

Assignment 2: Question 1

Sunday, 21 May 2023 8:44 pm

- Consider the time ordered, $\overline{T}(\varphi(x_1)\varphi(x_2))$ and the normal ordered, $:\varphi(x_1)\varphi(x_2):$ product of a field.

- a. Show that both of these products are symmetric under the interchange of x_1 and x_2

Solution:

$$\overline{T}(\varphi(x_1)\varphi(x_2)) = \begin{cases} \varphi(x_1)\varphi(x_2) & x_1^0 < x_2^0 \\ \varphi(x_2)\varphi(x_1) & x_1^0 > x_2^0 \end{cases}$$

If $x_1 \leftrightarrow x_2$:

$$\overline{T}(\varphi(x_2)\varphi(x_1)) = \begin{cases} \varphi(x_2)\varphi(x_1) & x_2^0 < x_1^0 \\ \varphi(x_1)\varphi(x_2) & x_2^0 > x_1^0 \end{cases}$$

- By inspection, it is evident that $\overline{T}(\varphi(x_1)\varphi(x_2)) = \overline{T}(\varphi(x_2)\varphi(x_1))$. Hence, the time ordered product of a field is symmetric under the interchange of particles

- Can write $\varphi(x_i)$ as:

$$\varphi(x_i) = \varphi^+(x_i) + \varphi^-(x_i)$$

where:

$$\varphi^+(x_i) = \int \frac{d^3 p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} a_p^+ e^{-ip \cdot x_i} \quad \text{and} \quad \varphi^-(x_i) = \int \frac{d^3 p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} a_p^- e^{ip \cdot x_i}$$

(and similarly for $\varphi(x_2)$)

$$\begin{aligned} :\varphi(x_1)\varphi(x_2): &= :(\varphi^+(x_1) + \varphi^-(x_1))(\varphi^+(x_2) + \varphi^-(x_2)): \\ &= :\varphi^+(x_1)\varphi^+(x_2) + \varphi^+(x_1)\varphi^-(x_2) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_1)\varphi^-(x_2): \\ &= \varphi^+(x_1)\varphi^+(x_2) + \varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_1)\varphi^-(x_2) \quad (*) \end{aligned}$$

- If we interchange x_1 and x_2 in (*), we obtain:

$$\varphi^+(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^+(x_2) + \varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^-(x_2)$$

- Since $[\varphi^+(x_1), \varphi^+(x_2)] = [\varphi^-(x_2), \varphi^-(x_1)] = 0$ and since $\varphi^-(x_2)\varphi^+(x_1) + \varphi^-(x_1)\varphi^-(x_2)$ is clearly symmetric under the interchange of x_1 and $x_2 \Rightarrow :\varphi(x_1)\varphi(x_2):$ is symmetric under $x_1 \leftrightarrow x_2$

- b. Deduce that the Feynman propagator has the same property

Solution:

- Since the Feynman propagator can be written as:

$$\Delta_F(x_1 - x_2) = \overline{T}(\varphi(x_1)\varphi(x_2)) - :\varphi(x_1)\varphi(x_2):$$

It is a difference of two products, both of which are symmetric under the interchange of x_1 and x_2 . Hence, it is symmetric under the interchange of x_1 and x_2 .

Assignment 2: Question 2

Saturday, 27 May 2023 12:10 pm

(2) Verify Wick's Theorem for the case of three scalar fields:

$$\overline{T}(\phi(x_1)\phi(x_2)\phi(x_3)) = : \phi(x_1)\phi(x_2)\phi(x_3) : + \phi(x_1)\Delta_F(x_2-x_3) + \phi(x_2)\Delta_F(x_3-x_1) + \phi(x_3)\Delta_F(x_1-x_2)$$

Solution:

- Consider the case that $x_1^0 < x_2^0 < x_3^0$, then:

LHS:

$$\begin{aligned} \overline{T}(\phi(x_1)\phi(x_2)\phi(x_3)) &= \phi^+(x_1)\phi^-(x_2)\phi^-(x_3) \\ &= (\phi^+(x_1) + \phi^-(x_1))(\phi^+(x_2) + \phi^-(x_2)) + (\phi^+(x_3) + \phi^-(x_3)) \\ &= [\phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)] [\phi^+(x_3) + \phi^-(x_3)] \\ &= \phi^+(x_1)\phi^+(x_2)\phi^+(x_3) + \overset{\textcircled{1}}{\phi^+(x_1)}\phi^+(x_2)\phi^-(x_3) + \phi^+(x_1)\phi^-(x_2)\phi^+(x_3) + \overset{\textcircled{2}}{\phi^+(x_1)}\phi^-(x_2)\phi^-(x_3) \\ &\quad + \phi^-(x_1)\phi^+(x_2)\phi^+(x_3) + \overset{\textcircled{4}}{\phi^-(x_1)}\phi^+(x_2)\phi^-(x_3) + \overset{\textcircled{5}}{\phi^-(x_1)}\phi^-(x_2)\phi^+(x_3) + \phi^-(x_1)\phi^-(x_2)\phi^-(x_3) + \overset{\textcircled{6}}{\phi^-(x_1)}\phi^-(x_2)\phi^-(x_3) \end{aligned}$$

One can normal order expressions ① - ⑦ as follows:

$$\begin{aligned} \textcircled{1} \quad & \phi^+(x_1)\phi^+(x_2)\phi^-(x_3) \\ &= \phi^-(x_2)\phi^+(x_1)\phi^-(x_3) + [\phi^+(x_1), \phi^-(x_3)]\phi^+(x_2) + \phi^+(x_1)[\phi^+(x_2), \phi^-(x_3)] \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad & \phi^+(x_1)\phi^-(x_2)\phi^+(x_3) \\ &= \phi^-(x_2)\phi^+(x_1)\phi^+(x_3) + [\phi^+(x_1), \phi^-(x_2)]\phi^+(x_3) \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad & \phi^+(x_1)\phi^-(x_2)\phi^-(x_3) \\ &= \phi^-(x_2)\phi^-(x_3)\phi^+(x_1) + \phi^-(x_2)[\phi^+(x_1), \phi^-(x_3)] + [\phi^+(x_1), \phi^-(x_2)]\phi^-(x_3) \end{aligned}$$

④ Already normal ordered

$$\begin{aligned} \textcircled{5} \quad & \phi^-(x_1)\phi^+(x_2)\phi^-(x_3) \\ &= \phi^-(x_1)\phi^+(x_2)\phi^-(x_3) + \phi^-(x_1)[\phi^+(x_2), \phi^-(x_3)] \end{aligned}$$

$$\textcircled{6} \quad \phi^-(x_1)\phi^-(x_2)\phi^+(x_3)$$

⑦ Already normal ordered:

$$\begin{aligned} & \phi^+(x_1)\phi^+(x_2)\phi^+(x_3) + \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + \textcircled{5} + \textcircled{6} + \textcircled{7} + \phi^-(x_1)\phi^-(x_2)\phi^-(x_3) \\ &= : \phi(x_1)\phi(x_2)\phi(x_3) : + [\phi^+(x_1), \phi^-(x_3)]\phi(x_2) + [\phi^+(x_2), \phi^-(x_3)]\phi(x_1) + [\phi^+(x_1), \phi^-(x_2)]\phi(x_3) \\ &= : \phi(x_1)\phi(x_2)\phi(x_3) : + \Delta_F(x_1-x_3)\phi(x_2) + \Delta_F(x_2-x_3)\phi(x_1) + \Delta_F(x_1-x_2)\phi(x_3) \end{aligned}$$

- Reordering x_1^0 , x_2^0 and $x_3^0 \Rightarrow$ relabelling the corresponding fields, etc. \Rightarrow Wick's theorem is verified for the above case without loss of generality.

Assignment 2: Question 3

Saturday, 27 May 2023 12:10 pm

(3) Consider the scalar Yukawa theory given by the Lagrangian density:

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - M^2 \varphi^* \varphi - \frac{1}{2} m^2 \varphi^* \varphi - g \varphi^* \varphi \varphi$$

- Compute the S-matrix amplitude for nucleon-meson scattering $\Psi \rightarrow \Psi \rightarrow \Psi + \Psi$ at order g^2

Solution:

$$|i\rangle = \sqrt{2E_{p_1}} \sqrt{2E_{p_2}} a_{p_1}^\dagger b_{p_2}^\dagger |0\rangle \equiv |p_1, p_2\rangle$$

$$= \sqrt{4E_{p_1} E_{p_2}} a_{p_1}^\dagger b_{p_2}^\dagger |0\rangle = |p_1, p_2\rangle$$

$$|f\rangle = \sqrt{4E_{p'_1} E_{p'_2}} a_{p'_1}^\dagger b_{p'_2}^\dagger |0\rangle = |p'_1, p'_2\rangle$$

$$\therefore \langle f | S - 1 | i \rangle$$

At order g^2 , we have the term:

$$\frac{(-ig)^2}{2} \int d^4x_1 d^4x_2 T [\varphi^*(x_1) \varphi(x_1) \phi(x_1) \varphi^*(x_2) \varphi(x_2) \phi(x_2)]$$

- By Wick's Theorem, the relevant components are:

$$\begin{aligned} & : \varphi^*(x_1) \phi(x_1) \varphi(x_2) \phi(x_2) : \overbrace{\varphi(x_1) \varphi^*(x_2)}^{(1)} \\ \text{and} \quad & : \varphi^*(x_2) \phi(x_2) \varphi(x_1) \phi(x_1) : \overbrace{\varphi(x_2) \varphi^*(x_1)}^{(2)} \end{aligned}$$

$$\textcircled{1} \quad \langle f | S - 1 | i \rangle = \langle p'_1, p'_2 | : \varphi^*(x_1) \phi(x_1) \varphi(x_2) \phi(x_2) : | p_1, p_2 \rangle$$

$$= \langle p'_1, p'_2 | \varphi^*(x_1) \phi(x_1) | 0 \rangle \langle 0 | \varphi(x_2) \phi(x_2) | p_1, p_2 \rangle \quad \textcircled{1}$$

$$\textcircled{2} \quad \langle p'_1, p'_2 | \varphi^*(x_1) \phi(x_1) | 0 \rangle :$$

$$= \langle 0 | \sqrt{4E_{p'_1} E_{p'_2}} a_{p'_1}^\dagger b_{p'_2}^\dagger \int \frac{d^3 q_1}{(2\pi)^3} \sqrt{2E_{q_1}} (b_{q_1}^\dagger e^{-iq_1 x_1} + c_{q_1}^\dagger e^{iq_1 x_1}) \int \frac{d^3 q_2}{(2\pi)^3} (a_{q_2}^\dagger e^{-iq_2 x_1} + a_{q_2}^\dagger e^{iq_2 x_1}) | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} \frac{\sqrt{4E_{p'_1} E_{p'_2}}}{\sqrt{4E_{q_1} E_{q_2}}} a_{p'_1}^\dagger b_{p'_2}^\dagger (b_{q_1}^\dagger e^{-iq_1 x_1} + c_{q_1}^\dagger e^{iq_1 x_1}) (a_{q_2}^\dagger e^{-iq_2 x_1} + a_{q_2}^\dagger e^{iq_2 x_1}) | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} \frac{\sqrt{4E_{p'_1} E_{p'_2}}}{\sqrt{4E_{q_1} E_{q_2}}} a_{p'_1}^\dagger b_{p'_2}^\dagger (b_{q_1}^\dagger e^{-iq_1 x_1} a_{q_2}^\dagger e^{-iq_2 x_1} + b_{q_1}^\dagger e^{-iq_1 x_1} a_{q_2}^\dagger e^{iq_2 x_1} + c_{q_1}^\dagger e^{-iq_1 x_1} a_{q_2}^\dagger e^{-iq_2 x_1} + c_{q_1}^\dagger e^{-iq_1 x_1} a_{q_2}^\dagger e^{iq_2 x_1}) | 0 \rangle \quad (\text{Terms containing } c \text{ will vanish})$$

$$= \langle 0 | \int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} \frac{\sqrt{4E_{p'_1} E_{p'_2}}}{\sqrt{4E_{q_1} E_{q_2}}} (a_{p'_1}^\dagger b_{p'_2}^\dagger b_{q_1}^\dagger a_{q_2}^\dagger) e^{i(-q_1 - q_2)x_1} + (a_{p'_1}^\dagger b_{p'_2}^\dagger b_{q_1}^\dagger a_{q_2}^\dagger) e^{i(q_1 - q_2)x_1} | 0 \rangle \quad \textcircled{11}$$

$$\textcircled{10} \quad \langle 0 | a_{p'_1}^\dagger b_{p'_2}^\dagger b_{q_1}^\dagger a_{q_2}^\dagger | 0 \rangle$$

$$= \langle 0 | a_{p'_1}^\dagger a_{q_2}^\dagger b_{p'_2}^\dagger b_{q_1}^\dagger | 0 \rangle$$

$$\text{Using } [b_{p'_1}, b_{q_1}^\dagger] = (2\pi)^3 \delta^{(3)}(p'_1 - q_1)$$

$$\Rightarrow b_{p'_1} b_{q_1}^\dagger - b_{q_1}^\dagger b_{p'_1} = (2\pi)^3 \delta^{(3)}(p'_1 - q_1)$$

$$\therefore b_{p'_1} b_{q_1}^\dagger = (2\pi)^3 \delta^{(3)}(p'_1 - q_1) + b_{q_1}^\dagger b_{p'_1}$$

$$\Rightarrow \langle 0 | a_{p'_1}^\dagger a_{q_2}^\dagger ((2\pi)^3 \delta^{(3)}(p'_1 - q_1) + b_{q_1}^\dagger b_{p'_1}) | 0 \rangle$$

$$\text{Using } [a_{p'_1}, a_{q_2}^\dagger] = a_{p'_1}^\dagger a_{q_2}^\dagger - a_{q_2}^\dagger a_{p'_1} = (2\pi)^3 \delta^{(3)}(p'_1 - q_2)$$

$$\Rightarrow a_{p'_1}^\dagger a_{q_2}^\dagger = (2\pi)^3 \delta^{(3)}(p'_1 - q_2) + a_{q_2}^\dagger a_{p'_1}$$

$$= \langle 0 | ((2\pi)^3 \delta^{(3)}(p'_1 - q_2) + a_{q_2}^\dagger a_{p'_1}) ((2\pi)^3 \delta^{(3)}(p'_2 - q_1) + b_{q_1}^\dagger b_{p'_2}) | 0 \rangle$$

$$= \langle 0 | (2\pi)^6 \delta^{(3)}(q_2 - p'_1) \delta^{(3)}(p'_2 - q_1) | 0 \rangle$$

$$= (2\pi)^6 \delta^{(3)}(q_2 - p'_1) \delta^{(3)}(p'_2 - q_1)$$

- The integral becomes:

$$\int \frac{d^3 q_1 d^3 q_2}{(2\pi)^6} (2\pi)^6 \delta^{(3)}(q_2 - p'_1) \delta^{(3)}(p'_2 - q_1) e^{i(q_2 - p'_1)x_1} e^{i(p'_2 - q_1)x_1} = \underline{\underline{e^{i(p'_1 - p'_2)x_1}}}$$

- By analogy, the element $\langle 0 | \varphi(x_2) \phi(x_2) | p_1, p_2 \rangle$ can be evaluated by complex conjugating $\textcircled{10}$ and relabelling $x_1 \rightarrow x_2$, $p'_1 \rightarrow p_1$ and $p'_2 \rightarrow p_2$.

$$\therefore \langle 0 | \varphi(x_2) \phi(x_2) | p_1, p_2 \rangle = e^{-i(p_1 - p_2)x_2}$$

$$\therefore e^{i(p'_1 - p'_2)x_1} e^{-i(p_1 - p_2)x_2}$$

$$\therefore e^{i((p'_1 - p'_2)x_1 - (p_1 - p_2)x_2)} = \langle p'_1, p'_2 | : \varphi^*(x_1) \phi(x_1) \varphi(x_2) \phi(x_2) : | p_1, p_2 \rangle \quad \textcircled{11}$$

$$\therefore \langle f | S - 1 | i \rangle$$

$$= \frac{(-ig)^2}{2} \int d^4x_1 d^4x_2 e^{i(p'_1 - p'_2)x_1 - i(p_1 - p_2)x_2} \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{i k(x_1 - x_2)}}{k^2 - m^2 + i\varepsilon}$$

$$= \frac{(-ig)^2}{2} \int d^4x_1 d^4x_2 \frac{i(2\pi)^8}{(2\pi)^4} \delta^{(4)}(p'_1 - p'_2 - k) \delta^{(4)}(p_2 - p_1 - k) | 0 \rangle$$

$$= \frac{(-ig)^2}{2} \int d^4x_1 d^4x_2 e^{i(p'_1 + p_2)x_1 - i(p'_2 + p_1)x_2} \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{i k(x_1 - x_2)}}{k^2 - m^2 + i\varepsilon}$$

$$= (-ig)^2 \int d^4x_1 d^4x_2 \frac{i(2\pi)^8}{(2\pi)^4} \delta^{(4)}((p'_1 + p_2) + k) \delta^{(4)}((-p'_2 - p_1) - k) | 0 \rangle$$

$$= (-ig)^2 (2\pi)^4 \frac{i}{(-p'_1 - p_2)^2 - m^2 + i\varepsilon} \delta^{(4)}((-p'_2 - p_1) - (-p'_1 - p_2))$$

$$= \frac{(2\pi)^4 g^2}{(-p'_1 - p_2)^2 - m^2 + i\varepsilon} \delta^{(4)}(p'_1 + p_2 - p_1 - p_2)$$

- The total S-matrix amplitude is given by

$$\frac{(2\pi)^4 g^2}{(-p'_1 - p_2)^2 - m^2 + i\varepsilon} \delta^{(4)}(p'_1 \rightarrow p_2 - p_1 - p'_2) + (2\pi)^4 g^2 \left[\frac{1}{(p'_1 + p_1)^2 - M^2 + i\varepsilon} \right] \delta^{(4)}(p_2 - p_1 - (p'_1 + p_1))$$

Assignment 2: Question 4

Saturday, 27 May 2023 12:10 pm

(4) The Weyl representation of the Clifford algebra is given by:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ matrix} \quad \text{2x2 identity matrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

a. Show that these indeed satisfy $\{\gamma^u, \gamma^v\} = 2\eta^{uv}$

Solution:

$$\{\gamma^u, \gamma^v\} = \gamma^u \gamma^v + \gamma^v \gamma^u$$

$$\gamma^u = \begin{pmatrix} 0 & \sigma^u \\ -\sigma^u & 0 \end{pmatrix}, \quad \gamma^v = \begin{pmatrix} 0 & \sigma^v \\ -\sigma^v & 0 \end{pmatrix}$$

$$\textcircled{1} \quad \gamma^u \gamma^v = \begin{pmatrix} 0 & \sigma^u \\ -\sigma^u & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^v \\ -\sigma^v & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^u \sigma^v & 0 \\ 0 & -\sigma^v \sigma^u \end{pmatrix}$$

$$\textcircled{2} \quad \gamma^v \gamma^u = \begin{pmatrix} 0 & \sigma^v \\ -\sigma^v & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^u \\ -\sigma^u & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^v \sigma^u & 0 \\ 0 & -\sigma^u \sigma^v \end{pmatrix}$$

$$\{\gamma^u, \gamma^v\} = \textcircled{1} + \textcircled{2} = \begin{pmatrix} -\sigma^u \sigma^v & 0 \\ 0 & -\sigma^u \sigma^v \end{pmatrix} + \begin{pmatrix} -\sigma^v \sigma^u & 0 \\ 0 & -\sigma^v \sigma^u \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^u \sigma^v - \sigma^v \sigma^u & 0 \\ 0 & -\sigma^u \sigma^v - \sigma^v \sigma^u \end{pmatrix}$$

$$= - \begin{pmatrix} \{\sigma^u, \sigma^v\} & 0 \\ 0 & \{\sigma^u, \sigma^v\} \end{pmatrix} = - \begin{pmatrix} 2\delta^{uv} & 0 \\ 0 & 2\delta^{uv} \end{pmatrix} = -2\delta^{uv} \underbrace{\frac{I}{2}}_{4 \times 4 \text{ identity matrix}}$$

b. Find a unitary matrix U such that $(\gamma^i)^u = U \gamma^u U^\dagger$, where $(\gamma^i)^u$ form the Dirac representation of the Clifford algebra.

Solution:

- Let U be defined as:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad A, B, C, D = 2 \times 2 \text{ matrices}$$

If U is unitary, then $UU^\dagger = I$

$$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow AA^\dagger + BB^\dagger = I_{2 \times 2} \quad CC^\dagger + DD^\dagger = I_{2 \times 2} \quad \left. \begin{array}{c} \\ \end{array} \right\} (\star)$$

$$AC^\dagger + BD^\dagger = 0$$

If $U \gamma^0 U^\dagger = \gamma^0$, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} \begin{pmatrix} A^\dagger & C^\dagger \\ B^\dagger & D^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow AA^\dagger - BB^\dagger = 0 \quad \left. \begin{array}{c} \\ \end{array} \right\} \Rightarrow AA^\dagger = BB^\dagger$$

$$AC^\dagger - BD^\dagger = 1 \quad \left. \begin{array}{c} \\ \end{array} \right\} (\star\star)$$

$$CA^\dagger - DB^\dagger = 1$$

$$CC^\dagger - DD^\dagger = 0 \quad \left. \begin{array}{c} \\ \end{array} \right\} \Rightarrow CC^\dagger = DD^\dagger$$

From the relations $(\star\star)$, the relations (\star) become:

$$\therefore 2AA^\dagger = 1, \quad 2CC^\dagger = 1, \quad 2BB^\dagger = 1, \quad 2DD^\dagger = 1$$

$$AC^\dagger = 1 + BD^\dagger$$

$$\therefore 1 + 2BD^\dagger = 0 \Rightarrow 2BD^\dagger = -1 \Rightarrow 2AC^\dagger = 1$$

- Rescale by setting $A = \frac{\sqrt{2}}{2} A'$, $B = \frac{\sqrt{2}}{2} B'$, $C = \frac{\sqrt{2}}{2} C'$, $D = \frac{\sqrt{2}}{2} D'$

(rescaling in such a way preserves unitary nature of A' , B' , C' and D').

From the above relations, $A' = C'$ and $B' = -D'$

$$\therefore U = \frac{\sqrt{2}}{2} \begin{pmatrix} A & B \\ A & -B \end{pmatrix}$$

- We need $U(\gamma^i)^\dagger U^\dagger = \gamma^i$

$$\Rightarrow \begin{pmatrix} A & B \\ A & -B \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} A^\dagger & A^\dagger \\ B^\dagger & -B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2} \begin{pmatrix} -B\sigma^i & A\sigma^i \\ B\sigma^i & A\sigma^i \end{pmatrix} \begin{pmatrix} A^\dagger & A^\dagger \\ B^\dagger & -B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -B\sigma^i A^\dagger + A\sigma^i B^\dagger & -B\sigma^i A^\dagger - A\sigma^i B^\dagger \\ B\sigma^i A^\dagger + A\sigma^i B^\dagger & B\sigma^i A^\dagger - A\sigma^i B^\dagger \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\therefore \text{The above is satisfied if } -B\sigma^i A^\dagger - A\sigma^i B^\dagger = \sigma^i. \text{ Can take } A = -B = 1 \text{ (2x2 identity)}$$

$$\therefore U = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Assignment 2: Question 6

Friday, 26 May 2023 5:30 pm

(6) The plane wave solutions to the Dirac equation are:

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix}$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s ($s=1, 2$) is a basis of orthonormal two-component spinors satisfying $(\xi^s)^+ \cdot \xi^{s'} = \delta^{ss'}$. Show that:

$$\textcircled{1} \quad \sum_{s=1}^2 u^s(p) \bar{u}^s(p) = p \cdot m$$

$$\textcircled{2} \quad \sum_{s=1}^2 v^s(p) \bar{v}_s(p) = p - m$$

Solution:

$$\begin{aligned} \textcircled{1} \quad \sum_{s=1,2} u^s(p) \bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix} \begin{pmatrix} \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \xi^{s+} \sqrt{p \cdot \sigma} \\ \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \xi^{s+} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &= \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \xi^s \xi^{s+} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \xi^s \xi^{s+} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &= \sum_s \xi^s \xi^{s+} \begin{pmatrix} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \end{pmatrix} \quad (\text{since } \sum_s \xi^s \xi^{s+} = \text{II}) \end{aligned}$$

identity

Using the identity $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2$, we have:

$$\begin{pmatrix} \sqrt{m^2} & (p \cdot \sigma) \\ p \cdot \bar{\sigma} & \sqrt{m^2} \end{pmatrix} = \begin{pmatrix} m & (p \cdot \sigma) \\ (p \cdot \bar{\sigma}) & m \end{pmatrix} = \gamma^\mu p_\mu \cdot m = p \cdot m \quad \text{since } \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\begin{aligned} \textcircled{2} \quad \sum_{s=1,2} v^s(p) \bar{v}_s(p) &= \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix} \begin{pmatrix} -\xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \xi^{s+} \sqrt{p \cdot \sigma} \\ \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & -\xi^{s+} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &= \sum_s \begin{pmatrix} -\sqrt{p \cdot \sigma} \xi^s \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & \sqrt{p \cdot \sigma} \xi^s \xi^{s+} \sqrt{p \cdot \sigma} \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \xi^{s+} \sqrt{p \cdot \bar{\sigma}} & -\sqrt{p \cdot \bar{\sigma}} \xi^s \xi^{s+} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &= \sum_s \xi^s \xi^{s+} \begin{pmatrix} -\sqrt{(p \cdot \sigma)(p \cdot \bar{\sigma})} & p \cdot \sigma \\ p \cdot \bar{\sigma} & -\sqrt{(p \cdot \bar{\sigma})(p \cdot \sigma)} \end{pmatrix} \\ &= \text{II} \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = \gamma^\mu p_\mu - m = p - m \end{aligned}$$