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General Relativity: Examination

(1). This is a reasonable ~~Eq~~ equation

- Reasonable equation
- Nonsense since the ~~o~~ operating with the covariant derivative twice on F^{α} would not lead to a ~~contravariant~~ (3,0) tensor.
- Reasonable equation.
- Reasonable equation.
- Reasonable equation.

(2) $ds^2 = xy\,dx^2 + x^2\,dy^2$

$$\therefore g_{\mu\nu} = \begin{pmatrix} xy & 0 \\ 0 & x^2 \end{pmatrix}$$

$\therefore$ ~~5~~ Possible Christoffel symbols:

$\Gamma^{x}_{xx}, \Gamma^{x}_{xy}, \Gamma^{y}_{xx}, \Gamma^{y}_{xy}$ (up to ~~anti~~symmetry)

$\Gamma^{y}_{yy}, \Gamma^{x}_{yy}, \cancel{\Gamma^{y}_{z}}$

$$\cancel{\Gamma^{xx}_{xx} = \frac{1}{2} \partial_g g^{x\alpha}\left(\partial_x g_{xx} + \partial g_{xx} \partial_x g_{xx}\right.}$$

Diagonal metric $\Rightarrow$ $\Gamma^{y}_{xx} = \frac{-1}{2g_{yy}} \partial_y g_{xx}$

$$= \frac{-1}{2x^2} \partial_y (xy) = \frac{-1}{2x}$$

$$\cancel{\Gamma^{x}_{xx} = \partial_x \ln\sqrt{|g_{xx}|}}$$

$$\cancel{= \partial_x\left(\ln\sqrt{xy}\right) = \frac{\left(\frac{1}{2}y\right)}{\sqrt{xy}} = \frac{y}{2y}}$$

(2) $ds^2 = xy\,dx^2 + x^2 dy^2$

$\therefore g_{\mu\nu} = \begin{pmatrix} xy & 0 \\ 0 & x^2 \end{pmatrix}$

$\therefore$ Metric is diagonal. Christoffel symbols are:

$\cancel{\Gamma^x_{xx}}, \Gamma^x_{xy}, \Gamma^x_{yx}, \Gamma^x_{yy}, \Gamma^y_{xx}, \Gamma^y_{xy}, \Gamma^y_{yx}, \cancel{\Gamma^y_{yy}}$

$\therefore \Gamma^x_{xx} = \partial_x \ln\sqrt{|g_{xx}|} = \partial_x \ln\sqrt{xy} = \frac{1}{(xy)^{1/2}} \cdot \frac{1}{2} x^{-1/2} y^{1/2}$

$= \frac{1}{2x}$

$\Gamma^y_{yy} = \partial_y \ln\sqrt{|g_{yy}|} = \partial_y (\ln\sqrt{|g_{yy}|}) = \underline{0}$

$\Gamma^x_{xy} = \partial_y \ln(\sqrt{g_{xx}}) = \partial_y \ln(\sqrt{xy}) = \frac{1}{2} y^{-1/2} \cdot \frac{1}{(xy)^{1/2}} = \underline{\frac{1}{2y}}$

$\Gamma^x_{yx} = \frac{1}{2y}$ (symmetry), $\Gamma^x_{yy} = \frac{-1}{2g_{xx}} \partial_x (g_{yy}) = \frac{-1}{2xy} \cdot 2x$

$= \underline{\frac{-1}{y}}$

$\Gamma^y_{xx} = \frac{-1}{2g_{yy}} \partial_y (g_{xx}) = \underline{\frac{-1}{2x}}$, $\Gamma^y_{xy} = \Gamma^y_{yx} = \partial_x \ln\sqrt{|g_{yy}|}$

$= \partial_x(\ln\sqrt{x^2})$

$= \partial_x(\ln(x)) = \underline{\frac{1}{x}}$

$\therefore$ To compute $R_{\mu\nu}$, we need $R^\mu{}_{\nu\rho\sigma}$ first: (Riemann tensor).

Compute $R^x{}_{yxy}$ and then R_{xyxy}. We can then obtain other components by symmetry

$$\therefore R^x{}_{yxy} = \cancel{\partial_x \Gamma^x_{yy}} - \partial_y \Gamma^x_{yx} - \Gamma^x_{xx}\Gamma^x_{yy} + \cancel{\Gamma^x_{xy}\Gamma^y_{yy}} - \Gamma^x_{yx} - \Gamma^x_{yy}\Gamma^y_{yx}$$

$$= \partial_x\left(x_y - \frac{1}{y}\right) = 0 \quad - \cancel{\partial_y \Gamma^x_{yx}}\left(\frac{1}{2y}\right) -$$

$$= -\partial_y\left(\frac{1}{2y}\right) - \left(\frac{1}{2x}\right)\left(-\frac{1}{y}\right) \mp \frac{1}{2y}\,\frac{1}{x}$$

$$= \frac{1}{2y^2} - \frac{1}{2xy} - \frac{1}{2yx} = \frac{1}{2y^2} - \frac{\cancel{2}}{\cancel{2}xy}$$

$$= \frac{1}{2y^2} - \frac{1}{xy} = \cancel{\frac{1}{2y^2x}} - \frac{x+2y}{4xy^2}$$

Since Riemann tensor is antisymmetric in first and last two indices we have

$$R^x{}_{yyx} = -R^x{}_{yxy} = \frac{-x+2y}{4xy^2}$$

Compute $R^y{}_{xxy} = -R^y{}_{xyx} =$

$= R^x{}_{yxyx} = -R \cdot \dfrac{x+2y}{4x^2y}$ (since the ~~partial de.~~ computation is identical to the above, up to $-\partial_y \to -\partial_x\left(\frac{1}{2x}\right)$

$\therefore R^y{}_{yxy} = -R^y{}_{yyx} = \dfrac{x+2y}{4x^2y}$ (it is symmetric with $R^x{}_{yyx}$

X (SEE ~~END OF~~ PAGE 7)

$R^{x}{}_{xxx}$ and $R^{y}{}_{yyy}$ vanish by definition of Riemann tensor.

$\cancel{R}\ \cancel{S}.\ \cancel{R^{x}{}_{yxx} = 0} \Rightarrow \cancel{R^{x}{}_{xyx}} = 0$

$$\therefore R_{xx} = g^{xx}\left(R^{x}{}_{xxx} + R^{y}{}_{xyx}\right) = g^{yy} R_{yxyx} = \frac{x+2y}{4x^2y}$$

Similarly, $R_{yy} = g^{x} R^{x}{}_{yxy} + R^{y}{}_{yyy} = \dfrac{x+2y}{4x^2y^2}$

$$\therefore R_{xy} = \begin{pmatrix} \dfrac{x+2y}{4x^2y} & 0 \\ 0 & \dfrac{x+2y}{4x^2y^2} \end{pmatrix}$$

$$\therefore R_{z} = g^{xy} R_{xy}$$

$$= \frac{1}{xy} \cdot \frac{x+2y}{4x^2y} + \frac{1}{x^2} \cdot \frac{x+2y}{4xy^2} = \frac{}{\cancel{4x^3y^2}}$$

$$= \frac{x+2y}{4x^3y^2} + \frac{x+2y}{4x^3y^2} = \frac{2x+4y}{4x^3y^2} = \frac{x+2y}{2x^3y^2}$$

(3) Claim:

$$\nabla_\mu T^{\mu}{}_{\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, T^{\mu}{}_{\nu}\right) - \Gamma^{\lambda}_{\mu\nu} T^{\mu}{}_{\lambda}$$

$$\cancel{\nabla_\mu T^{\mu}{}_{\nu} = \partial_\mu T^{\mu}{}_{\nu} + \Gamma^{\mu}_{\mu\sigma} T^{\sigma}{}_{\mu}}$$

$$\cancel{= \partial_\mu T^\mu{}_\nu + (\sqrt{-g})^{-1}\partial_\sigma(\sqrt{-g})\, T^{\sigma\mu}{}_\sigma - \Gamma}$$

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma$$

$$= \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + (\sqrt{-g})^{-1}\partial_\sigma \sqrt{-g}\, T^\sigma{}_\nu) - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma$$

$$= \partial_\mu T^\mu{}_\nu + \frac{1}{\sqrt{-g}}\partial_\sigma(\sqrt{-g}\, T^\mu{}_\nu) - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma \Rightarrow Q\ \underline{\underline{Ei)}}$$

(where $\lambda \equiv \sigma$)

Want to now show that:

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2}(\partial_\nu g_{\mu\lambda}) T^{\mu\lambda}$$

Case where $\sigma = \mu$:

$$= \frac{1}{\sqrt{-g}}\partial_\mu T^\mu{}_\nu + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^\mu{}_\nu) + \Gamma^\mu_{\mu\nu} T^\mu{}_\mu .$$

$$\therefore \Gamma^\mu_{\mu\nu} = \partial_\nu \ln\left(\sqrt{|g_{\mu\mu}|}\right) = \cancel{\partial_\nu \ln g} \cancel{= \frac{-1}{2g_{\mu\mu}}}$$

$$= \cancel{\ln 2}\ \cancel{\frac{1}{2\sqrt{g}}}\ \frac{1}{2g_{\mu\mu}}\cdot g^{\mu\lambda}\ T_\mu^\mu\ \cancel{\ln(\sqrt{x})}$$

$$\therefore \nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^\mu{}_\nu) + \partial_\nu(\ln(\sqrt{|g_{\mu\mu}|}))\, T^\mu{}_\mu$$

$$= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2}\partial_\nu(g_{\mu\lambda})\, T^\mu{}_\mu g^{\mu\lambda}$$

$$T^\mu{}_\mu g^{\mu\lambda} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2}\partial_\nu(g_{\mu\lambda})\, T^{\mu\lambda}$$

(4) a. Schwarzschild Metric:

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2$$

$$= -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2$$

∴ Geodesic eqn general form:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\therefore \cancel{\frac{d^2r}{d\tau^2}} + \cancel{\Gamma^r_{rr}}\,\cancel{\left(\frac{dx^r}{d\tau}\right)^2} + \Gamma$$

$$= \frac{d^2r}{d\tau^2} + \Gamma^r_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{tr}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0$$

$$\therefore \frac{d^2t}{d\tau^2} - \frac{2GM}{(2GM-r)r}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0 \qquad (1)$$

$$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{d\tau}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\Rightarrow \frac{d^2r}{d\tau^2} - \frac{GM(2GM-r)}{r^3}\left(\frac{dt}{d\tau}\right)^2 + \frac{GM}{(2GM-r)r}\left(\frac{dr}{d\tau}\right)^2 + 2GM-r\left(\frac{d\theta}{d\tau}\right)^2$$

$$+ (2GM-r)\sin^2(\theta)\left(\frac{d\varphi}{d\tau}\right)^2 = 0 \qquad (2)$$

$$\frac{d^2\theta}{d\tau^2} + 2\Gamma^\theta_{r\theta}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \Gamma^\theta_{\varphi\varphi}\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$= \frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \cos(\theta)\sin(\theta)\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\varphi}{d\tau^2} + 2\Gamma^{\varphi}_{\varphi r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + 2\Gamma^{\varphi}_{\theta\varphi}\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$= \frac{d^2\varphi}{d\tau^2} + \frac{2}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + 2\cot(\theta)\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0 \quad (4)$$

b. $R = R_s$ ...?

(2) Recalculating Riemann and Ricci tensors for Question 2.

By definition, the Riemann tensor is given by:

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\nu\alpha}$$

$$R^{x}{}_{xxx} = \partial_x \Gamma^{x}_{xx} - \partial_x \Gamma^{x}_{xx} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx} - \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx}$$

$$= \underbrace{\partial_x\left(\frac{1}{2x}\right) - \partial_x\left(\frac{1}{2x}\right)}_{0} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx} - \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx}$$

$$= 0$$

$\lambda = x \Rightarrow$ the above trivially vanishes.

$\lambda = y \Rightarrow \Gamma^{x}_{xy}\Gamma^{y}_{xx} - \Gamma^{x}_{xy}\Gamma^{y}_{xx} = 0$

Similarly for $R^{y}{}_{yyy} = 0$

$$R^{x}{}_{xxy} = \cancel{\partial_x \Gamma^{x}_{xy} - \partial_y \Gamma^{x}_{xy}} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xy} - \Gamma^{x}_{y\lambda}\Gamma^{\lambda}_{xx}$$

$\lambda = x \Rightarrow = 0$, $\lambda = y \Rightarrow = 0$ (given Christoffel symbols).

By symmetry $R^{x}{}_{xyx} = 0$ (Similarly, any entry with more $R^{y}{}_{yy-} = 0$

The non-vanishing components are:

$R^{yx}{}_{yxy}$ and $R^{x}{}_{yyx}$

~~Thes~~ $R^{x}{}_{yxy} =$

$= \cancel{\partial_x \Gamma^x_{yy}} - \partial_y \Gamma^x_{yx} + \Gamma^x_{x\lambda}\Gamma^\lambda_{yy} - \Gamma^x_{yx}\Gamma^{\lambda}_{yx}$

$= -\partial_y\left(\frac{1}{2y}\right) + \left(\frac{1}{2x}\right)\left(\frac{-1}{y}\right) - \frac{1}{2xy}$

$= \frac{1}{2y^2} - \frac{1}{xy} = \frac{x+2y}{4xy^2}$

By symmetry, $R^{x}{}_{yyx} = -\frac{x+2y}{4xy^2}$

Hence $R^{y}{}_{xxy} = -R^{y}{}_{xyx} = \frac{x+2y}{4x^2y}$ (computation is identical, up to $\partial_y \leftrightarrow \partial_x$) and ~~s~~ due to the symmetry in the form of the Christoffel symbols

Ricci tensor =

$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} = R^{x}{}_{yxx} + R^{x}{}_{yxy}$

$\therefore R_{xx} = R^{\sigma}{}_{x\sigma x} = R^{x}{}_{xxx} + R^{y}{}_{xyx} = 0 + \frac{x+2y}{4x^2y}$

$R_{xy} = R^{\sigma}{}_{x\sigma y} = R^{x}{}_{xxy} + R^{y}{}_{xyy} = 0$

$R_{yx} = -R_{xy} = 0$

$R_{yy} = R^{\sigma}{}_{y\sigma y} = R^{x}{}_{yxy} + R^{y}{}_{yyy}$

$$= \frac{x+2y}{4xy^2} + 0 = \frac{x+2y}{4xy^2}$$

$$\therefore R_{\mu\nu} = \begin{pmatrix} \frac{x+2y}{4y^2x^2} & 0 \\ 0 & \frac{x+2y}{4y^2x} \end{pmatrix}$$

(Ricci scalar was computed earlier = trace of $R_{\mu\nu}$)

$$= \frac{x+2y}{4yx^2} + \frac{x+2y}{4y^2x} = \frac{x+2y}{2x^2y^2}$$

(2) a. Want to write

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

$= \nabla_\mu A_\nu - \nabla_\nu A_\mu$ (Carroll Ch. 4 p. 153)

2 b. Y^μ = parallel transported along $x^\mu(\tau)$, $u^\mu = \frac{dx^\mu}{d\tau}$

It must obey the eqn of parallel transport

(i.e. $\nabla_{\underline{u}} Y = u^\lambda \nabla_\lambda Y^\mu = 0$)

Let $x^\mu(\tau)$ be an affine geodesic:

$\Rightarrow x^\mu(\tau) = \alpha \tau + \beta$, $\alpha, \beta \in \mathbb{R}$

$\therefore u^\mu = \alpha \Rightarrow u_\mu = g_{\mu\delta} u^\delta = g_{\mu\delta}(\alpha) =$

$$= \frac{x+2y}{4xy^2} + 0 = \frac{x+2y}{4xy^2}$$

$$\therefore R_{\mu\nu} = \begin{pmatrix} \frac{x+2y}{4y^2x^2} & 0 \\ 0 & \frac{x+2y}{4y^2x} \end{pmatrix}$$

(Ricci scalar was computed earlier = trace of $R_{\mu\nu}$)

$$= \frac{x+2y}{4yx^2} + \frac{x+2y}{4y^2x} = \frac{x+2y}{2x^3y^2}$$

(5) a. Want to write:

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

$= \nabla_\mu A_\nu - \nabla_\nu A_\mu$ (Carroll Ch. 4 p. 153)

b. Y^μ = parallel transported along $x^\nu(\tau)$, $u^\mu = \frac{dx^\mu}{d\tau}$

It must obey the eqn of parallel transport

(i.e. $\nabla_{\underset{\sim}{u}} \underset{\sim}{Y} = u^\lambda \nabla_\lambda Y^\mu = 0$)

Let $x^\mu(\tau)$ be an affine geodesic:

$\Rightarrow x^\mu(\tau) = \alpha\tau + \beta, \quad \alpha, \beta \in \mathbb{C}$

$u^\mu = \alpha \Rightarrow u_\mu = g_{\mu\delta} u^\delta = g_{\mu\delta}(\alpha) =$

$\therefore u^{\mu} u_{\mu} = \alpha^2 = \text{constant.}$

$Y^{\mu} u_{\mu} = Y^0 \alpha + \alpha Y^1 + \alpha Y^2 + \alpha Y^3 \ldots$

$= \alpha Y^{\mu}$ (dot product is conserved since $u_{\mu} =$ constant).

$\therefore \cancel{Y^{\mu} u_{\mu}} =$

Know that $\nabla_{\mu} Y^{\nu} = 0 \Rightarrow \cancel{\partial_{\mu} Y^{\nu} + \Gamma^{\nu}_{\mu\sigma}}$.

$\partial_{\mu} Y^{\nu} + \Gamma^{\nu}_{\mu\lambda} Y^{\lambda} = 0$

?

$\therefore Y^{\mu} u_{\mu} = Y^{\mu} \frac{dx_{\mu}}{d\tau}$

2 RG&TC-Code

In[54]:=
```
xCoord = {t, χ, θ, φ};
g = {
    {-x y, 0, 0, 0},
    {0, x y t, 0, 0},
    {0, 0, z, 0},
    {0, 0, 0, x t}
  };
RGtensors[g, xCoord]
```

$$gdd = \begin{pmatrix} -x\,y & 0 & 0 & 0 \\ 0 & t\,x\,y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & t\,x \end{pmatrix}$$

$$LineElement = -x\,y\,d[t]^2 + z\,d[\theta]^2 + t\,x\,d[\varphi]^2 + t\,x\,y\,d[\chi]^2$$

$$gUU = \begin{pmatrix} -\frac{1}{x\,y} & 0 & 0 & 0 \\ 0 & \frac{1}{t\,x\,y} & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & \frac{1}{t\,x} \end{pmatrix}$$

```
gUU computed in 0. sec
Gamma computed in 0. sec
Riemann(dddd) computed in 0. sec
Riemann(Uddd) computed in 0. sec
Ricci computed in 0. sec
Weyl computed in 0. sec
Einstein computed in 0. sec
```

Out[56]=
```
All tasks completed in 0.
```

In[57]:=
```
(* Ricci Scalar *)
```

In[58]:=
```
R
```

Out[58]=

$$-\frac{1}{2\,t^2\,x\,y}$$

In[59]:=
```
(* Einstein Tensor *)
```

In[60]:=
```
EUd
```

Out[60]=

$$\left\{\left\{-\frac{1}{4\,t^2\,x\,y}, 0, 0, 0\right\}, \left\{0, \frac{1}{4\,t^2\,x\,y}, 0, 0\right\}, \left\{0, 0, \frac{1}{4\,t^2\,x\,y}, 0\right\}, \left\{0, 0, 0, \frac{1}{4\,t^2\,x\,y}\right\}\right\}$$

In[61]:=
```
(* Christoffel Symbol *)
```

In[62]:=
```
GUdd // MatrixForm
```

Out[62]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2\,y} \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{1}{2\,t} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2\,t} \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2\,t} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2\,t} \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

In[63]:=
```
Part[GUdd, 1, 2, 2]
Part[GUdd, 2, 2, 1]
```

Out[63]=

$$\frac{1}{2}$$

Out[64]=

$$\frac{1}{2\,t}$$

In[65]:=
```
(* Riemann tensor *)
```

In[66]:= **RUddd**

Out[66]=

$$\left\{\left\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\left\{0,-\frac{1}{4t},0,0\right\},\left\{\frac{1}{4t},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\left\{0,0,0,-\frac{1}{4ty}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4ty},0,0,0\right\}\right\}\right\},\right.$$

$$\left\{\left\{\left\{0,-\frac{1}{4t^2},0,0\right\},\left\{\frac{1}{4t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{4ty}\right\},\{0,0,0,0\},\left\{0,-\frac{1}{4ty},0,0\right\}\right\}\right\},$$

$$\{\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\},$$

$$\left.\left\{\left\{\left\{0,0,0,-\frac{1}{4t^2}\right\},\{0,0,0,0\},\{0,0,0,0\},\left\{\frac{1}{4t^2},0,0,0\right\}\right\},\left\{\{0,0,0,0\},\left\{0,0,0,-\frac{1}{4t}\right\},\{0,0,0,0\},\left\{0,\frac{1}{4t},0,0\right\}\right\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\}\right\}\right\}$$

In[67]:= **(* Ricci Tensor *)**

In[68]:= **Rdd**

Out[68]=

$$\left\{\left\{\frac{1}{2t^2},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

In[69]:= **Part[Rdd, 1, 1]**

Out[69]=

$$\frac{1}{2t^2}$$

In[70]:= **xCoord = {t, r, φ};**

$$\mathbf{g} = \left\{\left\{-\left(1+2\,\frac{G*M}{r}\right),\ 0,\ 0\right\},\ \left\{0,\ \left(1+2\,\frac{G*M}{r}\right),\ 0\right\},\ \{0,\ 0,\ r\verb|^|2\}\right\}$$

Out[71]=

$$\left\{\left\{-1-\frac{2GM}{r},0,0\right\},\left\{0,1+\frac{2GM}{r},0\right\},\left\{0,0,r^2\right\}\right\}$$

In[72]:= **RGtensors[g, xCoord]**

$$\text{gdd} = \begin{pmatrix} -1-\frac{2GM}{r} & 0 & 0 \\ 0 & 1+\frac{2GM}{r} & 0 \\ 0 & 0 & r^2 \end{pmatrix}$$

$$\text{LineElement} = \frac{(2GM+r)\,d[r]^2}{r} - \frac{(2GM+r)\,d[t]^2}{r} + r^2\,d[\varphi]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{r}{2GM+r} & 0 & 0 \\ 0 & \frac{r}{2GM+r} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}$$

```
gUU computed in 0. sec

Gamma computed in 0. sec

Riemann(dddd) computed in 0. sec

Riemann(Uddd) computed in 0. sec

Ricci computed in 0. sec

Weyl computed in 0. sec
```

Testing for 3-dim conformal flatness...

Outer: Heads Times and List at positions 3 and 2 are expected to be the same.

```
Einstein computed in 0. sec
```

Out[72]=

```
All tasks completed in 0.
```

In[73]:= **GUdd**

Out[73]=

$$\left\{\left\{\left\{0, -\frac{GM}{r\,(2GM+r)}, 0\right\}, \left\{-\frac{GM}{r\,(2GM+r)}, 0, 0\right\}, \{0, 0, 0\}\right\},\right.$$
$$\left\{\left\{-\frac{GM}{r\,(2GM+r)}, 0, 0\right\}, \left\{0, -\frac{GM}{r\,(2GM+r)}, 0\right\}, \left\{0, 0, -\frac{r^2}{2GM+r}\right\}\right\},$$
$$\left.\left\{\{0, 0, 0\}, \left\{0, 0, \frac{1}{r}\right\}, \left\{0, \frac{1}{r}, 0\right\}\right\}\right\}$$

In[74]:=
```
xCoord = {ψ, θ, φ};
g = {{1, 0, 0},
  {0, Sin[ψ]^2, 0},
  {0, 0, Sin[ψ]^2 * Sin[φ]^2}}
```

Out[75]=

$$\left\{\{1, 0, 0\}, \left\{0, \text{Sin}[\psi]^2, 0\right\}, \left\{0, 0, \text{Sin}[\varphi]^2\,\text{Sin}[\psi]^2\right\}\right\}$$

In[76]:=
```
RGtensors[g, xCoord]
```

$$gdd = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Sin}[\psi]^2 & 0 \\ 0 & 0 & \text{Sin}[\varphi]^2\,\text{Sin}[\psi]^2 \end{pmatrix}$$

$$\text{LineElement} = d[\psi]^2 + d[\theta]^2\,\text{Sin}[\psi]^2 + d[\varphi]^2\,\text{Sin}[\varphi]^2\,\text{Sin}[\psi]^2$$

$$gUU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Csc}[\psi]^2 & 0 \\ 0 & 0 & \text{Csc}[\varphi]^2\,\text{Csc}[\psi]^2 \end{pmatrix}$$

```
gUU computed in 0. sec
Gamma computed in 0. sec
Riemann(dddd) computed in 0. sec
Riemann(Uddd) computed in 0. sec
Ricci computed in 0. sec
Weyl computed in 0. sec
```

Testing for 3-dim conformal flatness...

Outer: Heads Times and List at positions 3 and 2 are expected to be the same.

```
Einstein computed in 0. sec
```

Out[76]=

```
All tasks completed in 0.
```

In[77]:= **RUddd**

Out[77]=

$$\{\{\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\},\{\{0,\text{Sin}[\psi]^2,0\},\{-\text{Sin}[\psi]^2,0,0\},\{0,0,0\}\},$$
$$\{\{0,0,\text{Sin}[\varphi]^2\,\text{Sin}[\psi]^2\},\{0,0,0\},\{-\text{Sin}[\varphi]^2\,\text{Sin}[\psi]^2,0,0\}\}\},$$
$$\{\{\{0,-1,0\},\{1,0,0\},\{0,0,0\}\},\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\},$$
$$\{\{0,0,0\},\{0,0,-\text{Cos}[\psi]^2\,\text{Sin}[\varphi]^2\},\{0,\text{Cos}[\psi]^2\,\text{Sin}[\varphi]^2,0\}\}\},$$
$$\{\{\{0,0,-1\},\{0,0,0\},\{1,0,0\}\},\{\{0,0,0\},\{0,0,\text{Cos}[\psi]^2\},\{0,-\text{Cos}[\psi]^2,0\}\},$$
$$\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}\}\}$$

In[78]:= **GUdd**

Out[78]=

$$\{\{\{0,0,0\},\{0,-\text{Cos}[\psi]\,\text{Sin}[\psi],0\},\{0,0,-\text{Cos}[\psi]\,\text{Sin}[\varphi]^2\,\text{Sin}[\psi]\}\},$$
$$\{\{0,\text{Cot}[\psi],0\},\{\text{Cot}[\psi],0,0\},\{0,0,0\}\},$$
$$\{\{0,0,\text{Cot}[\psi]\},\{0,0,0\},\{\text{Cot}[\psi],0,\text{Cot}[\varphi]\}\}\}$$

In[79]:=

In[80]:=

In[81]:=

In[97]:=
```
xCoord = {x, y};
g = {{x * y, 0}, {0, x^2}}
```

Out[98]=

$$\{\{x\,y,0\},\{0,x^2\}\}$$

In[99]:=
```
RGtensors[g, xCoord]
```

$$gdd = \begin{pmatrix} x\,y & 0 \\ 0 & x^2 \end{pmatrix}$$

LineElement = $x\,y\,d[x]^2 + x^2\,d[y]^2$

$$gUU = \begin{pmatrix} \frac{1}{x\,y} & 0 \\ 0 & \frac{1}{x^2} \end{pmatrix}$$

gUU computed in 0. sec

Gamma computed in 0.015 sec

Riemann(dddd) computed in 0. sec

Riemann(Uddd) computed in 0. sec

Ricci computed in 0. sec

Weyl computed in 0. sec

Conformally Flat

Einstein computed in 0. sec

Einstein Space

Out[99]=

All tasks completed in 0.015625

In[85]:= **GUdd**

Out[85]=

$$\left\{\left\{\left\{\frac{1}{2\,x}, \frac{1}{2\,y}\right\}, \left\{\frac{1}{2\,y}, -\frac{1}{y}\right\}\right\}, \left\{\left\{-\frac{1}{2\,x}, \frac{1}{x}\right\}, \left\{\frac{1}{x}, 0\right\}\right\}\right\}$$

In[86]:= **RUdd**

Out[86]=

RUdd

In[100]:=

RUddd // MatrixForm

Out[100]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & \frac{x+2\,y}{4\,x\,y^2} \\ -\frac{x+2\,y}{4\,x\,y^2} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{x+2\,y}{4\,x^2\,y} \\ \frac{x+2\,y}{4\,x^2\,y} & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

In[88]:= **Rdd**

Out[88]=

$$\left\{\left\{\frac{x+2\,y}{4\,x^2\,y}, 0\right\}, \left\{0, \frac{x+2\,y}{4\,x\,y^2}\right\}\right\}$$

In[89]:= **R**

Out[89]=

$$\frac{x+2\,y}{2\,x^3\,y^2}$$

In[90]:= **xCoord = {t, r, θ, φ}**

Out[90]=

{t, r, θ, φ}

In[91]:= **g = {{-(1 - 2 * G * M / r), 0, 0, 0},**
{0, (1 - 2 * G * M / r) ^ (-1), 0, 0},
{0, 0, r^2, 0},
{0, 0, 0, r^2 * Sin[θ] ^2}}

Out[91]=

$$\left\{\left\{-1+\frac{2GM}{r},0,0,0\right\},\left\{0,\frac{1}{1-\frac{2GM}{r}},0,0\right\},\left\{0,0,r^2,0\right\},\left\{0,0,0,r^2\,\mathrm{Sin}[\theta]^2\right\}\right\}$$

In[92]:= **RGtensors[g, xCoord]**

$$\text{gdd} = \begin{pmatrix} -1+\frac{2GM}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2GM}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

$$\text{LineElement} = -\frac{r\,d[r]^2}{2GM-r}+\frac{(2GM-r)\,d[t]^2}{r}+r^2\,d[\theta]^2+r^2\,d[\varphi]^2\,\mathrm{Sin}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} \frac{r}{2GM-r} & 0 & 0 & 0 \\ 0 & -\frac{2GM-r}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

gUU computed in 0. sec

Gamma computed in 0. sec

Riemann(dddd) computed in 0. sec

Riemann(Uddd) computed in 0. sec

Ricci computed in 0. sec

Weyl computed in 0. sec

Ricci Flat

Out[92]=

All tasks completed in 0.

In[93]:= **GUdd**

Out[93]=

$$\left\{\left\{\left\{0,-\frac{GM}{(2GM-r)\,r},0,0\right\},\left\{-\frac{GM}{(2GM-r)\,r},0,0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\},\right.$$
$$\left\{\left\{-\frac{GM\,(2GM-r)}{r^3},0,0,0\right\},\left\{0,\frac{GM}{(2GM-r)\,r},0,0\right\},\right.$$
$$\left.\{0,0,2GM-r,0\},\left\{0,0,0,(2GM-r)\,\mathrm{Sin}[\theta]^2\right\}\right\},$$
$$\left\{\{0,0,0,0\},\left\{0,0,\frac{1}{r},0\right\},\left\{0,\frac{1}{r},0,0\right\},\{0,0,0,-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\}\right\},$$
$$\left.\left\{\{0,0,0,0\},\left\{0,0,0,\frac{1}{r}\right\},\{0,0,0,\mathrm{Cot}[\theta]\},\left\{0,\frac{1}{r},\mathrm{Cot}[\theta],0\right\}\right\}\right\}$$