# Assignment 2: Question 1

Sunday, 21 May 2023 8:44 pm

- Consider the time ordered, $T(\varphi(x_1)\varphi(x_2))$ and the normal ordered, $:\varphi(x_1)\varphi(x_2):$ product of a field.

a. Show that both of these products are symmetric under the interchange of $x_1$ and $x_2$

Solution:

$$T(\varphi(x_1)\varphi(x_2)) = \begin{cases} \varphi(x_1)\varphi(x_2) & x_1^0 < x_2^0 \\ \varphi(x_2)\varphi(x_1) & x_1^0 > x_2^0 \end{cases}$$

If $x_1 \leftrightarrow x_2$:

$$T(\varphi(x_2)\varphi(x_1)) = \begin{cases} \varphi(x_2)\varphi(x_1) & x_2^0 < x_1^0 \\ \varphi(x_1)\varphi(x_2) & x_1^0 < x_2^0 \end{cases}$$

- By inspection, it is evident that $T(\varphi(x_1)\varphi(x_2)) = T(\varphi(x_2)\varphi(x_1))$. Hence, the time ordered product of a field is symmetric under the interchange of particles

- Can write $\varphi(x_1)$ as:

- $\varphi(x_1) = \varphi^+(x_1) + \varphi(x_1^-)$

where:

$$\varphi^+(x_1) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} a_p^+ e^{-ip\cdot x_1} \quad \text{and} \quad \varphi^-(x_1) = \int \frac{d^3p}{(2\pi)^3} \times \frac{1}{\sqrt{2E_p}} a_p e^{ip\cdot x_1}$$

(and similarly for $\varphi(x_2)$)

$$\begin{aligned} \therefore\ :\varphi(x_1)\varphi(x_2): &= :(\varphi^+(x_1) + \varphi^-(x_1))(\varphi^+(x_2) + \varphi^-(x_2)): \\ &= :\varphi^+(x_1)\varphi^+(x_2) + \varphi^+(x_1)\varphi^-(x_2) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_1)\varphi^-(x_2): \\ &= \varphi^+(x_1)\varphi^+(x_2) + \varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_1)\varphi^-(x_2) \quad (*) \end{aligned}$$

- If we interchange $x_1$ and $x_2$ in $(*)$, we obtain:

$$\varphi^+(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^-(x_2)$$

- Since $[\varphi^+(x_1), \varphi^+(x_2)] = [\varphi^-(x_2), \varphi^-(x_2)] = 0$ and since $\varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi(x_2)$ is clearly symmetric under the interchange of $x_1$ and $x_2$ $\Rightarrow$ $:\varphi(x_1)\varphi(x_2):$ is symmetric under $x_1 \leftrightarrow x_2$.

b. Deduce that the Feynman propagator has the same property

Solution:

- Since the Feynman propagator can be written as:

$$\Delta_F(x_1 - x_2) = T(\varphi(x_1)\varphi(x_2)) - :\varphi(x_1)\varphi(x_2):$$

It is a difference of two products, both of which are symmetric under the interchange of $x_1$ and $x_2$. Hence, it is symmetric under the interchange of $x_1$ and $x_2$.

# Assignment 2: Question 2

Saturday, 27 May 2023 12:10 pm

(2) Verify Wick's Theorem for the case of three scalar fields:

$$T\left(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\right) = :\phi(x_1)\,\phi(x_2)\,\phi(x_3): + \phi(x_1)\,\Delta_F(x_2-x_3) + \phi(x_2)\,\Delta_F(x_3-x_1) + \phi(x_3)\,\Delta_F(x_1-x_2)$$

**Solution:**

- Consider the case that $x^0_1 < x^0_2 < x^0_3$, then:

LHS:

$$\begin{aligned}
T\left(\phi(x_1)\phi(x_2)\phi(x_3)\right) &= \phi(x_1)\,\phi(x_2)\,\phi(x_3) \\
&= \left(\phi^+(x_1) + \phi^-(x_1)\right)\left(\phi^+(x_2) + \phi^-(x_2)\right) + \left(\phi^+(x_3) + \phi^-(x_3)\right) \\
&= \left[\phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\right]\left[\phi^+(x_3) + \phi^-(x_3)\right] \\
&= \phi^+(x_1)\phi^+(x_2)\phi^+(x_3) + \underbrace{\phi^+(x_1)\phi^+(x_2)\phi^-(x_3)}_{1} + \underbrace{\phi^+(x_1)\phi^-(x_2)\phi^+(x_3)}_{2} + \underbrace{\phi^+(x_1)\phi^-(x_2)\phi^-(x_3)}_{3} \\
&\quad + \underbrace{\phi^-(x_1)\phi^+(x_2)\phi^+(x_3)}_{4} + \underbrace{\phi^-(x_1)\phi^+(x_2)\phi^-(x_3)}_{5} + \underbrace{\phi^-(x_1)\phi^-(x_2)\phi^+(x_3)}_{6} + \underbrace{\phi^-(x_1)\phi^-(x_2)\phi^-(x_3)}_{7}
\end{aligned}$$

- One can normal order expressions 1 – 7 as follows:

1 $\phi^+(x_1)\,\phi^+(x_2)\,\phi^-(x_3)$

$= \phi^-(x_3)\,\phi^+(x_1)\,\phi^+(x_2) + [\phi^+(x_1), \phi^-(x_3)]\,\phi^+(x_2) + \phi^+(x_1)\,[\phi^+(x_2), \phi^-(x_3)])$

2 $\phi^+(x_1)\,\phi^-(x_2)\,\phi^+(x_3)$

$= \phi^-(x_2)\,\phi^+(x_1)\,\phi^+(x_3) + [\phi^+(x_1), \phi^-(x_2)]\,\phi^+(x_3)$

3 $\phi^+(x_1)\,\phi^-(x_2)\,\phi^-(x_3)$

$= \phi^-(x_2)\,\phi^-(x_3)\,\phi^+(x_1) + \phi^-(x_2)\,[\phi^+(x_1), \phi^-(x_3)] + [\phi^+(x_1), \phi^-(x_2)]\,\phi^-(x_3)$

4 Already normal ordered

5 $\phi^-(x_1)\,\phi^+(x_2)\,\phi^-(x_3)$

$= \phi^-(x_1)\,\phi^+(x_2)\,\phi^-(x_3) + \phi^-(x_1)\,[\phi^+(x_2), \phi^-(x_3)]$

6 $\phi^-(x_1)\,\phi^-(x_2)\,\phi^+(x_3)$

7 Already normal ordered:

$\therefore\ \phi^+(x_1)\phi^+(x_2)\phi^+(x_3) + \phi^-(x_3)\phi^+(x_1)\phi^+(x_2) + [\phi^+(x_1), \phi^-(x_3)]\,\phi^+(x_2) + \phi^+(x_1)\,[\phi^+(x_2), \phi^-(x_3)]) + \phi^-(x_2)\phi^-(x_3)\phi^+(x_1) + \phi^-(x_2)\,[\phi^+(x_1), \phi^-(x_3)] + [\phi^+(x_1), \phi^-(x_2)]\,\phi^-(x_3)$

$+ \phi^-(x_1)\phi^+(x_2)\phi^+(x) + \phi^-(x_1)\phi^+(x_2)\phi^-(x_3) + \phi^-(x_1)\,[\phi^+(x_2), \phi^-(x_3)] + \phi^-(x_1)\phi^-(x_2)\phi^+(x_3) + \phi^-(x_1)\phi^-(x_2)\phi^-(x_3) + \phi^-(x_1)\phi^-(x_2)\phi^-(x_3)$

$= :\phi(x_1)\phi(x_2)\phi(x_3): + [\phi^+(x_1), \phi^-(x_3)]\,\phi(x_2) + [\phi^+(x_2), \phi^-(x_3)]\,\phi(x_1) + [\phi^+(x_1), \phi^-(x_2)]\,\phi(x_3)$

$= :\phi(x_1)\phi(x_2)\phi(x_3): + \Delta_F(x_1 - x_3)\,\phi(x_2) + \Delta_F(x_2 - x_3)\,\phi(x_1) + \Delta_F(x_1 - x_2)\,\phi(x_3) = \underline{\underline{RHS}}$

- Reordering $x^0_1$, $x^0_2$ and $x^0_3$ $\Rightarrow$ relabelling the corresponding fields, etc. $\Rightarrow$ Wick's theorem is verified for the above case without loss of generality.

# Assignment 2: Question 3

Saturday, 27 May 2023 12:10 pm

(3) Consider the scalar Yukawa theory given by the Lagrangian density:

$$\mathcal{L} = \partial_\mu \Psi^* \partial^\mu \Psi + \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - M^2 \Psi^* \Psi - \frac{1}{2} m^2 \varphi^2 - g \Psi^* \Psi \varphi$$

- Compute the S-matrix amplitude for nucleon-meson scattering $\varphi + \Psi \to \varphi + \Psi$ at order $g^2$

Solution:

$$|i\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a^\dagger_{p_1} b^\dagger_{p_2} |0\rangle \equiv |p_1 \cdot p_2\rangle$$

$$= \sqrt{4E_{p_1}E_{p_2}}\, a^\dagger_{p_1} b^\dagger_{p_2}|0\rangle = |p_1 \cdot p_2\rangle$$

$$|f\rangle = \sqrt{4E_{p_1'}E_{p_2'}}\, a^\dagger_{p_1'} b^\dagger_{p_2'}|0\rangle = |p_1' \cdot p_2'\rangle$$

$\therefore \langle f|S-1|i\rangle$

At order $g^2$, we have the term:

$$\frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\; T[\Psi^\dagger(x_1)\Psi(x_1)\phi(x_1)\Psi^\dagger(x_2)\Psi(x_2)\phi(x_2)]$$

- By Wick's Theorem, the relevant components are:

$$:\Psi^\dagger(x_1)\phi(x_1)\Psi(x_2)\phi(x_2):\ \overbrace{\Psi(x_1)\Psi^\dagger(x_2)} \quad ①$$

and

$$:\Psi^\dagger(x_2)\phi(x_2)\Psi(x_1)\phi(x_1):\ \overbrace{\Psi(x_2)\Psi^\dagger(x_1)} \quad ②$$

① $\langle f|S|i\rangle = \langle p_1', p_2'|:\Psi^\dagger(x_1)\phi(x_1)\Psi(x_2)\phi(x_2):|p_1, p_2\rangle$

$$= \langle p_1', p_2'|\underbrace{\Psi^\dagger(x_1)\phi(x_1)|0\rangle}_{\mathrm{I}} \underbrace{\langle 0|\Psi(x_2)\phi(x_2)|p_1, p_2\rangle}_{\mathrm{II}}$$

I $\langle p_1' \cdot p_2'|\Psi^\dagger(x_1)\phi(x_1)|0\rangle$:

$$= \langle 0|\sqrt{4E_{p_1'}E_{p_2'}}\, a_{p_1'} b_{p_2'} \int \frac{d^3q_1}{(2\pi)^3\sqrt{2E_{q_1}}}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_1} + c_{q_1}e^{iq_1\cdot x_1}\right)\int\frac{d^3q_2}{(2\pi)^3}\left(a_{q_2}e^{-iq_2\cdot x_1} + a^\dagger_{q_2}e^{iq_2\cdot x_1}\right)|0\rangle$$

$$= \langle 0|\int \frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\, a_{p_1'} b_{p_2'}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_1} + c_{q_1}e^{iq_1\cdot x_1}\right)\left(a_{q_2}e^{-iq_2\cdot x_1} + a^\dagger_{q_2}e^{iq_2\cdot x_1}\right)|0\rangle$$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q}E_{q_2}}}\, a_{p_1'}b_{p_2'}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_1}a_{q_2}e^{-iq_2\cdot x_1} + b^\dagger_{q_1}e^{-iq_1\cdot x_1}a^\dagger_{q_2}e^{iq_2\cdot x_1} + c_{q_1}e^{iq_1\cdot x_1}a_{q_2}e^{-iq_2\cdot x_1} + c_{q_1}e^{iq_1\cdot x_1}a^\dagger_{q_2}e^{iq_2\cdot x_1}\right)|0\rangle$$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\, a_{p_1'}b_{p_2'}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_1}a_{q_2}e^{-iq_2\cdot x_1} + b^\dagger_{q_1}e^{-iq_1\cdot x_1}a^\dagger_{q_2}e^{iq_2\cdot x_1}\right)|0\rangle \quad \text{(Terms containing } c \text{ will vanish)}$$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\underbrace{\left(a_{p_1'}\cancel{b_{p_2'}b^\dagger_{q_1}a_{q_2}}\right)}_{0}e^{i(-q_1-q_2)x_1} + \underbrace{\left(a_{p_1'}b_{p_2'}b^\dagger_{q_1}a^\dagger_{q_2}\right)}_{\mathrm{III}}e^{i(q_2-q_1)x_1}|0\rangle$$

III $\langle 0|a_{p_1'}b_{p_2'}b^\dagger_{q_1}a^\dagger_{q_2}|0\rangle$

$$= \langle 0|a_{p_1'}a^\dagger_{q_2}b_{p_2'}b^\dagger_{q_1}|0\rangle$$

Using $[b_{p_2'}, b^\dagger_{q_1}] = (2\pi)^3\delta^{(3)}(p_2'-q_1)$

$$\Rightarrow b_{p_2'}b^\dagger_{q_1} - b^\dagger_{q_1}b_{p_2'} = (2\pi)^3\delta^{(3)}(p_2'-q_1)$$

$$\therefore b_{p_2'}b^\dagger_{q_1} = (2\pi)^3\delta^{(3)}(p_2'-q_1) + b^\dagger_{q_1}b_{p_2'}$$

$$\Rightarrow \langle 0|a_{p_1'}a^\dagger_{q_2}\left((2\pi)^3\delta^{(3)}(p_2'-q_1) + b^\dagger_{q_1}b_{p_2'}\right)|0\rangle$$

Using $[a_{p_1'}, a^\dagger_{q_2}] = a_{p_1'}a^\dagger_{q_2} - a^\dagger_{q_2}a_{p_1'} = (2\pi)^3\delta^{(3)}(p_1'-q_2)$

$$\Rightarrow a_{p_1'}a^\dagger_{q_2} = (2\pi)^3\delta^{(3)}(p_1'-q_2) + a^\dagger_{q_2}a_{p_1'}$$

$$= \langle 0|\left((2\pi)^3\delta^{(3)}(p_1'-q_2) + a^\dagger_{q_2}a_{p_1'}\right)\left((2\pi)^3\delta^{(3)}(p_2'-q_1) + b^\dagger_{q_1}b_{p_2'}\right)|0\rangle$$

$$= \langle 0|(2\pi)^6\delta^{(3)}(q_2-p_1')\delta^{(3)}(p_2'-q_1)|0\rangle$$

$$= (2\pi)^6\delta^{(3)}(q_2-p_1')\delta^{(3)}(p_2'-q_1)$$

- The integral becomes:

$$\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}(2\pi)^6\delta^{(3)}(q_2-p_1')\delta^{(3)}(p_2'-q_1)\,e^{i(q_2-q_1)x_1} = e^{i(p_1'-p_2')x_1}$$

- By analogy, the element $\langle 0|\Psi(x_2)\phi(x_2)|p_1 p_2\rangle$ can be evaluated by complex conjugating III and relabelling $x_1 \to x_2$, $p_1' \to p_1$ and $p_2' \to p_2$

$$\therefore \langle 0|\Psi(x_2)\phi(x_2)|p_1 \cdot p_2\rangle = e^{-i(p_1-p_2)x_2}$$

$$\therefore e^{i(p_1'-p_2')x_1}\, e^{-i(p_1-p_2)x_2}$$

$$\therefore e^{i((p_1'-p_2')x_1 - (p_1-p_2)x_2)} = \langle p_1' \cdot p_2'|:\Psi^\dagger(x_1)\phi(x_1)\Psi(x_2)\phi(x_2):|p_1, p_2\rangle$$

$\therefore \langle f|S-1|i\rangle$

$$= \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\, e^{i((p_1'-p_2')x_1-(p_1-p_2)x_2)}\int\frac{d^4k}{(2\pi)^4}\frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\varepsilon}$$

$$= \frac{(-ig)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i(2\pi)^8}{k^2-m^2+i\varepsilon}\left[\delta^{(4)}(p_1'+p_2'-k)\,\delta^{(4)}(p_2-p_1-k)\right]$$

$$= \frac{(2\pi)^4 g^2}{2}\int\frac{d^4k}{k^2-M^2+i\varepsilon}\left[\delta^{(4)}(p_1'-p_2'+k)\,\delta^{(4)}(p_2-p_1-k)\right]$$

$$= \frac{(2\pi)^4 g^2}{2}\left[\frac{1}{(p_2'+p_1')^2-M^2+i\varepsilon}\right]\delta^{(4)}(p_2-p_1-k) \quad (+)$$

The computation for ② is identical up to relabelling. Hence overall contribution from s-channel (as above) is $= 2(+)$

$$= (2\pi)^4 g^2\left[\frac{1}{(p_2'+p_1')^2-M^2+i\varepsilon}\right]\delta^{(4)}(p_2-p_1-(p_2'+p_1'))$$

- Also have contribution arising from the following contractions:

$$:\Psi^\dagger(x_2)\phi(x_1)\Psi(x_1)\phi(x_2):\ \overbrace{\Psi^\dagger(x_1)\Psi(x_2)} \quad ③$$

$$+\ :\Psi^\dagger(x_1)\phi(x_2)\Psi(x_2)\phi(x_1):\ \overbrace{\Psi^\dagger(x_2)\Psi(x_1)} \quad ④$$

③ and ④ are identical upto relabelling of $x_2 \leftrightarrow x_1$. Compute ③:

③ $\langle f|S-1|i\rangle = \langle p_1' \cdot p_2'|:\Psi^\dagger(x_2)\phi(x_1)\Psi(x_1)\phi(x_2):|p_1, p_2\rangle$

$$= \langle p_1', p_2'|\Psi^\dagger(x_2)\phi(x_1)\Psi(x_1)\phi(x_2)|p_1, p_2\rangle$$

$$= \langle p_1', p_2'|\underbrace{\Psi^\dagger(x_2)\phi(x_1)|0\rangle}_{\mathrm{V}}\underbrace{\langle 0|\Psi(x_1)\phi(x_2)|p_1, p_2\rangle}_{\mathrm{VI}}$$

V $\langle p_1', p_2'|\Psi^\dagger(x_2)\phi(x_1)|0\rangle$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\, a_{p_1'}b_{p_2'}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_2} + c_{q_1}e^{iq_1\cdot x_2}\right)\left(a_{q_2}e^{-iq_2\cdot x_1} + a^\dagger_{q_2}e^{iq_2\cdot x_1}\right)|0\rangle$$

$$= \langle 0|\int\frac{d^3q_1 d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\, a_{p_1'}b_{p_2'}\left(b^\dagger_{q_1}e^{-iq_1\cdot x_2}a_{q_2}e^{-iq_2\cdot x_1} + b^\dagger_{q_1}e^{-iq_1\cdot x_2}a^\dagger_{q_2}e^{iq_2 x_1}\right)|0\rangle \quad (c \text{ terms vanish again})$$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\frac{\sqrt{4E_{p_1'}E_{p_2'}}}{\sqrt{4E_{q_1}E_{q_2}}}\, a_{p_1'}b_{p_2'}\underbrace{\cancel{b^\dagger_{q_1}a_{q_2}}}_{0}e^{-iq_1\cdot x_2}e^{-iq_2\cdot x_1} + a_{p_1'}b_{p_2'}b^\dagger_{q_1}e^{-iq_1\cdot x_2}a^\dagger_{q_2}e^{iq_2 x_1}|0\rangle$$

$$= \langle 0|\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}\, a_{p_1'}b_{p_2'}b^\dagger_{q_1}e^{-iq_1 x_2}a^\dagger_{q_2}e^{iq_2 x_1}|0\rangle$$

$\therefore$ From above, we have the result:

$$\int\frac{d^3q_1\, d^3q_2}{(2\pi)^6}(2\pi)^6\delta^{(3)}(q_2-p_1')\delta^{(3)}(p_2'-q_1)\,e^{-iq_1 x_2}e^{iq_2 x_1}$$

$\therefore q_2 = p_1'$ and $q_1 = p_2'$

$$\Rightarrow (2\pi)^6 e^{-ip_2' x_2}e^{ip_1' x_1} = e^{i(p_1' x_1 - p_2' x_2)}$$

VI $\equiv$ V up to complex conjugation and $x_2 \leftrightarrow x_1$ and $p_1' \to p_1$, $p_2' \to p_2$

$$\Rightarrow \mathrm{VI} = e^{-i(p_1 x_2 - p_2 x_1)}$$

$$\therefore e^{i(p_1' x_1 - p_2' x_2)}\, e^{-i(p_1 x_2 - p_2 x_1)}$$

$$= e^{i(p_1' x_1 - p_2' x_2 - p_1 x_2 + p_2 x_1)}$$

$$\therefore \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\, e^{i(p_1' x_1 - p_2' x_2) - i(p_1 x_2 - p_2 x_1)}\int\frac{d^4k}{(2\pi)^4}\frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\varepsilon}$$

$$= \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\, e^{i(p_1' x_1 - p_2' x_2 - p_1 x_2 + p_2 x_1)}\int\frac{d^4k}{(2\pi)^4}\frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\varepsilon}$$

$$= \frac{(-ig)^2}{2}\int d^4x_1\, d^4x_2\, e^{i(p_1'+p_2)x_1 - i(p_2'+p_1)x_2}\int\frac{d^4k}{(2\pi)^4}\frac{i e^{ik(x_1-x_2)}}{k^2-m^2+i\varepsilon}$$

$$= (-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i(2\pi)^8}{k^2-m^2+i\varepsilon}\left[\delta^{(4)}((p_1'+p_2)+k)\,\delta^{(4)}((p_2'-p_1)-k)\right]$$

$$= (-ig)^2(2\pi)^4\frac{i}{(-p_1'-p_2)^2-m^2+i\varepsilon}\delta^{(4)}(-p_2'-p_1-(-p_1'-p_2))$$

$$= \frac{-(2\pi)^4 g^2}{(p_1'-p_2)^2-m^2+i\varepsilon}\delta^{(4)}(-p_2'-p_1+p_1'+p_2)$$

$$= \frac{(2\pi)^4 g^2}{(p_1'-p_2)^2-m^2+i\varepsilon}\delta^{(4)}(-p_1'+p_2-p_1-p_2')$$

$\therefore$ Combining the two pairs of contractions we have:

$$\frac{-(2\pi)^4 g^2}{(p_1'-p_2)^2-M^2+i\varepsilon}\delta^{(4)}(-p_1'+p_2-p_1-p_2') - (2\pi)^4 g^2\left[\frac{1}{(p_2'+p_1')^2-M^2+i\varepsilon}\right]\delta^{(4)}(p_2+p_1-(p_2'+p_1'))$$

$$= (2\pi)^4\delta^{(4)}((p_1+p_2)-(p_1'+p_2'))\left[\frac{1}{(p_1'-p_2)^2-M^2+i\varepsilon} + \frac{1}{(p_1'+p_2')-M^2+i\varepsilon}\right]\times -ig^2$$

- Since $\langle f|S|i\rangle = i\mathcal{A}_{fi}$, the amplitude $\mathcal{A}_{fi}$ is given by

$$\mathcal{A}_{fi} = \boxed{-g^2\left[\frac{1}{(p_1'-p_2)^2-M^2+i\varepsilon} + \frac{1}{(p_1'+p_2')-M^2+i\varepsilon}\right]}$$

Assignment 2: Question 4

Saturday, 27 May 2023 12:10 pm

(4) The Weyl representation of the Clifford algebra is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \overset{\text{2x2 identity matrix}}{1} \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

a. Show that these indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Solution:

$$\{\gamma^\mu, \gamma^\nu\} = \underset{①}{\gamma^\mu\gamma^\nu} + \underset{②}{\gamma^\nu\gamma^\mu}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}, \quad \gamma^\nu = \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix}$$

① $$\gamma^\mu\gamma^\nu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^\mu\sigma^\nu & 0 \\ 0 & -\sigma^\nu\sigma^\mu \end{pmatrix}$$

② $$\gamma^\nu\gamma^\mu = \begin{pmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^\nu\sigma^\mu & 0 \\ 0 & -\sigma^\nu\sigma^\mu \end{pmatrix}$$

$$\{\gamma^\mu, \gamma^\nu\} = ① + ② = \begin{pmatrix} -\sigma^\mu\sigma^\nu & 0 \\ 0 & -\sigma^\mu\sigma^\nu \end{pmatrix} + \begin{pmatrix} -\sigma^\nu\sigma^\mu & 0 \\ 0 & -\sigma^\nu\sigma^\mu \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^\mu\sigma^\nu - \sigma^\nu\sigma^\mu & 0 \\ 0 & -\sigma^\mu\sigma^\nu - \sigma^\nu\sigma^\mu \end{pmatrix}$$

$$= \begin{pmatrix} -\{\sigma^\mu, \sigma^\nu\} & 0 \\ 0 & -\{\sigma^\mu, \sigma^\nu\} \end{pmatrix} = \begin{pmatrix} -2\delta^{\mu\nu} & 0 \\ 0 & -2\delta^{\mu\nu} \end{pmatrix} = -2\delta^\mu_\nu \underset{\text{4x4 identity matrix}}{\mathbb{I}}$$

$$= \underline{\underline{2\eta^{\mu\nu}}}$$

b. Find a unitary matrix $U$ such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra $(\gamma')^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $(\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Solution:

- Let $U$ be defined as:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad A, B, C, D = 2\times 2 \text{ matrices}$$

- If $U$ is unitary, then $UU^\dagger = \mathbb{I}$

$$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} AA^\dagger + BB^\dagger &= 1_{2\times2} \\ CC^\dagger + DD^\dagger &= 1_{2\times2} \\ AC^\dagger + BD^\dagger &= 0 \end{aligned}\right\} (*)$$

If $\underbrace{U\gamma'^0 U^\dagger}= \gamma^0$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A & -B \\ C & -D \end{pmatrix}\begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} AA^\dagger - BB^\dagger &= 0 \\ AC^\dagger - BD^\dagger &= 1 \\ CA^\dagger - DB^\dagger &= 1 \\ CC^\dagger - DD^\dagger &= 0 \end{aligned}\right\} (**) \quad \Rightarrow AA^\dagger = BB^\dagger, \quad \Rightarrow CC^\dagger = DD^\dagger$$

From the relations (**), the relations (*) become:

$\therefore$ $2AA^\dagger = 1$, $2CC^\dagger = 1$, $2BB^\dagger = 1$, $2DD^\dagger = 1$

$AC^\dagger = 1 + BD^\dagger$

$\therefore$ $1 + 2BD^\dagger = 0 \Rightarrow 2BD^\dagger = -1 \Rightarrow 2AC^\dagger = 1$

- Rescale by setting $A = \frac{\sqrt{2}}{2}A'$, $B = \frac{\sqrt{2}}{2}B'$, $C = \frac{\sqrt{2}}{2}C'$, $D = \frac{\sqrt{2}}{2}D'$

(rescaling in such a way preserves unitary nature of $A'$, $B'$, $C'$ and $D'$).
From the above relations, $A' = C'$ and $B' = -D'$

$$\therefore \quad U = \frac{\sqrt{2}}{2}\begin{pmatrix} A & B \\ A & -B \end{pmatrix}$$

- We need $U(\gamma')^i U^\dagger = \gamma^i$

$$\Rightarrow \begin{pmatrix} A & B \\ A & -B \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} A^\dagger & A^\dagger \\ B^\dagger & -B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix} -B\sigma^i & A\sigma^i \\ B\sigma^i & A\sigma^i \end{pmatrix}\begin{pmatrix} A^\dagger & A^\dagger \\ B^\dagger & -B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -B\sigma^i A^\dagger + A\sigma^i B^\dagger & -B\sigma^i A^\dagger - A\sigma^i B^\dagger \\ B\sigma^i A^\dagger + A\sigma^i B^\dagger & B\sigma^i A^\dagger - A\sigma^i B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- The above is satisfied if $-B\sigma^i A^\dagger - A\sigma^i B^\dagger = \sigma^i$. Can take $A = -B = 1$ } (2x2 identity)

$$\therefore \quad U = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

# Assignment 2: Question 5

Sunday, 21 May 2023 9:25 pm

(5) Consider a spin-$\frac{1}{2}$ field with the Dirac Lagrangian density:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

a. Derive the equation of motion, the Dirac equation for both $\psi$ and $\bar{\psi}$. Show that these equations are mathematically equivalent

**Solution:**

$\therefore \mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$

$\therefore$ Euler-Lagrange Eqns of motion for $\psi$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

$\therefore \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu \quad , \quad \dfrac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}$

$\therefore$ Equation of motion for $\psi$:

$$\partial_\mu\bar{\psi}i\gamma^\mu + m\bar{\psi} = 0 \qquad = \bar{\psi}(i\partial_\mu\gamma^\mu + m) = 0 \quad (1)$$

Similarly, equation of motion for $\bar{\psi}$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0$$

$\therefore \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \quad , \quad \dfrac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi$

$\therefore$ Eqn of motion for $\bar{\psi}$:

$$-(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0 \quad (2)$$

- To demonstrate the mathematical equivalence between (1) and (2) one can take (2) and take its Hermitian conjugate (i.e. conjugate transpose)

$$\left[(i\gamma^\mu\partial_\mu - m)\psi = 0\right]^\dagger$$

$$= \psi^\dagger(-i\gamma^{\mu\dagger}\partial_\mu - m) = 0$$

Given that $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$$\Rightarrow \psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\partial_\mu - m) = 0$$

$$= \psi^\dagger\gamma^0(-i\partial_\mu\gamma^\mu\gamma^0 - (\gamma^0)^{-1}m) = 0$$
$$= \bar{\psi}(-i\partial_\mu\gamma^\mu\gamma^0\gamma^0 - m) = 0$$

- By Clifford algebra, $\gamma^0\gamma^0 = \mathbb{1}$

$$\Rightarrow \bar{\psi}(-i\partial_\mu\gamma^\mu - m) = 0 \quad \Rightarrow \bar{\psi}(i\partial_\mu\gamma^\mu + m) = 0 = (1)$$

Hence, (1) and (2) are mathematically equivalent.

b. Show that the Lagrangian density is invariant under the global phase shift $\psi \to \psi' = e^{i\alpha}\psi$ where $\alpha$ is a constant transformation parameter

**Solution:**

$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$

$\psi' \to e^{i\alpha}\psi$ and $\bar{\psi} = e^{-i\alpha}\bar{\psi}$

$\therefore \mathcal{L}' = e^{-i\alpha}\bar{\psi}(i\gamma^\mu\partial_\mu - m)e^{i\alpha}\psi \Rightarrow \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \mathcal{L}$

$\Rightarrow \mathcal{L}$ is invariant under a transformation of the form $\psi \to \psi' = e^{i\alpha}\psi$.

c. Find the Nöther current associated with the above symmetry

**Solution:**

$$j^\mu = \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\overbrace{\Delta\phi}^{\text{change in field under transformation}} - \mathcal{J}_\mu}_{\text{from Peskin and Schröder (p.18)}} \quad , \quad \text{where } \Delta\mathcal{L} = \partial_\mu\mathcal{J}_\mu = 0$$

$\therefore j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi - \mathcal{J}_\mu$

$\delta\psi = i\psi \quad , \quad \delta\bar{\psi} = -i\bar{\psi}$

$\therefore \exists$ Two independent fields, $\psi$ and $\bar{\psi}$

$$\therefore j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi}$$

$$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = \bar{\psi}i\gamma^\mu(i\psi) = -\underline{\underline{\bar{\psi}\gamma^\mu\psi}}$$

d. Check that the above current is conserved using the equation of motion.

**Solution**

$\therefore \partial_\mu j^\mu = \partial_\mu(-\bar{\psi}\gamma^\mu\psi)$

$$= -\underset{(3)}{\partial_\mu\bar{\psi}\gamma^\mu\psi} - \underset{(4)}{\bar{\psi}\gamma^\mu\partial_\mu\psi}$$

The equations of motion are:

$$\partial_\mu\bar{\psi}i\gamma^\mu + m\bar{\psi} = 0 \quad (1)$$

and:

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad \Rightarrow i\gamma^\mu\partial_\mu\psi - m\psi = 0 \quad (2)$$

$\therefore \partial_\mu\bar{\psi}i\gamma^\mu = -m\bar{\psi} \Rightarrow -im\bar{\psi} = \partial_\mu\bar{\psi}\gamma^\mu$

(3) becomes: $im\bar{\psi}\psi$ and (4) becomes $-im\bar{\psi}\psi$

$\therefore \partial_\mu j^\mu = im\bar{\psi}\psi - im\bar{\psi}\psi = 0 \Rightarrow$ current is <u>conserved</u> $\Rightarrow$ <u>QED</u>

e. Find the condition under which the Lagrangian is symmetric under the axial transformation.

$\psi \to \psi' = e^{i\gamma_5}\psi \ , \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi}e^{i\gamma_5}$

**Solution:**

$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

$\mathcal{L}' = \bar{\psi}e^{i\gamma_5}(i\gamma^\mu\partial_\mu - m)e^{i\gamma_5}\psi$

$\quad = \bar{\psi}ie^{i\gamma_5}\gamma^\mu\partial_\mu e^{i\gamma_5}\psi - me^{i\gamma_5}\psi$

- Conserved current:

$$j^{\mu'} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi')}\delta\psi' + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}')}\delta\bar{\psi}'$$

$\therefore j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$

Since $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$, we also have: $j^\mu = -\bar{\psi}\gamma^5\gamma^\mu\psi$

$\therefore j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu\psi$

$$\therefore \partial_\mu j^\mu = (\bar{\psi}\gamma^\mu)\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi)$$
$$= 2im\bar{\psi}\gamma^5\psi \quad \text{(from eqns of motion)}$$

- In order for the Lagrangian to remain symmetric under the above transformation, $\partial_\mu j^\mu = 0$

$\Rightarrow 2im\bar{\psi}\gamma^5\psi = 0 \Rightarrow \underline{\underline{m = 0}}$

# Assignment 2: Question 6

Friday, 26 May 2023 5:30 pm

(6) The plane wave solutions to the Dirac Equation are:

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}$$

where $\sigma^\mu = (1, \underset{\sim}{\sigma})$, $\bar\sigma^\mu = (1, -\underset{\sim}{\sigma})$ and $\xi^s$ $(s=1,2)$ is a basis of orthonormal two-component spinors satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that:

① $$\sum_{s=1}^{2} u^s(p)\,\bar u^s(p) = \not{p} + m$$

② $$\sum_{s=1}^{2} v^s(p)\,\bar v_s(p) = \not{p} - m$$

**Solution:**

① $$\sum_{s=1,2} u^s(p)\,\bar u^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \sum_s \xi^s\xi^{s\dagger} \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix} \quad \Big(\text{since } \sum_s \xi^s\xi^{s\dagger} = \underbrace{\mathbb{I}}_{\text{identity}}\Big)$$

Using the identity $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2 = m^2$, we have:

$$\begin{pmatrix} \sqrt{m^2} & (p\cdot\sigma) \\ p\cdot\bar\sigma & \sqrt{m^2} \end{pmatrix} = \begin{pmatrix} m & (p\cdot\sigma) \\ (p\cdot\bar\sigma) & m \end{pmatrix} = \gamma^\mu p_\mu + m = \not{p} + m \qquad \text{since } \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

② $$\sum_{s=1,2} v^s(p)\,\bar v^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \begin{pmatrix} -\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \sum_s \begin{pmatrix} -\sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\bar\sigma}\,\xi^s\xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}$$

$$= \sum_s \xi^s\xi^{s\dagger} \begin{pmatrix} -\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)} & p\cdot\sigma \\ p\cdot\bar\sigma & -\sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} \end{pmatrix}$$

$$= \mathbb{I} \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix} = \gamma^\mu p_\mu - m = \underline{\underline{\not{p} - m}}$$