

$\therefore$ In position space, this can be written as:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\partial}{\partial x}\frac{\hbar}{m\omega}\right)$$

- Applying the definition of the coherent state: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ in position space

$$= \sqrt{\frac{m\omega}{2\hbar}}\left(x\Psi_\alpha(x) + \frac{\hbar}{m\omega}\frac{\partial\Psi}{\partial x}\right) = \alpha\Psi_\alpha(x)$$
$$= \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{\hbar}{m\omega}\Psi'(x) + x\Psi(x)\right) = \alpha\Psi(x)$$
$$\therefore \sqrt{\frac{m\omega}{2\hbar}}\cdot\frac{\hbar}{m\omega}\Psi'(x) + \sqrt{\frac{m\omega}{2\hbar}}\,x\Psi(x) = \alpha\Psi(x)$$
$$= \sqrt{\frac{\hbar^2 m\omega}{2\hbar m^2\omega^2}}\Psi'(x) + \sqrt{\frac{m\omega}{2\hbar}}\,x\Psi(x) = \alpha\Psi(x)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\Psi'(x) + \sqrt{\frac{m\omega}{2\hbar}}\,x\Psi(x) = \alpha\Psi(x)$$
$$\Rightarrow \Psi'(x) + \sqrt{\frac{m\omega}{2\hbar}}\cdot\sqrt{\frac{2m\omega}{\hbar}}\,x\Psi(x) = \alpha\sqrt{\frac{2m\omega}{\hbar}}\Psi(x)$$
$$\Rightarrow \Psi'(x) + \frac{m\omega}{\hbar}x\Psi(x) = \alpha\sqrt{\frac{2m\omega}{\hbar}}\Psi(x)$$
$$\Rightarrow \Psi'(x) + \frac{m\omega}{\hbar}x\Psi(x) = \alpha\frac{\sqrt{2}}{l}\Psi(x)$$
$$\Rightarrow \Psi'(x) = \left(\alpha\frac{\sqrt{2}}{l} - \frac{m\omega}{\hbar}x\right)\Psi(x)$$
$$\Rightarrow \Psi'(x) = \left(\frac{\sqrt{2}\alpha}{l} - \frac{x}{l^2}\right)\Psi(x)$$
$$\therefore \frac{\Psi'(x)}{\Psi(x)} = \frac{\sqrt{2}\alpha}{l} - \frac{x}{l^2}$$
$$\therefore \ln(\Psi(x)) = \frac{\sqrt{2}\alpha x}{l} - \frac{x^2}{2l^2}$$
$$\therefore \Psi(x) = \Psi_0\exp\left(\frac{\sqrt{2}\alpha x}{l} - \frac{x^2}{2l^2}\right) \qquad (\Psi_0 = \text{normalisation constant})$$

- Let $\alpha = \mathrm{Re}(\alpha) + i\,\mathrm{Im}(\alpha)$

$$\therefore \Psi(x) = \Psi_0\exp\left(\frac{\sqrt{2}\left(\mathrm{Re}(\alpha) + i\,\mathrm{Im}(\alpha)\right)x}{l} - \frac{x^2}{2l}\right)$$
$$= \Psi_0\exp\left(\frac{\sqrt{2}\,\mathrm{Re}(\alpha)}{l} + \frac{\sqrt{2}\,i\,\mathrm{Im}(\alpha)}{l} - \frac{x^2}{2l}\right)$$
$$= \Psi_0\exp\left(\frac{-x^2}{2l} + \frac{2\sqrt{2}\,\mathrm{Re}(\alpha)x}{2l} + \frac{\sqrt{2}\,i\,\mathrm{Im}(\alpha)x}{l}\right)$$
$$= \Psi_0\exp\left(\frac{-x^2 + 2\sqrt{2}\,\mathrm{Re}(\alpha)x + \sqrt{2}\,i\,\mathrm{Im}(\alpha)x}{2l}\right)$$
$$= \Psi_0\exp\left(-\frac{1}{2l^2}\overbrace{\left(x - \sqrt{2}\,l\,\mathrm{Re}(\alpha)\right)^2}^{\text{complete square of real part}} + \frac{1}{\hbar}\left(\sqrt{2}\hbar\,i\,\mathrm{Im}(\alpha)x\right)\right)$$
$$= \Psi_0\exp\left(-\frac{1}{2l^2}(x - x_0)^2 + \frac{1}{\hbar}(p_0 x i)\right)$$

- One can determine the normalisation constant by requiring that:

$$\int_{-\infty}^{\infty}|\Psi(x)|^2\,dx = 1$$
$$\Rightarrow \Psi_0^2\int_{-\infty}^{\infty}\left|e^{-1/2l^2(x-x_0)^2 + 1/\hbar(p_0 x i)}\right|^2 dx = 1$$

$\Rightarrow$ Evaluating this integral and combining the necessary constants yields that: (done computationally)

$$\Psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$\therefore$ The real space wavefunction is

$$\boxed{\Psi_\alpha(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{1}{2l^2}(x - x_0)^2 + \frac{1}{\hbar}p_0 x i\right)}$$

- The above expression for the position space wavefunction bears resemblance to the ground state wavefunction for the harmonic oscillator:

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}$$

Hence, it can be argued that the constants x_0 and p_0 can be interpreted as the initial position and momentum of the particle. Additionally, it may be argued that the quantum particle experiences classical motion by noting the similarity between the forms of the ground state wavefunction and the coherent state wave function in position space.