Assignment 2: Question 4

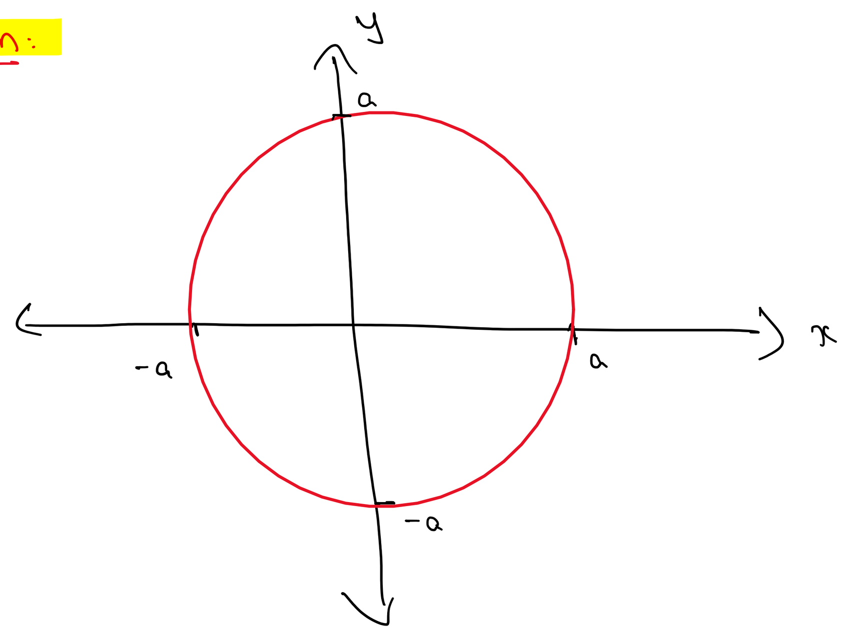
Saturday, 26 March 2022 11:51 am

(4) Thin conducting ring of radius a lies in the x-y plane and carries an alternating current $I = I_0 \cos(\omega t)$

a. Calculate the electric dipole moment
b. Calculate the magnetic dipole moment
c. Calculate the electric field in the radiation zone
d. Calculate the total time-averaged power radiated.

Solution.

a.



$\therefore$

a. The electric dipole moment is given by:

$$\vec{p} = \int \underset{\sim}{x}\, \rho(\underset{\sim}{x})\, d^3x$$

- Can use the result from the continuity equation to determine the charge density $\rho(\underset{\sim}{x})$:

$$i\omega\rho = \underset{\sim}{\nabla} \cdot \vec{J}$$

- The current density $\vec{J}$ only has a component in the azimuthal $(\hat{\underset{\sim}{\phi}})$ direction (where the problem is parametrised in cylindrical coordinates:

$$\vec{J} = I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{\phi}}$$

$\therefore\ \underset{\sim}{\nabla} \cdot \vec{J} = 0 \Rightarrow \rho = 0 \Rightarrow \underline{\underline{\underset{\sim}{p} = \underset{\sim}{0}}}$

b. Magnetic dipole moment:

$$\underset{\sim}{m} = \int M\, d^3x = \frac{1}{2}\int (\underset{\sim}{x} \times \vec{J})\, d^3x$$

$\therefore$

$\therefore\ \underset{\sim}{m} = -\frac{1}{2}\int \left(I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{\phi}}\right) \times \left(r\,\hat{\underset{\sim}{r}} + \phi\,\hat{\underset{\sim}{\phi}} + z\,\hat{\underset{\sim}{z}}\right) r\, dr\, d\phi\, dz$

$$= \begin{vmatrix} \hat{\underset{\sim}{r}} & \hat{\underset{\sim}{\phi}} & \hat{\underset{\sim}{z}} \\ 0 & I(t)\,\delta(r-a)\,\delta(z) & 0 \\ r & \phi & z \end{vmatrix}$$

$$= z\, I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{r}} + r\, I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{z}}$$

$$\Rightarrow \int \left(zr\, I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{r}} + r^2 I(t)\,\delta(r-a)\,\delta(z)\,\hat{\underset{\sim}{z}}\right) dr\, d\phi\, dz$$

$$= \left(2\pi a^2\, I(t)\right) \cdot \frac{1}{2} = \underline{\underline{\pi a^2 I(t)}} = \pi a^2 I_0 \cos(\omega t)\,\hat{\underset{\sim}{z}} = m_0 \cos(\omega t)\,\hat{\underset{\sim}{z}} \quad (\text{where } m_0 = \pi a^2 I_0)$$

c. The electric field is determined by:

$\vec{E}_{M_1} = \frac{-Z_0}{4\pi} k^2 (\underset{\sim}{n} \times \underset{\sim}{m}) \frac{e^{ikr}}{r}\left(1 + \frac{i}{kr}\right)$. In the radiation zone $\left(1 + \frac{i}{kr}\right) \sim 1$ (as $\frac{i}{kr}$ term is negligible)

Switch to spherical coordinates:

$$= \frac{-Z_0}{4\pi} k^2 I_0 e^{i\omega t} \pi a^2\, \underset{\sim}{n} \times \hat{\underset{\sim}{z}}\, \frac{e^{ikr}}{r}$$

$\hat{\underset{\sim}{n}} \times \hat{\underset{\sim}{z}}$ will point in the azimuthal direction

- $\hat{\underset{\sim}{n}} \times \hat{\underset{\sim}{z}} = \hat{\underset{\sim}{r}} \times \left(\hat{\underset{\sim}{r}} \cos(\theta) - \hat{\underset{\sim}{\theta}} \sin(\theta)\right) = -\sin(\theta)\,\hat{\underset{\sim}{\phi}}$

$$\therefore\ \vec{E}_{M_1} = \frac{Z_0}{4\pi} k^2 I_0 e^{i\omega t} \pi a^2 \sin(\theta) \frac{e^{ikr}}{r}\,\hat{\underset{\sim}{\phi}}$$

$$= \frac{Z_0}{4\pi} k^2 I_0 \pi a^2 \frac{e^{i(kr+\omega t)}}{r} \sin(\theta)\,\hat{\underset{\sim}{\phi}}$$

$$= \boxed{\frac{Z_0}{4\pi} k^2 I_0 \pi a^2 \cos(kr + \omega t) \sin(\theta)\,\hat{\underset{\sim}{\phi}}}$$

d. To compute the total time-averaged power radiated, one can use the following relation.

$$\frac{dP}{d\Omega} = \frac{c^2 k^4 Z_0}{32\pi^2} |\hat{\underset{\sim}{n}} \times \underset{\sim}{p}|^2 = \frac{c^2 k^4 Z_0}{32\pi^2} \left|\hat{\underset{\sim}{n}} \times \frac{\underset{\sim}{m}}{c}\right|^2$$

- Take $\hat{\underset{\sim}{n}} = \hat{\underset{\sim}{r}}$ and $\underset{\sim}{m} = \pi a^2 \cos(\omega t) I_0$

$$\therefore\ P = \frac{c^2 k^4 Z_0}{12\pi} \cdot \frac{\pi^2 a^4 \cos^2(\omega t) I_0}{c^2}$$

$$\therefore\ \boxed{P = \frac{k^4 Z_0 \pi a^4 I_0 \cos^2(\omega t)}{\underline{\underline{12}}}}$$