Assignment 1: Question 1

Monday, 8 May 2023 5:12 pm

(1) Parallel transport on the sphere:

Consider a sphere of unit radius:

a. Write down the metric and derive the connection coefficients:

Solution:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin(\theta) \end{pmatrix} \underset{(r=1)}{=} \begin{array}{c} \begin{array}{ccc} r & \theta & \phi \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2(\theta) \end{pmatrix} \end{array} = \begin{pmatrix} g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$

Connection coefficients:

Need $\partial_\theta(g_{\phi\phi})$ and g_{rr}, $g_{\theta\theta}$ or $g_{\phi\phi}$

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

$$\therefore \quad \Gamma^{r}_{\phi\phi} = \frac{1}{2} g^{r\rho}\left(\partial_\phi g_{\phi\rho} + \partial_\phi g_{\rho\phi} - \partial_\rho g_{\phi\phi}\right)$$

- Non-zero terms:

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} g^{\theta\rho}\left(\partial_\phi g_{\phi\rho} + \partial_\phi g_{\rho\phi} - \partial_\rho g_{\phi\phi}\right)$$

$$= \frac{1}{2} g^{\theta\theta}\left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right)$$

$$= \frac{1}{2}(1)\left(-\partial_\theta\left(\sin^2(\theta)\right)\right)$$

$$= \frac{1}{2}(1)\left(-\sin(2\theta)\right) = -\sin(\theta)\cos(\theta)$$

$$\Gamma^{\varphi}_{\theta\varphi} = \frac{1}{2} g^{\varphi\varphi}\left(\partial_\theta g_{\varphi\varphi} + \partial_\varphi g_{\varphi\theta} - \partial_\varphi g_{\theta\varphi}\right)$$

$$= \frac{1}{2}\frac{1}{\sin^2(\theta)}\partial_\theta\left(\sin^2(\theta)\right) = \frac{\cos(\theta)}{\sin(\theta)} = \cot(\theta)$$

By symmetry, $\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \cot(\theta)$. (All other connection coefficients vanish)

b. Explore what happens when you transport the tangent vector $\vec{A}$ around a parallel with fixed latitude θ. Show that the vector rotates by the angle $2\pi(1-\cos(\theta))$ = area on the sphere enclosed by the circle

Proceed as follows:

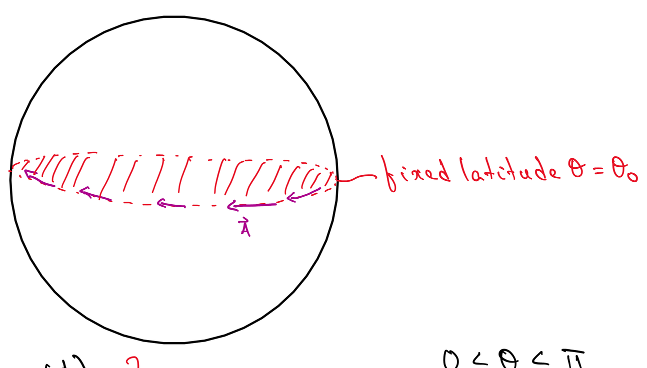
(i) Express the vector in the coordinate basis $\vec{A} = A^\varphi \partial_\varphi + A^\theta \partial_\theta$.

(ii) Derive how components evolve as one moves around the circle using $\nabla_\varphi \vec{A} = 0$

(iii) Define new variables $\hat{A}^\theta = A^\theta$ and $\hat{A}^\varphi = A^\varphi \sin(\phi)$. Using the results of (ii), derive how $\hat{A}^\theta$ and $\hat{A}^\varphi$ evolve as one moves around the circle of constant latitude. Show that this evolution is simply a rotation by angle $\varphi\cos(\theta)$.

(iv) Thus derive the angle by which the vector rotates as one transports it around the full circle

Solution:



i.
$$\left.\begin{aligned} x &= \cos(\theta)\cos(\phi) \\ y &= \cos(\theta)\sin(\phi) \\ z &= \cos(\theta) \end{aligned}\right\} r=1 \qquad \begin{aligned} 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{aligned}$$

i. Expressing the vector in the coordinate basis $\vec{A} = A^\varphi \partial_\varphi + A^\theta \partial_\theta$, one can derive how these components evolve as one moves around the circle at fixed latitude $\theta = \theta_0$:

- Parametrise the circle at $\theta = \theta_0$ by some parameter λ. Then, the components of $\vec{A}$ can be written as:

$$\vec{A} = A^\theta \partial_\theta + A^\varphi \partial_\varphi \quad \text{where } \theta = \theta(\lambda) \text{ and } \varphi = \varphi(\lambda)$$

$\therefore$ $A^\theta = \frac{d\theta}{d\lambda}$ and $A^\varphi = \frac{d\varphi}{d\lambda}$

$\therefore$ Since θ is fixed $\Rightarrow \frac{d\theta}{d\lambda} = 0 \Rightarrow \vec{A} = \frac{d\varphi}{d\lambda}\partial_\varphi$

Hence, using the relation $\nabla_\varphi \vec{A} = 0$, one can obtain that:

$$\nabla_\varphi A^\nu = 0 \Rightarrow \frac{dA^\nu}{d\lambda} + \Gamma^\nu_{\varphi\sigma}\frac{d\varphi}{d\lambda}A^\sigma = 0$$

$$\Rightarrow \frac{dA^\nu}{d\varphi}\frac{d\varphi}{d\lambda} + \Gamma^\nu_{\varphi\sigma}\frac{d\varphi}{d\lambda}A^\sigma = 0$$

$$\Rightarrow \frac{dA^\nu}{d\varphi} + \Gamma^\nu_{\varphi\sigma}A^\sigma = 0$$

Hence:

$$\frac{dA^\nu}{d\varphi} = -\Gamma^\nu_{\varphi\sigma}A^\sigma$$

$\therefore$ Using the Christoffel symbols computed above, one can deduce that:

$$\left.\begin{aligned} \frac{dA^\theta}{d\varphi} &= -\Gamma^\theta_{\varphi\sigma}A^\sigma = \sin(\theta_0)\cos(\theta_0)A^\varphi \quad (1) \\ \frac{dA^\varphi}{d\varphi} &= -\Gamma^\varphi_{\varphi\sigma}A^\sigma = -\cot(\theta_0)A^\theta \quad (2) \end{aligned}\right\} \text{Two coupled equations for how the components evolve}$$

- Define new variables where $\hat{A}^\theta = A^\theta$ and $\hat{A}^\varphi = A^\varphi\sin(\varphi)$

$\therefore$ The coupled equations from above become:

$$\therefore \quad \frac{d\hat{A}^\theta}{d\varphi} = 1\left\{\frac{d\hat{A}^\theta}{dA^\theta}\times\frac{dA^\theta}{d\varphi}\right. \Rightarrow \quad \frac{d\hat{A}^\theta}{d\varphi} = \sin(\theta_0)\cos(\theta_0)A^\varphi$$

$$\frac{d\hat{A}^\varphi}{d\varphi} = \frac{d\hat{A}^\varphi}{dA^\varphi}\frac{dA^\varphi}{d\varphi} = \sin(\varphi)\frac{dA^\varphi}{d\varphi} = -\cot(\theta_0)A^\theta$$

$$\frac{1}{\sin(\varphi)}\frac{d\hat{A}^\varphi}{d\varphi} = -\cot(\theta_0)\hat{A}_\theta$$

$$\sin(\varphi)\frac{dA^\varphi}{d\varphi} = \cot(\theta_0)A^\theta$$

$$\therefore \quad \frac{d\hat{A}^\varphi}{d\varphi} = -\cot(\theta_0)\sin(\varphi)\hat{A}_\theta$$

$$\therefore \quad \frac{d^2\hat{A}^\theta}{d\varphi^2} = \sin(\theta_0)\cos(\theta_0)\frac{dA^\varphi}{d\varphi}$$

$$\therefore \quad \frac{d^2\hat{A}^\theta}{d\varphi^2} = \sin(\theta_0)\cos(\theta_0)\cdot\left[-\cot(\theta_0)\hat{A}^\theta\right]$$

$$\therefore \quad \frac{d^2\hat{A}^\theta}{d\varphi^2} = -\cos^2(\theta_0)\hat{A}^\theta \quad \Rightarrow \quad \frac{d^2\hat{A}^\theta}{d\varphi^2} + \cos^2(\theta_0)\hat{A}^\theta = 0 \quad 3$$

$$\therefore \quad \hat{A}^\theta(\varphi) = A\cos\left(\cos(\theta_0)\varphi\right) + B\sin\left(\cos(\theta_0)\varphi\right)$$

$$\hat{A}^\varphi(\varphi) = C\cos\left(\cos(\theta_0)\varphi\right) + D\sin\left(\cos(\theta_0)\varphi\right)$$

In order to determine the constants A, B, C and D, impose the conditions $\underbrace{\hat{A}^\alpha}_{\alpha=\{\theta,\varphi\}} = \underbrace{(\hat{A}^\theta_0, \hat{A}^\varphi_0)}_{\text{constants}}$. Furthermore, we must have:

$$\left.\frac{\partial\hat{A}^\theta}{\partial\varphi}\right|_{\varphi=0} = \hat{A}^\varphi_0\sin(\theta_0)\cos(\theta_0)$$

$$\therefore \quad \left.\frac{\partial\hat{A}^\varphi}{\partial\varphi}\right|_{\varphi=0} = -\hat{A}^\theta_0\frac{\cos(\theta_0)}{\sin(\theta_0)}$$

- Hence, we have:

$$\hat{A}^\theta(\varphi) = \hat{A}^\theta_0\cos\left(\cos(\theta_0)\right) + \frac{\hat{A}^\varphi_0\sin(\theta_0)\cos(\theta_0)}{\cos(\theta_0)}\sin\left(\cos(\theta_0)\varphi\right)$$

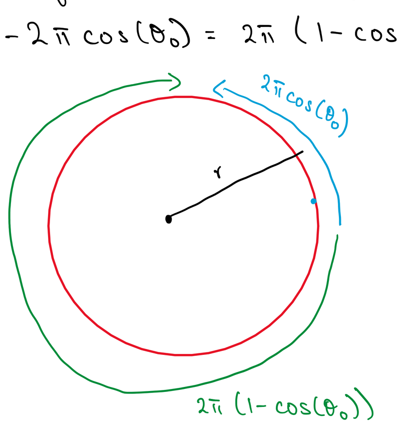
$$= \hat{A}^\theta_0\cos\left(\cos(\theta_0)\varphi\right) + \hat{A}^\varphi_0\sin(\theta_0)\sin\left(\cos(\theta_0)\varphi\right)$$

$$\therefore \quad \begin{aligned} \hat{A}^\theta(\varphi) &= \hat{A}^\theta_0\cos\left(\varphi\cos(\theta_0)\right) + \hat{A}^\varphi_0\sin(\theta_0)\sin\left(\varphi(\cos(\theta_0))\right) \\ \hat{A}^\varphi(\varphi) &= \hat{A}^\varphi_0\cos\left(\varphi\cos(\theta_0)\right) - \frac{\hat{A}^\theta_0}{\sin(\theta_0)}\left(\varphi\cos(\theta_0)\right) \end{aligned}$$

- Parallel transporting the vector $\vec{A}$ around the circle corresponds to setting $\varphi = 2\pi$. Under this transformation the vector components evolve as:

$$\begin{aligned} \hat{A}^\theta &= \hat{A}^\theta_0\cos\left(2\pi\cos(\theta_0)\right) + \hat{A}_0\sin(\theta_0)\sin\left(2\pi\cos(\theta_0)\right) \\ \hat{A}^\varphi &= \hat{A}^\varphi_0\cos\left(2\pi\cos(\theta_0)\right) - \frac{\hat{A}^\theta_0}{\sin(\theta_0)}\left(2\pi\cos(\theta_0)\right) \end{aligned}$$

- Consider the circle at fixed latitude $\theta = \theta_0$. A rotation by angle $2\pi\cos(\theta_0)$ in one direction is equivalent to a rotation by $2\pi - 2\pi\cos(\theta_0) = 2\pi(1-\cos(\theta_0))$ in the opposite direction. Hence, the vector is transported around the entire circle when rotated by either angle



$\therefore$ $\vec{A}$ is rotated by an angle of $2\pi(1-\cos(\theta_0)) \Rightarrow$ QED

Assignment 1: Question 2

Wednesday, 3 May 2023 2:51 pm

(2) Coordinate Transformations in Euclidean Space

Define coordinates $(x'^1, x'^2, x'^3) = (u, v, \phi)$ such that:

$$x = uv\cos(\phi)$$
$$y = uv\sin(\phi)$$
$$z = \frac{1}{2}(u^2 - v^2)$$

a. Find the Jacobian $\frac{\partial x^\alpha}{\partial x'^\beta}$ and the inverse of the Jacobian

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial \phi} \end{bmatrix} = \begin{bmatrix} v\cos(\phi) & u\cos(\phi) & -uv\sin(\phi) \\ v\sin(\phi) & u\sin(\phi) & uv\cos(\phi) \\ u & -v & 0 \end{bmatrix}$$

$$\therefore J^{-1} = \begin{bmatrix} v\cos(\phi) & u\cos(\phi) & -uv\sin(\phi) \\ v\sin(\phi) & u\sin(\phi) & uv\cos(\phi) \\ u & -v & 0 \end{bmatrix}^{-1} = \underbrace{\begin{bmatrix} \frac{v\cos(\phi)}{u^2+v^2} & \frac{v\sin(\phi)}{u^2+v^2} & \frac{u}{u^2+v^2} \\ \frac{u\cos(\phi)}{u^2+v^2} & \frac{u\sin(\phi)}{u^2+v^2} & \frac{-v}{u^2+v^2} \\ \frac{-\sin(\phi)}{uv} & \frac{\cos(\phi)}{uv} & 0 \end{bmatrix}}_{\text{Evaluated using Mathematica}}$$

b. Express the basis vectors $\frac{\partial}{\partial x'^\beta}$ in terms of Cartesian basis vectors:

Solution:

$$\partial_u = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} = v\cos(\phi)\,\partial_x + v\sin(\phi)\,\partial_y + u\,\partial_z$$
$$\partial_v = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z} = u\cos(\phi)\,\partial_x + u\sin(\phi)\,\partial_y - v\,\partial_z$$
$$\partial_\phi = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} = -uv\sin(\phi)\,\partial_x + uv\cos(\phi)\,\partial_y$$

$$\Rightarrow \begin{pmatrix} \partial_u \\ \partial_v \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} v\cos(\phi) & v\sin(\phi) & u \\ u\cos(\phi) & u\sin(\phi) & -v \\ -uv\sin(\phi) & uv\cos(\phi) & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

(where $\partial_x, \partial_y, \partial_z$ and $\partial_u, \partial_v, \partial_\phi$ represent the relevant basis vectors)

c. Find the metric and inverse metric in the new coordinates x'

Solution:

Euclidean line element in 3D space with Cartesian coordinates:

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

Change to (u, v, ϕ) coordinates:

$$x = uv\cos(\phi)$$
$$y = uv\sin(\phi)$$
$$z = \frac{1}{2}(u^2 - v^2)$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial \phi}d\phi$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial \phi}d\phi$$

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv$$

$$\Rightarrow dx = v\cos(\phi)\,du + u\cos(\phi)\,dv - uv\sin(\phi)\,d\phi$$
$$dy = v\sin(\phi)\,du + u\sin(\phi)\,dv + uv\cos(\phi)\,d\phi$$
$$dz = u\,du - v\,dv$$

$$\therefore (dx)^2 = v^2\cos^2(\phi)\,du^2 + u^2\cos^2(\phi)\,dv^2 + u^2v^2\sin^2(\phi)\,d\phi^2$$
$$(dy)^2 = v^2\sin^2(\phi)\,du^2 + u^2\sin^2(\phi)\,dv^2 + u^2v^2\cos^2(\phi)\,d\phi^2$$
$$(dz)^2 = u^2\,du^2 + v^2\,dv^2$$

$$\therefore ds^2 = (v^2\cos^2(\phi) + v^2\sin^2(\phi) + u^2)\,du^2 + (u^2\cos^2(\phi) + u^2\sin^2(\phi) + v^2)\,dv^2 + (u^2v^2\cos^2(\phi) + u^2v^2\sin^2(\phi))\,d\phi$$

$$= (v^2 + u^2)\,du^2 + (u^2 + v^2)\,dv^2 + u^2v^2\,d\phi^2$$ (cross terms cancel)

$$\therefore g_{\mu\nu} = \begin{pmatrix} v^2+u^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix}$$

Inverse metric $= g^{\mu\nu}$. By definition:

$$g^{\mu\nu} g_{\nu\vartheta} = \delta^\mu_\vartheta$$

$$\Rightarrow g^{\mu\nu} = \begin{pmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & u^2v^2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{u^2+v^2} & 0 & 0 \\ 0 & \frac{1}{u^2+v^2} & 0 \\ 0 & 0 & \frac{1}{u^2v^2} \end{pmatrix} = \text{inverse metric}$$

d. Calculate the divergence $\nabla_\mu V^\mu$ and Laplacian $\nabla_\mu \nabla^\mu J$

Solution:

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda$$
$$= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,V^\mu)$$

$$\therefore |g| = (u^2+v^2)^2u^2v^2$$
$$\therefore \sqrt{|g|} = (u^2+v^2)uv =$$

$$\therefore \nabla_\mu V^\mu = \frac{1}{(u^2+v^2)uv}\partial_\mu((u^2+v^2)uv\,V^\mu)$$

$$= \frac{1}{(u^2+v^2)uv}\left[\frac{\partial}{\partial u}(uv(u^2+v^2)V^u) + \frac{\partial}{\partial v}(uv(u^2+v^2)V^v) + \frac{\partial}{\partial \phi}(uv(u^2+v^2)V^\phi)\right]$$

$$= \frac{1}{(u^2+v^2)uv}\left[(2u^2v + v(u^2+v^2))V^u + uv(u^2+v^2)\frac{\partial V^u}{\partial u} + (2uv^2 + u(u^2+v^2))V^v + (u^2+v^2)uv\frac{\partial V}{\partial v} + uv(u^2+v^2)\frac{\partial V^\phi}{\partial \phi}\right]$$

$$= \left[\left(\frac{1}{u} + 2u^2v\right)V^u + \frac{\partial V^u}{\partial u} + \left(\frac{1}{v} + 2uv^2\right)V^v + \frac{\partial V^v}{\partial v} + \frac{\partial V^\phi}{\partial \phi}\right]$$

$$= \left[\frac{2u^3v+1}{u}V^u + \frac{\partial V^u}{\partial u} + \frac{2uv^3+1}{v}V^v + \frac{\partial V^v}{\partial v} + \frac{\partial V^\phi}{\partial \phi}\right]$$

The Laplacian is given by $\nabla_\mu \nabla^\mu J$:

= Covariant derivative of divergence of a vector field

The divergence $\nabla_\mu V^\mu$ is given by:

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,V^\mu)$$

$$\therefore \nabla_\mu \nabla^\mu J = \nabla_\mu(g^{\nu\mu}\nabla_\nu J)$$

$$= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}(g^{\nu\mu}\nabla_\nu J))$$

$$= \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\,g^{\mu\nu}\delta^\mu_\nu\nabla_\mu J)$$

$$= \frac{1}{\sqrt{|g|}}(\partial_\mu(\sqrt{|g|}\,g^{\mu\nu}(\partial_\nu J)\underbrace{\delta^\mu_\nu}_{\text{since the metric is diagonal}}))$$

$$= \frac{1}{uv(u^2+v^2)}(\partial_\mu(uv(u^2+v^2)\,g^{\mu\nu}(\partial_\nu J)\delta^\mu_\nu))$$

$$= \frac{1}{uv(u^2+v^2)}\left(\partial_\mu\left(uv(u^2+v^2)\left[\frac{1}{u^2+v^2}\partial_u(J) + \frac{1}{u^2+v^2}\partial_v(J) + \frac{1}{u^2v^2}\partial_\phi(J)\right]\right)\right)$$

$$= \frac{1}{uv(u^2+v^2)}\left(\underbrace{\partial_\mu}_{\mu = \{u, v, \varphi\}}\left[\frac{1}{uv}\partial_u(J) + \frac{1}{uv}\partial_v(J) + \frac{(u^2+v^2)}{uv}\partial_\phi(J)\right]\right)$$

$$= \frac{1}{uv(u^2+v^2)}\left(\partial_\mu\left[\frac{1}{uv}\frac{\partial J}{\partial u} + \frac{1}{uv}\frac{\partial J}{\partial v} + \frac{(u^2+v^2)}{uv}\frac{\partial J}{\partial \phi}\right]\right)$$

$$= \frac{1}{uv(u^2+v^2)}\left(\underbrace{\partial_\mu}_{\text{Greek letter "mu"}}\left[\frac{1}{uv}\left(\underbrace{\frac{\partial J}{\partial u}}_{\text{Latin letter "u"}} + \frac{\partial J}{\partial v} + (u^2+v^2)\frac{\partial J}{\partial \phi}\right)\right]\right)$$