MATH-3194, Written Component

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1 Motivation

In the paper *Functional Equations and Finite Groups of Substitutions*, Dr. Bessenyei discusses a common method for solving functional equations in math competitions by substituting a cyclic group of functions into the equation. While the method itself is straight-forward, the underlying construction and consequences of the method are rather intricate, and is what the paper focuses on. I believe such a focus is necessary when dealing with these common competition skills. Such intricacies are not only glossed over in preparation of competition skills, but studying these special conditions can lead to a deeper understanding of the method itself. As such, I believe that to be the primary motivation for investigating this method.

2 Content

Dr. Bessenyei begins by introducing the concept of the functional equation and its applications, including a basic example of the functional equations the method is interested in. With this, he moves into applying the method to the example, showing that it is indeed a solution to the example presented. It's important to note that there was a slight typo in his calculations of determinants, and so the solution he presented wasn't exactly right. A bit or recalculation showed that his method indeed yielding a solution, and did not change any of the results of the paper, as discussed in my preliminary report. After presenting this method, he explains the underlying intricacies that can lead to potential problems. This method of substitution, is essentially a change of variables to the equation. The issue with a change of variables via a substitution of functions, is that these functions might not necessarily be invertible; as such it isn't guaranteed that the method provided yields a solution to the problem. To combat this, Dr. Bessenyei presents a proof showing that, under some reasonable conditions, the method will guarantee a solution.

3 Execution

In his paper, Dr. Bessenyei focused on two forms of functional equations: a single-variable instance, formed through a linear combination of composition with functions, as well as a multivariable case. Due to time constrains and a lack of knowledge on functional analysis, I restricted my presentation to the linear case, where Cramer's rule could be applied to solve. To compenstate for the brevity of this decision, along with Dr. Daileda's encouragement, I included applications to number theory, by attempting the method to the Gamma and Zeta functions. Moreover, I showed that the methods of substitution could not provide explicit solutions to neither the Gamma nor the Zeta function. For the Gamma function, you simply cannot construct a cyclic group that allows the function to map back onto itself. Essentially, the substitution cannot preserve the domain of the function. As for the Zeta function, this included a more interesting result, but ended with the same problem. Upon inspection of the functional equation of the Zeta function, it is expressed in the same linear form mentioned in the paper, with functions

$\zeta(s)$ and $\zeta(1-s)$ as the unknowns. Moreover, the domains of these functions indicate that a cyclic group of order two can be generated; namely, with element $g(s) = s$, and $g(s) = 1-s$, under the operation of composition. In fact, an application of this transformation on the complex plane results in a half rotation of the domain about the axis $x = 1/2$. At first glance, it would seem the substitution preserves symmetry across the domain. However, the subtleties from the definition of the Zeta function itself results in some issues. For example, consider $\zeta(1)$. This function generates the harmonic series, which indeed diverges and cannot be evaluated. On the other hand, consider $\zeta(0)$, which through analytic continuation can be evaluated to $-1/2$. By applying this substitution, we will be interchanging these definitions, basically saying that $\zeta(1)$ can be evaluated. As such, the function does not map to itself, thus the transformation will not work. Perhaps some other approach could alleviate this issue, and could be a focus of future research.

4 Potential Improvements

As evident by my presentation, the approach for the proof isn't one that is exactly intuitive or clean. Dr. Bessenyei introduces functions defined as follows:

For a finite group G with elements $g_1, \ldots, g_n$, and with a fixed index $k \in \{1, \ldots, n\}$, define permutations σ_k and π_k as $\sigma_k(j) := l$, and $\pi_k(l) := j$, provided $g_j g_k = g_l$ holds true. Using these definitions, the proof turns into a relabeling of elements, until the identity $f_k = f \circ g_k$ holds true, indicating that the substitution is indeed invertible. Such methods leads to complicated and messy notation, as evident by my presentation, which leaves little clarity and intuition to gain from the proof. This was an issue both Dr. Daileda and I were concerned with, so I'd like to address it in this report.

These implementations of the σ_k and π_k functions are essentially a method of relabeling; indexing the elements of the group of arbitrary size, and is very tricky to dissect at first glance, especially in the context of an infinite linear system, decorated with composition and lots of sub-indexing. Instead, notice that σ_k is the permutation of G, induced by right multiplication by g_k. Similarly, π_k is the permutation of G, induced by right multiplication of g_k^{-1}. Indeed, σ_k is saying that for some $l = l_k(j) = \sigma_k(j)$, for any $l \in \{1, \ldots, n\}$. Essentially, consider xg_K, and by definition $xg_k = g_l$, or equivalently, $x = g_l g_k^{-1}$. This element x is simply g_j. Rather than this mess of indexing, we could define σ_k as follows: for $g \in G$, let $\sigma_g : G \to G$, by $\sigma_g(h) = hg$, and similarly, $\pi_g : G \to G$ by $\pi_g(h) = hg^{-1}$. These functions are both bijective by definition, and in fact are inverses of each other, as desired by the original definitions of the functions. The consequent lemma's then become much clearer, and in turn seems to be a much more intuitive representation.

Overall, the results of this paper were very interesting, and leave plenty of room for further expansion. Working on this project has a wonderful new experience, and I thoroughly enjoyed the content I learned to work on this. If possible, I would like to continue working on this project, perhaps rewriting the entirety of the proof in these new definitions previously outlined.

5 References

Ed Scheinerman, Mihaly Bessenyei. *Functional Equations and Finite Groups of Substitutions*. The American Mathematical Monthly, 117(10),921-927. https://doi.org/10.4169/000298910x523416

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