Assignment 1: Question 3

Saturday, 12 March 2022 12:07 pm

(3) a. Want to prove the commutation relations between components of kinematic ($\hat{\Pi} = \underset{\sim}{p} + e\vec{A}(\underset{\sim}{x})/c$) and magnetic $\vec{K} = \underset{\sim}{p} - e\vec{A}(\underset{\sim}{x})/c$ momenta: given that $A(\underset{\sim}{x}) = \left\{-\frac{By}{2}, \frac{Bx}{2}, 0\right\}$

(i) $[\hat{\Pi}_x, \hat{\Pi}_y] = -\frac{i\hbar^2}{l_B^2}$: Claim:

$$[\hat{\Pi}_x, \hat{\Pi}_y] = [p_x + eA_x(x)/c,\ p_y + eA_y(y)/c]$$
$$= [p_x, p_y] + [p_x, eA_y(y)/c] + [eA_x(x)/c, p_y] + [eA_x/c, eA_y/c]$$
$$= 0 - \left(i\hbar \frac{\partial A_y}{\partial x}\right)\cdot\frac{e}{c} + \frac{e}{c}\left(i\hbar\frac{\partial A_x}{\partial y}\right) + \frac{e}{c}(0)$$
$$= \left(-i\hbar\frac{\partial A_y}{\partial x}\right)\frac{e}{c} + \left(i\hbar\frac{\partial A_x}{\partial y}\right)\frac{e}{c}$$
$$= -\frac{i\hbar e}{c}\left(-\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y}\right)$$
$$= -\frac{i\hbar e}{c}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) = -\frac{i\hbar e}{c}B$$

$\vec{B} = \underset{\sim}{\nabla}\times\vec{A} \Rightarrow B_z = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$. Since B is uniform, $B_z = \underline{\underline{B}}$

Let $l_B = \sqrt{\frac{\hbar c}{eB}} \Rightarrow l_B^2 = \frac{\hbar c}{eB} \Rightarrow \frac{\hbar^2}{l_B^2} = \frac{\hbar^2 \cdot eB_z}{\hbar c} = \frac{e\hbar B_z}{c}$

$\therefore \Rightarrow [\hat{\Pi}_x, \hat{\Pi}_y] = \underline{\underline{-\frac{i\hbar^2}{l_B^2}}}$

(ii) $[\hat{K}_x, \hat{K}_y] = [p_x - eA_x/c,\ p_y - eA_y/c]$

$$= [p_x, p_y] - [p_x, eA_y/c] - [eA_x/c, p_y] + [eA_x/c, eA_y/c]$$
$$= [p_x, p_y] - \frac{e}{c}[p_x, A_y] - \frac{e}{c}[A_x, p_y] + \frac{e}{c}[A_x, A_y]$$
$$= 0 - \frac{e}{c}[p_x, A_y] - \frac{e}{c}[A_x, p_y] + \frac{e}{c}(0)$$
$$= 0 - \frac{e}{c}\left[-i\hbar\frac{\partial A_y}{\partial x} + i\hbar\frac{\partial A_x}{\partial y}\right] + 0$$
$$= -\frac{i\hbar e}{c}\left[\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right] = \frac{i\hbar e}{c}(B_z) \Rightarrow [\hat{K}_x, \hat{K}_y] = \underline{\underline{\frac{i\hbar^2}{l_B^2}}}$$

where $\underline{\underline{l_B = \sqrt{\frac{\hbar c}{eB}}}}$

$$[\hat{K}_x, \hat{\Pi}_x] = [p_x - e/c\,A_x,\ p_x + e/c\,A_x]$$
$$= [p_x, p_x] + [p_x, e/c\,A_x] - [e/c\,A_x, p_x] - [e/c\,A_x, e/c\,A_x]$$
$$= [\hat{p}_x, \tfrac{e}{c}\hat{A}_x] + [\hat{p}_x, \tfrac{e}{c}\hat{A}_x]$$
$$= \frac{e}{c}\left([\hat{p}_x, \hat{A}_x] + [\hat{p}_x, \hat{A}_x]\right) = \underline{\underline{0}}$$

$\therefore$ By symmetry $[\hat{K}_y, \hat{\Pi}_y] = 0$

- $[\hat{K}_x, \hat{\Pi}_y] = [p_x - e/c\,A_x,\ p_y + e/c\,A_y]$

$$= [p_x, p_y] - \left[p_x, \tfrac{e}{c}A_y\right] - \left[\tfrac{e}{c}A_x, p_y\right] + \left[\tfrac{e}{c}A_x, \tfrac{e}{c}A_y\right]$$
$$= -\left[p_x, \tfrac{e}{c}A_y\right] - [e/c\,A_x, p_y] + \left[\tfrac{e}{c}A_x, A_y\right]$$
$$= -\frac{e}{c}[p_x, A_y] - \frac{e}{c}[A_x, p_y] \qquad i\hbar\frac{\partial A_y}{\partial x} \quad + i\hbar\frac{\partial A_x}{\partial y}$$
$$= \frac{e}{c}[p_y, A_x] - \frac{e}{c}[p_x, A_y] \qquad =$$
$$= \frac{e}{c}\left[i\hbar\frac{\partial A_x}{\partial y} + i\hbar\frac{\partial A_y}{\partial x}\right] = \frac{i\hbar e}{c}\left(\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right) =$$

- Given that $A = \left\{-\frac{By}{2}, \frac{Bx}{2}, 0\right\}$, it is evident that $\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = \frac{B}{2} - \frac{B}{2} = 0$

$\therefore [\hat{K}_x, \hat{\Pi}_y] = \underline{\underline{0}} \Rightarrow \underline{\underline{QED}}$

$[\hat{K}_y, \hat{\Pi}_x] = 0$ by similar arguments

b. Introduce ladder operators: $\hat{a}$, $\hat{a}^+$, $\hat{b}$ and $\hat{b}^+$ as follows:

$$\hat{a} = \frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x - i\hat{\Pi}_y) \qquad \hat{a}^+ = \frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x + i\hat{\Pi}_y)$$

$$\hat{b} = \frac{l_B}{\sqrt{2\hbar}}(\hat{K}_y - i\hat{K}_x) \qquad \hat{b}^+ = \frac{l_B}{\sqrt{2\hbar}}(\hat{K}_y + i\hat{K}_x)$$

- Want to show that: $[\hat{a}, \hat{a}^+] = 1$ and $[\hat{b}, \hat{b}^+] = 1$ and $[\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^+] = [\hat{a}^+, \hat{b}] = [\hat{a}^+, \hat{b}^+] = 0$

$\therefore [\hat{a}, \hat{a}^+] = \left[\frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x - i\hat{\Pi}_y), \frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x + i\hat{\Pi}_y)\right]$

$$= \frac{l_B^2}{2\hbar^2}[\hat{\Pi}_x - i\hat{\Pi}_y, \hat{\Pi}_x + i\hat{\Pi}_y]$$
$$= \frac{l_B^2}{2\hbar^2}\left([\hat{\Pi}_x - i\hat{\Pi}_y, \hat{\Pi}_x] + [\hat{\Pi}_x - i\hat{\Pi}_y, i\hat{\Pi}_y]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-[\hat{\Pi}_x, \hat{\Pi}_x - i\hat{\Pi}_y] - [i\hat{\Pi}_y, \hat{\Pi}_x - i\hat{\Pi}_y]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-\left(\cancel{[\hat{\Pi}_x, \hat{\Pi}_x]}_{0} - [\hat{\Pi}_x, i\hat{\Pi}_y]\right) - \left([i\hat{\Pi}_y, \hat{\Pi}_x] - \cancel{[i\hat{\Pi}_y, i\hat{\Pi}_y]}_{0}\right)\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(i[\hat{\Pi}_x, \hat{\Pi}_y] - (i[\hat{\Pi}_y, \hat{\Pi}_x]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-i^2\frac{\hbar^2}{l_B^2} - i^2\frac{\hbar^2}{l_B^2}\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(\frac{\hbar^2 + \hbar^2}{l_B^2}\right) = \frac{l_B^2}{2\hbar^2}\left(\frac{2\hbar^2}{l_B^2}\right) = \underline{\underline{1}} \Rightarrow \underline{\underline{QED}}$$

$[\hat{b}, \hat{b}^+] = \left[\frac{l_B}{\sqrt{2\hbar}}(\hat{K}_y - i\hat{K}_x), \frac{l_B}{\sqrt{2\hbar}}(\hat{K}_y + i\hat{K}_x)\right]$

$$= \frac{l_B^2}{2\hbar^2}\left[(\hat{K}_y - i\hat{K}_x), (\hat{K}_y + i\hat{K}_x)\right]$$
$$= \frac{l_B^2}{2\hbar^2}\left([\hat{K}_y - i\hat{K}_x, \hat{K}_y] - [\hat{K}_y - i\hat{K}_x, \hat{K}_x]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-[\hat{K}_y, \hat{K}_y - i\hat{K}_x] + [\hat{K}_x, \hat{K}_y - i\hat{K}_x]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-\left([\hat{K}_y, \hat{K}_y] - [\hat{K}_y, i\hat{K}_x]\right) + \left([\hat{K}_x, \hat{K}_y] - [\hat{K}_x, -i\hat{K}_x]\right)\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(i[\hat{K}_x, \hat{K}_y] + i[\hat{K}_x, \hat{K}_y] + i[\hat{K}_x, \hat{K}_x]\right)$$
$$= \frac{l_B^2}{2\hbar^2}\left(-i\cdot\frac{i\hbar^2}{l_B^2} - i\cdot\frac{i\hbar^2}{l_B^2}\right) = \frac{l_B^2}{2\hbar^2}\left(\frac{2\hbar^2}{l_B^2}\right) = \underline{\underline{1}} \Rightarrow \underline{\underline{QED}}$$

- The remaining commutation relations $[\hat{a}, \hat{b}]$, $[\hat{a}, \hat{b}^+]$, $[\hat{a}^+, \hat{b}]$, $[\hat{a}^+, \hat{b}^+]$ all involve commutators of $\hat{K}$ and $\hat{\Pi}$, which are equal to zero. Hence, all of the above relations $= 0$ ($\Rightarrow [\hat{a}, \hat{b}] = [\hat{a}, \hat{b}^+] = [\hat{a}^+, \hat{b}] = [\hat{a}^+, \hat{b}^+] = 0$)

c. Let $\hat{N}_a = \hat{a}^+\hat{a}$ and $\hat{N}_b = \hat{b}^+\hat{b}$

- Want to show that $\hat{H}$ can be presented as $\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$

$:= \hbar\omega\left[\frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x + i\hat{\Pi}_y)\frac{l_B}{\sqrt{2\hbar}}(\hat{\Pi}_x - i\hat{\Pi}_y) + \frac{1}{2}\right] = \hat{H}$ (where the definitions of $\hat{a}$ and $\hat{a}^+$ have been substituted into the expression)

$$= \frac{\hbar\omega l_B^2}{2\hbar^2}\left[(\hat{\Pi}_x + i\hat{\Pi}_y)(\hat{\Pi}_x - i\hat{\Pi}_y) + \frac{1}{2}\right]$$
$$= \frac{\omega l_B^2}{2\hbar}\left[\hat{\Pi}_x^2 - i\hat{\Pi}_x\hat{\Pi}_y + i\hat{\Pi}_y\hat{\Pi}_x + \hat{\Pi}_y^2 + \frac{1}{2}\right]$$
$$= \frac{\omega l_B^2}{2\hbar}\left[\hat{\Pi}_x^2 - i[\hat{\Pi}_x, \hat{\Pi}_y] + \hat{\Pi}_y^2 + \frac{1}{2}\right]$$
$$= \frac{\omega l_B^2}{2\hbar}\left[\hat{\Pi}_x^2 - i\left(-\frac{i\hbar^2}{l_B^2}\right) + \hat{\Pi}_y^2 + \frac{1}{2}\right]$$
$$= \frac{\omega l_B^2}{2\hbar}\left[\hat{\Pi}_x^2 - \frac{\hbar^2}{l_B^2} + \hat{\Pi}_y^2 + \frac{1}{2}\right]$$
$$= \omega\left[\hat{\Pi}_x^2 - \frac{1}{2} + \frac{1}{2} + \hat{\Pi}_y^2\right]\cdot\frac{l_B^2}{2\hbar} \qquad \frac{\hbar c}{eB}\cdot\frac{1}{2\hbar} = \frac{c}{eB}$$

$\therefore \frac{eB}{2m}\cdot\frac{c}{eB} = \frac{c}{2m}$

- Set $c = 1$

$$\therefore \hat{H} = \frac{1}{2m}[\hat{\Pi}_x^2 + \hat{\Pi}_y^2] = \frac{1}{2m}\left(\left(p_x + \frac{e}{c}A_x\right)^2 + \left(p_y + \frac{e}{c}A_y\right)^2\right)$$
$$= \underline{\underline{\frac{1}{2m}\left(\underset{\sim}{p} + \frac{e}{c}A(\underset{\sim}{x})\right)^2}} = \hat{H} \Rightarrow \underline{\underline{QED}}$$

d. Since $[\hat{H}, \hat{N}_b] = 0$ and $[\hat{H}, \hat{N}_a] = 0$, this implies that $[N_b, N_a] = 0$ and thus, the eigenvalues of both of these operators constitute "good quantum numbers" and can therefore represent simultaneously measurable 'observables' (i.e. n and l represent observables that can be measured simultaneously). One can write the state $|n, l\rangle$ as follows:

$$|n, l\rangle = |n\rangle \otimes |l\rangle$$

$$\therefore \hat{N}_a|n, l\rangle = \hat{N}_a|n\rangle \otimes |l\rangle = n|n, l\rangle$$

$$\hat{N}_b|n, l\rangle = |n\rangle \otimes \hat{N}_b|l\rangle = l|n, l\rangle$$

- One can note that acting on the ground state with $\hat{a}$ or $\hat{b}$ $= 0$

$$\hat{a}|0, 0\rangle = \hat{b}|0, 0\rangle = 0$$

- Expressing $|n\rangle$ as:

$$|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$$

and similarly for $|l\rangle$

$$|l\rangle = \frac{(\hat{b}^+)^l}{\sqrt{l!}}|0\rangle$$

- One can rewrite the state $|n, l\rangle$ as:

$$|n, l\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle \otimes \frac{(\hat{a}^+)^l}{\sqrt{l!}}|0\rangle = \frac{(\hat{a}^+)^n(\hat{b}^+)^l}{\sqrt{n!\,l!}}|0, 0\rangle$$

- Hence, the spectrum can be constructed with the operators $\hat{a}^+$ and $\hat{b}^+$