An Intro to CGA

Conformal Geometric Algebra

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The extraordinary generality and simplicity of projective geometry led the English mathematician Cayley to exclaim: 'Projective Geometry is all of geometry'.

Source: "Geometric Algebra with Applications in Science and Engineering" by Corrochano & Sobczyk

The Outer Product

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Definition

Letting $\{w_k\}_{k=1}^m$ be the Gram-Schmidt orthonormalization of $\{v_k\}_{k=1}^m$, we have

$$\bigwedge_{k=1}^m v_k = \det \begin{bmatrix} v_1 \cdot e_1 & \dots & v_1 \cdot e_m \\ \vdots & \ddots & \vdots \\ v_m \cdot e_1 & \dots & v_m \cdot e_m \end{bmatrix} \bigwedge_{k=1}^m w_k,$$

where $\{e_k\}_{k=1}^m$ is any orthonormal basis for the m-dimensional vector sub-space represented by this blade.

Visualizing Euclidean Blades

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An m-blade is characterized by...

- Its attitude,
- Its m-dimensional hyper-volume,
- Its handedness.

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Generating all Elements of a Geometric Algebra

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The geometric algebra $\mathbb{G}(\mathbb{V}^n)$ is of dimension 2^n.

Adding Blades Together

Definition

For any $E \in \mathbb{G}(\mathbb{V}^n)$, we let $\langle E \rangle_k$ denote the grade k part of E, and so we may write $E = \sum_{k=0}^{n} \langle E \rangle_k$.

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All k-blades are k-vectors, but not all k-vectors are k-blades!

Example

The following 2-vector cannot be written as a 2-blade.

$$e_1 \wedge e_2 + e_3 \wedge e_4$$

The Inner Product

Definition

In a Euclidean geometric algebra, we define for all integers i and j,

$$e_i \cdot e_j = \delta_{ij},$$

where here, δ_{ij} is the Kronecker delta.

Definition

If for any vector $v \in \mathbb{V}^n$, we have $v \cdot v = 0$, we call v a null vector.

The Inner Product (Continued)

Definition

For any vector $v \in \mathbb{V}^n$ and any m-blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$v \cdot B = -\sum_{i=1}^{m} (-1)^i (v \cdot b_i) \bigwedge_{j=1, j \neq i}^{m} b_j,$$

where $B = \bigwedge_{k=1}^{m} b_k$. We also define

$$B \cdot v = -(-1)^m v \cdot B.$$

The Inner Product (Continued)

Lemma

For any $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we have

$$v \wedge B = v_\perp \wedge B$$
$$v \cdot B = v_\parallel \cdot B.$$

Example

Consider $v \cdot B$, where B is a 2-blade. WLOG, choose $a, b \in \mathbb{V}^n$ such that $B = a \wedge b$, $a \cdot b = 0$, $|b| = 1$ and $v \cdot b = 0$. We then have

$$v \cdot B = (v \cdot a)b - (v \cdot b)a = |B| \frac{v \cdot a}{|a|} b.$$

The Inner Product (Continued)

Definition

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ of grades i and j, respectively, we define

$$A \cdot B = \begin{cases} a_1 \cdot \cdots \cdot a_i \cdot B & \text{if } i \leq j, \text{ (R to L assoc.)} \\ A \cdot b_1 \cdot \cdots \cdot b_j & \text{if } i \geq j, \text{ (L to R assoc.)} \end{cases}$$

where $A = \bigwedge_{k=1}^{i} a_k$ and $B = \bigwedge_{k=1}^{j} b_j$.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any m-blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B,$$

and similarly, $Bv = B \cdot v + B \wedge v$.

Example

For any two vectors $a, b \in \mathbb{V}^n$, we have

$$ab = a \cdot b + a \wedge b = |a||b| \cos\theta + B|a||b| \sin\theta = |a||b| \exp(\theta B),$$

where $B = \frac{a \wedge b}{|a \wedge b|}$.

The Geometric Product (Continued)

Example

It can be shown that

$$v \cdot B = \frac{1}{2}(vB - (-1)^m Bv),$$

and

$$v \wedge B = \frac{1}{2}(vB + (-1)^m Bv).$$

The Geometric Product (Continued)

Definition

For any set of m vectors $\{v_k\}_{k=1}^m \subset \mathbb{V}^n$, a product

$$\prod_{k=1}^{m} v_k$$

is called a versor if for all integers k, v_k^{-1} exists.

If there exists an integer k such that v_k^{-1} does not exist, I call it a pseudo versor.

Lemma

In a Euclidean geometric algebra, any blade can be written as a versor by the Gram-Schmidt orthogonalization process.

Example

For any two multivectors $A, B \in \mathbb{G}(\mathbb{V}^n)$, we have

$$AB = \phi^{-1}(\phi(A)\phi(B)),$$

where ϕ is an algebraic transformation mapping a multivector to its multi-psuedo-versor form.

Lemma

The Gram-Schimdt process cannot always be used on blades taken from a non-Euclidean geometric algebra!

Proof.

Consider $a \wedge b$. If $a \cdot b \neq 0$ and a, b are null, then there does not exist a scalar λ such that $a \cdot (b + \lambda a) = 0$ or $(a + \lambda b) \cdot b = 0$. $\square$

Blade to Multi-Psuedo-Versor Form

Let $B \in \mathbb{G}(\mathbb{V}^n)$ be a blade of grade $m > 1$ where $B = \bigwedge_{k=1}^m b_k$. We then have

$$\begin{aligned}\phi(B) = B &= b_1 B^{(1)} - b_1 \cdot B^{(1)} \\ &= b_1 \phi(B^{(1)}) - \sum_{i=2}^{m} (-1)^i (b_1 \cdot b_i) \phi(B^{(1)(i)}),\end{aligned}$$

where $B^{(i)}$ is notation for the $(m-1)$-blade $\bigwedge_{k=1,k\neq i}^m b_k$.

Example

For the blade $a \wedge b$, we have $\phi(a \wedge b) = ab - a \cdot b$.

Psuedo-Versor to Multivector Form

Let $V \in \mathbb{G}(\mathbb{V}^n)$ be a versor of size $m > 1$ where $V = \prod_{k=1}^{m} v_k$. We then have

$$\phi^{-1}(V) = V = v_1 \sum_{k=0}^{m} \langle V^{(1)} \rangle_k =$$

$$\langle \phi^{-1}(V^{(1)}) \rangle_0 v_1 + \sum_{k=2}^{m} \left(v_1 \wedge \langle \phi^{-1}(V^{(1)}) \rangle_k + v_1 \cdot \langle \phi^{-1}(V^{(1)}) \rangle_k \right),$$

where $V^{(i)}$ is notation for the $(m-1)$-sized psuedo-versor $\prod_{k=1,k\neq i}^{m} v_k$.

Example

For the versor ab, we have $\phi^{-1}(ab) = a \cdot b + a \wedge b$.

The Geometric Product (Again)

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ *of grades* i *and* j*, respectively, it can be shown that*

$$A \cdot B = \langle AB \rangle_{|i-j|},$$

and

$$A \wedge B = \langle AB \rangle_{i+j}.$$

The Reverse

Definition

For any m-sized versor $V \in \mathbb{G}(\mathbb{V}^n)$ where $V = \prod_{k=1}^m v_k$, we define

$$\tilde{V} = \prod_{k=1}^{m} v_{m-k+1}.$$

We can extend this definition to any multivector if we let the reverse operator distribute over addition.

Definition

For any multivector $E \in \mathbb{G}(\mathbb{V}^n)$, we may write

$$\tilde{E} = \phi^{-1}(\tilde{\phi}(E)).$$

Lemma

For any m-sized versor $V \in \mathbb{G}(\mathbb{V}^n)$ *where* $V = \prod_{k=1}^m v_k$, *we have*

$$V^{-1} = \left(\prod_{k=1}^{m} |v_k|\right)^{-1} \tilde{V}.$$

Lemma

For any *Euclidean* *m-blade* $B \in \mathbb{G}(\mathbb{V}^n)$ *where* $B = \bigwedge_{k=1}^m b_k$, *we have*

$$B^{-1} = (-1)^{m(m-1)/2} \frac{\tilde{B}}{|B|}.$$

Example

If $v \in \mathbb{V}^n$ is a null vector, then v^{-1} does not exist.

Conjugation by Versors

Lemma

Conjugation by versors is outermorphic. That is, for any versor $V = \prod_{k=1}^{i} v_k$*, and any blade* $B = \bigwedge_{k=1}^{j} b_k$*, we have*

$$VBV^{-1} = \bigwedge_{k=1}^{j} Vb_kV^{-1}.$$

The proof of this is not too hard to get, but too big to put here.

Example

A given rotor $R \in \mathbb{G}(\mathbb{V}^n)$ is a versor that rotates points $v \in \mathbb{V}^n$ by versor conjugation. It therefore rotates blades as well!

Blades Can Represent Vector Spaces

Definition

For any vector $v \in \mathbb{V}^n$ and any m-blade $B \in \mathbb{G}(\mathbb{V}^n)$ where $B = \bigwedge_{k=1}^{m} b_k$, we say that

$$v \in B \text{ if and only if } v \in \operatorname{span}\{b_k\}_{k=1}^{m}$$

Lemma

We have $v \in B$ if and only if $v \wedge B = 0$.

Lemma

We have $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge BI)I$ if and only if $v \wedge BI = 0$. $\square$

How Blades Can Represent Geometry

Let $\mathbb{V}^n$ denote a Euclidean vector space.
Let $\mathbb{V}$ denote any other vector space.
Let $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V})$ be a blade-valued function of points.

Definition

Given any blade $B \in \mathbb{G}(\mathbb{V})$, we say that B directly represents the geometry that consists of all points

$$G(B) = \{x \in \mathbb{V}^n | p(x) \in B\}.$$

Definition

Given any blade $B \in \mathbb{G}(\mathbb{V})$, we say that B dually represents the geometry that consistent of all points

$$G^*(B) = \{x \in \mathbb{V}^n | p(x) \in B^*\}.$$

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V})$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff} \quad & p(x) \notin A \wedge B \\ \text{iff} \quad & p(x) \in (A \wedge B)^* \end{aligned}$$

□

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V})$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \Longrightarrow\ & p(x) \in A \wedge B \end{aligned}$$

$\square$

Let $C \subset A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subset A$ and $C \not\subset B$.

Finally, The Conformal Model

Let $\mathbb{V}^n$ be a vector-subpace of $\mathbb{V}$.
If $\{e_k\}_{k=1}^n$ is any basis for $\mathbb{V}^n$, let $\{e_k\}_{k=1}^n \cup \{o, \infty\}$ be a basis for $\mathbb{V}^n$.

Definition

For any vector $v \in \mathbb{V}^n$, we define $v \cdot o = v \cdot \infty = 0$. We define $o \cdot \infty = \infty \cdot o = -1$. Each of o and ∞ are defined as null.

Definition

We define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

What we can now discover about this model of geometry is almost endless!

Vectors In The Conformal Model

Lemma

For any $x \in \mathbb{V}^n$ *and any scalar* $r > 0$, *the vector*

$$p(x) - \frac{1}{2}r^2\infty$$

dually represents an n-dimensional hyper-sphere.

Lemma

For any $x \in \mathbb{V}^n$ *and any unit-length vector* $v \in \mathbb{V}^n$, *the vector*

$$v + (x \cdot v)\infty$$

dually represents an $(n-1)$*-dimensional hyper-plane.*

Generating All Rounds And Flats

Lemma

Let $\{\sigma_k\}_{k=1}^m$ *be a sequence of dual n-dimensional hyper-spheres. Then*

$$\bigwedge_{k=1}^{m} \sigma_k$$

may be a dual $(n-m+1)$*-dimensional hyper-sphere.*

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General Term For Coplanar

Definition

A set of $m+2$ points $\{x_k\}_{k=1}^{m+2}$ are co-*m*-hyper-planar if...

For $m = 0$, the points are identical,
For $m = 1$, the points are collinear,
For $m = 2$, the points are coplanar,
For $m = 3$, the points are co-hyper-planar,
etc...

Lemma

If $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

Lemma

For $m+1$ points $\{x_k\}_{k=1}^{m+1}$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade

$$\bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

Proof.

Let the $(n-m+1)$-blade B dually represent the sphere. Then for all k, we have $p(x_k) \in B^*$ and $\text{grade}(B^*) = n+2-(n-m+1) = m+1$. Lastly, $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set. □

Fitting Flats To Points

Lemma

For $m+2$ points $\{x_k\}_{k=1}^{m+2}$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade

$$\bigwedge_{k=1}^{m+2} p(x_k)$$

directly represents the m-dimensional hyper-plane.

Proof.

Let the $(n-m)$-blade B dually represent the plane. By (1) and (2), $\{p(x_k)\}_{k=1}^{m+2}$ is linearly independent. Then $\text{grade}(B^*) = n+2-(n-m) = m+2$. □

Fitting Flats To Points (Continued)

Lemma

For $m+1$ points $\{x_k\}_{k=1}^{m+1}$ on an m-dimensional hyper-plane that are non-co-$(m-1)$-hyper-planar, the blade

$$\infty \wedge \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-plane.

Lemma

Let $V \in \mathbb{G}(\mathbb{V})$ be a versor. Let the m-blade $B \in \mathbb{G}(\mathbb{V})$ directly represent any geometry of the conformal model, (except a flat point). Then the transformation

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vp(x_k)V^{-1}$$

is understood if $Vp(x_k)V^{-1}$ is understood.

Compare this idea to linear transformations determined by a basis of a vector space.