General Relativity: Workshop 2: Deriving the Schwarzschild Metric

Sai Pemmaraju

May 31, 2023

1 Introduction

In what follows, the angular terms θ and φ have been combined to form a total effective solid angle $d\Omega$. One must note that

$$d\Omega = \sin(\theta) d\theta d\varphi \tag{1}$$

and

$$d\Omega^2 = d\theta^2 + \sin^2(\theta) d\varphi^2 \tag{2}$$

2 Derivation

2.1 Assumptions

The first step is to make some assumptions. We assume that:

1. The metric is spherically symmetric
2. The metric is static; it does not depend on time
3. The metric is invariant under time reversal

Given the above assumptions, one can write an expression for the most general metric which satisfies the above criteria. Adopting spherical polar coordinates, owing to the requirement of the spherical symmetry of the metric, one can write:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + h(r)d\Omega^2 \tag{3}$$

where f, g, and h are arbitrary functions of the radial coordinate r, and the angular terms θ and φ, have been combined using the definition of $d\Omega^2$. Thus, the metric, $g_{\mu\nu}$ becomes:

$$g_{\mu\nu} = \begin{pmatrix} -f(r) & 0 & 0 & 0 \\ 0 & g(r) & 0 & 0 \\ 0 & 0 & h(r) & 0 \\ 0 & 0 & 0 & h(r)\sin^2(\theta) \end{pmatrix} \tag{4}$$

The off diagonal elements in (4) have been taken to be equal to zero. In general however, one can include off-diagonal elements and then perform coordinate transformations into a new system where the metric is diagonalised. Hence, without loss of generality, one can set the off-diagonal terms in the above metric to be equal to zero.

2.2 Exponential Formulation

In order to maintain the signature of the metric (i.e. its "mostly plus" convention), one can propose an ansatz for the form of the metric (4) with exponential functions as follows:

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha(r)} & 0 & 0 & 0 \\ 0 & e^{2\beta(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \tag{5}$$

thereby allowing for ds^2 to be rewritten as

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2 \tag{6}$$

Define a new coordinate

$$\overline{r} = e^{\gamma(r)} \tag{7}$$

such that we are able to physically interpret the original radial coordinate r, upon computing the solution. The basis one form associated with the new coordinate $\overline{r}$ is given by:

$$d\overline{r} = d(e^{\gamma(r)} = \gamma'(r)e^{\gamma(r)}rdr + e^{\gamma(r)}dr \tag{8}$$

$$\therefore d\overline{r} = (r\gamma'(r)+1)e^{\gamma(r)}dr \tag{9}$$

One can rewrite the metric (5) in terms of $\overline{r}$, and its basis one form (9). Starting with the metric (5)

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + e^{2\gamma(r)}r^2 d\Omega^2 \tag{10}$$

Making the substitution $\overline{r} = e^{\gamma(r)}$, and noting that $d\overline{r} = (r\gamma'(r)+1)e^{\gamma(r)}dr$, it is evident that

$$dr = \frac{d\overline{r}}{e^{\gamma(r)}(r\gamma'(r)+1)} \Rightarrow dr^2 = \frac{d\overline{r}^2}{e^{2\gamma(r)}(r\gamma'(r)+1)^2} \tag{11}$$

Thus, inserting (11) into (5), one obtains

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}\left(\frac{d\overline{r}^2}{e^{2\gamma(r)}(r\gamma'(r)+1)^2}\right) + \overline{r}^2 d\Omega^2 \tag{12}$$

$$\Rightarrow ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)-2\gamma(r)}(r\gamma'(r)+1)^{-2}d\overline{r}^2 + \overline{r}^2 d\Omega^2 \tag{13}$$

The variable $\bar{r}$ will henceforth be relabelled as r.

One can define a new variable:

$$e^{2\beta'(r)} \equiv (1 + r\gamma'(r))^{-2} e^{2\beta(r)-2\gamma(r)} \tag{14}$$

Under this substitution, the metric becomes

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta'(r)}dr^2 + r^2 d\Omega^2 \tag{15}$$

β' will henceforth be relabelled as β. Furthermore, the coordinate substitutions and relabelling has been done such that the metric no longer depends on the function $\gamma(r)$ despite being equivalent to the expression (13). Hence, one can set the function $\gamma(r) = 0$ without loss of generality.

3 Solving for α and β

In order to solve for $\alpha(r)$ and $\beta(r)$, one can employ the Einstein field equation. This requires the computation of the Christoffel symbols for the given metric, which have been computed in Section 4 of the Mathematica notebook in the Appendix. The non-zero Christoffel symbols are:

$$\begin{aligned}
\Gamma^t_{tr} = \Gamma^t_{rt} &= \alpha'(r) \\
\Gamma^r_{tt} &= \alpha'(r)e^{2\alpha(r)-2\beta(r)} \\
\Gamma^r_{rr} = \beta'(r), \Gamma^r_{\theta\theta} = -e^{-2\beta(r)}r, \Gamma^r_{\varphi\varphi} &= -e^{-2\beta(r)}r\sin^2(\theta) \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \Gamma^\theta_{\varphi\varphi} &= -\cos(\theta)\sin(\theta) \\
\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}, \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} &= \cot(\theta)
\end{aligned}$$

Although Mathematica has been used in order to compute the above Christoffel symbols, these can be calculated by hand. The symbol Γ^t_{tr} has been computed by hand below as an example.

The Chistoffel symbols are defined as:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{16}$$

Thus, for Γ^t_{tr}, this becomes

$$\Gamma^t_{tr} = \frac{1}{2}g^{t\rho}(\partial_t g_{r\rho} + \partial_r g_{\rho t} - \partial_\rho g_{tr}) \tag{17}$$

For (17) to not vanish, we require $\rho = t$. Thus, we have

$$\Gamma^t_{tr} = \frac{1}{2}g^{tt}(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr}) \tag{18}$$

Since the metric of interest is diagonal, the component of the inverse metric labelled g^{tt} is given by the reciprocal of the component g_{tt}. Hence, $g^{tt} = -e^{-2\alpha(r)}$. Additionally, since the metric is diagonal, $g_{rt} = g_{tr} = 0$. Hence, we have

$$\begin{aligned}
\Gamma^t_{tr} &= \frac{1}{2}(-e^{-2\alpha(r)})(0 + \partial_r g_{tt} - 0) \\
&= -\frac{1}{2}(\partial_r(-e^{2\alpha(r)})) \\
&= -\frac{1}{2}(e^{-2\alpha(r)})(-2\alpha'(r)e^{2\alpha(r)}) \\
&= \frac{1}{2}(2\alpha'(r)) = \alpha'(r)
\end{aligned}$$

These Christoffel symbols have been computed in order to determine the components of the Riemann curvature tensor $R^{\mu}{}_{\nu\rho\sigma}$, as well as the Ricci tensor $R_{\mu\nu}$. These have been computed in Section 4 of the Mathematica notebook in the Appendix. In order to calculate these tensors, the computer implements the definition of each of these, which are defined in terms of the partial derivatives of the Christoffel symbols, which are in turn derived from the metric, as follows:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{19}$$

and

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} \tag{20}$$

The relevant expressions from the Ricci tensor that are necessary for the following steps are those of the components R_{tt}, R_{rr}, and $R_{\theta\theta}$. These are given by:

$$\boxed{R_{tt} = e^{2\alpha(r)-2\beta(r)}\left(\frac{2\alpha'(r) + r\alpha'(r)^2 - r\alpha'(r)\beta'(r) + r\alpha''(r)}{r}\right)} \tag{21}$$

$$\boxed{R_{rr} = -\frac{r\alpha'(r)^2 - 2\beta'(r) - r\alpha'(r)\beta'(r) + r\alpha''(r)}{r}} \tag{22}$$

$$\boxed{R_{\theta\theta} = e^{-2\beta(r)}(-1 + e^{2\beta(r)} - r\alpha'(r) + r\beta'(r))} \tag{23}$$

At this point, it must be noted that the metric applies for the spacetime in the vicinity of the Earth. Nonetheless, one can proceed by setting the stress energy tensor $T_{\mu\nu} = 0$, despite the Earth being composed of matter. This is because we can regard the immediate vicinity of the Earth as a vacuum, and therefore be devoid of matter (i.e. we are sufficiently far away from the Earth such that our space can be regarded as a vacuum, but we are still in relatively close proximity to the Earth).

The Einstein equation in vacuum can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} = 0 \tag{24}$$

Since $T_{\mu\nu} = 0$, then we can write that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{25}$$

Since $T_{\mu\nu} = 0$, this can be regarded as a vacuum. The fact that the Ricci scalar, R vanishes can be shown by contracting the $\mu\nu$ indices of the above equation as follows:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 0 \\ R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} &= 0 \\ R - \frac{R}{2} &= 0 \\ \Rightarrow \frac{R}{2} = 0 \Rightarrow R &= 0 \end{aligned}$$

Consequently, the Ricci tensor $R_{\mu\nu} = 0$. One can combine the expressions for R_{tt} and R_{rr} described in Equations (21) and (22) respectively to obtain

$$\begin{aligned} R_{tt} + R_{rr} = (e^{2\alpha(r)-2\beta(r)} &\left(\frac{2\alpha'(r) + r\alpha'(r)^2 - r\alpha'(r)\beta'(r) + r\alpha''(r)}{r}\right)) \\ &+ \left(-\frac{r\alpha'(r)^2 - 2\beta'(r) - r\alpha'(r)\beta'(r) + r\alpha''(r)}{r}\right) \end{aligned}$$

Hence, one can write that

$$\begin{aligned} e^{2\beta(r)-2\alpha(r)}R_{tt} + R_{rr} = &\left(\frac{2\alpha'(r) + r\alpha'(r)^2 - r\alpha'(r)\beta'(r) + r\alpha''(r)}{r}\right) \\ &+ \left(\frac{-r\alpha'(r)^2 + 2\beta'(r) + r\alpha'(r)\beta'(r) - r\alpha''(r)}{r}\right) \\ &= \frac{1}{r}(2\alpha'(r) + 2\beta'(r)) = \frac{2}{r}(\alpha'(r) + \beta'(r)) \end{aligned}$$

Since $R_{\mu\nu} = 0$ in this case, one can set the above result to be equal to zero, as both R_{tt} and R_{rr} vanish independently. Hence, one can write that

$$\frac{2}{r}(\alpha'(r) + \beta'(r)) = 0 \tag{26}$$

From Equation (26), it is evident that

$$\alpha'(r) = -\beta'(r) \tag{27}$$

Solving the above equation for $\alpha(r)$ yields that

$$\alpha(r) = -\beta(r) + c \tag{28}$$

where c is a constant of integration that can be absorbed in the definition of time. This can be done since the absorption of the constant represents a relative scaling of the time coordinate by a constant amount, $t \to e^{-c}t$.

One can use the result obtained above in conjunction with the definition of $R_{\theta\theta}$ (i.e. Equation (23)) in order to derive an expression for $e^{2\alpha(r)}$. This is done by first recalling that

$$R_{\theta\theta} = e^{-2\beta(r)}(-1 + e^{2\beta(r)} - r\alpha'(r) + r\beta'(r))$$

Owing to the scaling of the time coordinate by a factor of e^{-c}, where c is the constant of integration obtained in the previous step, one can set $c = 0$ in Equation (28), thereby implying that

$$\alpha(r) = -\beta(r) \tag{29}$$

Making the above substitution into the expression for $R_{\theta\theta}$, one obtains:

$$\begin{aligned} R_{\theta\theta} &= e^{2\alpha(r)}(-1 + e^{-2\alpha(r)} - r\alpha'(r) - r\alpha'(r)) \\ &= -e^{2\alpha(r)}(1 - e^{-2\alpha(r)} + 2r\alpha'(r)) \\ &= -e^{2\alpha(r)}2r\alpha'(r) - e^{2\alpha(r)} + 1 \end{aligned}$$

Since $R_{\mu\nu} = 0$ in this case, the above expression for $R_{\theta\theta}$ can be set to be equal to zero. Doing so yields

$$\begin{aligned} -e^{2\alpha(r)}2r\alpha'(r) - e^{2\alpha(r)} + 1 &= 0 \\ \Rightarrow e^{2\alpha(r)}2r\alpha'(r) + e^{2\alpha(r)} - 1 &= 0 \\ \Rightarrow e^{2\alpha(r)}(2r\alpha'(r) + 1) - 1 &= 0 \\ \Rightarrow e^{2\alpha(r)}(2r\alpha'(r) + 1) &= 1 \end{aligned}$$

One can recognise the left-hand side as being the covariant derivative with respect to r of $re^{2\alpha(r)}$, and therefore arrive at:

$$\Rightarrow \partial_r(re^{2\alpha(r)}) = 1 \tag{30}$$

Solving the above differential equation for $e^{2\alpha(r)}$ yields that

$$\begin{aligned} re^{2\alpha(r)} &= r + s \\ e^{2\alpha(r)} &= 1 + \frac{s}{r} \end{aligned}$$

where s is an arbitrary constant of integration. Setting $s = -R_S$, one obtains that

$$e^{2\alpha(r)} = 1 - \frac{R_S}{r} \tag{31}$$

The constant R_S represents the Schwarzschild radius.

4 Exploring the Schwarzschild Metric

Given the expressions obtained for $\alpha(r), \beta(r)$, and $\gamma(r)$ in the previous section, one can write an expression for the Schwarzschild metric by substituting these definitions into Equation (15). Doing so yields

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{32}$$

One can proceed further by deriving an expression for the Schwarzschild radius, R_S by investigating the behaviour of the metric in the Newtonian regime. For this, one can consider the geodesic equation for a slow- moving particle in the limit that $r \to \infty$. The general form of the geodesic equation is given by

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} \tag{33}$$

Since the particles are assumed to be moving slowly (i.e. $\frac{dx^i}{dt} << \frac{dt}{d\tau}$) the Christoffel symbols simplify as follows.

$$\Gamma^\mu_{tt} = \frac{1}{2} g^{\mu\lambda} (\partial_t g_{\lambda t} + \partial_t g_{t\lambda} - \partial_\lambda g_{tt}) \tag{34}$$

These can be further simplified by noting that $\partial_t g_{\mu\nu} = 0$ since the metric is static (i.e. unchanging with time). Thus, the above expression simplifies to

$$\Gamma^\mu_{tt} = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{tt} \tag{35}$$

Furthermore, since the metric is static and the particles are moving slowly, the geodesic equation simplifies to

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{tt} \left(\frac{dt}{d\tau}\right)^2 = 0 \tag{36}$$

Inserting the expression (35) into (36), one obtains:

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{tt} \left(\frac{dt}{d\tau}\right)^2 = 0 \tag{37}$$

Since the metric is diagonal in this case, it is evident that $\lambda = \mu$, Thus, one obtains:

$$\frac{d^2 x^\mu}{d\tau^2} - \frac{1}{2} g^{\mu\mu} \partial_\mu g_{tt} \left(\frac{dt}{d\tau}\right)^2 = 0 \tag{38}$$

For the coordinates, t, θ, and φ, it is evident upon inspection of the metric that the geodesic equation for these reduces to

$$\frac{d^2 x^i}{d\tau^2} = 0 \tag{39}$$

where $i = \{t, \theta, \varphi\}$. On the other hand, the equation for r becomes

$$\frac{d^2r}{d\tau^2} - \frac{1}{2}g^{rr}\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{40}$$

Inserting the relevant components of the metric into the above expression, one obtains:

$$\begin{aligned}
\frac{d^2r}{d\tau^2} - \frac{1}{2}g^{rr}\partial_r g_{tt}\left(\frac{dt}{d\tau}\right)^2 &= 0 \\
= \frac{d^2r}{d\tau^2} - \frac{1}{2}\left(1 - \frac{R_S}{r}\right)\partial_r\left(\left(1 - \frac{R_S}{r}\right)^{-1}\right)\left(\frac{dt}{d\tau}\right)^2 &= 0 \\
= \frac{d^2r}{d\tau^2} - \frac{1}{2}\left(\frac{r - R_S}{r}\right)\left(\frac{r - R_S - r}{(r - R_S)^2}\right)\left(\frac{dt}{d\tau}\right)^2 &= 0 \\
= \frac{d^2r}{d\tau^2} + \frac{1}{2}\left(\frac{R_S}{r(r - R_S)}\right)\left(\frac{dt}{d\tau}\right)^2 &= 0
\end{aligned}$$

Dividing both sides of the above by $\left(\frac{dt}{d\tau}\right)^2$ yields

$$\frac{d^2r}{dt^2} + \frac{1}{2}\left(\frac{R_S}{r(r - R_S)}\right) = 0 \tag{41}$$

In the weak field limit (i.e. in the case that $r >> R_S$), the above relation becomes

$$\frac{d^2r}{dt^2} + \frac{R_S}{2r^2} = 0 \Rightarrow \frac{d^2r}{dt^2} = -\frac{R_S}{2r^2} \tag{42}$$

In this limit, the Newtonian result for gravitational acceleration from outside a spherical symmetric mass should be reproduced. This is given by:

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} \tag{43}$$

In order for Equation (42) to be consistent with Equation (43), R_S must be equal to $2GM$.

The Schwarzschild metric can therefore be written as

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{44}$$

Taking the limit as $M \to 0$ in the expression (44), it is evident that $\left(1 - \frac{2GM}{r}\right) \to 1$. In addition to this, it can be observed that

$$\left(1 - \frac{2GM}{r}\right)^{-1} = \frac{1}{\left(1 - \frac{2GM}{r}\right)} = \frac{1}{\left(\frac{r - 2GM}{r}\right)} = \frac{r}{r - 2GM} \tag{45}$$

Taking $M \to 0$ in the expression (45), it is evident that $\left(1 - \frac{2GM}{r}\right)^{-1} \to 1$ as $M \to 0$. Combining these results, one can deduce that in the limit that $M \to 0$, the metric (44) behaves as:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{46}$$

which is the definition of the Minkowski metric in spherical polar coordinates. Hence, in the limit that $M \to 0$, the Schwarzschild metric behaves like Minkowski space. Taking the limit as $r \to \infty$ of the expressions (45) and $\left(1 - \frac{2GM}{r}\right)$ also leads to the Minkowski metric being recovered, alluding to the asymptotically flat nature of the Schwarzschild metric.

2 RG&TC-Code

In[54]:=
```
xCoord = {t, χ, θ, φ};
g = {
    {-x y, 0, 0, 0},
    {0, x y t, 0, 0},
    {0, 0, z, 0},
    {0, 0, 0, x t}
  };
RGtensors[g, xCoord]
```

$$\text{gdd} = \begin{pmatrix} -x\,y & 0 & 0 & 0 \\ 0 & t\,x\,y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & t\,x \end{pmatrix}$$

$$\text{LineElement} = -x\,y\,d[t]^2 + z\,d[\theta]^2 + t\,x\,d[\varphi]^2 + t\,x\,y\,d[\chi]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{x\,y} & 0 & 0 & 0 \\ 0 & \frac{1}{t\,x\,y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{t\,x} \end{pmatrix}$$

```
gUU computed in 0. sec
Gamma computed in 0. sec
Riemann(dddd) computed in 0. sec
Riemann(Uddd) computed in 0. sec
Ricci computed in 0. sec
Weyl computed in 0. sec
Einstein computed in 0. sec
```

Out[56]=
```
All tasks completed in 0.
```

In[57]:= `(* Ricci Scalar *)`

In[58]:= `R`

Out[58]=

$$-\frac{1}{2\,t^2\,x\,y}$$

In[59]:= `(* Einstein Tensor *)`

In[60]:= `EUd`

Out[60]=

$$\left\{\left\{-\frac{1}{4\,t^2\,x\,y}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\,t^2\,x\,y}, 0, 0\right\}, \left\{0, 0, \frac{1}{4\,t^2\,x\,y}, 0\right\}, \left\{0, 0, 0, \frac{1}{4\,t^2\,x\,y}\right\}\right\}$$

In[61]:= `(* Christoffel Symbol *)`

In[62]:=
```
GUdd // MatrixForm
```

Out[62]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2\,y} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{1}{2\,t} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2\,t} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2\,t} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2\,t} \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

In[63]:=
```
Part[GUdd, 1, 2, 2]
Part[GUdd, 2, 2, 1]
```

Out[63]=

$$\frac{1}{2}$$

Out[64]=

$$\frac{1}{2\,t}$$

In[65]:=
```
(* Riemann tensor *)
```

In[66]:= **RUddd**

Out[66]=

$$\left\{\left\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\right.\right.$$
$$\left\{\left\{0,-\frac{1}{4\,t},0,0\right\},\left\{\frac{1}{4\,t},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\left.\left\{\left\{0,0,0,-\frac{1}{4\,t\,y}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4\,t\,y},0,0,0\right\}\right\}\right\},$$
$$\left\{\left\{\left\{0,-\frac{1}{4\,t^2},0,0\right\},\left\{\frac{1}{4\,t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\right.$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\left.\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{4\,t\,y}\right\},\{0,0,0,0\},\left\{0,-\frac{1}{4\,t\,y},0,0\right\}\right\}\right\},$$
$$\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\},$$
$$\left\{\left\{\left\{0,0,0,-\frac{1}{4\,t^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4\,t^2},0,0,0\right\}\right\},\right.$$
$$\left\{\{0,0,0,0\},\left\{0,0,0,-\frac{1}{4\,t}\right\},\{0,0,0,0\},\left\{0,\frac{1}{4\,t},0,0\right\}\right\},$$
$$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$$
$$\left.\left.\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\right\}\right\}$$

In[67]:= **(* Ricci Tensor *)**

In[68]:= **Rdd**

Out[68]=

$$\left\{\left\{\frac{1}{2\,t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

In[69]:= **Part[Rdd, 1, 1]**

Out[69]=

$$\frac{1}{2\,t^2}$$

In[70]:=

```
xCoord = {t, r, θ, φ};
g = {-Exp[2 * a[r]], 0, 0, 0,
  {0, Exp[2 * b[r]], 0, 0, 0},
  {0, 0, r^{2} * Sin[θ], 0},
 }
```

Out[71]=

$$\left\{-e^{2\,a[r]},0,0,0,\left\{0,e^{2\,b[r]},0,0,0\right\},\left\{0,0,\left\{r^2\,\mathrm{Sin}[\theta]\right\},0\right\},\mathrm{Null}\right\}$$

1. Introduction

4. Solving for α and β

In[73]:=
```
xCoord = {t, r, θ, φ}
```
Out[73]=
```
{t, r, θ, φ}
```

In[87]:=
```
g = {{-Exp[2*a[r]], 0, 0, 0},
  {0, Exp[2*b[r]], 0, 0},
  {0, 0, r^2, 0},
  {0, 0, 0, r^2*(Sin[θ])^2}}
```
Out[87]=

$$\left\{\left\{-e^{2\,a[r]}, 0, 0, 0\right\}, \left\{0, e^{2\,b[r]}, 0, 0\right\}, \left\{0, 0, r^2, 0\right\}, \left\{0, 0, 0, r^2\,\mathrm{Sin}[\theta]^2\right\}\right\}$$

In[74]:=
```
g
```

In[85]:=

$$\left\{-e^{2\,a[r]}, 0, 0, 0, \left\{0, e^{2\,b[r]}, 0, 0, 0\right\}, \left\{0, 0, \left\{r^2\,\mathrm{Sin}[\theta]\right\}, 0\right\}, \mathrm{Null}\right\}$$

Out[85]=

$$\left\{-e^{2\,a[r]}, 0, 0, 0, \left\{0, e^{2\,b[r]}, 0, 0, 0\right\}, \left\{0, 0, \left\{r^2\,\mathrm{Sin}[\theta]\right\}, 0\right\}, \mathrm{Null}\right\}$$

In[75]:=
```
GUdd
```
Out[75]=

$$\left\{\left\{\{0, 0, 0, 0\}, \left\{0, \frac{1}{2}, 0, 0\right\}, \{0, 0, 0, 0\}, \left\{0, 0, 0, \frac{1}{2\,y}\right\}\right\},\right.$$
$$\left\{\left\{0, \frac{1}{2\,t}, 0, 0\right\}, \left\{\frac{1}{2\,t}, 0, 0, 0\right\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\right\},$$
$$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},$$
$$\left.\left\{\left\{0, 0, 0, \frac{1}{2\,t}\right\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \left\{\frac{1}{2\,t}, 0, 0, 0\right\}\right\}\right\}$$

In[88]:= **RGtensors[g, xCoord]**

$$gdd = \begin{pmatrix} -e^{2 a[r]} & 0 & 0 & 0 \\ 0 & e^{2 b[r]} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$\mathrm{LineElement} = e^{2 b[r]} d[r]^2 - e^{2 a[r]} d[t]^2 + r^2 d[\theta]^2 + r^2 d[\varphi]^2 \mathrm{Sin}[\theta]^2$$

$$gUU = \begin{pmatrix} -e^{-2 a[r]} & 0 & 0 & 0 \\ 0 & e^{-2 b[r]} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

```
gUU computed in 0. sec

Gamma computed in 0. sec

Riemann(dddd) computed in 0. sec

Riemann(Uddd) computed in 0. sec

Ricci computed in 0. sec

Weyl computed in 0. sec

Einstein computed in 0. sec
```

Out[88]=

All tasks completed in 0.

In[89]:= **GUdd**

Out[89]=

$$\{\{\{0, a'[r], 0, 0\}, \{a'[r], 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},$$
$$\{\{e^{2 a[r]-2 b[r]} a'[r], 0, 0, 0\}, \{0, b'[r], 0, 0\},$$
$$\{0, 0, -e^{-2 b[r]} r, 0\}, \{0, 0, 0, -e^{-2 b[r]} r \mathrm{Sin}[\theta]^2\}\},$$
$$\{\{0, 0, 0, 0\}, \{0, 0, \frac{1}{r}, 0\}, \{0, \frac{1}{r}, 0, 0\}, \{0, 0, 0, -\mathrm{Cos}[\theta] \mathrm{Sin}[\theta]\}\},$$
$$\{\{0, 0, 0, 0\}, \{0, 0, 0, \frac{1}{r}\}, \{0, 0, 0, \mathrm{Cot}[\theta]\}, \{0, \frac{1}{r}, \mathrm{Cot}[\theta], 0\}\}\}$$

In[90]:= **RUddd**

Out[90]=

$$
\begin{array}{l}
\{\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,-a'[r]^2+a'[r]\,b'[r]-a''[r],0,0\}, \\
\quad \{a'[r]^2-a'[r]\,b'[r]+a''[r],0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,-e^{-2b[r]}\,r\,a'[r],0\},\{0,0,0,0\},\{e^{-2b[r]}\,r\,a'[r],0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,-e^{-2b[r]}\,r\,\mathrm{Sin}[\theta]^2\,a'[r]\},\{0,0,0,0\}, \\
\quad \{0,0,0,0\},\{e^{-2b[r]}\,r\,\mathrm{Sin}[\theta]^2\,a'[r],0,0,0\}\}\}, \\
\{\{\{0,-e^{2a[r]-2b[r]}\left(a'[r]^2-a'[r]\,b'[r]+a''[r]\right),0,0\}, \\
\quad \{e^{2a[r]-2b[r]}\left(a'[r]^2-a'[r]\,b'[r]+a''[r]\right),0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,e^{-2b[r]}\,r\,b'[r],0\},\{0,-e^{-2b[r]}\,r\,b'[r],0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,e^{-2b[r]}\,r\,\mathrm{Sin}[\theta]^2\,b'[r]\}, \\
\quad \{0,0,0,0\},\{0,-e^{-2b[r]}\,r\,\mathrm{Sin}[\theta]^2\,b'[r],0,0\}\}\}, \\
\{\{\{0,0,-\dfrac{e^{2a[r]-2b[r]}\,a'[r]}{r},0\},\{0,0,0,0\},\{\dfrac{e^{2a[r]-2b[r]}\,a'[r]}{r},0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,-\dfrac{b'[r]}{r},0\},\{0,\dfrac{b'[r]}{r},0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,e^{-2b[r]}\left(-1+e^{b[r]}\right)\left(1+e^{b[r]}\right)\mathrm{Sin}[\theta]^2\}, \\
\quad \{0,0,-e^{-2b[r]}\left(-1+e^{b[r]}\right)\left(1+e^{b[r]}\right)\mathrm{Sin}[\theta]^2,0\}\}\}, \\
\{\{\{0,0,0,-\dfrac{e^{2a[r]-2b[r]}\,a'[r]}{r}\},\{0,0,0,0\},\{0,0,0,0\},\{\dfrac{e^{2a[r]-2b[r]}\,a'[r]}{r},0,0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,-\dfrac{b'[r]}{r}\},\{0,0,0,0\},\{0,\dfrac{b'[r]}{r},0,0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,-e^{-2b[r]}\left(-1+e^{b[r]}\right)\left(1+e^{b[r]}\right)\}, \\
\quad \{0,0,e^{-2b[r]}\left(-1+e^{b[r]}\right)\left(1+e^{b[r]}\right),0\}\}, \\
\quad \{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\}\}
\end{array}
$$

In[91]:= **Rdd**

Out[91]=

$$
\begin{array}{l}
\{\{\dfrac{e^{2a[r]-2b[r]}\left(2\,a'[r]+r\,a'[r]^2-r\,a'[r]\,b'[r]+r\,a''[r]\right)}{r},0,0,0\}, \\
\{0,-\dfrac{r\,a'[r]^2-2\,b'[r]-r\,a'[r]\,b'[r]+r\,a''[r]}{r},0,0\}, \\
\{0,0,e^{-2b[r]}\left(-1+e^{2b[r]}-r\,a'[r]+r\,b'[r]\right),0\}, \\
\{0,0,0,e^{-2b[r]}\,\mathrm{Sin}[\theta]^2\left(-1+e^{2b[r]}-r\,a'[r]+r\,b'[r]\right)\}\}
\end{array}
$$