An Extension Of The Quadric Model

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Abstract. An extension is found for the model set forth in [2] which expands the set of all transformations that can be applied to quadric surfaces to the set of all conformal transformations.

Mathematics Subject Classification (2010). Primary 14J70; Secondary 14J29.

Keywords. Quadric Surface, Geometric Algebra, Quadric Model.

1. The Expansion Of $\mathbb{G}$ to $\mathbb{G}^*$

We assume here from the beginning that the reader is familiar with all definitions and results set forth in [2] as this paper will make use of that material without recounting any of it. That said, we begin with an extension of $\mathbb{G}$ to the geometric algebra $\mathbb{G}^*$. We will let $\mathbb{G}$ be a proper sub-algebra of $\mathbb{G}^*$ by adding the following four basis vectors.

$$\begin{array}{ll} o & \text{The null-vector at the origin.} \\ \infty & \text{The null-vector at infinity.} \\ \overline{o} & \text{The counter-part of } o. \\ \overline{\infty} & \text{The counter-part of } \infty. \end{array} \tag{1.1}$$

Recall that the counter-part of any vector $v \in \mathbb{V}$ is the vector $\overline{v} \in \overline{\mathbb{V}}$, and vice-versa. At the moment, however, the over-bar notation used in table (1.1) is nothing more than notation. In [2], the over-bar notation refers to the application of an outermorphic function. We will see shortly that we can overload this notation to refer to an extension of this outermorphic function.

The following is an inner product table for the basis vectors in table (1.1).

$$\begin{array}{c|cccc} \cdot & o & \infty & \overline{o} & \overline{\infty} \\ \hline o & 0 & -1 & 0 & 0 \\ \infty & -1 & 0 & 0 & 0 \\ \overline{o} & 0 & 0 & 0 & -1 \\ \overline{\infty} & 0 & 0 & -1 & 0 \end{array} \tag{1.2}$$

We will let $\mathbb{V}^*$ contain $\mathbb{V}$ as a proper vector-subspace, adding to it the basis vectors o and ∞. Similarly, we will let $\overline{\mathbb{V}^*}$ contain $\overline{\mathbb{V}}$ as a proper vector-space, adding to it the basis vectors $\overline{o}$ and $\overline{\infty}$. We will let $\mathbb{W}^*$ denote the smallest vector space containing $\mathbb{V}^*$ and $\overline{\mathbb{V}*}$ as vector sub-spaces. For all vector $v \in \mathbb{W}$, (not $v \in \mathbb{W}^*$), we will define $0 = v \cdot b$, where b is any basis vector in table (1.1).

What we have now with $\mathbb{G}^*$ is simply a geometric algebra containing two isomorphic Minkowski sub-algebras $\mathbb{G}(\mathbb{V}^*)$ and $\mathbb{G}(\overline{\mathbb{V}^*})$. To preserve the use of the over-bar notation in our extended model, we will want to develop it as an outermorphic isomorphism between these two sub-algebras. To that end, we will find it useful to refer to [1] in defining the following vectors.

$$e_- = \frac{1}{2}\infty + o \tag{1.3}$$

$$e_+ = \frac{1}{2}\infty - o \tag{1.4}$$

As the reader can check, e_- is a unit-length anti-Euclidean vector, (having an inner-product square of -1), while e_+ is a unit-length Euclidean vector. We will define $\overline{e_-}$ and $\overline{e_+}$ similarly with $\overline{o}$ and $\overline{\infty}$. We can now define, for any element $E \in \mathbb{G}^*$, the element $\overline{E}$ as

$$\overline{E} = S^* E \tilde{S^*}, \tag{1.5}$$

where S^* is defined in terms of S as

$$S^* = \frac{1}{2}(1 + e_-\overline{e_-})(1 - e_+\overline{e_+})S. \tag{1.6}$$

It now follows that for any vector $v \in \mathbb{V}^*$, the vector $\overline{v}$ is the counter-part of v in $\overline{V^*}$.

We now introduce the conformal mapping $P : \mathbb{V} \to \mathbb{V}^*$ as

$$P(p) = o + p + \frac{1}{2}p^2\infty, \tag{1.7}$$

and then realize that for any bivector $E \in \mathbb{G}$ representative of an n-dimensional quadric surface in our original model, we have

$$P(p) \wedge \overline{P(p)} \cdot E = p \wedge \overline{p} \cdot E \tag{1.8}$$

showing that the bivectors of the form E in [2] are conveniently the very bivectors in our extended model that are also presentative of n-dimensional quadric surfaces. To see this, it is convenient to make use of the vectors e_- and e_+; rewriting the conformal mapping in terms of them as

$$P(p) = \alpha e_- + p + \beta e_+, \tag{1.9}$$

where $\alpha = \frac{1}{2}(p^2 + 1)$ and $\beta = \frac{1}{2}(p^2 - 1)$. Doing so, we see that

$$\begin{aligned} P(p) \wedge \overline{P(p)} &= (\alpha e_- + \beta e_+) \wedge \overline{(\alpha e_- + \beta e_+)} && (1.10)\\ &+ (\alpha e_- + \beta e_+) \wedge \overline{p} && (1.11)\\ &+ p \wedge \overline{(\alpha e_- + \beta e_+)} && (1.12)\\ &+ p \wedge \overline{p}. && (1.13) \end{aligned}$$

It is now easy to see that E, when taken in the inner product with each of (1.10), (1.11) and (1.12), vanishes to zero.

2. Transformations Of The Extended Model

At this point we have extended the framework of the quadric model to a higher dimensional algebra $\mathbb{G}^*$ in which all previously known results of $\mathbb{G}$ are preserved. In this extended framework we can now discover a larger set transformations applicable to quadrics as versors. Indeed, what we'll now show is that the entire set of conformal transformations are available to us in the extended model. To see this, we start by making the simple observation that for any versor $V \in \mathbb{G}(\mathbb{V}^*)$, we can recognize the algebraic variety generated by the set of all projective points $p \in \mathbb{V}$, such that

$$0 = V^{-1}P(p)V \wedge \overline{V^{-1}P(p)V} \cdot E, \tag{2.1}$$

as the transformation of the quadric $E \in \mathbb{G}$ by the versor V, provided that $e_0 = V^{-1}e_0V$. Indeed, what we'll find is that the transformation E' of E by V is given by

$$E' = V\overline{V}E(V\overline{V})^{-1}. \tag{2.2}$$

To see this, let us first write E in the form

$$E = \sum_{i=1}^{k} a_i \wedge \overline{b_i}, \tag{2.3}$$

where each of $\{a_i\}_{i=1}^k$ and $\{b_k\}_{k=1}^i$ is a sequence of k vectors taken from $\mathbb{V}$. Then, by the linearity of all the products of geometric algebra, there is no loss in generality here if we, for convenience, consider only the case $k = 1$, and write E as simply the 2-blade

$$E = a \wedge \overline{b}, \tag{2.4}$$

where $a, b \in \mathbb{V}$. Having done this, it is easy to establish that the quadric represented by E' is the very quadric represented in equation (2.1) by the following chain equalities.

$$V^{-1}P(p)V \wedge \overline{V^{-1}P(p)V} \cdot a \wedge \overline{b} \tag{2.5}$$
$$= -(V^{-1}P(p)V \cdot a)(V^{-1}P(p)V \cdot b) \tag{2.6}$$
$$= (P(p) \cdot VaV^{-1})(P(p) \cdot VbV^{-1}) \tag{2.7}$$
$$= P(p) \wedge \overline{P(p)} \cdot VaV^{-1} \wedge \overline{VbV^{-1}} \tag{2.8}$$
$$= P(p) \wedge \overline{P(p)} \cdot V\overline{V}a(V\overline{V})^{-1} \wedge V\overline{V}\overline{b}(V\overline{V})^{-1} \tag{2.9}$$
$$= P(p) \wedge \overline{P(p)} \cdot V\overline{V}(a \wedge \overline{b})(V\overline{V})^{-1} \tag{2.10}$$

This is a nice result, because we can now find in $\mathbb{G}^*$ versors representative of the rigid by transformations that are applicable to any quadric surface. But as we know, of course, the conformal model offers many more types of transformations, and these can also be applied to quadric surfaces.

The biggest gap that remains, however, between our extended model and the conformal model is the lack of conformal operations such as intersecting geometries, fiting geometries to a set of points, and so on. It isn't too surprising that that these features do not naturally present themselves, however, because the quadrics are not closed under the intersection operation, and there may not be a unique quadric fitting a given set of points in a certain way. In any case, the jury is still out on what the best model for quadrics is, but until a better model comes along, this one appears to show promise.

References

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