An Intro to CGA
Conformal Geometric Algebra

Spencer T. Parkin

Avalanche Software

September 19, 2012

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

Proofs will often freely mix synthetic and algebraic arguments.

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

Proofs will often freely mix synthetic and algebraic arguments.
Geometries are elements of computation.

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

Proofs will often freely mix synthetic and algebraic arguments.

Geometries are elements of computation.

The versor group is a double cover for the group of conformal transformations.

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

Proofs will often freely mix synthetic and algebraic arguments.

Geometries are elements of computation.

The versor group is a double cover for the group of conformal transformations.

Invented by David Hestens in his paper, "Old Wine in New Bottles: A new algebraic framework for computational geometry."

In this presentation, we will...

In this presentation, we will...

- Introduce concepts from GA only as necessary,

Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

Let $\mathbb{V}^n$ denote an *n*-dimensional vector space.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from $\mathbb{V}^n$.

Blades

Let $\mathbb{V}^n$ denote an *n*-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of *m* vectors taken from $\mathbb{V}^n$.

Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero *m*-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from $\mathbb{V}^n$.

Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $\mathrm{grade}(B) = m \leq n$.

Visualizing Euclidean Blades

Imagine an infinite m-dimensional hyper-plane.

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.
Attitude matters, position does not.

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.
Attitude matters, position does not.
Non-Euclidean blades require more imagination!

Imagine an infinite m-dimensional hyper-plane.

Think of B as a finite m-dimensional hyper-plane.

Attitude matters, position does not.

Non-Euclidean blades require more imagination!

Our geometric arguments will not require us to visualize the homogeneous representation space.

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \notin B \text{ if and only if } v \in B^*,$$

where B^* represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

Let $v_{\parallel}$ denote the orthogonal projection of v down onto B.

Building Intuition About Euclidean Blades

Let $v_\parallel$ denote the orthogonal projection of v down onto B.

Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.

Let $v_\parallel$ denote the orthogonal projection of v down onto B.
Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.
Notice that $v_\parallel \in B$, but $v_\perp \notin B$.

Let $v_\parallel$ denote the orthogonal projection of v down onto B.

Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.

Notice that $v_\parallel \in B$, but $v_\perp \notin B$.

For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_\parallel + v_\perp) \wedge B = v_\perp \wedge B,$$
$$v \cdot B = (v_\parallel + v_\perp) \cdot B = v_\parallel \cdot B.$$

Let $v_\parallel$ denote the orthogonal projection of v down onto B.

Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.

Notice that $v_\parallel \in B$, but $v_\perp \notin B$.

For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_\parallel + v_\perp) \wedge B = v_\perp \wedge B,$$
$$v \cdot B = (v_\parallel + v_\perp) \cdot B = v_\parallel \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Building Intuition About Euclidean Blades

Let $v_\parallel$ denote the orthogonal projection of v down onto B.
Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.
Notice that $v_\parallel \in B$, but $v_\perp \notin B$.
For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_\parallel + v_\perp) \wedge B = v_\perp \wedge B,$$
$$v \cdot B = (v_\parallel + v_\perp) \cdot B = v_\parallel \cdot B.$$

$$\mathrm{grade}(v \wedge B) = \mathrm{grade}(B) + 1$$
$$\mathrm{grade}(v \cdot B) = \mathrm{grade}(B) - 1$$

We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the complement of B with respect to $\mathbb{V}^n$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\square$

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\qquad\square$

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\qquad \square$

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. $\qquad \square$

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.
Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.
Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.
Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Definition

We say that B dually represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.
Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Definition

We say that B dually represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$p(x) \in A \text{ or } p(x) \in B$$
$$\implies p(x) \in A \wedge B$$

$\square$

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$p(x) \in A \text{ or } p(x) \in B$$
$$\implies p(x) \in A \wedge B$$

□

Why doesn't the converse hold?

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

We Can Intersect Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$p(x) \in A^* \text{ and } p(x) \in B^*$$
$$\text{iff } p(x) \notin A \text{ and } p(x) \notin B$$
$$\text{iff } p(x) \notin A \wedge B$$
$$\text{iff } p(x) \in (A \wedge B)^*$$

$\square$

Fundamental construction methods in CGA include...

Fundamental construction methods in CGA include...

- Intersecting geometry – $G^*(A \wedge B)$,

Fundamental construction methods in CGA include...

- Intersecting geometry – $G^*(A \wedge B)$,
- Combining geometry – $G(A \wedge B)$,

Fundamental construction methods in CGA include...

- Intersecting geometry – $G^*(A \wedge B)$,
- Combining geometry – $G(A \wedge B)$,
- Reinterpreting geometry – $G(B)$ versus $G^*(B)$

Fundamental Construction Methods

Fundamental construction methods in CGA include...

- Intersecting geometry – $G^*(A \wedge B)$,
- Combining geometry – $G(A \wedge B)$,
- Reinterpreting geometry – $G(B)$ versus $G^*(B)$

With a fixed geometry, we have freedom in representation – dually or directly represented.

Fundamental construction methods in CGA include...

- Intersecting geometry – $G^*(A \wedge B)$,
- Combining geometry – $G(A \wedge B)$,
- Reinterpreting geometry – $G(B)$ versus $G^*(B)$

With a fixed geometry, we have freedom in representation – dually or directly represented.

With a fixed representation, we have freedom in geometry – dually or directly interpreted.

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.
For any blade B, there is a scalar λ such that λB is a homogenized form.

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.

For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operation, then such a λ has geometric signficance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Versors

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^m v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

Versors

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^m v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

Notice that $v^{-1} = v/v^2$.

Definition

Given the versor $V = v_1 \ldots v_m$, we define

$$\tilde{V} = \prod_{k=1}^{m} v_{m-k+1}.$$

The Inverse And The Reverse Of Versors

Definition

Given the versor $V = v_1 \ldots v_m$, we define

$$\tilde{V} = \prod_{k=1}^{m} v_{m-k+1}.$$

The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

Versors form a group under the geometric product.

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.

The scalar 1 is the identity versor.

For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

Conjugation by versors is outermorphic!

Conjugation by versors is outermorphic!

Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Properties Of Versors

Conjugation by versors is outermorphic!
Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Conjugation by versors is grade preserving!

Conjugation by versors is outermorphic!
Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.

Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space of a non-Euclidean vector space.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.

Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space of a non-Euclidean vector space.

Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.

Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space of a non-Euclidean vector space.

Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.

Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space of a non-Euclidean vector space.

Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Having invented this specific model, what we are now able to discover about it is almost endless!

For any $c \in \mathbb{V}^n$, the vector $p(c)$ both dually and directly represents the point c in space.

For any $c \in \mathbb{V}^n$, the vector $p(c)$ both dually and directly represents the point c in space.

That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

Points are degenerate spheres, or spheres with radius zero.

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a non-empty and non-degenerate intersection.

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a non-empty and non-degenerate intersection.

Then the blade B, given by

$$B = \bigwedge_{k=1}^m \sigma_k,$$

dually represents an $(n - m + 1)$-dimensional hyper-sphere.

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a non-empty and non-degenerate intersection.

Then the blade B, given by

$$B = \bigwedge_{k=1}^m \sigma_k,$$

dually represents an $(n - m + 1)$-dimensional hyper-sphere.

Rounds with zero radius give us tangent points!
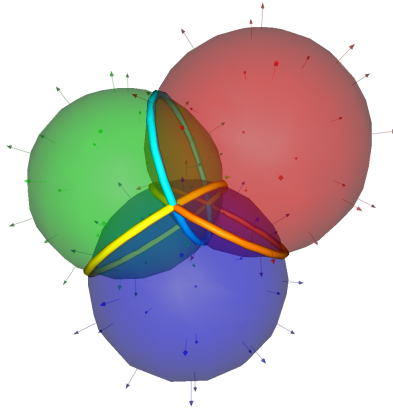
Figure : 3 Rounds, 3 Circles and 1 Point-Pair

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection.

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection.

Then the blade B, given by

$$B = \bigwedge_{k=1}^{m} \pi_k,$$

dually represents an $(n-m)$-dimensional hyper-plane.

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection.

Then the blade B, given by

$$B = \bigwedge_{k=1}^{m} \pi_k,$$

dually represents an $(n-m)$-dimensional hyper-plane.

Flats at infinity are free blades.

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection.

Then the blade B, given by

$$B = \bigwedge_{k=1}^m \pi_k,$$

dually represents an $(n-m)$-dimensional hyper-plane.

Flats at infinity are free blades.

0-dimensional flats are called flat points.
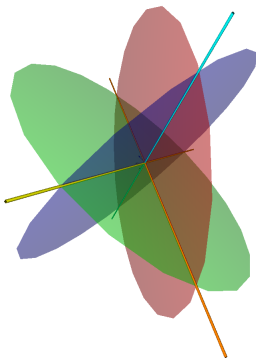
Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of points are co-m-hyper-planar if...

> For $m = 0$, the points are identical,
> For $m = 1$, the points are collinear,
> For $m = 2$, the points are coplanar,
> For $m = 3$, the points are co-hyper-planar,
> etc...

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Let $m \geq 1$. For $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

Let $m \geq 1$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

Proof.

Let the $(n - m + 1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^{m}$ is a linearly independent set. Lastly, $\text{grade}(B) = m + 1 = n + 2 - (n - m + 1) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of points are co-m-hyper-spherical if...

> For $m = 1$, the points are co-point-pair,
> For $m = 2$, the points are co-circular,
> For $m = 3$, the points are co-spherical,
> For $m = 4$, the points are co-hyper-spherical,
> etc...

Let $m \geq 1$. For $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m-dimensional hyper-plane.

Let $m \geq 1$. For $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m-dimensional hyper-plane.

Proof.

By (1), and WLOG, $p(x_1) \wedge \cdots \wedge p(x_{m+1})$ is a direct m-dimensional round. By (2), $B \neq 0$. Let the $(n-m)$-blade A dually represent the m-dimensional hyper-plane. Then $\mathrm{grade}(B) = m + 2 = n + 2 - (n-m) = n + 2 - \mathrm{grade}(A) = \mathrm{grade}(A^*)$. $\qquad\square$

Let $m \geq 0$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

$$B = \infty \wedge \bigwedge_{k=1}^{m+1} p(x_k),$$

directly represents the m-dimensional hyper-plane.

Let $m \geq 0$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

$$B = \infty \wedge \bigwedge_{k=1}^{m+1} p(x_k),$$

directly represents the m-dimensional hyper-plane.
The proof, again, is not hard, but can't fit here.

Quiz Time!

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?

Answer: $N = (P \wedge L^*)^* = P \cdot L$.

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?

Answer: $N = (P \wedge L^*)^* = P \cdot L$.

Question: Let S be a dual sphere that intersects a dual plane N in more than one point, and let P be a point on S but not on N. Then, if P' is the reflection of P about N, what is the dual sphere reflection S' of S about N?

Quiz Time!

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?

Answer: $N = (P \wedge L^*)^* = P \cdot L$.

Question: Let S be a dual sphere that intersects a dual plane N in more than one point, and let P be a point on S but not on N. Then, if P' is the reflection of P about N, what is the dual sphere reflection S' of S about N?

Answer:

$$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$$

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?

Answer: $N = (P \wedge L^*)^* = P \cdot L$.

Question: Let S be a dual sphere that intersects a dual plane N in more than one point, and let P be a point on S but not on N. Then, if P' is the reflection of P about N, what is the dual sphere reflection S' of S about N?

Answer:

$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S$.

Better Answer: $S' = NSN^{-1}$.

The Fun Just Doesn't Stop!

Spencer T. Parkin An Intro to CGA

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Answer:
$$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$$

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Answer:

$$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$$

Question: Given a dual sphere S and a point P not on S, find the dual circle C that is the countour S as seen from P.

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Answer:

$$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$$

Question: Given a dual sphere S and a point P not on S, find the dual circle C that is the countour S as seen from P.

Answer: Let $Q = S \wedge (P \wedge \infty)^*$, then $C = Q^* \wedge S$.
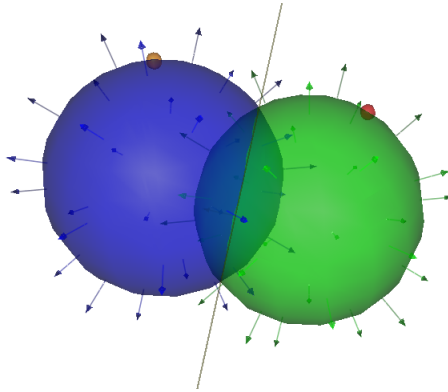
Figure : Reflection of sphere with aide of point.
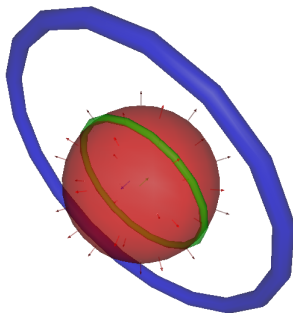
Figure : Saturn's Fallen Ring
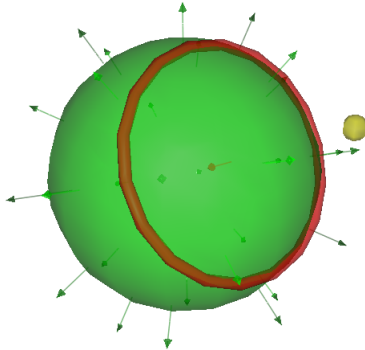
Figure : The contour of a circle as seen from a point.

Recall the outermorphic property of versors!

Recall the outermorphic property of versors!

Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V\left(\bigwedge_{k=1}^m p(x_k)\right) V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

Transformations Of Direct Geometry By Versors

Recall the outermorphic property of versors!

Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V\left(\bigwedge_{k=1}^m p(x_k)\right)V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

Therefore, if we understand how V transforms $p(x)$ for any $x \in \mathbb{V}^n$, then we can predict how V transforms any geometry, (except flat points)!

Recall the outermorphic property of versors!

Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V\left(\bigwedge_{k=1}^m p(x_k)\right)V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

Therefore, if we understand how V transforms $p(x)$ for any $x \in \mathbb{V}^n$, then we can predict how V transforms any geometry, (except flat points)!

If we know how a desired transformation transformas a point, then we have found that transformation for any conformal geometry, (except flats points)!

Transformations Of Direct Geometry By Versors

Recall the outermorphic property of versors!

Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V\left(\bigwedge_{k=1}^m p(x_k)\right)V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

Therefore, if we understand how V transforms $p(x)$ for any $x \in \mathbb{V}^n$, then we can predict how V transforms any geometry, (except flat points)!

If we know how a desired transformation transformas a point, then we have found that transformation for any conformal geometry, (except flats points)!

A versor may or may not leave ∞ invariant under conjugation.

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

Types Of Transformations By Versors

All conformal transformations can be represented by versors!
Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

Surprise!

Blades representative of geometry are also versors representative of transformations!

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!
- Translations & Rotations are two successive planar reflections in well chosen planes.

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!
- Translations & Rotations are two successive planar reflections in well chosen planes.
- Dilations & Transversions are two successive spherical reflections in well chosen spheres.

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!
- Translations & Rotations are two successive planar reflections in well chosen planes.
- Dilations & Transversions are two successive spherical reflections in well chosen spheres.

Corollary: We can use versors to transform transformations!

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!
- Translations & Rotations are two successive planar reflections in well chosen planes.
- Dilations & Transversions are two successive spherical reflections in well chosen spheres.

Corollary: We can use versors to transform transformations!
Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.
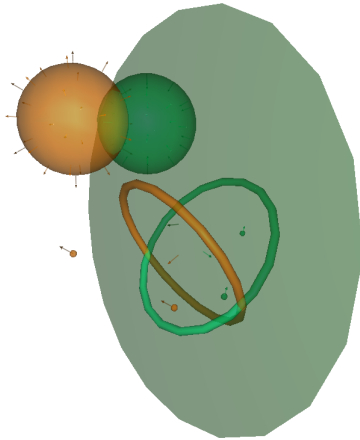
Figure : Planar reflections of a sphere, circle and point-pair.

Figure : Planar reflections of a sphere, circle and point-pair.
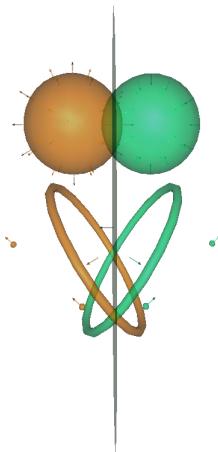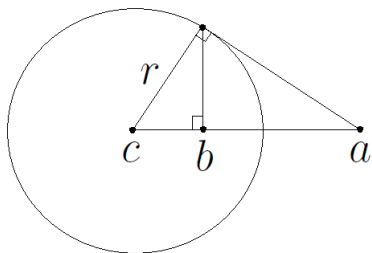
Figure : How we define spherical reflections.
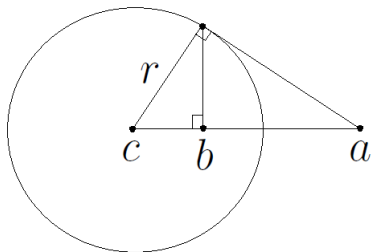
Figure : How we define spherical reflections.

It can be shown that $b = c + \lambda(a - c)$, where $\lambda = r^2/(a - c)^2$.
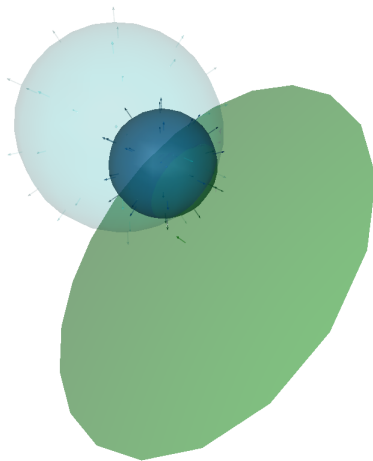
Figure : The spherical reflection of a plane about a sphere is a sphere.
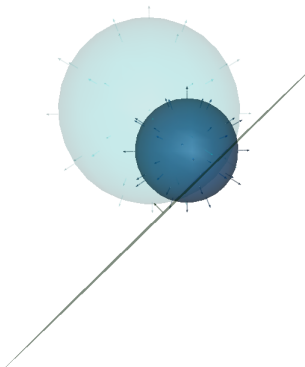
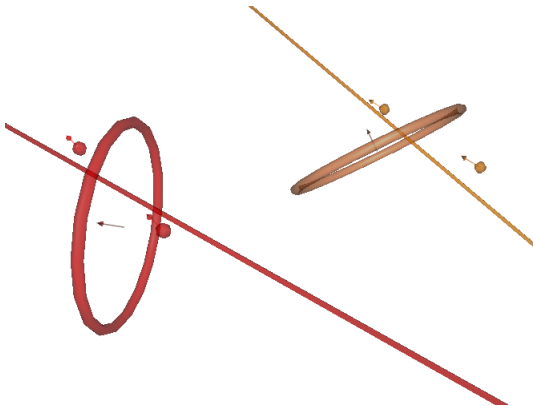Figure : The spherical reflection of a plane about a sphere is a sphere.

Rigid Body Motions

Figure : A line, circle and point-pair rotated/translated.
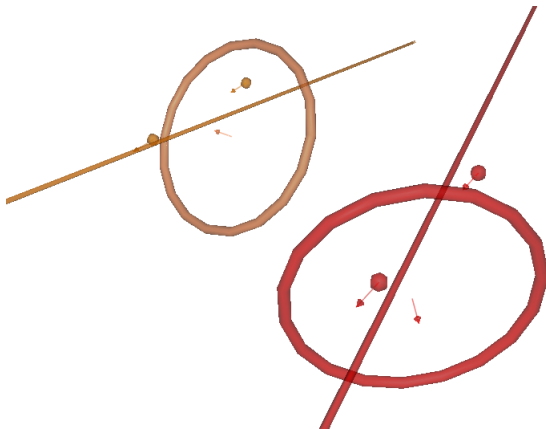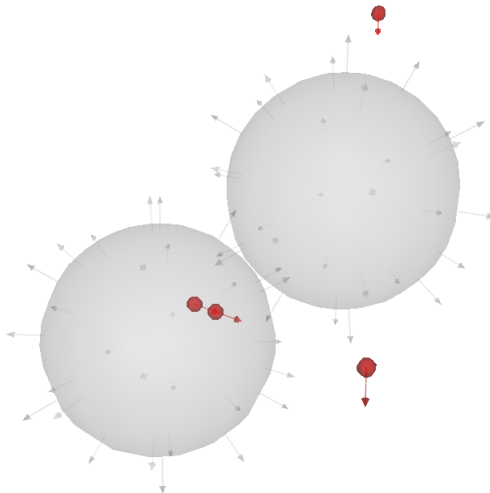
Figure : A line, circle and point-pair rotated/translated.
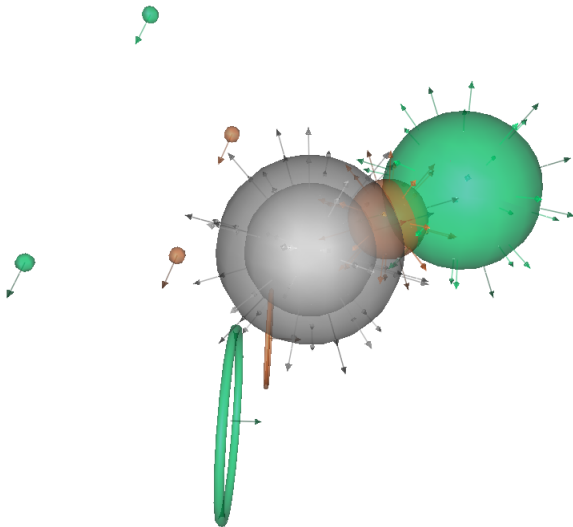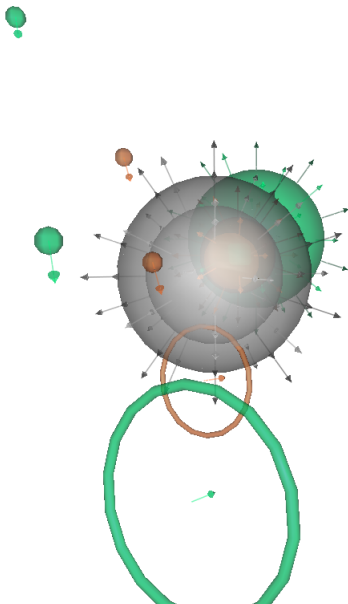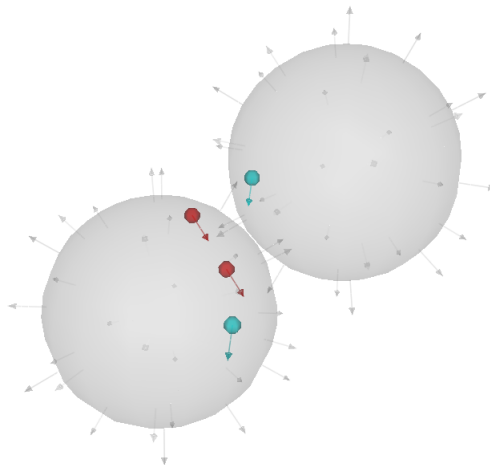
Figure : The transversion of a point-pair.

Figure : The transversion of some other point-pair.

Thank you for your time. Any questions?