Ball & Beam: Controller Design in Simulation

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***Abstract*— A ball-and-beam is a classic nonlinear system for testing observer and control designs. We present an Extended Kalman Filter, tracking LQR, and approximate feedback linearization with LQR and discuss the performance following sinusoidal and square wave references. Our best performing implementation starts with the tracking LQR then switches to the feedback linearization when the error goes below a threshold (score = 0.83 for sinusoidal, 3.20 for square). The github repository link is `https://github.com/spencer-schutz/EE222-Nonlinear-Systems-Ball-and-Beam-Project`**

I. Problem Formulation

The goal of this project is to control a ball-and-beam system, making the ball follow desired sine and square wave reference trajectories. An illustration of the system is below.

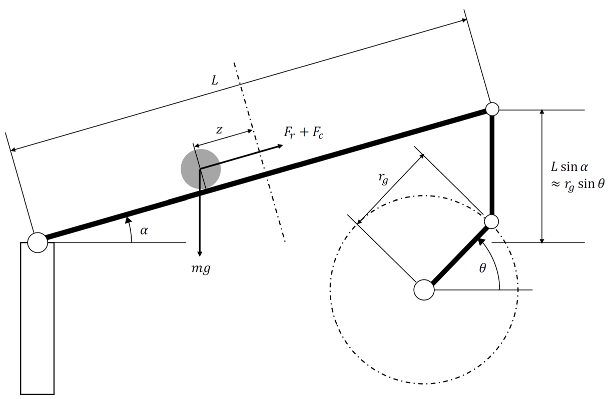

Fig. 1. Ball and beam setup

The system state is $x = [z, \dot{z}, \theta, \dot{\theta}]^T$, where z is the deviation of the ball from the center of the beam and θ is the angle of the arm of the servo motor with respect to the horizontal axis. The control input is u, the voltage applied to the motor. With this choice of x and u, the continuous-time nonlinear dynamics are given by:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{5g}{7}\frac{r_g}{L}\sin x_3 - \frac{5}{7}\left(\frac{L}{2} - x_1\right)\left(\frac{r_g}{L}\right)^2 x_4^2 \cos^2 x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{x_4}{\tau} + \frac{K}{\tau}u
\end{aligned} \tag{1}$$

These state equations are written generally as

$$\dot{x}(t) = f_c(x(t), u(t)). \tag{2}$$

For the Extended Kalman Filter in Section II and the tracking LQR controller in Section III, the dynamics in (2) will be manipulated further by introducing additive noises, discretizing, and linearizing.

Only measurements of the coordinate z and the angle θ are available:

$$\begin{aligned}
y[k] &= Hx[k] \\
H &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned} \tag{3}$$

The controller performance is scored on a cost function that sums tracking error, energy consumption, and safety constraint violation with respective weights w_t, w_e, and w_s:

$$J_{score} = w_t J_{tracking} + w_e J_{energy} + w_s J_{safety} \tag{4}$$

The goal is to design a state estimator and controller to achieve the lowest score. Section II describes our observer, Section III presents two different controllers, and Section IV discusses the results.

II. Observer

A. Extended Kalman Filter

We use an Extended Kalman Filter (EKF) for state estimation. The EKF is composed of a prediction step, propagating an internal state estimate through the dynamics equation, and a measurement update step, correcting the internal estimate based on incoming measurement [1], [2]. We denote with the p subscript the quantities involved in the prediction steps and with the m subscript the ones involved in the measurement update step. We discretize the nonlinear dynamics in (2) using Forward Euler with a timestep of T_s, and augment the model with a random process noise v with mean zero and covariance Σ_v. Furthermore, the measurement equation is augmented with a random measurement noise w with mean zero and covariance Σ_w.

$$\begin{aligned}
x[k] &= f_d(x[k-1], u[k-1], v[k-1]) \\
&= x[k-1] + T_s * f_c(x[k-1], u[k-1]) + v[k-1]
\end{aligned} \tag{5a}$$

$$y[k] = Hx[k] + w[k] \tag{5b}$$

The nonlinear discrete dynamics are linearized about the previous estimate $x_m[k-1]$, known $u[k-1]$, and zero variance as follows:

$$A[k-1] = \frac{\partial f_d}{\partial x}(x_m[k-1], u[k-1], 0) \tag{6}$$

The EKF is initialized with a state estimate $x_m[0]$ and a covariance matrix $P_m[0]$. The prediction step is then performed as follows:

$$\begin{aligned} x_p[k] &= f_d(x[k-1], u[k-1], 0) \\ P_p[k] &= A[k-1]P_m[k-1]A[k-1]^T + \Sigma_v. \end{aligned} \tag{7}$$

When a measurement $y[k]$ is received, the measurement update step is performed as follows:

$$\begin{aligned} K[k] &= P_p[k]H^T(HP_p[k]H^T + \Sigma_w)^{-1} \\ x_m[k] &= x_p[k] + K[k](y[k] - Hx_p[k]) \\ P_m[k] &= (I - K[k]H)P_p[k], \end{aligned} \tag{8}$$

where $K[k]$ is the Kalman Filter gain at step k.

III. Controllers

A. LQR

The first controller we present is a reference-tracking LQR [3]. The reference state x^* is constructed using the given reference for the desired ball position (p_{ref}) and ball velocity (v_{ref}). The reference input $u^* = 0$ for all steps.

$$x^*[k] = [p_{ref}[k], v_{ref}[k], 0, 0]^T \tag{9}$$

At each step, the discrete nonlinear dynamics in (5a) is linearized about the current reference (ignoring the process noise).

$$A[k] = \frac{\partial f_d}{\partial x}(x^*[k], u^*[k]) \quad B[k] = \frac{\partial f_d}{\partial u}(x^*[k], u^*[k]) \tag{10}$$

The controller chooses a time-varying state feedback gain $K[k]$ to minimize the following cost:

$$\begin{aligned} J = \sum_{k=0}^{\infty} &(x_m[k] - x^*[k])^T Q (x_m[k] - x^*[k])^T + \\ &(u[k] - u^*[k])^T R (u[k] - u^*[k]) \end{aligned} \tag{11}$$

The optimal cost is

$$J^* = (x_m[k]^T - x^*[k])P[k](x_m[k]^T - x^*[k]) \tag{12}$$

where $P[k]$ is the solution to the discrete time algebraic Ricatti equation (DARE) at step k ([k] dropped from right side for brevity):

$$P[k] = A^T PA + Q - A^T PB \left(R + B^T PB\right)^{-1} B^T PA \tag{13}$$

At each step, $P[k]$ is used to calculate the feedback gain K[k] ([k] dropped from right side for brevity),

$$K[k] = (R + B^T PB)^{-1} B^T PA \tag{14}$$

and the applied input is

$$u[k] = u^*[k] - K[k](x_m[k] - x^*[k]). \tag{15}$$

The linearized dynamics are controllable and observable, so a unique $P[k] \succ 0$ exists for all k and the controller is locally asymptotically stabilizing for the nonlinear system.

B. Approximate Feedback Linearization + LQR

The second controller we present combines approximate feedback linearization with Linear Quadratic Regulator (LQR). Feedback linearization works by applying a control input that cancels nonlinear terms and adds linear state feedback, making the closed-loop system linear [4]. In our approach, we utilize an approximate full state feedback linearization by treating certain terms as disturbances. Then, we design the feedback gains by using LQR to ensure asymptotic stability in the closed-loop system.

A system is fully feedback linearizable when its relative degree equals the order of the system. In our case, we view $-\frac{5}{7}(\frac{L}{2} - x_1)(\frac{r_g}{L})^2 x_4^2 \cos^2 x_3$ term in $\dot{x}_2$ in (1) as a disturbance to the system and ignore it to make the system fully feedback linearizable. We show the reason behind this choice in detail later. We define the output as $y = x_1$ and the tracking error as

$$e(t) := y(t) - y_d(t) \tag{16}$$

where $y_d(t)$ is the reference position trajectory. Then, we can write the system as

$$\begin{aligned} e^{(1)} &= y^{(1)} - y_d^{(1)} = x_2 - y_d^{(1)} \\ e^{(2)} &= y^{(2)} - y_d^{(2)} = a \sin x_3 - y_d^{(2)} \\ e^{(3)} &= y^{(3)} - y_d^{(3)} = a x_4 \cos x_3 - y_d^{(3)} \\ e^{(4)} &= y^{(4)} - y_d^{(4)} \\ &= -\frac{a}{\tau} x_4 \cos x_3 - a x_4^2 \sin x_3 + \frac{ak}{\tau} \cos x_3 u - y_d^{(4)} \end{aligned} \tag{17}$$

where the superscript denotes the order of time derivatives and constant $a = \frac{5g}{7}\frac{r_g}{L}$. Notice that $\frac{ak}{\tau}\cos x_3 \neq 0$ when $x_3 \neq \pm\frac{\pi}{2}$. Since x_3's safety bound is $[-\frac{\pi}{3}, \frac{\pi}{3}]$, we can conclude that $\frac{ak}{\tau}\cos x_3 \neq 0$. Therefore, the relative degree for (17) is 4, so the system is now fully feedback linearizable. With this, we define $e = \left[e, e^{(1)}, e^{(2)}, e^{(3)}\right]^{\intercal}$, $f(x) = -\frac{a}{\tau} x_4 \cos x_3 - a x_4^2 \sin x_3$, $g(x) = \frac{ak}{\tau}\cos x_3$, and the system becomes

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= e_4 \\ \dot{e}_4 &= f(x) - y_d^{(4)} + g(x)u \end{aligned} \tag{18}$$

If we include the previously ignored term $-\frac{5}{7}(\frac{L}{2} - x_1)(\frac{r_g}{L})^2 x_4^2 \cos^2 x_3$, we get

$$g(x) = -\frac{10}{7}\frac{k}{\tau}(\frac{L}{2} - x_1)(\frac{r_g}{L})^2 x_4 \cos^2 x_3 \tag{19}$$

It is hard to achieve $g(x) \neq 0$, since we would need to make x_4, the angular velocity, to be nonzero. Even if we treat $-\frac{5}{7}(\frac{L}{2} - x_1)(\frac{r_g}{L})^2 x_4^2 \cos^2 x_3$ as disturbance, our controller still results in satisfactory control performance.

We separate $u = u_{cancel} + u_{LQR}$ where

$$\begin{aligned} u_{cancel} &= g(x)^{-1}\left(-f(x) + y_d^{(4)}\right) \\ u_{LQR} &= -k_1 e_1 - k_2 e_2 - k_3 e_3 - k_4 e_4 \end{aligned} \tag{20}$$

TABLE I
COMPARING CONTROLLER PERFORMANCE

Reference	Controller	Tracking Cost	Energy Cost	Total Score
Sine Wave	LQR	0.71	0.16	0.87
	FL + LQR	0.73	0.14	0.86
	Switching at 2cm	0.67	0.16	**0.83**
Square Wave	LQR	2.69	0.51	**3.20**
	FL + LQR	2.86	0.38	3.24
	Switching at 1.5cm	2.89	0.50	3.40

Here, u_{cancel} cancels the nonlinear terms and u_{LQR} provides feedback control using LQR. After applying u_{cancel} to the system, the system can be shown in linear form:

$$\dot{e} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{LQR} \tag{21}$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} e$$

This system is controllable and observable, so LQR can be applied. We discretize A and B in (21) with zero-order hold (ZOH) and apply discrete LQR described in Section III-A.

Notice that when calculating e, we need up to fourth derivative of y_d. Given a second-order reference, we fit a polynomial to obtain the third and fourth derivatives. In practice, however, polynomial fitting may be unreliable or computationally expensive for certain trajectories. In such cases, we simply set the third and fourth derivative of y_d to be zero and demonstrate that this approximation has minimal impact on performance.

IV. RESULTS AND DISCUSSION

A. Simulation results

We test on two types of reference trajectories: a sine wave and a square wave. We compare between two controllers:

- **LQR**: controller we describe in III-A. We used gain values of $Q = diag(1000, 500, 1, 1)$ and $R = 0.1$.
- **FL + LQR**: controller we describe in III-B. We used gain values of $Q = diag(800, 50, 1, 1)$, $R = 0.3$.

We tested on sine waves and square waves as references. We used $Var[v[k]] = diag(0.05, 0.001, 0.0175, 0.00175)$, $Var[w[k]] = diag(0.05, 0.0175)$ for EKF. Table I summarizes the results. LQR and FL + LQR shows similar total score in both references. There is a tendency that LQR has a better tracking cost and FL + LQR has a better energy cost.

B. Discussion

Robustness to noise in estimated states

While tuning the gain values, we found out robustness property in the two controllers. If the gains are tuned wrongly in the observer, "chattering" might happen in the estimated states. Since LQR linearizes the system with the estimated states, this causes a big drop in performance. The total score goes up to 2.48 when chattering occurs. However, FL + LQR is robust in this cases, since it doesn't have a strong dependency in the estimated states. The scores for FL + LQR does have a drop of performance of about 0.02, but it is very small compared to LQR.

Switching

During experiments, we noticed that the proposed LQR controller has a more aggressive behavior during the transient response, while the FL + LQR controller has a lower steady-state tracking error. We experimented implementing a piecewise control law combining these two approaches through a switching mechanism. We heuristically selected a threshold for the tracking error $|z - z_{des}| = 2\,\mathrm{cm}$ below which we use FL, and above which we use LQR. With this modification, we managed to slightly decrease the score obtained to $\mathbf{0.83}$ for the sine wave and $\mathbf{3.40}$ for the square wave.

C. Future Work

Even though our controller shows good performance, we can have some improvements to make it better. LQR is a proportional controller, so it cannot drive the steady-state tracking error to zero. Introducing integral action could improve the tracking performance. For some tests, the LQR input (15) was augmented with a manually-tuned integral term:

$$u_{new}[k] = u[k] + K_e \sum_{i=k_0}^{k} (x_m[k] - x^*[k]) * T_s \tag{22}$$

where K_e and k_0 are tuning parameters. The delayed start of the integral term prevents windup during the initial period when the ball is far from the reference. With $K_e = [100, 10, 0, 0]$ and k_0 being the first step after $t > 3$, the total score was 0.83 for the sine wave and 3.98 for the square wave. Clearly, the tuning of this integral term is very sensitive to the experiment. While we can continue to try integral action on the hardware setup, it may not be a robust solution to improving the LQR performance.

Another improvement can be made with adding a disturbance observer with FL + LQR. FL + LQR shows some error in tracking the reference. This might be caused by posing disturbance terms and ignoring them. To overcome this, we can use an observer that also estimates the disturbance in the system. For example, an extended high gain observer [1] can be used to observe the disturbance as $\sigma_z = -\frac{5}{7}(\frac{L}{2} - x_1)(\frac{r_g}{L})^2 x_4^2 \cos^2 x_3$ and arbitrary σ_θ and cancel these disturbances in the control input u.

V. CONCLUSION

We presented an EKF, tracking LQR, and approximate feedback linearization with LQR for controlling the ball-and-beam. While the performance was improved by switching between controllers (best score = 0.83 for sinusoidal, 3.20 for square), these heuristic strategies will require careful tuning when deploying on the real system.

REFERENCES

[1] H. K. Khalil, "Extended high-gain observers as disturbance estimators," *SICE Journal of Control, Measurement, and System Integration*, vol. 10, no. 3, pp. 125–134, 2017.
[2] M. Mueller, "C231b lecture notes.".

[3] F. Borrelli, “C231a lecture notes.”.
[4] M. Arcak, “C222 lecture notes.”.