

An Intro to CGA

Conformal Geometric Algebra

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Invented by **David Hestenes** in his paper, "Old Wine in New Bottles: A new algebraic framework for computational geometry."

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- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

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Definition

We say the blade B , given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m -blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

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is a non-zero m -blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $\text{grade}(B) = m \leq n$.

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Our geometric arguments will not require us to visualize the homogeneous representation space.

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For any $v \in \mathbb{V}^n$, we say that

$$v \notin B \text{ if and only if } v \in B^*,$$

where B^* represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

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For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{\parallel} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$

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We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the **complement** of B with respect to \mathbb{V}^n .

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The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. □

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Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

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Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

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For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

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Why doesn't the converse hold?

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Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff } & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff } & p(x) \notin A \wedge B \\ \text{iff } & p(x) \in (A \wedge B)^* \end{aligned}$$



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If B is the result of some geometric operation, then such a λ has geometric significance WRT to that operation.

The Geometric Product

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Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

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Notice that $v^{-1} = v/v^2$.

The Inverse And The Reverse Of Vectors

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The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

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The Versor Group

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Proof.

Associativity follows from the associativity of the geometric product.

The scalar 1 is the **identity** versor.

For every versor V , there exists an **inverse** V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

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For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

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Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

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Having **invented** this specific model, what we are now able to **discover** about it is almost endless!

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That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

n -dimensional Dual Hyper-Spheres

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2}r^2\infty \right) = 0.$$

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We may refer to $p(c)$ as a **round** point.

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Rounds with zero radius give us tangent points!

All Rounds Of CGA For 3-dimensional Space

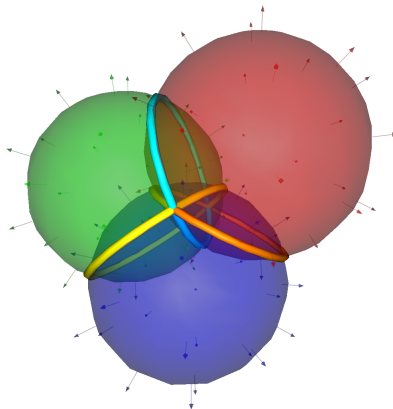


Figure : 3 Rounds, 3 Circles and 1 Point-Pair

$(n - 1)$ -dimensional Dual Hyper-Planes

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The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

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0-dimensional **flats** are called **flat points**.

All Flats Of CGA For 3-dimensional Space

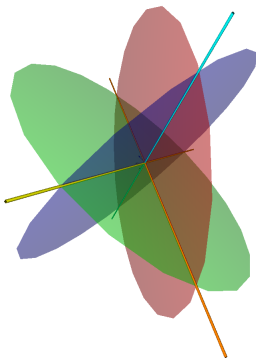


Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

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Definition

For $m \geq 0$, a set of points are **co- m -hyper-planar** if...

For $m = 0$, the points are identical,

For $m = 1$, the points are collinear,

For $m = 2$, the points are coplanar,

For $m = 3$, the points are co-hyper-planar,

etc...

A Condition For Linear Independents Of Points

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Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co- $(m - 1)$ -hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

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The proof is not hard, but too big for this slide.

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Let $m \geq 1$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m -dimensional hyper-sphere that are non-co- $(m - 1)$ -hyper-planar, the blade B , given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m -dimensional hyper-sphere.

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directly represents the m -dimensional hyper-sphere.

Proof.

Let the $(n - m + 1)$ -blade A **dually** represent the m -dimensional hyper-sphere determined by the points. If A **dually** represents this sphere, then A^* **directly** represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k , we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^m$ is a linearly independent set. Lastly, $\text{grade}(B) = m + 1 = n + 2 - (n - m + 1) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

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Definition

For $m \geq 1$, a set of points are **co- m -hyper-spherical** if...

For $m = 1$, the points are co-point-pair,

For $m = 2$, the points are co-circular,

For $m = 3$, the points are co-spherical,

For $m = 4$, the points are co-hyper-spherical,
etc...

Generating Almost All Direct Flats Of CGA

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Let $m \geq 1$. For $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m -dimensional hyper-plane that are (1) non-co- $(m - 1)$ -hyper-planar and (2) non-co- m -hyper-spherical, the blade B , given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m -dimensional hyper-plane.

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Proof.

By (1), and WLOG, $p(x_1) \wedge \cdots \wedge p(x_{m+1})$ is a direct m -dimensional round. By (2), $B \neq 0$. Let the $(n - m)$ -blade A dually represent the m -dimensional hyper-plane. Then $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. \square

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The proof, again, is not hard, but can't fit here.

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Better Answer: $S' = NSN^{-1}$.

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Answer: Let $Q = S \wedge (P \wedge \infty)^*$, then $C = Q^* \wedge S$.

Pictures Of Quiz Answers

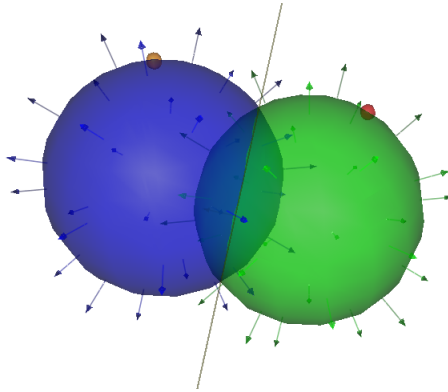


Figure : Reflection of sphere with aide of point.

Pictures Of Quiz Answers (Continued)

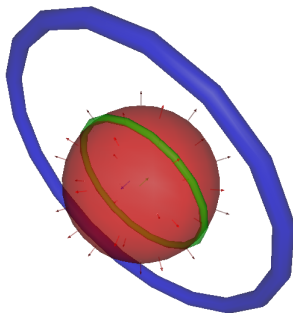


Figure : Saturn's Fallen Ring

Pictures Of Quiz Answers (Continued)

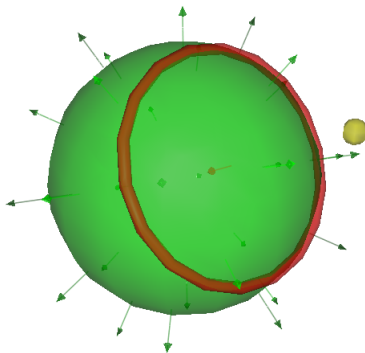


Figure : The contour of a circle as seen from a point.

Transformations Of Direct Geometry By Vectors

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A versor may or may not leave ∞ invariant under conjugation.

Transformations Of Dual Geometry By Versors

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If the m -blade B *dually* represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

Types Of Transformations By Vectors

Types Of Transformations By Versors

All conformal transformations can be represented by **versors**!
Some of these include...

- **Translations**,
- **Rotations**,
- **Dilations**,
- **Transversions**.

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Corollary: We can use versors to transform transformations!

Note: Points are null, (non-invertible), and therefore, planar and spherical reflections **generate** the versor group of all transformations.

Planar Reflections

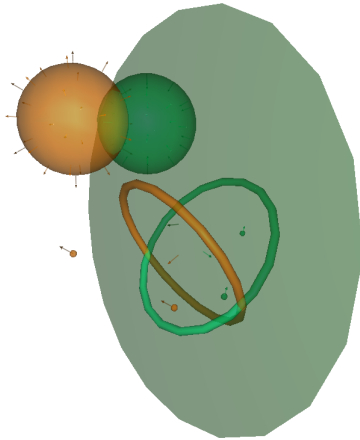


Figure : Planar reflections of a sphere, circle and point-pair.

Planar Reflections (Again)

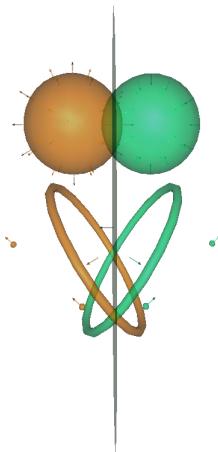


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Spherical Reflections

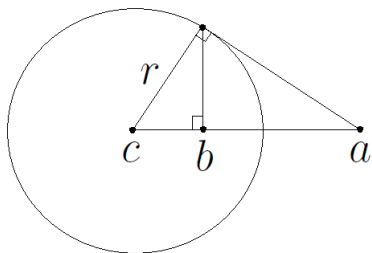


Figure : How we define spherical reflections.

Spherical Reflections

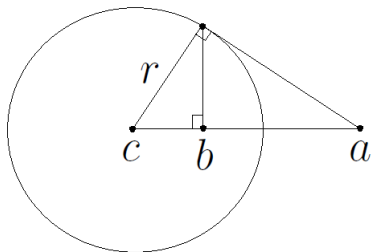


Figure : How we define spherical reflections.

It can be shown that $b = c + \lambda(a - c)$, where $\lambda = r^2/(a - c)^2$.

Spherical Reflections (Again)

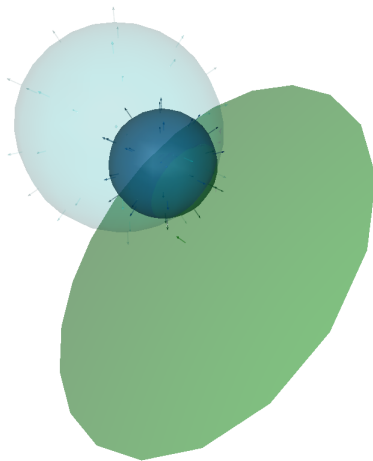


Figure : The spherical reflection of a plane about a sphere is a sphere.

Spherical Reflections (Yet Again)

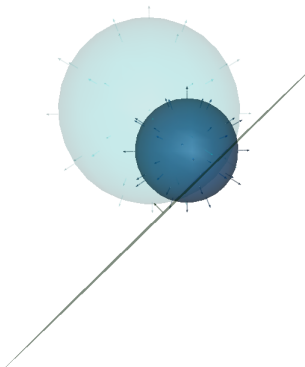


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Rigid Body Motions

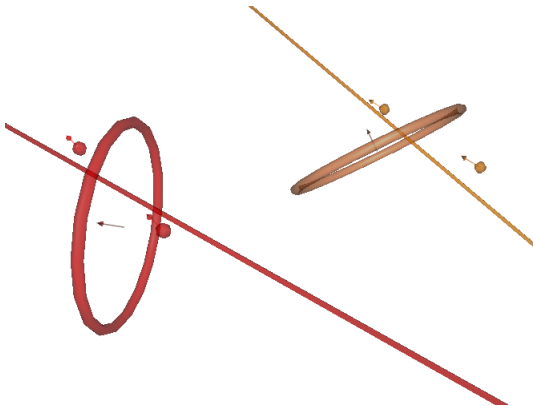


Figure : A line, circle and point-pair rotated/translated.

Rigid Body Motions (Again)

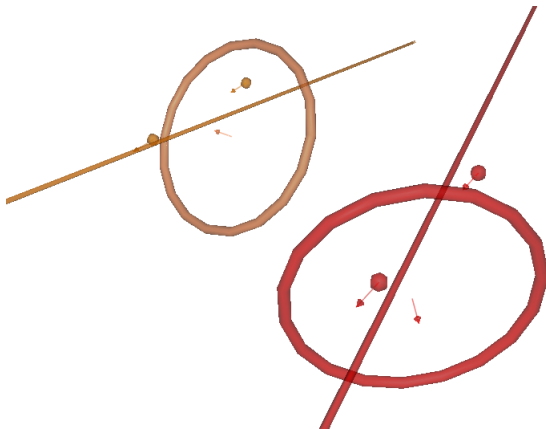
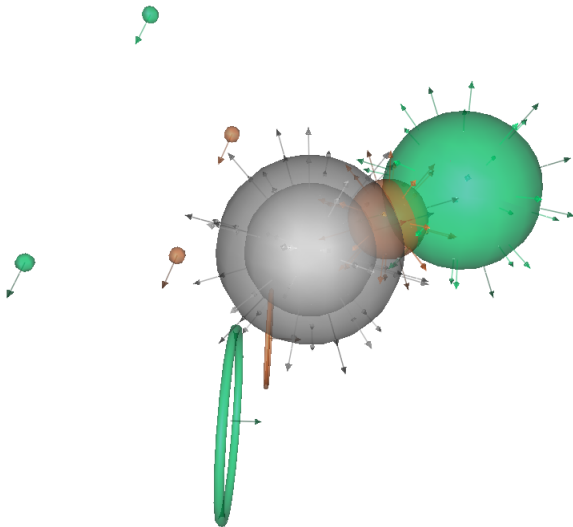
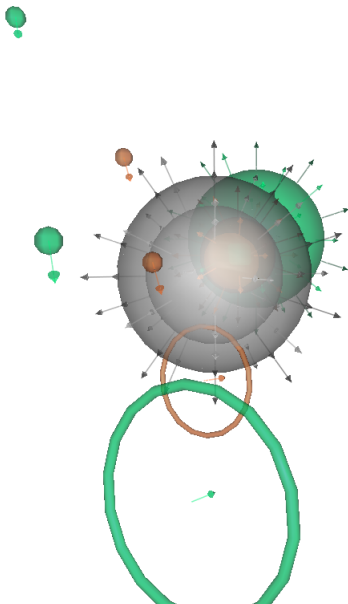


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Dilations



Dilations (Again)



Transversions

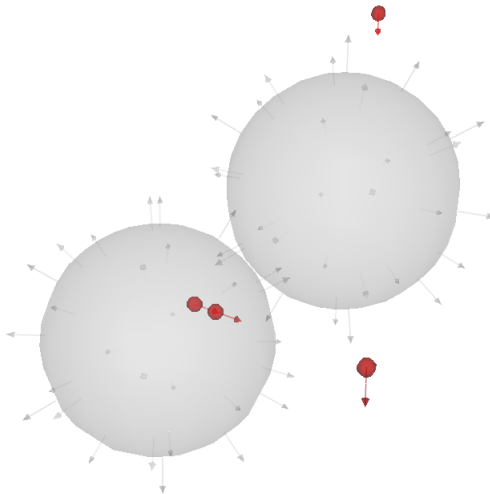


Figure : The transversion of a point-pair.

Transversions (Again)

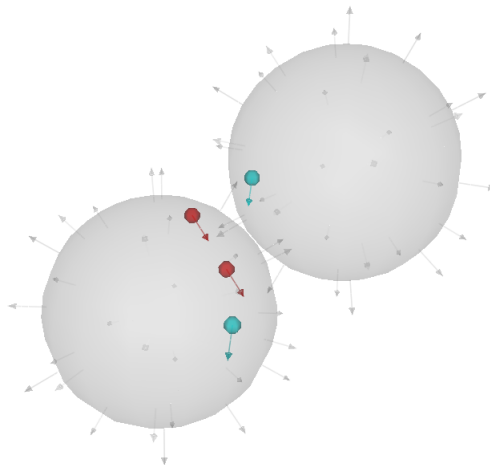


Figure : The transversion of some other point-pair.

The End

Thank you for your time. Any questions?