Notes On Geometric Calculus

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This paper is a formal compilation of all my notes on geometric calculus. At the times when I finally make a breakthrough in understanding what's being said in a technical paper by Hestenes or anyone else, (which times, admittedly, are few and far between), I make note of it here and try to give a good explanation. See the reference section for the list of sources from which I am pulling information to be reiterated in simpler terms here.

1 Outermorphism

Let $f : \mathbb{A} \to \mathbb{B}$ be a linear transformation from the vector space $\mathbb{A}$ to the vector space $\mathbb{B}$. Every such transformation f can be extended to what we call an outermorphism $\underline{f}$ if for all vectors $a \in \mathbb{A}$, we have $\underline{f}(a) = f(a)$ and for all blades $A \in \mathbb{G}(\mathbb{A})$, we define $\underline{f}$ for A as preserving the outer product. Notice that by preserving the outer product, this does not necessarily mean that $\underline{f}$ preserves grade. For any k-blade $A \in \mathbb{G}(\mathbb{A})$, letting $A = \bigwedge_{i=1}^k a_k$ with each $a_k \in \mathbb{A}$, we may write

$$\underline{f}(A) = \underline{f}\left(\bigwedge_{i=1}^{k} a_k\right) = \bigwedge_{i=1}^{k} \underline{f}(a_k),$$

yet while $A \neq 0$, we may have $\underline{f}(A) = 0$, showing that while $\{a_i\}_{k=1}^k$ is a linearly independent set, $\{\underline{f}(a_i)\}_{i=1}^n$ may not be such a set. As will become clear later on, if a given $\underline{f}$ is always grade preserving, then $\underline{f}^{-1}$ must exist.

Of particular interest is how $\underline{f}$ maps the unit psuedo-scalar of $\mathbb{G}(\mathbb{A})$, which we'll denote by $I_{\mathbb{A}}$. Clearly this will be some scalar multiple of the unit psuedo-scalar of $\mathbb{G}(\mathbb{B})$, which we'll denote by $I_{\mathbb{B}}$. We define this scalar multiple as the determinant of $\underline{f}$ and write

$$\underline{f}(I_{\mathbb{A}}) = \left(\det \underline{f}\right) I_{\mathbb{B}}.$$

Associated with every outermorphism is a function $\overline{f}$ denoting what we call the adjoint of $\underline{f}$. We define $\overline{f} : \mathbb{A} \to \mathbb{B}$ as an outermorphism with the property that for any pair of vectors $a, b \in \mathbb{A}$, we have

$$a \cdot \underline{f}(b) = \overline{f}(a) \cdot b.$$

Using the k-blade A given earlier, this leads to the following result.

$$\begin{aligned}
a \cdot \underline{f}(A) &= -\sum_{i=1}^{k} (-1)^i (a \cdot \underline{f}(a_i)) \bigwedge_{j=1, j \neq i}^{k} \underline{f}(a_j) \\
&= -\sum_{i=1}^{k} (-1)^i (\overline{f}(a) \cdot a_i) \bigwedge_{j=1, j \neq i}^{k} \underline{f}(a_j) \\
&= \underline{f}\left(-\sum_{i=1}^{k} (-1)^i (\overline{f}(a) \cdot a_i) \bigwedge_{j=1, j \neq i}^{k} a_j \right) \\
&= \underline{f}(\overline{f}(a) \cdot A)
\end{aligned}$$

Now since $\overline{f}$ is an outermorphism, we may interchange underbars and overbars in the above equation.

Letting $A, B \in \mathbb{G}(\mathbb{A})$ be i and j-blades, respectively, recall that if $i \leq j$, we have the identity

$$A \cdot B = \left(\bigwedge_{k=1}^{i-1} a_k \right) \cdot (a_i \cdot B),$$

which is not hard to show. Recursively applying this identity, we get

$$A \cdot B = a_1 \cdot \cdots \cdot a_i \cdot B,$$

where here, right to left associativity of the inner product is understood. It then follows that

$$A \cdot \underline{f}(B) = \underline{f}\left(\overline{f}(a_1) \cdot \cdots \cdot \overline{f}(a_i) \cdot B\right) = \underline{f}(\overline{f}(A) \cdot B), \tag{1}$$

where here again we recursively applied the identity above.

We can now use the result in equation (1) to show that $\underline{f}$ and $\overline{f}$ have the same determinant in the case that $\mathbb{A} = \mathbb{B}$. In the case that $\mathbb{A} = \mathbb{B}$, let I denote the unit psuedo-scalar of $\mathbb{G}(\mathbb{A}) = \mathbb{G}(\mathbb{B})$. Recalling that for any k-blade A, we have $\tilde{A} = (-1)^{k(k-1)/2}A$, it follows that $I^{-1} = \lambda I$, where $\lambda = \pm 1$, depending on the dimension of $\mathbb{A}$. We then see that

$$\begin{aligned}
\det \underline{f} &= I^{-1} \cdot \underline{f}(I) \\
&= \underline{f}(\overline{f}(I^{-1}) \cdot I) \\
&= \underline{f}(\overline{f}(\lambda I) \cdot I) \\
&= \underline{f}(\overline{f}(I) \cdot \lambda I) \\
&= \underline{f}(\overline{f}(I) \cdot I^{-1}) \\
&= \underline{f}(\det \overline{f}) \\
&= \det \overline{f}.
\end{aligned}$$

We also have enough at this point to find a formula for the inverse of the outermorphism $\underline{f}$. Letting A be a blade in $\mathbb{G}(\mathbb{A})$, we have

$$(\det \underline{f})A \cdot I_{\mathbb{B}} = A \cdot \underline{f}(I_{\mathbb{A}}) = \underline{f}(\overline{f}(A) \cdot I_{\mathbb{A}}).$$

From this we get

$$(\det \underline{f})\underline{f}^{-1}(A \cdot I_{\mathbb{B}}) = \overline{f}(A) \cdot I_{\mathbb{A}}.$$

We can then make the substitution $B = A \cdot I_{\mathbb{B}}$ to get

$$\underline{f}^{-1}(B) = \frac{\overline{f}(B \cdot I_{\mathbb{B}}^{-1}) \cdot I_{\mathbb{A}}}{\det \underline{f}}. \tag{2}$$

We'll now show that $\underline{f}^{-1}$ is an outermorphism. Do that here...

Show that the outermorphism inverse is the inverse outermorphism. Do that here...

Using some calculus, we can find a formula for the adjoint $\overline{f}$ in terms of $\underline{f}$. Do that here...

References

[1] David Hestenes. The design of linear algebra and geometry. *Acta Applicandae Mathematicae*, 1991.

[2] Alan Macdonald. A survey of geometric algebra and geometric calculus, 2012.