An Extension Of The Quadric Model

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Abstract. An extension is found for the model set forth in [2] which expands the set of all transformations that can be applied to quadric surfaces to the set of all conformal transformations.

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1. The Expansion Of $\mathbb{G}$ to $\mathbb{G}^*$

We assume here from the beginning that the reader is familiar with all definitions and results set forth in [2] as this paper will make use of that material without recounting it. That said, we begin with an extension of $\mathbb{G}$ to the geometric algebra $\mathbb{G}^*$. We will let $\mathbb{G}$ be a proper sub-algebra of $\mathbb{G}^*$ by adding the following four basis vectors.

$$\begin{array}{ll} o & \text{The null-vector at the origin.} \\ \infty & \text{The null-vector at infinity.} \\ \overline{o} & \text{The friend of } o. \\ \overline{\infty} & \text{The friend of } \infty. \end{array} \tag{1.1}$$

What was referred to in [2] as the counter-part of a vector will be referred to here in this paper as the friend of a vector. In group theory terms, the vectors o and $\overline{o}$ are conjugates of one another, which is always a mutual relationship. Friendship, however, as it will be defined shortly, and as we'll find, is not always such a relationship.

At the moment, the over-bar notation used in table (1.1) is nothing more than notation. In [2], the over-bar notation refers to the application of an outermorphic function. We will see shortly that we can overload this notation to also refer to an extension of this outermorphic function.

The following is an inner product table for the basis vectors in table (1.1).

$$\begin{array}{c|cccc} \cdot & o & \infty & \bar{o} & \overline{\infty} \\ \hline o & 0 & -1 & 0 & 0 \\ \infty & -1 & 0 & 0 & 0 \\ \bar{o} & 0 & 0 & 0 & -1 \\ \overline{\infty} & 0 & 0 & -1 & 0 \end{array} \tag{1.2}$$

We will let $\mathbb{V}^*$ contain $\mathbb{V}$ as a proper vector-subspace, adding to it the basis vectors o and ∞. Similarly, we will let $\overline{\mathbb{V}}^*$ contain $\overline{\mathbb{V}}$ as a proper vector-space, adding to it the basis vectors $\bar{o}$ and $\overline{\infty}$. We will let $\mathbb{W}^*$ denote the smallest vector space containing $\mathbb{V}^*$ and $\overline{\mathbb{V}}^*$ as vector sub-spaces. For all vectors $v \in \mathbb{W}$, (not $v \in \mathbb{W}^*$), we will define $0 = v \cdot b$, where b is any basis vector in table (1.1).

What we have now with $\mathbb{G}^*$ is simply a geometric algebra containing two isomorphic Minkowski sub-algebras $\mathbb{G}(\mathbb{V}^*)$ and $\mathbb{G}(\overline{\mathbb{V}}^*)$. To preserve the use of the over-bar notation in our extended model, we will want to develop it as an outermorphic isomorphism between these two sub-algebras. To that end, we will find it useful to refer to [1] in defining the following vectors.

$$e_- = \frac{1}{2}\infty + o \tag{1.3}$$

$$e_+ = \frac{1}{2}\infty - o \tag{1.4}$$

As the reader can check, e_- is a unit-length anti-Euclidean vector, (having an inner-product square of -1), while e_+ is a unit-length Euclidean vector. We will define $\bar{e}_-$ and $\bar{e}_+$ similarly with $\bar{o}$ and $\overline{\infty}$. We can now define, for any element $E \in \mathbb{G}^*$, the friend $\overline{E}$ of E as

$$\overline{E} = S^* E \tilde{S^*}, \tag{1.5}$$

where S^* is defined in terms of S as

$$S^* = \frac{1}{2}(1 + e_-\bar{e}_-)(1 - e_+\bar{e}_+)S. \tag{1.6}$$

It now follows that for any vector $v \in \mathbb{V}^*$, the vector $\bar{v}$ is the friend of v in $\overline{V}^*$. Similarly, for any vector $v \in \overline{\mathbb{V}}^*$, the vector $\bar{v}$ as the friend of v in $\mathbb{V}^*$. We must be careful, however, because it does not now follow, as it did in our original model, that the friend of $v \in \mathbb{V}^*$ in $\overline{\mathbb{V}}^*$ is also the friend of v, and vice-versa. This is because o is not the friend of $\bar{o}$, nor is ∞ the friend of $\overline{\infty}$. Rather, $-o$ is the friend of $\bar{o}$, and $-\infty$ is the friend of $\overline{\infty}$. In the extended model, friendship is not always a mutual relationship as it was in the original model. This will not present a problem for us, however, if we only use the over-bar function in one direction; from $\mathbb{V}^*$ to $\overline{\mathbb{V}}^*$.

We now introduce the conformal mapping $P : \mathbb{V} \to \mathbb{V}^*$ as

$$P(p) = o + p + \frac{1}{2}p^2\infty, \tag{1.7}$$

and then realize that for any bivector $E \in \mathbb{G}$ representative of an n-dimensional quadric surface by our original model, that we have

$$P(p) \wedge \overline{P(p)} \cdot E = p \wedge \overline{p} \cdot E \tag{1.8}$$

showing that the bivectors of the form E in [2] are conveniently the very bivectors in our extended model that are also presentative of n-dimensional quadric surfaces. To see this, it is convenient to make use of the vectors e_- and e_+; rewriting the conformal mapping in terms of them as

$$P(p) = \alpha e_- + p + \beta e_+, \tag{1.9}$$

where $\alpha = \frac{1}{2}(p^2 + 1)$ and $\beta = \frac{1}{2}(p^2 - 1)$. Doing so, we see that

$$\begin{aligned} P(p) \wedge \overline{P(p)} &= (\alpha e_- + \beta e_+) \wedge \overline{(\alpha e_- + \beta e_+)} && (1.10)\\ &+ (\alpha e_- + \beta e_+) \wedge \overline{p} && (1.11)\\ &+ p \wedge \overline{(\alpha e_- + \beta e_+)} && (1.12)\\ &+ p \wedge \overline{p}. && (1.13)\end{aligned}$$

It is now easy to see that E, when taken in the inner product with each of (1.10), (1.11) and (1.12), vanishes to zero, leaving just (1.13).

2. Transformations Of The Extended Model

At this point we have extended the framework of the quadric model to a higher dimensional algebra $\mathbb{G}^*$ in which all previously known results of $\mathbb{G}$ are preserved. In this extended framework we can now discover a larger set of transformations applicable to quadrics as versors. Indeed, what we'll now show is that the entire set of conformal transformations are available to us in the extended model. To see this, we start by making the simple observation that for any versor $V \in \mathbb{G}(\mathbb{V}^*)$, we can recognize the algebraic variety generated by the set of all projective points $p \in \mathbb{V}$, such that

$$0 = V^{-1}P(p)V \wedge \overline{V^{-1}P(p)V} \cdot E, \tag{2.1}$$

as the transformation of the quadric $E \in \mathbb{G}$ by the versor V, provided that $e_0 = V^{-1}e_0 V$. Indeed, what we'll find is that the transformation E' of E by V is given by

$$E' = V\overline{V}E(V\overline{V})^{-1}. \tag{2.2}$$

To see this, let us first write E in the form

$$E = \sum_{i=1}^{k} a_i \wedge \overline{b_i}, \tag{2.3}$$

where each of $\{a_i\}_{i=1}^k$ and $\{b_k\}_{k=1}^i$ is a sequence of k vectors taken from $\mathbb{V}$. Then, by the linearity of all the products of geometric algebra, there is no loss in generality here if we, for convenience, consider only the case $k = 1$, and write E as simply the 2-blade

$$E = a \wedge \overline{b}, \tag{2.4}$$

where $a, b \in \mathbb{V}$. Having done this, it is easy to establish that the quadric represented by E' is the very quadric represented in equation (2.1) by the equality of (2.5) with (2.10).

$$\begin{aligned}
& V^{-1}P(p)V \wedge \overline{V^{-1}P(p)V} \cdot a \wedge \overline{b} && (2.5)\\
= & -(V^{-1}P(p)V \cdot a)(V^{-1}P(p)V \cdot b) && (2.6)\\
= & (P(p) \cdot VaV^{-1})(P(p) \cdot VbV^{-1}) && (2.7)\\
= & P(p) \wedge \overline{P(p)} \cdot VaV^{-1} \wedge \overline{VbV^{-1}} && (2.8)\\
= & P(p) \wedge \overline{P(p)} \cdot V\overline{V}a(V\overline{V})^{-1} \wedge V\overline{V}\overline{b}(V\overline{V})^{-1} && (2.9)\\
= & P(p) \wedge \overline{P(p)} \cdot V\overline{V}(a \wedge \overline{b})(V\overline{V})^{-1} && (2.10)
\end{aligned}$$

To see the step from (2.8) to (2.9), realize that for any vector $v \in \mathbb{V}$, the conjugation of $\overline{v}$ by V leaves $\overline{v}$ invariant, up to scale; and likewise, the conjugation of v by $\overline{V}$ leaves v invariant, up to scale. A change in sign depends upon the parity of the versor V, but it doesn't matter, because only zero or two sign changes, if any, will happen, a cancelation occuring in the latter case.

Of course, the requirement that V keep e_0 invariant under versor conjugation is only necessary if we wish to easily visualize the newly transformed point $V^{-1}P(p)V$ in Euclidean space. Removing this constraint, a versor V transforms points in homogeneous space, the results of which are harder to visualize, but which may provide us with the ability to project the quadrics.

3. Concluding Remarks

The biggest gap that seems to remain between our extended model and the conformal model is the lack of conformal operations such as intersecting geometries, fitting geometries to a set of points, and so on. It isn't too surprising that that these features do not naturally present themselves, however, because the quadrics are not closed under the intersection operation, and there may not be a unique quadric fitting a given set of points in a certain way. In any case, the jury is still out on what the best model for quadrics is, but until a better model comes along, this one appears to show some promise.

References

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