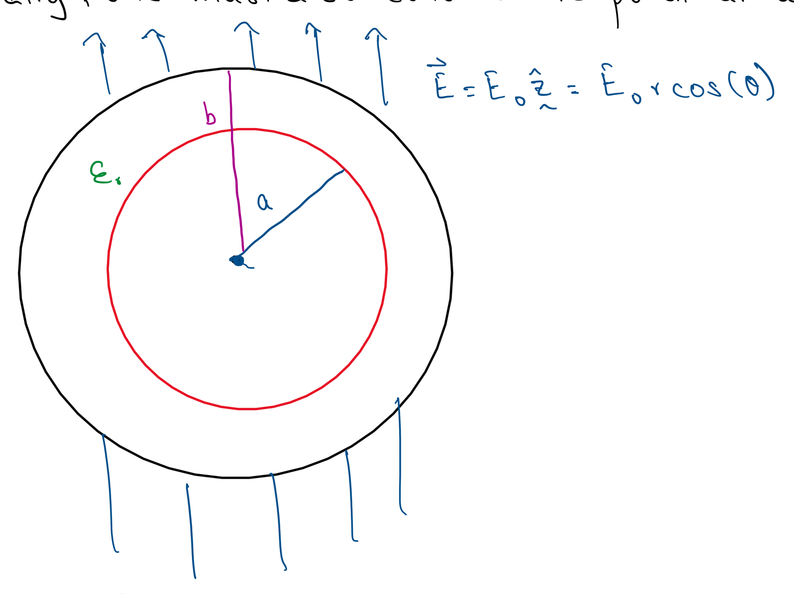
Assignment 2: Question 3

Tuesday, 22 March 2022 6:44 am

Uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ε_r) out to radius b. Object is now placed in an otherwise uniform electric field E_0. Seek to find electric field within the insulator.

Solution:

- One can first consider the potential both within and outside the dielectric. Let V_{med} denote the potential within the dielectric and V_{out} denote the potential outside it. Additionally, one must also consider the potential within the sphere and denote this by V_{in}



- One can note that the following boundary conditions must be obeyed:

$V_{out} = V_{med}$ (at $r = b$) (1) (potentials must be equal at the boundary between the dielectric and the outer region)

$$\varepsilon \frac{\partial V_{med}}{\partial r}\bigg|_{r=b} = \varepsilon_0 \frac{\partial V_{out}}{\partial r}\bigg|_{r=b} \quad (2)$$ (potentials must be continuous across the boundary b/w the dielectric and the outer region)

$V_{med} = V_{in} = 0$ (at $r = a$) (3) (potential must be equal at the boundary between the dielectric and the sphere)

- $V_{in} = 0$ as the sphere is uncharged and conducting
- Since there are no free charges within the dielectric, it is evident that the potential is a solution to Laplace's Equation:

$$V_{med}(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos(\theta))$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty}\left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}}\right)P_\ell(\cos(\theta))$$

As $r \gg b$, $V \to -E_0 r\cos(\theta)$ and $r^{-(\ell+1)}$ terms are negligible

$$\therefore -E_0 r\cos(\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos(\theta))\, C_\ell r^\ell$$

$\therefore C_0 + C_1 r P_1(\cos(\theta)) + \ldots = -E_0 r\cos(\theta)$

$\therefore C_1 = -E_0$ (all other $C_\ell = 0$)

$$\therefore V_{out}(r,\theta) = \sum_{\ell=0}^{\infty}\frac{D_\ell}{r^{\ell+1}}P_\ell(\cos(\theta)) - E_0 r\cos(\theta)$$

$\therefore$ Applying the aforementioned boundary conditions to V_{med} and V_{out} yields:

- $V_{med}(b,\theta) = V_{out}(b,\theta)$

$$\Rightarrow \sum_{\ell=0}^{\infty}\frac{D_\ell}{b^{\ell+1}}P_\ell(\cos(\theta)) - E_0 b\cos(\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}}\right)P_\ell(\cos(\theta)) \quad (1)$$

By the second boundary condition:

$$\varepsilon \frac{\partial}{\partial r}\left[\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos(\theta))\right] = \varepsilon_0\frac{\partial}{\partial r}\left[\sum_{\ell=0}^{\infty}\frac{D_\ell}{r^{\ell+1}}P_\ell(\cos(\theta)) - E_0 r\cos(\theta)\right]$$

$$\therefore \frac{\varepsilon}{\varepsilon_0}\left[\sum_{\ell=0}^{\infty}\ell A_\ell b^{\ell-1} - (\ell+1)\frac{B_\ell}{b^{\ell+2}}P_\ell(\cos(\theta))\right] = -\left[\sum_{\ell=0}^{\infty}\frac{(\ell+1)D_\ell}{b^{\ell+2}}P_\ell(\cos(\theta)) - E_0\cos(\theta)\right]$$

$$\therefore \varepsilon_r\left(\sum_{\ell=0}^{\infty}\ell A_\ell b^{\ell-1} - (\ell+1)\frac{B_\ell}{b^{\ell+2}}P_\ell(\cos(\theta))\right) = -\left(\sum_{\ell=0}^{\infty}\frac{(\ell+1)D_\ell}{b^{\ell+2}}P_\ell(\cos(\theta)) - E_0\cos(\theta)\right) \quad (2)$$

- By the third boundary condition:

$$\therefore V_{med}(a,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}}\right)P_\ell(\cos(\theta)) = 0$$

$\therefore A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = 0$ (as this is true $\forall \ell \in \mathbb{Z}$)

$\therefore A_\ell a^\ell = -\frac{B_\ell}{a^{\ell+1}} \Rightarrow B_\ell = -A_\ell a^{2\ell+1}$

- First boundary condition for $\ell \neq 1$ yields:

$$\frac{D_\ell}{b^{\ell+1}}P_\ell(\cos(\theta)) = \left(A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}}\right)P_\ell(\cos(\theta))$$

$\therefore \frac{D_\ell}{b^{\ell+1}} = A_\ell b^\ell + \frac{B_\ell}{b^{\ell+1}}$

$\therefore \frac{D_\ell}{b^{\ell+1}} = A_\ell b^\ell - \frac{A_\ell a^{2\ell+1}}{b^{\ell+1}}$

$\therefore A_\ell b^{2\ell+1} - A_\ell a^{2\ell+1} = D_\ell$

$\therefore D_\ell = A_\ell(b^{2\ell+1} - a^{2\ell+1})$

$\therefore$ Second boundary condition becomes (for $\ell \neq 1$)

$$\varepsilon_r\left(A_\ell \ell b^{\ell-1} - (\ell+1)\cdot\frac{-A_\ell a^{2\ell+1}}{b^{\ell+2}}\right) = -(\ell+1)\frac{D_\ell}{b^{\ell+2}}$$

$$\therefore D_\ell = -\varepsilon_r A_\ell\left(\left(\frac{\ell}{\ell+1}\right)b^{2\ell+1} + a^{2\ell+1}\right) \Rightarrow D_\ell = A_\ell = 0$$

- For $\ell = 1$, boundary condition (1) yields:

$$-E_0 b + \frac{D_1}{b^2} = A_1 b - \frac{A_1 a^3}{b^2}$$

$\therefore D_1 - E_0 b^3 = A_1 b^3 - A_1 a^3$

$\therefore D_1 - E_0 b^3 = A_1(b^3 - a^3) \quad (4)$

- For $\ell = 1$, condition (2) yields:

$$\varepsilon_r\left(A_1 + \frac{2a^3 A_1}{b^3}\right) = -E_0 - \frac{2D_1}{b^3}$$

$$\therefore \varepsilon_r A_1\left(1 + \frac{2a^3}{b^3}\right) = \frac{-E_0 b^3 - 2D_1}{b^3}$$

$\therefore \varepsilon_r A_1(b^3 + 2a^3) = -E_0 b^3 - 2D_1$

$\therefore -E_0 b^3 - 2D_1 = \varepsilon_r A_1(b^3 + 2a^3) \quad (5)$

$\therefore$ From relations (4) and (5) one obtains

$$\begin{aligned}
2(4) + (5) &= 2(B_1 - E_0 b^3) + (-E_0 b^3 - 2B_1) = 2A_1(b^3 - a^3) + \varepsilon_r A_1(b^3 + 2a^3) \\
&= 2B_1 - 2E_0 b^3 - E_0 b^3 - 2B_1 = 2A_1(b^3 - a^3) + \varepsilon_r A_1(b^3 + 2a^3) \\
&= -3E_0 b^3 = 2A_1(b^3 - a^3) + \varepsilon_r A_1(b^3 + 2a^3) \\
&= -3E_0 b^3 = A_1\left[2(b^3 - a^3) + \varepsilon_r(b^3 + 2a^3)\right]
\end{aligned}$$

$$A_1 = \frac{-3E_0 b^3}{2(b^3 - a^3) + \varepsilon_r(b^3 + 2a^3)} = \frac{-3E_0}{\left(\frac{2(b^3 - a^3) + \varepsilon_r(b^3 + 2a^3)}{b^3}\right)}$$

$$= \underbrace{\frac{3E_0}{\left(2\left(1 - \left(\frac{a}{b}\right)^3\right) + \varepsilon_r\left(1 + 2\left(\frac{a}{b}\right)^3\right)\right)}}_{C}\left(r - \frac{a^3}{r^2}\right)\cos(\theta)$$

- Electric field in the insulator can be computed as follows:

$E(r,\theta) = -\nabla V_{med}$:

$$= C\nabla\left(r - \frac{a^3}{r^2}\cos(\theta)\right)$$

$$= C\left(1 + \frac{2a^3}{r^3}\cos(\theta)\hat{r} - \left(r - \frac{a^3}{r^2}\right)\sin(\theta)\hat{\theta}\right)$$

$\therefore$ Electric field is given by:

$$\vec{E}(r,\theta) = C\left(1 + \frac{2a^3}{r^3}\cos(\theta)\hat{r} - \left(r - \frac{a^3}{r^2}\right)\sin(\theta)\hat{\theta}\right) \text{ where}$$

$$C = \frac{3E_0}{\left(2\left(1 - \left(\frac{a}{b}\right)^3\right) + \varepsilon_r\left(1 + 2\left(\frac{a}{b}\right)^3\right)\right)}$$