Assignment 2: Question 2

Monday, 21 March 2022 10:29 pm

(2) Spherical shell of radius R has negligible thickness and charge distribution:

$$\sigma = \sigma_0 \sin^2(\theta)\sin(2\phi)$$

a. Rewrite σ in terms of spherical harmonics
b. Find the dipole moment in the shell.
c. Write a general expression for the potential inside and outside the sphere and the boundary conditions that should be satisfied by your expressions
d. Determine the electric field inside the shell
e. Is there sufficient information to decide if the shell is conducting or insulating

Solution.

a. $Y_2^2(\theta,\varphi) = \frac{1}{4} e^{2i\varphi}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)$ $\qquad Y_2^{-2} = \frac{1}{4} e^{-2i\varphi}\sin^2(\theta)\cdot\sqrt{\frac{15}{2\pi}}$

$$\therefore Y_2^2 + Y_2^{-2} = \frac{1}{4} e^{2i\varphi}\sqrt{\frac{15}{2\pi}}\sin^2(\theta) + \sqrt{\frac{15}{2\pi}}\cdot\frac{1}{4} e^{-2i\varphi}\sin^2(\theta)$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\left(e^{2i\varphi} + e^{-2i\varphi}\right)$$

$$= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\cdot 2i\sin(2\varphi)$$

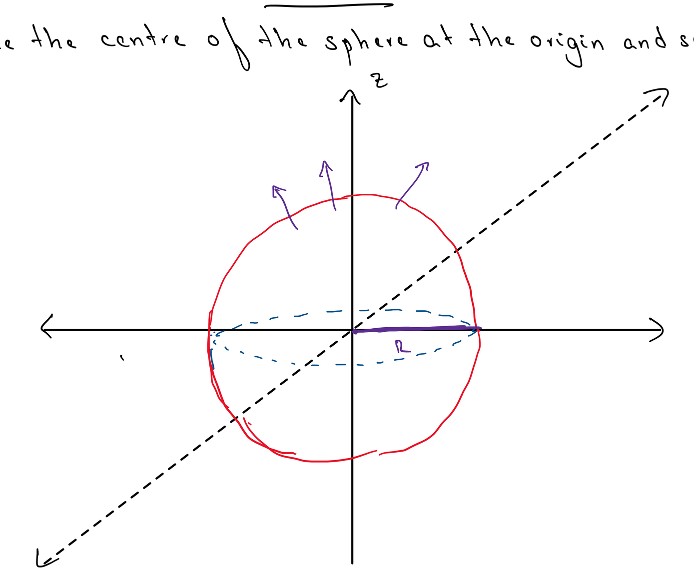
$$= \frac{i}{2}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\sin(2\phi)$$

$$= i\sqrt{\frac{15}{8\pi}}\sin^2(\theta)\sin(2\varphi)$$

$$(Y_2^2 + Y_2^{-2})\cdot -\sqrt{\frac{8\pi}{15}}\, i = \sin^2(\theta)\sin(2\phi)$$

$$\therefore -\sigma_0\sqrt{\frac{8\pi}{15}}\, i\,(Y_2^2 + Y_{-2}^{-2}) = \sigma_0\sin^2(\theta)\sin(2\phi)$$

b. Place the centre of the sphere at the origin and set $\theta = 0$ in the positive z-direction



The dipole moment $\vec{P}$ is given by:

$$\vec{P} = \iint_S \sigma(\underset{\sim}{x}')\,\underset{\sim}{x}'\,dS$$

Parametrise $\underset{\sim}{x}'$ in spherical polar coordinates for a sphere of radius R.

$$\underset{\sim}{x}' = R\sin(\theta)\cos(\phi)\,\hat{\underset{\sim}{x}} + R\sin(\theta)\sin(\phi)\,\hat{\underset{\sim}{y}} + R\cos(\theta)\,\hat{\underset{\sim}{z}}$$
$$= R\left(\sin(\theta)\cos(\phi)\,\hat{\underset{\sim}{x}} + \sin(\theta)\sin(\phi)\,\hat{\underset{\sim}{y}} + \cos(\theta)\,\hat{\underset{\sim}{z}}\right)$$

$$\therefore \vec{P} = \iint_S R\left(\sin(\theta)\cos(\phi)\,\hat{\underset{\sim}{x}} + \sin(\theta)\sin(\phi)\,\hat{\underset{\sim}{y}} + \cos(\theta)\,\hat{\underset{\sim}{z}}\right)\cdot R^2\sin(\theta)\,\sigma_0\sin^2(\theta)\sin(2\phi)\,d\theta\,d\phi$$

$0 \le \phi \le 2\pi$
$0 \le \theta \le \pi$

$$= \sigma_0\int_0^{2\pi}\int_0^{\pi}\left[R^3\sin^2(\theta)\cos(\phi)\,\hat{\underset{\sim}{x}} + R^2\sin^2(\theta)\sin(\phi)\,\hat{\underset{\sim}{y}} + R^2\sin(\theta)\cos(\theta)\right]\sin^2(\theta)\sin(2\phi)\,d\theta\,d\phi\Big]$$

$$= \sigma_0\int_0^{2\pi}\int_0^{\pi} R^3\sin^4(\theta)\sin(2\phi)\cos(\phi)\,\hat{\underset{\sim}{x}} + R^2\sin^4(\theta)\sin(\phi)\sin(2\phi)\,\hat{\underset{\sim}{y}} + R^2\sin^3(\theta)\cos(\theta)\sin(2\phi)\,\hat{\underset{\sim}{z}}\,d\theta\,d\phi\Big]$$

$$= \sigma_0\int_0^{2\pi}\int_0^{\pi} 2R^3\sin^4(\theta)\sin(\phi)\cos^2(\phi)\,\hat{\underset{\sim}{x}} + 2R^2\sin^4(\theta)\sin^2(\phi)\cos(\phi)\,\hat{\underset{\sim}{y}} + R^2\sin^3(\theta)\cos(\theta)\sin(2\phi)\,\hat{\underset{\sim}{z}}\,d\theta\,d\phi\Big]$$

$$= \sigma_0\int_0^{2\pi}\int_0^{\pi} 2R^3\sin^4(\theta)\sin(\phi)\cos^2(\phi)\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi} 2R^2\sin^4(\theta)\sin^2(\phi)\cos(\phi)\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi} R^2\sin^3(\theta)\cos(\theta)\sin(2\phi)\,d\theta\,d\phi\Big]$$

$$= \sigma_0\left(\int_0^{2\pi}\sin(\phi)\cos^2(\phi)\overset{d\phi}{}\int_0^{\pi}2R^3\sin^4(\theta)\,d\theta\right)\hat{x} + \left(\int_0^{2\pi}\sin^2(\phi)\cos(\phi)\int_0^{\pi}2R^2\sin^4(\theta)\,d\theta\right)\hat{y} + \left(\int_0^{2\pi}\sin(2\phi)\overset{d\phi}{}\int_0^{\pi}R^2\sin^3(\theta)\cos(\theta)\,d\theta\Big]\right)\hat{\underset{\sim}{z}}$$

$$= \sigma_0\int_0^{2\pi}\sin(\phi)\cos^2(\phi)\,d\phi\int_0^{\pi}2R^3\sin^4(\theta)\,d\theta + \int_0^{2\pi}\sin^2(\phi)\cos(\phi)\,d\phi\int_0^{\pi}2R^2\sin^4(\theta)\,d\theta + \int_0^{2\pi}\sin(2\phi)\,d\phi\int_0^{\pi}R^2\sin^3(\theta)\cos(\theta)\,d\theta\Big]$$

Since $\int_0^{2\pi}\sin(\phi)\cos^2(\phi)\,d\phi = \int_0^{2\pi}\sin^2(\phi)\cos(\phi)\,d\phi = 0 \Rightarrow \hat{x}$ and $\hat{\underset{\sim}{y}}$ components $= 0$

$$\vec{P} = \sigma_0\int_0^{2\pi}\sin(2\phi)\,d\phi\int_0^{\pi}R^2\sin^3(\theta)\cos(\theta)\,d\theta$$

$$\vec{P} = \sigma_0\left[-\frac{1}{2}\cos(2\phi)\right]_0^{2\pi}\int_0^{\pi}R^2\sin^3(\theta)\cos(\theta)\,d\theta$$

$$\Rightarrow = \sigma_0\left(\frac{-1}{2} + \frac{1}{2}\right) = \sigma_0(0) = \underline{\underline{0}}$$

Hence $\vec{P} = 0$

c. The potential inside and outside the sphere is given by the solution to Laplace's Equation for the case where there is azimuthal dependence.

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}r^l + B_{l,m}r^{-l-1}\right)P_l^m(\cos(\theta))\,e^{im\phi}$$

The solution can be separated into two parts, namely the potentials inside and outside the sphere denoted $V_{in}(r,\theta,\phi)$ and $V_{out}(r,\theta,\phi)$ respectively

$$V_{in} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{l,m}r^l + B_{l,m}r^{-l-1}\right)P_l^m(\cos(\theta))\,e^{im\phi}$$
$$V_{out} = \sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'}\left(C_{l,m'}r^{l'} + D_{l,m'}r^{-l'-1}\right)P_{l'}^{m'}(\cos(\theta))\,e^{im'\phi}$$

Boundary conditions

$V_{in}(r,\theta,\phi) =$ finite at $r = 0$ (1)
$V_{out}(r\to\infty,\theta,\phi) = 0$ (2)

$$\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = \frac{\sigma}{\varepsilon_0} \quad (3)$$
$$= \frac{\sigma_0\sin^2(\theta)\sin(2\phi)}{\varepsilon_0}$$

$V_{in}\big|_{r=R} = V_{out}\big|_{r=R}$ (4)

By condition (1) $B_{l,m} = 0$ (as V_{in} must be finite at $r = 0$)
" (2) $C_{l,m} = 0$ (as V_{out} must be 0 as $r\to\infty$)

$$\therefore V_{in} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{l,m}r^l P_l^m(\cos(\theta))\,e^{im\phi} \qquad V_{out} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}D_{l,m}r^{-l-1}P_l^m(\cos(\theta))\,e^{im\phi}$$

d. $$\frac{\partial V_{in}}{\partial r} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} l A_{l,m} r^{l-1}P_l^m(\cos(\theta))\,e^{im\phi} \qquad , \quad \frac{\partial V_{out}}{\partial r} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-l-1)D_{l,m}R^{-l-2}P_l^m(\cos(\theta))\,e^{im\phi}$$
$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} l A_{l,m} r^{l-1}Y_l^m(\theta,\phi) \qquad , \quad \frac{\partial V_{out}}{\partial r} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-l-1)D_{l,m}r^{-l-2}Y_l^m(\theta,\phi)$$

$\therefore$ Imposing boundary condition (3) yields:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} l A_{l,m}R^{l-1}Y_l^m(\theta,\phi) - \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-l-1)D_{l,m}R^{-l-2}Y_l^m(\theta,\phi) = \frac{-\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i\,(Y_2^2 + Y_2^{-2})$$

$\therefore$ Only $l = 2$ and $m = \pm 2$ terms remain in each summation (by comparing coefficients)

$$\therefore (2A_{2,2}RY_2^2 + 2A_{2,-2}RY_{-2}^2) - (-3D_{2,2}R^{-4}Y_2^2 + (-3D_{2,-2}R^{-4}Y_{-2}^2)) = \frac{-\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i\,(Y_2^2 + Y_{-2}^2)$$

$$\therefore 2A_{2,2}RY_2^2 + 3D_{2,2}R^{-4}Y_2^2 = -\frac{\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi i}{15}}\,Y_2^2 \quad (1)$$
$$2A_{2,-2}RY_{-2}^2 + 3D_{2,-2}R^{-4}Y_{-2}^2 = \frac{\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i\,Y_{-2}^2 \quad (2)$$

$$(1')\quad 2A_{2,2}R + 3D_{2,2}R^{-4} = \frac{-\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$
$$(2')\quad 2A_{2,-2}R + 3D_{2,-2}R^{-4} = \frac{\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$

Imposing the condition that $V_{in}\big|_{r=R} = V_{out}\big|_{r=R}$ yields that,

$$A_{22}R^2Y_2^2 + A_{2,-2}R^2Y_{-2}^2 = D_{2,2}R^{-3}Y_2^2 + D_{-2}^2R^{-3}Y_{-2}^2$$

$$\therefore A_{22}R^2 = D_{22}R^{-3} \quad (3)$$
$$A_{2,-2}R^2 = D_{2,-2}R^{-3} \quad (4)$$

$$\therefore D_{22} = A_{2,2}R^5$$

$$\Rightarrow D_{2,-2} = A_{2,-2}R^5$$

Substituting the above into (1') and (2') yields:

$$\therefore 2A_{22}R + 3A_{22}R = -\frac{\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$
$$2A_{2,-2}R + 3A_{2,-2}R = \frac{+\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$

$$\Rightarrow 5A_{22}R = -\frac{\sigma_0}{\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$

$$\therefore A_{2,2} = \frac{-\sigma_0}{5R\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i \quad \text{and} \quad A_{2,-2} = \frac{+\sigma_0}{5R\varepsilon_0}\sqrt{\frac{8\pi}{15}}\,i$$

$$\therefore V_{in}(r,\theta,\phi) = \left(\frac{-\sigma_0}{5\varepsilon_0R}\sqrt{\frac{8\pi}{15}}\,i\cdot r^2\cdot\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\,e^{2i\varphi} + \frac{\sigma_0 r^2}{5\varepsilon_0R}\sqrt{\frac{8\pi}{15}}\,i\cdot\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2(\theta)\,e^{-2i\varphi}\right)$$
$$= \frac{-\sigma_0}{5\varepsilon_0R}\cdot\frac{i}{2}\sin^2(\theta)\,e^{2i\varphi} + \frac{\sigma_0}{5\varepsilon_0R}\cdot\frac{i}{2}\sin^2(\theta)\,e^{-2i\varphi}$$
$$= \frac{\sigma_0 r^2}{5\varepsilon_0R}\sin^2(\theta)\sin(2\phi)$$

$$\vec{E}_{in} = -\nabla V_{in} - \frac{2r\sigma_0}{5R\varepsilon_0}\left[\sin^2(\theta)\sin(2\phi)\,\hat{\underset{\sim}{r}} + \sin(\theta)\cos(\theta)\sin(2\varphi)\,\hat{\underset{\sim}{\theta}} + \frac{\sin(\theta)}{r}\cos(2\varphi)\,\hat{\underset{\sim}{\phi}}\right]$$ (evaluated computationally)

e. Since the electric field inside the sphere is non-zero, it is evident that the sphere is non-conducting