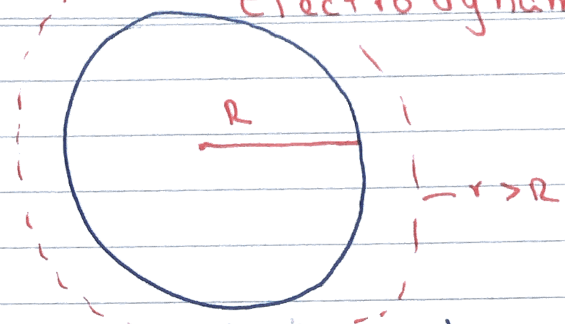
Electrodynamics: Final Examination

(1) a.



· Spherical conducting shell, radius R total charge Q

∴ $\vec{E}_{inside} = 0$ (as the shell is conducting)

& $\vec{E}_{out} = ?$

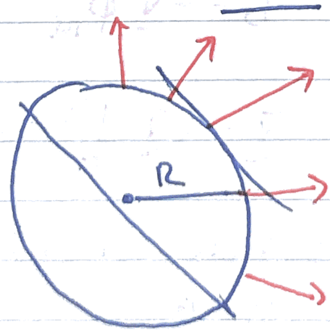
Apply Gauss's Law:

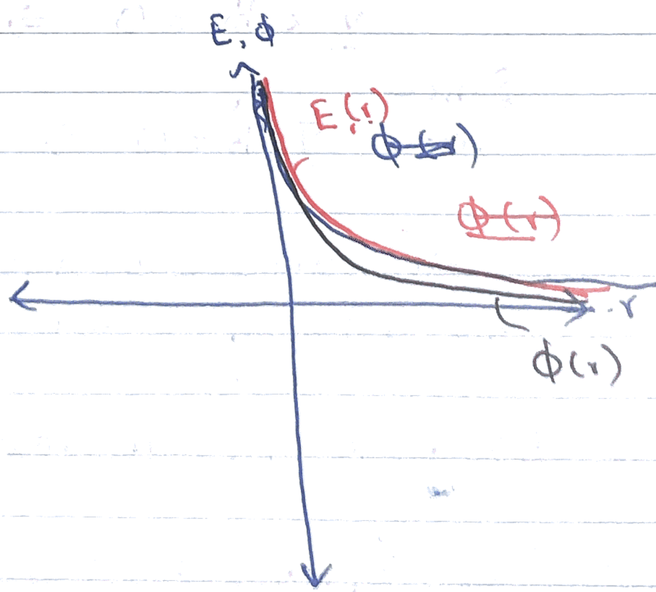
$$\oint \vec{E} \cdot d\underset{\sim}{a} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$$

∴ $\vec{E} = \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\underset{\sim}{r}}$ $\qquad \phi_{out}(r) = -\dfrac{q}{4\pi\varepsilon_0 r}$

$\phi_{in}(r) = C$ (c = constant)

b.





c.

~~d.~~ d. $\nabla^2 \phi_M = 0$ has the ~~solution~~ following solution:

$$\nabla^2 \phi_M = \sum_{l=0}^{\infty} (a_l r^l + b_l r^{-l-1}) P_l(\cos(\theta))$$

where $P_l(\cos(\theta))$ = Legendre polynomials. Since there is azimuthal symmetry, there is no ϕ-dependence (as this is a sphere) in the solution. This ~~can be applied as~~ solution applies as there is no free ~~charge~~ current.

$\therefore$ Since ~~$\nabla \times$~~ ~~$\nabla \times \vec{H} = 0$~~ Since $\nabla \times \vec{B} = 0$ in this case, it follows that $\vec{B}$ can be written as the gradient of a scalar potential:

$$\nabla \times \vec{B} = 0 \Rightarrow \vec{B} = -\nabla \phi_M$$

Since ~~Since~~ $\nabla \cdot \vec{B} = 0 \Rightarrow -\nabla \cdot (\nabla \phi_M) = 0 \Rightarrow \nabla^2 \phi_M = 0$

At the surface of the sphere, $(r = R)$ $\nabla \times \vec{B} \neq 0 \Rightarrow$ this solution will not hold.

e. At $r = R$, the normal component of $\vec{B}$ is continuous

(4) $(\vec{B}_2 - \vec{B}_1) \cdot n = 0$ (where $\vec{B}_2$ and $\vec{B}_1$ are ~~soln~~ obtained from the ~~soln~~ ϕ_M in each case (where $r > R$ and $r < R$) (from slides)

~~Since Assuming the~~

$\therefore$ We have that $(-\nabla \phi_2 + \nabla \phi_1) \cdot \hat{n} = 0$

$\therefore \cancel{(\underset{\sim}{\nabla}\phi_1 - \underset{\sim}{\nabla}\phi_2) = 0}$ Additionally, since $\nabla \cdot \vec{B} = 0$

$\therefore \cancel{\phi_1 = \phi_2 \text{ at } r = R}$ $\Rightarrow B_{r}$ radial component of $\vec{B}$

$$-\frac{\partial \phi_1}{\partial r}\bigg|_{r=R} = \frac{\partial \phi_2}{\partial r}\bigg|_{r=R}$$

d. ~~(Addendum to~~ (Addendum to part (d))

- For $r<R$, we seek $\vec{B}$ to be finite
For $r>R$, we seek $\vec{B}$ to $\rightarrow 0$

$$\therefore \Phi_{r<R} = \sum_{l=0}^{\infty} a_l r^l P_l(\cos(\theta)),$$

$$\phi_{r>R} = \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos(\theta))$$

e. (contd.) Using the condition obtained above, one can write that:

$$\cancel{\sum_{l=0}^{\infty} a_l R^l P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{b_l}{R^{l+1}} P_l(\cos(\theta))}$$

f. Tangential component is discontinuous ~~by~~ across the surface due to the surface current density $\vec{K}$:

$$\underset{\sim}{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K} \quad (\text{where } \vec{H} = \mu_r \vec{B})$$

$$\therefore \underset{\sim}{n} \times (\mu_2 \vec{B}_2 - \mu_1 \vec{B}_1) = \vec{K} \quad (\text{from lecture slides})$$

$$\underset{\sim}{n} \times (-\mu_2 \underset{\sim}{\nabla}\phi_2 + \mu_1 \underset{\sim}{\nabla}\phi_1) = \vec{K}$$

$\therefore$

g. Surface current density $\vec{K} = k_0 \sin(\theta)\,\hat{\underset{\sim}{\phi}}$ can be

e. Applying the boundary condition:

$$-\frac{\partial \Phi_1}{\partial r}\Big|_{r=R} = -\frac{\partial \Phi_2}{\partial r}\Big|_{r=R}$$

$$\Rightarrow -\sum_{l=0}^{\infty} l a_l R^{l-1} P_l(\cos(\theta)) = -(\cancel{\mp 1})\sum_{l=0}^{\infty} (-l-1) \frac{b_l}{R^{l+2}} P_l(\cos(\theta))$$

$$\therefore \sum_{l=0}^{\infty} l a_l R^{l-1} P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{(l+1)}{R^{l+2}} P_l(\cos(\theta))$$

$$\therefore l a_l \cancel{R} = -\frac{(l+1)}{R\,\cancel{l}} b_l \Rightarrow a_l = -\frac{(l+1)}{l} b_l$$

h. By Ampere's Law:

$$\nabla \times \vec{B} = \frac{4\pi \vec{K}}{c}$$

for an Amperian Loop surrounding the surface where ϕ is constant:

$$\frac{4\pi}{c} k_0 \sin(\theta) = \underbrace{B_{2\theta}(r=R^+) - B_{1\theta}(r=R^-)}_{\text{(polar components of B)}}:$$

$$= -\frac{1}{R}\frac{\partial \Phi_2(r=R)}{\partial \theta} + \frac{1}{R}\frac{\partial \Phi_1(r=R)}{\partial \theta}$$

g. $$\sum_{l=0}^{\infty} l a_l R^{l-1} P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{(l+1)}{R^{l+2}} P_l(\cos(\theta))$$

$$\therefore -\frac{4\pi k_0}{c}\frac{d}{d\theta}(P_1(\cos(\theta))) = \sum_{l=0}^{\infty} a_l r^l P_l'(\cos(\theta)) + \sum_{l=0} b_l r^{-l-1} P_l'(\cos(\theta))$$

$\therefore$ Only $l=1$ term will contribute

$\therefore \frac{4\pi k_0}{c} = b_1 - a_1 = 3b_1$ (~~by Mathematica~~)

$\therefore$ ~~$a_1 = \frac{-2}{3c} k_0 \cdot 4\pi$~~ , ~~$b_1 = \frac{4\pi k_0}{3c}$~~

$\therefore$ ~~$\phi(r<a) = -\frac{8\pi k_0 r}{3Rc} P_1$~~

~~$\phi(r>a) = \frac{4\pi k_0 R^2}{3c} \frac{\cos(\theta)}{r^2}$~~

~~By Mathematica.~~

$a_1 R P_1(\cos(\theta)) - \frac{b_1}{R^2} P_1(\cos(\theta)) = -\frac{4\pi k_0}{c} \frac{d}{d\theta}(P_1(\cos(\theta$

$\therefore \frac{4\pi k_0}{c} = b_1 - a_1$

$\therefore a_1 = -\frac{8\pi k_0}{c}$, $b_1 = \frac{4\pi k_0}{3c}$

$\therefore \phi(r<R) = -\frac{8\pi k_0}{3Rc} r P_1$, $\phi(r>R) = \frac{k_0 R^2}{3c} \frac{\cos(\theta)}{r^2}$

~~$\phi(r>R)$~~:

b. The magnetic dipole moment of the sphere ~~charge~~ is given by:

$\underset{\sim}{m} = \int \frac{1}{2} (\underset{\sim}{x} \times \vec{J})$ (Week 5 slides)

$\therefore \underset{\sim}{m} = \int \frac{1}{2} (\underset{\sim}{x} \times \vec{K})$ in this case.

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} k_0 \sin(\theta)\ r^2 \sin(\theta)\, dr\, d\theta\, d\phi$$

$$\therefore \frac{k_0\ \pi\ R^3}{\cancel{2}\ 3\ 8} \int_0^{\pi} \sin^2(\theta)\, d\theta \quad \frac{k_0 \pi^2 R^3}{6}$$