Micro to Macro Equation-Free Bifurcation Analysis of Neuronal Random Graphs: Symmetry Breaking of Majority Rule Dynamics

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We address how the Equation-Free approach can be exploited to bridge in a computational rigorous way the micro to macro scales of the dynamics of stochastic individualistic neuronal models evolving on complex random graphs. In particular, we show how bifurcation analysis can be performed bypassing the need to extract macroscopic models in a closed form. The analysis targets on the majority rule model developing on Regular Random (RRN), Erdős–Rényi, and Watts-Strogatz (small-world) networks. We construct the coarse-grained bifurcation diagrams with respect to the switching probability and we show how the connectivity distribution may result to symmetry breaking of the underlying macroscopic dynamics.

1. Introduction

Symmetry breaking of majority rule dynamics has been associated to phenomena such as herd behaviour under panic, the emergence of cooperation dynamics and public opinion formation. For individualistic/ stochastic models whose dynamics evolve on complex networks, the extraction of closed coarse-grained models in the form of ordinary (ODEs) and/ or partial-integro-differential (PIDEs) equations is not an easy task. Due to the stochastic, nonlinear nature, multi-scale character and complexity of the network-deployed interactions, such equations are simply not available, or overwhelming difficult to derive. Without the existence of such models, what is usually done for analysis purposes is simple brute-force simulations: starting from different initial conditions, run in time and average over many ensembles to get the required statistics. Even if we try to exploit the tools of Statistical Physics in order to derive some closures, these are just approximations that may introduce biases in the modelling and therefore in the analysis of the actual emergent dynamics.

This imposes a major impediment in our ability to analyse in a rigorous way the system's behaviour. In order to analyse the way the network topology influences the emergent dynamics we exploit the Equation-Free framework (Kevrekidis et al., 2003) bypassing the construction of explicit coarse-grained models. In particular, we construct the coarse-grained bifurcation diagrams of the basic majority rule model, for Regular Random, Erdős–Rényi, and Watts-Strogatz (small-world) networks (Newman, 2003) with respect to the switching probability and analyze the stability of the computed stationary solutions.

2. The Majority Rule Model

Each neuron is labeled as i ($i = 1,2,...,N$), and its state gets two values: the value "1" if is it activated and the value "0" if it is not. We describe the state of the $i-th$ neuron in time t with the function $a_i(t) \in \{0,1\}$. Let $\Lambda(i)$ be the set of the neighbors (i.e. the neurons connected to $i-th$ neuron, with self loop included). Also consider the summation

$$\sigma_i(t) = \sum_{j \in \Lambda(i)} a_i(t)$$

which gives the number of activated neighbors of the i-th neuron. At each time step each neuron interacts with its neighboring neurons, and changes its state-value according to the following stochastic way (Kozma et al., 2005):

1. An inactive neuron becomes activated with probability ε, if $\sigma_i(t) \le \frac{k_i}{2}$ (k_i is the degree of the $i-th$ neuron). If $\sigma_i(t) > \frac{k_i}{2}$ the neuron becomes activated with probability $1-\varepsilon$.
2. An activated neuron becomes inactive with probability ε, if the $\sigma_i(t) > \frac{k_i}{2}$. If $\sigma_i(t) \le \frac{k_i}{2}$ the neuron becomes inactivate with probability $1-\varepsilon$.

ε takes values in the interval $(0, 0.5)$.

3. The Equation-Free Approach

The Equation-free approach can be used to bypass the need for extracting explicit continuum models in closed form (Makeev et al., 2002; Gear et al., 2002; Kevrekidis et al., 2003; Siettos et al., 2003). The main assumption of the framework is that macroscopic models in principle exist and close in terms of a few coarse-grained variables, which are usually the first moments of the underlying microscopic distributions; all the other higher-order moments become very-fast in the macroscopic time, functionals of the lower-order ones. What the methodology does, is to provide these closures "on demand" in a strict computational manner. A caricature of the method is described in the following steps:

(a) Choose the coarse-grained statistics, say $\boldsymbol{x}$, for describing the emergent behavior of the system and an appropriate representation for them (for example the mean value of the underlying evolving distribution).

(b) Choose an appropriate lifting operator μ that maps $\boldsymbol{x}$ to a detailed distribution $\boldsymbol{U}$ on the network. (For example, μ could make random state assignments over the networks which are consistent with the densities).

(c) Prescribe a continuum initial condition at a time t_k, say, $\boldsymbol{x}_{t_k}$.

(d) Transform this initial condition through lifting to N_r consistent individual-based realizations $\boldsymbol{U}_{t_k} = \mu \boldsymbol{x}_{t_k}$.

(e) Evolve these N_r realizations for a desired time *T*, generating the $\boldsymbol{U}_{t_{k+1}}$, where $t_k = kT$.

(f) Obtain the restrictions $\boldsymbol{x}_{t_{k+1}} = \aleph \boldsymbol{U}_{t_k}$.

The above steps, constitute the so called *coarse timestepper*, which, given an initial coarse-grained state of the system $\boldsymbol{x}_{t_k}$ at time t_k reports the result of the integration of the model over the network after a given time-horizon *T (at time* t_{k+1}), i.e.

$\boldsymbol{x}_{t_{k+1}} = \boldsymbol{\Phi}_T(\boldsymbol{x}_{t_k}, \boldsymbol{p})$, where $\boldsymbol{\Phi}_T : R^n x R^m \to R^n$ having $\boldsymbol{x_k}$ as initial condition.

4. Numerical Analysis Results

The numerical analysis was obtained using networks of $N = 10000$ neurons. We performed a coarse-grained analysis for RRN, Erdős–Rényi, Watts-Strogatz (small-world) networks. The bifurcation diagrams, with respect to the activation probability parameter ε, were constructed exploiting the Equation-free framework as described in the previous section. Our coarse-grained variable was the density d of the active individuals. At time t_0, we created N_r different distribution realizations consistent with the macroscopic variable d denoting the density of activated neurons. The coarse timestepper is constructed as the map:

$$d_{t+1} = \Phi_T(d, \varepsilon) \tag{1}$$

The derived coarse-grained bifurcation diagrams are depicted in Fig. 1-3. These are obtained using the detailed stochastic majority-rule simulator as a black-box timestepper and wrapping around it the Newton-Raphson iterative procedure in order to find the fixed points of the map (1). In the figures, dotted lines correspond to unstable solutions, while solid ones to stable stationary solutions. Figure 1 shows the coarse-grained bifurcation diagram as derived using a RRN with a connectivity degree equal to five. As it is shown the coarse-grained stationary solutions are symmetric with respect to the solution $d = 0.5$. Figure 2 illustrates the coarse-grained bifurcation diagram when the underlying structure follows an Erdős–Rényi topology constructed using a with connectivity probability $p = 0.0008$. Figure 3 shows the derived coarse-grained bifurcation diagram in the case of a Watts-Strogatz network constructed with a rewiring probability $p = 0.2$ starting from a ring lattice with eight neighbours per node (four left and four right). As it is shown in Fig. 2, 3 the heterogeneity in the connectivity distribution results to a symmetry breaking of the stationary solutions.

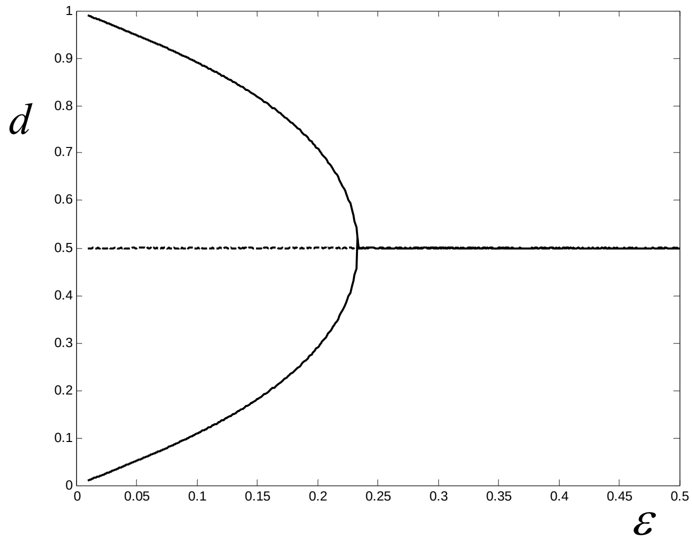

Figure 1: Coarse-grained bifurcation diagram of the density of activated neurons vs. ε for a RRN with constant degree distribution equal to five.

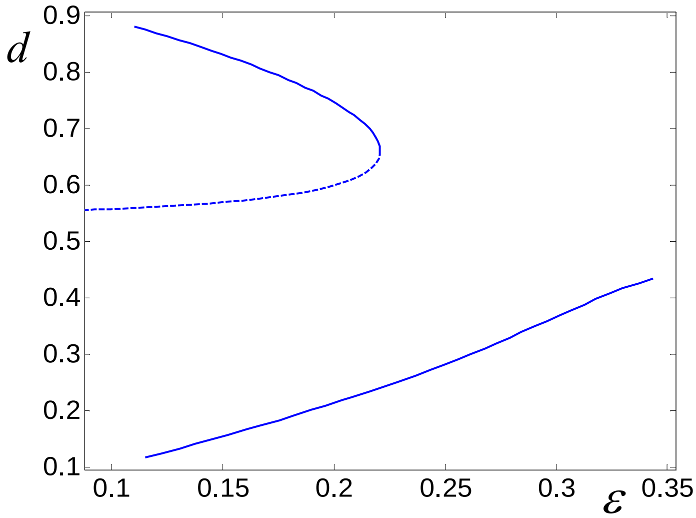

Figure 2: Coarse-grained bifurcation diagram of the density of activated neurons vs. ε for a Erdős–Rényi network constructed using with connectivity probability $p = 0.0008$.

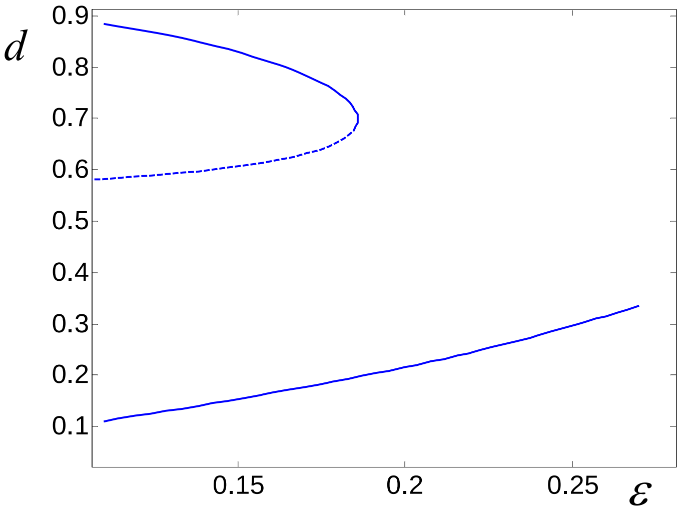

Figure 3: Coarse-grained bifurcation diagram of the density of activated neurons vs. ε *for a Watts-Strogatz network constructed with a rewiring probability* $p = 0.2$ *starting from a ring lattice with eight neighbours per node (four left and four right).*

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