Cofinal Types and Cohen Forcing

AMS 2024 Fall Central Sectional Meeting

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Motivating Directed Sets and Nets

The Weakness of Sequences in Topology

Let f be a map between topological spaces X and Y.
Are the following equivalent?

1. The map f is continuous in the topological sense.
2. Given any point x in X, and a sequence in X converging to x, the composition of f with this sequence converges to $f(x)$.

The spaces for which these conditions are equivalent are called **sequential spaces**. Not all spaces are sequential however.

Informally, sequences cannot encode sufficient information about functions between ”large” topological spaces.

Directed Sets and Nets

We say that a partially ordered set $\mathbb{D}$ is **directed** if every pair of elements in $\mathbb{D}$ has an upper bound. A **net** in a space X is then a function $f : \mathbb{D} \to X$. A sequence is then just a net from $\mathbb{N}$ into X.

We say f **converges** to a point $x \in X$ iff for every open neighborhood U containing x there exists $d_0 \in D$ such that $f(d) \in U$ whenever $d \geq d_0$. This notion of Moore-Smith convergence is sufficient for describing topological spaces. If we replace the sequences in our definition of sequential continuity with nets, it becomes equivalent to topological continuity.

A Problem in Moore-Smith Convergence

The following result shows that nets can entirely characterize the topology of a space.

Proposition.

A subset $S \subseteq X$ is open in X if and only if every net converging to an element of S is eventually contained in S.

A natural question is to understand *which* nets are sufficient to describe a space. In a paper in 1940, Tukey created notation for comparing how two directed sets look ”cofinally” as a means of seeing when a certain directed set suffices to describe a space.

The Tukey Order and Cofinal Types

The Tukey Order

For two directed sets D and E, we say that D is **cofinally finer** than E, written $E \leq_T D$, if and only if there exists a convergent map from D into E. That is, a function $f : D \to E$ such that for all e in E, there is a d in D such that $f(c) \geq e$ for all $c \geq d$. There are several ways to characterize this relation. The following are equivalent:

1. $E \leq_T D$.
2. There exists a map $f : D \to E$ mapping cofinal subsets to cofinal subsets.
3. There exists a map $g : E \to D$ mapping unbounded subsets to unbounded subsets.

It turns out that to understand which directed sets are needed to understand a space, it suffices to understand the equivalence classes induced by the Tukey order.

A Theorem of Tukey

Let D and E be directed sets. We say $D =_T E$ if and only if $D \leq_T E$ and $E \leq_T D$. The following result tells us a bit about the "direction" of the Tukey order.

Theorem (Tukey, 1940)

Let D and E be directed sets. Then $D =_T E$ if and only if there exists a directed set X such that D and E are isomorphic to cofinal subsets of X.

The equivalence classes on $=_T$ are called "Tukey" or "Cofinal" Types. This result reveals a bit about the structure of the Tukey order:

- The singleton poset is the minimal cofinal type.
- Given two directed sets D and E, the $D \times E$ is the least upper bound of D and E.

The study of the structure and classification of these cofinal types then becomes a combinatorial question...

Classifying Cofinal Types of Size $\aleph_1$

For a cardinal κ, let $\mathcal{D}_\kappa$ denote the set of distinct cofinal types of directed sets of size at most κ. Observe that $\mathcal{D}_\omega = \{1, \omega\}$. The first non-trivial problem is counting $\mathcal{D}_{\aleph_1}$.

We know that $1 <_T \omega$. It may be shown that for a directed set D, if $|D| \leq \kappa$, then $D \leq_T [\kappa]^{<\omega}$. Indeed, enumerate D by indexing in κ, and take the map $f : [\kappa]^{<\omega} \to D$ where each entry is bounded by the image of its indices.

Lastly, we observe that ω and ω_1 are incomparable. Indeed, $\omega_1 \leq_T D$ if and only if D contains an uncountable subset with each uncountable subset unbounded. Every countable subset of ω_1 is bounded, and ω has no uncountable subsets.

So, $\{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\} \subseteq \mathcal{D}_{\aleph_1}$. Anything else?

Classifying Cofinal Types of Size $\aleph_1$

$$\begin{tikzcd}
 & \omega \arrow[dr] & & & \\
1 \arrow[ur] \arrow[dr] & & \omega \times \omega_1 \arrow[r] & D \arrow[r] & {[\omega_1]^{<\omega}} \\
 & \omega_1 \arrow[ur] & & &
\end{tikzcd}$$

Figure 1: A diagram of the cofinal types of size $\aleph_1$.

Consistency of Counting Cofinal Types

The work of Todorčević revealed that it is consistent with ZFC for $\mathrm{Card}\mathcal{D}_{\aleph_1}$ to vary.

Theorem (Todorčević, 1983)

If ZFC is consistent, then so is ZFC plus MA plus

$$\mathcal{D}_{\aleph_1} = \{1, \omega, \omega_1, \omega \times \omega_1, [\omega_1]^{<\omega}\}.$$

Additionally, if CH holds, then $\mathrm{Card}\mathcal{D}_{\aleph_1} = 2^{\aleph_1}$.

The results are phenomenal, but the proofs are overwhelming.

- The first uses finite support iteration of a proper poset to force $D =_T [\omega_1]^{<\omega}$ in some elementary submodel of $H_{\aleph_2}$.
- Todorčević also showed that this follows directly from PFA.
- The second result is more straightforward, and uses Ulam's family of stationary sets to construct many cofinal types.

Can we get somewhere in between? How much strength do we need?

Forcing a Sixth Cofinal Type

Forcing a Directed Partial Order on ω_1

Let $\mathbb{P}$ be the set of ordered pairs $p = (A, \leq_p)$ such that

1. A is a finite subset of ω_1,
2. $\leq_p$ partially orders A and respects ordinals.

We say that $(A, \leq_p) < (B, \leq_q)$ if and only if $B \subset A$ and $\leq_p$ restricted to B is $\leq_q$. It may be shown that forcing with $\mathbb{P}$ is equivalent to forcing with $\mathrm{Fin}(\omega_1, 2)$.

For a ground model M, if G is an M-generic filter on $\mathbb{P}$, then a density argument will show that $\bigcup G = \omega_1 \times \leq_{\mathbb{D}}$, where $\leq_{\mathbb{D}}$ is an ordinal-respecting partial order on ω_1, distinct from the ground model. Let us denote this directed set by $\mathbb{D}$.

The poset $\mathbb{P}$ is recognized as an unpublished folk result, but the bulk of our work is showing it is cofinally distinct.

ω_1-Knaster Condition and Distinction of Types

It is unsurprising that, due to its equivalence to Cohen forcing, that $\mathbb{P}$ satisfies ccc. In particular, it satisfies the following stronger condition, ω_1-Knaster.

Definition: The κ-Knaster Condition

Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is **κ-Knaster** if, for every κ-sequence $\langle p_\alpha : \alpha < \kappa \rangle$ in $\mathbb{P}$, there is an unbounded set $X \subset \kappa$ such that the elements of $\langle p_\beta : \beta \in X \rangle$ are pairwise compatible.

Proof Idea: For an ω_1-sequence of conditions in $\mathbb{P}$, apply the Δ-System Lemma to the family of domains. The root is finite, and there are finitely many choices for partial orders on a finite condition, so there are uncountably many conditions extending this root.

It turns out that $\mathbb{P}$ being ccc is sufficient for showing that $\omega \times \omega_1 <_T \mathbb{D}$, since $\mathbb{D}$ is not a countable union of ω-bounded subsets.

Elementary Submodels and Distinction of Types

The previous result combined with the following will show that $\mathbb{D}$ is cofinally distinct from the five famous types in the forcing extension.

Theorem:

Let $\dot{\mathbb{D}}$ be a $\mathbb{P}$-name for $\mathbb{D}$. Then $\mathbb{P} \Vdash$ "$\dot{\mathbb{D}} \neq_T [\omega_1]^{<\omega}$"

Proof: It suffices to show that for all conditions p and names $\dot{X}$ such that p forces "$\dot{X} \in [\omega_1]^{\omega_1}$", there exists q stronger than p such that q forces that $\dot{X}$ contains an infinite bounded set.

Idea: Take a sufficiently large countable elementary submodel M of H_κ containing $\mathbb{P}$,p, and $\dot{X}$. We will find a stronger condition that forces a subset of $\dot{X}$ to be unbounded in M.

Elementary Submodels and Distinction of Types

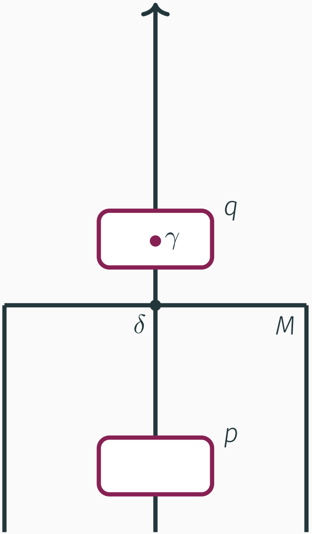
Let κ be a sufficiently large regular cardinal, and M a countable set such that $\mathbb{P}, p, \dot{X} \in M \prec H_\kappa$.
We will denote $\delta = M \cap \omega_1$.

Since $\dot{X}$ is forced to be unbounded, we can find $q \leq p$ and $\gamma > \delta$ such that

1. $q \Vdash$ "$\gamma \in \dot{X}$,
2. $\gamma \in \operatorname{dom} q$.

Claim:

q forces that $\{\xi < \delta : \xi \in \dot{X} \wedge \xi \leq_{\dot{\mathbb{D}}} \gamma\}$ is unbounded in δ.



Let $r \leq q$ and $\alpha < \delta$. We want to find ξ and $\bar{r} \leq r$ such that

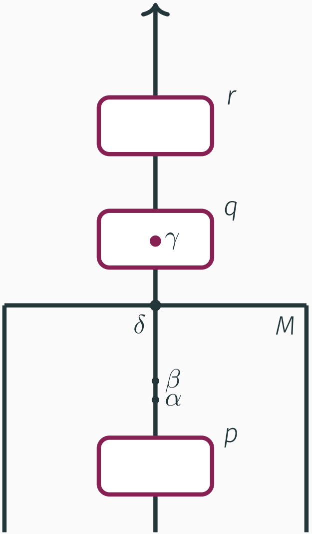
1. $\alpha < \xi < \delta$,
2. $\bar{r} \Vdash$ "$\xi \in \dot{X} \wedge \xi \leq_{\dot{\mathbb{D}}} \gamma$"

Pick β such that $\beta < \delta$ and $\alpha, \max(\operatorname{dom} r \cap \delta) < \beta$.

The idea is to **copy γ inside δ**.

We let $r_0 = (R_0, \leq_{r_0})$ where $R_0 = R \cap \delta$ and $\leq_{r_0}$ is $\leq_r$ restricted to R_0. Note that $r \leq r_0 \in \mathbb{P}$ and that $r_0 \in M$ since R_0 and $\leq_{r_0}$ are finite.

We will now extend r_0 within M.

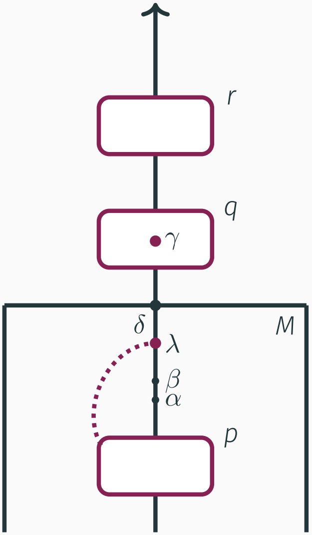


Elementary Submodels and Distinction of Types

Let $A = \{\xi \in R_0 : \xi \leq_r \gamma\}$, and $B = \{\xi \in R_0 : \not\leq_r \gamma\}$. Note that $H_\kappa \models \exists d \in \mathbb{P}$ such that

1. $d \leq r_0$,
2. $\operatorname{dom} d \cap \beta = R_0$,
3. There is $\lambda \in \operatorname{dom} d$ such that
    - $\lambda > \beta$
    - $d \Vdash$ "$\lambda \in \dot{X}$
    - If $\xi \in A$, then $\xi \leq_d \lambda$
    - If $\xi \in B$, then $\xi \not\leq_d \lambda$.

These all hold since r and γ are witnesses of this. By elementarity, $d, \gamma \in M$ as above. We note that $\operatorname{dom} d \subseteq \delta$ as $d \in M$.



Elementary Submodels and Distinction of Types

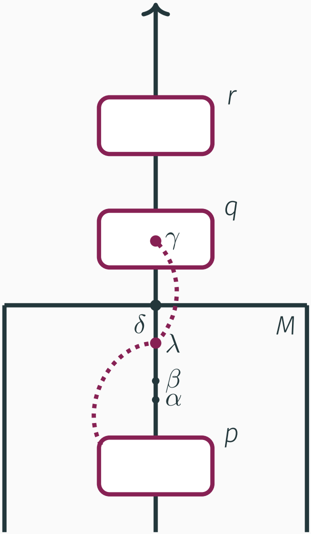
Why is this useful?

(d, λ) and (r, γ) are isomorphic regarding R_0. That is, for $\xi \in R_0$, $\xi \leq_d \lambda$ if and only if $\lambda \leq_r \gamma$.

Lastly, we find $e = (E, \leq_e)$ such that:

- $E = \operatorname{dom} r \cup \operatorname{dom} d$,
- $\lambda \leq_e \gamma$,
- $e \leq r, d$.

So, we have $e \leq r$ such that $e \Vdash$ "$\lambda \in \dot{X}$" and $\lambda > \beta \geq \alpha$.



Future Work

Future Work: Ground Model Distinction and The Cohen Model

Our main result presented here is that forcing ω_1-many Cohen reals into a ground model of ZFC creates a sixth cofinal type.

However, this may not be the *only type* that we've added.
A result of Todorčević hints at this:

Theorem (Todorčević)

If $\mathfrak{b} = \omega_1$, there is a sublattice $D_{\mathfrak{b}}$ of $\mathbb{N}^{\mathbb{N}}$ such that

$$\omega \times \omega_1 <_T D_{\mathfrak{b}} <_t [\omega_1]^{<\omega}.$$

However, we conjecture that this forcing will always adjoin a distinct cofinal type to the ground model.

We will also be investigating the cofinal types within the Cohen model, in an attempt to see how our forcing extension behaves under CH.

Thank you!