An Intro to CGA

Conformal Geometric Algebra

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Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find the forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from $\mathbb{V}^n$.

Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^{m} b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Visualizing Euclidean Blades

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.
Non-Euclidean blades require more imagination!
Our geometric arguments will not require us to visualize the homogeneous representation space.

Building Intuition About Euclidean Blades

Let $v_{||}$ denote the orthogonal projection of v down onto B.
Let $v_{\perp} = v - v_{||}$ denote the orthogonal rejection of v from B.
For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{||} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
$$v \cdot B = (v_{||} + v_{\perp}) \cdot B = v_{||} \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \operatorname{span}\{b_k\}_{k=1}^m.$$

Definition

If $v \notin B$, then $v \in B^*$, which represents the complement $(\mathbb{V}^n - \operatorname{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\square$

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. $\square$

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Definition

We say that B dually represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \implies & p(x) \in A \wedge B \end{aligned}$$

$\square$

Let $C \subseteq A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subseteq A$ and $C \not\subseteq B$.

We Can Intersect Geometries

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$\begin{aligned}
& p(x) \in A^* \text{ and } p(x) \in B^* \\
\text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\
\text{iff} \quad & p(x) \notin A \wedge B \\
\text{iff} \quad & p(x) \in (A \wedge B)^*
\end{aligned}$$

$\square$

The Homogeneous Nature Of The Model

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.
For any blade B, there is a scalar λ such that λB is a homogenized form.
If B is the result of some geometric operations, then such a λ has geometric signficance WRT to that operation.

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Versors

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^{m} v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

The Inverse And The Reverse Of Versors

Definition

Given the versor $V = v_1 \dots v_m$, we define

$$\tilde{V} = \prod_{k=1}^{m} v_{m-k+1}.$$

The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

The Versor Group

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.
The scalar 1 is the identity versor.
For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = 1$. $\square$

Properties Of Versors

Conjugation by versors is outermorphic!
Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.

The Specifics Of The Conformal Model

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.
Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space.
Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Having invented this specific model, what we are now able to discover about it is almost endless!