Neville's Algorithm

- Consider fitting a cubic to 4 points
- We want to interpolate at point x
- Start with simplest approximation: four zero order polynomials:

$$P_i = f_0(x_i) = y_i, \quad i = 0 \ldots 3$$

- Next highest order is linear interpolation:

$$\begin{aligned}
P_{01}(x) &= \frac{(x - x_1)P_0 + (x_0 - x)P_1}{x_0 - x_1} \\
P_{12}(x) &= \frac{(x - x_2)P_1 + (x_1 - x)P_2}{x_1 - x_2} \\
P_{23}(x) &= \frac{(x - x_3)P_2 + (x_2 - x)P_3}{x_2 - x_3}
\end{aligned}$$

- Next highest order is linear interpolation of the linear interpolants:

$$\begin{aligned}
P_{012}(x) &= \frac{(x - x_2)P_{01} + (x_0 - x)P_{12}}{x_0 - x_2} \\
P_{123}(x) &= \frac{(x - x_3)P_{12} + (x_1 - x)P_{23}}{x_1 - x_3}
\end{aligned}$$

- Next highest order is linear interpolation of the linear interpolation of the linear interpolants:

$$P_{0123}(x) = \frac{(x - x_3)P_{012} + (x_0 - x)P_{123}}{x_0 - x_3}$$

- But consider this in terms of corrections. At the top level:

$$\begin{aligned}
C_{1,0} &= P_{01} - P_0 = \frac{(x_0 - x)(P_1 - P_0)}{x_0 - x_1} \\
D_{1,0} &= P_{01} - P_1 = \frac{(x - x_1)(P_0 - P_1)}{x_0 - x_1}
\end{aligned}$$

- So in general:

$$
\begin{aligned}
C_{1,i} &= P_{(i)(i+1)} - P_i = \frac{(x_i - x)(P_{i+1} - P_i)}{x_i - x_{i+1}} \\
D_{1,i} &= P_{(i)(i+1)} - P_{i+1} = \frac{(x - x_{i+1})(P_i - P_{i+1})}{x_i - x_{i+1}}
\end{aligned}
$$

- At the next level

$$
\begin{aligned}
C_{2,0} &= P_{012} - P_{01} = \frac{(x_0 - x)(P_{12} - P_{01})}{x_0 - x_2} \\
D_{2,0} &= P_{012} - P_{12} = \frac{(x_2 - x)(P_{12} - P_{01})}{x_0 - x_2}
\end{aligned}
$$

- But these can be related to the previous level:

$$
\begin{aligned}
C_{2,0} &= \frac{(x_0 - x)(C_{1,1} - D_{1,0})}{x_0 - x_2} \\
D_{2,0} &= \frac{(x_2 - x)(C_{1,1} - D_{1,0})}{x_0 - x_2}
\end{aligned}
$$

- Or in general:

$$
\begin{aligned}
C_{m+1,i} &= \frac{(x_i - x)(C_{m,i+1} - D_{m,i})}{x_i - x_{i+m+1}} \\
D_{m+1,i} &= \frac{(x_{i+m+1} - x)(C_{m,i+1} - D_{m,i})}{x_i - x_{i+m+1}}
\end{aligned}
$$

- So the algorithm is:
  1. Order interpolation points in x.
  2. Set $P_i = y_i$
  3. Determine $C_{1,i}, D_{1,i}$ (linear interpolation)
  4. Check convergence
  5. If not adaquate construct P_{01}
  6. Construct $C_{m,i}, D_{m,i}$
  7. Check for convergence, if too large
  8. Construct $P_{01\ldots m}$; loop to 6.
  9. If OK, Stop and get error estimate.