General Relativity: Worksheet 1

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1 Surface of a Sphere

The metric for the manifold described by the two-dimensional surface of the sphere with spherical coordinates (θ, φ) where $0 \leq \theta \leq \pi$, and $0 \leq \varphi \leq 2\pi$ is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix} \tag{1}$$

The manifold described by the above metric is curved as some of the components of the Riemann curvature tensor, $R^{\mu}_{\nu\rho\sigma}$ do not vanish (as demonstrated in page 4 of the Mathematica notebook in the Appendix). However, the Einstein tensor, defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{2}$$

vanishes on this manifold, as described on page 4 of the Mathematica notebook. A two dimensional metric with a non-vanishing Einstein tensor cannot be constructed. This is due to the definition of the Ricci tensor in two dimensions being

$$R_{\mu\nu} = \frac{R}{2} g_{\mu\nu} \tag{3}$$

Inserting (3) into (2), it is evident that

$$G_{\mu\nu} = \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \tag{4}$$

There is only one independent component of the Riemann curvature tensor $R^{\mu}_{\nu\rho\sigma}$ which is completely determined by the Ricci scalar. Despite this, it is known that a two-dimensional surface can exhibit stronger curvature in one direction compared to another (e.g. a folded sheet of paper). This is because the Riemann curvature is a part of the intrinsic geometry of the manifold, as measured by beings that are confined to that space (for whom the manifold would look flat), whereas the intuitive notion of the curvature of the manifold describes it as being dependent on the extrinsic geometry of the space (i.e. how it is embedded in a larger space).

Although the Ricci scalar is constant on the sphere, this is not a universal property of two-dimensional manifolds. This is evident upon computing the Ricci scalar of a 2-torus, whose metric in terms of its coordinates u and v is given by:

$$g_{\mu\nu} = \begin{pmatrix} c + a\cos(v) & 0 \\ 0 & a \end{pmatrix} \tag{5}$$

The Ricci scalar for this metric is computed in Section 1 of the Mathematica notebook in the Appendix on page 5, and is given by:

$$R = \frac{2c\cos(v) + 2a\cos^2(v) + a\sin^2(v)}{2(c + a\cos(v))^2} \tag{6}$$

The expression (6) is not constant, as it is dependent upon the coordinates u and v.

There also exist metrics exhibiting constant negative curvature on a two dimensional metric. An example of such a metric is the Poincare metric, which can be written in terms of the Cartesian coordinates, x and y as:

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \tag{7}$$

The Ricci scalar corresponding to the metric (7) has a value of $R = -2$, as demonstrated in Section 1 in page 6 of the Mathematica notebook in the Appendix.

1.1 Three-dimensional Manifolds

A three-dimensional metric for which the Einstein tensor vanishes does not exist, since in 3 dimensions, the Ricci tensor is proportional to the Ricci scalar, thereby leading to the vanishing of the Einstein tensor based on the definition given in Equation (2).

2 Space-Time Metrics

2.1 A Time-dependent Metric

Adapting a diagonal Minkowski metric by introducing a dependence for one of the coefficients on the time coordinate as follows

$$ds^2 = -a(t)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{8}$$

does not alter the metric (i.e. the metric described in Equation 8 is the Minkowski metric). In order to illustrate this, one can consider a transformation of coordinates of the form $(t', x', y', z') = (a(t), x, y, z)$ such that

$$ds'^2 = -dt'^2 + dx'^2 + dy'^2 + dz'^2 \tag{9}$$

Given that $t' = a(t)$, one can deduce that

$$dt' = a'(t)dt \Rightarrow \frac{dt'}{dt} = a'(t) \tag{10}$$

Solving the above differential equation yields

$$t' = \int a'(t)dt = a(t) \tag{11}$$

Since $ds'^2 = ds^2$, due to the invariance of the spacetime interval, we must have that

$$-dt'^2 + dx'^2 + dy'^2 + dz'^2 = -a(t)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{12}$$

From this one can conclude that $dt' = a(t)dt$, and as such, one can perform the transformation $t' = a(t)$ to preserve the time-independent nature of the metric. Consequently, the resulting metric is the Minkowski metric, which by definition, is independent of time.

Altering the metric such that one of the spatial components evolve with time, such that the metric is of the form

$$ds^2 = -dt^2 + a(t)^2 dx^2 + dy^2 + dz^2 \tag{13}$$

does not make it a vacuum solution, as demonstrated by the non-vanishing Einstein tensor (computed on page 8 of the Appendix). The above metric exhibits all forms of spatial symmetries (such as spatial translations and rotations), but not temporal symmetries due to the dependence of one of the coordinates on t. It would manifest itself through the time-dependent expansion of the "universe" in the direction of the x-coordinate.

Upon inspecting the Christoffel symbols for the above metric, it is evident that the non-vanishing coefficients are

$$\Gamma^x_{xt} = \Gamma^x_{tx} = \frac{a'(t)}{a(t)} \quad \text{and} \quad \Gamma^t_{xx} = a(t)a'(t) \tag{14}$$

The general form of the geodesic equation is:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} \tag{15}$$

Hence, the geodesic equation for the symbol Γ^t_{xx} is:

$$\frac{d^2t}{d\lambda^2} + a'(t)a(t)\left(\frac{dx}{d\lambda}\right)^2 = 0 \tag{16}$$

while that for the symbol $\Gamma^x_{xt} = \Gamma^x_{tx}$ is given by

$$\begin{aligned} \frac{d^2x}{d\lambda^2} + \frac{a'(t)}{a(t)}\frac{dt}{d\lambda}\frac{dx}{d\lambda} + \frac{a'(t)}{a(t)}\frac{dx}{d\lambda}\frac{dt}{d\lambda} &= 0 \\ = \frac{d^2x}{d\lambda^2} + \frac{2a'(t)}{a(t)}\left(\frac{dt}{d\lambda}\frac{dx}{d\lambda}\right) &= 0 \end{aligned}$$

A particle at rest has $\frac{dx}{d\lambda} = 0$. Thus, given the form of the geodesic equation for the above metric, it is evident that a test particle at rest would not experience an acceleration, signified by the simplification of the above expression, resulting in

$$\frac{d^2x}{d\lambda^2} = 0 \tag{17}$$

Since photons travel along null geodesics, where $ds^2 = 0$, it is possible to solve the geodesic equation for photons travelling in the x-direction. In this case, $dx = 0$. The line element then becomes

$$ds^2 = -dt^2 + dy^2 + dz^2 \tag{18}$$

thereby leading to

$$-dt^2 + dy^2 + dz^2 = 0 \tag{19}$$

Rearranging the above expression leads to the relation

$$\left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 1 \tag{20}$$

The equation (20) represents the constraint that must be in place on the velocities of photons travelling perpendicular to the x-direction. The geodesic equation can be solved with the imposition of the above constraint. Thus, the geodesic equation can be written for y and z. Since the Christoffel symbols associated with these coordinates vanish, the equations become:

$$\frac{d^2y}{d\lambda^2} = 0 \quad \text{and} \quad \frac{d^2z}{d\lambda^2} = 0 \tag{21}$$

Solving both of the above second-order differential equations for $y(\lambda)$ and $z(\lambda)$ yields

$$y(\lambda) = p + q\lambda \quad \text{and} \quad z(\lambda) = r + s\lambda \tag{22}$$

for constants p, q, r and s.

2.2 Non-Constant Coefficients in Space

Consider allowing the metric coefficients to vary in space, e.g.

$$ds^2 = -a(x)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{23}$$

For the metric (23), the non-zero Christoffel symbols, $\Gamma^{\mu}_{\rho\sigma}$ have been computed on page 10 in Section 2.2 of the Mathematica notebook in the Appendix. These are:

$$\Gamma^{t}_{tx} = \Gamma^{t}_{xt} = \frac{a'(x)}{a(x)} \tag{24}$$

and

$$\Gamma^{x}_{tt} = a'(x)a(x) \tag{25}$$

Hence, the geodesic equation for the symbols $\Gamma^t_{xt} = \Gamma^t_{tx}$ becomes:

$$\frac{d^2t}{d\lambda^2} + \Gamma^t_{tx}\frac{dx^t}{d\lambda}\frac{dx^x}{d\lambda} + \Gamma^t_{xt}\frac{dx^x}{d\lambda}\frac{dx^t}{d\lambda} = 0 \tag{26}$$

$$\begin{aligned}
&= \frac{d^2t}{d\lambda^2} + \frac{a'(x)}{a(x)}\frac{dt}{d\lambda}\frac{dx}{d\lambda} + \frac{a'(x)}{a(x)}\frac{dx}{d\lambda}\frac{dt}{d\lambda} = 0 \\
&= \frac{d^2t}{d\lambda^2} + \frac{a'(x)}{a(x)}\left(\frac{dt}{d\lambda}\frac{dx}{d\lambda} + \frac{dx}{d\lambda}\frac{dt}{d\lambda}\right) = 0 \\
&= \frac{d^2t}{d\lambda^2} + \frac{a'(x)}{a(x)}\left[2\frac{dt}{d\lambda}\frac{dx}{d\lambda}\right] = 0
\end{aligned}$$

The above expression can be written as

$$\frac{d^2t}{d\lambda^2} + \frac{2a'(x)}{a(x)}\frac{dt}{d\lambda}\frac{dx}{d\lambda} = 0 \tag{27}$$

Similarly, for the symbol Γ^x_{tt}, the geodesic equation becomes

$$\frac{d^2x}{d\lambda^2} + a'(x)a(x)\left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{28}$$

Particles that are initially at rest in this metric can still exhibit an acceleration, as $\frac{dx}{d\lambda} = 0$ does not imply that $\frac{d^2x}{d\lambda^2} = 0$ in this case, unlike in the time-dependent case. Equations can also be deduced for the spatial coordinates y and z as well. Since the Christoffel symbols associated with these coordinates vanish, the geodesic equation for these coordinates reduces to

$$\frac{d^2y}{d\lambda^2} = 0 \quad \text{and} \quad \frac{d^2z}{d\lambda^2} = 0 \tag{29}$$

Integrating each of the equations (29) above with respect to λ twice yields

$$y(\lambda) = a\lambda + b \tag{30}$$

and

$$z(\lambda) = k\lambda + \ell \tag{31}$$

for arbitrary constants a, b, k, and ℓ.

In the case of the above metric, one has the acceleration of a particle being non-zero despite it being initially at rest. Symmetries that prevent this are essential to prevent the propagation of this particle of its own accord.

The metric (23) appears to have a simple form, but it already illustrates that it can be very difficult to intuitively judge whether a metric is "well-behaved". One can compare the following metrics as an example

$$ds^2 = -x^2dt^2 + dx^2 + dy^2 + dz^2 \tag{32}$$

$$ds^2 = -x^4 dt^2 + dx^2 + dy^2 + dz^2 \tag{33}$$

While the metric (32) is "well-behaved" in that the components of its Riemann tensor vanish, the same is not true for the metric (33), which contains components that diverge at $x = 0$ (e.g. $\frac{2}{x^2}$). Consequently, this metric contains a singularity, and this leads to the breaking down of Einstein's field equations.

One can now consider the case of the following spherically symmetric metric

$$ds^2 = -dt^2 + e^{-(x^2+y^2+z^2)}(dx^2 + dy^2 + dz^2) \tag{34}$$

This metric does not contain singularities, as none of the Riemann tensor components diverge. This is neither a viable metric for a mass configuration in flat space, nor is it a viable metric for a cosmological model, as for both of the abovementioned conditions to be met, the metric must resemble the Minkowski metric in the limit that x, y, and z approach infinity. This is not the case with the above metric. Furthermore, the Ricci scalar for this metric, computed in Section 2.2 on page 14 of the Mathematica notebook in the Appendix, given by

$$R = -2e^{x^2+y^2+z^2}(-6x^2 + y^2 + z^2) \tag{35}$$

appears to diverge as x, y, and z approach infinity. This alludes to a divergent curvature of spacetime in the limit of large values of x, y, and z, and thereby reinforces the argument that the above metric cannot be a viable metric for a mass configuration in flat space.

2.3 Wave Solutions

One can attempt to construct wave-like solutions in a vacuum. For a wave travelling in the positive x-direction, one can attempt to construct a solution of the form

$$g_{\mu\nu} = \begin{pmatrix} -1 + f(x-t) & 0 & 0 & 0 \\ 0 & 1 + f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

However, such a metric does not satisfy the vacuum field equations (as demonstrated by the non-vanishing Einstein tensor in Section 3.3 on page 16 of the Mathematica notebook). Consequently, off-diagonal elements of the metric must be permitted.

$$g_{\mu\nu} = \begin{pmatrix} -1 + f(x-t) & -f(x-t) & 0 & 0 \\ -f(x-t) & 1 + f(x-t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{37}$$

The above is a special case of the Peres metric, and it can be demonstrated that this metric satisfies the vacuum field equations. This is illustrated below.

In general, one can write Einstein's field equations as

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \kappa T^{\mu\nu} \tag{38}$$

In vacuum, this simplifies to

$$G^{\mu\nu} = 0 \Rightarrow R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 0 \tag{39}$$

The metric in Equation (37) satisfies the Einstein field equations, as demonstrated by the vanishing of the Ricci tensor $R^{\mu\nu}$, as well as the Ricci scalar, and consequently the Einstein tensor, $G^{\mu\nu}$, computed in Section 2.3 of the Mathematica notebook in the Appendix). Furthermore, the non-zero components of the Riemann curvature tensor $R^{\mu}_{\nu\rho\sigma}$ have equal value, and are given by

$$\frac{-f(x-t)f'(x-t)^2 - f''(x-t) + f(x-t)^2 f''(x-t)}{(-1+f(x-t))^2(1+f(x-t))} \tag{40}$$

This value appears four times within the metric and corresponds to the components $R^t_{xtx}, R^t_{xxt}, R^x_{ttx}$ and R^x_{txt}.

The metric (37) can be generalised to also depend on y and z as follows

$$g_{\mu\nu} = \begin{pmatrix} -1+f(x-t,y,z) & -f(x-t,y,z) & 0 & 0 \\ -f(x-t,y,z) & 1+f(x-t,y,z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

It can be shown that the metric (41) is a solution if f is a harmonic function of y and z (i.e. if the following relation holds)

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{42}$$

This is demonstrated below.

Noting that the Einstein tensor can be written in the form described by Equation (2), one can compute the Ricci scalar, R and see that this vanishes (as demonstrated in Section 3.2 of the Mathematica notebook in the Appendix). Thus, the second term in Equation (2). It remains to compute the Ricci tensor, $R_{\mu\nu}$ for the metric given in (41). As demonstrated in Section 3.2 on page 20 of the Mathematica notebook in the Appendix, the Ricci tensor can be written as:

$$R_{\mu\nu} = \begin{pmatrix} \frac{1}{2}(-\partial_y^2 f - \partial_z^2 f) & \frac{1}{2}(\partial_z^2 f + \partial_y^2 f) & 0 & 0 \\ \frac{1}{2}(\partial_z^2 + \partial_y^2 f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{43}$$

$$= \frac{1}{2}\begin{pmatrix} (-\partial_y^2 f - \partial_z^2 f) & (\partial_z^2 f + \partial_y^2 f) & 0 & 0 \\ (\partial_z^2 f + \partial_y^2 f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{44}$$

where the notation $\partial_z^2 f$ signifies the second partial derivative of f with respect to z. In other words

$$\partial_z^2 f \equiv \frac{\partial^2 f}{\partial z^2}$$

Given that the function f is harmonic, and therefore obeys the relation described in Equation 42, one can deduce that

$$\partial_z^2 + \partial_y^2 f = 0 \Rightarrow \partial_z^2 f = -\partial_y^2 f \tag{45}$$

Hence, replacing $\partial_z^2 f$ with $-\partial_y^2 f$ in (43) yields:

$$R_{\mu\nu} = \frac{1}{2}\begin{pmatrix} (-\partial_y^2 f + \partial_y^2 f) & (-\partial_y^2 f + \partial_y^2 f) & 0 & 0 \\ (-\partial_y^2 f + \partial_y^2 f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \tag{46}$$

Hence, it is evident that the Ricci tensor, $R_{\mu\nu}$ vanishes for a function f that is harmonic in y and z. This leads to the vanishing of the Einstein tensor, and as a result, the metric satisfies the vacuum equations if $f(t,x,y,z)$ is harmonic in y and z.

An example of a metric wherein $f(x,y,z)$ is harmonic in y and z is one where $f(x,y,z) = x+y+z$. It is evident that f is harmonic in y and z since it satisfies the relation described in (42). Under this substitution, the metric (41) becomes:

$$g_{\mu\nu} = \begin{pmatrix} -1+t-x-y-z & t-x-y-z & 0 & 0 \\ t-x-y-z & 1+t-x-y-z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{47}$$

The Riemann tensor and consequently the Ricci tensor vanish for the above metric. The Einstein tensor also vanishes for this reason. Hence, it is evident that the above metric satisfies the vacuum equations. This spacetime is flat, as evidenced by the vanishing of the Riemann curvature tensor (as demonstrated in Section 2.3 on page 21 of the Mathematica notebook in the Appendix). Thus, as $y, z \to \infty$, the spacetime exhibits zero curvature.