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Modeling a large population of traders: Mimesis and stability ☆

Ahmet Omurtag *, Lawrence Sirovich

Laboratory of Applied Mathematics, Box 1012, Mt. Sinai School of Medicine, 1 Gustave L. Levy place, New York, NY 10029, United States

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Abstract

We introduce a method of accurately and efficiently modeling a large population of participants in a financial market. Each participant is modeled as having an internal preference state affected by the continual arrival of exogenous information and by the behavior of others. In order to describe a community of traders, we introduce a population equation that is derived rigorously from the underlying single-agent model. The population equation is used to investigate collective behavior with mimetic interactions. We observe and study the sharp transitions in parameter space from a stable time-independent regime to instability where the demand and supply diverge sharply.

JEL classification: C63; D79; G10

Keywords: Herd behavior; Mimesis; Collective decision making; Financial markets; Stability

1. Introduction

Behavioral traits of market participants and their effects on the properties of markets have been under growing investigation in recent years. Much of this work (e.g., Arthur et al., 1997; Challet and Zhang, 1997; Chan et al., 1998; Lux and Marchesi, 1999; Shiller, 1999; Cont and Bouchaud, 2000; Farmer, 2002; Iori, 2002), which emphasizes deviations from efficient markets theory, tries to elicit an internal market dynamics via computational approaches and is loosely

☆ The views expressed herein are those of the author and not necessarily those of his employer.

* Corresponding author. Tel.: +1 212 241 3948; fax: +1 212 426 5037.
E-mail address: aomurtag@gmail.com (A. Omurtag).

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linked to the subject known as behavioral finance. These interdisciplinary studies try to explain market phenomena by formalizing a diverse array of behavioral and psychological considerations.

Despite the emphasis on the characteristics of real market participants, models of individual behavior in agent-based models have remained rather simple. An agent's psychology as distinct from manifest behavior is typically not separately represented in an agent-based model. Further, an agent is often modeled as a unit capable of being in one of a small number of states. It is generally believed that agent-based modeling is in its early stages (Farmer, 2001), but the simplicity of most existing agent models may also derive from a number of other (valid) reasons: (i) attempts to extract minimal conditions responsible for the "stylized facts" of price statistics (Cont, 2001); agent-based modeling has been very successful in this regard; (ii) computational costs and the difficulty of drawing insights from simulating a very large number of realistic agents and their interactions; these have become considerable especially with multiple agent types and interactions, so that methods beyond direct simulation need to be specified; and (iii) lack of well-accepted quantitative models of deliberation and the behaviors associated with it. In this study we try to address the last two items. In particular we develop a framework for analyzing and efficiently simulating large populations of participants in a financial market whose psychology and behavior can be specified with greater precision than so far allowed.

We will first introduce a single-agent model capable of describing the process of deliberation and how it leads to action. In our model, each agent is described by an internal preference state with an intrinsic dynamics and influenced by external events within or outside the population. Actions result when the agent's state crosses a predetermined threshold in state space. Specifically, we consider a population consisting of traders repeatedly deciding whether to buy or sell an asset. We next formulate a dynamical equation describing the entire population of agents. Analogues of this equation are referred to as kinetic equations in statistical mechanics. We derive the kinetic (or population) equation rigorously from the single-agent model with the addition of a reasonable statistical assumption and with *no free parameters*. We investigate the implications of the kinetic equation for the rates of buy and sell orders that result from a steady input of information.

In addition to information that arrives from the outside and generates exogenous shocks, a population is also affected by the endogenous exchange of information. In fact, direct interaction and personal communication can exert a powerful influence on the individuals in humans groups in general. The ability of groups to act cohesively and respond collectively to information may have had a high adaptive value in evolutionary history. Cohesive human behavior resulting from mimesis or herding has been studied extensively (see Shiller, 2000, chapter 8, for a review and references). While possibly associated with emotional or "irrational" drives, mimesis can also be interpreted as a rational strategy under conditions of imperfect information. Recent economic literature continues to emphasize complex informational influences on behavior and their potential to bring about unexpected patterns (Bannerjee, 1992; Topol, 1991; Bikhchandani et al., 1992; Kirman, 1993; Orléan, 1995). In the relatively simplified context of agent-based financial market models, mimesis has been proposed as a major contributor to the strongly non-Gaussian nature of short to middle term price changes and to excess volatility (Lux and Marchesi, 1999; Cont and Bouchaud, 2000; Iori, 2002).

As an application of our method we will use the kinetic equation to investigate the effects of mimetic behavior. For mimetic agents, the kinetic equation takes the form of a nonlinear partial differential equation having a kinship to the Boltzmann equation of kinetic theory (Cercignani, 1988). This formulation allows us to analyze stability boundaries in parameter space theoretically.

By analysis and simulations we focus attention on the circumstances where mimetic behavior causes a transition to instability and consequent abnormal discrepancies between the aggregate demand and supply.

The formal analysis that is presented in the following is largely based on experience gained from treating problems in neuroscience, which deal with the behavior of populations of neurons (Knight, 1996, 2000; Sirovich et al., 2000; Omurtag et al., 2000; Sirovich, 2003).

2. A model of trader behavior

Here we try to capture quantitatively a simple, perhaps superficial, part of a financial agent's psychology as well as its connection to actual behavior. This is kept to a bare minimum in order not to obscure our general approach.

Our model is based partly on results of so-called decision field theory developed by quantitative psychologists (Diederich, 1997). Another significant inspiration for the present model is the *integrate-and-fire* representation of neuronal cells. Integrate-and-fire models describe the behavior of a neuron's membrane potential by an equivalent circuit made up of a resistor in parallel with a capacitor that resets to baseline when the potential reaches a threshold (Tuckwell, 1988). In a recent financial model, Iori introduced thresholds for action in order to study stylized facts under trade friction and imperfect information. Threshold models of individual behavior can be traced back to the work of Granovetter (1978). Watts (2003, chapter 8) discusses threshold models of decision making in relation to information cascades.

We assume that the state of an agent at time t is fully determined by the value of a variable $x(t)$. This is the agent's momentary internal condition, which we will refer to as its preference state. It may be viewed, in general, as the agent's disposition or tendency to behave in a number of specific ways. x determines the coordinates of state (or phase) space where each agent is represented as a point. The agent acts only when its representative point crosses a predetermined boundary or threshold in state space. In our model the agent's ongoing process of deliberation is described by the dynamical system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\gamma x + I(t), \tag{1}$$

supplemented by the condition that at $|x| = 1$, the preference state is reset to $x = 0$. The reset condition constrains the preference state to remain within a finite interval, $-1 \leq x \leq 1$. $I(t)$ describes the effect of external influence on the agent. This may originate from all kinds of sources information such as the news media or personal contacts. Note that, despite appearances, Eq. (1) is non-linear due to the reset condition. We regard x as the agent's fraction of certainty, and choose $x = 1$ and $x = -1$ as the thresholds for buying and selling, respectively.

The effect of information arrival is represented by a series of instantaneous jumps of size $\varepsilon^{(k)}$ in the preference state received at arrival times $t^{(k)}$, $k = 1, 2, \ldots$ This formulation is valid as long as the time scale of the impact of information is much shorter than that of other changes in x. The agent's preference state is therefore driven by a term

$$I(t) = \sum_k \varepsilon^{(k)} \delta(t - t^{(k)}).$$

We assume a large number of uncorrelated sources of information and accordingly take arrivals as being Poisson distributed in time.

For simplicity each jump will be taken as positive, ε^+, or negative, ε^-, with arrival frequencies $\nu^+(t)$ and $\nu^-(t)$, respectively. Following this formulation we will refer below to positive and negative pieces of information. Generalizations to jumps with stochastic size or jump size that depends on the current preference state are straightforward. We take the magnitudes $\varepsilon^\pm$ as measures of informational impact on a typical agent. In the special case of $\varepsilon^+ = -\varepsilon^-$, one can speak of informational impact being symmetric.

The first term in Eq. (1) represents the evolution of the preference state of an hypothetical *isolated* agent. In the present model this is simple forgetfulness and consists of temporal reversion to the neutral preference state, $x = 0$, with a speed determined by γ. Hence, in the absence of any incoming information, the agent has no intrinsic tendency either to buy or sell. Our reset condition specifies that immediately after the decision threshold is crossed, the preference state be set at the neutral position. This implies that by placing an order the agent has acted on all the information received thus far and thereafter embarks on fresh deliberation. However, memory effects or "regret" may be easily incorporated into this model by placing the reset at a non-neutral position. Clearly, when $x(t)$ is in the positive (negative) half of the domain, the agent has a greater tendency to buy (sell) and may be viewed as being "optimistic" ("pessimistic").

3. Interactions

We next consider a large collection of N agents trading the same asset. Each agent is indexed by $i = 1, \ldots, N$, and its preference state described by Eq. (1). Market participants clearly do not collectively listen to the same news and are not exposed to the same channels of information. In addition, any input to a real agent undergoes intensive processing before it is interpreted as meaningful information relevant to an asset. As the end result of this process, inter-agent differences in exogenous information are to be expected. In our model we allow every agent i to be driven by its private stream of information $I_i(t)$. Informational heterogeneity, along with nonlinear dynamical evolution and possibly different initial conditions, injects a great variety of individual behaviors into the population.

The average rate of arrival of exogenous information is taken to be the same for all agents. Every agent, in addition, is affected by the behavior of other agents in the population. We denote by g the average number of agents affecting any one agent. The size of the influence group of each agent is chosen randomly from a Poisson distribution with mean g, and the group is populated by drawing agents randomly from the rest of the population. Hence mutual interactions in the population occur through the influence of the *actions* of agents on the *preference states* of other agents. Specifically, if an agent places an order to buy (sell), the preference state of another agent is instantaneously incremented by $\varepsilon^{+(-)}$ if that agent is affected by the action at that moment. This formalizes the intuition that the more a trader sees others buying (selling), the more bullish (bearish) he or she will become.

Existence of mutual interactions raises the possibility of correlated inputs to agents. In the presence of a steady input each agent in our model is figuratively an "oscillator," and it is well known that interconnected oscillators are capable of global synchrony (Kuramoto, 1991). (The recent popular account of synchrony by Strogatz, 2003, contains many references.) In certain ranges of parameter values global synchrony is observed in our population. In the asynchronous regime, on the other hand, local oscillations are possible if the connectivity is fixed in time. Our numerical experience indicates that with sparse coupling, $g \ll N$, such correlated inputs do not arise. In our direct simulations, we use sparse coupling, and connectivity remains fixed in time.

We define the rates of buy and sell orders per agent, R^+ and R^-, respectively, as

$$R^{\pm} = \frac{1}{N\Delta t}\sum_{n}\sum_{i=1}^{N}\int_{t}^{t+\Delta t} dt'\delta(t' - T_i^{(n)\pm}), \tag{2}$$

where $T_i^{(n)\pm}$ is the "instants" at which agent i crosses the decision boundary at ± 1 for the nth time. Since trading is not synchronized in our model, we work with order rates rather than the aggregate demand and supply. Δt is a small time interval adjusted to reveal the momentary mean of $R^{\pm}$ along with its fluctuations about the mean. We refer to a calculation involving the simultaneous numerical solution of all N single-agent equations as a *direct simulation*. Note that no agent has greater influence on the rest of the population than any other.

4. Population dynamics

It is possible to reach a concise description of the dynamics of a large assembly of agents by keeping track of only the *number* of agents in each preference state rather than the preference states of every agent in the population. This is accomplished by deriving an equation for $\rho(x, t)$, the density of agents in preference state x at time t. The probability density states the probability $\rho\mathrm{d}x$ that we can find an agent of the population within the range of states $(x, x + \mathrm{d}x)$ at time t. We imagine $\rho(x, t)$ as the density of agents averaged over a large number of replicas of the population, all with the same total number of agents, mean exogenous arrival rate, and feedback strength g. As explained in the previous sections, the exogenous inputs are drawn from a distribution, as are the members of influence groups. Both of these can be considered as generating the ensemble of replicas of the population.

To define buy and sell order rates, first note that any agent in the interval $[1 - \varepsilon^+, 1]$, upon receiving a positive impulse, crosses the buy decision threshold. Similarly, any agent in the interval $[-1, -1 + \varepsilon^-]$ upon receiving a negative impulse crosses the sell decision threshold. Recall that the jump arrival rates are denoted by $\nu^{\pm}$. Then, if we assume that an agent's preference state is not correlated with the fact that it is about to receive information[1] we can write the rate of buy and sell orders as

$$R^{\pm} = \pm\nu^{\pm}\int_{\pm 1\mp\varepsilon^{\pm}}^{\pm 1} \rho(x, t)\mathrm{d}x = \nu^{\pm}(P^{\pm}, \rho), \tag{3}$$

where $P^{\pm}$ is unity only in the appropriate subintervals, zero elsewhere, and $(\cdot, \cdot)$ denotes the inner product in $[-1, 1]$. We denote the order rates by $R^{\pm}$ in the direct simulation as well as in the population equation since the distinction will be clear from context. When each agent is influenced by g other agents, the rate of input per agent is clearly $\nu^{\pm} = \nu^{\pm}_{\mathrm{ex}} + gR^{\pm} = \nu^{\pm}_{\mathrm{ex}} + g\nu^{\pm}(P^{\pm}, \rho)$ that, solving for $\nu^{\pm}$, leads to an expression for total input:

$$\nu^{\pm} = \frac{\nu^{\pm}_{\mathrm{ex}}}{1 - g(P^{\pm}, \rho)}. \tag{4}$$

[1] Assumptions of this type, known as *Stosszahlansatz*, are used in kinetic theory (Cercignani, 1988).

It can be shown (Omurtag et al., 2000) that for agents governed by Eq. (1) the conservation of probability leads to the following equation for the evolution of the density:

$$\frac{\partial\rho}{\partial t} = \gamma\frac{\partial}{\partial x}(x\rho) + \sum_{k=+,-}\frac{\nu_{\rm ex}^{(k)}}{1-g(P^{(k)},\rho)}\left(\rho(x-\varepsilon^{(k)},t)-\rho(x,t)\right) + (R^{+}+R^{-})\delta(x). \quad (5)$$

See Appendix B, available on the JEBO website for a brief derivation and discussion of Eq. (5). This equation contains a non-linearity in the density variable brought about by the mimetic gain, g. A general derivation of population equations as applied to neuronal systems can be found in the cited references, which also discuss numerical methods for its solution. Gardiner (1994) provides further general background in the context of Markov processes with jumps. The last term in Eq. (5) represents the fact that agents crossing the decision boundaries are reinstated at the origin. The first term by itself would result in the density asymptotically accumulating at the neutral position $x=0$ as a result of forget-fulness, and as a result of jumps associated with information arrival, there are loss and gain terms, $\rho(x, t)$ and $\rho(x-\varepsilon^{(k)}, t)$, respectively.

Simple considerations dictate that $\rho(x=\pm 1, t)=0$. This and the fact that agents leaving at $x=\pm 1$ are restored at the origin assure that $\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\rho\,\mathrm{d}x = 0$. Since probability is conserved, this condition must be respected.

For small $\varepsilon^{(k)}$ we can expand the offset terms $\rho(x-\varepsilon^{(k)}, t)$ about $\rho(x, t)$ in powers of $\varepsilon^{(k)}$ and, on truncating terms higher than second order in $\varepsilon^{(k)}$, one obtains

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x}(\mu\rho) + \frac{1}{2}\sigma^2\frac{\partial^2\rho}{\partial x^2} + (R^{+}+R^{-})\delta(x), \quad (6)$$

where $\mu(x, t)=\gamma x-(\nu^{+}\varepsilon^{+}+\nu^{-}\varepsilon^{-})$ and $\sigma^2(t)=\nu^{+}(\varepsilon^{+})^2+\nu^{-}(\varepsilon^{-})^2$. (A similar expansion can also be introduced for ν.) This has the form of a diffusion equation, sometimes referred to as a Fokker–Planck equation (van Kampen, 1987). Parameter μ is associated with the deterministic ("advection" or "drift") part of changes in the agent's preference state, and σ is a measure of the magnitude of its random fluctuations. In order to conserve probability we define $R^{\pm} = \mp\frac{1}{2}\sigma^2\partial\rho/\partial x|_{x=\pm 1}$. If the diffusion term is neglected in Eq. (5) we obtain a purely deterministic system. We will refer to this last case as the deterministic approximation. Note that Eq. (5) remains non-linear due to the dependence of μ and σ on the density. The diffusion and deterministic approximations are discussed further in Appendix B.

5. Equilibrium

In the presence of a steady input of information, the population of agents usually approaches an equilibrium where the buy and sell order rates become time-independent. Such an equilibrium state, $\rho_{\rm o}(x)$, corresponds to the solution of Eq. (5) with $\partial\rho/\partial t=0$. To explore the properties of the equilibrium state, we have numerically integrated Eq. (5) until the system attains time-independence. At equilibrium, $\nu^{\pm}$ are each constant. It follows from Eq. (4) that each equilibrium solution actually generates a class of solutions for varying $\nu_{\rm ex}^{\pm}$ and g such that $\nu^{\pm}$ is fixed. One such solution is the case of $g=0$, and we will speak of this as the representative case. We compare these results with those from direct simulations.

Fig. 1 provides a summary of the relationships that hold between the rates of incoming information and buy and sell orders at equilibrium. For purposes of comparison and contrast, Fig. 1 also shows the order rates from the analytical solutions of the Fokker–Planck Eq. (6) and of the deterministic approximation. Expressions for the last two cases are given in Appendix B.

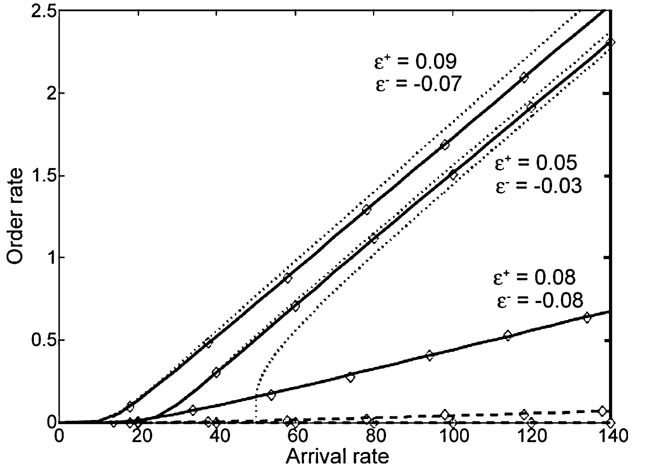

Fig. 1. Rate of buy and sell orders, $R^{\pm}$, as a function of the rate of information arrival, $\nu^{+} = \nu^{-}$. Solid curves represent R^{+} from numerical integration of the population equation. R^{-} is shown by the dashed curves. The dotted curve with a sharp threshold is the deterministic approximation for R^{+} and the other dotted curves are the Fokker–Planck approximations. Diamonds represent the corresponding results from direct simulation with 10^4 agents.

Our main interest lies in the dynamics of the model under some suitably defined *ordinary* exogenous circumstances. This presumably excludes large differences between positive and negative arrivals. We have therefore taken the mean rates of arrival of positive and negative exogenous information to be equal, $\nu_{\mathrm{ex}}^{+} = \nu_{\mathrm{ex}}^{-}$. Despite this equality the typical agent continues to place orders, and this is due to two independent factors.

First, the input to the system is "noisy" as a result of the fact that the arrival times are random. This is a result of our assumption that information is generated by a large number of external sources and its arrival is consequently Poisson distributed in time. This causes the system to have a positive probability that either kind of information will temporarily dominate and push agents across the decision threshold. In fact, this alone accounts for having nonzero order rates when the informational impact is symmetric, $\varepsilon^{+} = -\varepsilon^{-}$. The case with $|\varepsilon^{\pm}| = 0.08$ shown in Fig. 1 has equal buy and sell order rates that slowly increase with the rate of incoming information. It is interesting to note that in the deterministic approximation, the order rate vanishes if the exogenous arrival rate is less than a sharply defined threshold at $\gamma/(\varepsilon^{+} + \varepsilon^{-})$ (cf. Appendix B). This threshold diverges when the informational impact is symmetric, and the corresponding deterministic system generates no orders at any value of the rate of incoming information.

Second, buying and selling may occur because of a possible asymmetry in the informational impact. That is, one kind of information may have a greater impact on agents' preference state than the other. We have investigated cases with $\varepsilon^{+} > -\varepsilon^{-}$. Fig. 1 shows that an apparently small asymmetry in the values of the jump size introduces a relatively large difference in the buy and sell order rates. The two cases with asymmetric informational impact in Fig. 1 ($\varepsilon^{+} = 0.05$, $\varepsilon^{-} = -0.03$ and $\varepsilon^{+} = 0.09$, $\varepsilon^{-} = -0.07$) also illustrate how the randomness in arrival time affects the order rates. Both cases correspond to the same deterministic approximation since they have the same value of the parameter μ. However the case with higher $|\varepsilon^{\pm}|$, where according to the Fokker–Planck approximation the typical agent's preference state undergoes greater random fluctuations (larger σ), displays higher rates of orders.

As the exogenous input increases, the order rates shown by solid curves in Fig. 1 are initially zero, then they become positive at what appears to be an approximate threshold value of $\nu_{\text{ex}}^{\pm}$. To understand the presence of this threshold note that at low values of $\nu_{\text{ex}}^{\pm}$, equilibrium density is shaped approximately like a Gaussian with variable σ in Eq. (3) as its standard deviation. The boundaries at $x = \pm 1$ are sufficiently far away from the bulk of the distribution so that the order rate given by Eq. (3) vanishes. As $\nu_{\text{ex}}^{\pm}$ increases, so does σ, and the density widens. As the width of the density becomes comparable to the range of the preference state, $-1 \le x \le 1$, the order rates are expected to become significantly greater than zero. On following this reasoning with an order of magnitude evaluation, we obtain $\nu_{\text{ex}}^{\pm} \sim 10^{-1}\varepsilon^{-2}$, which agrees with Fig. 1.

We do not expect any differences between the order rates from the population equation and those from the direct simulation. In Fig. 1 there are virtually no differences between the direct simulation and the population equation, and these are due to imperfect convergence of the order rate obtained by Eq. (2). The results of the Fokker–Planck approximation differ from both the direct simulations and the population equation.

In brief, we have found that a large portion of the orders results from the stochastic nature of the arrival of information and that a small endogenous bias in the informational impact may result in a large deviation between the buy and sell order rates. These conclusions, although established here for equilibrium, are expected to be valid more generally since their explanation makes no specific references to equilibrium.

6. Mimetic dynamics

When agents are mimetic, each buy or sell order is an event that serves as a piece of information for other agents and modifies their preference states accordingly. In our model, the recipient of feedback information becomes more predisposed to place an order of the same kind. In general, feedback information could be public (prices) or private (conversations among subgroups). In our model, mimetic behavior is associated with the latter.

The population equation, Eq. (5), contains a term for each effect included in the single-agent model. To emphasize this fact and for brevity we write the population equation in terms of separate linear operators on the density. These consist of the isolated-agent operator, Q^{o}, which refers to the advection term, and the informational impact operators, $Q^{\pm}$. Eq. (5) thus becomes

$$\frac{\partial \rho}{\partial t} = \left(Q^{\text{o}} + \sum_{k=+,-} \nu^{(k)} Q^{(k)} \right) \rho \equiv Q\rho. \tag{7}$$

We envision the equilibrium state being disturbed by a small perturbation so that

$$\rho(x, t) = \rho_{\text{o}}(x) + \varepsilon r(x, t), \tag{8}$$

where $r(x, t)$ is an unknown function, ε is the usual small parameter, and ρ_{o} denotes the equilibrium, $Q\rho_{\text{o}} = 0$. We seek to understand the circumstances under which such disturbances will not decay in time, thus preventing a stable equilibrium from becoming established.

Inserting Eq. (8) into Eq. (7) and retaining only terms linear in ε leads to a dynamical equation for the perturbation

$$\frac{\partial r}{\partial t} = Qr + \sum_{k=+,-} \frac{g\nu_o^{(k)}}{1 - g(P^{(k)}, \rho_o)} Q^{(k)} \rho_o(P^{(k)}, r) \equiv \tilde{Q}r, \tag{9}$$

where $\nu_{\rm o}^{(k)}$, is the total input at equilibrium. A generic property of a linear operator such as $\tilde{Q}$ is to possess a set of eigenfunctions, ϕ_n, with corresponding eigenvalues, λ_n, such that $\tilde{Q}\phi_n = \lambda_n\phi_n$. It is straightforward to demonstrate that one eigenvalue, say $\lambda_{\rm o}$, will always be zero. In addition, the eigenvalues are real or occur in complex conjugate pairs. To see this, note that eigenvalues are roots of the polynomial $\det(Q-\lambda I)=0$ where λ is the only possibly complex quantity. Complex conjugation of this equation simply replaces λ by its complex conjugate, λ^*. Hence whenever λ is a root, so is λ^*. We have found that the largest eigenvalues are real unless ν^+ and ν^- differ greatly and that in the absence of mimetic feedback, nonzero eigenvalues are negative or have negative real parts.

An important property of the set of eigenfunctions is that any function in the state-space may be expressed as a weighted sum of them. In particular the perturbation, r, can be expanded in terms of ϕ_n and substituted into Eq. (9). By formally solving the resulting equations, it can be shown the perturbation must have the form

$$r(x,t) = \sum_n a_n e^{\lambda_n t}\phi_n(x) \tag{10}$$

where a_n are constants. Thus, in the neighborhood of the equilibrium, the perturbation can be expressed as a sum of independently evolving terms, sometimes called normal modes, each consisting of an eigenfunction of $\tilde{Q}$ multiplied by a changing amplitude. Eq. (10) indicates that the stability of the equilibrium requires $\mathrm{Re}[\lambda_{n\geq 1}(g)]<0$.

The eigenvalues depend on the parameters of the problem, in particular on the mimetic gain. We have indicated this fact explicitly by writing $\lambda_n(g)$. Typically, as g increases there will be a critical value $g_{\rm cr}$ at which one of the eigenvalues λ_n, $n\geq 1$ (or a pair of such eigenvalues simultaneously) has real part equal to zero. This is a condition for *marginal stability* where, if the feedback increases any further, a normal mode will begin to grow exponentially in time and carry the system away from equilibrium. Consequently, we define marginal stability as

$$\max_{n\geq 1}\; Re[\lambda_n(g_{\rm cr})] = 0 \tag{11}$$

and determine $g_{\rm cr}$ by numerically solving for it from Eq. (11). A key factor in our implementation of the above method is that $\nu_{\rm ex}^{\pm}$ and g occur in the population equation only in the combination given by Eq. (4). This enables us initially to choose a pair of values of the total input, $\nu^{\pm}$, calculate the equilibrium density from Eq. (7), and determine the critical mimetic gain from Eq. (11). The corresponding exogenous input is then calculated from Eq. (4). When informational impact is asymmetric but the positive and negative exogenous inputs are equal, this procedure needs to be performed in a loop of iterations. A special case of this problem is treated analytically in Sirovich et al. (2006, 22–27).

7. Stability results

Fig. 2 shows that as $\nu_{\rm ex}^{\pm}$ increases, the values of $g_{\rm cr}$ initially drop dramatically. This occurs when exogenous input begins to overcome forgetfulness and $\nu_{\rm ex}^{\pm}$ approximately reaches a value where the order rates become noticeably greater than zero, as seen in Fig. 1. Therefore, for low levels of activity equilibrium is nearly impossible to destabilize. This is understandable since in our model agents are influenced by the behavior of others and, as a result, mimesis does not spread if the agents are placing orders at a low rate. In addition, Fig. 2 shows that higher the asymmetry in the informational impact, the easier it is to destabilize the equilibrium state.

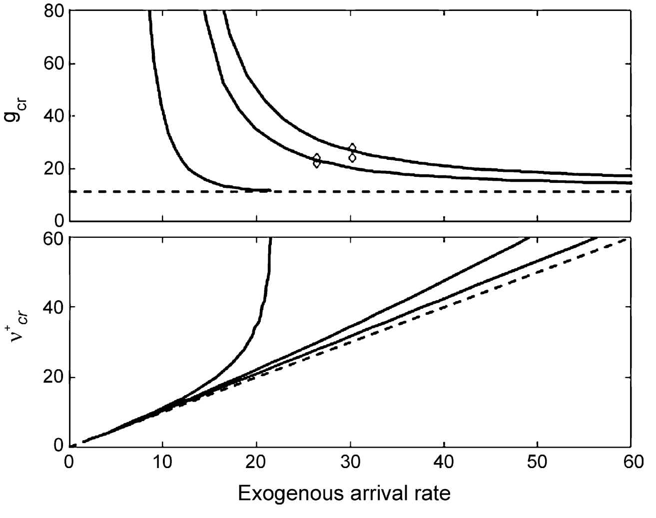

Fig. 2. Top panel: solid curves show the stability boundaries. Dashed line shows the level of $(\varepsilon^+)^{-1}$. In order of decreasing $g_{\rm cr}$, the curves correspond to $\varepsilon^+ = -\varepsilon^- = 0.08$; $\varepsilon^+ = 0.081$, $\varepsilon^- = -0.079$; and $\varepsilon^+ = 0.09$, $\varepsilon^- = -0.07$. Bottom panel: corresponding total input rates at marginal stability as a function of exogenous input rate. Dashed curve is the 45° line.

Why do the marginal stability curves, after sharply falling with increasing information arrival rate, appear to approach horizontal lines asymptotically? Consider a typical agent placing buy orders at the rate R^+. Since each agent receives feedback from g others, the preference state of a typical agent is being incremented by the value ε^+ at the rate gR^+. Assuming that buy orders dominate and forgetfulness is negligible, for an agent starting at the origin it takes about $(\varepsilon^+)^{-1}$ increments to reach the buy decision boundary. More specifically we need to assume that the time it takes for that many increments to arrive is much smaller than the time scale of forgetfulness, $(\varepsilon^+ \nu_{\rm ex}^{\pm})^{-1} \ll \gamma^{-1}$. Then we can conclude that the agent is placing $\varepsilon^+ gR^+$ orders due *only* to the feedback. Now if we set $g = (\varepsilon^+)^{-1}$ then the typical agent is placing buy orders at the rate R^+ and *all* of them are due to the feedback. This represents a self-sustaining or "runaway" activity, in the sense that the population of traders is able to generate an arbitrarily large order rate with no exogenous input at all. The value of g where this begins to occur corresponds to the horizontal line shown in Fig. 2. This is the onset of an instability distinct from the linear instability of the small perturbations we have analyzed. However, the results shown in Fig. 2 indicate that linear instability boundaries merge smoothly with those of runaway instability.

It is instructive, in this regard, to examine the total input to the population that is a sum of the exogenous arrivals and feedback. The bottom panel of Fig. 2 displays the total input at marginal stability as a function of the exogenous input. As the exogenous input increases, total input also increases.

In addition, the probability increases for finding a member of the population at equilibrium near the buy threshold. In other words the equilibrium density, $\rho_{\rm o}$, is displaced further toward the right. Consequently, the value of the integral given by $(P^+, \rho_{\rm o})$, although small initially, increases with the exogenous input. (Remember that P^+ is the characteristic function of the interval $[1 - \varepsilon^+, 1]$.) As (P^+, ρ_o) approaches the value ε^+ the total input given by Eq. (4) will diverge if $g = (\varepsilon^+)^{-1}$. The bottom panel of Fig. 2 shows this occurring for the case that has the highest asymmetry in

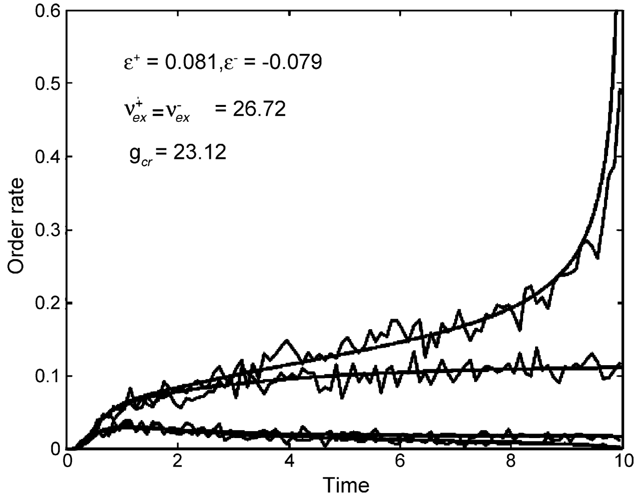

Fig. 3. Order rates calculated by direct simulation (irregular curves, 10^4 agents) and by the population equation (from top, R^+ with $g = 24$, R^+ with $g = 22$, R^- with $g = 22$, R^- with $g = 24$). $\nu_{\text{ex}}^+ = \nu_{\text{ex}}^- = 26.72$, $\varepsilon^+ = 0.081$, $\varepsilon^- = -0.079$.

informational impact. For the other cases this is expected to occur at values of $\nu_{\text{ex}}^{\pm}$ higher than the largest ones we have examined.

We have verified the validity of the foregoing analysis by numerous simulations. Two selected sets of such studies are displayed in Figs. 3 and 4. These show the time course of order rates from the numerical solution of Eq. (5) together with those from direct simulations (irregular curves). The parameter values in each simulation correspond to one of the crosses in Fig. 2.

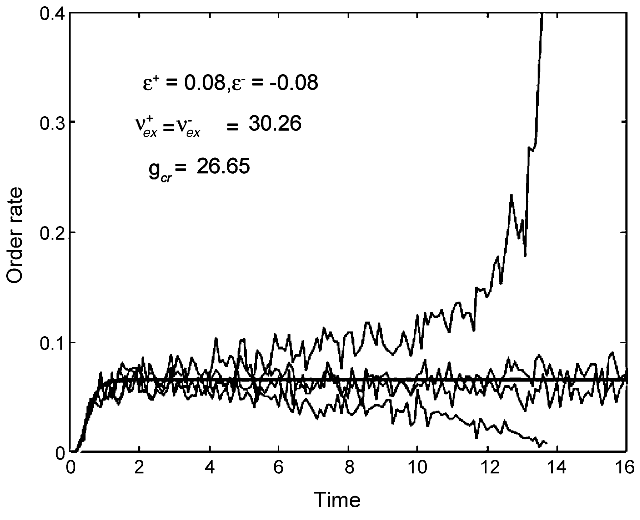

Fig. 4. Order rates calculated by direct simulation (irregular curves, 10^4 agents) and by the population equation ($R^{\pm}$ with $g = 24$ and $g = 28$ appear together as a solid horizontal line for time > 2). R^+ in the direct simulation with $g = 28$ increases sharply while the corresponding R^- decreases. $\nu_{\text{ex}}^+ = \nu_{\text{ex}}^- = 30.26$, $\varepsilon^+ = 0.08$, $\varepsilon^- = -0.08$.

The initial transients in Figs. 3 and 4 are due to all agents starting in the neutral state at the origin. In Fig. 3, while the stable population ($g < g_{cr}$) attains equilibrium, the unstable population gradually drifts then diverges steeply away from it. A somewhat different situation is displayed by Fig. 4 where $\varepsilon^+ = -\varepsilon^-$. The population equation, since the simulation is perfectly symmetric and contains no perturbations about equilibrium, remains at equilibrium in both the stable and unstable cases. The direct simulation of the unstable case, however, breaks the symmetry due to asymmetries in the stochastic input. We find that the closer the unstable parameter values to marginal stability, the longer it takes for divergence finally to occur.

8. Further observations

In a market destabilized by agent interactions, the deviations from equilibrium would eventually lead to a complete breakdown in the absence of any restoring influences. However such influences, in the form of value investors, are present in reality, and their trades exert stabilizing forces on the price when it deviates from perceived (fundamental) value. Additionally, in real markets one would expect the average group size, g, and jump sizes, $\varepsilon^{\pm}$, to be dynamically related to the rest of the variables and change endogenously instead of being held constant as adopted here. Cont and Bouchaud suggest that g may settle near its critical value (also Corcos et al., 2002). Coupled with endogenous mechanisms for dynamically changing the mimetic gain, the present model could give rise to periods of temporary instability that induce temporal correlations in the volatility. Regarding this phenomenon, usually referred to as *clustered volatility*, sudden transient periods of instability have been proposed as an explanation (Youssefmir and Huberman, 1997; Lux and Marchesi, 2001).

Our simulation of the trader population in the unstable regime furnishes examples of a loosely interacting group suddenly behaving as a coherent mass. When this occurs under symmetric conditions, $\varepsilon^+ = -\varepsilon^-$ and $\nu_{ex}^+ = \nu_{ex}^-$ in our model, the outcome depends on the random details of the arriving information. Indeed, in different trials with such symmetry, as depicted in part in Fig. 4, we see that R^+ diverges in some simulations and R^- in others. In financial markets, by contrast, no exactly symmetric counterpart of panic selling is observed. Whatever the underlying reasons for this (perhaps fundamental differences between the spread of fear and that of greed), it can be incorporated into our model by distinguishing between negative and positive mimetic gains and letting $g^+ < g^-$.

The steep divergence of order rates in simulations of unstable systems occur after a period of gradual build up. During this period the overall output of the system does not look appreciably different than that of a stable one. This has the interesting implication that a market that is already doomed to suffer a disruptive move may continue to behave normally for some time. The typical length of such a period is not possible to deduce from the arbitrary units of our variables. Our analysis also indicates that a reduction in the level of incoming financial information, or of attention to it, would stabilize the market and reduce the likelihood of large movements. This corresponds to horizontally crossing a stability boundary toward the left in Fig. 2. These predictions are consistent with the well-known fact that large market moves do not typically occur on days with significant news in the media (Cutler et al., 1989).

In Kirman, as in Lux and Marchesi (1999), the state of a population is defined as the fraction of agents that have one of two possible opinions. Let us point out some similarities between our results and those of Kirman. Kirman postulates a stochastic process of random meetings between pairs of agents such that in each meeting one of the agents is possibly converted or recruited by the other. An agent can also spontaneously self-convert in isolation, which can be considered as being due to the arrival of exogenous information. Probabilities of recruitment and self-conversion are parameters of the system. Kirman demonstrates that the population has bimodal equilibrium states when the level of recruitment is sufficiently high and self-conversion low. In such equilibria, the majority of agents have the same opinion most of the time while the dominant opinion continually and randomly switches. With maximal recruitment and minimal self-conversion, all agents in Kirman's model are permanently absorbed into one of the two opinions.

In our model, under the symmetric conditions mentioned above, random fluctuations of input cause temporarily unequal buy and sale rates, although these remain equal on average. Eq. (10) implies that such deviations decay at a rate given by the eigenvalue of the slowest decaying normal mode. As g approaches g_{cr}, this decay becomes slower. In direct simulations such transient states of varying duration tend to persist longer as instability is approached. The resulting behavior is reminiscent of the random switches in the "mood" of the market found by Kirman. In a further analogy with Kirman's population, all our agents permanently accumulate in either the optimistic or pessimistic side of state space when $g > g_{\mathrm{cr}}$. We would like to emphasize that these qualitative results are generic and not limited to the narrow set of choice of values of our parameters.

9. Concluding remarks

We have shown that a psychologically oriented single-agent model combined with a population approach brings about a wide range of possibilities for analyzing and efficiently simulating a population of financial agents. Computations with the population equation are faster than direct simulations by an order of magnitude or more. This advantage is greater when agents are interacting. In this preliminary study we have investigated only the buy and sell order rates as functions of information input. We have shown that sharp transitions exist between stable and unstable regimes in parameter space for mimetic agents. Our simulations indicate the accuracy of the population equation in representing the behavior of a large assembly of agents. We see in the results that the order rates from direct simulations fluctuate closely around the order rate given by the population equation throughout the simulations at equilibrium as well as during transients, with stable or unstable systems.

Potential extensions of the present work include studying the effects of multiple subgroups such as value investors, the inclusion of price formation mechanisms, single agent-models with further components such as capital and stock holdings in addition to the preference state, and various patterns of connectivity and forms of informational impact. As more realistic descriptions of agents and their interactions become available, our population approach will remain directly applicable.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at doi:10.1016/j.jebo.2004.07.016.

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