

# Unrolling a Recurrence

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Work with your team to write a report showing your knowledge of the unrolling technique to solve a recurrence.

Submit the team's report of Canvas. Include a matrix indicating who did what.

*Sums*

Give a closed form expression for the sums below.

1. Sigma notation:

$$\sum_{i=0}^{i=n-1} 1 \quad (1)$$

Closed form expression:

$$n \quad (2)$$

2. Sigma notation:

$$\sum_{i=0}^{i=n-1} 2^i \quad (3)$$

Closed form expression:

$$2 - 2^{n-1} \quad (4)$$

3. Sigma notation:

$$\sum_{k=0}^{k=n} (n!/k!(n-k)!) \quad (5)$$

Closed form expression:

$$2^n \quad (6)$$

4. Sigma notation:

$$\sum_{k=0}^{k=n-1} r^k \quad (7)$$

Closed form expression:

$$(r^n - 1)/(r - 1) \quad (8)$$

5. Sigma notation:

$$\sum_{i=0}^{i=n-1} kr^{k-1} \quad (9)$$

Closed form expression:

$$knr^{k-1} \quad (10)$$

## Unrolling a Recurrence

I like this recurrence solving technique because it only involves simple algebraic substitutions and an ability to compute closed form expressions for common sums. Here is an example of unrolling that you can use as a template for other problems.

Consider the recurrence:

$$T(n) = T(n/2) + 1, T(0) = 1$$

The idea is simple: if

$$T(n) = T(n/2) + 1, T(1) = 0$$

then by substitution (unrolling)

$$T(n/2) = T(n/4) + 1, T(1) = 0$$

n can be divided by 2 a only a finite number of times before 1 is reached ending the unrolling. (It is okay to assume  $n = 2^p$  or  $p = \log n$  and the unrolling ends after p stages.) In another step

$$T(n) = T(n/4) + 1 + 1, T(1) = 0$$

Convince yourself this results in adding 1 to itself p times computing the sum of values in the first column of Pascal's triangle.

Therefore,

$$T(n) = p = \log(n)$$

Make the connection that the recurrence models the binary search algorithm.

Here are some additional problems to flex and sharpen your skills.

1. Solve the recurrence

$$T(n) = T(n-1) + n, T(0) = 1$$

$$T(n) = T(n-1)^2 + n$$

$$T(n) = n^2 - 2n + 1 + n$$

$$T(n) = n^2 - (2-1)n + 1$$

$$T(n) = n^2 - (2-1)n + 1$$

$$T(n) = n^2$$

Therefore,

$$T(n) = n^2$$

Mention the algorithm that is described by this recurrence.

Bubble sort is an algorithm that has a recurrence of  $O(n^2)$ .

2. Solve the recurrence

$$T(n) = 2T(n/2) + n, T(0) = 1$$

Assume,

$$n/2^k = 1$$

$$n = 2^k$$

$$k = \log(n)$$

$$nk = n\log(n)$$

$$T(n) = 2T(n/2) + n$$

$$T(n/2) = 2T(n/4) + n/2$$

$$T(n) = 2T(n/2) + n$$

$$T(n) = 2[2T(n/4) + n/2] + n$$

$$T(n) = 2^k T(n/2^k) + kn$$

$$T(n/2^k) = T(1)$$

$$n/2^k = 1$$

$$n = 2^k$$

$$k = \log(n)$$

$$nk = n \log(n)$$

Therefore,

$$T(n) = n \log(n)$$

Mention the algorithm that is described by this recurrence.

Mergesort is a recursive algorithm that has a time complexity of  $O(n \log(n))$ .

3. Solve the recurrence

$$T(n) = 2T(n-1) + 1, T(1) = 1$$

Assume,

$$n/2^k = 1$$

$$n = 2^k$$

$$k = \log(n)$$

$$nk = n \log(n)$$

$$T(n) = 2T(n/2) + n$$

$$T(n/2) = 2T(n/4) + n/2$$

$$T(n) = 2T(n/2) + n$$

$$T(n) = 2[2T(n/4) + n/2] + n$$

$$T(n) = 2^k T(n/2^k) + kn$$

$$T(n/2^k) = T(1)$$

$$n/2^k = 1$$

$$n = 2^k$$

$$k = \log(n)$$

$$nk = n \log(n)$$

Therefore,

$$T(n) = n \log(n)$$

What famous problem is described by this recurrence?

<b>Name</b>	<b>Section</b>
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Tyler Gutowski	Closed form expression sums
Spencer Hirsch	Unrolling a recurrence