ECE311 Introduction to Linear Control System

Lab 3:	Speed Control of a Simplified Car Model
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Video link:
https://drive.google.com/file/d/1oLU3A_vsSTMDWeuCdVdUMivvEjhBznAG/view?usp=sharing

Part 3: Asymptotic Tracking With Disturbance Rejection: Theoretical Analysis

The Internal Model Principle (IMP):

Suppose R(s) and D(s) are rational and strictly proper functions with poles in {Re(s) ≥ 0}, a controller C(s) solves the Basic Control Problem(BCP) if and only if:

1) C(s) makes CLS BIBO Stable (i.e. r(t) and d(t) bounded => e(t) and u(t) bounded)
    a) The roots of 1+C(s)G(s) are in OLHP.
    b) The product C(s)G(s) does not have pole-zero cancellation with real part ≥ 0.
2) Asymptotic tracking: The product of C(s)G(s) has the poles of R(s).
3) Disturbance rejection: C(s) has the poles of D(s).

Output 1.1

Appealing to the Internal Model Principle, show that in order to meet SPEC 1, C(s) must have a pole at zero.

SPEC 1: The output y(t) of the closed-loop system should asymptotically track reference signals of the form $r(t) = v_{des}\mathbf{1}(t)$ (where v_{des} is the desired speed in m/s), despite the presence of the unknown disturbance $d(t) = \bar{d}\mathbf{1}(t)$.

Answer:

Since we want $e(t) = r(t) - y(t) \to 0$ when both r(t) and d(t) are not 0, we need to meet both Asymptotic Tracking and Disturbance Rejection. This is because each of them assumes one of d(t) and r(t) is zero, so in order to meet SPEC 1, we need to superpose these two conditions.

1. First, let us look at Asymptotic Tracking:
    - To meet Asymptotic Tracking, we need to meet condition 2) in the above IMP. Therefore, C(s)G(s) must has the poles of $R(s) = \frac{v_{des}}{s}$, a pole at s=0.
    - Since $G(s) = \frac{1}{s+a}$ does not have poles at s=0, C(s) must have a pole at s=0 to meet condition 2).
2. Then, let us look at Disturbance Rejection:
    - We need to meet condition 3) in IMP, i.e. C(s) has the poles of D(s). Since $D(s) = \frac{\bar{d}}{s}$ has a pole at 0, then C(s) must also have a pole at s= 0.

Therefore, from both Asymptotic Tracking and Disturbance Rejection, C(s) need to have a pole at s=0.

Output 1.2
The PI controller has a pole at zero. Give a clear explanation of the Internal Model Principle, and clearly show that it implies that if the gang of four transfer functions are BIBO stable, then SPEC 1 and SPEC 2 are met.

SPEC 2: The closed-loop system (input R(s), output Y(s), and no disturbance, D(s) =0) should be BIBO stable.

Answer:
The Basic Control Problem (BCP) for a Standard Feedback Loop is to design a controller C(s) to make the system have all the three specifications of 1) stability (BIBO Stable), 2) asymptotic tracking, and 3) disturbance rejection. The Internal Model Principles are three sufficient and necessary conditions (listed on the previous page), that the system must meet to solve the BCP.

The system can also meet each of the three properties in the BCP individually. However, to meet Asymptotic Tracking and Disturbance Rejection individually, we must first make the closed-loop system BIBO stable. Otherwise, 2) and 3) may not hold.

Therefore, to meet SPEC 1, we need both C(s) to have a pole at zero and the CLS to be BIBO stable:

If C(s) has a pole at zero and the CLS is BIBO stable $\rightarrow$ SPEC1 is met

As the PI controller already has a pole at zero, we now only need the CLS to be BIBO stable. Moreover, the stability says for any bounded r(t) and d(t), u(t) and e(t) are bounded. Since E(s) and U(s) are computed by adding up the multiplication between the gang of four TFs with R(s) and D(s), we say the CLS is BIBO stable if the gang of four TFs are BIBO stable (from lecture notes). This implies that:

If the gang of four TFs are BIBO stable $\rightarrow$ The CLS is BIBO stable

So SPEC 2 is met (the disturbance D(s) = 0 can be seen as a bounded signal). Putting them all together, we have:

If the gang of four TFs are BIBO stable $\rightarrow$ SPEC1 is met

Therefore, if the gang of four transfer functions are BIBO stable, then SPEC 1 and SPEC 2 are met.

Output 1.3

A necessary and sufficient condition for the gang of four transfer functions to be BIBO stable is that:

(a) all poles of the transfer function 1/(1+CG) have the negative real part, and
(b) the product C(s)G(s) has no pole-zero cancellations in the closed right-half plane.

Show that for any $K, T_I > 0$ the PI controller above satisfies conditions (a) and (b). Present clear and concise arguments.

Answer:

(a) All poles of the transfer function 1/(1+CG) have the negative real part.

Since our C(s) and G(s) are:

$$C(s) = K\frac{T_I s + 1}{T_I s} \qquad G(s) = \frac{1}{s+a}$$

Therefore,

$$1 + C(s)G(s) = 1 + \frac{K(T_I s + 1)}{T_I s(s+a)} = 0$$
$$T_I s(s+a) + K(T_I s + 1) = 0$$
$$T_I s^2 + aT_I s + KT_I s + K = 0$$
$$s^2 + (a+k)s + \frac{K}{T_I} = 0$$

Since $a + k > 0, \frac{K}{T_I} > 0$, roots are in OLHP (negative real part).

(b) The product C(s)G(s) has no pole-zero cancellations in the closed right-half plane.

$$C(s)G(s) = K\frac{T_I s + 1}{T_I s} \cdot \frac{1}{s+a}$$

We found that the only possibility of cancellation is cancelling $(T_I s + 1)$ with $(s+a)$.

However, since both $a = \frac{B}{M} > 0, T_I > 0$, and:

$$(1)\ T_I s + 1 = 0 \rightarrow s = -\frac{1}{T_I} < 0$$
$$(2)\ s + a = 0 \rightarrow s = -a < 0$$

If there is a cancellation, the cancellation must happen in OLHP.

Output 1.4
Now we need pick $K, T_I > 0$ to meet SPECS 3, 4 and 5. We begin with SPEC 3. Find the transfer function T(s) = Y(s) / R(s) (assuming D(s) = 0). In order for this transfer function to have negative real poles at $-p_1, -p_2$, its denominator must have the forms $s^2 + (p_1 + p_2)s + p_1 p_2$. By equating coefficients, find the unique values of $K, T_I > 0$ such that the poles of T(s) are $-p_1, -p_2$. Present a clear derivation of the relationship between (K, T_I) and (p_1, p_2).

SPEC 3: All poles of the above closed-loop system should lie on the real axis so that the output y(t) does not have oscillatory behaviour.

Answer:
From the gang of four TFs:

$$E(s) = R(s) - Y(s) = \frac{1}{1+CG}R(s) + 0(D(s) = 0)$$

$$\begin{aligned} Y(s) &= (1 - \frac{1}{1+CG})R(s) \\ &= \frac{CG}{1+CG}R(s) \end{aligned}$$

Therefore,

$$\begin{aligned} T(s) = Y(s)/R(s) &= \frac{CG}{1+CG} \\ &= \frac{K\frac{T_I s+1}{T_I s} \cdot \frac{1}{s+a}}{1 + K\frac{T_I s+1}{T_I s} \cdot \frac{1}{s+a}} \\ &= \frac{K(T_I s + 1)}{T_I s(s+a) + K(T_I s + 1)} \\ &= \frac{\frac{K}{T_I}(T_I s + 1)}{s^2 + (a+k)s + \frac{K}{T_I}} \end{aligned}$$

By equating coefficients:

$$s^2 + (a+k)s + \frac{K}{T_I} = s^2 + (p_1 + p_2)s + p_1 p_2$$

$$\Rightarrow \begin{cases} p_1 + p_2 = a + K \\ p_1 p_2 = \frac{K}{T_I} \end{cases}$$

$$\Rightarrow \begin{cases} K = p_1 + p_2 - a \\ T_I = \frac{p_1 + p_2 - a}{p_1 p_2} \end{cases}$$

Output 1.5

Now we investigate SPEC 5, The car is initialized at zero speed (because we are modelling it as a transfer function), and is asked to accelerate and reach a speed v_{des} . Is it natural to assume that the largest acceleration will occur at t=0. Assuming that this is the case (you will verify this assumption later via simulation), we need to guarantee that $|u(t)| < 30m/s^2$. For this we use the following result.

> Initial Value Theorem (IVT): Suppose that F : C -> C is a rational and strictly proper function, and denotes $f(t) = L^{-1}(F(s))$. Then $f(0) = \lim_{s\to\infty} sF(s)$.

Find the transfer function U(s)/R(s), and setting R(s) = vdes/s = 14/s, compute U(s). Then, using the IVT, compute u(0) as a function of K and TI. In turn, K and TI depend on p1 and p2. Find the constraint that p1 and p2 must satisfy in order that |u(0)| < 30. Write out this constraint neatly. It should be an upper bound on the sum p1 + p2. This upper bound should depend on the constant a.

Answer:

$$\begin{aligned} Y(s) &= \frac{CG}{1+CG}R(s)E(s) = R(s) - Y(s) \\ &= (1 - \frac{CG}{1+CG})R(s) = \frac{1}{1+CG}R(s) \end{aligned}$$

$$\begin{aligned} \frac{U(s)}{R(s)} &= \frac{E(s)C(s)}{R(s)} \\ &= \frac{\frac{K(T_I S+1)}{T_I S}}{1+\frac{K(T_I S+1)}{T_I S}\frac{1}{s+a}} \\ &= \frac{K(S+a)(T_I S+1)}{T_I S(S+a)+K(T_I S+1)} \\ &= \frac{K}{T_I}\frac{T_I S^2+(1+aT_I)s+a}{s^2+(a+K)s+K} \\ &= \frac{Ks^2}{s^2+(a+K)s+K} + \frac{K}{T_I}\frac{(1+aT_I)s}{s^2+(a+K)s+K} + \frac{K}{T_I}\frac{a}{s^2+(a+K)s+K} \end{aligned}$$

The above expression is in the form of the sum of three components. We use A, B, C to denote the three terms.

When R(s) = 14 / s, U(s) = 14 / s * (A + B + C)

$$\begin{aligned} u(0) &= \lim_{s\to\infty} sU(s) \\ &= 14 \lim_{s\to\infty} A+B+C \\ &= 14(\lim_{s\to\infty} A + \lim_{s\to\infty} B + \lim_{s\to\infty} C) \end{aligned}$$

$$\lim_{s\to\infty} A = \lim_{s\to\infty} \frac{2Ks}{2s+a+k} = \lim_{s\to\infty} \frac{2K}{2} = K$$
$$\lim_{s\to\infty} B = \frac{K}{T_I} \lim_{s\to\infty} \frac{1+aT_I}{2s+a+K} = \frac{K}{T_I} \lim_{s\to\infty} \frac{0}{2} = 0$$
$$\lim_{s\to\infty} C = \lim_{s\to\infty} \frac{0}{2S+a+K} = 0$$

$u(0) = 14K = 14(p_1 + p_2 - a)$.
Because $k > 0$, $0 < u(0) = 14(p_1 + p_2 - a) < 30$
Therefore, $p_1 + p_2 < 30 / 14 + a = 15/7 + a$

$p1 + p2 < \frac{15}{7} + a$ is the constraint for p_1 and p_2 to satisfy SPEC 5.

Part 4: Asymptotic Tracking With Disturbance Rejection: Simulation

Output 2.1 Print your initial choices of p1, p2, and the corresponding values of K and TI . Produce three plots corresponding to the three scopes in the Simulink diagram, and estimate the settling time based on the system response over the first 15 seconds. Then, estimate the settling time after the disturbance kicks in at t = 15 s. Your plots should have titles describing their content.

1. The initial choices of p1 and p2:

```
>> p1

p1 =

     1
```

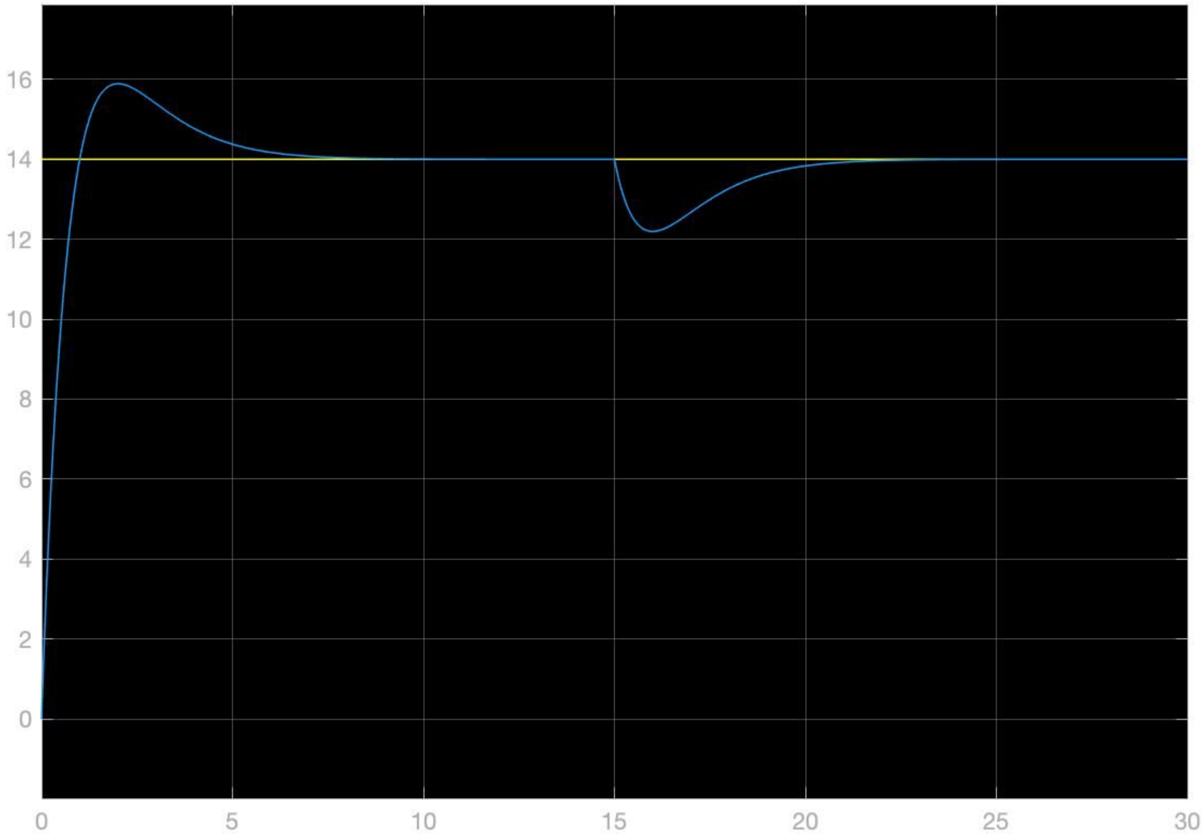
```
>> p2

p2 =

     1
```

2. The corresponding values of K and TI:

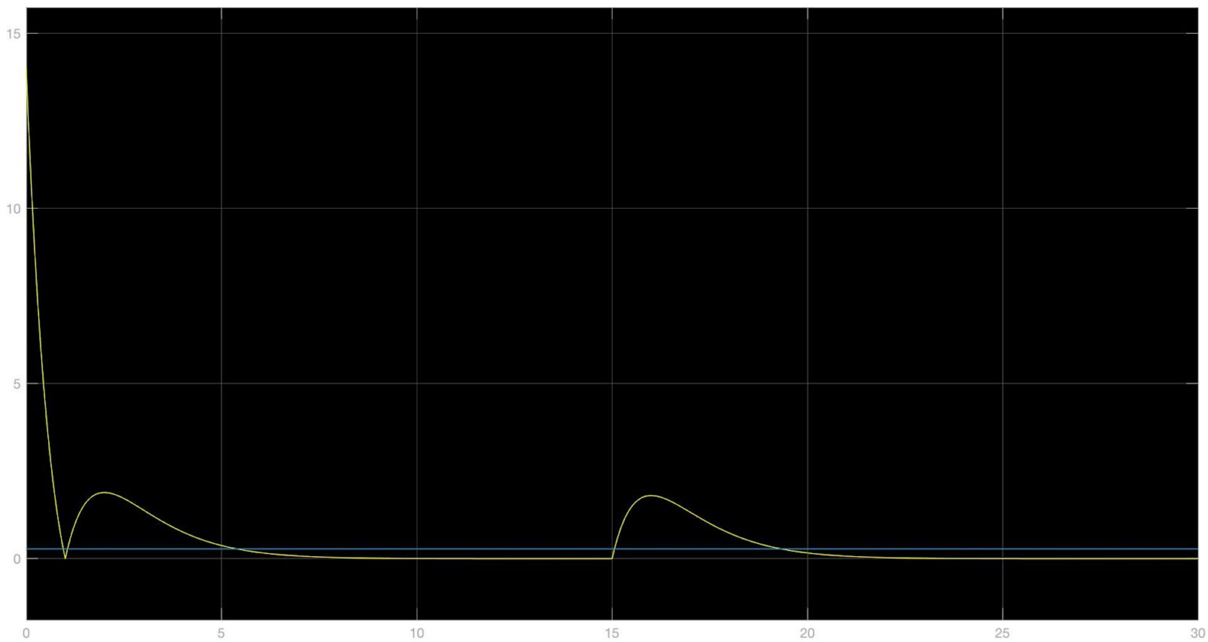
```
>> K

K =

    1.9990
```

```
>> TI

TI =

    1.9990
```

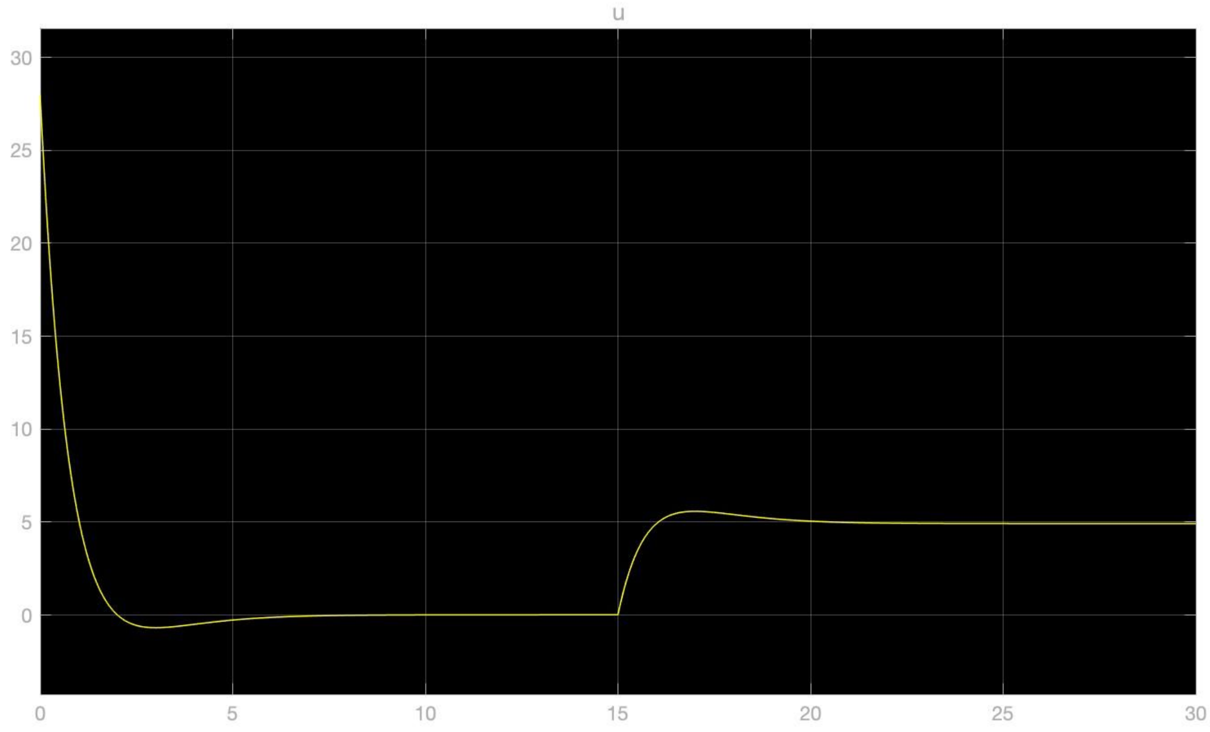
3. The plot corresponding to the output and reference signal scope:



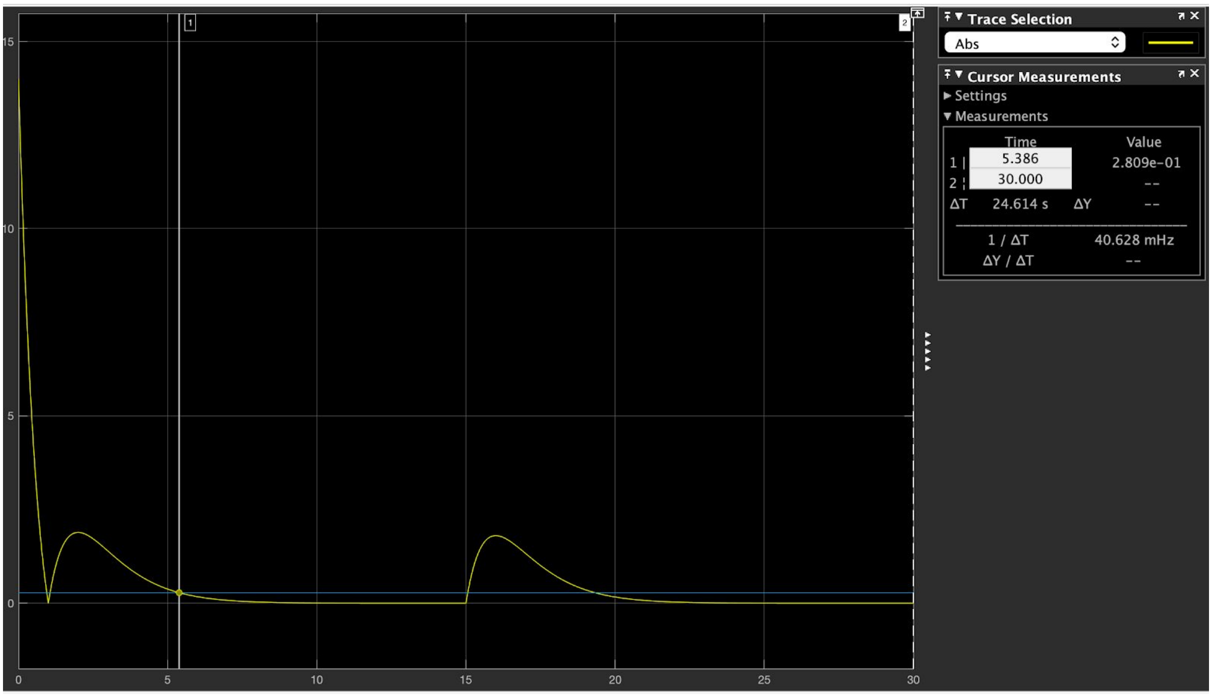
4. The plot corresponding to the tracking error scope:



5. The plot corresponding to the control signal scope u:

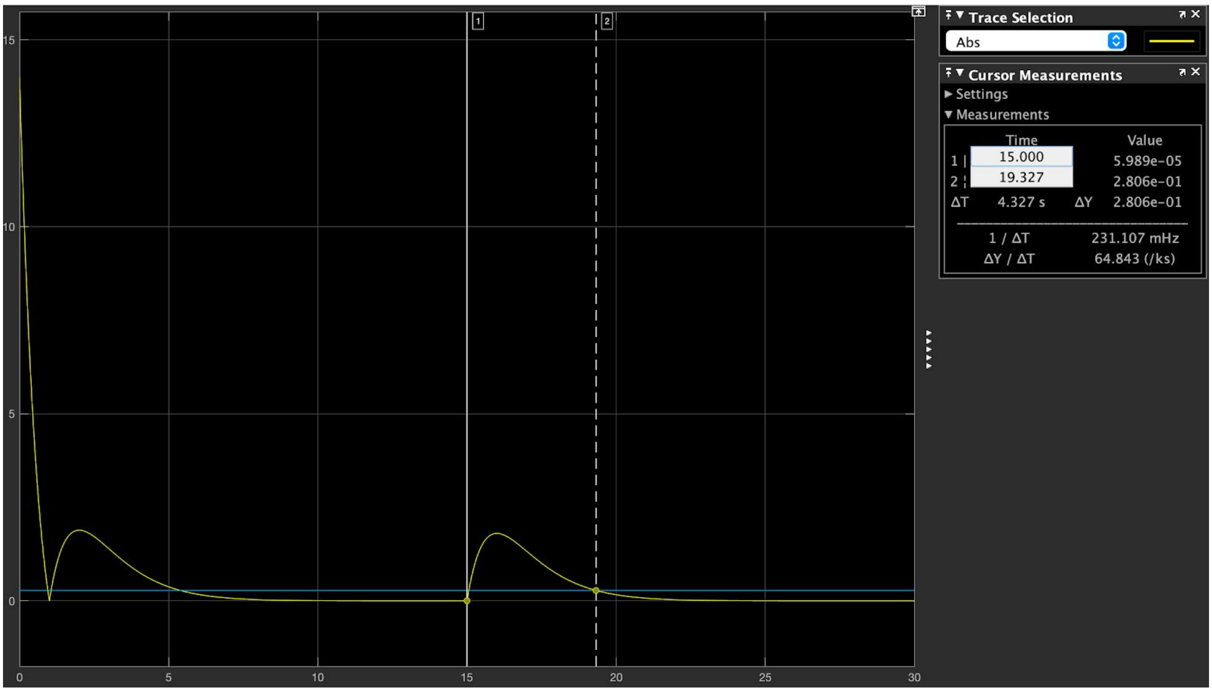


6. Estimate the settling time based on the system response over the first 15 seconds:



The settling time over the first 15 seconds is measured/estimated to be 5.386 sec.

7. Estimate the settling time after the disturbance kicks in at t = 15s:



The settling time after the disturbance kicks in at t = 15 seconds is measured/estimated to be 19.327 - 15.000 = 4.327 sec.

Output 2.2 Comment on your initial findings before tuning. Are all the five specs met? Which ones aren't? Guide the reader of your report through the verification of which specs are met and which ones aren't. Explain the tuning you plan to do to improve the performance and show your reasoning behind it.

According to Output 1, all five SPECs are met:

1. In order to meet SPEC 1, C(s) must have a pole at zero.

The following equation is the function of C when p1 = p2 = 1 and the controller C has a pole at zero. So, SPEC 1 is met.

```
>> C

C =

  3.996 s + 1.999
  ---------------
      1.999 s

Continuous-time transfer function.
```

```
>> pole

pole =

     0
```

2. In order to meet SPEC 1 and SPEC 2, the gang of four transfer functions needs to be BIBO stable. And there is two necessary and sufficient condition for the four transfer functions to be BIBO stable is provided below:

a) all poles of the transfer function gang=1/(1+CG) have the negative real part

```
gang =

     1.999 s^2 + 0.001999 s
  ---------------------------
  1.999 s^2 + 3.998 s + 1.999

Continuous-time transfer function.


pole =

   -1.0000
   -1.0000
```

As we can see, all the poles of the transfer function 1/(1+CG) have the negative real part at -1.

b) the product C(s)*G(s) has no pole-zero cancellations in the closed right-half plane.

```
>> C

C =

  3.996 s + 1.999
  ---------------
      1.999 s

Continuous-time transfer function.
```

```
>> G

G =

      1
  ---------
  s + 0.001

Continuous-time transfer function.
```

As we can see, there is no cancellation of pole and zero existing based on the two functions.

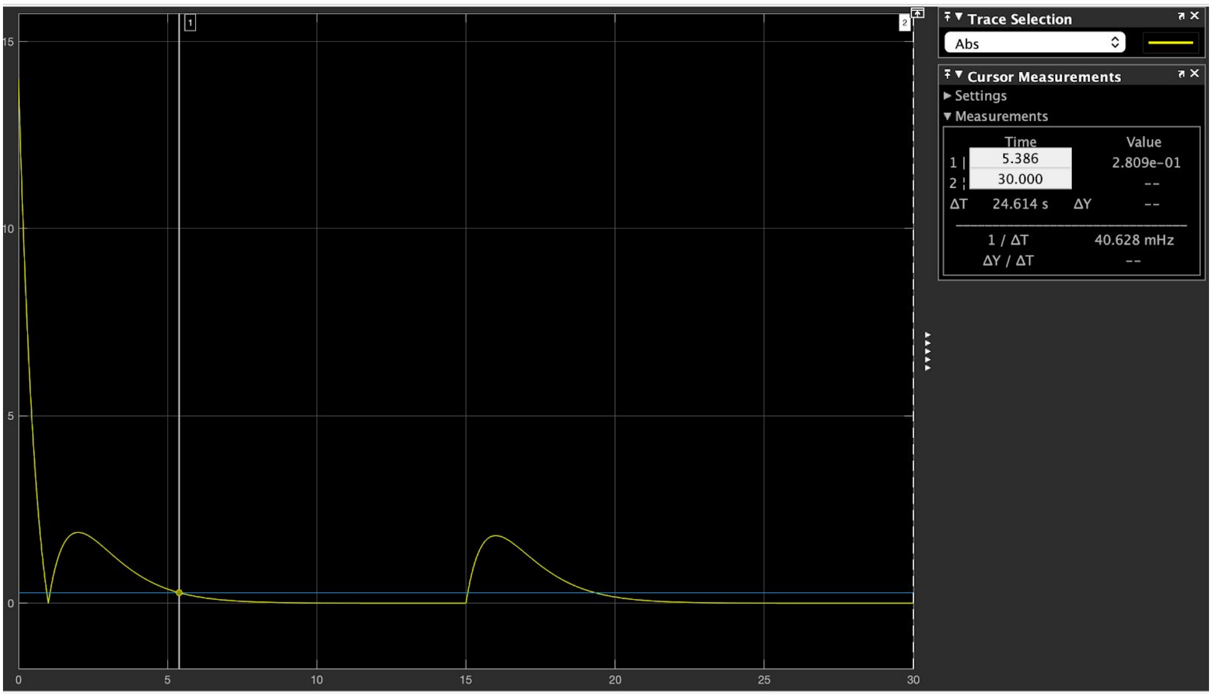
Based on a) and b), the CLS is BIBO stable which met SPEC 1 and SPEC 2.

3. To meet SPEC 3, all poles of the above closed-loop system should lie on the real axis, so that the output y(t) does not have oscillatory behaviour.

The poles of the above closed-loop system (T(s)) are all on the negative real axis, and R(s) has a pole at zero. So, Y(s) = T(s)R(s) have all poles lying on the real axis with {Re(S) ≤ 0}, so it will not have oscillatory behaviour, which met SPEC 3.

```
>> pole

pole =

   -1.0000
   -1.0000
```

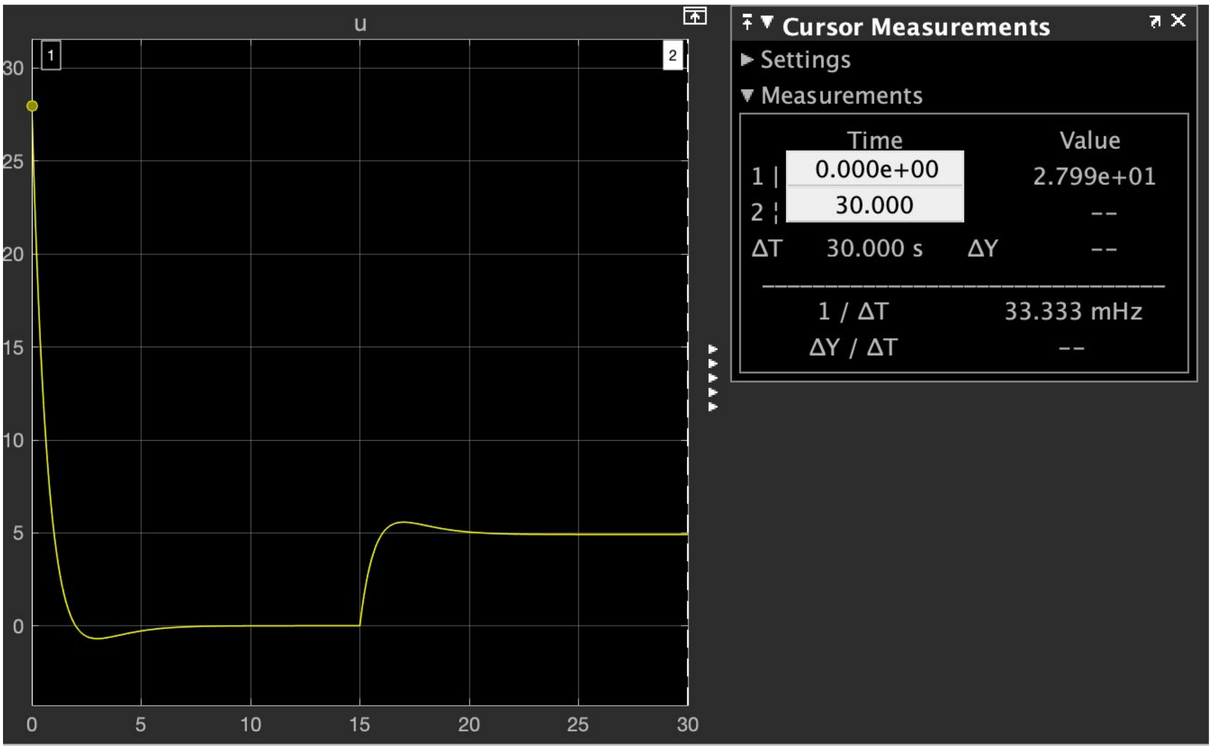
4. To meet SPEC 4, when D(s) = 0, the settling time Ts should be less than 6 seconds.



According to output2.1, the estimation/measurement of the settling time within the tracking error scope over the first 15 seconds (when D(s) = 0) is 5.386 sec, which is less than 6 seconds and met SPEC 4.

5. In order to have the control input signal u(t) not exceed the bound $|u(t)| < 30m/s^2$, when the desired reference speed is $v_{des} = 14m/s$ and $D(s) = 0$, the p1 and p2 values should satisfied the range $p1 + p2 \ < \frac{15}{7} + a, \ p1 > 0, \ p2 > 0$ to meet SPEC 5.

According to the diagram, $|u(t)| < 30m/s^2$ is satisfied with maximum value at t=0, with value 27.99 m/s^2 .



a = B/M = 0.001
p1 = 1
p2 = 1
$p1 + p2 \ = \ 2 \ < \ \frac{15}{7} + 0.001$
So, $p1 + p2 \ < \frac{15}{7} + a, \ p1 > 0, \ p2 > 0$ is satisfied and met SPEC 5.

6. Explain the tuning you plan to do to improve the performance and show your reasoning behind it.

In order to improve the performance and reduce the settling time, we have to let the e(t) converge faster by tuning the p1 and p2 parameters.

$$
\begin{aligned}
E(s) &= \frac{1}{1+C(s)G(s)}R(s) + \frac{G(s)}{1+C(s)G(s)}D(s) \\
&= \frac{1}{1+\frac{KT_Is+K}{T_Is}\cdot\frac{1}{s+a}}\cdot\frac{1}{s} \\
&= \frac{T_Is(s+a)}{T_Is(s+a)+K(T_Is+1)}\cdot\frac{1}{s} \\
&= \frac{T_Is^2+T_Ias}{T_Is^2+(T_Ia+KT_I)s+K}\cdot\frac{1}{s} \\
&= \frac{s^2+as}{s^2+(a+K)s+\frac{K}{T_I}}\cdot\frac{1}{s} \\
&= \frac{s+a}{s^2+(a+K)s+\frac{K}{T_I}}
\end{aligned}
$$

with the equating denominator:

$$s^2+(a+K)s+\frac{K}{T_I}$$

with $a+k = p_1 + p_2$ and $\frac{K}{T_I} = p_1 \times p_2$

The poles of E(s) are the roots of the polynomial $s^2 + (a+k)s \ + \frac{K}{T_I}$. By choosing K, a, TI > 0, the poles are in the OLHP. Thus, we can make e(t) converge to zero arbitrarily fast by making the real parts of the poles be at far left in the OLHP. By doing so, we increased both of the p values to let it to be in the far left half plane, while it still satisfying with the p1+p2 value satisfied the constraint in SPEC 5 $p1 + p2 \ < \frac{15}{7} + a$.

Since $a = \frac{B}{M} = 0.001$ from the lab, so the desired p1 and p2 values are be chosen to $p_{1,} p_2 < \frac{1}{2}(\frac{15}{7} + 0.001)$, so $p_1 = p_2 = 1.071$.

Output 2.3 After tuning, print the values of p1, p2 minimizing the settling time while respecting the bound on u(t). Repeat again your analysis verifying whether all the specs are met. What is the best settling time you could get over the time interval 0 to 15 seconds? What is the settling time over the time interval of 15 to 30 seconds? Print the settling time you found using "stepinfo" and verify that it's the same as the one over the time interval 0 to 15 seconds that you deduced from the plots.

1. After tuning, print the values of p1, p2 minimizing the settling time while respecting the bound on u(t).

```
>> p1

p1 =

    1.0710
```

```
>> p2

p2 =

    1.0710
```

2. Repeat again your analysis verifying whether all the specs are met :

2.1 In order to meet SPEC 1, C(s) must have a pole at zero.
The following equation is the function of C when p1 = 1.071 and p2 = 1.071 and the controller C has a pole at zero.

```
>> C

C =

  3.996 s + 2.141
  ---------------
      1.867 s

Continuous-time transfer function.
```

```
>> pole

pole =

     0
```

2.2 In order to meet SPEC 1 and SPEC 2, the gang of four transfer functions needs to be BIBO stable. And there is two necessary and sufficient condition for the four transfer functions to be BIBO stable is provided below:

a) all poles of the transfer function gang=1/(1+CG) have the negative real part

```
>> gang

gang =

    1.867 s^2 + 0.001867 s
  --------------------------
  1.867 s^2 + 3.998 s + 2.141

Continuous-time transfer function.

>> pole1

pole1 =

   -1.0710
   -1.0710
```

As we can see, all the poles of the transfer function 1/(1+CG) have the negative real part at -1.071 and -1.071.

b) the product C(s)*G(s) has no pole-zero cancellations in the closed right-half plane.

```
>> C

C =

  3.996 s + 2.141
  ---------------
      1.867 s

Continuous-time transfer function.
```

```
>> G

G =

      1
  ---------
  s + 0.001

Continuous-time transfer function.
```

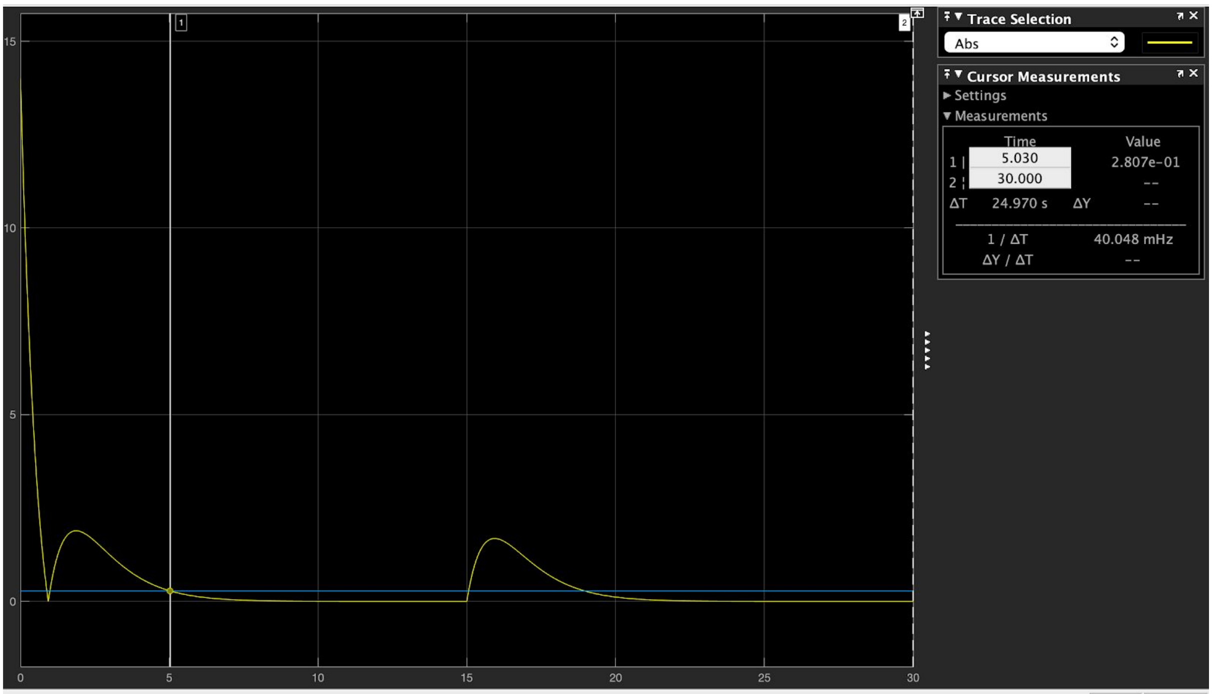
As we can see, there is no cancellation of pole and zero existing based on the two functions.

Based on a) and b), the CLS is BIBO stable which met SPEC 1 and SPEC 2.

2.3 To meet SPEC 3, all poles of the above closed-loop system should lie on the real axis, so that the output y(t) does not have oscillatory behaviour.
The poles of the above closed-loop system (T(s)) are all on the negative real axis, and R(s) has a pole at zero. So, Y(s) = T(s)R(s) have all poles lying on the real axis with {Re(S) ≤ 0}, so it will not have oscillatory behaviour, which met SPEC 3.

```
>> pole

pole =

   -1.0710
   -1.0710
```

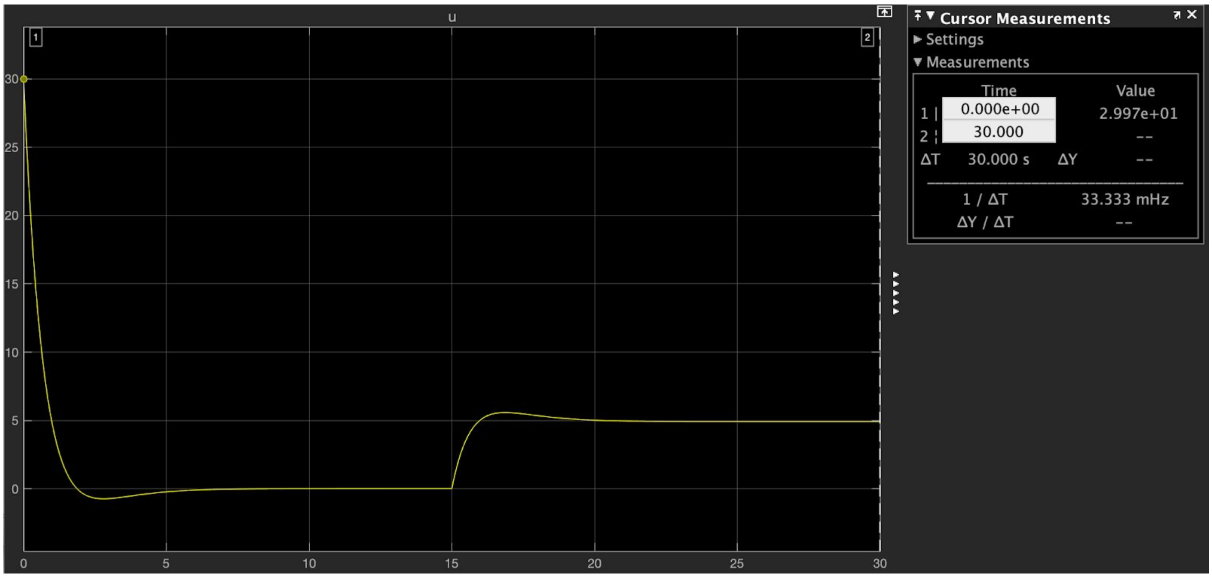
2.4 To meet SPEC 4, when D(s) = 0, the settling time Ts should be less than 6 seconds.



The estimation/measurement of the settling time within the tracking error scope over the first 15 seconds (when D(s) = 0) is 5.030 sec, which is less than 6 seconds.

2.5 In order to have the control input signal u(t) not exceed the bound $|u(t)| < 30m/s^2$, when the desired reference speed is $v_{des} = 14m/s$ and $D(s) = 0$, the p1 and p2 values should satisfied the range $p1 + p2 \ < \ \frac{15}{7} + a, \ p1 > 0, \ p2 > 0$ to meet SPEC 5.

According to the diagram, $|u(t)| < 30m/s^2$ is satisfied with maximum value at t=0, with value 29.97 m/s^2 .



a = B/M = 0.001

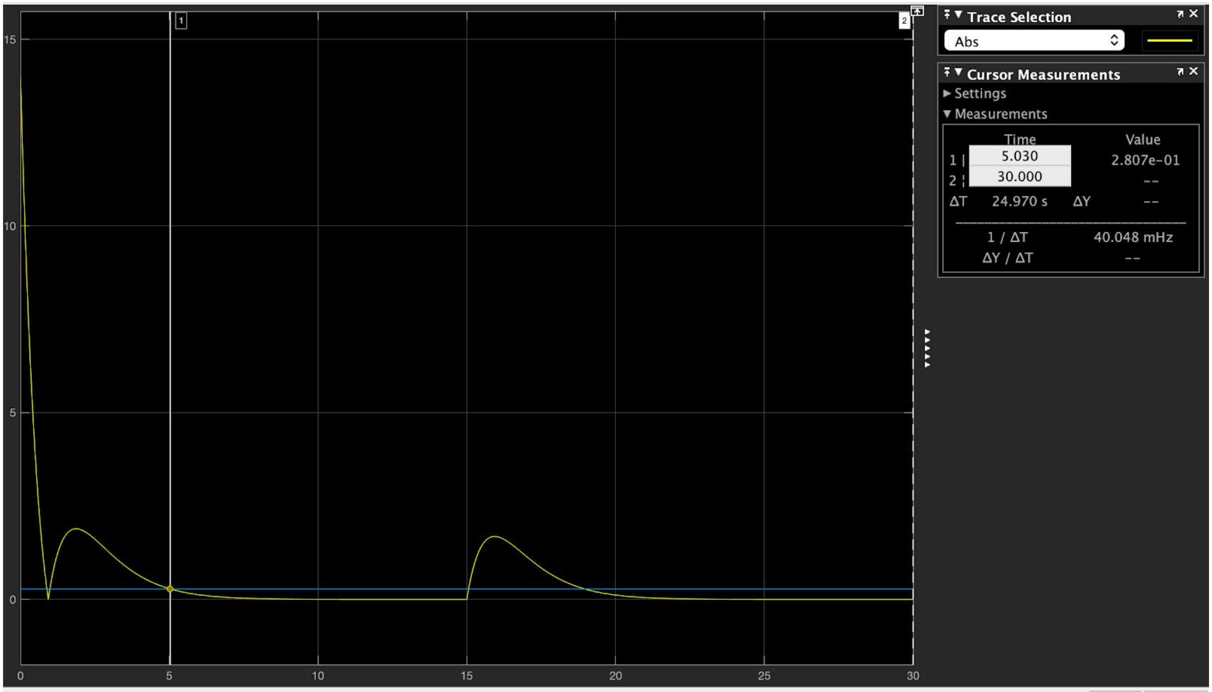
p1 = 1.071

p2 = 1.071

$p1 + p2 \ = \ 2.142 \ < \ \frac{15}{7} + 0.001$

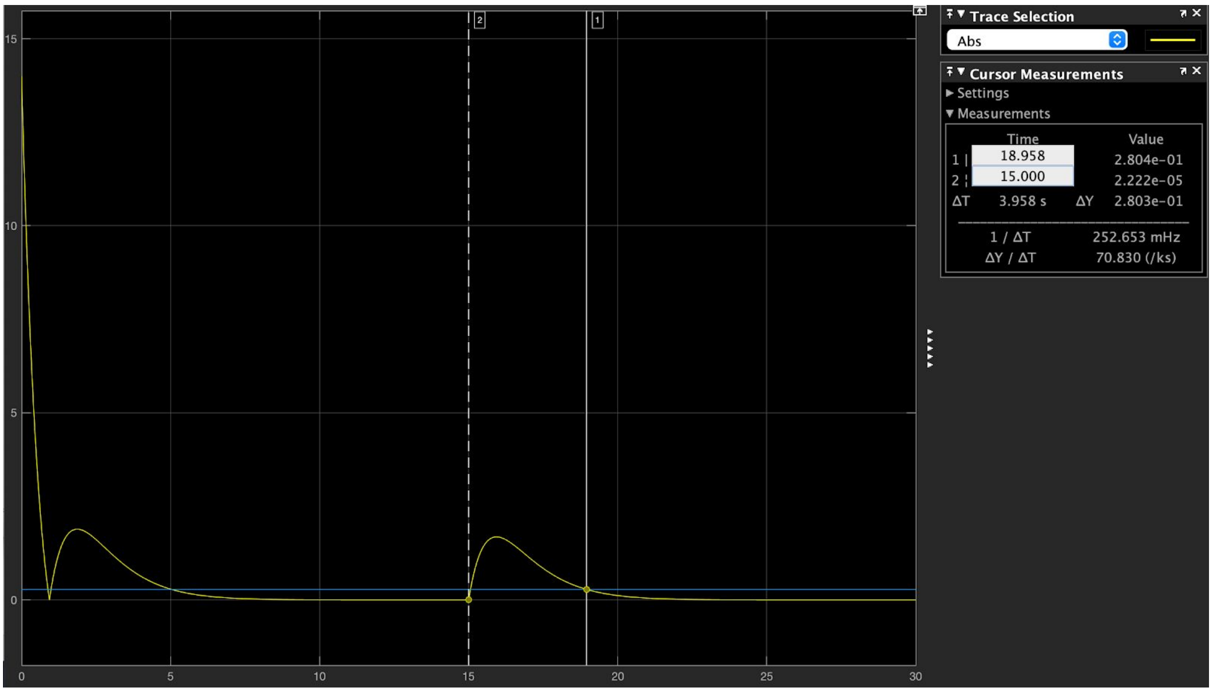
So, $p1 + p2 < \frac{15}{7} + a,\ p1 > 0,\ p2 > 0$ is satisfied and met SPEC 5.

3. What is the best settling time you could get over the time interval 0 to 15 seconds?



The estimation/measurement of the settling time within the tracking error scope over the first 15 seconds (when D(s) = 0) is 5.030 sec, which is the fastest settling time with best performance that the team tied by tuning p1 and p2 values.

4. What is the settling time over the time interval of 15 to 30 seconds?



The settling time after the disturbance kicks in at t = 15 seconds is measured/estimated to be 18.958 - 15.000 = 3.958 sec.

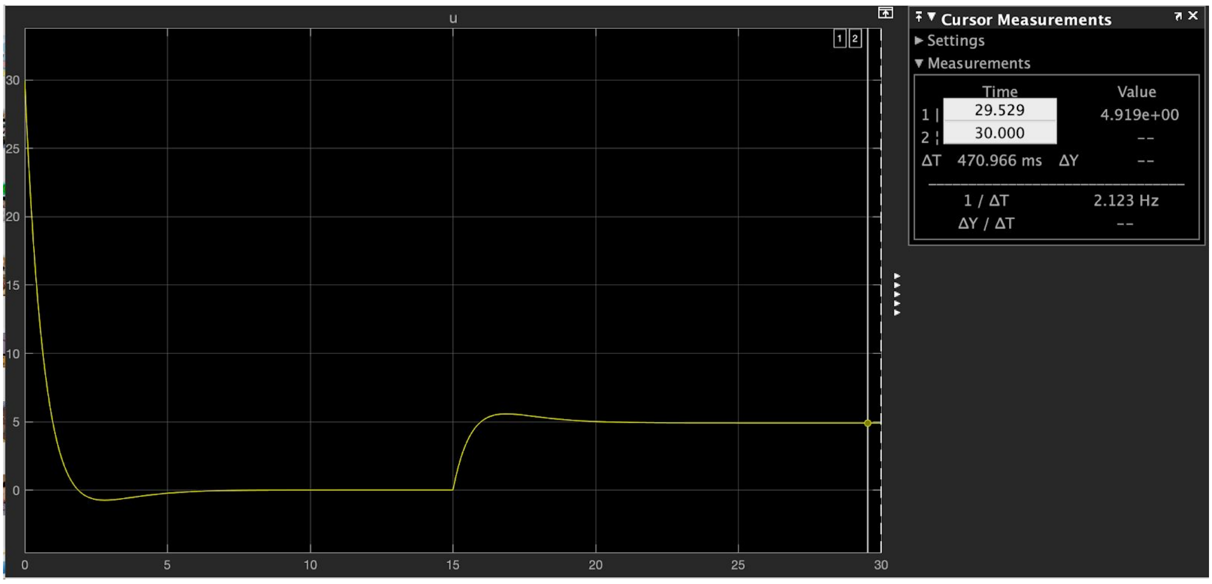
5. Print the settling time you found using “stepinfo” and verify that it’s the same as the one over the time interval 0 to 15 seconds that you deduced from the plots.

```
        RiseTime: 0.6817
    SettlingTime: 5.0329
     SettlingMin: 0.9057
     SettlingMax: 1.1351
       Overshoot: 13.5082
      Undershoot: 0
            Peak: 1.1351
        PeakTime: 1.8700
```

The settling time that we computed by the “stepinfo” function is 5.0329s. The approximated time we measured from the graph is 5.030s (measured using cursor), which is approximately the same.

Output 2.4 You will notice that, after the disturbance signal is enabled, the control signal u(t) will settle to a nonzero value. Find this value, and explain how it conforms with the physics of the problem (a mass on an inclined plane subject to gravity).

Since the mass is on an inclined plane, by analyzing the forces existing on the mass, there is a component of the gravity is along the inclined plane pointing downwards. So in order for it to be moving at constant speed after the disturbance signal is enabled, the control signal u(t) has to settle to a nonzero value and work against that disturbance signal.



The u(t) value is approximately to be 4.919 m/s^2 by measuring using the cursor from the diagram.