An Intro to CGA

Conformal Geometric Algebra

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Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from $\mathbb{V}^n$.

Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^{m} b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $m \leq n$.

Visualizing Euclidean Blades

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.
Non-Euclidean blades require more imagination!
Our geometric arguments will not require us to visualize the homogeneous representation space.

Building Intuition About Euclidean Blades

Let $v_{||}$ denote the orthogonal projection of v down onto B.
Let $v_{\perp} = v - v_{||}$ denote the orthogonal rejection of v from B.
For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{||} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
$$v \cdot B = (v_{||} + v_{\perp}) \cdot B = v_{||} \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Definition

If $v \notin B$, then $v \in B^*$, which represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

Membership in Vector Spaces and Dual Vector Spaces

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. □

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Definition

We say that B dually represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \implies & p(x) \in A \wedge B \end{aligned}$$

□

Let $C \subseteq A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subseteq A$ and $C \not\subseteq B$.

We Can Intersect Geometries

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ *such that* $A \wedge B \neq 0$, *we have*

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$\begin{aligned}
& p(x) \in A^* \text{ and } p(x) \in B^* \\
\text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\
\text{iff} \quad & p(x) \notin A \wedge B \\
\text{iff} \quad & p(x) \in (A \wedge B)^*
\end{aligned}$$

□

The Homogeneous Nature Of The Model

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.

For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operations, then such a λ has geometric signficance WRT to that operation.

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^{m} v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

Definition

Given the versor $V = v_1 \dots v_m$, we define

$$\tilde{V} = \prod_{k=1}^{m} v_{m-k+1}.$$

The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

The Versor Group

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.
The scalar 1 is the identity versor.
For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

Conjugation by versors is outermorphic!
Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.
Given $G(B)$, it will be interesting to investigate $G(VBV^{-1})$.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.
Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space.
Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Having invented this specific model, what we are now able to discover about it is almost endless!

For any $c \in \mathbb{V}^n$, the vector $p(c)$ both dually and directly represents the point c in space.
That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

n-dimensional Dual Hyper-Spheres

The function $p(x)$ factors out of the equation

$$(x-c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2}r^2\infty\right) = 0.$$

Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

Generating All Dual Rounds Of CGA

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a non-empty and non-degenerate intersection. Then the blade B, given by

$$B = \bigwedge_{k=1}^{m} \sigma_k,$$

dually represents an $(n - m + 1)$-dimensional hyper-sphere. *Rounds* with zero radius give us tangent points!

All Rounds Of CGA For 3-dimensional Space

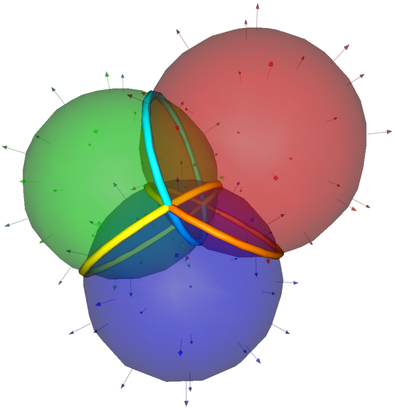

Figure : 3 Rounds, 3 Circles and 1 Point-Pair

$(n-1)$-dimensional Dual Hyper-Planes

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

Generating All Dual Flats Of CGA

Let $\{\pi_k\}_{k=1}^{m}$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection. Then the blade B, given by

$$B = \bigwedge_{k=1}^{m} \pi_k,$$

dually represents an $(n-m)$-dimensional hyper-plane.
Flats at infinity are free blades.

All Flats Of CGA For 3-dimensional Space

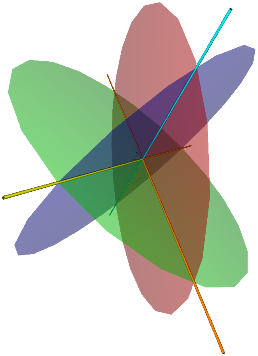

Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are co-m-hyper-planar if...

For $m = 0$, the points are identical,
For $m = 1$, the points are collinear,
For $m = 2$, the points are coplanar,
For $m = 3$, the points are co-hyper-planar,
etc...

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Let $m \geq 1$. For $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

Proof.

Let the $(n-m+1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^{m}$ is a linearly independent set. Lastly, $\text{grade}(B) = m+1 = n+2-(n-m-1) = n+2-\text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are co-m-hyper-planar if...

For $m = 1$, the points are co-point-pair (distinct),
For $m = 2$, the points are co-circular,
For $m = 3$, the points are co-spherical,
For $m = 4$, the points are co-hyper-spherical,
etc...

Let $m \geq 1$. For $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m-dimensional hyper-plane.

Proof.

By (1), there exists the $(n-m)$-blade A dually representative of the m-dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. $\square$

Let $m \geq 1$. For m points $\{x_k\}_{k=1}^{m} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

$$B = \infty \wedge \bigwedge_{k=1}^{m+1} p(x_k),$$

directly represents the m-dimensional hyper-plane.
The proof, again, is not hard, but can't fit here.

Quiz Time!

Question: Given a dual line L and a point P not on L, how do I find the dual plane N containing L and P?
Answer: $N = (P \wedge L^*)^* = P \cdot L$.
Question: Given a dual circle C and a point P not on C or on the plane determined by C, how do I find the dual sphere S containing C and P?
Answer: $S = (P \wedge C^*)^* = P \cdot C$.
Question: Let S be a dual sphere that intersects a dual plane N in more than one point, and let P be a point on S but not on N. Then, if P' is the reflection of P about N, what is the dual sphere reflection S' of S about N?
Answer:
$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S$.

The Fun Just Doesn't Stop!

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Answer:

$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$

Question: Let $\{N_k\}_{k=0}^3$ be 4 dual planes forming the sides of a non-degenerate tetrahedron, and let S be a dual sphere containing all 4 vertices of the tetrahedron. If all planes are known, how do we find S? If S is known and all planes save one, how do we find the remaining plane?

Answer: Not very pretty!

Transformations Of Direct Geometry By Versors

Recall the outermorphic property of versors!

Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V\left(\bigwedge_{k=1}^{m} p(x_k)\right)V^{-1} = \bigwedge_{k=1}^{m} Vp(x_k)V^{-1}.$$

If we understand how V transforms $p(x)$ for any $x \in \mathbb{V}^n$, then we can predict how V transforms any geometry!

A versor may or may not leave ∞ invariant under conjugation.

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

Types Of Transformations By Versors

All conformal transformations can be represented by versors!
Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

Surprise!

Blades representative of geometry are also versors representative of transformations!

- Planar reflections are represented by a dual planes!
- Spherical reflections are represented by dual spheres!
- Translations & Rotations are two successive planar reflections in well chosen planes.
- Dilations & Transversions are two successive spherical reflections in well chosen spheres.

Corollary: We can use versors to transform transformations!
Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.

Planar Reflections

Translations

Rotations

Spherical Reflections

Dilations

Transversions

The End

Thank you for your time. Any questions?