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General Relativity: Examination

①

(1). This is a reasonable ~~eq~~ equation

- Reasonable equation
- Nonsense since the operating with the covariant derivative twice on F^{α} would not lead to a ~~contravariant~~ (3,0) tensor.
- Reasonable equation.
- Reasonable equation.
- Reasonable equation.

(2) $ds^2 = xy\,dx^2 + x^2\,dy^2$

$$\therefore g_{\mu\nu} = \begin{pmatrix} xy & 0 \\ 0 & x^2 \end{pmatrix}$$

$\therefore$ Possible Christoffel symbols:

Γ^{x}_{xx}, Γ^{x}_{xy}, Γ^{y}_{xx}, Γ^{y}_{xy} (up to ~~anti~~symmetry)

Γ^{y}_{yy}, Γ^{x}_{yy}, $\cancel{\Gamma^{y}_{x}}$

$$\cancel{\Gamma^{xx}_{xx} = \frac{1}{2} \partial_g g^{x\alpha} (\partial_x g_{xx} + \partial g_{xx} \partial_x g_{xx}}$$

Diagonal metric $\Rightarrow$ $\Gamma^{y}_{xx} = \frac{-1}{2g_{yy}} \partial_y g_{xx}$

$$= \frac{-1}{2x^2} \partial_y (xy) = \frac{-1}{2x}$$

$$\cancel{\Gamma^{x}_{xx} = \partial_x \ln\sqrt{|g_{xx}|}}$$

$$\cancel{= \partial_x (\ln\sqrt{xy}) = \frac{(\frac{1}{2}y)}{\sqrt{xy}} = \frac{y}{2y}}$$

(2) $ds^2 = xy\,dx^2 + x^2 dy^2$

$$\therefore g_{\mu\nu} = \begin{pmatrix} xy & 0 \\ 0 & x^2 \end{pmatrix}$$

$\therefore$ Metric is diagonal. Christoffel symbols are:

$\cancel{\Gamma^x_{xx}}, \Gamma^x_{xy}, \Gamma^x_{yx}, \Gamma^x_{yy}, \Gamma^y_{xx}, \Gamma^y_{xy}, \Gamma^y_{yx}, \cancel{\Gamma^y_{yy}}$

$$\therefore \Gamma^x_{xx} = \partial_x \ln\sqrt{|g_{xx}|} = \partial_x \ln\sqrt{xy} = \frac{1}{(xy)^{1/2}} \cdot \frac{1}{2} x^{-1/2} y^{1/2}$$

$$= \frac{1}{2x}$$

$$\Gamma^y_{yy} = \partial_y \ln\sqrt{|g_{yy}|} = \partial_y (\ln\sqrt{|g_{yy}|}) = \underline{0}$$

$$\Gamma^x_{xy} = \partial_y \ln(\sqrt{g_{xx}}) = \partial_y \ln(\sqrt{xy}) = \frac{1}{2} y^{-1/2} \cdot \frac{1}{(xy)^{1/2}} = \underline{\frac{1}{2y}}$$

$$\Gamma^x_{yx} = \frac{1}{2y} \text{ (symmetry)}, \quad \Gamma^x_{yy} = \frac{-1}{2g_{xx}} \partial_x (g_{yy}) = \frac{-1}{2xy} \cdot 2x$$

$$= \frac{-1}{y}$$

$$\Gamma^y_{xx} = \frac{-1}{2g_{yy}} \partial_y (g_{xx}) = \underline{\frac{-1}{2x}}, \quad \Gamma^y_{xy} = \Gamma^y_{yx} = \partial_x \ln\sqrt{|g_{yy}|}$$

$$= \partial_x (\ln\sqrt{x^2})$$

$$= \partial_x (\ln(x)) = \underline{\frac{1}{x}}$$

$\therefore$ To compute $R_{\mu\nu}$, we need $R^{\mu}{}_{\nu\rho\sigma}$ first: (Riemann tensor).

Compute $R^x{}_{yxy}$ and then R_{xyxy}. We can then obtain other components by symmetry

$$\therefore R^x{}_{yxy} = \cancel{\partial_x \Gamma^x_{yy}} - \partial_y \Gamma^x_{yx} - \Gamma^x_{xx}\Gamma^x_{yy} + \cancel{\Gamma^x_{xy}\Gamma^y_{yy}} - \Gamma^x_{yx} - \Gamma^x_{yy}\Gamma^y_{yx}$$

$$= \partial_x \left(x_y - \frac{1}{y}\right) = 0 \quad \cancel{-\partial_y \Gamma^x_{yx}} \left(\frac{1}{2y}\right) -$$

$$= -\partial_y\left(\frac{1}{2y}\right) - \left(\frac{1}{2x}\right)\left(-\frac{1}{y}\right) \mp \frac{1}{2y}\frac{1}{x}$$

$$= \frac{1}{2y^2} - \frac{1}{2xy} - \frac{1}{2yx} = \frac{1}{2y^2} - \frac{\cancel{2}}{\cancel{2}xy}$$

$$= \frac{1}{2y^2} - \frac{1}{xy} = \cancel{\frac{1 -}{2y^2x}} \quad \frac{x+2y}{4xy^2}$$

Since Riemann tensor is antisymmetric in first and last two indices we have

$$R^x{}_{yyx} = -R^x{}_{yxy} = \frac{-x+2y}{4xy^2}$$

Compute $R^y{}_{xxy} = -R^y{}_{xyx} =$

$$= R^x{}_{yxyx} = -\cancel{R}. \ \frac{x+2y}{4x^2y}$$ (since the computation ~~partial de.~~ is identical to the above, up to $-\partial_y \to -\partial_x\left(\frac{1}{2x}\right)$

$$\therefore R^y{}_{yxy} = -R^y{}_{yyx} = \frac{x+2y}{4x^2y}$$ (it is symmetric with $R^x{}_{yyx}$

$\times$ (SEE ~~END OF~~ PAGE 7)

$R^{x}{}_{xxx}$ and $R^{y}{}_{yyy}$ vanish by definition of Riemann tensor.

R_{xx} S. $R^{x}{}_{yxx} = 0 \Rightarrow R^{x}{}_{xyx} = 0$

$\therefore R_{xx} = g^{xx}(R^{x}{}_{xxx} + R^{y}{}_{xyx}) = g^{yy} R_{yxyx} = \frac{x+2y}{4x^2 y}$

Similarly, $R_{yy} = g^{x} R^{x}{}_{yxy} + R^{y}{}_{yyy} = \frac{x+2y}{4x^2 y^2}$

$$\therefore R_{xy} = \begin{pmatrix} \frac{x+2y}{4x^2 y} & 0 \\ 0 & \frac{x+2y}{4x^2 y^2} \end{pmatrix}$$

$$\therefore R_{z} = g^{xy} R_{xy}$$

$$= \frac{1}{xy} \cdot \frac{x+2y}{4x^2 y} + \frac{1}{x^2} \cdot \frac{x+2y}{4xy^2} = \frac{\cancel{4x^3y^2}}{\cancel{4x^3y^2}}$$

$$= \frac{x+2y}{4x^3 y^2} + \frac{x+2y}{4x^3 y^2} = \frac{2x+4y}{4x^3 y^2} = \frac{x+2y}{2x^3 y^2}$$

(3) Claim:

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, T^\mu{}_\nu\right) - \Gamma^\lambda_{\mu\nu} T^\mu{}_\lambda$$

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma{}_\mu$$

$$\cancel{= \partial_\mu T^\mu{}_\nu + (\sqrt{-g})^{-1} \partial_\sigma (\sqrt{-g}) T^{\sigma\mu}{}_\sigma - \Gamma}$$

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma$$

$$= \nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + (\sqrt{-g})^{-1} \partial_\sigma \sqrt{-g}\, T^\sigma{}_\nu) - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma$$

$$= \partial_\mu T^\mu{}_\nu + \frac{1}{\sqrt{-g}} \partial_\sigma (\sqrt{-g}\, T^\mu{}_\nu) - \Gamma^\sigma_{\mu\nu} T^\mu{}_\sigma \Rightarrow \underline{(E i)}$$

(where $\lambda \equiv \sigma$)

Want to now show that:

$$\nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2} (\partial_\nu g_{\mu\lambda}) T^{\mu\lambda}$$

Case where $\sigma = \mu$:

$$= \frac{1}{\sqrt{-g}} \partial_\mu T^\mu{}_\nu + \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu{}_\nu) + \Gamma^\mu_{\mu\nu} T^\mu{}_\mu .$$

$$\therefore \Gamma^\mu_{\mu\nu} = \partial_\nu \ln(\sqrt{|g_{\mu\mu}|}) = \cancel{\partial_\nu t g} = \cancel{\frac{-1}{2 g_{\mu\mu}} 2g}$$

$$= \cancel{\ln 2}\ \cancel{\frac{1}{2\sqrt{g}}}\quad \frac{1}{2 g_{\mu\mu}} \cdot g^{\mu\lambda} \quad T^\mu{}_\mu \quad \cancel{\ln(\sqrt{x})}$$

$$\therefore \nabla_\mu T^\mu{}_\nu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu{}_\nu) + \partial_\nu (\ln(\sqrt{|g_{\mu\mu}|})) T^\mu{}_\mu$$

$$= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2} \partial_\nu (g_{\mu\lambda}) T^\mu{}_\mu g^{\mu\lambda}$$

$$T^\mu{}_\mu g^{\mu\lambda} = \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu{}_\nu) - \frac{1}{2} \partial_\nu (g_{\mu\lambda}) T^{\mu\lambda}$$

(4)a. Schwarzschild Metric:

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$= -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

$\therefore$ Geodesic eqn general form:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\therefore \cancel{\frac{d^2r}{d\tau^2}} + \cancel{\Gamma^r_{rr}}\cancel{\left(\frac{dx^r}{d\tau}\right)^2} + \Gamma$$

$$= \frac{d^2r}{d\tau^2} + \Gamma^r_{rt}\frac{dr}{d\tau}\frac{dt}{d\tau} + \Gamma^r_{tr}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0$$

$$\therefore \frac{d^2t}{d\tau^2} - \frac{2GM}{(2GM-r)r}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0 \qquad (1)$$

$$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{d\tau}\right)^2 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{d\tau}\right)^2$$

$$\Rightarrow \frac{d^2r}{d\tau^2} - \frac{GM(2GM-r)}{r^3}\left(\frac{dt}{d\tau}\right)^2 + \frac{GM}{(2GM-r)r}\left(\frac{dr}{d\tau}\right)^2 + (2GM-r)\left(\frac{d\theta}{d\tau}\right)^2$$

$$+ (2GM-r)\sin^2(\theta)\left(\frac{d\varphi}{d\tau}\right)^2 = 0 \qquad (2)$$

$$\frac{d^2\theta}{d\tau^2} + 2\Gamma^\theta_{r\theta}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \Gamma^\theta_{\varphi\varphi}\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$= \frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \cos(\theta)\sin(\theta)\left(\frac{d\varphi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\varphi}{d\tau^2} + 2\Gamma^{\varphi}_{\varphi r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + 2\Gamma^{\varphi}_{\theta\varphi}\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0$$

$$= \frac{d^2\varphi}{d\tau^2} + \frac{2}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + 2\cot(\theta)\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau} = 0 \quad (4)$$

b. $R = R_s$ ...?

(2) Recalculating Riemann and Ricci tensors for Question 2.

By definition, the Riemann tensor is given by:

$$R^{\mu}{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^{\mu}_{\nu\beta} - \partial_\beta \Gamma^{\mu}_{\nu\alpha} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\sigma}_{\nu\beta} - \Gamma^{\mu}_{\beta\sigma}\Gamma^{\sigma}_{\nu\alpha}$$

$$R^{x}{}_{xxx} = \partial_x \Gamma^{x}_{xx} - \partial_x \Gamma^{x}_{xx} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx} - \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx}$$

$$= \underbrace{\partial_x\left(\frac{1}{2x}\right) - \partial_x\left(\frac{1}{2x}\right)}_{0} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx} - \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xx}$$

$$= 0$$

$\lambda = x \Rightarrow$ the above trivially vanishes.

$\lambda = y \Rightarrow \Gamma^{x}_{xy}\Gamma^{y}_{xx} - \Gamma^{x}_{xy}\Gamma^{y}_{xx} = 0$

Similarly for $R^{y}{}_{yyy} = 0$

$$R^{x}{}_{xxy} = \cancel{\partial_x \Gamma^{x}_{xy}} - \underbrace{\partial_y \Gamma^{x}_{xy}}_{0} + \Gamma^{x}_{x\lambda}\Gamma^{\lambda}_{xy} - \Gamma^{x}_{y\lambda}\Gamma^{\lambda}_{xx}$$

$\lambda = x \Rightarrow = 0$, $\lambda = y \Rightarrow = 0$ (given Christoffel symbols).

By symmetry $R^{x}{}_{xyx} = 0$ (Similarly, any entry with more $R^{y}{}_{yy\cdot} = 0$

The non-vanishing components are:

$R^{yx}{}_{yxy}$ and $R^{x}{}_{yyx}$

~~Thes~~ $R^{x}{}_{yxy}$ =

$= \cancel{\partial_x \Gamma^x_{yy}} - \partial_y \Gamma^x_{yx} + \Gamma^x_{x\lambda}\Gamma^\lambda_{yy} - \Gamma^x_{y\lambda}\Gamma^\lambda_{yx}$

$= -\partial_y\left(\frac{1}{2y}\right) + \left(\frac{1}{2x}\right)\left(\frac{-1}{y}\right) - \frac{1}{2xy}$

$= \frac{1}{2y^2} - \frac{1}{xy} = \frac{x+2y}{4xy^2}$

By symmetry, $R^{x}{}_{yyx} = -\frac{x+2y}{4xy^2}$

Hence $R^{y}{}_{xxy} = -R^{y}{}_{xyx} = \frac{x+2y}{4x^2y}$ (computation is identical, up to $\partial_y \leftrightarrow \partial_x$) and due to the symmetry in the form of the Christoffel symbols

Ricci tensor =

$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} = R^{x}{}_{yxx} + R^{x}{}_{yxy}$

$\therefore R_{xx} = R^{\sigma}{}_{x\sigma x} = R^{x}{}_{xxx} + R^{y}{}_{xyx} = 0 + \frac{x+2y}{4x^2y}$

$R_{xy} = R^{\sigma}{}_{x\sigma y} = R^{x}{}_{xxy} + R^{y}{}_{xyy} = 0$

$R_{yx} = -R_{xy} = 0$

$R_{yy} = R^{\sigma}{}_{y\sigma y} = R^{x}{}_{yxy} + R^{y}{}_{yyy}$

$$= \frac{x+2y}{4xy^2} + 0 = \frac{x+2y}{4xy^2}$$

$$\therefore R_{\mu\nu} = \begin{pmatrix} \frac{x+2y}{4y^2x^2} & 0 \\ 0 & \frac{x+2y}{4y^2x} \end{pmatrix}$$

(Ricci scalar was computed earlier = trace of $R_{\mu\nu}$)

$$= \frac{x+2y}{4yx^2} + \frac{x+2y}{4y^2x} = \frac{x+2y}{2x^2y^2}$$

(5)a. Want to write

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

$= \nabla_\mu A_\nu - \nabla_\nu A_\mu$ (Carroll Ch. 4 p. 153)

b. Y^μ = parallel transported along $x^\mu(\tau)$, $u^\mu = \frac{dx^\mu}{d\tau}$

It must obey the eqn of parallel transport

(i.e. $\nabla_u Y = u^\lambda \nabla_\lambda Y^\mu = 0$)

Let $x^\mu(\tau)$ be an affine geodesic:

$\Rightarrow x^\mu(\tau) = \alpha \tau + \beta$, $\alpha, \beta \in \mathbb{C}$

$\therefore u^\mu = \alpha \Rightarrow u_\mu = g_{\mu\delta} u^\delta = g_{\mu\delta}(\alpha) =$

$$= \frac{x+2y}{4xy^2} + 0 = \frac{x+2y}{4xy^2}$$

$$\therefore R_{\mu\nu} = \begin{pmatrix} \frac{x+2y}{4y^2x^2} & 0 \\ 0 & \frac{x+2y}{4y^2x} \end{pmatrix}$$

(Ricci scalar was computed earlier = trace of $R_{\mu\nu}$)

$$= \frac{x+2y}{4yx^2} + \frac{x+2y}{4y^2x} = \frac{x+2y}{2x^3y^2}$$

(5) a. Want to write:

$\partial_\mu A_\nu - \partial_\nu A_\mu$ in covariant form

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$u^\mu = \alpha \Rightarrow u_\mu = g_{\mu\delta} u^\delta = g_{\mu\delta}(\alpha) =$

$\therefore u^{\mu} u_{\mu} = \alpha^2 = \text{constant}.$

$$Y^{\mu} u_{\mu} = Y^0 \alpha_0 + \alpha Y^1 + \alpha Y^2 + \alpha Y^3 \ldots$$

$$= \alpha Y^{\mu}$$ (dot product is conserved since $u_{\mu} =$ constant).

$\therefore$ ~~$Y^{\mu} u_{\mu} =$~~

Know that $\nabla_{\mu} Y^{\nu} = 0 \Rightarrow \cancel{Y}\ \partial_{\mu} Y^{\nu} + \Gamma^{\nu}_{\mu\sigma}$.

$$\partial_{\mu} Y^{\nu} + \Gamma^{\nu}_{\mu\lambda} Y^{\lambda} = 0$$

?

$$\therefore Y^{\mu} u_{\mu} = Y^{\mu} \frac{dx_{\mu}}{d\tau}$$