ECEN 5407 Homework #6

Spencer Egan

Problem 1.

(i)

We can quickly see that $\dot{x} = 0$ when either $x = 0$ or $(1 - \frac{x}{\kappa}) = 0$. This second term is zero when $x = \kappa$. $\dot{x} = 0$ at no other points so these are the two equilibrium points.

(ii)

We wish to solve the ODE

$$\frac{dx}{dt} = rx(1 - \frac{x}{\kappa}).$$

Rearranging and taking the integral of both sides we have

$$\int \frac{1}{x(1 - \frac{x}{\kappa})} dx = \int r dt. \tag{1}$$

Now, $\frac{1}{x(1-\frac{x}{\kappa})}$ can be decomposed into two fractions of the form $\frac{a}{x} + \frac{b}{1-\frac{x}{\kappa}}$ such that by equating coefficients in x we have $b - \frac{a}{\kappa} = 0$ and $a = 1$. From this we see that $b = \frac{1}{\kappa}$, which allows us to write (1) as

$$\int \frac{1}{x} dx + \int \frac{1}{\kappa(1 - \frac{x}{\kappa})} dx = \int r dt.$$

Integrating, expoentiating and rearranging in terms of x we have

$$\begin{aligned}
\ln x - \ln(\kappa - x) &= rt + C \\
\frac{x}{(\kappa - x)} &= C' e^{rt} \\
x &= \kappa C' e^{rt} - x C' e^{rt} \\
x(1 + C' e^{rt}) &= \kappa C' e^{rt} \\
x &= \frac{\kappa C' e^{rt}}{1 + C' e^{rt}} \\
x &= \frac{\kappa e^{rt}}{\frac{1}{C'} + e^{rt}}.
\end{aligned}$$

Solving for our constant C' in terms of $x(0)$ we see that

$$\begin{aligned}
x(0) &= \kappa C' - x(0) C' \\
C' &= \frac{x(0)}{\kappa - x(0)}.
\end{aligned}$$

Plugging this into our general expression for x we see that

$$\begin{aligned} x &= \frac{\kappa e^{rt}}{\frac{\kappa - x(0)}{x(0)} + e^{rt}} \\ &= \frac{\kappa x(0) e^{rt}}{\kappa - x(0) + x(0)e^{rt}} \\ &= \frac{\kappa x(0) e^{rt}}{\kappa + x(0)(e^{rt} - 1)} \end{aligned}$$

(iii)

We know that $x = 0$ and $x = \kappa$ are the two equilibrium points. Checking the value of $\dot{x}$ for $0 < x < \kappa$ we see that both terms of $\dot{x}$ are positive and therefore $x(t)$ will monotonically increase towards κ for $0 < x(0) < \kappa$. Since $\dot{x}$ is continuous and differentiable in this region, and equal to 0 at $x = \kappa$ we know that $x(t)$ asymptotically approaches zero if $x(0)$ is within these bounds.

(iv)

for $\kappa < x$, rx will of course always be greater than zero, and $(1 - \frac{x}{\kappa})$ will always be less than zero, meaning that in this region $x(t)$ will be monotonically decreasing. Since again $\dot{x}$ is continuous and differentiable in this region, $x(t)$ asymptotically approaches κ for $k < x(0)$.

Problem 2.

We consider the non-linear system:

$$\begin{aligned} \dot{x_1} &= -2x_1 - 2x_2 - 4x_1^3 x_2^2 \\ \dot{x_2} &= -2x_1 - 2x_2 - 2x_1^4 x_2 \end{aligned}$$

where we have assumed that the coefficient of the $x_1^4 x_2$ term in $\dot{x_2}$ is not a 4 as stated in the problem.

Noticing the symmetry of the system, we note that can be expressed as a negative gradient dynamical system $\dot{x} = -\nabla V(x)$ where $V(x) = x_1^2 + x_2^2 + 2x_1x_2 + x_1^4 x_2^2$ which we can rewrite as $V(x) = (x_1 + x_2)^2 + x_1^4 x_2^2$. Clearly, since all terms have even powers, we see that

$$V(x) > 0 \ \ \forall \ \ x \in \mathbb{R}^2 \setminus \{0\}. \tag{2}$$

Taking the Lie derivative of V with respect to the function $f(x) = \dot{x}$ we see that

$$\begin{aligned} \mathcal{L}_f V(x) &= (2x_1 + 2x_2 + 4x_1^3 x_2^2)(-2x_1 - 2x_2 - 4x_1^3 x_2^2) + (2x_2 + 2x_1 + 2x_1^4 x_2)(2x_2 + 2x_1 + 2x_1^4 x_2) \\ &= -(2x_1 + 2x_2 + 4x_1^3 x_2^2)^{-}(2x_2 + 2x_1 + 2x_1^4 x_2)^2 \end{aligned}$$

Since both terms are negatives of squared numbers, we see that

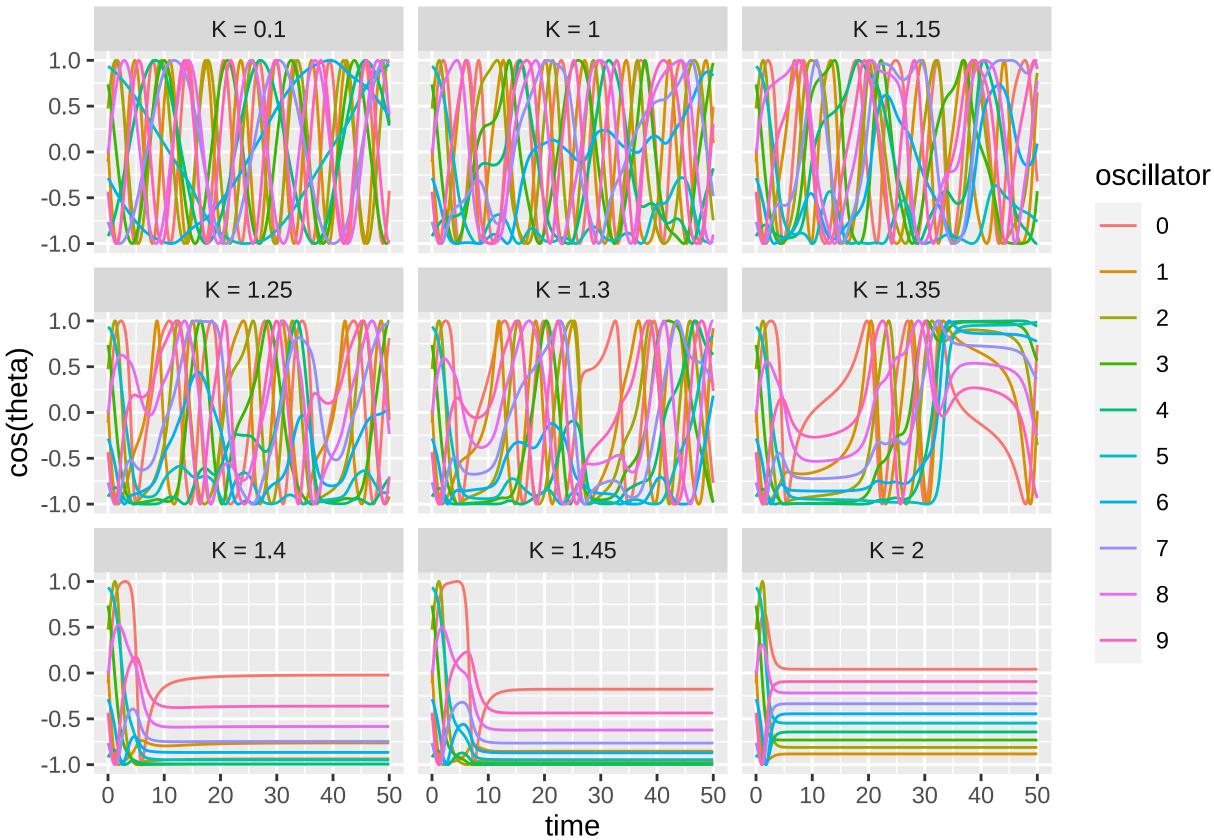
$$\mathcal{L}_f V(x) = \dot{V}(x) < 0 \ \ \forall \ \ x \in \mathbb{R}^2 \setminus \{0\}. \tag{3}$$

From (2) and (3), combined with the fact that $\lim_{x \to \infty} V(x) = \lim_{x \to -\infty} V(x) = \infty$ we know that our dynamical system is globally asymptotically stable with a region of attraction of all of $\mathbb{R}^2$.

Problem 3.

We model the coupled Kuramoto oscillators across a range of different values for the coupling constant K. We notice that partial synchronization occurs for values of K greater than 1, and strong synchronization occurs for values of K greater than about 1.4.

```r
N <- 10
Klist <- c(0.1,1,1.15,1.25,1.3,1.35,1.4,1.45,2)

Kuramoto <- function(time, theta, pars) {
  with(as.list(c(theta, pars)),{
    n <- length(theta)
    th_dot <- rep(0,5)
    for(i in 1:n){
      th_diff <- theta[i] - theta
      th_dot[i] <- omega[i] - (Kpar/n) * sum(sin(th_diff))
    }
  return(list(th_dot))
  })
}


times <- seq(0,50,0.025)
theta0 <- sample(seq(-pi,pi,0.02),N,TRUE)
omega = seq(-1,1, length.out=N)

sims = data.frame(matrix(ncol = N+2, nrow = 0))
colnames(sims) <- c('time', 'K', paste('X', 1:N, sep = ''))
facet_lbl <- rbind(t(Klist), t(paste('K =',Klist)))
colnames(facet_lbl) <- facet_lbl[1,]
facet_lbl <- facet_lbl[2,]

for(k in Klist){
  params <- list(Kpar = k, omega = omega)
  sim <- ode(theta0, times, Kuramoto, params)
  sim <- data.frame(sim) %>% mutate(K = k)
  sims <- bind_rows(sims, sim)
}

plt_kuramoto <- sims %>%
  pivot_longer(!c('time', 'K'), names_to = 'oscillator', values_to = 'phase') %>%
  mutate(phase = cos(phase), oscillator = as.integer(str_sub(oscillator, -1))) %>%
  ggplot(aes(time, phase, color = as.factor(oscillator))) + geom_line() +
  facet_wrap(facets = vars(K), labeller = as_labeller(facet_lbl)) +
  labs(y='cos(theta)', color = 'oscillator')

plt_kuramoto
```



The code below animates the dynamics of the particles phase around the unit circle for $K = 1.3$

```
params_crit <- list(Kpar = 1.3, omega = omega)
theta0 <- sample(seq(-pi,pi,0.02),N,TRUE)
times_anim <- seq(0,200,0.2)
sim_anim <- ode(theta0, times, Kuramoto, params_crit)

anm_kuramoto <- data.frame(sim_anim) %>%
  pivot_longer(!c('time'), names_to = 'oscillator', values_to = 'theta') %>%
  mutate(X = cos(theta), Y = sin(theta),
         oscillator = as.integer(str_sub(oscillator, -1))) %>%
  select(-c('theta')) %>%
  ggplot(aes(X, Y, color = as.factor(oscillator))) + geom_point() +
  labs(x='cos(theta)', y='sin(theta)', color = 'oscillator') +
  theme(aspect.ratio = 1) + transition_time(time) + ease_aes('linear')

anm_kuramoto
```

Problem 4.

(1)

We see that the transition matrix A is given by

$$A = \begin{bmatrix} 1 & 0 \\ \alpha & (1-\alpha) \end{bmatrix}$$

Clearly A is row-stochastic since $\alpha + (1-\alpha) = 1$.

(ii)

Since A is row stochastic, we immediately know that 1 is an eigenvalue and that $\begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ is the associated right eigenvector. To solve for the left eigenvector associated with the eigenvalue of 1, we solve the system of equations $a + b = a$ and $b(1 - \alpha) = b$. Clearly $b = 0$ and so the left eigenvector associated with the eigenvalue of 1 to be $\begin{bmatrix} 1 & 0 \end{bmatrix}^\top$.

The characteristic equation for this matrix is given by $(\lambda - 1)(\lambda - (1 - \alpha)$. Clearly the roots are given by $\{1, (1 - \alpha)\}$. For the second eigenvalue $(1 - \alpha)$ we can see that right eigenvector is given by $\begin{bmatrix} 0 & 1 \end{bmatrix}^\top$ and the left eigenvector is given by $\begin{bmatrix} 1 & -1 \end{bmatrix}^\top$.

(iii)

The graph associated with this algorithm has self loops at every node, and has a directed edge from node 2 to node 1, but no directed edge from node 1 to node 2. The graph is drawn below in figure 1.

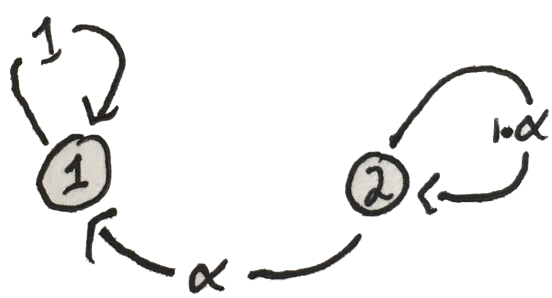


Figure 1: (ii) digraph of A

(iv)

The condensation graph is drawn below in figure 2.

Figure 2: (ii) condensation graph of A

(v)

Since $\alpha > 0$ we know that the eigenvalue 1 is simple, and therefore from Theorem 5.1 of [Bullo, 2022] we know that

$$\lim_{k\to\infty} x(k) = \left(\begin{bmatrix}1 & 0\end{bmatrix} x(0)\right)\mathbb{1} \begin{bmatrix}x(0)_1 & x(0)_1\end{bmatrix}^\top$$

where we have defined $x(0) = \begin{bmatrix}x(0)_1 & x(0)_2\end{bmatrix}^\top$.