An Intro to CGA

Conformal Geometric Algebra

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What is CGA?

$$\left\{\begin{array}{c}\text{Synthetic}\\ \text{geometry}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}\text{Computational}\\ \text{geometry}\end{array}\right\}$$

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Invented by David Hestens in his paper, "Old Wine in New Bottles: A new algebraic framework for computational geometry."

Presentation Outline

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In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.

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Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^{m} b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

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is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $\text{grade}(B) = m \leq n$.

Visualizing Euclidean Blades

Imagine an infinite m-dimensional hyper-plane.

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Non-Euclidean blades require more imagination!
Our geometric arguments will not require us to visualize the homogeneous representation space.

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

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For any $v \in \mathbb{V}^n$, we say that

$$v \notin B \text{ if and only if } v \in B^*,$$

where B^* represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

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For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{||} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
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$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
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We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the complement of B with respect to $\mathbb{V}^n$.

Membership in Vector Spaces and Dual Vector Spaces

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

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Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\square$

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Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

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Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

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$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \Longrightarrow\ & p(x) \in A \wedge B \end{aligned}$$

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$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \implies & p(x) \in A \wedge B \end{aligned}$$

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Why doesn't the converse hold?

We Can Intersect Geometries

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Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff} \quad & p(x) \notin A \wedge B \\ \text{iff} \quad & p(x) \in (A \wedge B)^* \end{aligned}$$

$\square$

The Homogeneous Nature Of The Model

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For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operation, then such a λ has geometric signficance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

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Notice that $v^{-1} = v/v^2$.

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The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

The Versor Group

Versors form a group under the geometric product.

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Proof.

Associativity follows from the associativity of the geometric product.
The scalar 1 is the identity versor.
For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. $\square$

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Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

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Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

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Having invented this specific model, what we are now able to discover about it is almost endless!

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For any $c \in \mathbb{V}^n$, the vector $p(c)$ both dually and directly represents the point c in space.
That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

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Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

Generating All Dual Rounds Of CGA

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dually represents an $(n - m + 1)$-dimensional hyper-sphere.

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Rounds with zero radius give us tangent points!

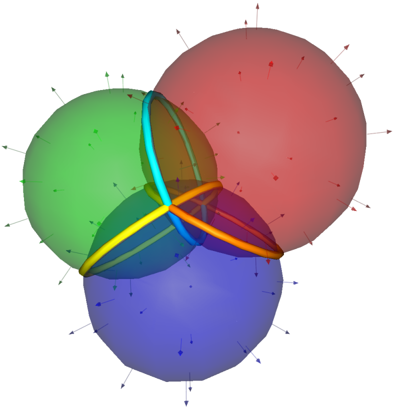

Figure : 3 Rounds, 3 Circles and 1 Point-Pair

$(n-1)$-dimensional Dual Hyper-Planes

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

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Flats at infinity are free blades.
0-dimensional flats are called flat points.

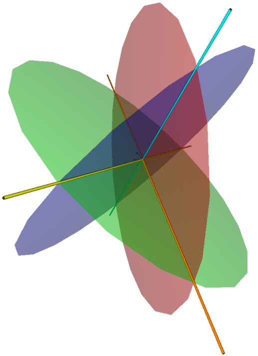

Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of points are co-m-hyper-planar if...

For $m = 0$, the points are identical,
For $m = 1$, the points are collinear,
For $m = 2$, the points are coplanar,
For $m = 3$, the points are co-hyper-planar,
etc...

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$*, if* $m+1$ *points* $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ *are non-co-*$(m-1)$*-hyper-planar, then* $\{p(x_k)\}_{k=1}^{m+1}$ *is a linearly independent set.*

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Generating All Direct Rounds Of CGA

Let $m \geq 1$. For $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

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Proof.

Let the $(n-m+1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^{m}$ is a linearly independent set. Lastly, $\text{grade}(B) = m+1 = n+2-(n-m+1) = n+2-\text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of points are co-m-hyper-spherical if...

For $m = 1$, the points are co-point-pair,
For $m = 2$, the points are co-circular,
For $m = 3$, the points are co-spherical,
For $m = 4$, the points are co-hyper-spherical,
etc...

Generating Almost All Direct Flats Of CGA

Let $m \geq 1$. For $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

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Proof.

By (1), there exists the $(n-m)$-blade A dually representative of the m-dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. $\square$

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directly represents the m-dimensional hyper-plane.
The proof, again, is not hard, but can't fit here.

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Answer:
$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S$.

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Answer: Not very pretty!

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A versor may or may not leave ∞ invariant under conjugation.

Transformations Of Dual Geometry By Versors

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

Types Of Transformations By Versors

All conformal transformations can be represented by versors!
Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

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Corollary: We can use versors to transform transformations!
Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.

Planar Reflections

Translations

Rotations

Dilations

Transversions

The End

Thank you for your time. Any questions?