Change Of Basis Transformations In Geometric Algebra

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July 27, 2012

Let $\mathbb{V}^{2n}$ denote a $2n$-dimensional Euclidean vector space, and $\mathbb{A}$ denote any n-dimensional Euclidean vector sub-space of $\mathbb{V}^{2n}$. Let $\mathbb{B}$ denote the complement of $\mathbb{A}$ with respect to $\mathbb{V}^{2n}$. Let $\{e_k\}_{k=1}^n$ be a set of n Euclidean vectors forming an orthonormal basis for $\mathbb{A}$. Let $\{e_{k+n}\}_{k=1}^n$ be a set of n Euclidean vectors forming an orthonormal basis for $\mathbb{B}$. We will work in the geometric algebra $\mathbb{G}(\mathbb{V}^{2n})$.

Defining the function r as the rotor $r(a,b) = \frac{\sqrt{2}}{2}(1 - a \wedge b)$, where a and b are vectors, we see that

$$R = \prod_{k=1}^{n} r(e_k, e_{k+n})$$

is a unit-rotor rotating the blade $\bigwedge_{k=1}^n e_k$ into $\bigwedge_{k=1}^n e_{k+n}$, or vice-versa. More to the point, R also effectly performs a change of basis transformation of any vector taken from $\mathbb{A}$ to its counter-part in $\mathbb{B}$, or vice-versa. This is illustrated in the following equation.

$$R\left(\sum_{k=1}^{n}(a\cdot e_k)e_k\right)\tilde{R} = \sum_{k=1}^{n}(a\cdot e_k)e_{k+n}$$

Here, a is any vector taken from $\mathbb{A}$ and $Ra\tilde{R}$ is in $\mathbb{B}$.

Unfortunately, we cannot employ the same technique we have used here in the formulation of R to formulate an element E such that for any vector

$a \in \mathbb{A}$, the element E takes a to any given basis as EaE^{-1}. Whether there is such an element in a geometric algebra will be left as an open question for now, but what will be said here is that clearly E cannot be a rotor, because rotors are angle preserving. We want to be able to perform change of basis transformations between basis sets that are not necessarily rotations of one another.

Defining the function $f : \mathbb{A} \to \mathbb{B}$ as $f(a) = Ra\tilde{R}$, we see that f is an outermorephism between $\mathbb{A}$ and $\mathbb{B}$. Now let $\{m_k\}_{k=1}^n$ be any set of n linearly independent vectors taken from $\mathbb{A}$ and let M be the bivector

$$M = \sum_{k=1}^{n} e_{k+n} \wedge m_k.$$

It then follows that the function $F : \mathbb{A} \to \mathbb{A}$ defined as

$$F(a) = f(a) \cdot M$$

performs any change of basis transformation. This includes those that perform shear and non-uniform scale transformations. To be more specific, the vector $a \in \mathbb{A}$ written as

$$a = \sum_{k=1}^{n} (a \cdot e_k) e_k$$

is transformed to

$$F(a) = \sum_{k=1}^{n} (a \cdot e_k) m_k.$$

There are a few problems with F, however. The inner product is not an invertible product and M does not have an inverse with respect to the geometric product. This means that when we go to solve the problem of finding the bivector M' such that $F^{-1}(a) = M' \cdot f(a)$, geometric algebra offers no obvious way to solve for M'. Never-the-less, we will make do with what we have here.

Taking a lesson from linear algebra, we will define $\det(M)$ as

$$\det(M) = \bigwedge_{k=1}^{n} F(e_k) \cdot I_{\mathbb{A}},$$

where $I_{\mathbb{A}}$ is the unit psuedo-scalar of $\mathbb{G}(\mathbb{A})$. Similarly, we will define $I_{\mathbb{B}}$ as the unit psuedo-scalar of $\mathbb{G}(\mathbb{B})$.