On Functional Equations and Finite Groups of Substitutions

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Outline

1. Recap of Previous Presentation
2. Establishing the Problem
3. Setup
4. Proofs
5. Application of Results

What are functional equations?

- Functional equations are equations with functions as the unknown.
- We can think of these equations as relations between different domains.
- Several outside fields and mathematical notions can be traced back to functional equations.
- Functional equations appear frequently in math competitions, the focus of this paper.

$$3f(x) + xf(1-x) = sinx$$

$$f(n+1) = f(n) + f(n-1)$$

$$\Gamma(1-z) = z\Gamma(z)$$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

Exercise

Find all solutions $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ *of the functional equation*

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = x^2$$

Let

$$g_1(x) = x, \;\; g_2(x) = (x-1)/x, \;\; g_3(x) = 1/(1-x).$$

These functions form a cyclic group under composition.

Notice the following:

$$x^2 f(x) = g_1^2(x) f\left(g_1(x)\right), \;\; f\left(\frac{x-1}{x}\right) = f(g_2(x)), \;\; x^2 = g_1^2(x).$$

A Typical Exercise

So the functional equation becomes

$$g_1^2(x)f(g_1(x)) + f(g_2(x)) = g_1^2(x).$$

Composing with $g_2(x)$ twice we obtain the system of equations

$$\begin{array}{rcrcrcl}
g_1^2(x)f(g_1(x)) & + & f(g_2(x)) & & & = & g_1^2(x), \\
 & & g_2^2(x)f(g_2(x)) & + & f(g_3(x)) & = & g_2^2(x), \\
f(g_1(x)) & + & & & g_3^2(x)f(g_3(x)) & = & g_3^2(x).
\end{array}$$

This is a linear system in the functions $f_k = f \circ g_k$ with functions as coefficients, so we apply Cramer's rule.

A Typical Exercise

Expressing the left hand as a coefficient matrix yields the following:

$$D = \begin{vmatrix} x^2 & 1 & 0 \\ 0 & (\frac{x-1}{x})^2 & 1 \\ 1 & 0 & (\frac{1}{1-x})^2 \end{vmatrix} = 2$$

$$D_1 = \begin{vmatrix} x^2 & 1 & 0 \\ (\frac{x-1}{x})^2 & (\frac{x-1}{x})^2 & 1 \\ (\frac{1}{1-x})^2 & 0 & (\frac{1}{1-x})^2 \end{vmatrix} = \frac{(x-1)^2(x^2-1)+x^2}{x^2(x-1)^2}$$

Cramer's Rule then allows us to express the unknown function as a quotient of D_1 and D.

A Typical Exercise

Therefore, for $f : \mathbb{R} \setminus 0, 1 \to \mathbb{R}$, given by

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = x^2$$

Our solution f to the functional equation is defined as

$$f(x) = \frac{(x-1)^2(x^2-1) + x^2}{2x^2(x-1)^2},$$

The Problem

In general, we are consider equations of the type

$$\alpha_1 f \circ g_1 + \cdots + \alpha_n f \circ g_n = h, \tag{1}$$

where g_k, α_k, h are given functions.

As we've seen in the first exercise, these types of equations reduce to a system of equations with unknown functions $f_k = f \circ g_k$.

While we can verify that our previous result is indeed a solution to the equation, this doesn't solve the problem underlying all equations solved by this method.

The Problem

It is not always clear that the methods sketched will yield solutions to the original functional equations.

We've made a change of variables that may not be invertible.

This is due to the cyclic group we've introduced. The functions g within this group represent transformations of the domains. That is, these g's tell how the functions are related to the symmetries of the domain. However, applying these g's in concoction with our unknown function doesn't guarantee its invertibility, and in turn a solution is not guaranteed.

As such, that will be the focus of this paper; the main results of this note will not only achieve representation and existential conditions for solutions, but will guarantee, under some reasonable conditions, that the solution sets will be *compatible*.

To begin, we will establish some notation.

- Let $(G, \circ)$ be a finite group, with elements $g_1, \ldots, g_n$, and fix an index $k \in \{1, \ldots, n\}$
- Next, define the permutations σ_k,π_k by the equations $\sigma_k(j) := l$ and $\pi_k(l) := j$, provided that $g_j g_k = g_l$ holds true.
    - Emphasize that these equations are well defined, and are in fact inverses of each other

Now we will introduce a lemma that reveal some simple, yet crucial properties needed to reach our results.

Lemma

Let G be a finite group with elements $g_1, \ldots, g_n$.
Then for all indices j, k, the following identity holds:

$$\pi_{\pi_k(j)} = \pi_j \sigma_k$$

Proof.

Let $l \in \{1, \ldots, n\}$ be arbitrary, and set $s := \pi_{\pi_k(j)}(l)$.
Then $g_s g_{\pi_k(j)} = g_l$ by definition. Multiply both sides by g_k, and using the same definition, $g_l g_k = g_s g_{\pi_k(j)} g_k = g_s g_j$ follows, which implies $\sigma_j(s) = \sigma_k(l)$. Therefore, $s = \pi_j \sigma_k(l)$. $\square$

Recall the Problem

Our result is concerned with equations where the functions g_k are assumed to form an arbitrary finite group. Recall the form of the equation (1):

$$\alpha_1 f \circ g_1 + \cdots + \alpha_n f \circ g_n = h$$

A consequence of linearity is that Cramer's rule guarantees an explicit representation for the solution. The following theorem will create conditions for our equations that ensure our substitution is invertible.

Theorem

Let $H \subset \mathbb{R}$ be a nonempty set, and $g_1, \ldots, g_n : H \to H$ be functions forming a group. Also let $\alpha_1, \ldots, \alpha_n; h : H \to \mathbb{R}$ be given functions and assume that the determinant

$$D := \begin{vmatrix} \alpha_{\pi_1(1)} \circ g_1 & \cdots & \alpha_{\pi_1(n)} \circ g_1 \\ \vdots & \ddots & \vdots \\ \alpha_{\pi_n(1)} \circ g_n & \cdots & \alpha_{\pi_n(n)} \circ g_n \end{vmatrix}$$

is nonzero on H. Then there exists a unique function $f : H \to \mathbb{R}$ satisfying the functional equation (1).

For simplicity, assume g_1 to be the identity element of the group. Recall equation (1):

$$\alpha_1 f \circ g_1 + \cdots + \alpha_n f \circ g_n = h$$

Compose both sides of (1) with g_k for $k = 1, \ldots, n$. As $g_j \circ g_k = g_{\sigma_k(j)}$, we yield the following equation:

$$(\alpha_1 \circ g_k) f \circ g_{\sigma_k(1)} + \cdots + (\alpha_n \circ g_k) f \circ g_{\sigma_k(n)} = h \circ g_k$$

Recall that $\sigma_k(j) := l$ and $\pi_k(l) := j$, provided $g_j g_k = g_l$.
Then the coefficient of $f \circ g_j$ is $\alpha_{\pi_k(j)} \circ g_k$. We then rearrange the equation into the form

$$(\alpha_{\pi_k(1)} \circ g_k) f \circ g_1 + \cdots + (\alpha_{\pi_k(n)} \circ g_k) f \circ g_n = h \circ g_k$$

Proof of Theorem 1

$$(\alpha_{\pi_k(1)} \circ g_k) f \circ g_1 + \cdots + (\alpha_{\pi_k(n)} \circ g_k) f \circ g_n = h \circ g_k$$

Denote the determinant of the coefficient matrix of this system as D, which from assumption is nonsingular on H. Next, replace the kth column of D by the vector with components $h \circ g_1, \ldots, h \circ g_n$, and denote this obtained determinant as D_k. Then, define the function $f_k : H \to \mathbb{R}$ by

$$f_k := \frac{D_k}{D}$$

We expect that the function $f := f_1$ is the unique solution of (1). To show this, we must verify that the system is not contradictory; that is, the compatibility equation $f_1 \circ g_k = f_k$ holds on H.

To begin, we will evaluate the numerator of $f_1 \circ g_k$:

$$D_1 \circ g_k = \begin{vmatrix} h \circ g_1 \circ g_k & \alpha_{\pi_1(2)} \circ g_1 \circ g_k & \cdots & \alpha_{\pi_1(n)} \circ g_1 \circ g_k \\ h \circ g_2 \circ g_k & \alpha_{\pi_2(2)} \circ g_2 \circ g_k & \cdots & \alpha_{\pi_2(n)} \circ g_2 \circ g_k \\ \vdots & \vdots & \ddots & \vdots \\ h \circ g_{\sigma_k(n)} & \alpha_{\pi_n(2)} \circ g_{\sigma_k(n)} & \cdots & \alpha_{\pi_n(n)} \circ g_{\sigma_k(n)} \end{vmatrix}$$

$$= \begin{vmatrix} h \circ g_{\sigma_k(1)} & \alpha_{\pi_1(2)} \circ g_{\sigma_k(1)} & \cdots & \alpha_{\pi_1(n)} \circ g_{\sigma_k(1)} \\ h \circ g_{\sigma_k(2)} & \alpha_{\pi_2(2)} \circ g_{\sigma_k(2)} & \cdots & \alpha_{\pi_2(n)} \circ g_{\sigma_k(2)} \\ \vdots & \vdots & \ddots & \vdots \\ h \circ g_{\sigma_k(n)} & \alpha_{\pi_n(2)} \circ g_{\sigma_k(n)} & \cdots & \alpha_{\pi_n(n)} \circ g_{\sigma_k(n)} \end{vmatrix}.$$

Proof of Theorem 1

Recall Lemma 1: $\pi_{\pi_k(j)} = \pi_j \sigma_k$.
If j_i denotes the index that satisfies $\sigma_k(j_l) = l$, then $j_l = \pi_k(l)$.
Hence, we may rearrange the rows into the natural order, and apply Lemma 1. From this, the determinant $D_1 \circ g_k$ previously shown can be transformed up to a possible change in sign, as follows:

$$
\begin{aligned}
D_1 \circ g_k &= \begin{vmatrix} h \circ g_1 & \alpha_{\pi_{\pi_k(1)}(2)} \circ g_1 & \cdots & \alpha_{\pi_{\pi_k(1)}(n)} \circ g_1 \\ h \circ g_2 & \alpha_{\pi_{\pi_k(2)}(2)} \circ g_2 & \cdots & \alpha_{\pi_{\pi_k(2)}(n)} \circ g_2 \\ \vdots & \vdots & \ddots & \vdots \\ h \circ g_n & \alpha_{\pi_{\pi_k(n)}(2)} \circ g_n & \cdots & \alpha_{\pi_{\pi_k(n)}(n)} \circ g_n \end{vmatrix} \\
&= \begin{vmatrix} h \circ g_1 & \alpha_{\pi_1\sigma_k(2)} \circ g_1 & \cdots & \alpha_{\pi_1\sigma_k(n)} \circ g_1 \\ h \circ g_2 & \alpha_{\pi_2\sigma_k(2)} \circ g_2 & \cdots & \alpha_{\pi_2\sigma_k(n)} \circ g_2 \\ \vdots & \vdots & \ddots & \vdots \\ h \circ g_n & \alpha_{\pi_n\sigma_k(2)} \circ g_n & \cdots & \alpha_{\pi_n\sigma_k(n)} \circ g_n \end{vmatrix}.
\end{aligned}
$$

Proof of Theorem 1

As σ_k is defined as a permutation, $\sigma_k(1), \ldots, \sigma_k(n)$ are pairwise distinct elements of the set $\{1, \ldots, n\}$. However, $\sigma_k(1) = k$, as $g_1 g_k = g_k$.

Hence, the determinant of the last previous expression contains all columns of components $\alpha_{\pi_1(j)} \circ g_1, \ldots, \alpha_{\pi_n(j)} \circ g_n$, except when $j = k$.

Therefore, $D_1 \circ g_k$ consists of the same columns as D_k.
In other words, the absolute values of these determinants coincide.

Proof of Theorem 1

Lastly, a similar argument can be made to show that the absolute value of the determinants $D \circ g_k$ and D are equal. We see that $D \circ g_k$ is also nonsingular on H, and applying the same permutations among the rows and columns of $D_1 \circ g_k$ to obtain D_k can be done to obtain D from $D \circ g_k$. Therefore, we've reached the following property:

$$f_1 \circ g_k := \frac{D_1 \circ g_k}{D \circ g_k} = \frac{D_k}{D} = f_k.$$

Thus, we've proved that the compatibility equation holds true, and so D_k/D is in fact a solution to the equation.

Application of Proof

Let's use our result against another example.
Solve the functional equation

$$3f(x) + xf(1-x) = sin(x)$$

Consider the group of functions $g_1(x) = x$, and $g_2(x) = 1 - x$. This group is indeed a finite cyclic group, and we shall substitute these into the equation:

$$\begin{array}{rcccl} 3f(g_1(x)) & + & g_1(x)f(g_2(x)) & = & sin(g_1(x)) \\ 3f(g_2(x)) & + & g_2(x)f(g_1(x)) & = & sin(g_2(x)) \end{array}$$

$D = 3\sin(x) - x\sin(1-x)$
$D_1 = x^2 - x + 9$

$$f(x) = \frac{3\sin x - x\sin(1-x)}{x^2 - x + 9}$$

Applications Outside of Competition?

Can we apply these results to functional equations from other fields? How about the Gamma Function?

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

The integral converges to an analytic function on $\mathrm{Re}(z) > 0$.

Furthermore, integration by parts shows that $\Gamma(z+1) = z\Gamma(z)$. Rewrite as $\Gamma(z+1) - z\Gamma(z) = 0$, and the equation is of the form we've investigated. However, we cannot apply our new method to this equation due to the domains involved. We see that $g(z) = z+1$ generates an infinite group, and so this cannot preserve our domain.

Applications Outside of Competition?

How about the Zeta function? The functional equation of the Zeta function is

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

,

Which is indeed of the form we've investigated. Furthermore, we know that $g(s) = 1 - s$ indeed creates a cyclic group. However, our method still does not work for this equation. This is due to the definition of Zeta.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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