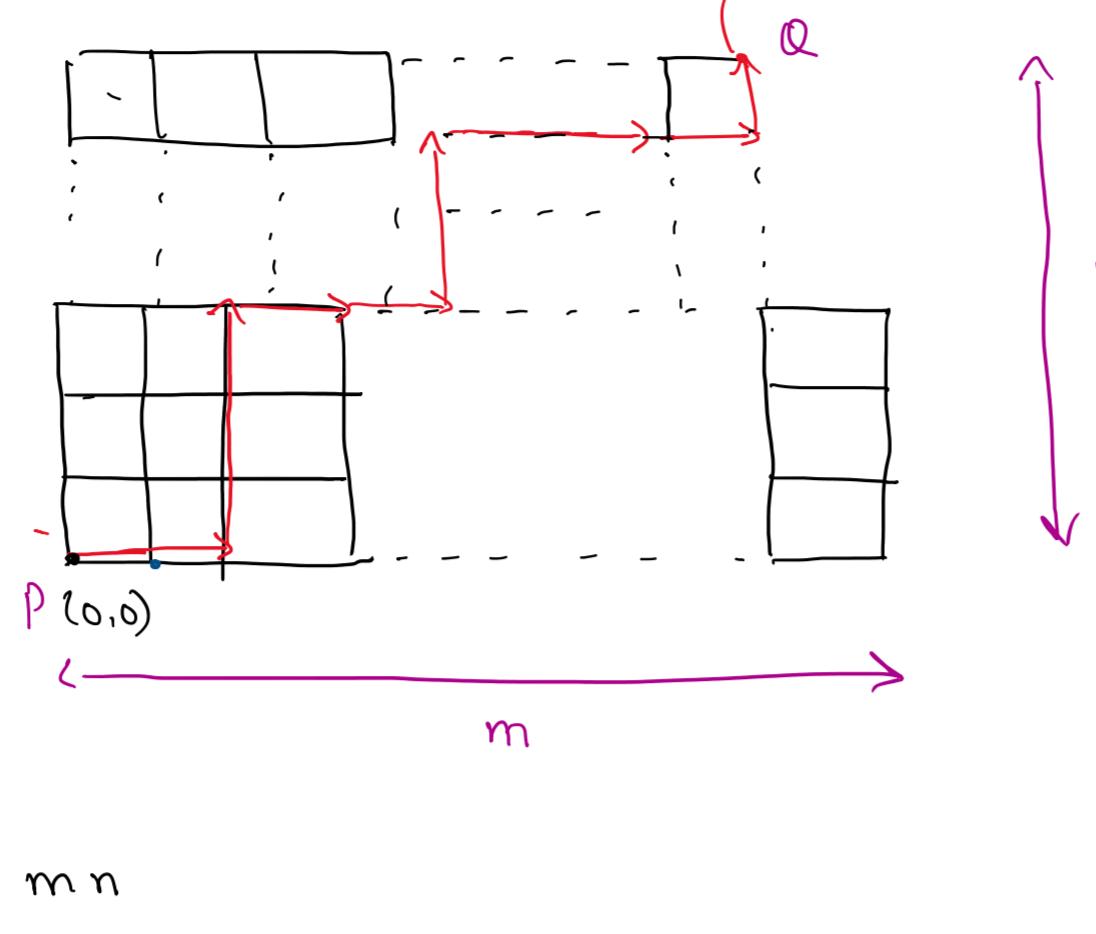


Assignment 1: Question 2

Saturday, 13 May 2023 3:47 pm

- (2) A physics student lives in a location P in a city. She walks to the lecture theatre at point Q. At each street, she has equal probability of going east or north. Because she wants to arrive on time for her lectures, she never doubles back.

- a. If there are m streets east and n streets north of P, how many different ways can the student go from P to Q?



Solution:

- a. Total number of streets: $m+n$

- ∴ Total number of ways in which the student can go from P to Q:

$$= \frac{(m+n)!}{n! m!}$$

- b. We now relax the requirement that the student is not allowed to double back. She has equal probability of going east, west, north or south. Calculate the probability that after a large number of streets, the student has taken n_E streets eastward

Solution:

- This can be interpreted as two independent one-dimensional random walks, one in the east-west direction and the other in the north-south direction. Let there be k steps in the east-west direction and $N-k$ steps in the north-south direction. Then:

① East-West direction:

$$m = a \underbrace{(n_E - n_W)}_{\text{Step size}} = as \quad (n_E + n_W = k) \quad \underbrace{n_E - n_W = s}_{2n_E - s = k} \Rightarrow \frac{k+s}{2}$$

$$\therefore N(n_E) = \frac{k!}{n_E! (k-n_E)!} \Rightarrow P(n_E) = \frac{k!}{n_E! (k-n_E)!} \left(\frac{1}{2}\right)^k$$

$$\therefore n_E + n_W = k \quad \text{and} \quad n_E = \frac{1}{2}(k+s) \quad \text{and} \quad n_W = \frac{1}{2}(k-s)$$

$$\therefore P(s) = \frac{k!}{\left[\frac{1}{2}(k+s)\right]! \left[\frac{1}{2}(k-s)\right]!} \left(\frac{1}{2}\right)^k \quad \left. \begin{array}{l} \text{Apply Stirling's approximation and simplify} \\ \text{Since } k \text{ is presumably large, the continuum limit can be taken.} \end{array} \right\}$$

$$\therefore \text{From lectures} \Rightarrow P(x) = e^{-x^2/2ka^2} \quad (\text{where } m \rightarrow x = as)$$

- Similarly, given that we must take $N-k$ steps in the north-south direction, we have:

$$P(y) = e^{-y^2/2(N-k)a^2}$$

- Hence, the probability distribution can be written as:

$$P(n_E) = \exp\left(-\frac{x^2}{2(2n_E-s)a^2}\right) \cdot \exp\left(-\frac{y^2}{2(N-(2n_E-s))a^2}\right)$$

- Impose normalisation condition as follows:

$$\int_{-\infty}^{\infty} P(n_E) dx dy = 1$$

$$= \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2(2n_E-s)a^2}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2(N-(2n_E-s))a^2}\right) dy = 1$$

$$= \left(\sqrt{2\pi} a \sqrt{2n_E-s}\right) \left(\sqrt{2\pi} a \sqrt{N-(2n_E-s)}\right) = 1$$

$$= \left(\sqrt{2\pi} a \sqrt{2n_E-s}\right) \left(\sqrt{2\pi} a \sqrt{N-(2n_E-s)}\right) = 1$$

$$= 2\pi a^2 \sqrt{(2n_E-s)(N-2n_E-s)} = 1$$

$$\Rightarrow \text{Normalisation constant} = \frac{1}{2\pi a^2 \sqrt{(2n_E-s)(N-2n_E-s)}}$$

$$\therefore P(x,y) = \frac{1}{2\pi a^2 \sqrt{(2n_E-s)(N-2n_E-s)}} \exp\left(-\frac{x^2}{2(n_E+n_W)a^2} - \frac{y^2}{2(N-(n_E+n_W))a^2}\right)$$

- c. Discuss the relationship between this random walk and diffusion in two dimensions

Solution:

- If a time dependence is introduced into the above probability distribution (i.e. $P(x,y) \rightarrow P(x,y,t)$), then this distribution must satisfy the two-dimensional diffusion equation of the form:

$$\frac{\partial P(x,y,t)}{\partial t} = D \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right)$$

where D is the diffusion coefficient

Assignment 1: Question 3

Friday, 19 May 2023 10:34 am

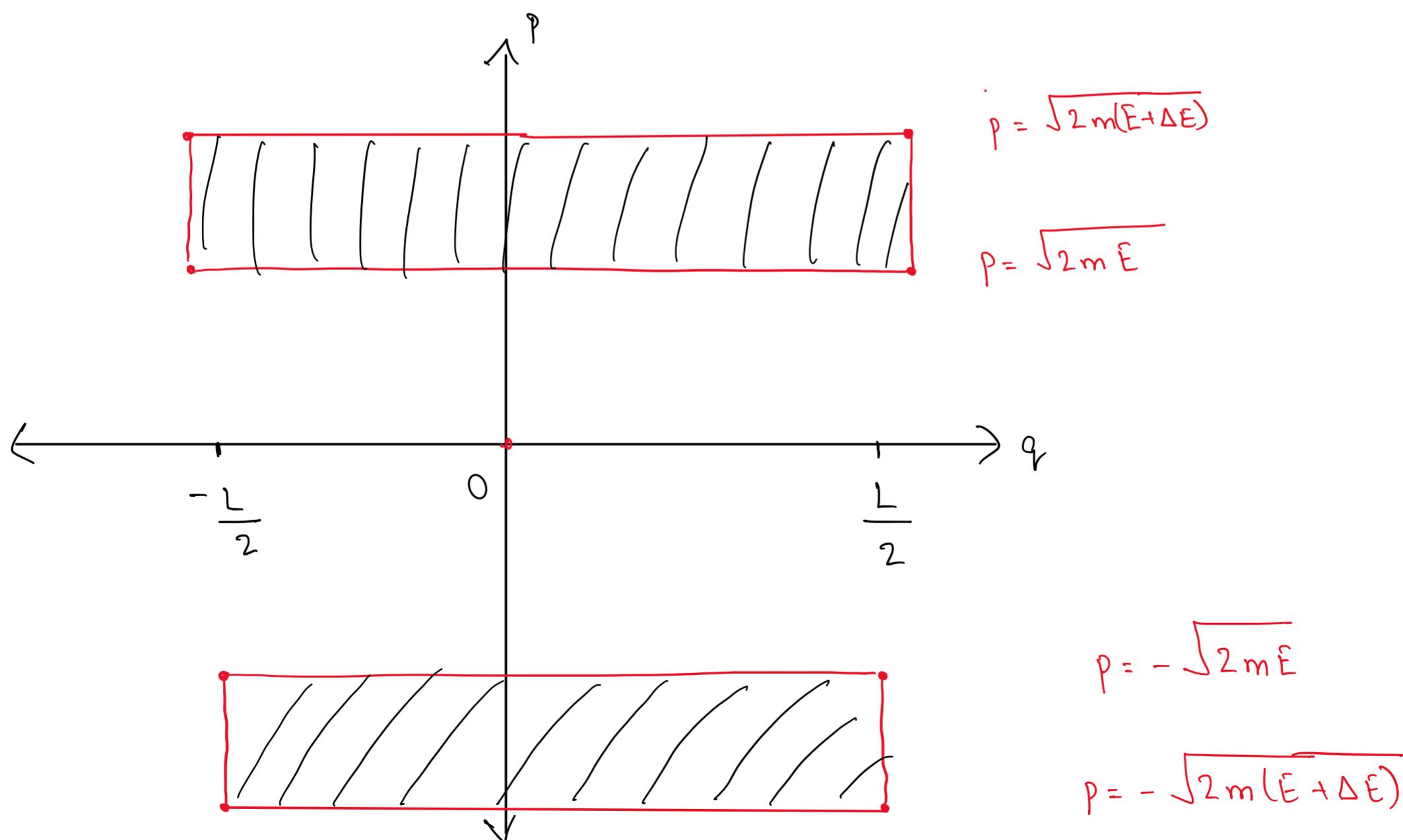
- (3) Construct ^{the} classical phase space diagram for a particle of mass m confined to a one-dimensional box of size L . Assume that the energy of particle lies between E and $E + \Delta E$. Clearly label regions of phase space accessible to the particle.

Solution:

- Hamiltonian for particle of mass m in a one-dimensional box of length L :

$$E = \frac{p^2}{2m} \quad 0 < x \leq L$$

$$\Rightarrow p = \pm \sqrt{2mE}, \quad 0 \leq x \leq L$$



Shaded region = region of phase space accessible to particle

- (4) Calculate the thermodynamic properties (F, p, S and u) of an ultrarelativistic ideal gas whose Hamiltonian is given by:

$$H(q, p) = \sum_{i=1}^N E_i \quad \text{where } E_i = \sqrt{p_i^2 c^2 + m_0 c^4} \rightarrow |p_i|c \quad (|p_i| \gg m_0 c^2)$$

Solution: $= \omega(E)$

$$H(q, p) = \sum_{i=1}^N |p_i|c$$

$$\begin{aligned} Z(\beta, V, N) &= \frac{1}{N! h^{3N}} \int_{\Gamma(E)} e^{-\beta H(q, p)} d^n q d^n p \\ &= \frac{1}{N! h^{3N}} \int_{\Gamma(E)} e^{-\beta |p_i|c} d^n q d^n p \\ &= \frac{V^N}{N! h^{3N}} \int_{\Gamma(E)} e^{-\beta |p_i|c} d^n p \end{aligned}$$

$$\Rightarrow Z(\beta, V, N) = \frac{V^N}{N! h^{3N}} \int_{-\infty}^{\infty} e^{-\beta |p_i|c} dp_i = \frac{V^N}{N! h^N} \cdot \prod_{i=1}^N \frac{2}{\beta c} = \frac{V^N}{N! h^N} \left[\frac{2k_B T}{c} \right]^N = \frac{1}{N!} \left[\frac{2V}{\beta h c} \right]^N$$

$$\begin{aligned} F(T, V, N) &= -k_B T \ln(Z(\beta, V, N)) \\ &= -k_B T \ln \left[\frac{1}{N!} \left[\frac{2k_B T}{h c} \right]^N \right] \\ &= -k_B T \left[\ln \left(\left(\frac{2k_B T}{h c} \right)^N \right) - \ln(N!) \right] \\ &= -N k_B T \ln \left(\frac{2k_B T V}{h c} \right) + k_B T \ln(N!) \\ &= k_B T \left(\ln(N!) - N \ln(V) - N \ln \left(\frac{2k_B T}{h c} \right) \right) \\ &= k_B T \left(N \ln(N) - N \ln(V) - N \ln \left(\frac{2k_B T}{h c} \right) - 1 \right) \\ &= N k_B T \left[\ln \left(\frac{N}{V} \right) - \ln \left(\frac{2k_B T}{h c} \right) \right] \\ &= N k_B T \left[\ln \left(\frac{N}{V} \times \frac{h c}{2k_B T} \right) - 1 \right] \\ &= N k_B T \left[\ln \left(\frac{N h c}{2k_B T V} \right) - 1 \right] \\ &= -N k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] = F(T, V, N) \end{aligned}$$

$$\begin{aligned} p &= - \left(\frac{\partial f}{\partial V} \right)_{T, N} = - \frac{\partial}{\partial V} \left(-N k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] \right) \\ &= \frac{\partial}{\partial V} \left(N k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] \right) \\ &= \frac{\partial}{\partial V} \left[-\ln \left(\frac{N h c}{2k_B T V} \right) \right] \times N k_B T \\ &= \frac{\partial}{\partial V} \left[\ln \left(\frac{2k_B T V}{N h c} \right) \right] \times N k_B T \\ &\Rightarrow \frac{\partial}{\partial V} \left[\ln(\alpha V) \right] \times N k_B T \end{aligned}$$

$$p = \frac{N k_B T}{V} \Rightarrow \boxed{pV = N k_B T}$$

$$S = - \left(\frac{\partial f}{\partial T} \right)_{V, N}$$

$$= - \frac{\partial}{\partial T} \left[-N k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] \right]$$

$$= \frac{\partial}{\partial T} \left(N k_B T \left(1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right) \right)$$

$$= N k_B \left(1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right) + N k_B T \frac{\partial}{\partial T} \left(1 - \ln \left(\frac{2k_B T V}{N h c} \right) \right)$$

$$= N k_B \left(1 + \ln \left(\frac{2k_B T V}{N h c} \right) \right) + N k_B T \times \frac{1}{T}$$

$$= N k_B \left(1 + \ln \left(\frac{2k_B T V}{N h c} \right) \right) + N k_B$$

$$= \boxed{N k_B \left(2 + \ln \left(\frac{2k_B T V}{N h c} \right) \right)} = S =$$

$$\begin{aligned} u &= \left(\frac{\partial f}{\partial N} \right)_{T, V} = \frac{\partial}{\partial N} \left[-N k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] \right] \\ &= -k_B T \left[1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right] - N k_B T \left[\frac{\partial}{\partial N} \left\{ 1 - \ln \left(\frac{N h c}{2k_B T V} \right) \right\} \right] \\ &= -k_B T + k_B T \ln \left(\frac{N h c}{2k_B T V} \right) - N k_B T \left(-\frac{1}{N} \right) \\ &= -k_B T + k_B T \ln \left(\frac{N h c}{2k_B T V} \right) + k_B T \end{aligned}$$

$$= \boxed{k_B T \ln \left(\frac{N h c}{2k_B T V} \right) + k_B T} = u$$

Assignment 1: Question 5

Thursday, 11 May 2023 7:52 pm

(5) The grand canonical partition function is defined as:

$$\Xi(\beta, V, N) = \sum_N z^N Z(\beta, V, N)$$

where $z = e^{\beta\mu}$ = fugacity, μ = chemical potential and $Z(\beta, V, N)$ = canonical partition function for N particles

a. Defining $D = z \frac{\partial}{\partial z}$, prove that the grand canonical average is given by:

$$\langle (N - \langle N \rangle)^2 \rangle = D^2 \ln(\Xi(\beta, V, N))$$

Solution:

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$$

$$\therefore \langle N \rangle = \frac{\sum_{N=0}^{\infty} N z^N Z(\beta, V, N)}{\sum_{N=0}^{\infty} z^N Z(\beta, V, N)} = \frac{1}{\Xi(\beta, V, N)} \sum_{N=0}^{\infty} \frac{\partial}{\partial z} (\Xi(\beta, V, N)) \cdot z = \frac{1}{\Xi(\beta, V, N)} z \frac{\partial}{\partial z} (\Xi(\beta, V, N)) = z \frac{\partial}{\partial z} (\ln(\Xi(\beta, V, N))) = D \ln(\Xi(\beta, V, N)) \quad ①$$

$$\therefore \langle N^2 \rangle = \frac{\sum_{N=0}^{\infty} N^2 z^N Z(\beta, V, N)}{\sum_{N=0}^{\infty} z^N Z(\beta, V, N)} = \frac{1}{\Xi(\beta, V, N)} \left(z \frac{\partial}{\partial z} \right) \left(z \frac{\partial}{\partial z} \right) (\Xi(\beta, V, N)) \quad ②$$

$$\begin{aligned} & \therefore \langle N^2 \rangle - \langle N \rangle^2 \\ &= \frac{1}{\Xi(\beta, V, N)} \left(z \frac{\partial}{\partial z} \right) \left(z \frac{\partial}{\partial z} \right) (\Xi(\beta, V, N)) - \frac{1}{\Xi(\beta, V, N)} z^2 \left(\frac{\partial \Xi}{\partial z} \right)^2 \\ &= \frac{1}{\Xi(\beta, V, N)} \left(z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2} \right) (\Xi(\beta, V, N)) - \frac{1}{\Xi(\beta, V, N)} z^2 \left(\frac{\partial \Xi}{\partial z} \right)^2 \\ &= \frac{1}{\Xi(\beta, V, N)} \left[\left(z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2} \right) (\Xi(\beta, V, N)) \right] - \left[\frac{1}{\Xi(\beta, V, N)} z^2 \left(\frac{\partial \Xi}{\partial z} \right)^2 \right] \\ &= \frac{1}{\Xi(\beta, V, N)} \left[\left(z \frac{\partial}{\partial z} + z^2 \frac{\partial^2}{\partial z^2} \right) (\Xi(\beta, V, N)) - \frac{1}{\Xi(\beta, V, N)} z^2 \left(\frac{\partial \Xi}{\partial z} \right)^2 \right] \\ &= \left(z \frac{\partial}{\partial z} \right)^2 \ln(\Xi(\beta, V, N)) = D^2 \ln(\Xi(\beta, V, N)) \Rightarrow \underline{\underline{QED}} \end{aligned}$$

b. Does the result in Part (a) generalise? Is the following result valid:

$$\langle (N - \langle N \rangle)^n \rangle = D^n \ln(\Xi(\beta, V, N))$$

Solution:

- The result does not generalise. A counterexample is the $n=4$ case, where $\left(z \frac{\partial}{\partial z} \right)^4 \ln(\Xi(\beta, V, N)) = D^4 \ln(\Xi(\beta, V, N)) \neq \langle (N - \langle N \rangle)^4 \rangle$

c. Prove that the internal energy is given by:

$$U = -\frac{\partial}{\partial \beta} \ln(\Xi(\beta, V, N))$$

Solution:

$$\begin{aligned} \Xi(\beta, V, N) &= \sum_{N=0}^{\infty} \sum_i e^{-\beta(E_i - \mu N)} \\ \therefore -\frac{\partial}{\partial \beta} (\ln(\Xi(\beta, V, N))) &= \frac{1}{\Xi(\beta, V, N)} \cdot -\frac{\partial}{\partial \beta} \left(\sum_{N=0}^{\infty} \sum_i e^{-\beta(E_i - \mu N)} \right) \\ &= \frac{1}{\Xi(\beta, V, N)} \times \sum_{N=0}^{\infty} \sum_i -\frac{\partial}{\partial \beta} (e^{-\beta(E_i - \mu N)}) \\ &= \frac{1}{\Xi(\beta, V, N)} \sum_{N=0}^{\infty} \sum_i (E_i - \mu N) e^{-\beta(E_i - \mu N)} \\ &= \frac{1}{\Xi(\beta, V, N)} \left(\sum_{N=0}^{\infty} \sum_i E_i e^{-\beta(E_i - \mu N)} - \sum_{N=0}^{\infty} \sum_i \mu N e^{-\beta(E_i - \mu N)} \right) \\ &= \langle E \rangle - \mu \langle N \rangle \\ &= \underline{\underline{U - \mu \langle N \rangle}} \end{aligned}$$

Assignment 1: Question 6

Thursday, 11 May 2023 7:54 pm

- (6) Consider a one dimensional Einstein solid with anharmonicity defined by a cubic term in the potential of the oscillator:

$$V(x) = \frac{1}{2} m \omega_0^2 x^2 - \frac{1}{3} \lambda x^3$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and λ is small. The thermal expansion coefficient of this one-dimensional system is given by:

$$\alpha_L = \frac{1}{L_0} \left(\frac{\partial \langle x \rangle}{\partial T} \right)_P$$

where L_0 = unperturbed length of one-dimensional solid. Calculate α_L and comment on the case where $\lambda=0$

Solution:

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x e^{-\beta V(x)} dx}{\int_{-\infty}^{\infty} e^{-\beta V(x)} dx} \quad (1)$$

$$(1) \int_{-\infty}^{\infty} x e^{-\beta V(x)} dx = \int_{-\infty}^{\infty} x e^{-\beta \left(\frac{1}{2} m \omega_0^2 x^2 - \frac{1}{3} \lambda x^3 \right)} dx \\ = \int_{-\infty}^{\infty} x e^{-\beta \left(\frac{1}{2} m \omega_0^2 x^2 \right)} \cdot e^{\frac{\beta \lambda x^3}{3}} dx$$

∴ Noting that λ is small, one can write:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{\frac{\beta \lambda x^3}{3}} = \left(1 + \frac{\beta \lambda x^3}{3} + O(x^6) \right)$$

∴ The integral (1) becomes:

$$(1) \int_{-\infty}^{\infty} x e^{-\beta \left(\frac{1}{2} m \omega_0^2 x^2 \right)} \left(1 + \frac{\beta \lambda}{3} x^3 + O(x^6) \right) dx \\ = \int_{-\infty}^{\infty} x e^{-\beta \frac{m \omega_0^2 x^2}{2}} \left(1 + \frac{\beta \lambda}{3} x^3 \right) dx \quad (\text{neglecting higher order terms}) \\ = \frac{\sqrt{2\pi} \lambda \beta}{(m \beta \omega_0^2)^{5/2}} = \left(\frac{2\pi \lambda^2 \beta^2}{(m \beta \omega_0^2)^5} \right)^{1/2}$$

$$(2) \int_{-\infty}^{\infty} e^{-\beta V(x)} dx = \int_{-\infty}^{\infty} e^{-\beta \left(\frac{1}{2} m \omega_0^2 x^2 - \frac{\lambda x^3}{3} \right)} dx = \int_{-\infty}^{\infty} e^{-\frac{\beta m \omega_0^2 x^2}{2}} \cdot e^{\frac{\beta \lambda x^3}{3}} dx = \int_{-\infty}^{\infty} e^{-\beta \frac{m \omega_0^2 x^2}{2}} \left(1 + \frac{\beta \lambda x^3}{3} + O(x^6) \right) dx \\ = \int_{-\infty}^{\infty} e^{-\beta \frac{m \omega_0^2 x^2}{2}} \left(1 + \frac{\beta \lambda x^3}{3} \right) dx \quad (\text{neglecting higher order terms}) \\ = \left(\frac{2\pi}{m \beta \omega_0^2} \right)^{1/2} = \frac{1}{\omega_0} \left(\frac{2\pi}{m \beta} \right)^{1/2} \quad (\text{by Mathematica})$$

$$\therefore \langle x \rangle = \frac{(1)}{(2)} = \frac{\sqrt{2\pi} \lambda \beta}{(m \beta \omega_0^2)^{5/2}} = \left(\frac{2\pi}{m \beta \omega_0^2} \right)^{1/2} \times \left(\frac{m \beta \omega_0^2}{2\pi} \right)^{1/2} \\ = \left(\frac{2\pi \lambda^2 \beta^2}{(m \beta \omega_0^2)^5} \right)^{1/2} \times \left(\frac{m \beta \omega_0^2}{2\pi} \right)^{1/2} \\ = \left(\frac{\lambda^2}{m^5 \beta^3 \omega_0^{10}} \right)^{1/2} (m \beta \omega_0^2)^{1/2} \\ = \frac{\lambda}{m^{5/2} \beta^{3/2} \omega_0^5} \cdot m^{1/2} \beta^{1/2} \omega_0 \\ = \frac{\lambda}{m^2 \beta \omega^4} = \frac{\lambda k_B T}{m^2 \omega^4} \quad (3)$$

$$\therefore \alpha_L = \frac{1}{L_0} \left(\frac{\partial \langle x \rangle}{\partial T} \right)_P = \frac{1}{L_0} \frac{\partial}{\partial T} \left(\frac{\lambda k_B T}{m^2 \omega^4} \right) = \boxed{\frac{1}{L_0} \frac{\lambda k_B}{m^2 \omega^4}}$$

- When $\lambda=0$, $\alpha_L=0 \Rightarrow$ no thermal expansion occurs when $\lambda=0$.

Assignment 1: Question 7

Thursday, 11 May 2023 9:01 pm

- (7) Evaluate the classical and quantum partition functions for a simple harmonic oscillator. Show that the quantum oscillator reproduces the classical result in the case $T \rightarrow \infty$

Solution:

- Quantum harmonic oscillator in one dimension has quantised energies given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad (n=0, 1, 2, \dots)$$

$$\therefore Z(\beta, V, 1) = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\beta\hbar\omega/2} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}$$

$$\therefore Z(\beta, V, 1) = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} = \frac{1}{2 \sinh\left(\frac{\beta\hbar\omega}{2}\right)} = \boxed{\frac{1}{2 \sinh\left(\frac{\hbar\omega}{2k_B T}\right)}} \quad (1)$$

- Classical harmonic oscillator has a continuum of possible energies, and is described by the Hamiltonian:

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 q^2$$

- Partition function is given by:

$$\begin{aligned} Z(\beta, V, 1) &= \frac{1}{h^3} \int_{\Gamma(E)} e^{-\beta H(p, q)} d^3p d^3q = \frac{1}{h^3} \int_{-\infty}^{\infty} e^{-\beta p^2/2m} d^3p \int_{-\infty}^{\infty} e^{-\beta m\omega_0^2 q^2/2} d^3q \\ &= \frac{1}{h^3} \left(\frac{2\pi\hbar}{\beta}\right)^{1/2} \left(\frac{2\pi}{\beta m\omega_0^2}\right)^{1/2} = \frac{2\pi}{\beta\omega_0 h} = \frac{2\pi k_B T}{\omega_0 h} = \boxed{\frac{k_B T}{\hbar\omega_0}} \quad (\text{where } \hbar = \frac{h}{2\pi}) \end{aligned} \quad (2)$$

- Seek to show that in the limit $T \rightarrow \infty$, the quantum oscillator reproduces the classical result

$$Z(\beta, V, 1) = \frac{1}{2 \sinh\left(\frac{\hbar\omega}{2k_B T}\right)} = \frac{1}{2 \sinh\left(\frac{\beta\hbar\omega}{2}\right)}$$

- $T \rightarrow \infty \Rightarrow \hbar\omega \ll k_B T \Rightarrow$ Taylor expand denominator in β :

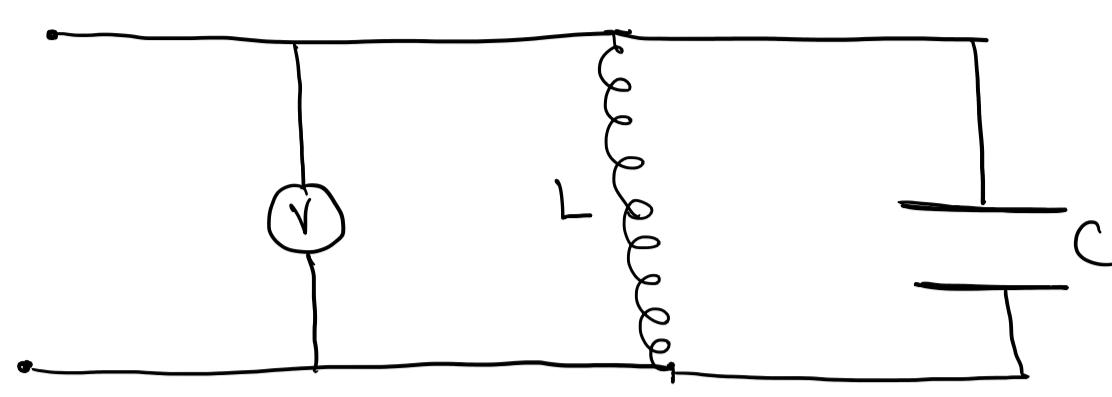
$$\begin{aligned} \therefore Z(\beta, V, 1) &= \frac{1}{2 \left[\left(\frac{\beta\hbar\omega}{2}\right) + \frac{1}{6} \left(\frac{\beta\hbar\omega}{2}\right)^3 + \dots \right]} \quad \left(\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \right) \\ &= \frac{1}{2 \left[\left(\frac{\beta\hbar\omega}{2}\right) + O\left(\left(\frac{\beta\hbar\omega}{2}\right)^3\right) \right]} \approx \frac{1}{\beta\hbar\omega_0} = \boxed{\frac{k_B T}{\hbar\omega_0}} \quad (\text{to leading order}) = (2) \end{aligned}$$

Hence, in the high temperature limit, the quantum oscillator reproduces the classical result $\Rightarrow \underline{\text{QED}}$

Assignment 1: Question 8

Saturday, 13 May 2023 1:33 pm

- (8) An LC circuit can be used as a thermometer by measuring the noise voltage across the inductor and capacitor in parallel. Determine the relationship between the root mean square (RMS) noise voltage and the absolute temperature. You may assume the classical limit $k_B T \gg \hbar\omega$, where $\omega = (LC)^{-1/2}$ = frequency of the oscillator.



Solution:

- Hamiltonian describing oscillations in the LC circuit is:

$$H = \frac{1}{2} L \left(\frac{dQ(t)}{dt} \right)^2 + \frac{Q^2(t)}{2C}$$

Given that $V = \frac{Q}{C}$ (from circuit theory), one can deduce that $\frac{Q^2}{2C} = \frac{CV^2}{2}$

Voltage

Additionally, letting $I = \frac{dQ}{dt}$ (from the definition of electric current), one can rewrite the Hamiltonian as:

$$H = \frac{1}{2} LI^2 + \frac{CV^2}{2}, \text{ where } I = I(t) \text{ and } V = V(t)$$

- Hence, one can compute the partition function for the circuit:

$$\begin{aligned} Z &= \int_{-\infty}^{\infty} e^{-\beta \left(\frac{CV^2}{2} + \frac{LI^2}{2} \right)} dV dI \\ &= \int_{-\infty}^{\infty} e^{-\beta CV^2/2} dV \int_{-\infty}^{\infty} e^{-\beta LI^2/2} dI \\ &= \int_{-\infty}^{\infty} e^{-\frac{1}{2} \alpha V^2} dV \int_{-\infty}^{\infty} e^{-\frac{1}{2} s I^2} dI \\ &\quad \downarrow \qquad \downarrow \\ a &= \beta C \qquad s = \beta L \\ &= \left(\frac{2\pi}{\beta C} \right)^{1/2} \left(\frac{2\pi}{\beta L} \right)^{1/2} = \frac{2\pi}{\beta (LC)^{1/2}} = \frac{2\pi}{\beta \omega} \end{aligned}$$

- Noting that $U = -\frac{1}{Z} \frac{\partial Z}{\partial \beta}$

$$\begin{aligned} &= -\frac{1}{Z} \frac{\partial}{\partial \beta} \left(\frac{2\pi}{\beta \omega} \right) \\ &= -\frac{1}{Z} \left(\frac{-2\pi}{\beta^2 \omega} \right) \\ &= \left(\frac{\beta \omega}{2\pi} \right) \left(\frac{2\pi}{\beta^2 \omega} \right) = \frac{1}{\beta} = k_B T \quad \textcircled{1} \end{aligned}$$

- Furthermore, one can note that:

$$\begin{aligned} U &= \frac{1}{Z} \frac{\partial}{\partial \beta} \int e^{-\beta \left(\frac{CV^2}{2} + \frac{LI^2}{2} \right)} dV dI \\ &= \frac{1}{Z} \int \left(\frac{CV^2}{2} + \frac{LI^2}{2} \right) e^{-\beta \left(\frac{CV^2}{2} + \frac{LI^2}{2} \right)} dV dI \\ &= \frac{C}{2} \langle V^2 \rangle + \frac{L}{2} \langle I^2 \rangle \end{aligned}$$

- Since there exist two quadratic degrees of freedom in this system, one in V and the other in I ,

- the equipartition theorem states that each of these must contribute equally to the internal energy U of the system:

$$\Rightarrow \frac{C}{2} \langle V^2 \rangle = \frac{L}{2} \langle I^2 \rangle = \frac{U}{2}$$

$$\therefore \text{Since } \frac{C}{2} \langle V^2 \rangle = \frac{U}{2} \Rightarrow C \langle V^2 \rangle = U \quad \textcircled{2}$$

$$\therefore \text{Hence } V_{\text{rms}} = \sqrt{\langle V^2 \rangle} = \sqrt{\frac{U}{C}}$$

$$\therefore \text{From } \textcircled{1}, U = k_B T \Rightarrow \boxed{V_{\text{rms}} = \sqrt{\frac{k_B T}{C}}} \Rightarrow V_{\text{rms}} \propto \underline{\underline{T^{1/2}}}$$