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A MODEL FOR QUADRIC SURFACES USING GEOMETRIC ALGEBRA

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Abstract. Inspired by the conformal model of geometric algebra, a similar model of geometry is developed for the set of all quadric surfaces in n-dimensional space. Bivectors of the geometric algebra are found to be representative of quadric surfaces. Coordinate free canonical forms of such bivectors are found for common quadric surfaces. The model is investigated for usefulness and compared to the conformal model.

1. The Construction Of The Model

The stage for this model of n-dimensional quadric surfaces is set in the geometric algebra we'll denote by $\mathbb{G}$ that is generated by a vector space $\mathbb{W}$ of dimension $2(n+1)$. Letting $\{e_i\}_{i=0}^{2n+1}$ be an orthonormal set of basis vectors generating $\mathbb{W}$, we let $\{e_i\}_{i=0}^{n}$ be such a set of vectors generating the $(n+1)$-dimensional vector subspace $\mathbb{V}$ of $\mathbb{W}$ in which we'll impose the usual interpretation of $(n+1)$-dimensional homogeneous space. Specifically, a vector $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ represents the point given by[1]

$$e_0 \cdot \frac{e_0 \wedge v}{e_0 \cdot v} \tag{1.1}$$

in n-dimensional Euclidean space, imposing the usual correlation between n dimensional vectors and n-dimensional points[2]. We will take the liberty of letting vectors $v \in \mathbb{V}$ with $v \cdot e_0 = 0$ represent points under the same interpretation of which has just been spoken, as well as pure directions with magnitude. The intended interpretation will be made clear in the context of our usage. We will refer to all vectors $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ as projective points, and such vectors with $v \cdot e_0 = 0$ as non-projective points or sometimes directions.

We now introduce a function defined on $\mathbb{G}$ having the outermorphic property. This means that it is a linear function and that it preserves the outer product. We will use over-bar notation to denote the use of this function. Doing so, for any element $E \in \mathbb{G}$, we define $\overline{E}$ as

$$\overline{E} = RE\tilde{R}, \tag{1.2}$$

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[1]Throughout this paper we let the outer product take precedence over the inner product, and the geometric product take precedence over both the inner and outer products.

[2]The correlation between vectors and points spoken of here is that of having a vector represent the point at its tip when its tail is placed at the origin.

where the rotor R is given by

$$R = 2^{-(n+1)/2} \prod_{i=0}^{n} (1 - e_i e_{i+n+1}) . \tag{1.3}$$

As the reader can check, for any integer $i \in [0, n]$, we have $\overline{e_i} = e_{i+n+1}$. The rotor R simply rotates any k-vector taken from the geometric algebra generated by $\mathbb{V}$ and rotates it into the identical geometric algebra generated by the vector space complement to $\mathbb{V}$ with respect to $\mathbb{W}$, thereby creating an isomorphism between the two geometric algebras. This idea can be found in [1]. We will find the over-bar notation convenient when perform algebraic manipulations in our model.

We are now ready to give the definition by which we will interpret bivectors in $\mathbb{G}$ as n-dimensional quadric surfaces.

Definition 1.1. For any element $E \in \mathbb{G}$, we say that E is representative of the n-dimensional quadric surface generated by the set of all projective points $p \in \mathbb{V}$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{1.4}$$

Notice that when $\mathrm{grade}(E) > 1$, there is no ambiguity, despite the non-associativity of the inner product, in rewriting equation (1.4) as

$$0 = p \cdot E \cdot \overline{p}, \tag{1.5}$$

which resembles a sort of conjugation of E by p. This may perhaps be a more familiar form for readers familiar with the study of quadric surfaces in projective geometry. Also notice that we have not required that E be a bivector in Definition 1.1, because we may find this condition useful and meaningful for any element of $\mathbb{G}$. For now, however, we will restrict our attention to the case when E is a bivector.

To see why Definition 1.1 works, simply notice that when E is a bivector, we have

$$p \wedge \overline{p} \cdot E = \sum_{i=0}^{n} \sum_{j=i}^{n} \lambda_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.6}$$

which we can recognize as a homogeneous polynomial of degree 2 in the vector components of p. The scalars λ_{ij}, with $0 \leq i \leq j \leq n$, may be formulated in terms of E by

$$\lambda_{ij} = \begin{cases} e_i \overline{e_j} \cdot E & \text{if } i = j, \\ (e_i \overline{e_j} - \overline{e_i} e_j) \cdot E & \text{if } i \neq j. \end{cases} \tag{1.7}$$

It should be noted that bivectors do not uniquely represent quadric surfaces, not even up to scale. This is apparent from equation (1.7) when we see that for $i \neq j$, we can freely choose certain components of the bivector without changing the represented quadric so long as that their sum is still $-\lambda_{ij}$. The problem this may pose in our model comes from a very important result in the conformal model. In the conformal model, if two blades are known to represent the same geometry, then it can be shown that the two blades are equal, up to scale. In our present model, it may take more than just homogenization to get a bivector known to represent a certain geometry in a known canonical form.

Another important difference to point out here between our present model and the conformal model is that, unlike what we can analogously expect from the point-definition of the conformal model, here the 2-blade form $a \wedge \overline{a}$ found in Definition 1.1, for any projective point $a \in \mathbb{V}$ not at origin, does not represent the projective point a under Definition 1.1. In homogenized form, the projective point represented by $a \wedge \overline{a}$ is given by

$$e_0 - \left(e_0 \cdot \frac{e_0 \wedge a}{e_0 \cdot a} \right)^{-1}, \tag{1.8}$$

which is the reflection about the origin of the spherical reflection of the projective point a about the unit-sphere centered at the origin. The projective point e_0 at the origin simply represents the empty point-set geometry, or the geometry of nothing. It is also easy to see that $a \wedge \overline{a}$ cannot represent itself, because there are no null blades in our purely Euclidean geometric algebra $\mathbb{G}$.

2. The Construction Of Quadric Surfaces In The Model

Having constructed our model, we are now ready to find canonical forms of bivectors representing a variety of well-known quadric surfaces. Let us begin with the spheroid, (a special case of ellipsoid), the circular cylinder, and the circular hyperboloid of one sheet. We will find that all of these surfaces share the same canonical form, because they may all be characterized as the non-projective point solution set of the equation

$$0 = -r^2 + (x - c)^2 + \lambda((x - c) \cdot v)^2 \tag{2.1}$$

in the non-projective point $x \in \mathbb{V}$, where $c \in \mathbb{V}$ is a non-projective point denoting the center of the surface, $v \in \mathbb{V}$ is a unit-length direction vector, $r \in \mathbb{R}$ is the radius of the geometry about the axis v at c, and $\lambda \in \mathbb{R}$ is a scalar indicating the type and extremity of the surface. Specifically, if $\lambda < -1$, we get a circular hyperboloid of one sheet, if $\lambda = -1$, we get a circular cylinder, if $-1 < \lambda < 0$, we get a stretched sphere, if $\lambda = 0$, a sphere, and if $\lambda > 0$, a squished sphere. Interestingly, when $r = 0$ and $\lambda < -1$, we get circular conical surfaces; a right-circular conical surface if $\lambda = -2$.

Expanding equation (2.1), we get

$$0 = x^2 + \lambda(x \cdot v)^2 - 2x \cdot (c + \lambda(c \cdot v)v) + c^2 + \lambda(c \cdot v)^2 - r^2, \tag{2.2}$$

from which it is possible to factor out $-p \wedge \overline{p}$ in terms of the inner product, where $p = e_0 + x$ is a homogenized projective point. Doing so, we see that the bivector E given by

$$E = \Omega + \lambda v \wedge \overline{v} - 2(c + \lambda(c \cdot v)v) \wedge \overline{e_0} + (c^2 + \lambda(c \cdot v)^2 - r^2)A, \tag{2.3}$$

is representative of the three surface types by Definition 1.1, where the constant Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i \overline{e_i}, \tag{2.4}$$

and A is the constant defined as $A = e_0 \overline{e_0}$. We will find each of these useful as frequently recurring constants in our calculations.

Such forms as that in equation (2.3) are useful, not only for composition, but especially decomposition in the cases where we have formulated what may, for example, be a spheroid by some means other than composition. This gives the model power as an analytical tool. If we can solve a problem whose solution is a bivector known to represent a spheroid, then we can use this canonical form to answer questions about that spheroid. Where is its center? What is its axis? What is its radius about that axis? As is often the case in mathematics, however, decomposition is harder than composition. A general sequence of decomposition steps for the form (2.3) is not obvious, if it exists, but we will proceed now to give such a sequence for the case when E is known to be a cylinder. That is, when $\lambda = -1$.

The first thing to notice is that the canonical form E in equation (2.3) is in a homogenized form, because the coeficient of Ω is 1. If our given bivector is not already homogenized, then we'll want to divide it through by $-\Omega \cdot E/n$.

We then notice that for $1 \leq i < j \leq n$, we have the system of equations

$$(v \cdot e_i)(v \cdot e_j) = -e_i\overline{e_j} \cdot E, \tag{2.5}$$

from which we can deduce the magnitudes of the components of v and the direction of v, up to sign. For example, if $(v \cdot e_i)(v \cdot e_j) > 0$, then $\text{sign}(v \cdot e_i) = \text{sign}(v \cdot e_j)$, and so on. It is also helpful to notice that for all $i = j$, we have

$$(v \cdot e_i)^2 = 1 - e_i\overline{e_j} \cdot E. \tag{2.6}$$

It is unfortunate that we had to refer to a basis to obtain v; nevertheless, it is done. The rest of the decomposition will proceed with greater satisfaction.

There is no way to recover c for cylinders, which is quite obvious. The choice for the point c, the center of the cylinder, may be arbitrarily chosen as any point along its spine. This information is lost in composition, so we may therefore arbitrarily choose $c = -A \cdot (E \wedge e_0)/2$ as the cylinder's center, which, incidentally, will also be the point on the spine of the cylinder closest to the origin.

Lastly, we may find the radius of the cylinder from the simple equation

$$r^2 = c^2 + A \cdot E. \tag{2.7}$$

A generalization of equation (2.1) should be mentioned before moving on. It is given by

$$0 = -r^2 + (x - c)^2 + \sum_{i=1}^{k} \lambda_i((x - c) \cdot v_i)^2, \tag{2.8}$$

which would probably give us the general set of ellipsoids, provided the set of k direction vectors in $\{v_i\}_{i=1}^k$ are linearly independent.

The following table summarizes a few additional canonical forms.

(2.9)

Geometry	Canonical/Homogenized Form
Plane	$v \wedge \overline{e_0} - (c \cdot v)A$
Sphere	$\Omega - 2c \wedge \overline{e_0} + (c^2 - r^2)A$
Point	$\Omega - 2c \wedge \overline{e_0} + c^2 A$

3. Making Use Of The Model

Admittedly, there is really nothing interesting about this model unless we can prove that it has some utility. The conformal model, for example, has at least two great features. The first is the utility of the wedge product in generating intersections between geometries in dual form, or point-fitting between geometries in direct form. A good user of the conformal model can even make use of dual imaginary intersections by reinterpreting them as real geometries in direct form. The second great feature of the conformal model is the surprising fact that all geometries in the conformal model are also, as versors, conformal transformations with geometric significance relative to the simultaneously represented geometry. Then, realizing that all conformal geometries, (with the exception of flat points), have a factorization in direct form as an outer product of points, the outermorphic property of versor conjugation allows us to predict the action of any versor transformation on almost any conformal geometry.

These are great features! But what can the model at present do for us? Well, the first observation we must make is that the set of all known quadrics is represented by the set of all bivectors in $\mathbb{G}$, under-which the inner and outer products are obviously not closed. Only addition and subtraction are closed in this set, and so we're left to wonder what we might be able to prove about the addition and subtraction of n-dimensional quadric surfaces. Letting $B_a, B_b \in \mathbb{G}$ be bivectors, it is not hard to see that $B_a \pm B_b$, under Definition 1.1, must represent at least the intersection, if any, of the quadric surfaces B_a and B_b, but this is not an exact answer to the question of what surface $B_a \pm B_b$ represents.

Let's try an example. Suppose B_a and B_b are both homogenized spheres with a real intersection and having non-projective centers $c_a, c_b \in \mathbb{V}$, respectively. Let $r_a, r_b \in \mathbb{R}$ be the respective radii of B_a and B_b. It then follows from table (2.9) that $B_a - B_b$, in homogenized form, is given by

$$\frac{v}{|v|} \wedge \overline{e_0} - \left(\frac{v}{|v|} \cdot \frac{c_a + c_b + (r_b^2 - r_a^2)v^{-1}}{2} \right) A, \tag{3.1}$$

where v is the vector $c_a - c_b$, which, again by table (2.9), tells us that this is a plane with normal v. A point on the plane is also apparent from (3.1). Then, knowing that $B_a - B_b$ must contain the intersection of the two spheres, we can conclude that this point must be in the plane containing the circle that is the intersection of the two spheres, because $B_a - B_b$ must be the said plane. Notice that even if the spheres don't intersect, we still get a meaningful result. A picture of $B_a - B_b$ is given in Figure 1.

At first sight, the sum of a sphere and a plane may not seem that interesting. However, the sum of a homogenized sphere and a non-homogenized plane is interesting, because the result is always a sphere in homogenized form. The scalar amount at which the plane is non-homogenized simply indicates half the length along the normal of the plane that the center of the original sphere is displaced in the direction of that normal to find a sphere intersecting the plane in the same circle as that of the original sphere.

Interestingly, the difference of spheres generalizes to the idea of subtracting spheroids. A picture of this is given in Figure 2. Of course, there is undoubtedly a geometric significance in the difference between any two homogenized quadric surfaces containing Ω. It would be interesting to find out exactly what that is.

4. A Brief Consideration Of m-Vectors

Specifically, what is meant here is the consideration of m-vectors where $m \neq 2$. What we find in this section, unfortunately, is that such vectors with $m > 2$ are unlikely to play an interesting role in the model. What is immediately obvious is

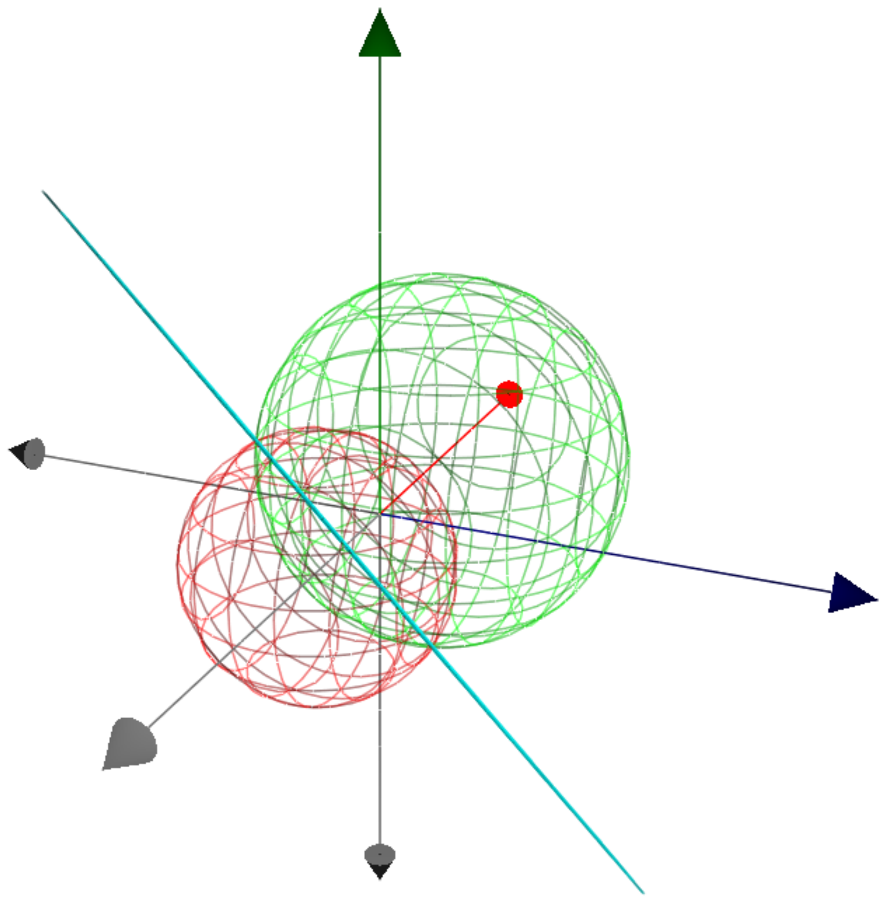

Figure 1. The difference of two spheres gives a plane, shown here on edge. The spheres were rendered as a number of traces in various parallel planes.

that for any such m-vector M, the $(m-2)$-vector $p \wedge \overline{p} \cdot M$, when set to zero, creates a system of equations whose solution set is the intersection of all geometries represented by each individual equation. The problem with this is that M, as an element of $\mathbb{G}$, does not characterize this intersection directly, but only indirectly as the characterization of each individual geometry taken in the intersection. It follows that no decomposition of M is likely to easily reveal any information about the intersection that is M. One redeeming idea in all of this, however, stems from a common theme found throughout various models of geometry based in geometric algebra as the act of performing geometry being the process of simply going from one characterization of some piece of geometry to another, and then viewing that geometry in a different light. While M may not directly characterize the geometry it represents, it may offer a characterization alternative to the one used in the

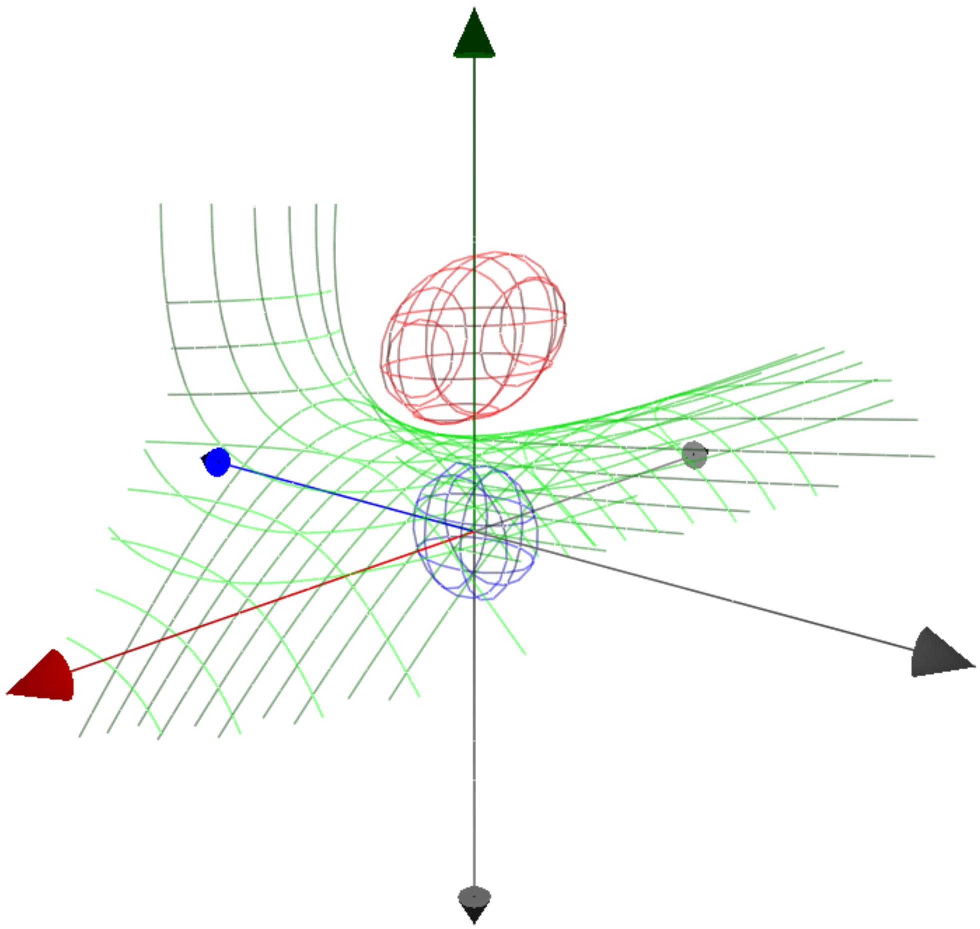

FIGURE 2. The difference of two spheroids gives a hyperbolic paraboloid. Traces in various planes were used to render the surfaces.

formulation of M. And sometimes that is what doing geometry is all about in models of geometry based in geometric algebra.

To illustrate an earlier point, realize that any trivector $T \in \mathbb{G}$ has the form

$$T = \sum_{i=1}^{k} v_i \wedge B_i, \tag{4.1}$$

where $\{v_i\}_{i=1}^k \subset \mathbb{W}$ is a set of k vectors and $\{B_i\}_{i=1}^k \subset \mathbb{G}$ is a set of k bivectors. Applying Definition 1.1, we get the equation

$$0 = p \wedge \overline{p} \cdot T = p \cdot \sum_{i=1}^{k} (\overline{p} \cdot v_i) B_i - \overline{p} \cdot \sum_{i=1}^{k} (p \cdot v_i) B_i + \sum_{i=1}^{k} (p \wedge \overline{p} \cdot B_i) v_i. \tag{4.2}$$

Now if $\{v_i\}_{i=1}^k$ was a linearly independent set, and for all projective points $p \in \mathbb{V}$, we have $p \cdot v_i = \overline{p} \cdot v_i = 0$ for any integer $i \in [1, k]$, (which is not possible anyway without expanding the dimension our geometric algebra), then it is clear from equation (4.2) that T represents the intersection of all quadrics in $\{B_i\}_{i=1}^k$. Now while this may be true, T certainly does not characterize the intersection directly, but only indirectly through the characterizations of each B_i. This result is therefore useless unless perhaps the formulation of T was made through some means other than that of taking the intersection of the B_i quadrics. But as pointed out, this condition cannot be satisfied in $\mathbb{G}$, so it may, in any case, be interesting to formulate trivectors this way, and see what we get.

As for any vector $v \in \mathbb{W}$, under Definition 1.1 this represents the set of all projective points p such that $0 = \overline{p} \cdot v$. Clearly, only those vectors v found in the complement of $\mathbb{V}$ with respect to $\mathbb{W}$ will hold any interest for us.

Realizing that general trivectors may be too difficult at the moment to consider, let us now narrow our scope to that of 3-blades. Doing so, we see that what might motivate us to investigate the set of quadrics that are 2-blades, or to find a better model where all quadrics are 2-blades, is the following lemma.

Lemma 4.1. *For any given non-zero 3-blade $T \in \mathbb{G}$, given by $T = a \wedge b \wedge c$, the geometry represented by this 3-blade under Definition 1.1 is the intersection of the 3 quadrics $a \wedge b$, $a \wedge c$ and $b \wedge c$.*

Proof. The proof follows directly from the following identity, which the reader can easily verify.

$$p \wedge \overline{p} \cdot a \wedge b \wedge c = (p \wedge \overline{p} \cdot a \wedge b)c - (p \wedge \overline{p} \cdot a \wedge c)b + (p \wedge \overline{p} \cdot b \wedge c)a \tag{4.3}$$

Now realize that since $a \wedge b \wedge c \neq 0$, $\{a, b, c\}$ is a linearly independent set, and therefore, $0 = p \wedge \overline{p} \cdot T$ if and only if p is on $a \wedge b$, $a \wedge c$ and $b \wedge c$. □

This, of course, would generalize to blades of higher grade.

So why is it hard to represent quadrics with blades? One possible answer lies in the combinatorics of blades versus symmetric matrices. For n-dimensional quadric surfaces, the 2-blades of our algebra $\mathbb{G}$ generates $2(n+1)$ independent variables as the components of each vector in the blade product. Also for n-dimensional quadric surfaces, we see that square $(n+1) \times (n+1)$ symmetric matrices generate $(n+1)(n+2)/2$ independent variables as the elements of such matrices. So for 3-dimensional surfaces, our 2-blades generate 8 independent variables, while the minimum number needed is 10. It follows that 2-blades cannot possibly represent all quadric surfaces. On the other hand, for 2-dimensional quadric surfaces, (curves), there is a possibility, because in both cases, 6 independent variables are generated. Of course, the 3-way intersection of 3 curves in the plane really doesn't sound that interesting. Come to think of it, the 3-way intersection of any n-dimensional quadric surfaces may not always be interesting either.

5. Transformations In The Model

Explore the idea of applying versors to bivectors here.

6. Concluding Remarks

That $\mathbb{G}$ was not something fancy like a Minkowski space or some other type of non-Euclidean geometric algebra was perhaps our first clue from the beginning that the potential for great things coming out of this model was, let's say, less than likely. On the other hand, it is very hard to see all ends, and so perhaps there are deep results to be found or new insights to be had using this method of studying quadric surfaces. In any case, geometric algebra has proven to be a fundamental, versatile and unifying language that perhaps most naturally extends mathematics beyond the real number line. Perhaps there is a much better way to use geometric algebra to study quadric surfaces.

While it has been shown that elements of $\mathbb{G}$ do indeed, under a given definition, represent quadric surfaces, there really is nothing more or less interesting about adding and subtracting these elements than adding and subtracting vector equations whose solution sets represent the quadric surfaces. There might not be any advantage in using the elemental form over the functional form.

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