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THE QUADRIC MODEL OF GEOMETRIC ALGEBRA

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Abstract. A great achievement of the conformal model of geometric algebra is the realization of geometries as elements of calculation. What's further astonishing is the realization that these same elements simultaneously represent versor transformations with geometric significance relative to the represented geometry. These features, combined with the outermorphic property of versor conjugation, make the conformal model an interesting find, to say the least. It is then no wonder that we would hope to find a similar model capable of representing more than just a small subset of the set of all quadric surfaces. This paper presents a model, based upon geometric algebra, that is capable of representing all quadrics, and comparable to the conformal model. While not all of the great features of the conformal model are preserved in this new model, we will find that it may have some utility.

1. Finding The Quadric Equation

Taking our que from [1], the n-dimensional quadric surfaces may be characterized as the set of all projective points in an $(n+1)$-dimensional homogeneous space satisfying a matrix equation involving a symmetric matrix. We will let $\mathbb{V}^{n+1}$ be an $(n+1)$-dimensional vector space and identify vectors in this space with projective points of n-dimensional space in the usual manner. That is, letting $\{e_i\}_{i=0}^n$ be an orthonormal basis for $\mathbb{V}^{n+1}$, we identify the n-dimensional point represented by any $p \in \mathbb{V}^{n+1}$ as the point $p/(p \cdot e_0)$ in the $e_0 = 1$ plane, provided $p \cdot e_0 \neq 0$.

Letting $\{\alpha_{ij}\} \subset \mathbb{R}$ with $0 \leq i \leq j \leq n$ be the scalar elements of a symmetric matrix M, an n-dimensional quadric surface is the projective solution set to the matrix equation

$$0 = pMp^{\mathsf{T}}, \tag{1.1}$$

where here we have abused notation by interpreting the vector p taken from $\mathbb{V}^{n+1}$ as a row-vector with p^{T} as the corresponding column-vector, and where M is the $(n+1) \times (n+1)$ symmetric matrix given by

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ \alpha_{01} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{1n} & \dots & \alpha_{nn} \end{bmatrix}. \tag{1.2}$$

Written another way without abusing notation, we have

$$0 = \sum_{i=0}^{n} \sum_{j=i}^{n} \sigma_{ij} \alpha_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.3}$$

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where σ_{ij} is defined as

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \neq j. \end{cases} \tag{1.4}$$

The form (1.1) lends itself to the study of quadrics using matrix algebra, while the form (1.3) may be thought of as a low-level form of the equation in geometric algebra. What we might think of as a high-level form in geometric algebra, coming from a framework of compuation, may provide a better means of studying quadrics using geometric algebra. We procede now to develop such a form.

Let $\mathbb{W}^{2(n+1)}$ denote a $2(n+1)$-dimensional vector space having $\{e_i\}_{i=0}^{2n+1}$ as a set of orthonormal basis vectors generating it. The vector space $\mathbb{V}^{n+1}$ is therefore a vector sub-space of $\mathbb{W}^{2(n+1)}$ and we will let $\overline{\mathbb{V}}^{n+1}$ denote the $(n+1)$-dimensional vector sub-space of $\mathbb{W}^{2(n+1)}$ that is complement to $\mathbb{V}^{n+1}$. It is then helpful to introduce the notation $\overline{p}$ as the vector in $\overline{\mathbb{V}}^{n+1}$ related to the vector $p \in \mathbb{V}^{n+1}$ by the equation

$$\overline{p} = Rp\tilde{R}, \tag{1.5}$$

where R is a rotor defined as

$$R = 2^{-n/2} \prod_{i=0}^{n} (1 - e_i e_{i+n+1}) \,. \tag{1.6}$$

This idea comes from [2], and it is easy to see that for any integer $i \in [0, n]$, we have $\overline{e}_i = e_{i+n+1}$ and $\overline{e}_{i+n+1} = e_i$. Notice that the over-bar operator is also an outermorphic function that we may apply to any element of the geometric algebra $\mathbb{G}(\mathbb{W}^{2(n+1)})$.

We are now ready to give the high-level form of equation (1.3) as[1]

$$0 = p \wedge \overline{p} \cdot B, \tag{1.7}$$

where $B \in \mathbb{G}(\mathbb{W}^{2(n+1)})$ is a bivector of the form

$$B = -\frac{1}{2} \sum_{i=0}^{n} \sum_{j=i}^{n} \alpha_{ij} (e_i \overline{e}_j + (-1)^{\sigma_{ij}} \overline{e}_i e_j). \tag{1.8}$$

Here, as in the form (1.1) where we may think of the symmetric matrix as representative of the quadric, the bivector B may also be thought of as a representative of the quadric.

Realizing that we need to be careful, because the inner product is not associative, it is interesting to write equation (1.7) in a form similar to that of equation (1.1). Doing so, we get

$$0 = p \cdot B \cdot \overline{p}. \tag{1.9}$$

We can get away with this, because the choice of associativity here only changes the sign of the right-hand side, and the sign of the left-hand side clearly doesn't matter. Considering $\overline{p}$ a type of conjugate to p, we may refer to equation (1.9) as the inner product conjugation of B by p.

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[1]Here and throughout this paper, we assume that the outer product takes precedence over the inner product. We also assume that the geometric product takes precedence over the inner and outer products.

2. Using The Quadric Equation

Having developed the quadric equation (1.7) in geometric algebra, we can now benefit from the language of geometric algebra in using it to answer questions about quadric geometry.

Notice that in our model we can make a distinction between members of $\mathbb{V}^{n+1}$ that are representative of points and those representative of directions. Specifically, a vector $v \in \mathbb{V}^{n+1}$ is a direction if and only if $v \cdot e_0 = 0$. Although we will use an arrow accent to distinguish between direction vectors and position vectors, there should be no confusion on the form of a vector and what we intend it to represent when we refer to it as a direction or a position. Similarly, we will take the liberty of referring to bivectors taken from $\mathbb{G}(\mathbb{V}^{2(n+1)})$ as quadrics. This helps eliminate phrases that would otherwise sound a bit too pedantic.

2.1. Characterizing Flat Quadrics.

Letting $f : \mathbb{V}^{n+1} \to \mathbb{R}$ be the function defined as

$$f(x) = x \wedge \overline{x} \cdot B, \tag{2.1}$$

we arrive at our first lemma.

Lemma 2.1. *Given any quadric B, if for all direction vectors $\vec{v} \in \mathbb{V}^{n+1}$, we have $f(\vec{v}) = 0$, then B is a linear (flat) quadric.*

Proof. For any pair of homogenized points $p, x \in \mathbb{V}^{n+1}$, there is a direction $\vec{x} \in \mathbb{V}^{n+1}$ such that $x = p + \vec{x}$. We then find that

$$f(x) = f(p + \vec{x}) = \nabla_{\vec{x}} f(p), \tag{2.2}$$

in the case that p is on B, where $\nabla_{\vec{x}} f(p)$ is the directional derivative of f at p in the direction of $\vec{x}$. It follows that the tangent space of any point on the quadric is also in the quadric. The quadric is therefore flat at any point upon its surface. □

Recaling that for any $x \in \mathbb{V}^{n+1}$, the definition of $\nabla f(x)$ is given by

$$\nabla f(x) = \sum_{i=0}^{n} e_i \nabla_{e_i} f(x), \tag{2.3}$$

it is not hard to show that for any vector $y \in \mathbb{V}^{n+1}$, we have $y \cdot \nabla f(x) = \nabla_y f(x)$. Seeing that $\vec{x} \cdot \nabla f(p) = \nabla_{\vec{x}} f(p)$ in the light of Lemma 2.1, it follows that the direction $\vec{v} = e_0 \cdot e_0 \wedge \nabla f(p)$ is normal to the surface of the quadric B at p. We can then formulate the quadric A that is the plane tangent to B at p as follows.

$$A = (p \cdot \vec{v}) e_0 \overline{e}_0 + e_0 \overline{\vec{v}} + \overline{e}_0 \vec{v} \tag{2.4}$$

(Needs verification.) An altnerative way to formulate planes that is more in keeping with the tradition of the conformal model will be found in the next section.

2.2. Quadric 2-Blades.

All quadrics are bivectors, but not all bivectors are 2-blades. Here we study the class of quadrics that are 2-blades. For any four points $a, b, c, d \in \mathbb{V}^{n+1}$, such a quadric B has the form

$$B = (a + \overline{b}) \wedge (c + \overline{d}) \tag{2.5}$$

$$= a \wedge c + a \wedge \overline{d} + \overline{b} \wedge c + \overline{b \wedge d}. \tag{2.6}$$

It is curious to think what geometric significance the quadric B has in relation to these four points. Whatever the case may be, it is clear from equation (2.6) that the quadric B contains the intersection, if any, of the four quadrics appearing in the sum. Considering the three forms of 2-blades found in the expansion of equation (2.5) to be more fundamental, (namely, $a \wedge c$, $\overline{b \wedge d}$ and the identical forms $a \wedge \overline{d}$ and $-c \wedge \overline{b}$), we'll start with a treatment of each of these forms.

We first notice that the quadrics of the form $a \wedge c$ and $\overline{b \wedge d}$ trivially represent the quadric of all space. They therefore contribute nothing to the shape of B. The remaining form $a \wedge \overline{d}$, therefore, deserves our full attention. We break this form into two cases, the first being the case when $a = d$, and the second when $a \neq d$.

In the first case, a quick application of equation (1.7) reveals the type of quadric represented by $a \wedge \overline{a}$. Doing so, we see that it represents the set of all projective points $p \in \mathbb{V}^{n+1}$ such that

$$0 = p \wedge \overline{p} \cdot a \wedge \overline{a} = -(p \cdot a)^2, \tag{2.7}$$

which holds if and only if $p \cdot a = 0$. Letting $p = e_0 + \vec{p}$ and $a = e_0 + \vec{a}$, and dividing through by $|\vec{a}|$, our equation becomes

$$\vec{p} \cdot \frac{\vec{a}}{|\vec{a}|} = -\frac{1}{|\vec{a}|}, \tag{2.8}$$

where it is now clear that $a \wedge \overline{a}$ is a plane having a unit-normal of $\vec{a}/|\vec{a}|$ and containing the point $e_0 - \vec{a}/|\vec{a}|^2$ as the point on $a \wedge \overline{a}$ closest to the origin.

In the second case, our use of equation (1.7) barrows from what we learned in the first case.

$$0 = p \wedge \overline{p} \cdot a \wedge \overline{d} = -(p \cdot a)(p \cdot d) \tag{2.9}$$

Clearly each point in the set of all points satisfying this equation is in one of two non-parallel planes. We can now conclude that all quadric 2-blades are single or double planes. Then since all bivectors are sums of 2-blades, we can deduce that all quadrics are formed as the sum of single and double planes. Exactly what geometric significance a given quadric has in relation to such a sum of geometries remains to be seen.

At this point we are also able to see that there may be a new and significant interpretation to equation (1.7) in the realization that $p \wedge \overline{p}$ is a plane. Specifically, this brings up the question: if A and B are quadrics, what geometric relation can we draw between these two quadrics in the case that $A \cdot B = 0$? Clearly, we already know that the general quadric B is characterized by the set of all planes satisfying the condition of this relationship, whatever this relationship may be.

References

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