

An Intro to CGA

Conformal Geometric Algebra

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The Outer Product

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Letting $\{w_k\}_{k=1}^m$ be the Gram-Schmidt orthonormalization of $\{v_k\}_{k=1}^m$, we have

$$\bigwedge_{k=1}^m v_k = \det \begin{bmatrix} v_1 \cdot e_1 & \dots & v_1 \cdot e_m \\ \vdots & \ddots & \vdots \\ v_m \cdot e_1 & \dots & v_m \cdot e_m \end{bmatrix} \bigwedge_{k=1}^m w_k,$$

where $\{e_k\}_{k=1}^m$ is any orthonormal basis for the m -dimensional vector sub-space represented by this blade.

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- Its **attitude**,
- Its m -dimensional **hyper-volume**,
- Its **handedness**.

Generating all Elements of a Geometric Algebra

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The geometric algebra $\mathbb{G}(\mathbb{V}^n)$ is of dimension 2^n .

Adding Blades Together

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Example

The following 2-vector cannot be written as a 2-blade.

$$e_1 \wedge e_2 + e_3 \wedge e_4$$

The Inner Product

Definition

In a Euclidean geometric algebra, we define for all integers i and j ,

$$e_i \cdot e_j = \delta_{ij},$$

where here, δ_{ij} is the Kronecker delta.

Definition

If for any vector $v \in \mathbb{V}^n$, we have $v \cdot v = 0$, we call v a null vector.

The Inner Product (Continued)

Definition

For any vector $v \in \mathbb{V}^n$ and any m -blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$v \cdot B = - \sum_{i=1}^m (-1)^i (v \cdot b_i) \bigwedge_{j=1, j \neq i}^m b_j,$$

where $B = \bigwedge_{k=1}^m b_k$. We also define

$$B \cdot v = -(-1)^n v \cdot B.$$

The Inner Product (Continued)

Example

Consider $v \cdot B$, where B is a 2-blade. WLOG, choose $a, b \in \mathbb{V}^n$ such that $B = a \wedge b$, $a \cdot b = 0$, $|b| = 1$ and $v \cdot b = 0$. We then have

$$v \cdot B = (v \cdot a)b - (v \cdot b)a = |B| \frac{v \cdot a}{|a|} b.$$

The Inner Product (Continued)

Definition

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ of grades i and j , respectively, we define

$$A \cdot B = \begin{cases} a_1 \cdots a_i \cdot B & \text{if } i \leq j, \text{ (R to L assoc.)} \\ A \cdot b_1 \cdots b_j & \text{if } i \geq j, \text{ (L to R assoc.)} \end{cases}$$

where $A = \bigwedge_{k=1}^i a_k$ and $B = \bigwedge_{k=1}^j b_k$.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any m -blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B,$$

and similarly, $Bv = B \cdot v + B \wedge v$.

Example

For any two vectors $a, b \in \mathbb{V}^n$, we have

$$ab = a \cdot b + a \wedge b = |a||b| \cos \theta + B|a||b| \sin \theta = |a||b| \exp(\theta B),$$

where $B = \frac{a \wedge b}{|a \wedge b|}$.

The Geometric Product (Continued)

Example

It can be shown that

$$v \cdot B = \frac{1}{2}(vB - (-1)^m Bv),$$

and

$$v \wedge B = \frac{1}{2}(vB + (-1)^m Bv).$$

The Geometric Product (Continued)

Definition

For any set of m vectors $\{v_k\}_{k=1}^m \subset \mathbb{V}^n$, a product

$$\prod_{k=1}^m v_k$$

is called a versor if for all integers k , v_k^{-1} exists.

If there exists an integer k such that v_k^{-1} does not exist, I call it a pseudo versor.

Lemma

*In a **Euclidean** geometric algebra, any blade can be written as a versor by the Gram-Schmidt orthogonalization process.*

The Geometric Product (Continued)

Example

For any two multivectors $A, B \in \mathbb{G}(\mathbb{V}^n)$, we have

$$AB = f^{-1}(f(A)f(B)),$$

where f is a function mapping a multivector to its multi-pseudo-vector form.

Lemma

*The Gram-Schmidt process cannot always be used on blades taken from a **non-Euclidean** geometric algebra!*

Proof.

Consider $a \wedge b$. If $a \cdot b \neq 0$ and a, b are null, then there does not exist a scalar λ such that $a \cdot (b + \lambda a) = 0$ or $(a + \lambda b) \cdot b = 0$. \square

Blade to Multi-Pseudo-Versor Form

Let $B \in \mathbb{G}(\mathbb{V}^n)$ be a blade of grade $m > 1$ where $B = \bigwedge_{k=1}^m b_k$.
We then have

$$\begin{aligned} f(B) &= B = b_1 B^{(1)} - b_1 \cdot B^{(1)} \\ &= b_1 f(B^{(1)}) - \sum_{k=2}^m (-1)^i (b_1 \cdot b_i) f(B^{(1)(i)}), \end{aligned}$$

where $B^{(i)}$ is notation for the $(m-1)$ -blade $\bigwedge_{k=1, k \neq i}^m b_k$.

Example

For the blade $a \wedge b$, we have $f(a \wedge b) = a \wedge b = ab - a \cdot b$.

Pseudo-Versor to Multivector Form

Let $V \in \mathbb{G}(\mathbb{V}^n)$ be a versor of size $m > 1$ where $V = \prod_{k=1}^m v_k$. We then have

$$f^{-1}(V) = V = v_1 \sum_{k=0}^m \langle V^{(1)} \rangle_k = \\ \langle f^{-1}(V^{(1)}) \rangle_0 v_1 + \sum_{k=2}^m \left(v_1 \wedge \langle f^{-1}(V^{(1)}) \rangle_k + v_1 \cdot \langle f^{-1}(V^{(1)}) \rangle_k \right),$$

where $V^{(i)}$ is notation for the $(m-1)$ -sized pseudo-versor $\prod_{k=1, k \neq i}^m v_k$.

Example

For the versor ab , we have $f(ab) = ab = a \cdot b + a \wedge b$.

The Geometric Product (Again)

Example

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ of grades i and j , respectively, it can be shown that

$$A \cdot B = \langle AB \rangle_{|i-j|},$$

and

$$A \wedge B = \langle AB \rangle_{i+j}.$$

The Reverse

Definition

For any m -sized versor $V \in \mathbb{G}(\mathbb{V}^n)$ where $V = \prod_{k=1}^m v_k$, we define

$$\tilde{V} = \prod_{k=1}^m v_{m-k+1}.$$

We can extend this definition to any multivector if we let the reverse operator distribute over addition.

Definition

For any multivector $E \in \mathbb{G}(\mathbb{V}^n)$, we may write

$$\tilde{E} = f^{-1}(\tilde{f}(E)).$$

The Inverse

Lemma

For any m -sized versor $V \in \mathbb{G}(\mathbb{V}^n)$ where $V = \prod_{k=1}^m v_k$, we have

$$V^{-1} = \left(\prod_{k=1}^m |v_k| \right)^{-1} \tilde{V}.$$

Lemma

For any **Euclidean** m -blade $B \in \mathbb{G}(\mathbb{V}^n)$ where $B = \bigwedge_{k=1}^m b_k$, we have

$$B^{-1} = (-1)^{n(n-1)} \tilde{B}.$$

Example

If $v \in \mathbb{V}^n$ is a null vector, then v^{-1} does not exist.

Conjugation by Versors

Lemma

Conjugation by versors is outermorphic. That is, for any versor $V = \prod_{k=1}^i v_k$, and any blade $B = \bigwedge_{k=1}^j b_k$, we have

$$VBV^{-1} = \bigwedge_{k=1}^j Vb_kV^{-1}.$$

The proof of this is not too hard to get, but too big to put here.

Example

A given rotor $R \in \mathbb{G}(\mathbb{V}^n)$ is a versor that rotates points $v \in \mathbb{V}^n$ by versor conjugation. It therefore rotates blades as well!

How Blades Can Represent Geometry

Let \mathbb{V}^n denote a **Euclidean** vector space.

Let \mathbb{V} denote any other vector space.

Let $p : \mathbb{V}^n \rightarrow \mathbb{G}(\mathbb{V})$ be a blade-valued function of points.

Definition

Given any blade $B \in \mathbb{G}(\mathbb{V})$, we say that B **directly** represents the geometry that consists of all points

$$G(B) = \{x \in \mathbb{V}^n | p(x) \in B\}.$$

Definition

Given any blade $B \in \mathbb{G}(\mathbb{V})$, we say that B **dually** represents the geometry that consistent of all points

$$G^*(B) = \{x \in \mathbb{V}^n | p(x) \in B^*\}.$$

Finally, The Conformal Model

Let \mathbb{V}^n be a vector-subspace of \mathbb{V} .

If $\{e_k\}_{k=1}^n$ is any basis for \mathbb{V}^n , let $\{e_k\}_{k=1}^n \cup \{o, \infty\}$ be a basis for \mathbb{V}^n .

Definition

For any vector $v \in \mathbb{V}^n$, we define $v \cdot o = v \cdot \infty = 0$. We define $o \cdot \infty = \infty \cdot o = -1$. Each of o and ∞ are defined as null.

Definition

We define $p : \mathbb{V}^n \rightarrow \mathbb{G}(\mathbb{V})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

What we can now discover about this model of geometry is almost endless!