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A MODEL FOR QUADRIC SURFACES USING GEOMETRIC ALGEBRA

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Abstract. Inspired by the conformal model of geometric algebra, a similar model of geometry is developed for the set of all quadric surfaces in n-dimensional space. Bivectors of the geometric algebra are found to be representative of quadric surfaces. Coordinate free canonical forms of such bivectors are found for common quadric surfaces. The model is investigated for usefulness and compared to the conformal model.

1. The Construction Of The Model

The stage for this model of n-dimensional quadric surfaces is set in the geometric algebra we'll denote by $\mathbb{G}$ that is generated by a vector space $\mathbb{W}$ of dimension $2(n+1)$. Letting $\{e_i\}_{i=0}^{2n+1}$ be an orthonormal set of basis vectors generating $\mathbb{W}$, we let $\{e_i\}_{i=0}^{n}$ be such a set of vectors generating the $(n+1)$-dimensional vector subspace $\mathbb{V}$ of $\mathbb{W}$ in which we'll impose the usual interpretation of $(n+1)$-dimensional homogeneous space. Specifically, a vector $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ represents the point given by[1]

$$e_0 \cdot \frac{e_0 \wedge v}{e_0 \cdot v} \tag{1.1}$$

in n-dimensional Euclidean space, imposing the usual correlation between n dimensional vectors and n-dimensional points[2]. We will take the liberty of letting vectors $v \in \mathbb{V}$ with $v \cdot e_0 = 0$ represent points under the same interpretation of which has been just spoken, as well as pure directions with magnitude. The intended interpretation will be made clear in the context of our usage. We will refer to all vectors $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ as projective points, and such vectors with $v \cdot e_0 = 0$ as non-projective points.

We now introduce a function defined on $\mathbb{G}$ having the outermorphic property. This means it is a linear function and that it preserves the outer product. We will use over-bar notation to denote the use of this function. Doing so, for any element $E \in \mathbb{G}$, we define $\overline{E}$ as

$$\overline{E} = RE\tilde{R}, \tag{1.2}$$

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[1]Throughout this paper we let the outer product take precedence over the inner product, and the geometric product take precedence over both the inner and outer products.

[2]The correlation between vectors and points spoken of here is that of having a vector represent the point at its tip when its tail is placed at the origin.

where the rotor R is given by

$$R = 2^{-(n+1)/2} \prod_{i=0}^{n} (1 - e_i e_{i+n+1}) . \tag{1.3}$$

As the reader can check, for any integer $i \in [0, n]$, we have $\overline{e_i} = e_{i+n+1}$. The rotor R simply rotates any k-vector taken from the geometric algebra generated by $\mathbb{V}$ and rotates it into the identical geometric algebra generated by the vector space complement to $\mathbb{V}$ with respect to $\mathbb{W}$. This idea can be found in [1]. We will find the over-bar notation convenient when perform algebraic manipulations in our model.

We are now ready to give the definition by which we will interpret bivectors in $\mathbb{G}$ as n-dimensional quadric surfaces.

Definition 1.1. For any element $E \in \mathbb{G}$, we say that E is representative of the n-dimensional quadric surface generated by the set of all projective points $v \in \mathbb{V}$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{1.4}$$

Notice that when grade$(E) > 1$, there is no ambiguity, despite the non-associativity of the inner product, in rewriting equation (1.4) as

$$0 = p \cdot E \cdot \overline{p}, \tag{1.5}$$

which resembles a sort of conjugation of E by p. This may perhaps be a more familiar form for readers familiar with the study of quadric surfaces in projective geometry. Also notice that we have not required that E be a bivector in Definition 1.1, because we may find this condition useful and meaningful for any element of $\mathbb{G}$. For now, however, we will restrict our attention to the case when E is a bivector.

To see why Definition 1.1 works, simply notice that when E is a bivector, we have

$$p \wedge \overline{p} \cdot E = \sum_{i=0}^{n} \sum_{j=i}^{n} \lambda_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.6}$$

which we can recognize as a homogeneous polynomial of degree 2 in the vector components of p. The scalars λ_{ij}, with $0 \leq i \leq j \leq n$, may be formulated in terms of E by

$$\lambda_{ij} = \begin{cases} e_i \overline{e_j} \cdot E & \text{if } i = j, \\ (e_i \overline{e_j} - \overline{e_i} e_j) \cdot E & \text{if } i \neq j. \end{cases} \tag{1.7}$$

It should be noted that bivectors do not uniquely represent quadric surfaces, not even up to scale. This is apparent from equation (1.7) when we see that for $i \neq j$, we can freely choose certain components of the bivector without changing the represented quadric so long as that their sum is still $-\lambda_{ij}$.

An important difference to point out here between this model and the conformal model is that, unlike what we can analogously expect from the point-definition of the conformal model, here the 2-blade form $a \wedge \overline{a}$ found in Definition 1.1, for any projective point $a \in \mathbb{V}$, does not represent the projective point a under Definition 1.1. In homogenized form, the projective point represented by $a \wedge \overline{a}$ is given by

$$e_0 - \left(e_0 \cdot \frac{e_0 \wedge a}{e_0 \cdot a} \right)^{-1}, \tag{1.8}$$

which is the reflection about the origin of the spherical reflection of the projective point a about the unit-sphere centered at the origin. The only point that represents itself in the form $a \wedge \overline{a}$ appears to be e_0.

2. The Construction Of Quadric Surfaces In The Model

Having constructed our model, we are now ready to find canonical forms of bivectors representing a variety of well-known quadric surfaces. Let us begin with the spheroid, (a special case of ellipsoid), the circular cylinder, and the circular hyperboloid of one sheet. We will find that all of these surfaces share the same canonical form, because they may all be characterized as the non-projective point solution set of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2 \tag{2.1}$$

in the non-projective point $x \in \mathbb{V}$, where $c \in \mathbb{V}$ is a non-projective point denoting the center of the surface, $v \in \mathbb{V}$ is a unit-length direction vector, $r \in \mathbb{R}$ is the radius of the geometry about the axis v at c, and $\lambda \in \mathbb{R}$ is a scalar indicating the type and extremity of the surface. Specifically, if $\lambda < 1$, we get a spheroid, if $\lambda = 1$, we get a circular cylinder, and if $\lambda > 1$, we get a circular hyperboloid.

Expanding equation (2.1), we get

$$0 = x^2 - \lambda(x \cdot v)^2 - 2x \cdot (c + \lambda(c \cdot v)v) + c^2 + \lambda(c \cdot v)^2 - r^2, \tag{2.2}$$

from which it is possible to factor out $p \wedge \overline{p}$ in terms of the inner product, where $p = e_0 + x$ is a homogenized projective point. Doing so, we see that the bivector E given by

$$E = -\Omega + \lambda v \wedge \overline{v} + 2(c + \lambda(c \cdot v)v) \wedge \overline{e_0} - (c^2 + \lambda(c \cdot v)^2 - r^2)A, \tag{2.3}$$

is representative of the three surface types by Definition 1.1, where the constant Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i \overline{e_i}, \tag{2.4}$$

and A is the constant defined as $A = e_0 \overline{e_0}$. We will find each of these useful as frequently recurring constants in our calculations.

Such forms as that in equation (2.3) are useful, not only for composition, but especially decomposition in the cases where we have formulated what may, for example, be a spheroid by some other means. This gives the model power as an analytical tool. If we can solve a problem whose solution is a bivector representative of a spheroid, then we can use this canonical form to answer questions about that spheroid. Where is its center? What is its axis? What is its radius about that axis? As is often the case in mathematics, decomposition is harder than composition.

Referring to equation (2.3), we can deduce...give a decomposition of the spheroid once I know the form is correct. Consider E^2.

3. Making Use Of The Model

Admittedly, there is really nothing interesting about this model unless we can prove that it has some utility. The conformal model, for example, has at least two great features. The first is the utility of the wedge product in generating intersections between geometries in dual form, or point-fitting between geometries in direct form. A good user of the conformal model can even make use of dual imaginary intersections by reinterpreting them as real geometries in direct form. The second great feature of the conformal model is the surprising fact that all geometries in the conformal model are also versor transformations with geometric significance relative to the simultaneously represented geometry. Realizing that all conformal geometries, (with the exception of flat points), have a factorization in direct form as an outer product of points, the outermorphic property of versor conjugation allows us to predict the action of any versor transformation on almost any conformal geometry.

These are great features! But what can the model at present do for us? Well, the first observation we must make is that the set of all known quadrics is represented by the set of all bivectors in $\mathbb{G}$, underwhich the inner and outer products are obviously not closed. Only addition and subtraction are closed in this set, and so we're left to wonder what we might be able to prove about the addition and subtraction of n-dimensional quadric surfaces. Letting $A, B \in \mathbb{G}$ be bivectors, it is not hard to see that $A + B$, under Definition 1.1, must represent at least the intersection, if any, of the quadric surfaces A and B, but this is not an exact answer to the question of what surface $A + B$ represents.

References

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