A Compilation Of Fun GA Problems

November 9, 2012

1 The First Set Of Problems

For this first set of problems, assume that all elements come from a *Euclidean* geometric algebra.

1.1 The Inverse Of A Vector

Given a non-zero vector v, determine its inverse v^{-1} with respect to the geometric product. Show that this inverse is unique. Does v have an inverse with respect to the inner or outer products? Do all vectors have an inverse with respect to the geometric product?

1.2 Reflecting A Vector About A Vector

Given a unit-length vector n and a vector v, find the vector v' which is the orthogonal reflection of v about n.

1.3 Reflecting A Vector About A Blade

Given a 2-blade B and a vector v, find the vector v' which is the orthogonal reflection of v about B.

1.4 The Angle Between Blades

Given two blades A and B, each of the same grade $k \geq 1$, find a formula for the product $A \cdot B$ in terms of the angle θ made between the planes containing each of these blades.

1.5 The Magnitude Of The Outer Product Of Two Blades

Given two blades A and B, each of the same grade $k \geq 1$, find a formula for the quantity $|A \wedge B|$ in terms of the angle θ made between the planes containing each of these blades.

1.6 The Inner Product Of A Vector And A Blade

Given a vector v and a k-blade B, which may be written in terms of the k vectors in $\{b_i\}_{i=1}^k$ as

$$B = \bigwedge_{i=1}^{k} b_k, \tag{1}$$

we may define the inner product of v and B as

$$v \cdot B = -\sum_{i=1}^{k} (v \cdot b_i) B_i, \tag{2}$$

where we define B_i as the $(i-1)$-blade given as

$$B_i = \bigwedge_{\substack{j=1 \\ j \neq i}} b_j. \tag{3}$$

Given this definition of $v \cdot B$, show that

$$v \cdot B = (v \cdot a_k)B_1 - a_k \cdot (v \wedge B_1). \tag{4}$$

1.7 Versors

A versor is any geometric product of a finite sequence $\{v_i\}_{i=1}^k$ of invertible vectors. That is, if V is a versor, then we may write

$$V = a_1 a_2 a_3 \dots a_k = \prod_{i=1}^{k} a_i \tag{5}$$

The reverse of V, written as $\tilde{V}$, is defined as

$$\tilde{V} = a_k a_{k-1} a_{k-2} \dots a_1 = \prod_{i=1}^{k} a_{k-i+1}. \tag{6}$$

Given all of this information, determine the inverse V^{-1} of V.

1.8 Converting Between Versors And Blades

Show that any versor can be rewritten as a blade. Then show that any blade can be rewritten as a versor.

1.9 The Inverse Of a Blade

Given a blade B, determine when B^{-1} exists, given a formula for B^{-1} in terms of B, and show that if B^{-1} exists, then it is unique.

1.10 The Conjugation Of Vectors By Versors

Given any vector v and a versor V, show that VvV^{-1} is also a vector.

1.11 The Conjugation Of Blades By Versors

Given any k-blade B and a versor V, show that

$$VBV^{-1} = \bigwedge_{i=1}^{k} V b_k V^{-1}, \tag{7}$$

in the case that B may be factored in terms of the vectors in $\{b_i\}_{i=1}^k$ as

$$B = \bigwedge_{i=1}^{k} b_k. \tag{8}$$

Also show that VBV^{-1} has the same grade as B.

1.12 Versors And The Inner Product

Given any two vectors a and B, and a versor V, show that

$$a \cdot b = (VaV^{-1}) \cdot (VbV^{-1}), \tag{9}$$

and that

$$(V^{-1}aV) \cdot b = a \cdot (VbV^{-1}). \tag{10}$$

(Hint: One of these follows trivially from the other.)

2 The Second Set Of Problems

In this second set of problems, assume that all elements come from a *non-Euclidean* geometric algebra.

2.1 Null Vectors

In a non-Euclidean geometric algebra, we allow vectors v having an inner product square $v \cdot v$ of zero. How does this impact the invertibility of non-zero vectors?

2.2 Converting Between Versors And Blades

Re-examine Problem 1.8. Does your proof work in a non-Euclidean geometric algebra?! What can you conclude?

2.3 Conjugation By Versors

Re-examine Problem 1.10, Problem 1.11 and Problem 1.12. Do your proofs for these problems still go through in a non-Euclidean geometric algebra? If not, can an alternative proof by found that does work in a non-Euclidean geometric algebra, or if so, could you have found an easier proof that takes advantage of properties available in a Euclidean geometric algebra?