CGAUtil Math

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This document is formal documentation for the CGAUtil Lua module, supplementary to the GALua module. Here in is explained the mathematics used by the CGAUtil module to compose and decompose each geometric primitive of 3-dimensional CGA, the conformal model of geometric algebra for 3-dimensional euclidean space. It is assumed the reader is already familiar with CGA. In this document we let the outer product take precedence over the inner product, and the geometric product take precedence over the inner and outer products.

It should be noted that the CGA system is patented by David Hestenes, Hongbo Li, et. al.

1 Composition

We begin with an explanation of the composition of each geometric primitive. The geometric primitives of CGA are listed in Table 1, along with the parameters characterizing each geometry.[1] Each geometry can be represented in a direct or dual form. Table 2 lists each geometry again, this time giving the grade of blade representing each geometrying in dual form. This table is not a comprehensive listing for each grade. For example, the null pseudo-vectors of CGA are also dually representative of points. The table does, however, completely describe the CGAUtil module's choice of which grades of blades it uses to dually represent the geometric primitives of CGA. The grade g of a

[1]The reader may notice that tangent points and free blades are missing from Table 1. As points are simply degenerate spheres, tangent points arrise as the degenerate point-pairs and circles. Free blades are the intersection of distinct and parallel planes. Tangent points and free blades are, as of this writing, not explicitly supported, but only implicitly in the form of these special cases.

	weight	center	normal	radius	real/imaginary
point	yes	yes	no	no	no
flat-point	yes	yes	no	no	no
point-pair	yes	yes	yes	yes	yes
line	yes	yes	yes	no	no
circle	yes	yes	yes	yes	yes
plane	yes	yes	yes	no	no
sphere	yes	yes	no	yes	yes

Table 1: The parameters characterizing each CGA geometry.

grade 1	grade 2	grade 3
point	circle	point-pair
sphere	line	flat-point
plane		

Table 2: CGAUtil's grades for dual geometries.

blade used by CGAUtil to directly represent each of the geometric primitives in this table is given by $5 - g$.

The composition (and decomposition) of each CGA geometry is given in terms of its dual form, because a such form naturally presents the parameters (the columns of Table 1) characterising each geometry (a row of Table 1).

For the compositions (and decompositions) of this document, we will let the variables w, c, n and r denote the weight, center, normal and radius of each geometry, respectively. Each of w and r are scalars while each of c and n are euclidean vectors. The normal vector n is always of unit-length. We always require $r \geq 0$. Then, letting $\{e_1, e_2, e_2, o, \infty\}$ be a set of basis vectors generating the vector space $\mathbb{V}$ that in turn generates the geometric algebra $\mathbb{G}$, and letting I denote the unit-pseudo scalar of G and i denote the unit-pseudo scalar of the largest euclidean geometric sub-algebra of G, we can now procede to give the composition of each geometry.

1.1 Point Composition

A point ρ being characterized by a weight w and center c, is given by

$$\rho = w\left(o + c + \frac{1}{2}c^2\infty\right). \tag{1}$$

1.2 Sphere Composition

A sphere σ being characterized by a weight w, center c and radius r, is given by

$$\sigma = w\left(o + c + \frac{1}{2}(c^2 - sr^2)\infty\right), \tag{2}$$

where s is a scalar being 1 or -1. We have $s = 1$ in the case that σ is a real sphere, and $s = -1$ in the case that σ is an imaginary sphere.

1.3 Plane Composition

A plane π being characterized by a weight w, center c and normal n, is given by

$$\pi = w\left(n + (c \cdot n)\infty\right). \tag{3}$$

1.4 Circle Composition

A circle γ being characgterized by a weight w, center c, normal n and radius r, is given by

$$\begin{align}
\gamma &= w\left(o + c + \frac{1}{2}(c^2 - sr^2)\infty\right) \wedge (n + (c \cdot n)\infty) \tag{4}\\
&= w\left(o \wedge n + (c \cdot n)o \wedge \infty + c \wedge n + \left((c \cdot n)c - \frac{1}{2}(c^2 - sr^2)n\right) \wedge \infty\right), \tag{5}
\end{align}$$

where s is a scalar being 1 or -1. We have $s = 1$ in the case that γ is a real circle, and $s = -1$ in the case that γ is an imaginary circle. Our choice to multiply the sphere left of the plane in the outer product is arbitrary, but a choice warrenting documentation as this affects the sign of w.

1.5 Line Composition

A line λ being characterized by a weight w, center c and normal n, is given by

$$\lambda = w\,(n + (c \wedge n)\infty)\,i. \tag{6}$$

1.6 Point-Pair Composition

A point-pair β being characterized by a weight w, center c, normal n and radius r, is given by

$$\beta = w\left(o + c + \frac{1}{2}(c^2 - sr^2)\infty\right) \wedge (n + (c \wedge n)\infty)i \tag{7}$$

$$= w\left(o \wedge n + c \wedge n \wedge o \wedge \infty + c \cdot n + \left((c \cdot n)c - \frac{1}{2}(c^2 + sr^2)n\right) \wedge \infty\right) i, \tag{8}$$

where s is a scalar being 1 or -1. We have $s = 1$ in the case that β is a real point-pair, and $s = -1$ in the case that β is an imaginary piont-pair. Our choice to multiply the sphere left of the line in the otuer product is arbitrary, but a choice warrenting documentation as this affects the sign of w.

1.7 Flat-Point Composition

A flat-point ϕ being characterized by a weight w and center c, is given by

$$\phi = w(n + (c \cdot n)\infty) \wedge (n + (c \wedge n)\infty)i \tag{9}$$

$$= w(1 - c \wedge \infty)i, \tag{10}$$

where here the unit-length normal n cancels. Our choice to multiply the plane left of the line is arbitrary, but a choice warrenting documentation as this affects the sign of w.

2 Decomposition

Decomposition begins with a treatment of identification. In CGA, it is possible to show that the blades dually representative of real non-degenerate rounds are also directly representative of imaginary rounds. Given such a blade, we cannot uniquely determine a method of decomposition. Between the two possible geometric interpretations of the blade, we must choose which one we want, and then apply the corresponding decomposition method. There may be other instances were such a choice needs to be made. Any decomposition method of CGAUtil, therefore, does not attempt to identify blades as being of any particular type of CGA geometry. Instead, the decomposition methods of CGAUtil attempt to decompose a given blade under the assumption that it is of the a specific geometric type. Such an attempt will either pass or fail. An attempt has been made, however, in a provided routine, to identify a given blade as being one of a number returned geometric types. The user can then use this information to select a decomposition method.

2.1 Point Decomposition

A point ρ, given in equation (1), may be decomposed as follows.

$$w = -\rho \cdot \infty \tag{11}$$

$$c = o \wedge \infty \cdot \frac{\rho}{w} \wedge o \wedge \infty \tag{12}$$

If the weight w is zero, the decomposition fails. Furthermore, to insure a correct decomposition, the validity of the following equation should be checked.

$$-2\frac{\rho}{w} \cdot o = c^2 \tag{13}$$

The decomposition also fails if this equation is not satisified.

2.2 Sphere Decomposition

A sphere σ, given in equation (2), may be decomposed as follows.

$$w = -\sigma \cdot \infty \tag{14}$$

$$c = o \wedge \infty \cdot \frac{\sigma}{w} \wedge o \wedge \infty \tag{15}$$

If the weight w is zero, the decomposition fails. Realizing that $s^{-1} = s$, the radius r may be found as

$$r^2 = s\left(c^2 + 2\frac{\sigma}{w} \cdot o\right), \tag{16}$$

where here we may choose $s = 1$ or $s = -1$ so that $r^2 \geq 0$.

Note that given any vector σ, we should first attempt to decompose it as a point before decomposing it as a sphere. This is because every point is a sphere, but the converse of this statement is not true.

2.3 Plane Decomposition

A plane π, given in equation (3), may be decomposed as follows.

$$w = |o \cdot \pi \wedge \infty| \tag{17}$$

$$n = o \cdot \frac{\pi}{w} \wedge \infty \tag{18}$$

Then, assuming $c \wedge n = 0$, we may write

$$c = \left(-\frac{\pi}{w} \cdot o\right) n. \tag{19}$$

Such an assumption is necessary, because the center of a plane is arbitrary and not recoverable from composition. Our choice of center c here gives us a position on the plane closest to the origin.

Note that given any vector π, we should first try to decompose it as a point or sphere before decomposing it as a plane. If $\pi \cdot \infty \neq 0$, then we consider the above decomposition as undefined.

2.4 Circle Decomposition

2.5 Line Decomposition

2.6 Point-Pair Decomposition

2.7 Flat-Point Decomposition