Programming Project: The Frequencies of Musical Notes
CPSC-298-6 Programming in C++
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Introduction

Audio programming, a key subfield in computer game development, relies on C++ for performance and for the language's facilities for low-level interaction with hardware. Signal processing applications, such as those for acoustic signal processing and radio frequency, or R/F, signal processing, often depend on C++ for the same reasons.

In this assignment, you'll investigate the frequencies of musical notes and their mathematical relation.

In particular, you'll implement the formula for computing the frequencies of musical notes given a reference frequency.

$f_{k,\nu} = f_R \times 2^{(\nu)+ k/12}$

where

$\boldsymbol{f_R}$ is the Reference Frequency, in this case 16.35 Hz (cycles per second), the frequency of the note C in octave 0 (denoted by C_0).
ν is the octave number (which ranges from 0 to 9 for our purposes)
k is the half-step (or semitone number) within the octave, it has values between 0 and 11 inclusive.
$\boldsymbol{f_{k,\nu}}$ is the frequency of the note in octave ν whose half-step within the octave is *k*. [1]

The reference frequency typically used is 440Hz, corresponding to note A_4, which is the A note in (piano) octave 4. The equation when this value is used for f_R is a bit more complex and less intuitive; we'll stick with C_0 and use 16.35 Hz as our reference frequency.

The next section provides you with some background on the assignment. You can skip this section and go directly to the Assignment section if you'd like. However, you might find the background information helpful and possibly even interesting.

Background

You may be more familiar with music than signal processing; however, both have their foundations in mathematics.

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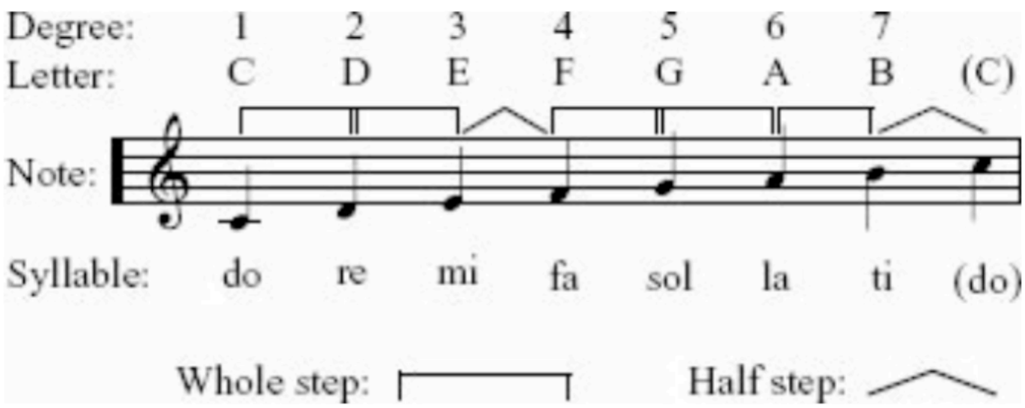
[1] More technically, it is frequency of equal-tempered interval *k* in octave **ν**. The octaves are named in the order of their appearance on a standard 88-key piano keyboard, beginning with octave 0 (or the "zero'th octave").

The musical notes you're familiar with have very specific frequencies that are mathematically related to each other. (Incidentally, we perceive frequency as pitch; the higher the frequency the higher the pitch.)

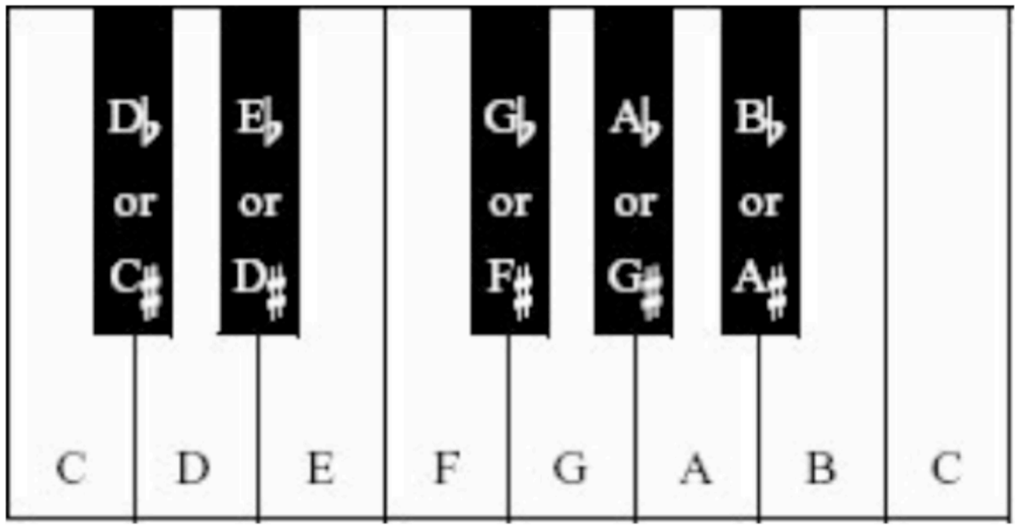
For a piano or other string instrument, frequency is the rate of oscillation (or. more accurately, vibration) of the strings used to produce the sound. Frequency is measured in cycles per second, or cycles/second. The unit of frequency is called the Hertz, and is abbreviated Hz. Its dimensions are inverse seconds because cycles (as in cycles per second) is just a count - we call it a "pure" number; it has no dimensions, so we're left with seconds in the denominator or s^{-1} (inverse seconds) for Hz.

The notes whose familiar syllables are "do", "re", "me", "fa", "so", "la", and "ti" have letter designations, C, D, E, F G, A, and B, respectively, as show in the diatonic scale below. (You don't need to know this to do the assignment, but you may find it interesting.)



The Diatonic Scale

You may recognize these notes on the piano keyboard as shown in the figure below.



A Portion of the Piano Keyboard

But piano keyboards have a lot more keys that just what is shown above; what's going on?

The notes sort of repeat themselves. They're arranged in octaves where the notes in the next higher octave have twice the frequency.

Let's look at an example, C_0, the note with the lowest frequency, a meager 16.35 cycles per second (16.35 Hz). C_0 is in the zero'th octave. The next octave is 1 and the C note in that octave, denoted as C_1, has a frequency of 32.70 Hz, twice as much. The C in octave 2, C2, has a frequency twice that of C_1, or 65.41 (there's a little rounding error). Each time you go up an octave, the frequency doubles.

C_0	16.35 Hz	(Too low for the piano)
C_1	32.70 Hz	
C_2	65.41 Hz	
C_3	130.81 Hz	
C_4	261.626 Hz	("Middle C")
C_5	523.251 Hz	
C_6	1046.502 Hz	
C_7	2093.005 Hz	
C_8	4186.009 Hz	(highest note of piano)
C_9	8372.018 Hz	

The same holds true for the other notes too; take A for example.

A in octave 0, or A_0, has a frequency of 27.5 Hz; it's the lowest note on the piano.

A_0	27.5 Hz	(Lowest note of piano)
A_1	55 Hz	
A_2	110 Hz	
A_3	220 Hz	
A_4	440 Hz	(Tuning Reference Note)
A_5	880 Hz	
A_6	1760 Hz	
A_7	3520 Hz	
A_8	7040 Hz	
A_9	14080 Hz	

So, for any note with a frequency, f, its equivalent in the next higher octave has a frequency of 2*f. And the frequency of its equivalent two octaves up is 2*2*f. Three octaves up, it's 2*2*2*f. The frequency of its equivalent one octave down is f/2. Two octaves down, it's f/(2*2) or f/4.

But what about the other notes, the ones on the black keys, $C^{\#}$ (C Sharp) and $D^{\flat}$ (D Flat, sometime written Db). The sharp symbol in $C^{\#}$ means the note "is a little higher in pitch (or frequency)" than C and the flat symbol in D♭ means "a little power in pitch (or frequency) than D." $C^{\#}$ (C Sharp) and $D^{\flat}$ (D Flat) have the same frequency - they're just different perspectives of the same thing.

The frequency of $C^{\#}$ in the zero'th octave, $C^{\#}_0$, is 17.32 Hz, just a little higher than C0 at 16.35. (And this is the frequency of $D^{\flat}_0$, too.)

Let's take a look at the frequencies of all the notes in the zero'th octave.

Note	Frequency (Hz)
C_0	16.35
$C^{\#}_0/D^b_0$	17.32
D_0	18.35
$D^{\#}_0/E^b_0$	19.45
E_0	20.60
F_0	21.83
$F^{\#}_0/G^b_0$	23.12
G_0	24.50
$G^{\#}_0/A^b_0$	25.96
A_0	27.50
$A^{\#}_0/B^b_0$	29.14
B_0	30.87

Frequencies of Notes in the Zero'th Octave in Hertz

You learned earlier that to go from C_0 (C in the zero'th octave) to C_1 (C in the first octave), you just double the frequency. (C0 has a frequency of 16.35 Hz and C1 has a frequency of 32.70 Hz - exactly double.) But what about going from C_0 to $C^{\#}_0$, the next note right after C_0?

That's a little harder. Each transition (from C_0 to $C^{\#}_0$, for instance) is called a half step. (Transitioning from one row to the next row in the table above is one half step; to go from the top to the bottom is 11 half steps.) It turns out the formula is:

$f_n = f_R * (a)^n$

where
* indicates multiplication, as in a C++ program.
f_R = the frequency of one fixed note, the reference frequency, which must be defined.
n = the number of half steps away from the fixed note you are. If you are at a higher note, n is positive. If you are on a lower note, n is negative.
f_n = the frequency of the note n half steps away.
$a = (2)^{1/12}$ = the twelfth root of 2 = the number which when multiplied by itself 12 times equals 2 = 1.059463094359...

So, let's see if this works for single half step from C_0 to $C^{\#}_0$.

In this case:

$\boldsymbol{f_R}$ is the frequency of C_0, which is 16.35 Hz
n is the number of half steps from C_0 to $C^{\#}_0$, which is 1.
a is approximately 1.059463094359
$\boldsymbol{f_n}$ is just f_1 (one step from 0), the frequency of $C^{\#}_0$, the value we want to compute, which from our table should be 17.32 Hz.

$f_n = f_R * (a)^n$
f_1 = 16.35 Hz $* (1.059463094359)^1$
f_1 = 16.35 Hz $*$ 1.059463094359
f_1 = 17.32 Hz (approximately)

17.32 Hz is exactly what we were looking for.

Now, let's see if this works if we go up a full octave. Recall that there are 11 half steps on the zero'th octave (count them if you'd like to double check this). So, to get to the next octave we just need to go one more, or 12 half steps. The table below shows the frequencies of the notes in octave 0 and the frequency of the first note in octave 1, C_0 , which is what we're looking for.

Note	Frequency (Hz)
C_0	16.35
$C^{\#}_0/D^{b}_0$	17.32
D_0	18.35
$D^{\#}_0/E^{b}_0$	19.45
E_0	20.60
F_0	21.83
$F^{\#}_0/G^{b}_0$	23.12
G_0	24.50
$G^{\#}_0/A^{b}_0$	25.96
A_0	27.50
$A^{\#}_0/B^{b}_0$	29.14
B_0	30.87
C_1	32.70

Frequencies of Notes in the Zero'th Octave in Hertz as well as C_1, the C of the 1st Octave

For this case:
$\boldsymbol{f_R}$ is the frequency of C0, which is 16.35 Hz
n is the number of half steps from C_0 to C_1, which is 12.
a is approximately 1.059463094359
$\boldsymbol{f_n}$ is just f_{12} (twelve step from 0), the frequency of $C^\#_0$, the value we want to compute, which from our table should be 17.32 Hz.

$f_{12} = f_R * (a)^{12}$

But **a** is $(2)^{1/12}$ = the twelfth root of 2, and $((2)^{1/12})^{12}$ is $(2)^{12/12}$, which is $(2)^1$ or 2.

$f_{12} = f_0 * 2$
f_{12} = (16.35 Hz) * 2
f_{12} = 32.70 Hz

The frequency of C_1 is 32.70, exactly double that of C_0, exactly what it should be.

Let's simplify our formula to make "jumping to the next octave easier. We'll introduce a new variable, nu, , which is the octave number. Instead of n for the number of half steps in total, we'll use k, which is the number of half steps within an octave, and so has values from 0 to 11, inclusive.

$f_{k,v} = f_R \times 2^{(\nu)+ k/12}$

$\boldsymbol{f_R}$ is the Reference Frequency, in this case 16.35 Hz (cycles per second), the Frequency of the note C in octave 0, the "zero'th octave, (denoted by C_0).
ν is the octave number (which ranges from 0 to 9 for our purposes)
k is the half-step (or semitone number) within the octave, it has values between 0 and 11 inclusive.
$\boldsymbol{f_{k,v}}$ is the frequency of the note in octave ν whose half-step within the octave is *k*. [2]

Our original formula,

$f_{k,v} = f_R \times 2^{(\nu)+ k/12}$

can be rewritten as (we converted exponent addition to multiplication)

$f_{k,v} = f_R \times 2^{(\nu)} \times 2^{k/12}$

Let's compute the frequency of D_3, the D note in octave 3, whose frequency is 146.83. In this case **ν** is 3 (D_3 is in octave 3), k is 2 (D_3 is two half tones from the start of the octave).

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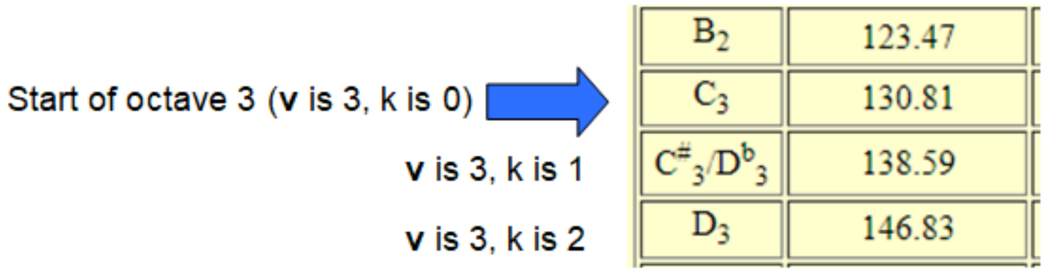
[2] More technically, it is frequency of equal-tempered interval *k* in octave **ν**. The octaves are named in the order of their appearance on a standard 88-key piano keyboard, beginning with octave 0 (or the "zero'th octave").



Annotated Excerpt of Table of Notes and Frequencies

$f_{k,\nu} = f_R \times 2^{(\nu)} \times 2^{k/12}$
$f_{k,\nu} = f_R \times 2^{(3)} \times 2^{2/12}$
$f_{k,\nu} = (16.35 \text{ Hz}) \times 2^{(3)} \times 2^{2/12}$
$f_{k,\nu} = (16.35 \text{ Hz}) \times 2*2*2 \times 2^{2/12}$
$f_{k,\nu} = (16.35 \text{ Hz}) \times 8 \times 2^{2/12}$

Remember that $2^{1/12}$ is the twelfth root of 2 which is approximately 1.059463094359.

$f_{k,\nu} = (16.35 \text{ Hz}) \times 8 \times (1.059463094359)^2$
$f_{k,\nu} = (16.35 \text{ Hz}) \times 8 \times (1.059463094359) \times (1.059463094359)$
$f_{k,\nu} = 146.818$ Hz which is very close to 146.83 Hz.

Rather than doing this by hand, let's write a C++ program to compute $f_{k,\nu}$.

You haven't studied loops or functions in C++ yet, so writing the program isn't straightforward.

Assignment

Use the formula below to compute the frequencies of the following notes: , $C^{\#}_0$, D_0, D_3 , C_4, $D^{\#}_7$, and C_8.

$f_{k,\nu} = f_R \times 2^{(\nu)+ k/12}$

$\boldsymbol{f_R}$ is the Reference Frequency, in this case 16.35 Hz (cycles per second), the Frequency of the note C in octave 0, the "zero'th octave, (denoted by C_0).
ν is the octave number (which ranges from 0 to 9 for our purposes)
k is the half-step (or semitone number) within the octave, it has values between 0 and 11 inclusive.
$\boldsymbol{f_{k,\nu}}$ is the frequency of the note in octave ν whose half-step within the octave is *k*. [3]

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[3] More technically, it is the frequency of the equal-tempered interval *k* in octave **ν**. The octaves are named in the order of their appearance on a standard 88-key piano keyboard, beginning with octave 0 (or the "zero'th octave").

It's easier to code the formula if you place it in this form:

$f_{k,\nu} = f_R \times 2^{(\nu)} \times 2^{k/12}$

Or, even better, if you put it in this form:

$f_{k,\nu} = f_R \times 2^{(\nu)} \times (2^{1/12})^k$

Assume that the twelfth root of 2 ($2^{1/12}$)is 1.059463094359.
Also assume the reference frequency, f_R , is that of note C_0 and is exactly 16.35 Hz.

For each calculation, display the name of the note (e.g. "C#0", "D0", "C4", "D#7" and "C8"), the value of nu (**ν**, the octave number), and the value of k (the half-tone or interval relative to the start of the octave) and the computed frequency, $f_{k,\nu}$.

At the start of your program, display the reference frequency, f_R, 16.35 Hz.
Use a double data type to hold the value of the twelfth root of 2, 1.059463094359 (e.g. `double dTwelfthRootOfTwo`).

Remember if you need to square 1.059463094359, you can just multiply it by itself: (1.059463094359 * 1.059463094359). Cubing it is just as easy: (1.059463094359 * 1.059463094359 * 1.059463094359)

Use an integer data type to store values such as 2, 2^2, 2^3 (e.g. `long iTwoRaisedToPowerNu`); of course, these values are just 2, 4, and 8.

The table below lists the notes, their octave number and the half-tone offset number from the start of the octave. It also lists their frequency (and wavelength) for you to check your answers.

Musical Note	**ν** Octave Number	k Half-tone offset	Frequency (Hz)	Wavelength (cm)
C_0 Reference	0	0	16.35 Hz	2109.89 cm
$C^{\#}{}_0$	0	1	17.32 Hz	1991.47 cm
D_0	0	2	18.35 Hz	1879.69 cm
C4 (Middle C)	4	0	261.63 Hz	131.87 cm
D#7	7	3	2489.02 Hz	13.86 cm
C8 (highest piano note)	8	0	4186.01 Hz	8.24 cm

Incidentally, what is 2 raised to the zero'th power (2^0)? 1 of course.

For each frequency you compute, $f_{k,\nu}$, you'll also compute and display the wavelength using the following equation.

$W_{k,v} = c / f_{k,v}$

where

$W_{k,v}$ is the wavelength, c is the speed of sound in air (at room temperature), and $f_{k,v}$ is the frequency.

The speed of sound in air at room temperature is (roughly) 345 meters per second (345 m/s). Near the start of your program, display the value of the speed of sound.

Also, you'll display the value for the wavelength in centimeters per second (cm/s). Remember that there are 100 centimeters in a meter, so use a conversion factor (100 cm/1 m).

The output of your program will appear similar to that shown in the following screen capture.

```
Reference Frequency: 16.35 Hz
Speed of Sound: 345 m/s
Note:  C0; nu: 0; k: 0; frequency: 16.35 Hz; wavelength: 2110.09 cm
Note: C#0; nu: 0; k: 1; frequency: 17.3222 Hz; wavelength: 1991.66 cm
Note:  D0; nu: 0; k: 2; frequency: 18.3523 Hz; wavelength: 1879.88 cm
Note:  C4; nu: 4; k: 0; frequency: 261.6 Hz; wavelength: 131.881 cm
Note: D#7; nu: 7; k: 3; frequency: 2488.77 Hz; wavelength: 13.8623 cm
Note:  C8; nu: 8; k: 0; frequency: 4185.6 Hz; wavelength: 8.24255 cm
```

Notice that the computed frequency and wavelength values don't match the expected values exactly (they're very close though). Why is that?

The next section provides a few references in case you like to explore further. After this, you'll find a full table of the notes with their frequencies and wavelengths.

References

Loy, Gareth. *Musimathics*, Volume 1 (p. 41). MIT Press, 2011.

Frequencies of Musical Notes, A4 = 440 Hz (mtu.edu)
https://pages.mtu.edu/~suits/notefreqs.html

Formula for frequency table (mtu.edu)
https://pages.mtu.edu/~suits/NoteFreqCalcs.html

Table of Musical Notes and Their Frequencies and Wavelengths (liutaiomottola.com)

https://www.liutaiomottola.com/formulae/freqtab.htm

Appendix: Table of Musical Notes and their Frequencies

Note	Frequency (Hz)	Wavelength (cm)
C_0	16.35	2109.89
$C^{\#}_0/D^b_0$	17.32	1991.47
D_0	18.35	1879.69
$D^{\#}_0/E^b_0$	19.45	1774.20
E_0	20.60	1674.62
F_0	21.83	1580.63
$F^{\#}_0/G^b_0$	23.12	1491.91
G_0	24.50	1408.18
$G^{\#}_0/A^b_0$	25.96	1329.14
A_0	27.50	1254.55
$A^{\#}_0/B^b_0$	29.14	1184.13
B_0	30.87	1117.67
C_1	32.70	1054.94
$C^{\#}_1/D^b_1$	34.65	995.73
D_1	36.71	939.85
$D^{\#}_1/E^b_1$	38.89	887.10
E_1	41.20	837.31
F_1	43.65	790.31
$F^{\#}_1/G^b_1$	46.25	745.96
G_1	49.00	704.09
$G^{\#}_1/A^b_1$	51.91	664.57
A_1	55.00	627.27

$A^{\#}_1/B^b_1$	58.27	592.07
B_1	61.74	558.84
C_2	65.41	527.47
$C^{\#}_2/D^b_2$	69.30	497.87
D_2	73.42	469.92
$D^{\#}_2/E^b_2$	77.78	443.55
E_2	82.41	418.65
F_2	87.31	395.16
$F^{\#}_2/G^b_2$	92.50	372.98
G_2	98.00	352.04
$G^{\#}_2/A^b_2$	103.83	332.29
A_2	110.00	313.64
$A^{\#}_2/B^b_2$	116.54	296.03
B_2	123.47	279.42
C_3	130.81	263.74
$C^{\#}_3/D^b_3$	138.59	248.93
D_3	146.83	234.96
$D^{\#}_3/E^b_3$	155.56	221.77
E_3	164.81	209.33
F_3	174.61	197.58
$F^{\#}_3/G^b_3$	185.00	186.49
G_3	196.00	176.02
$G^{\#}_3/A^b_3$	207.65	166.14

A_3	220.00	156.82
$A^\#_3/B^b_3$	233.08	148.02
B_3	246.94	139.71
C_4	261.63	131.87
$C^\#_4/D^b_4$	277.18	124.47
D_4	293.66	117.48
$D^\#_4/E^b_4$	311.13	110.89
E_4	329.63	104.66
F_4	349.23	98.79
$F^\#_4/G^b_4$	369.99	93.24
G_4	392.00	88.01
$G^\#_4/A^b_4$	415.30	83.07
A_4	440.00	78.41
$A^\#_4/B^b_4$	466.16	74.01
B_4	493.88	69.85
C_5	523.25	65.93
$C^\#_5/D^b_5$	554.37	62.23
D_5	587.33	58.74
$D^\#_5/E^b_5$	622.25	55.44
E_5	659.25	52.33
F_5	698.46	49.39
$F^\#_5/G^b_5$	739.99	46.62
G_5	783.99	44.01

$G^{\#}_5/A^b_5$	830.61	41.54
A_5	880.00	39.20
$A^{\#}_5/B^b_5$	932.33	37.00
B_5	987.77	34.93
C_6	1046.50	32.97
$C^{\#}_6/D^b_6$	1108.73	31.12
D_6	1174.66	29.37
$D^{\#}_6/E^b_6$	1244.51	27.72
E_6	1318.51	26.17
F_6	1396.91	24.70
$F^{\#}_6/G^b_6$	1479.98	23.31
G_6	1567.98	22.00
$G^{\#}_6/A^b_6$	1661.22	20.77
A_6	1760.00	19.60
$A^{\#}_6/B^b_6$	1864.66	18.50
B_6	1975.53	17.46
C_7	2093.00	16.48
$C^{\#}_7/D^b_7$	2217.46	15.56
D_7	2349.32	14.69
$D^{\#}_7/E^b_7$	2489.02	13.86
E_7	2637.02	13.08
F_7	2793.83	12.35
$F^{\#}_7/G^b_7$	2959.96	11.66

G_7	3135.96	11.00
$G^\#_7/A^b_7$	3322.44	10.38
A_7	3520.00	9.80
$A^\#_7/B^b_7$	3729.31	9.25
B_7	3951.07	8.73
C_8	4186.01	8.24
$C^\#_8/D^b_8$	4434.92	7.78
D_8	4698.63	7.34
$D^\#_8/E^b_8$	4978.03	6.93
E_8	5274.04	6.54
F_8	5587.65	6.17
$F^\#_8/G^b_8$	5919.91	5.83
G_8	6271.93	5.50
$G^\#_8/A^b_8$	6644.88	5.19
A_8	7040.00	4.90
$A^\#_8/B^b_8$	7458.62	4.63
B_8	7902.13	4.37