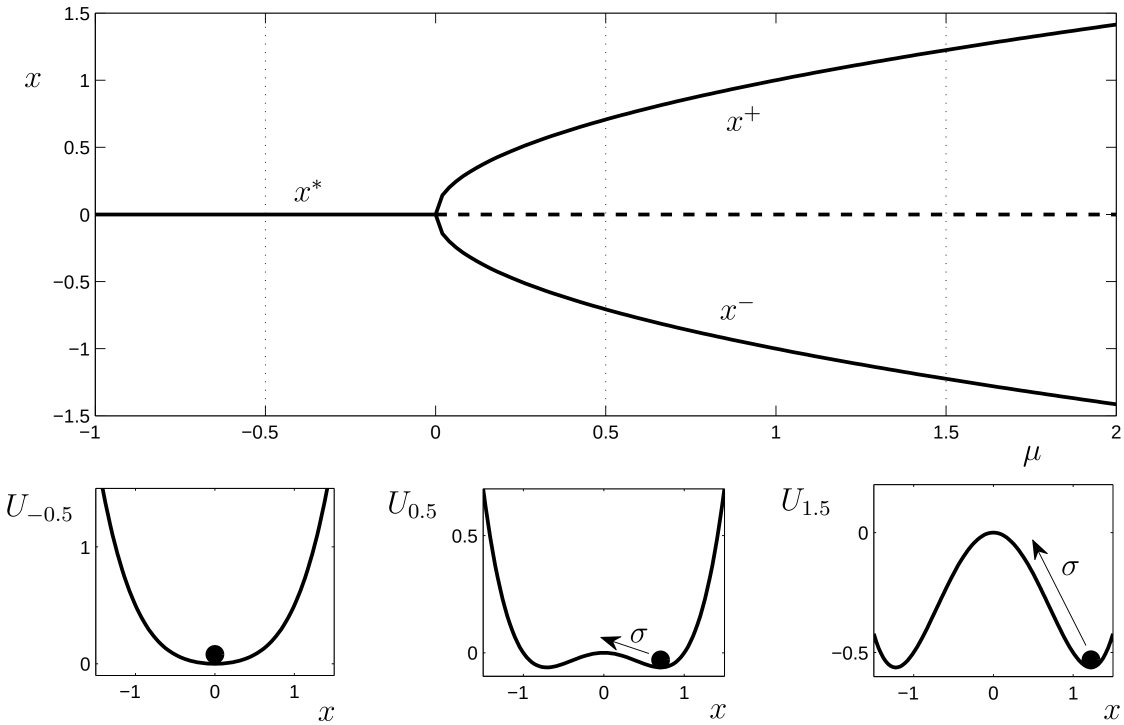

Figure 1: Bifurcation diagram (top) for the deterministic part of (2)-(3). The potentials are shown as well (bottom). In the bistable regime for $\mu > 0$ and $\sigma > 0$ noise-induced transitions between the metastable equilibria occur.

analytically as $\{(x, \mu) \in \mathbb{R}^2 : x = 0\}$ and $\{(x, \mu) \in \mathbb{R}^2 : x = \pm\sqrt{\mu}\}$ one has to use numerical techniques, such as numerical continuation, for more general systems.

Interesting noise-induced dynamics occurs in the bistable regime for $\mu > 0$. Fix any $\mu > 0$, $\sigma > 0$ and initial condition x_0. Then consider the first hitting times $t^\pm := \inf\{t \geq 0 : x_t = x^\pm\}$. A standard result from probability [36] is that

$$\mathbb{P}(t^\pm < \infty) = 1 \tag{4}$$

i.e. no matter where we start, we will eventually visit both deterministically stable equilibrium points with probability one. Although the result (4) is of importance from a theoretical viewpoint it is of very limited practical use. In particular, the time scale on which the stochastic switching between the potential minima occurs is of major interest. Suppose we start the process x_t at $x_0 = x^+$. If $\sigma \gg 1$ frequent switching occurs and we will quickly visit x^- while for $0 < \sigma \ll 1$ switching is rare; see Figure 2. The theory of large deviations [36] considers the first-exit time over the saddle point x^* given by $\tau^+ := \inf\{t \geq 0 : x_0 = x^+, x_t < x^*\}$ and shows that the mean first exit time is

$$\mathbb{E}[\tau^+] = \mathcal{O}\left(e^{2[U_\mu(x^*) - U_\mu(x^+)]/\sigma^2}\right) \qquad \text{as } \sigma \to 0. \tag{5}$$

The result (5) is also known as Arrhenius' law [6] and the rate $1/\mathbb{E}[\tau^+]$ is called Eyring-Kramers rate [34, 63]; see also Section 9. Furthermore observe that the potential difference