Change Of Basis Transformations In Geometric Algebra

Spencer T. Parkin

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Abstract

A method of representing and performing change-of-basis transformations in geometric algebra is given. This type of transformation is equivilant to matrix multiplication. Since shear and non-uniform scale operations can be represented by change-of-basis transformations, it follows that the method provides a way to perform these transformations using geometric algebra. The change-of-basis transformation is developed for 2-dimensional space. A generalization to higher-dimensional spaces is considered. A generalization to tensor products is also considered.

In this paper we will let $\mathbb{V}^4$ denote a 4-dimensional Euclidean vector space spanned by the set of orthonormal basis vectors $\{e_0, e_1, e_2, e_3\}$. $\mathbb{G}(\mathbb{V}^4)$ will denote the geometric algebra generated by this vector space, and we will let $I = e_0e_1e_2e_3$ be the unit-psuedo scalar of $\mathbb{G}(\mathbb{V}^4)$.

So what is a change-of-basis transformation? Consider the following matrix equation.

$$\begin{bmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{bmatrix} \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix} = \gamma_x \begin{bmatrix} \alpha_x \\ \beta_x \end{bmatrix} + \gamma_y \begin{bmatrix} \alpha_y \\ \beta_y \end{bmatrix}$$

The reader will recognize this as ordinary matrix multiplication, but might not immediately recognize this as a change-of-basis transformation. Let $a, b, c \in \mathbb{V}^4$ be the vectors $a = \alpha_x e_0 + \alpha_y e_1$, $b = \beta_x e_0 + \beta_y e_1$ and $c = \gamma_x e_0 + \gamma_y e_1$. If we can then be allowed a few small abuses of matrix notation, the above equation becomes much clearer with these variables. For example, in the context of matrices, let c denote the row-vector matrix $[\ \gamma_x \quad \gamma_y\ ]$. Let us now rewrite the above equation as follows.

$$\begin{bmatrix} a \\ b \end{bmatrix} ((c \cdot e_0)e_0^T + (c \cdot e_1)e_1^T) = (c \cdot e_0)a^T + (c \cdot e_1)b^T$$

We see now that the 2×2 matrix above, acting on c^T, gives us a result that changes the basis upon which the coordinates of c are based. That is, the basis $\{e_0, e_1\}$ is swapped out in favor of $\{a, b\}$. Using the language of geometric algebra, we can achieve the same effect.

We begin with an idea set forth in [1]. In matrix algebra the change-of-basis transformation is represented by a matrix whose rows or columns contain the desired basis. With geometric algebra we may let a bivector M represent the same transformation.

$$M = ae_2 + be_3$$

Let us now define $\mathbb{A}$ as the 2-dimensional Euclidean vector space spanned by the vectors $\{e_0, e_1\}$, and $\mathbb{B}$ as the 2-dimensional Euclidean vector space that is the complement of $\mathbb{A}$ with respect to $\mathbb{V}^4$. Clearly, $\mathbb{B}$ is spanned by $\{e_2, e_3\}$.

What we'll show now is that any change-of-basis transformation can be performed in $\mathbb{A}$ if we provide a way to perform the change-of-basis transformation of a vector taken from $\mathbb{A}$ to a vector taken from $\mathbb{B}$. Such a transformation is simply given by an isomorphism between these two spaces that maps e_0 to e_2 and e_1 to e_3. To see why, let us define $f : \mathbb{A} \to \mathbb{B}$ to be such an isomorphism, and then $T : \mathbb{A} \to \mathbb{A}$ as follows.

$$T(c) = M \cdot f(c)$$

As the reader can easily verify, $T(c) = (c \cdot e_0)a + (c \cdot e_1)b$, which is the desired transformation of c using M.

One way to define f is using a rotor that rotates the 2-blade $e_0 \wedge e_1$ into $e_2 \wedge e_3$. As such, f is not only an isomorphism, but also an outermorphism. This property will become essential to finding inverse change-of-basis transformations. For any $g \in \mathbb{G}(\mathbb{V}^4)$, we may define $f : \mathbb{G}(\mathbb{V}^4) \to \mathbb{G}(V^4)$ as $f(g) = Rg\tilde{R}$, where R is the unit rotor

$$R = \frac{1}{2}\left(1 - e_0e_2 - e_1e_3 - I\right).$$

The reader can check that $f(e_0) = e_2$ and $f(e_1) = e_3$, that f is a linear transformation from $\mathbb{A}$ to $\mathbb{B}$, and that f preserves the outer product.

We now make the observation that while $T(g)$ gives us the desired transformation, it does not benefit from the invertability of the geometric product. Let us therefore, for any $g \in \mathbb{G}(\mathbb{V}^4)$, define $F : \mathbb{G}(\mathbb{V}^4) \to \mathbb{G}(\mathbb{V}^4)$ as

$$F(g) = Mf(g).$$

For all vectors $c \in \mathbb{V}^4$, we see that $F(c) = T(c) + M \wedge f(c)$. Interestingly, this can be rewritten as

$$F(c) = T(c) + (T(c) \cdot Me_2 \wedge Me_3)\, i_{\mathbb{B}},$$

where we will let $i_{\mathbb{A}}$ denote the unit psuedo-scalar of $\mathbb{G}(\mathbb{A})$ and $i_{\mathbb{B}}$ denote the unit psuedo-scalar of $\mathbb{G}(\mathbb{B})$. Specifically, $i_{\mathbb{A}} = e_0 e_1$ and $i_{\mathbb{B}} = e_2 e_3$. Here it is easy to see that the vector $T(c) \cdot Me_2 \wedge Me_3$ is a $\pi/2$ radians rotation of $T(c)$ in the plane determined by the basis in M.

As we know from linear algebra, a 2-dimensional change-of-basis transformation is invertible if and only if the two axes are non-parallel.

References

[1] David Hestenese. Hamiltonian mechanics with geometric calculus. *Journal?*, 1993.