Documentation on Octane's Geometric Algebra

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July 7, 2012

1 Motivation and Introduction

Many people have come to me with questions about the geometric algebra types supported by the Octane Engine. The purpose of this document is to educate people on some basic geometric algebra so that they can make better use of these data types in their programming work. Perhaps the most important data type is the rotor. A lot of background is covered before a treatment of rotors is given, but it is worth the effort. Some problems you face might be more approachable with geometric algebra than traditional methods. We do not discuss any homogeneous models invented for geometric algebra, such as the conformal model. Though Octane may one day make use of such a model, it is beyond the scope of this document. At the time of this writing, Octane implements a geometric algebra generated by a 3-dimensional vector space, and this will be our focus. Anything learned here will still apply to future expansions of Octane's geometric algebra support.

Lastly, let me say here that the author has gone to great lengths to insure the accuracy of this document. Even still, if you find what you believe to be an error, please inform the author so that we can get it fixed.

2 Blades and the Wedge Product

We will let $\mathbb{V}^3$ denote our 3-dimensional vector space. This is the space of vectors you are familiar with in your programming work. You know how to add them, you know how to scale them, and you've done lots of other algebra with them. You're familiar with the dot product and the cross product. Maybe you've even done some vector calculus. In any case, we're going to keep all of these things as tools we can use in geometric algebra. Surprise! You already know some geometric algebra! We won't really need the cross product, however, because what we'll find is that it is a special case of what we can already do in geometric algebra. Of course, the cross product is still supported in code, but after coming to know the wedge product, it is possible that you may find cases where you're using the wedge product when otherwise you might have used the cross product.

2.1 What is the wedge product?

The wedge product is used to wedge vectors together to form what we call blades. Formally, if $\{v_1, v_2, \dots, v_k\}$ is a set of k vectors taken from $\mathbb{V}^3$, then we define the wedge product

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k$$

to be a non-zero k-blade if and only if $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors. Recall from your linear algebra class that in our case this implies that $k \leq 3$. Any set of four or more vectors taken from $\mathbb{V}^3$ must be linearly dependent. We define the wedge product of any set of linearly dependent vectors to be zero.

2.2 What is a k-blade?

A k-blade has a geometric interpretation just like vectors do. In fact, we call non-zero vectors 1-blades, non-zero scalars 0-blades, and we let zero just be zero. You might think of zero as the absence of any geometry whatsoever. The interpretation we choose to give k-blades will explain what they are for our purposes. Let's start with a pair of non-parallel vectors $a, b \in \mathbb{V}^3$. It follows by definition that their wedge product, $a \wedge b$, is non-zero. Just as the name blade implies, we think of $a \wedge b$ as something flat. Being non-parllel, the vectors a and b determine the plane containing a and b, and we think of $a \wedge b$ as a flat area in that plane. The amount of area, denoted $|a \wedge b|$, is given by $|a||b| \sin\theta$, where θ comes from the familiar equation $a \cdot b = |a||b|\cos\theta$. So while 1-blades (vectors) have length and direction, 2-blades have area and orientation, the latter refering to the orientation of a plane, and the former to the amount of area in that plane.

The keen observer at this point may realize that there are many pairs of vectors $a, b \in \mathbb{V}^3$ whose wedge product meet this description for a particular instance of a 2-blade. Indeed, this is not an accident, and turns out to be one of the strengths of geometric algebra. Geometrically, this means that we do not put a constraint on the shape of the area of $a \wedge b$, so we can think of it as any shape we want. Algebraically, this means that 2-blades do not have a unique factorization in terms of the wedge product. Often we will choose a factorization that is most convenient for our purposes. You'll see many examples of this in the sections to come.

Lastly, there is another characteristic of the blade $a \wedge b$ taken on to differentiate it from the blade $b \wedge a$. This isn't so much a characteristic of a single blade as it is something that is manifested in the comparison of two such blades with the same area and in the same plane. Think of $a \wedge b$ as oriented the opposite way of $b \wedge a$. What we might call the front and back of $a \wedge b$ is reversed for $b \wedge a$. We also say that $a \wedge b$ and $b \wedge a$ have opposite handedness.

Let's now consider a linearly independent subset $\{a, b, c\}$ of $\mathbb{V}^3$. This non-zero 3-blade $a \wedge b \wedge c$ has volume and orientation. Notice, however, that the hyper-plane containing $a \wedge b \wedge c$ isn't very interesting, because all 3-blades generated from $\mathbb{V}^3$ will have the same orientation. (They all determine the same hyper-plane.) If we let $\{x, y, z\}$ be a set of vectors forming an orthonormal basis for $\mathbb{V}^3$, then it is instructive to express $a \wedge b \wedge c$ as a scalar multiple of $x \wedge y \wedge z$ as follows.

$$a \wedge b \wedge c = \det \begin{bmatrix} a \cdot x & a \cdot y & a \cdot z \\ b \cdot x & b \cdot y & b \cdot z \\ c \cdot x & c \cdot y & c \cdot z \end{bmatrix} x \wedge y \wedge z \tag{1}$$

Here, the absolute value of the determinant gives us the volume $|a \wedge b \wedge c|$, while the sign of the the determinant would indicate the handedness of $a \wedge b \wedge c$ relative to $x \wedge y \wedge z$. From this point on we will think of $x \wedge y \wedge z$ as being right-handed. This is an arbitrary decision, but important so that we're all thinking about it in the same way. Being right-handed, x is the index finger of your right hand, y is the middle finger, and z is the thumb.

In closing, we'll note here that just as vectors are not thought of as residing at any particular location in space, the same is said for blades in general. This is yet another strength of the algebra, and of which you are already familiar as no doubt you have often experienced the convenience of visualizing vectors in various locations for various reasons.

2.3 How do k-blades add?

You already know how to add vectors geometrically. 2-blade addition can be defined in terms of vector addition. Given a 2-blade $c \wedge a$ and a 2-blade $c \wedge b$, we define their sum to be $c \wedge (a + b)$. (Try to visualize this. Draw a picture if necessary.) Similarly, we define $a \wedge c + b \wedge c$ as $(a + b) \wedge c$. (Notice that this implies the distributivity of the wedge product over addition.) At this point you might be asking: how do we add the blades $a \wedge b$ and $c \wedge d$ in general? It might seem that our definition of 2-blade addition is inadiquite to the task, but it isn't! We need only make three observations. First, realize that $\{a, b, c, d\}$ is a subset of $\mathbb{V}^3$ and therefore linearly dependent. Secondly, notice that we can rotate $a \wedge b$ and $c \wedge d$ in their respective planes without altering the blades in question. Lastly, realize that we can lengthen a while shortening b without altering $|a \wedge b|$. These last two observations take advantage of the non-unique factorizations of blades. The first observation assures us that if we rotate $a \wedge b$ and $c \wedge d$ appropriately, we can get the vectors a and c into alignment. We then adjust the length of a to be that of c while adjusting the length of b to maintain the area of $a \wedge b$. Then once $a = c$, we have $a \wedge b + c \wedge d = a \wedge (b + d)$. Realize that the vectors a, b, c, and d in this equation are not necessarily the ones we started with!

Our discussion of 2-blade addition brings up an important point. That is, if $\{a, b, c, d\}$ is a set of 4 linearly independent vectors, then the 2-blades $a \wedge b$ and $c \wedge d$ do not add together! This is because no refactorizations of the blades can be found that allow us to find a common vector factor. If we were to imagine each blade as a circle of area at the origin, notice that these circles do not intersect, because they exist in disjoint vector sub-spaces. (Strictly speaking, no two vector sub-spaces are disjoint, because they both contain zero. But for our purposes, we will say that two such spaces are disjoint if there does not exist a non-zero vector in both spaces.) In such a case, the sum of these blades is just what it is, and we tote them about as a pair throughout our calculations. This is not a new idea. Recall your study of complex numbers.

Now notice that no set of 4 linearly independent vectors can be found in $\mathbb{V}^3$. It follows that all 2-blades generated from $\mathbb{V}^3$ add together as 2-blades. This is an important result that we will refer to again in subsection (2.5).

In the previous section we saw that all 3-blades are scalar multiples of one another, because they're all scalar multiples of $x \wedge y \wedge z$. We will give $x \wedge y \wedge z$ a special symbol I, and call it the unit-psuedo scalar. We'll find this useful as you'll see in sections to come. Start likening it to the imaginary unit i from complex analysis.

So what about adding an i-blade with a j-blade when $i \neq j$? These don't add to a single blade. Again, we simply tote these about as a pair. Sums of blades of various grade, (k being what we refer to as the grade of a k-blade), are the form of the general element in our geometric algebra, as we'll see in subsection (2.5).

2.4 What of the commutativity of the wedge product?

We've seen that the wedge product is left and right distributive over addition, and we have implied its associativity in our discussion of it, which really follows directly from our definition of the product. But what can we say about its commutativity? At this point, we now have enough to prove something about the commutativity of the wedge product. Let $a \wedge b$ be a non-zero 2-blade. Now convince yourself that $a \wedge b = b \wedge (-a)$. (Don't hesitate to draw a picture of this.) These have the same area, orientation, and handedness. It now follows that $a \wedge b + b \wedge a = b \wedge (-a) + b \wedge a = b \wedge (a - a) = 0$. So we see that $a \wedge b$ and $b \wedge a$ are additive inverses of one another. We can express this as

$$a \wedge b = -(b \wedge a),$$

showing that the wedge product is anti-commutative. Furthermore, we've shown that $-(b \wedge a) = b \wedge (-a)$, and it is easy to see that $b \wedge (-a) = (-b) \wedge a$. Together, these show that the sign commutes across the wedge product so that there is no ambiguity in the statement $a \wedge b = -b \wedge a$, which is what we'll prefer to write when making use of the anti-commutative property of the wedge product. In general, we will take for granted the fact that scalars commute and distribute with all other elements and operations of our algebra.

Notice that the anti-commutativity of the wedge product in the context of equation (1) is consistent with what we know in linear algebra about what happens to the sign of determinents when we interchange rows and columns of a matrix. This is no accident!

2.5 Putting it all Together

We now have enough geometric algebra under our belts to make a complete picture of every element in the algebra currently supported by the Octane Engine. We'll do this by comming up with a basis for the algebra. While such a basis is needed to do geometric algebra on the computer, it's important to realize that in practice, when we're doing geometric algebra on paper, we do not need to refer to a basis as we do in linear algebra. This is another one of the strengths of geometric algebra, and interestingly, what this often means is that formulas and results we find using geometric algebra are independent of dimension. That is, if we derive a formula that holds in 3 dimenions, it will likely generalize to holding in all dimensions without any additional work! This happens if we don't make any assumptions along the way that depend on the dimensionality of the generating vector space.

To generate a basis, we start with the orthonormal basis $\{x, y, z\}$ of $\mathbb{V}^3$ used earlier, and then consider what terms show up in the expansion of the product

$$\bigwedge_{i=1}^{k}(\alpha_i x + \beta_i y + \gamma_i z), \tag{2}$$

for $k = 1, 2, 3$. (The scalars α_i, β_i, and γ_i are not important here.) When $k = 1$, we get x, y, and z, which are simply the basis elements for $\mathbb{V}^3$. When $k = 2$, we get $x \wedge y$, $y \wedge z$, and $x \wedge z$, after collecting terms. When $k = 3$, we get $x \wedge y \wedge z$, after collecting terms. (Try using (2) to prove (1). Let $\alpha_1 = a \cdot x$, $\beta_1 = a \cdot y$, $\gamma_1 = a \cdot z$, let $\alpha_2 = b \cdot x$, $\beta_2 = b \cdot y$, $\gamma_2 = b \cdot z$, and let $\alpha_3 = c \cdot x$, $\beta_3 = c \cdot y$, and $\gamma_3 = c \cdot z$.) Throwing in 1 as the basis element for the non-zero scalars, this suggests the following set of basis elements.

$$\{1, x, y, z, y \wedge z, z \wedge x, x \wedge y, x \wedge y \wedge z\}.$$

Here we have used a variation of the 2-blade basis elements that we'll find convenient. Notice that the geometric algebra is $2^3 = 8$ dimensional.

We now see that the general element of our algebra is some linear combination of these basis elements. These are the multivectors in the Octane Engine. The other vector types are special cases of this. Outside of the computer we could simply describe the general element as a sum of blades of grades zero, one, two, and three, realizing that zero takes on any grade we want, and that we can absorb scalars into the lengths, areas, and volumes of blades. This brings us to an important discussion about the geometric types supported by the Octane Engine. To understand why they're named the way they are, we need to introduce a bit of notation and terminology.

Letting $\mathbb{G}^3$ denote Octane's 8-dimensional geometric algebra, we can say that for all $g \in \mathbb{G}^3$, that

$$g = \sum_{k=0}^{3} \langle g \rangle_k,$$

where $\langle g \rangle_k$ denotes the grade k part of g. The grade k part of g is the sum of all k-blades in g. If there were no other blades of a different grade in g other than k, then we could express this as $g = \langle g \rangle_k$, and we would say that g is homogeneous of grade k. An element $g \in \mathbb{G}^3$ homogeneous of grade 1 is called a vector. An element homogeneous of grade 2 is called a bivector, and an element homogeneous of grade 3 is called a trivector.

We now come to an important point about our algebra $\mathbb{G}^3$. In our algebra, all bivectors are 2-blades and all tivectors are 3-blades. This is because all 2-blades add together as 2-blades and all 3-blades add together as 3-blades. Do not make the mistake of believing that in general, all k-blades add together as k-blades. This is not true! In general, all k-blades are k-vectors, but not all k-vectors are k-blades. We saw an example of this in subsection (2.3) when we considered the sum $a \wedge b + c \wedge d$. This is a 2-vector, but it is not a 2-blade. These distinctions come up in geometric algebras generated by vector spaces of dimensions greater than 3. So when using bivectors and trivectors in Octane, realize that these are blades, but only because Octane currently implements $\mathbb{G}^3$ and not $\mathbb{G}^n$ for some $n > 3$. A reminder of this fact is also embedded in the naming scheme Bryant Collard (the software architect of Octane's geometric algebra system) has given to these types. The "3d" appended to the name "bivector" reminds you that these are bivectors taken from $\mathbb{G}^3$, which again, are 2-blades.

In closing, everything that we have described up to this point in the document is a Grassman algebra, sometimes called an algebra of an outter product. Our task now is to take it to a Clifford algebra by definning a dot product on it, sometimes called an inner product. This is the goal of the following sections.

3 The Geometric Product

Here we will introduce the geometric product initially as a tool for studying a generalization of the dot product, and then go on to see that it is a fundamental product in its own right. The geometric product will become our main tool in formulas for common geometric operations such as reflections, projections, rejections, and rotations. That's right, I said rejections, but thankfully it doesn't have anything to do with getting rejected by women.

3.1 What is the Geometric Product?

Given any two vectors $a, b \in \mathbb{V}^3$, we define

$$ab = a \cdot b + a \wedge b \tag{3}$$

to be the geometric product of a and b, where $(\cdot)$ is the familiar dot product, and $(\wedge)$ is the now familiar wedge product. The reader can easily check that this product commutes whenever a and b are parallel vectors, and anti-commutes whenever a and b are perpendicular vectors. The distributivity of this product follows from the distributivity of the dot and wedge products. (We will find that the product is distributive over addition with respect to any pair of elements in $\mathbb{G}^3$.) As it stands now, there is nothing we can say about the associativity of this product. To see why, consider the product aab. If we were to disambiguate this as $a(ab)$, then we get $(a \cdot b)a + a(a \wedge b)$. The term $(a \cdot b)a$ is defined, but $a(a \wedge b)$ is not. Seeing that our definition of the geometric product does not predetermine its associativity, we are free to simply define it as associative. It then follows that $a(ab) = (aa)b = |a|^2 b$, and we see that $a(a \wedge b) = |a|^2 b - (a \cdot b)a$. We now have a formulation for the geometric product of a and $a \wedge b$. As one of the main results of this section, we will find a formula for the geometric product between a vector and a k-blade.

Later on we'll see that the geometric product of two vectors is what we'll call a rotor. Rotors can be used to represent transformations that scale and rotate any element of the algebra, even rotors themselves. A treatment of rotors will be given in subsection (4.7).

3.2 How can we Extend the Dot Product?

We do this by defining it in terms of the geometric product. Possible motivations for our definition are found in the results we get out of it, most notably those results that are consistent with other ways in which people have defined the dot product between blades. These nice results are a credit to Clifford who invented the geometric product.

Letting $A_i = a_1 \wedge a_2 \wedge \cdots \wedge a_i$ be an i-blade and $B_j = b_1 \wedge b_2 \wedge \cdots \wedge b_j$ be a j-blade, we will define their dot product as

$$A_i \cdot B_j = \langle A_i B_j \rangle_{|i-j|}.$$

In other words, $A_i \cdot B_j$ is the grade $|i-j|$ part of the geometric product between A_i and B_j. Now, this may seem silly, because at first glance, such a geometric product appears to be undefined, but it isn't! The reason comes from two observations. The first is that we have defined the geomtric product to be associative, and the second is that we can find refactorizations of A_i and B_j such that they may be expressed in terms of the geometric product. That is, without loss of generality, we can assume that the sets $\{a_k\}_{k=1}^{i}$ and $\{b_k\}_{k=1}^{j}$ each form an orthogonal basis. It follows that

$$A_i \cdot B_j = \langle a_1 a_2 \ldots a_i b_1 b_2 \ldots b_j \rangle_{|i-j|}.$$

The key to seeing this is realizing that the geometric product *is* the wedge product among a set of pair-wise orthogonal vectors. Admittedly, in this case, using (3) to rewrite $a_1 \wedge a_2 \wedge \cdots \wedge a_i$ as $a_1 a_2 \dots a_i$ is still a bit of a stretch for us at this point if we were to try to prove it formally, but in order to build our algebra from the ground up, it is something we'll have to take for granted. Consider an inductive proof later on in your study.

Let's do an example. Suppose we want to find the dot product between a vector $a \in \mathbb{G}^3$ and a 2-blade $B \in \mathbb{G}^3$. Now, since there exist vectors $b, c \in \mathbb{G}^3$ such that $B = b \wedge c$ and $b \cdot c = 0$, we can write $B = bc$. It follows that

$$a \cdot B = \langle abc \rangle_1.$$

Reducing the problem of finding their dot product to that of finding the grade 1 part of a geometric product is nice, because, as we'll see now and continue to see, the geometric product has a lot of nice algebraic properties. Consider now writing a in terms of the parts $a_\parallel$ and $a_\perp$ of a parallel and perpendicular to B, respectively. Being in the plane of B, it is clear that $a_\parallel$ is a linear combination of b and c. We can express this as

$$\langle abc \rangle_1 = \langle (a_\perp + a_\parallel) bc \rangle_1 = \left\langle \left[a_\perp + \left(\frac{a_\parallel \cdot b}{|b|} \right) \frac{b}{|b|} + \left(\frac{a_\parallel \cdot c}{|c|} \right) \frac{c}{|c|} \right] bc \right\rangle_1. \tag{4}$$

(Recall your study of vector projections.) Simplifying this, we get $\langle abc \rangle_1 = \langle a_\perp bc \rangle_1 + (a_\parallel \cdot b)c - (a_\parallel \cdot c)b$. Now notice that $a_\perp bc = a_\perp \wedge b \wedge c$, which is a 3-blade if $a_\perp \neq 0$. In any case, $\langle a_\perp bc \rangle_1 = 0$. It now follows that

$$a \cdot B = (a_\parallel \cdot b)c - (a_\parallel \cdot c)b, \tag{5}$$

and we have our first result about the dot product between a vector and a blade! What this equation is telling us is that the operation $a \cdot B$ produces a vector that is a scaled rotation of the orthogonal projection of a down onto the plane of B. The rotation is $\pi/2$ radians with respect to the handedness of B, and the scale is the magnitude of B. To see this, multiply and divide the right hand side of (5) by $|B| = |b||c|$.

$$a \cdot B = |B| \left[\left(\frac{a_\parallel \cdot b}{|b|} \right) \frac{c}{|c|} - \left(\frac{a_\parallel \cdot c}{|c|} \right) \frac{b}{|b|} \right]$$

Comparing the change of basis between this and that shown in (4), we can recognize the $\pi/2$ rotation. If this rotation is hard to see, realize that without loss of generality, we could have also chosen b and c such that the projection of a down onto the plane of B is parallel to b with $a_\parallel \cdot b \geq 0$. Furthermore, we could have required that $|b| = 1$ and $|c| = |B|$. We now see that

$$a \cdot B = \langle abc \rangle_1 = \langle a_\parallel bc \rangle_1 = \langle |a_\parallel| bbc \rangle_1 = |a_\parallel| c = |a_\parallel||B| \frac{c}{|c|}. \tag{6}$$

This makes it even easier to see how the direction of the $\pi/2$ rotation is determined by the handedness of B. The reader would not be wrong at this point to complain that we should have done this in the first place, but going through the more extensive steps to begin with has only strengthened our geometric algebra skills!

Returning to (5), notice that replacing $a_\parallel$ with $a - a_\perp$, this equation simplifies to $a \cdot B = (a \cdot b)c - (a \cdot c)b$. We know that this holds if the 2-blade $B = bc$, which implies that $b \cdot c = 0$. But does it hold when $B = b \wedge c$ and $b \cdot c \neq 0$? To investigate this, lets find a refactorization of B that lets us apply our current result aboug $a \cdot B$. Assuming $b \cdot c \neq 0$, but $B = b \wedge c \neq 0$, notice that there exists a scalar λ such that $(b + \lambda c) \cdot c = 0$. (We could use $\lambda = -(b \cdot c)/|c|^2$, but there is no need to complicate our calculations with the extra symbols.) Now notice that $B = (b + \lambda c) \wedge c = (b + \lambda c)c$. It then follows by our earlier result that

$$\begin{aligned} a \cdot (b \wedge c) &= a \cdot ((b + \lambda c)c) \\ &= (a \cdot (b + \lambda c))c - (a \cdot c)(b + \lambda c) \\ &= (a \cdot b)c + \lambda(a \cdot c)c - (a \cdot c)b - \lambda(a \cdot c)c \\ &= (a \cdot b)c - (a \cdot c)b. \end{aligned} \tag{7}$$

So the answer is yes, and we have a nice formula for $a \cdot (b \wedge c)$ as a linear combination of b and c!

3.3 Formulas for the Dot and Wedge Products

In this section we will derive useful formulas for the dot and wedge products between vectors and blades in terms of the geometric product. After reading this section, the reader is encouraged to prove that, given the blades A_i and B_j of the previous section, $A_i \wedge B_j = \langle A_i B_j \rangle_{i+j}$.

Let a be a vector and $A_k = a_1 a_2 \dots a_k$ be a non-zero k-blade. (Again, we can satisfy this condition if for all $i \neq j$, we let $a_i \cdot a_j = 0$.) Now consider the geometric product aA_k. If we write a as $a_\parallel + a_\perp$, where $a_\parallel$ and $a_\perp$ are the parts of a parallel and perpendicular to the blade A_k, respectively, then the calculation of aA_k falls through fairly easily. Let us begin by writing

$$aA_k = a_\parallel A_k + a_\perp A_k. \tag{8}$$

Seeing that for all $1 \leq i \leq k$, we have $a_\perp \cdot a_i = 0$, it is clear that $a_\perp A_k$ is a $(k+1)$-blade, and therefore homogeneous of grade $k+1$. It is not too hard to show that $a_\parallel A_k$ is homogeneous of grade $k-1$. To see this, write

$$a_\parallel A_k = ((a_\parallel \cdot a_1)a_1^{-1} + (a_\parallel \cdot a_2)a_2^{-1} + \cdots + (a_\parallel \cdot a_k)a_k^{-1})A_k, \tag{9}$$

where for all $1 \leq i \leq k$, we let a_i^{-1} denote the vector a_i / a_i^2. Notice that $(a_\parallel \cdot a_i)a_i^{-1}A_k$ is a $(k-1)$-blade, since $a_i^{-1}A_k = -(-1)^i A_k^i$, where A_k^i denotes the product $a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_k$. Notice that we do not immediately claim here that $a_\parallel A_k$ is a $(k-1)$-blade, because we do not know if the $(k-1)$-blades in (9) add as a single blade, but we can say that $a_\parallel A_k$ is a $(k-1)$-vector. The reader is encouraged to later prove that $a_\parallel A_k$ is indeed a $(k-1)$-blade. (The key is in realizing that $a_\parallel$ is in the vector space represented by A_k. Now let $a_\parallel$ form part of the basis for that vector space.)

What we have now shown is that (8) can be rewritten as

$$aA_k = \langle aA_k \rangle_{k-1} + \langle aA_k \rangle_{k+1},$$

showing that aA_k expands as a multivector having grade $k-1$ and $k+1$ parts. Furthermore, by our definitions of the dot and wedge products, it is immediately clear that this becomes

$$aA_k = a \cdot A_k + a \wedge A_k.$$

Notice how this nicely generalizes our definition of the geometric product in (3) to vectors and k-blades.

Now convince yourself of the following identities.

$$a_\perp A_k = (-1)^k A_k a_\perp \tag{10}$$

$$a_\parallel A_k = -(-1)^k A_k a_\parallel \tag{11}$$

To see (10), imagine $a_\perp$ as it anti-commutes its way to the other side of A_k. Use the same idea with (11) for each a_i^{-1} in (9) and notice that in each case, exactly one of the swaps is commutative instead of anti-commutative.

Using (10), then (11), we see that

$$a \wedge A_k = \frac{1}{2}(a_\perp A_k + (-1)^k A_k a_\perp) = \frac{1}{2}(aA_k + (-1)^k A_k a). \tag{12}$$

To see the last step, replace $a_\perp$ with $a - a_\parallel$. Using (11), then (10), we see that

$$a \cdot A_k = \frac{1}{2}(a_\parallel A_k - (-1)^k A_k a_\parallel) = \frac{1}{2}(aA_k - (-1)^k A_k a). \tag{13}$$

Again, replace $a_\parallel$ with $a - a_\perp$ to see the last step. Together, (12) and (13) give us a way to calculate $a \wedge A_k$ and $a \cdot A_k$ without resorting to the definitions of these products. These equations also reveal the commutativity of these products in terms of the parity of k.

Let's now do a fun example that uses (13) to prove (7). Replacing $b \wedge c$ with $bc - b \cdot c$, we see that

$$a \cdot (b \wedge c) = \frac{1}{2}(a(b \wedge c) - (b \wedge c)a) = \frac{1}{2}(abc - bca). \tag{14}$$

Expanding ab and ca as geometric products, this becomes

$$2a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b + (a \wedge b)c - b(c \wedge a).$$

We leave it as an exercise for the reader to prove that $(a \wedge b)c - b(c \wedge a) = -c \cdot (a \wedge b) - b \cdot (c \wedge a)$. (Try it! It's not as hard as you might think.) All that remains to be shown now is that

$$a \cdot (b \wedge c) + b \cdot (c \wedge a) + c \cdot (a \wedge b) = 0.$$

This is easy to prove by simply applying the identity in (14) three times. Give it a try!

3.4 The Magnitude of Dot, Wedge and Geometric Products

In this section we study the magnitudes of $a \cdot A_k$ and $a \wedge A_k$ of the previous section. In so doing, we gain insight into what these products are doing geometrically. For example, we can think of the wedge product as extruding a blade into a new dimension, creating a higher dimensional volume. The wedge product is a grade raising operation. Being a grade lowering operation, how might we interpet the geometrical actions of the dot product? As we already saw in section (3.2), the dot product of a vector and a 2-blade gives us a vector in the plane of the 2-blade most unlike our original vector. It's worth trying to understand this idea for k-blades in general.

Beginning with $a \wedge A_k$, it is possible to see that $|a \wedge A_k| = |a_\perp||A_k| = |a||A_k| \sin\theta$, where θ is the angle between a and A_k. (If this is hard for you to see when $k > 2$, you're not alone. I feel the same way.) To be clear about θ and $a_\perp$, these come from the equation $a \cdot a_\parallel = |a||a_\parallel| \cos\theta$, where again, $a_\parallel$ is the orthogonal projection of a down onto the hyper-plane of A_k, and $a = a_\parallel + a_\perp$.

Now consider the magnitude of $a \cdot A_k$ when $k > 1$. To investigate this, we'll employ the ideas used in (6). Choose a factorization $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ of A_k such that $a_\parallel \wedge a_1 = 0$, $a_\parallel \cdot a_1 > 0$, $|a_k| = |A_k|$, and for all $i \neq j$, we have $a_i \cdot a_j = 0$. It then follows that $1 = \prod_{i \neq k} |a_i|$. Notice that without loss of generality, we can require that $|a_1| = 1$. Calculating $a \cdot A_k$, we then get

$$\begin{aligned}
a \cdot A_k &= \langle a a_1 a_2 \ldots a_k \rangle_{k-1} \\
&= \langle a_\parallel a_1 a_2 \ldots a_k \rangle_{k-1} \\
&= \langle |a_\parallel| a_1 a_1 a_2 \ldots a_k \rangle_{k-1} \\
&= |a_\parallel| a_2 a_3 \ldots a_k \\
&= |a_\parallel| A_k^1,
\end{aligned}$$

where here we're making use of the notation $A_k^i = a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_k$. Notice that $|A_k^1| = |A_k|$. We see now that $|a \cdot A_k| = |a_\parallel||A_k| = |a||A_k| \cos\theta$, where again, θ is the angle between a and A_k. What are we to make of A_k^1? Geometrically, and informally, we might think of this as A_k rotated in its own hyper-plane until one of its vectors comes into alignment with the projection of a onto it, and then having this aligned vector removed from it. In this way, the dot product does the reverse of what the wedge product does. Let's try to be a bit more precise about this.

When we calculate $a \wedge A_k$, the only real contribution a gives to the result is the part of a perpendicular to A_k. This part of a determines which new dimension we extrude A_k into. Similarly, when we calculate $a \cdot A_k$, the only real contribution a gives to the result is the part of a parallel to A_k. (Put another way, this is the part of a in the vector space represented by A_k.) This part of a determines which dimension A_k already spans and of which we no longer want it to extrude into. This may not be the way that most people think about the dot product, but it's the way I like to think about it.

We now have enough to prove something about the magnitude of the geometric product between a vector and a blade, provided we make the following definition. For all $g \in \mathbb{G}^3$, let

$$|g|^2 = \sum_{k=0}^{3} |\langle g \rangle_k|^2.$$

We now see that

$$\begin{aligned} |aA_k|^2 &= |a_\perp A_k + a_\parallel A_k|^2 \\ &= |a_\perp A_k|^2 + |a_\parallel A_k|^2 \\ &= |a_\perp|^2 |A_k|^2 + |a_\parallel|^2 |A_k|^2 \\ &= |a|^2 |A_k|^2. \end{aligned}$$

The first step is trivial, the second invokes our definition, the third follows from our study of $|a \cdot A_k|$ and $|a \wedge A_k|$, and the last step invokes the Pythogorean Theorem. Notice that this implies that

$$|aA_k| = |a||A_k|. \tag{15}$$

In particular, notice that the magnitude of a geometric product of vectors is the product of the magnitudes of those vectors.

While on the subject of magnitudes, the reader is incouraged to prove that $A_k^2 = (-1)^{T(k-1)}|A_k|^2$, where $T(k) = k(k+1)/2$ is the k^{th} triangle number. This lets us express the magnitude of a blade in terms of the geometric product. It also lets us use geometric squares of blades as denominators without there being any confusion about commutativity.

3.5 The Zero Product Property of the Geometric Product

We prove here an important result about the geometric product between a vector a and a k-blade A_k. What we want to show is that if $aA_k = 0$, then at least one of a and A_k is zero. We will refer to this as the zero product property of the geometric product. To begin, notice that if $aA_k = 0$, then $a \cdot A_k = 0$ and $a \wedge A_k = 0$, because no two non-zero blades of differing grades add to zero. It then becomes sufficient to show that if both a and A_k are non-zero, then at least one of $a \cdot A_k$ and $a \wedge A_k$ is non-zero. If $a \wedge A_k \neq 0$, then we're done. If not, then it's easy to see that $a \cdot A_k = aA_k$ reduces to a non-zero $(k-1)$-blade.

Notice that we could have used (15) to prove this property. If $aA_k = 0$, then $|aA_k| = 0$. This implies that $|a||A_k| = 0$. Now by the zero-product property of the scalar product, we see that at least one of $|a|$ and $|A_k|$ is zero. If $|a| = 0$, then $a = 0$. If not, then $|A_k| = 0$ and therefore $A_k = 0$.

Lastly, notice that the converse of our result is even easier to prove. If at least one of a and A_k is zero, then clearly $aA_k = 0$. We can now say that $aA_k = 0$ if and only if at least one of a and A_k is zero.

3.6 The Invertability of the Geometric Product

A really great algebraic property of the geometric product is that it is invertable. Let's explore this with k-blades.

We say that a k-blade A_k is invertable if there exists an element we'll denote by A_k^{-1} such that $A_k A_k^{-1} = 1$, which is the multiplicative identity. What we'll show now is that given any non-zero k-blade A_k, such an element always exists and is unique. Furthermore, it will become immediately apparent that $A_k^{-1} A_k = 1$, showing that they commute and come in pairs. That is, if A_k^{-1} is the multiplicative inverse of A_k, then A_k is the multiplicative inverse of A_k^{-1}.

To show existence, we need only come up with one example. Let $A_k^{-1} = \tilde{A}_k / |A_k|^2$, where $\tilde{A}_k$ denotes what we'll define as the reverse of A_k. Choosing a factorization of A_k such that it can be written as the geometric product of vectors $a_1 a_2 \dots a_k$, we define the reverse of A_k, denoted $\tilde{A}_k$, as the product $a_k a_{k-1} \dots a_1$. Leaving it to the reader to prove that $A_k \tilde{A}_k = |A_k|^2$, it is now clear that $A_k A_k^{-1} = |A_k|^2/|A_k|^2 = 1$. (We will define the reverse of any geometric product of vectors $a_1 a_2 \dots a_k$ to be $a_k a_{k-1} \dots a_1$, whether or not they form a set of pair-wise orthogonal vectors.)

To show uniqueness, suppose there exists an element $g \neq A_k^{-1}$ such that $A_k g = 1$. It follows that $A_k g = A_k A_k^{-1}$, which implies that $0 = A_k g - A_k A_k^{-1} = A_k(g - A_k^{-1})$. But by the zero product property of the geometric product, we must have at least one of A_k and $g - A_k^{-1}$ be zero. Since $A_k \neq 0$, we have $g = A_k^{-1}$, which is a contradiction.

Before leaving the subject of blade inverses, the reader should now prove that $A^{-1} = A_k/A_k^2$. Hint: First show that $\tilde{A}_k = (-1)^{T(k-1)} A_k$.

4 Common Geometric Operations

Here we talk about a number of geometric operations that you might run across in your programming work. The first thing we notice is that this list suffers from a lack of support for affine transformations, which motivates us to learn one of the homogeneous models of geometric algebra, but that is beyond the scope of this document.

4.1 Projections and Rejections

Given a vector a and a k-blade A_k, we know that $a \cdot A_k = a_{\|} A_k$ and $a \wedge A_k = a_{\perp} A_k$, where $a = a_{\|} + a_{\perp}$. To make this procise, we need to describe either $a_{\|}$ as the orthogonal projection of a down onto the blade A_k, or $a_{\perp}$ as a vector with $a_{\perp} \cdot a \geq 0$ and the property that for all vectors v such that $v \wedge A_k = 0$, we have $v \cdot a_{\perp} = 0$. That $a_{\|}$ is a projection is clear. The term rejection was invented to describe $a_{\perp}$, and we say that $a_{\perp}$ is the rejection of a from A_k. Using what we know so far we can easily come up with formulas for $a_{\|}$ and $a_{\perp}$ in terms of a and A_k. We immediately get $a_{\|} = (a \cdot A_k) A_k^{-1}$ and $a_{\perp} = (a \wedge A_k) A_k^{-1}$, but we can do better than this. Refering back to equation (10), we see that

$$a_{\perp} = a_{\perp} A_k A_k^{-1} = (-1)^k A_k a_{\perp} A_k^{-1} = \frac{A_k a_{\perp} A_k}{(-1)^k A_k^2}.$$

Now realize that $a_{\perp} A_k = a \wedge A_k$, and $A_k(a \wedge A_k) = A_k \cdot (a \wedge A_k)$. Similarly, we can claim that $A_k a_{\perp} A_k = (A_k \wedge a) \cdot A_k$. Our formula for $a_{\perp}$ now becomes

$$a_{\perp} = \frac{A_k \cdot (a \wedge A_k)}{(-1)^k A_k^2} = \frac{(A_k \wedge a) \cdot A_k}{(-1)^k A_k^2}, \tag{16}$$

where either form works. Refering back to equation (11), we see that

$$a_{\|} = a_{\|} A_k A_k^{-1} = -(-1)^k A_k a_{\|} A_k^{-1} = \frac{A_k a_{\|} A_k}{-(-1)^k A_k^2}.$$

Now just as before, we can show that the numerator takes on the form of either $A_k \cdot (a \cdot A_k)$ or $(A_k \cdot a) \cdot A_k$. Interestingly, this shows that in this case, the dot product is associative. (Be careful. The dot product is not associative in general. See if you can find an example.) Our formula for $a_{\|}$ now becomes

$$a_{\|} = \frac{A_k \cdot a \cdot A_k}{-(-1)^k A_k^2}. \tag{17}$$

It is a worth-while exercise for the reader at this point to verify formulas (16) and (17) for $k = 1, 2, 3$. Geometric arguments based upon what we know about certain operations are just as good as purely algebraic proofs. Use a combination of both.

4.2 The Cross Product

Remember the unit psuedo-scalar I we defined in subsection (2.3)? Well, we're going to start putting it to use. Let's first prove an interesting fact about it in $\mathbb{G}^3$. Like scalars, I commutes with all other elements in $\mathbb{G}^3$. That I commutes with other 3-blades is trivial. That I commutes with scalars is even more trivial. So let's consider aI for any vector $a \in \mathbb{G}^3$ and BI for any bivector (2-blade) $B \in \mathbb{G}^3$. Starting with aI, we have

$$\begin{aligned} aI &= ((a \cdot x)x + (a \cdot y)y + (a \cdot z)z)xyz \\ &= (a \cdot x)yz + (a \cdot y)zx + (a \cdot z)xy \\ &= xyz((a \cdot x)x + (a \cdot y)y + (a \cdot z)z) = Ia. \end{aligned}$$

The proof of BI is similar, and so I'll leave it to you to prove that $BI = IB$. Notice from our calculations that aI is a 2-blade and BI is a 1-blade! This begs the question of what geometrical relationship aI might have with a and BI with B.

Let us first consider a and aI. One way to go about investigating their relationship is to simply compute their geometric product. Seeing that $aaI = |a|^2 I$ isn't particularly enlightening, but if we instead write $aaI = a \cdot (aI) + a \wedge (aI)$, then our knowledge about the geometrical interpretations of the dot and wedge products can help us relate a to aI. Seeing that aaI is homogeneous of grade 3, it immediately follows that $a \cdot (aI) = 0$, showing that a is orthogonal to the blade aI. The act of multiplying a by the unit psuedo-scalar is therefore a way of finding a 2-blade having a as a normal vector. It then becomes natural to ask: how will the area and handedness of this blade relate to the length and direction of the a? Well, using our result in (15), we see that $|aI| = |a||I| = |a|$, showing that this blade's area is the length of a. To investigate the handedness of aI, we compare the handedness of $a \wedge (aI)$ to I. One way to do this is to see that $a \wedge (aI)/|a|^2 = I$. The reader can check that these are equal. Being equal, it is clear that these 3-blades have the same handedness. What this tells us is that if we image x as pointing to the front side of $y \wedge z$, then we can think of a as pointing to the front of aI.

An analysis of the relationship between B and BI is left as an exersize for the reader. After you've done this, it becomes clear that we can define the cross product between vectors a and b as

$$a \times b = -(a \wedge b)I.$$

We will not find any need to use the cross product in this paper.

When we take any element $g \in \mathbb{G}^3$ and compute gI, we call this taking the dual of g. Since rotors are elements in $\mathbb{G}^3$ having parts of grade zero and two, the duals of rotors are those elements of $\mathbb{G}^3$ having parts of grade one and three. These are refered to as rotor-duals in code.

4.3 Finding Angles

Finding the angle between two vectors is something we often do. What about finding the angle between two 2-blades? One way we can come up with this angle is to use what we know about duals. Convince yourself that if θ is the angle between the 2-blades A and B, then $(AI) \cdot (BI) = |AI||BI| \cos(\pi - \theta)$. (Don't hesitate to draw a picture of this.) The right-hand side of this simplifies as $-|A||B| \cos\theta$. For the left-hand side, we see that

$$(AI) \cdot (BI) = \langle AIBI \rangle_0 = \langle AI^2 B \rangle_0 = -\langle AB \rangle_0 = -A \cdot B.$$

This gives us a nice result! The angle θ between the 2-blades A and B is therefore given by

$$A \cdot B = |A||B| \cos\theta.$$

4.4 Finding Intersections

Consider the intersection between a line and a plane. Since any such problem can be translated for convenience, we will assume here that the plane contains the origin. Bivectors in $\mathbb{G}^3$, (2-blades), are natural candidates as representatives of planes. Let P be a 2-blade representing our plane, and let vectors a and b represent a point and the direction of our line, respectively. We can now reason that the point of intersection, if it exists, between our line and the plane is the point $a + \lambda b$ where $(a + \lambda b) \wedge P = 0$. In the course of solving for λ, we come to $\lambda b \wedge P = -a \wedge P$. Now since we can multiply both sides on the right or left by $(b \wedge P)^{-1}$ to isolate λ, there should be no confusion when we write

$$\lambda = -\frac{a \wedge P}{b \wedge P}.$$

This is really just the ratio of two volumes, seeing that both terms are psuedo-scalars. It's curious to think that the geometry of our problem is solved by such a ratio. Clearly there is no solution when $b \wedge P = 0$, and this makes sense since in this case the direction of our line is parallel to the plane represented by P.

Suppose now that we wanted to find the intersection between two planes. We can represent our two planes with the 2-blades A and B. We'll assume that our planes both contain the origin, since the general problem can be reduced to this case. In the case that the planes being represented are not the same plane, a vector at the origin contained in both planes suffices to represent their intersection. It is not hard to see that $((AI) \wedge (BI))I$ is such a vector. Notice that this goes to zero in the case that A and B are parallel. There is a problem with this solution, however. The validity of the result depends on the fact that both A and B are in $\mathbb{G}^3$, and therefore must intersect. If, however, A and B represented disjoint vector sub-spaces, then $((AI) \wedge (BI))I$ cannot be their intersection, because they don't intersect. A better solution would be the expression $(AI) \cdot B$. This is also zero in the case that A and B are parallel, and it is zero when A and B are wedged from disjoint vector sub-spaces. The result is an expression that works not just in $\mathbb{G}^3$, but any geometric algebra containing $\mathbb{G}^3$, provided we use the appropriate psuedo-scalar for that algebra. Remember that $I = xyz$ for $\mathbb{G}^3$ only.

4.5 Finding Distances

Suppose we wished to find the shortest distance from a point to a plane. Again, without loss of generality, we will assume our plane contains the origin. Letting B be a unit 2-blade representing our plane and p being a vector representing our point, what we're looking for is simply the length of the rejection of p from B. Using (16), this is given by $|B \cdot (p \wedge B)|$. If the reader can simplify this further, then that would be great, but as it is now, this appears to be a reasonable calculation one might perform using Octane's geometric algebra system. This would not be as efficient as using the plane equation, but if there can be something said about writing code quickly, then this method has merit.

4.6 Reflections

Given a vector a and a vector b, we define the vector reflection of b about a as the vector $b_{\|} - b_{\perp}$, where $b_{\|}$ and $b_{\perp}$ are the vector components of b parallel and perpendicular to a, respectively. Denoting such a reflection as b', it is easy to derive a formula for b' in terms of vector projections. We simply have $b' = 2(b \cdot a)a/a^2 - b$. We can get a nicer result, however, if we use the geometric product. Notice that $b = ba^2/a^2 = (b \cdot a)a/a^2 + (b \wedge a)a/a^2$. Seeing that $b_{\|} = (b \cdot a)a/a^2$ and $b_{\perp} = (b \wedge a)a/a^2$, we write

$$b' = (b \cdot a)a/a^2 - (b \wedge a)a/a^2 = (a \cdot b + a \wedge b)a/a^2 = aba/a^2,$$

and there we have it. Notice that it can also be written as $b' = aba^{-1}$.

At this point it is a worth-while exercise for the reader to investigate the idea of reflecting a vector about a 2-blade. How could such a reflection be defined? How could you go about finding a formula for it?

4.7 Rotations

Here we finally give a treatment of rotors in the subject of geometric algebra. With the background that we now have, it is almost trivial! Let us begin with the observation that vectors and 2-blades are both good candidates for representing rotations. And not surprisingly, if just one of these clearly represents a rotation, then naturally, so does the other by virtual of duality. What we'll find, however, is that the most convenient way for us to represent a rotation will be by using neither vectors nor 2-blades, but elements of $\mathbb{G}^3$ we'll call unit-rotors. In the next section, we'll discover a special relationship between unit-rotors and the vectors or 2-blades representing the same rotation.

Our development of the rotor will be similar to that given in [4]. Considering what we know about reflecting vectors about vectors, it is not hard to convince yourself that a rotation can be achieved by two vector reflections. Let a and b be a pair of linearly independent vectors taken from $\mathbb{G}^3$. Given a vector $v \in \mathbb{G}^3$, let v' denote the vector that is the reflection of v about a, then about b. If $v \wedge a \wedge b = 0$, then drawing a quick picture of this is enough to convince yourself that v was rotated in the plane of $a \wedge b$ by an angle 2θ, where θ is the angle between a and b. Now consider the case when $v \wedge a \wedge b \neq 0$. A bit of trigonometry will convince you that

$$\frac{b\left(\frac{ava}{|a|^2}\right)b}{|b|^2} = \frac{bavab}{|a|^2|b|^2} \tag{18}$$

is also a rotation of v by an angle of 2θ and parallel to the plane of $a \wedge b$. We can also easily prove this by writing

$$\frac{bavab}{|a|^2|b|^2} = \frac{ba(v_{\|} + v_{\perp})ba}{|a|^2|b|^2} = \frac{bav_{\|}ab}{|a|^2|b|^2} + v_{\perp},$$

seeing that the vector component $v_{\perp}$ of v perpendicular to $a \wedge b$ is not rotated, but the vector component $v_{\|}$ of v parallel to $a \wedge b$ is rotated. Again, all of this is simply based on our previous study of vector reflections!

Looking again at equation (18), we now notice that this suggests $ba/|ab|$ as an element encoding a rotation. (Recall our proof that $|ab| = |a||b|$ in (15).) Letting $R = ba/|ab|$, a formula for v' in terms of v and R is then given by

$$v' = Rv\tilde{R}. \tag{19}$$

Elements in $\mathbb{G}^3$ that are the geometric product of any two non-zero vectors are what we call rotors. Unit-magnitude rotors, refered to as unit-rotors in Octane, are the geometric elements we use to represent rotations instead of the traditional quaternion. Be aware that (19) is a rotation formula for unit-rotors. For non-unit-rotors, it is easy to see that it performs a scaled rotation. The reader should identify the scale in terms of the magnitude of the rotor.

At this point it is worth showing that the set of all rotors in $\mathbb{G}^3$ forms a group under the geometric product, and then show that the set of unit-rotors is a sub-group. We'll then review a number of corollaries to this result. We refer the reader to $[1, p.43]$ for the definition of a group. (Equivilant definitions can also be found on the internet.) Under this definition, to show that the set of all rotors forms a group, we need to show a number of things. First, we need to show that our rotor multiplication is associative. But this is easy since it follows immediately from the associativity of the geometric product. Secondly, we must show that there exists an identity rotor. Clearly, the scalar 1 is a rotor since it is the geometric product of many example pairs of vectors, and it has the property of being a multiplicative identity in our case. Thirdly, we need to show that every rotor has an inverse. To that end, let R be a rotor. It then follows by definition that there exist non-zero vectors a and b such that $R = ab$. Noticing that $Rba/|ab|^2 = 1$, the multiplicative identity, we see that $R^{-1} = \tilde{R}/|R|^2$, which proves the existance of R^{-1} since $|R| \neq 0$. Lastly, we must show that the set of all rotors is closed under the operation of the geometric product. What this means is that for any pair of rotors A and B, there must exist a pair of vectors a and b such that $ab = AB$. The key to our proof of this is in the realization that our rotors are taken from $\mathbb{G}^3$. It follows from this that there must exist vectors $x, y, z, w \in \mathbb{G}^3$ such that $A = xy$, $B = zw$, $y = z$, and $|y| = |z| = 1$. Letting $a = x$ and $b = z$, we now see that

$$ab = xz = x(1^2)z = xyzw = AB,$$

which is what we wanted to show. What remains to be shown is that the set of unit-rotors in $\mathbb{G}^3$ forms a sub-group. Being a sub-set of the set of all rotors in $\mathbb{G}^3$, we need only use the theorem given in $[1, p.61]$ to prove our result. That said, if A and B are unit-rotors, then there exist vectors $x, y, z, w \in \mathbb{G}^3$ such that $A = xy$, $B = zw$, $y = w$, and $|y| = |w| = 1$. We then have $AB^{-1} = A\tilde{B} = xywz = xz$, and clearly xz is a unit-rotor, since $|x| = |z| = 1$.

Showing that the set of all unit-rotors froms a group teaches us a number of things. The first and most useful result is that of rotor concatination. Suppose we want to perform the rotation represented by the unit-rotor R followed by the unit-rotor S. Using (19), we have

$$v' = SRv\tilde{R}\tilde{S} = SRv(SR)\tilde{},$$

but realizing that unit-rotors form a group, this reduces to the application of just one rotor, namely SR.

We've also learned that rotors are invertable, which is a good thing since rotations are clearly reversable. It shouldn't be hard to convince yourself that the unit-rotor $R^{-1} = \tilde{R}$ represents the rotation that undoes the rotation represented by the unit-rotor R, by simply rotating in the opposite direction.

Before continuing to the next subsection, we mention here what you may have already noticed. The rotation represented by the unit-rotor R is the same rotation represented by $-R$. This is easy to see. Just plug $-R$ into (19). This is what some authors are refering to when that talk about unit-rotors as being a double cover for the set of all rotations.

4.8 The Polar Decomposition of a Rotor

Let $R = ab$ be a rotor, not necessarily of unit-magnitude. By the geometric product, we may write

$$R = a \cdot b + a \wedge b = |a||b| \cos\theta + \frac{a \wedge b}{|a \wedge b|}|a||b| \sin\theta = |R| \left(\cos\theta + \frac{a \wedge b}{|a \wedge b|} \sin\theta \right), \qquad (20)$$

where θ is the angle between a and b. It is not too hard to prove that the geometric square of $a \wedge b/|a \wedge b|$ is -1. You can do this by either applying a number of identities or by simply finding a refactorization of any unit-2-blade in terms of a pair of orthogonal unit-vectors. In any case, seeing that this 2-blade has this property is a rationalization for re-writing (20) as

$$R = |R| \exp\left(\theta \frac{a \wedge b}{|a \wedge b|} \right).$$

This comes from the Tayler series expansions of $\exp(x)$, $\cos(x)$ and $\sin(x)$, each extended to geometric algebra and using the geometric product. If R is a unit-rotor, we can now write (19) as

$$v' = \exp(\theta B/2) v \exp(-\theta B/2),$$

where B is a unit-2-blade representing the plane and direction of rotation, and θ is the desired angle of rotation. (It is a worth-while exercise to show that this formula reduces to $v' = v \exp(\theta B)$ when $v \wedge B = 0$. We will use this result in the next subsection, but leave it to you to prove it.)

From the polar decomposition of a rotor, we can now see how to pick apart the axis and angle of a rotation, which is a common operation. We also frequently construct rotations from axis/angle pairs. Given an axis a and an angle θ, the desired rotor is given by

$$R = \exp(-\theta a I/2),$$

provided that positive angles of rotation are to represent counter-clock-wise rotations when viewing the axis as pointing toward us. So that we don't contradict the 2-blade form of this equation, namely $R = \exp(\theta B/2)$, we have to be clear about the dual of vectors and 2-blades. We define the dual of a vector a to be aI, and the dual of a 2-blade B to be $-BI$. It should now be easy to show that the dual operation on vectors or 2-blades is its own inverse. With this convention, how we think of rotations in terms of 2-blades, vectors, or rotors, will all be consistent.

4.9 Spherical Linear Interpolation

In this subsection we show how geometric algebra can be used to derive the formula for performing the well-known spherical linear interpolation.

Given a pair of linearly independent vectors a and b, each of unit length and taken from our 3-dimensional vector space $\mathbb{V}^3$, we wish to uniformly interpolate between them along the shortest arc of the unit sphere connecting the tips of a and b by the normalized index $0 \leq t \leq 1$. Notice that such an arc exists by our requirement that a and b are not parallel to one another. Using what we know about rotors, it is easy to construct the desired interpolation. We begin by constructing the rotor that will rotate a toward b so that the ratio of the angle made between the interpolated vector and a with the angle made between a and b is exactly t. The desired rotor is given by

$$R(a, b, t) = \exp\left(-\frac{a \wedge b}{|a \wedge b|}(t\theta/2)\right),$$

where θ is the angle made between a and b. Here we have chosen the proper plane of rotation using the blade $a \wedge b$, and then rescaled it to be of a magnitude equal to half the desired angle of rotation. The sign of the blade is chosen so that the equation

$$\text{slerp}(a, b, t) = R(a, b, t)a\tilde{R}(a, b, t) \tag{21}$$

rotates a in the plane of $a \wedge b$ toward b as required. This is also a formula for the desired interpolation. Noticing that $a \wedge a \wedge b = 0$, we leave it to the reader to prove that (21) reduces to

$$\text{slerp}(a, b, t) = a \exp\left(\frac{a \wedge b}{|a \wedge b|} t\theta\right) = a\cos(t\theta) + a\frac{a \wedge b}{|a \wedge b|}\sin(t\theta).$$

From this we see that we can express the interpolated vector as a linear combination of a and $a \cdot (a \wedge b)/|a \wedge b|$, a unit-vector perpedicular to a. The linear combination of these vectors consists of $\cos(t\theta)$ and $\sin(t\theta)$, making our result clear from basic trigonometry. We, however, will want to find a formula for the interpolated vector as a linear combination of a and b. To this end, we come up with the following matrix equation

$$\begin{bmatrix} \cos(t\theta) & \sin(t\theta) \end{bmatrix} \begin{bmatrix} a \\ a\frac{a \wedge b}{|a \wedge b|} \end{bmatrix} = \begin{bmatrix} f(t) & g(t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix},$$

where the elements are members of $\mathbb{G}^3$ and matrix multiplication is as usual, but using the geometric product instead of scalar multiplication. Before we proceed, it is helpful to put the row-oriented matrices on a common basis. It is convenient to use the components of each row-vector parallel and perpendicular to a. Our matrix equation then becomes

$$\begin{bmatrix} \cos(t\theta) & \sin(t\theta) \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a\frac{a \wedge b}{|a \wedge b|} \end{bmatrix} = \begin{bmatrix} f(t) & g(t) \end{bmatrix} \begin{bmatrix} a & 0 \\ (a \cdot b)a & -(a \wedge b)a \end{bmatrix}.$$

To see this, recall how the geometric product can be used to find parallel and perpendicular vector components.

$$b = ba^2 = (a \cdot b - a \wedge b)a = (a \cdot b)a - (a \wedge b)a.$$

We now solve for the inverse of the row-oriented matrix on the right-hand side.

$$\begin{bmatrix} a & 0 \\ (a \cdot b)a & -(a \wedge b)a \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $w = a$ and $x = 0$. Solving for y, we get

$$\begin{aligned} & (a \cdot b)a^2 - (a \wedge b)ay = 0 \\ \implies & (a \wedge b)ay = a \cdot b \\ \implies & (a \wedge b)^2 y = a(a \wedge b)(a \cdot b) \\ \implies & y = \frac{a(a \wedge b)(a \cdot b)}{(a \wedge b)^2}. \end{aligned}$$

Solving for z, we get

$$\begin{aligned} & -(a \wedge b)az = 1 \\ \implies & -(a \wedge b)^2 z = a(a \wedge b) \\ \implies & z = \frac{a(a \wedge b)}{-(a \wedge b)^2}. \end{aligned}$$

Having the inverse of the matrix, we're now ready to solve our original matrix equation.

$$\begin{aligned} \begin{bmatrix} f(t) & g(t) \end{bmatrix} &= \begin{bmatrix} \cos(t\theta) & \sin(t\theta) \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a\frac{a \wedge b}{|a \wedge b|} \end{bmatrix} \begin{bmatrix} a & 0 \\ \frac{a(a \wedge b)(a \cdot b)}{(a \wedge b)^2} & \frac{a(a \wedge b)}{-(a \wedge b)^2} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t\theta) & \sin(t\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{a \cdot b}{|a \wedge b|} & \frac{1}{|a \wedge b|} \end{bmatrix} \\ &= \begin{bmatrix} \cos(t\theta) - \frac{a \cdot b}{|a \wedge b|}\sin(t\theta) & \frac{1}{|a \wedge b|}\sin(t\theta) \end{bmatrix}. \end{aligned}$$

We can now write $g(t) = \sin(t\theta)/\sin(\theta)$, and we are not far from a similar result for $f(t)$.

$$f(t) = \frac{\sin(\theta)\cos(t\theta) - \cos(\theta)\sin(t\theta)}{\sin(\theta)} = \frac{\sin((1-t)\theta)}{\sin(\theta)}.$$

Our formula for the spherical linear interpolation now becomes

$$\text{slerp}(a, b, t) = \frac{a\sin((1-t)\theta) + b\sin(t\theta)}{\sin(\theta)},$$

which is what we wanted to show. That the interpolated vector is always of unit-length follows directly from our construction of the interpolation formula, as do the other properties of the interpolation.

Notice that our derivation did not depend on the dimensionality of the vectors a and b. This means that our formula works for any pair of unit-length, linearly independent vectors pointing to the surface of an n-dimensional hyper-sphere. This generalization came for free in our derivation using geometric algebra.

5 The Dot Product of k-Blades

If you've read this far, then let's go ahead and prove the following result for fun. If A_k and B_k are k-blades, not necessarily wedged from vectors taken from the same k-dimensional vector sub-space, then their dot product is given by

$$A_k \cdot B_k = \begin{vmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \vdots & \ddots & \vdots \\ a_k \cdot b_1 & \dots & a_k \cdot b_k \end{vmatrix}, \tag{22}$$

where $A_k = a_1 \wedge \cdots \wedge a_k$ and $B_k = b_1 \wedge \cdots \wedge b_k$. We first prove (22) for the case when $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ each form an orthogonal basis. (Again, these need not be basis for the same k-dimensional vector sub-space.) Let us proceed by induction. Checking (22) for the case $k = 1$ is trivial. Now assume that (22) holds for a fixed integer $k - 1 \geq 0$. It follows by definition that

$$\begin{aligned}
A_k \cdot B_k &= \langle A_k B_k \rangle_0 \\
&= \langle A_k^k (a_k^{\|} + a_k^{\perp}) B_k \rangle_0 \\
&= \langle A_k^k ((a_k \cdot b_1) b_1^{-1} + \cdots + (a_k \cdot b_k) b_k^{-1}) B_k \rangle_0 + \langle A_k^k a_k^{\perp} B_k \rangle_0 \\
&= \langle A_k^k ((-1)^0 (a_k \cdot b_1) B_k^1 + \cdots + (-1)^{k-1} (a_k \cdot b_k) B_k^k) \rangle_0 \\
&= (-1)^0 (a_k \cdot b_1) \langle A_k^k B_k^1 \rangle_0 + \cdots + (-1)^{k-1} (a_k \cdot b_k) \langle A_k^k B_k^k \rangle_0 \\
&= (-1)^0 (a_k \cdot b_1) A_k^k \cdot B_k^1 + \cdots + (-1)^{k-1} (a_k \cdot b_k) A_k^k \cdot B_k^k \\
&= \begin{vmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \vdots & \ddots & \vdots \\ a_k \cdot b_1 & \dots & a_k \cdot b_k \end{vmatrix},
\end{aligned}$$

where here we're again making use of the notation $A_k^i = a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_{k-1} a_k$. The vectors $a_k^{\|}$ and $a_k^{\perp}$ are the vector components of a_k parallel and perpendicular to B_k, respectively. What we've done here is invoke the inductive hypothesis k times to perform a cofactor expansion along the last row of the matrix. To see that $\langle A_k^k a_k^{\perp} B_k \rangle_0 = 0$ when $a_k^{\perp} \neq 0$, realize that $A_k^k \cdot (a_k^{\perp} B_k)$ is the lowest grade part of the geometric product $A_k^k a_k^{\perp} B_k$ that has the potential to be non-zero. This part has grade $|k - 1 - (k + 1)| = 2$. We can now claim by the principle of mathematical induction that (22) holds for all positive integers k.

To complete our proof, all that remains to be shown is that (22) holds even when either $\{a_i\}_{i=1}^k$ or $\{b_i\}_{i=1}^k$ or both do not form an orthogonal basis. Our argument will be based on the Gram-Schmidt orthogonalization process. Write A_k as the geometric product $a_1' \dots a_k'$, where $a_1' = a_1$, and

$$a_{i>1}' = a_i - \sum_{j=1}^{i-1} (a_i \cdot a_j')(a_j')^{-1}.$$

We can now apply our earlier result and write

$$A_k \cdot B_k = \begin{vmatrix} a_1' \cdot b_1' & \dots & a_1' \cdot b_k' \\ \vdots & \ddots & \vdots \\ a_k' \cdot b_1' & \dots & a_k' \cdot b_k' \end{vmatrix}.$$

To see that this equation still holds when we remove the prime tick marks, realize that each step of the Gram-Schmidt orthogonalization process translates into a row (or column) operation where a scalar multiple of one row (or column) is added to another. By Theorem 3 of [2, p.192], this does not change the value of the determinant. What we've done here is similar to what we did in equation (7).

6 Concluding Remarks

More than enough material has been covered to this point that the reader should be able to understand the geometric algebra support offered by the Octane Engine. If nothing else, hopefully our treatment of rotors was enough to give those readers familiar with quaternions a solid understanding of how rotations are represented in Octane using unit rotors. Geometric algebra is a very difficult subject and we have only scratched the surface here. I have only attempted to write what I think I know about it, and there still remain many of my own unanswered questions. But this is a good thing as it is what drives a personal pursuit of the field.

Geometric algebra does not appear to me to be the end-all be-all of mathematical tools for solving computational geometry problems. Of course, this is coming from someone who is far from a reasonably complete understanding of its applications to computer graphics. Even still, I don't think it's ready to stand on its own. I'll continue to use linear algebra and explore other mathematical models. As far as geometric algebra goes, perhaps in the future, even better algebraic structures will be found for use in solving geometry problems, but until then, geometric algebra has proven to be a promising tool.

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