

A Systematic Study of Altruism

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Abstract

Analysis of the Altruism model using macroscopic techniques via equation-free bifurcation analysis.

systematic analysis reveals: small band of bi-stability with no stable mixed state, tipping points, large stable selfish domain, altruists stable in tough conditions.

dynamics are Gaussian in general (except $D=0$, $H=0$ where there are Dirac functions. Even Gaussian near the tipping point and shows an increase in variance as you approach the tipping point -¿ what about the other regime shifts?

1 Introduction

The emergence of individuals working together within a collective is a wide spread phenomenon observed in a wide range of areas of science from social science, mathematics and biology.

Specifically, the development of altruism [4, 22] has sparked interest in social science [14, 23, 24], game theory [5] and biology [28].

Despite this breadth of importance, there has been limited investigation into the systematic behaviour of altruistic models. In particular, questions regarding the conditions for altruism to arise still remain open. Studies are typically focussed on specific societies and groups can not provide a global understanding of this behaviour.

Specifically, under what settings do competitive (selfish) agents succeed and at what point does this change to favour, if at all, co-operative (altruistic) agents.

One particular use of ABMs is to model complex systems in biology, social science and game theory.

2 Agent-based Models

Agent-based or individual-based models are extremely useful for simulating the behaviour of complex systems in the absence of a mathematical description of their macroscopic behaviour. It is possible to simulate the dynamics of these highly non-linear systems by defining the interactions on the level of individual agents. The use of probabilistic rules enable the individuals to make ‘decisions’ and ultimately give rise to stochastic dynamics at the system level.

Although agent-based models (ABMs) have been used in a wide range of disciplines to simulate the behaviour of complex systems in this way, their analysis is not straight forward. The non-linear interactions and (generally) stochastic nature obfuscate the effects of the model parameters limiting the insight individual simulations can provide. As a consequence most investigations run multiple configurations of parameters and aggregate the outcomes usually under the assumptions of and underlying Gaussian behaviour. This simplistic analytical technique can obscure behaviours typically associated with complex systems, such as multiple stability; tipping points; regime shifts (e.g. bifurcations); and path dependency.

3 Equation-free Techniques

Continuation (path following) and bifurcation analysis are well established mathematical techniques for the analysis of non-linear systems [7, 26]. These are traditionally applied to deterministic systems with an equation describing its macroscopic behaviour. However, the development of so called equation-free methods [30] have provided a platform for applying continuation methods directly to ‘black box’ simulations. This is achieved by replacing the equation governing the macroscopic behaviour with a collection of appropriately initialised microscopic simulations [18]. Utilising equation-free techniques allows the systematic study of fine grain simulators by side stepping the requirements of an equation of state. Previous investigations have used equation-free methods to analyse stochastic systems [3], chemical systems [17], fluid mechanics [21], biological models [8], experimental setups [27] and have been used with ABMs [1].

The generalised framework involves a three step process; 1) initialising the micro-level system from the macro-level (*Lifting*); 2) running the simulation forward in time (*Evolving*); and 3) estimating the macro-level state from the ensemble of micro-level simulations (*Restricting*). The use of a root-finding algorithm and parameter step processes between the *Lift* and *Restrict* operators extend this framework to continuation of the macro-level system based on the micro-level simulation. Thus we can perform systematic analysis of a systems dynamics including tasks that are inappropriate for direct simulation only, such as the analysis of unstable states in the system. Moreover, continuation methods are much more computationally efficient than repeatedly performing long time simulations under parameter variation. **cite both.**

4 A Model of Altruism

The altruism model [32] simulates the competition between selfish and altruistic agents for survival in a 2D world. In general the outcome of this battle depends on the environmental conditions though there is a stochastic element coming from the random initial configuration of the system. For this specific model, the system is randomly initialised where each agent is a square in a 2D lattice and has a probability p_A, p_S or p_E of being either selfish, altruistic or empty respectively. At each unit of time n , the state of each agent is updated based on the state of itself and its four neighbours with probabilities

$$p_{A_{i,j}}^{n+1} = \frac{F_{A_{i,j}}^n}{F_{T_{i,j}}^n} \quad (4.1)$$

$$p_{S_{i,j}}^{n+1} = \frac{F_{S_{i,j}}^n}{F_{T_{i,j}}^n} \quad (4.2)$$

$$p_{E_{i,j}}^{n+1} = \frac{F_{E_{i,j}}^n + D}{F_{T_{i,j}}^n} = 1 - p_{A_{i,j}}^{n+1} - p_{S_{i,j}}^{n+1}, \quad (4.3)$$

where $F_{A_{i,j}}^n, F_{S_{i,j}}^n$ represent the ‘fitness’ of altruist and selfish traits, $F_{E_{i,j}}^n$ is the minimum fitness required to survive in the environment, i.e. the fitness of an empty state, and $F_{T_{i,j}}^n$ is the total fitness. System parameters D, C, B and H represent the disease rate of altruist and selfish agents, the cost and benefits of altruism and the harshness of the environment respectively. These are used to calculate the fitness of each state as

$$F_{A_{i,j}}^n = \frac{N_{A_{i,j}}^n}{5} (1 - C + B), \quad (4.4)$$

$$F_{S_{i,j}}^n = \frac{N_{S_{i,j}}^n}{5} (1 + B), \quad (4.5)$$

$$F_{E_{i,j}}^n = \frac{N_{E_{i,j}}^n}{5} H. \quad (4.6)$$

Here $N_{Y_{i,j}}^n$ is the number of Y type agents in the neighbourhood of i, j . The total fitness is given by

$$F_{T_{i,j}}^n = F_{A_{i,j}}^n + F_{S_{i,j}}^n + F_{E_{i,j}}^n. \quad (4.7)$$

There is a conservation in the model due to the total number of lattice sites, N_T . Each site can be either altruists or selfish agents, or empty squares. The relative number of each agent (including the empty squares) type must sum to unity, $(N_A + N_S + N_E)/N_T = 1$, therefore the number of agents under any settings is confined to a plane in $f(N_A, N_S, N_E)$ space. It is also worth noting that this model does not allow for regeneration as agents are updated based on their neighbours type.

This model does not address the behaviors of individuals, only the relative weights of genetic traits. A next step in considering the evolution of altruism is to model altruistic behaviors. (See the related model: Cooperation.) [32] This model is based on a paper by Mitteldorf and Wilson, 2000, "Population Viscosity and the Evolution of Altruism", Journal of Theoretical Biology, v.204, pp.481-496. [32]

5 Results

5.1 Bifurcations in Disease

In Fig. 1 it is clear that several bifurcations occur in this model under the variation of the disease parameter D . Bifurcations occur in the equilibrium branches of both Altruist and Selfish agents where when a new, unstable, path emerges connecting the two branches. This new branch connects the two agents in the plane of $F(N_A, N_S, N_E)$ space where the total number of agents is conserved. This branch corresponds to a state where both agents co-exist in the environment. However, this

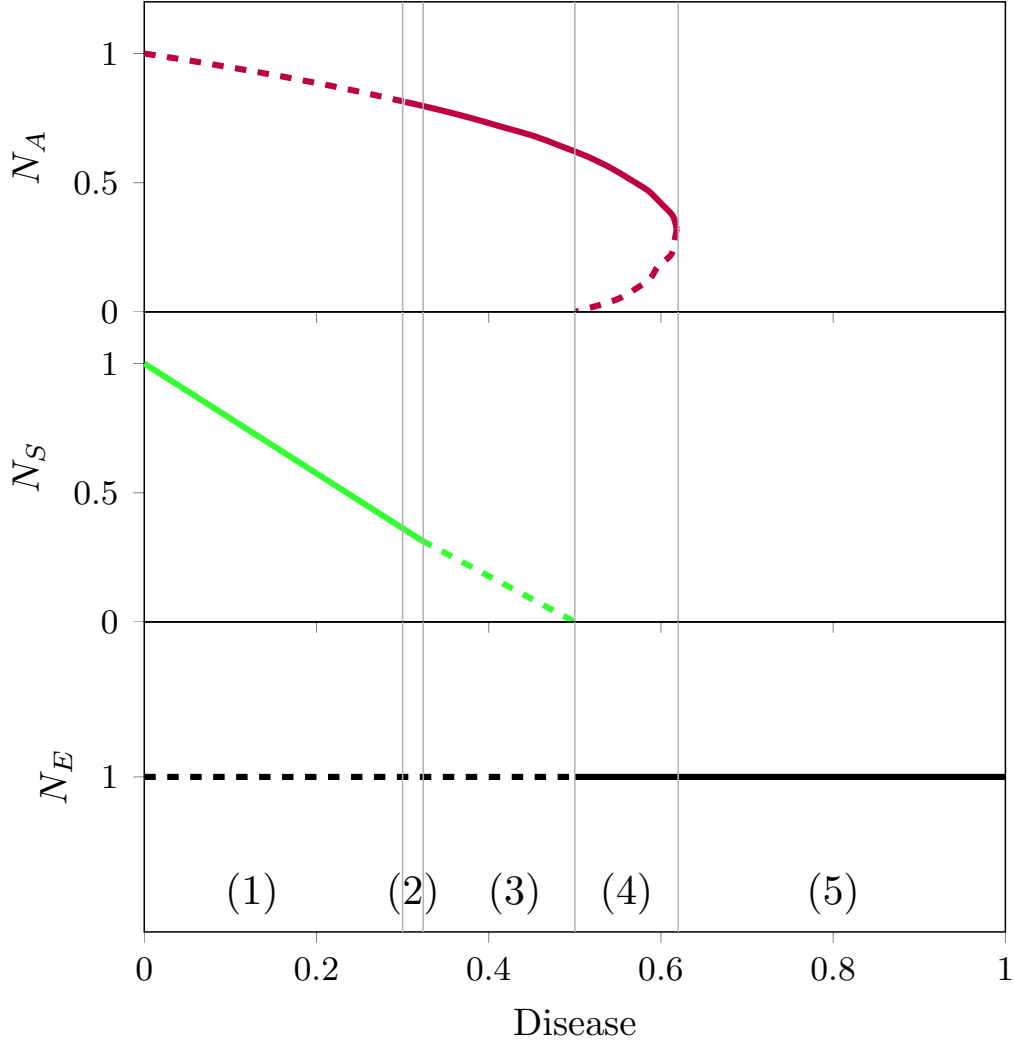


Figure 1: *Bifurcation curve for the altruism model. Stability is indicated by solid (dashed) lines for stable (unstable) branches and numbers indicate regions of behaviour separated by bifurcation points. Parameters: $C_A = 0.13$, $B_A = 0.48$, $H = 0.85$.*

branch is highly unstable, inheriting the stability of the Altruist branch, and quickly collides with the stable Selfish branch causing it to become unstable. The existence of this unstable state results in a narrow window of D where a stable state exists for both agents, region 2 in Fig. 1.

If we consider the macroscopic state of this model under the same parameter settings as in Fig. 1, but with $H = 1.0$, we observe a different bifurcation. Here instead of a change in stability and the creating of a new branch linking the equilibria of the two agents we observe a saddle node bifurcation in the Altruism branch, and no fixed points (stable or unstable) for the Selfish agents. In bifurcation curve under this configuration is given Fig. 2. Clearly H is also a bifurcation parameter and appears to push the collision point of the Altruist and Empty branch (the start of region 5 in Fig. 1) towards zero. This indicates that several bifurcations may occur between Fig. 1 and Fig. 2. We shall further investigate this in the next section. The stable branch in Fig. 2 at zero Altruists corresponds to the 'Dead' state where there are no Altruists or Selfish agents left and is similar to the region 5 in Fig. 1, where the collision of the Altruist and Empty branch is pushed towards the origin.

what about the variance of the alt / selfish branches when $H=0.85$? or when continuing in H ?

At $H = 1$ the system undergoes a global bifurcation and the selfish fixed point has vanished leaving only a saddle-node in the altruist. The altruist branch now begins stable and only changes stability at the limit point $(N_A, D) = (0.27, 0.49)$. The unstable branch collides with the branch of the empty agents at $(N_A, D) = (0, 0)$ causing it to be stable for all value of D . Consequently there are two regimes in this state, a bi-stable regime where either altruists or empty agents survive dependent on initial number of altruistic agents (characterised by the unstable branch), and a regime where only the empty agents survive, a dead state. The boundary between these regimes, the limit point, corresponds to a catastrophic loss in the altruist population. As this model has no regeneration, once the altruists die out they can not be recovered, i.e. the path dependence is irreversible. The random initial conditions, i.e. spatial orientation of agents, cause some variation in the output of the simulations. The exception is when $D = 0$ where all simulations reveal the same outcome, either an entirely altruistic population, or a state of empty agents. The latter of which is effectively unstable due to the unstable altruist branch which approaches zero. As a consequence even a small number of altruists (due to the finite number of individuals) is sufficient to position the system above the unstable branch in Fig. 2 repelling it towards the stable branch at $N_A \approx 1$. The deterministic come of $D = 0$ is due to the lack of any competition, as $N_S = 0$, and the environmental conditions unable to kill off individuals, therefore $N_E = 0$. As D is increased the environment becomes more difficult and individuals begin to die. The initial conditions cause variation in the outcome of simulations as groups of altruists are more likely to survive than small groups or isolate individuals (altruistic benefits are reaped by neighbouring agents). The distribution of the fixed points across the independent simulations is well characterised by a Gaussian distribution (red and blue curves in Fig. 2 respectively). This is true for both branches and near the limit point (see Fig. 2). The variance of the distributions is a function of D and increases from zero due to the system effectively being deterministic here leading to a delta function with a mean of 0 or 1 and a variance of zero. The variance increases with D until it reaches a maximum value at the limit point, then begins to decrease with D as the system pass around the fold. This property, alongside other measures, has been used as an indication of a regime shift or limit point [6]. Interestingly the unstable branch exhibits higher than the stable branch for a given value of D resembling path-dependence-like behaviour. This is intuitive when considering that, for the stable branch it is only the noise (both the initial configuration and probabilistic rules) in the system causing the variance, whereas for the unstable branch, the dynamics are also moving the system away from the fixed point. The increased variance also corresponds to a decrease in the density at the mean value of the Gaussian fits (data not show).

5.2 Bifurcations in Harshness

The shift from Fig. 1 to Fig. 2, as well as the disappearance of the Selfish branch, indicate that H is also a bifurcation parameter. Performing the same analysis as in Fig. ?? with a fixed value of disease and varying the Harshness yields Fig. 3. Clearly the selfish agents depend non-linearly on the Harshness, unlike the Disease where the relationship is approximately linear. The non-zero Disease value means that even at $H = 0$ neither strategy can sustain a full population. Compared to their dependence on D , both the altruists and selfish agents are not strongly dependent on H below $H \approx 0.6$. Above this value however, both vary strongly with H and again the altruistic branch experiences a limit point.

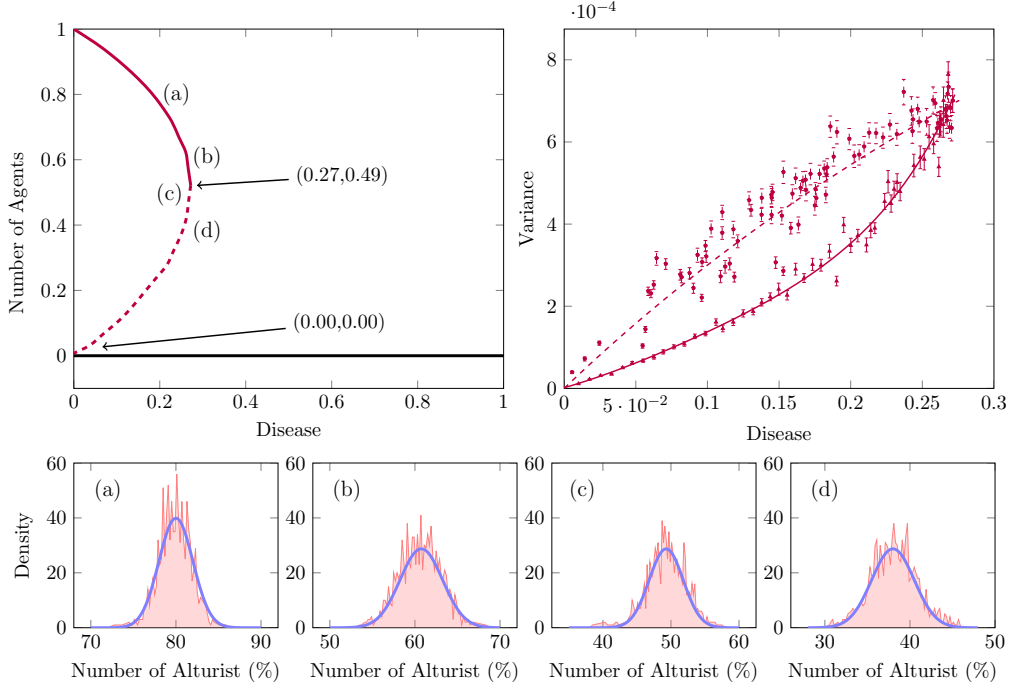


Figure 2: **Saddle node bifurcation in the altruism model.** Stability is indicated by solid (dashed) lines for stable (unstable). Under the same parameters settings as Fig. 1 with $H = 1.0$. add an arrow to show the direction of continuation to link in with the Gaussian and variance plots. Probability density function of realisations after the restrict step. Except the initial delta function, all distributions are well poised Gaussian (blue curve). Check what these plots are called in other papers. Variance of PDF of restricted realisations. Solid red (dashed blue) points/lines correspond to PDFs along the stable (unstable) branches.

5.3 Bifurcations in 2D

In Fig. 4 we observe the behaviour of each type of player in the model and how this varies with environmental conditions. Clearly the selfish strategy is dominant for the majority of the parameter space as at low disease and harshness values it is not necessary for altruistic behaviour to support the group thus the cost of altruism is too high and ultimately altruists are eliminated. At low values of D the selfish agents are dominant (i.e. stable) for all values of H where a fixed point exists. As the conditions become more difficult there is now a transition in the system where altruists can now survive and the selfish strategy is now unsustainable. In even more difficult environments (top right of the plots in Fig. 4) we can see that there is no longer a fixed point for the selfish strategy. Eventually even the altruists can not survive in these conditions leaving only the empty agents, a dead state. Interestingly the saddle-node observed in Fig. 1 is present through all values of disease. We do however divide this in to two distinct regions in Fig.4 as below $D \approx 0.3$ the limit point is at $H > 1$ and therefore no longer in the feasible region of this model. Unlike the selfish population, which constantly reduces in size until it is unsustainable, the altruism population experiences an abrupt collapse at a limit point. That is, the boundary between the red (saddle node) and white (no solution) regions in Fig.4 of the altruists is at $N_A > 0$.

In Fig. 5 the bifurcation diagrams in Fig. 4 are combined to give a the behaviour of the entire system. This representation clearly shows a region where both altruists and selfish agents are stable. This narrow region indicates the presences of an unstable state where both agents can co-exist. Such a state was observed in [31] and was found to be highly unstable. This unstable state

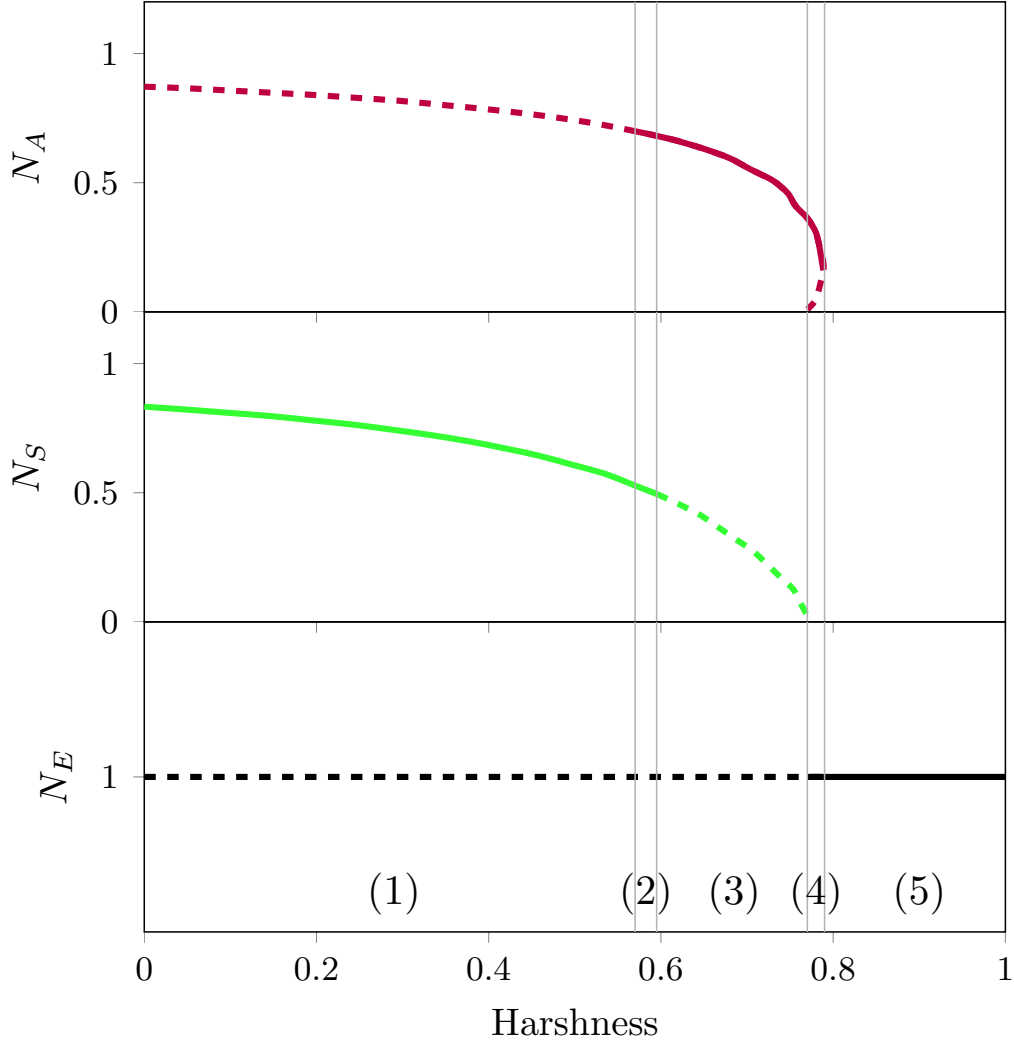


Figure 3: *Bifurcation curve for the altruism model. Stability is indicated by solid (dashed) lines for stable (unstable) branches and numbers indicate regions of behaviour separated by bifurcation points. Parameters: $C_A = 0.13$, $B_A = 0.48$, $D = 0.8$.*

quickly converges to one of the either stable branch depending on the initial conditions. This and the lack of a stable state where selfish and altruists co-exists highlights that although mixed states are possible in theory, they are not observable in practice in this system.

Due to the high levels of noise, countinuation around the fold is difficult with a stadndard predictor corrector method. The algorithm would oftern fail at the limit point, however as introduced in [31], the use of a corrector repeat algorithm enables the progression around the limit point and along the unstable branch even in the presents of high levels of noise.

6 Discussion

Acknowledgements

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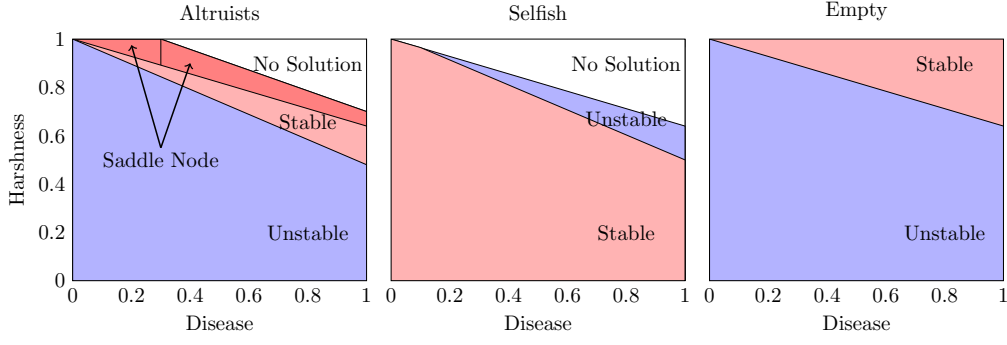


Figure 4: 2D bifurcation diagrams for the players in the altruism model. Several bifurcations are observed indicated by the boundaries in each plot.

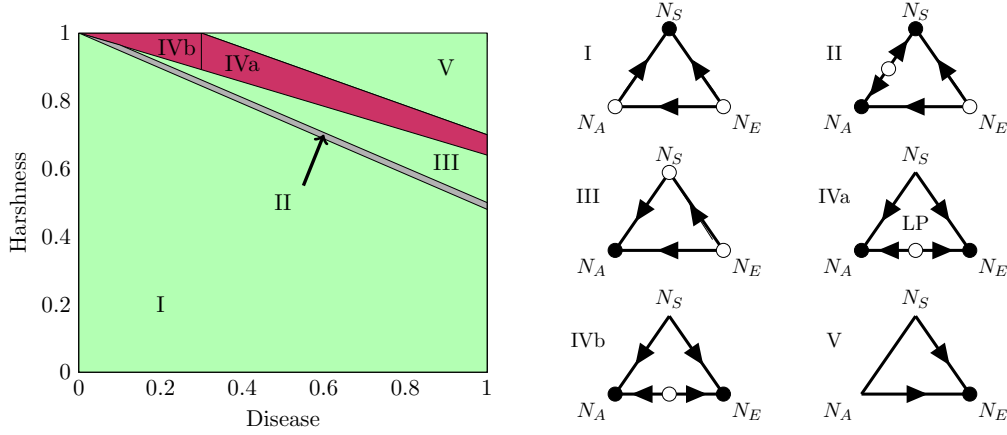


Figure 5: 2D bifurcations for the system in the altruism model. (left) Several bifurcations are observed and labelled. Regions I, II and V (green) denote a single stable fixed point in the system, region II (grey) and region IV (purple) indicate the existence of a bistable system. (right) phase diagrams of system dynamics in each region. The existence of a saddle-node is present in region IV, though the limit point in IVb is at $H > 1$ therefore is distinguished from IV.

The authors confirm that data underlying the findings are available without restriction. Details of the data and how to request access are available from the University of Surrey publications repository: [symplectic_doi](#)

The code and models are also available, along with a user guide, for download at <https://feps-web.eps.surrey.ac.uk/gitlab/st0028/EFNC/tree/master>.

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