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THE QUADRIC MODEL OF GEOMETRIC ALGEBRA

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Abstract. A great achievement of the conformal model of geometric algebra is that the elements of computation are representatives of geometry, and may therefore be thought of as geometry. A limitation of the model, however, is its inability to represent all quadric surfaces. Set forth in this paper, the quadric model of geometric algebra, while maintaining the idea of geometries as elements of computation, overcomes this limitation at the expense of added complexity and dimension.

1. Finding The Quadric Equation

Taking our que from [1], the n-dimensional quadric surfaces may be characterized as the set of all projective points in an $(n+1)$-dimensional homogeneous space satisfying a matrix equation involving a symmetric matrix. We will let $\mathbb{V}^{n+1}$ be an $(n+1)$-dimensional vector space and identify vectors in this space with projective points of n-dimensional space in the usual manner. That is, letting $\{e_i\}_{i=0}^n$ be an orthonormal basis for $\mathbb{V}^{n+1}$, we identify the n-dimensional point represented by any $p \in \mathbb{V}^{n+1}$ as the point $p/(p \cdot e_0)$ in the $e_0 = 1$ plane.

Letting $\{\alpha_{ij}\} \subset \mathbb{R}$ with $0 \leq i \leq j \leq n$ be the scalar elements of a symmetric matrix, an n-dimensional quadric surface is the projective solution set to the matrix equation

$$0 = p \begin{bmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ \alpha_{01} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{2n} & \dots & \alpha_{nn} \end{bmatrix} p^T, \tag{1.1}$$

where here we have abused notation by interpreting the vector p taken from $\mathbb{V}^{n+1}$ as a row-vector with p^T as the corresponding column-vector. Written another way without abuse of notation, we have

$$0 = \sum_{i=0}^{3} \sum_{j=i}^{3} \sigma_{ij} \alpha_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.2}$$

where σ_{ij} is defined as

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \neq j. \end{cases} \tag{1.3}$$

The form (1.1) lends itself to the study of quadrics using matrix algebra, while the form (1.2) may be thought of as a low-level form of the equation in geometric

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algebra. What we might think of as a high-level form in geometric algebra, coming from a framework of compuation, may provide a better means of studying quadrics using geometric algebra. We procede now to develop such a form.

Let $\mathbb{W}^{2(n+1)}$ denote a $2(n+1)$-dimensional vector space having $\{e_i\}_{i=0}^{2n+1}$ as a set orthonormal basis vectors generating it. The vector space $\mathbb{V}^{n+1}$ is therefore a vector sub-space of $\mathbb{W}^{2(n+1)}$ and we will let $\bar{\mathbb{V}}^{n+1}$ denote the $(n+1)$-dimensional vector sub-space of $\mathbb{W}^{2(n+1)}$ that is complement to $\mathbb{V}^{n+1}$. It is then helpful to introduce the notation $\bar{p}$ as the vector in $\bar{\mathbb{V}}^{n+1}$ related to the vector $p \in \mathbb{V}^{n+1}$ by the equation

$$\bar{p} = Rp\tilde{R}, \tag{1.4}$$

where R is a rotor defined as

$$R = 2^{-n/2} \prod_{i=0}^{n} (1 - e_i e_{i+n+1}) \,. \tag{1.5}$$

This idea comes from [2], and it is easy to see that for any integer $i \in [0, n]$, we have $\bar{e}_i = e_{i+n+1}$ and $\bar{e}_{i+n+1} = e_i$. Notice that the over-bar operator is an outermorphic function and that we may apply it to any element of the geometric algebra $\mathbb{G}(\mathbb{W}^{2(n+1)})$.

We are now ready to give the high-level form of equation (1.2) as

$$0 = p \wedge \bar{p} \cdot B, \tag{1.6}$$

where $B \in \mathbb{G}(\mathbb{W}^{2(n+1)})$ is a bivector of the form

$$B = -\frac{1}{2} \sum_{i=0}^{n} \sum_{j=i}^{n} \alpha_{ij} (e_i \bar{e}_j + (-1)^{\sigma_{ij}} \bar{e}_i e_j). \tag{1.7}$$

Here, as in the form (1.1) where we may think of the symmetric matrix as representative of the quadric, the bivector B may also be thought of as representative of this quadric.

Realizing that we need to be careful, because the inner product is not associative, it is interesting to write equation (1.6) in a form similar to that of equation (1.1), which resembles conjugation. Doing so, we get

$$0 = p \cdot B \cdot \bar{p}. \tag{1.8}$$

2. Using The Quadric Equation

Having developed the quadric equation (1.6) in geometric algebra, we can now benefit from the language of geometric algebra in using it to answer questions about quadric geometry.

Notice that in our model we can make a distinction between members of $\mathbb{V}^{n+1}$ that are representative of points and those representative of directions. Specifically, a vector $v \in \mathbb{V}^{n+1}$ is a direction if and only if $v \cdot e_0 = 0$. While we will use an arrow accent to distinguish between direction vectors and position vectors, there should be no confusion on the form of a vector and what we intend it to represent when we refer to it as a direction or a vector. Similarly, we will take the liberty of refering to bivectors taken from $\mathbb{G}(\mathbb{V}^{2(n+1)})$ as quadrics. This helps elliminate phrases that would otherwise sound a bit too pedantic.

With all of that said, letting $f : \mathbb{V}^{n+1} \to \mathbb{R}$ be the function defined as

$$f(x) = x \wedge \bar{x} \cdot B, \tag{2.1}$$

we arrive now at our first result given in the following lemma.

Lemma 2.1. *Given any quadric B, if for all direction vectors $\vec{v} \in \mathbb{V}^{n+1}$, we have $f(\vec{v}) = 0$, then B is a linear (flat) quadric.*

Proof. For any point $p \in \mathbb{V}^{n+1}$, we find that

$$f(x) = f(p + \vec{x}) = \nabla_{\vec{x}} f(p), \tag{2.2}$$

in the case that p is on B, where $\nabla_{\vec{x}} f(p)$ is the directional derivative of f at p in the direction of $\vec{x}$. It follows that the tangent space of any point on the quadric is also in the quadric. □

References

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2. C. Doran and D. Hestenes, *Lie groups as spin groups*, J. Math. Phys. **34** (1993), 8.

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