A Better Model For Quadric Surfaces Using Geometric Algebra

Spencer T. Parkin

Abstract. A new model of n-dimensional quadric surfaces, based upon geometric algebra, is found in which all conformal transformations of the conformal model are supported in the form of versors that may be applied to elements of the algebra representative of such surfaces. The model is used to find interesting results, such as the spherical reflection of an infinite cylinder in a sphere.

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1. Introduction

In the original paper [2], a model for quadric surfaces was presented based upon the ideas of projective geometry. What was unfortunate about this model, however, was it's lack of transformational features. It was predicted in the conclusion of [2] that a better model for quadric surfaces may exist that is more like the conformal model of geometric algebra. The present paper details what may be such a model, and shows that all conformal transformations are featured in the new model.

2. The Geometric Algebra

An explanation of any model of geometry based upon geometric algebra must begin with a description of the structure of the geometric algebra upon which the model is imposed. The model set forth in this paper is set in a geometric algebra containing the following nested vector spaces.

$$\begin{array}{ll} \text{Notation} & \text{Basis} \\ \hline \mathbb{V}^e & \{e_i\}_{i=1}^n \\ \mathbb{V}^o & \{o\} \cup \{e_i\}_{i=1}^n \\ \mathbb{V} & \{o\} \cup \{e_i\}_{i=1}^n \cup \{\infty\} \end{array} \tag{2.1}$$

The set of vectors $\{e_i\}_{i=1}^n$ forms an orthonormal set of basis vectors for the n-dimensional Euclidean vector space $\mathbb{V}^e$, which we'll use to represent n-dimensional Euclidean space. The vectors o and ∞ are the familiar null-vectors at origin and infinity taken from the conformal model of geometric algebra. An inner-product table for these basis vectors is given as follows.

$$\begin{array}{c|ccc} \cdot & o & e_i & \infty \\ \hline o & 0 & 0 & -1 \\ e_i & 0 & 1 & 0 \\ \infty & -1 & 0 & 0 \end{array} \tag{2.2}$$

We will now let $\mathbb{G}(\mathbb{V})$ denote the Minkowski geometric algebra generated by $\mathbb{V}$. For each vectors space in table (2.1), we will let an over-bar above this vector space denote an identical copy of that vector space. The vector space $\mathbb{W}$ will denote the smallest vector space containing each $\mathbb{V}$ and $\overline{\mathbb{V}}$ as vector subspaces. In symbols, one may write

$$\mathbb{G}(\mathbb{W}) = \mathbb{G}(\mathbb{V}) \oplus \mathbb{G}(\overline{\mathbb{V}}) \tag{2.3}$$

to illustrate the structure of $\mathbb{G}(\mathbb{W})$ in terms of its two isomorphic Minkowski geometric sub-algebras $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$.

We will use over-bar notation to distinguish between vectors taken from $\mathbb{V}$ with vectors taken from $\overline{\mathbb{V}}$. Though not necessary, we can work exclusively in $\mathbb{G}(\mathbb{V})$ by defining the over-bar notation as an outermorphic ismorphism between $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. Doing so, we see that for any element $E \in \mathbb{G}(\mathbb{V})$, we may define $\overline{E} \in \mathbb{G}(\overline{\mathbb{V}})$ as

$$\overline{E} = SE\tilde{S}, \tag{2.4}$$

where S is the versor given by

$$S = 2^{-n/2}(1 + e_-\overline{e}_-)(1 - e_+\overline{e}_+)\prod_{i=0}^{n}(1 - e_i\overline{e}_i). \tag{2.5}$$

This definition is non-circular if we let the over-bars in equation (2.5) be purely notation. The vectors e_- and e_+, taken from [1], are defined as

$$e_- = \frac{1}{2}\infty + o \tag{2.6}$$

$$e_+ = \frac{1}{2}\infty - o. \tag{2.7}$$

The vectors $\overline{e}_-$ and $\overline{e}_+$ are defined similarly in terms of $\overline{o}$ and $\overline{\infty}$. Defined this way, it is important to realize that, unlike the over-bar function defined in [2], here we do not have the property that for any vector $w \in \mathbb{W}$, we have $\overline{\overline{w}} = w$. This is because $\overline{\overline{o}} = -o$ and $\overline{\overline{\infty}} = -\infty$.

3. The Form Of Quadric Surfaces In $\mathbb{G}(\mathbb{W})$

We now give a formal definition under which elements $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces.

Definition 3.1. Referring to an element $E \in \mathbb{G}(\mathbb{W})$ as a quadric surface, it is representative of such an n-dimensional surface as the set of all points $p \in \mathbb{V}^o$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{3.1}$$

From this definition it can be seen that the general form of a quadric $E \in \mathbb{G}(\mathbb{W})$ is given by

$$E = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} e_i \overline{e}_j + \sum_{i=1}^{n} \lambda_i (e_i \overline{\infty} + \infty \overline{e}_j) + \lambda \infty \overline{\infty}. \tag{3.2}$$

This is because an element of the form (3.2), when substituted into equation (3.1), produces a polynomial equation of degree 2 in the vector components of $p - o$. Doing so with $p = o + x$, where $x \in \mathbb{V}^e$, we get equation

$$0 = -\sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij} (x \cdot e_i)(x \cdot e_j) + \sum_{i=1}^{n} 2\lambda_i (x \cdot e_i) - \lambda, \tag{3.3}$$

which we may recognize as the equation for an n-dimensional quadric surface. Of course, using geometric algebra, it is undesirable and unecessary to think of quadrics in terms of polynomial equations. A better way to think of quadrics is in terms of an element of the algebra whose decomposition produces the parameters characterizing the quadric surface. For example, many common quadrics are the solution set in $\mathbb{V}^e$ of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2, \tag{3.4}$$

in the variable x. (An explanation of the parameters r, c, v and λ was given in [2].) Then, factoring out $-p \wedge \overline{p}$, we see that the element $E \in \mathbb{G}(\mathbb{W})$, given by

$$\Omega + \lambda v \overline{v} + 2(c + \lambda(c \cdot v)v)\overline{\infty} + (c^2 + \lambda(c \cdot v)^2 - r^2)\infty\overline{\infty} \tag{3.5}$$

is representative of this very same quadric by Definition 3.1, where Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i \overline{e}_i. \tag{3.6}$$

Give table of canonical forms here...

4. Transformations Supported By The Model

Here we begin with the following lemma to be used in the main result of this paper, which shows that all conformal transformations exist as versors applicable to quadrics in the extended model.

Lemma 4.1. *For any versor $V \in \mathbb{G}(\mathbb{W})$, and any four vectors $a, b, c, d \in \mathbb{V}$, we have*

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \tag{4.1}$$

Proof. Give proof here... □

The main result, as promised, can now be given as follows.

Theorem 4.2. *Let $E \in \mathbb{G}(\mathbb{W})$ be a bivector of the form* (3.2)*. Then, for any versor $V \in \mathbb{G}(\mathbb{V})$ of the conformal model, the element E', given by*

$$E' = V\overline{V}E(V\overline{V})^{-1}, \tag{4.2}$$

is representative of the n-dimensional quadric surface by Definition 3.1 *that is the transformation of E by V.*

Proof. Let $P : \mathbb{V}^{\mathrm{e}} \to \mathbb{V}$ be the conformal mapping defined by

$$P(v) = o + v + \frac{1}{2}v^2\infty. \tag{4.3}$$

We then make the observation that if $p' \in \mathbb{V}^o$ is representative of the point that is the transformation of $p \in \mathbb{V}^o$ by the transformation of V^{-1}, then the transformation of E by V is described as the set of all points $p \in \mathbb{V}^o$ such that $0 = p' \wedge \overline{p}' \cdot E$. We then see that

$$\frac{p' \wedge \overline{p}'}{(\infty \cdot p')^2} \cdot E \tag{4.4}$$

$$= P\left(\frac{p'}{-\infty \cdot p'} - o\right) \wedge \overline{P}\left(\frac{p'}{-\infty \cdot p'} - o\right) \cdot E \tag{4.5}$$

$$= V^{-1}P\left(\frac{p}{-\infty \cdot p} - o\right)V \wedge \overline{V^{-1}P}\left(\frac{p}{-\infty \cdot p} - o\right)\overline{V} \cdot E \tag{4.6}$$

$$= P\left(\frac{p}{-\infty \cdot p} - o\right) \wedge \overline{P}\left(\frac{p}{-\infty \cdot p} - o\right) \cdot E' \tag{4.7}$$

$$= \frac{p \wedge \overline{p}}{(\infty \cdot p)^2} \cdot E', \tag{4.8}$$

showing that E' is representative of this very same set of points under Definition 3.1. Lemma 4.1 was employed in the step taken from (4.6) to (4.7). □

5. Putting Theory Into Practice

While Theorem 4.2 tells us that the quadrics are closed under the group of conformal transformations, it is still not entirely obvious, given a quadric surface, what the image of this surface is under a given conformal transformation. To explore this, software has been written to implement the above quadric model. This section exhibits the interesting results.

References

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Spencer T. Parkin
2113 S. Claremont Dr.
Bountiful, Utah 84010
USA
e-mail: spencer.parkin@gmail.com