

An Intro to CGA

Conformal Geometric Algebra

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Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Let \mathbb{V}^n denote an n -dimensional vector space.

Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from \mathbb{V}^n .

Definition

We say the blade B , given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m -blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $m \leq n$.

Visualizing Euclidean Blades

Imagine an **infinite** m -dimensional hyper-plane.

Think of B as a **finite** m -dimensional hyper-plane.

Non-Euclidean blades require more imagination!

Our geometric arguments will not require us to visualize the homogeneous representation space.

Building Intuition About Euclidean Blades

Let v_{\parallel} denote the orthogonal **projection** of v down onto B .

Let $v_{\perp} = v - v_{\parallel}$ denote the orthogonal **rejection** of v from B .

For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{\parallel} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$

$$v \cdot B = (v_{\parallel} + v_{\perp}) \cdot B = v_{\parallel} \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$

$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Definition

If $v \notin B$, then $v \in B^*$, which represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

Membership in Vector Spaces and Dual Vector Spaces

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. □

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

Blades May Represent Geometries

Let \mathbb{R}^n denote n -dimensional Euclidean space. Let $p : \mathbb{R}^n \rightarrow \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B **directly** represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n \mid p(x) \in B\}.$$

Definition

We say that B **dually** represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n \mid p(x) \in B^*\}.$$

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} p(x) \in A \text{ or } p(x) \in B \\ \implies p(x) \in A \wedge B \end{aligned}$$



Let $C \subseteq A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subseteq A$ and $C \not\subseteq B$.

We Can Intersect Geometries

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff } & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff } & p(x) \notin A \wedge B \\ \text{iff } & p(x) \in (A \wedge B)^* \end{aligned}$$



The Homogeneous Nature Of The Model

For any non-zero scalar λ , we have $G(B) = G(\lambda B)$.

For any blade B , there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operations, then such a λ has geometric significance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^m v_k,$$

is a versor if and only if for all k , the vector v_k^{-1} exists.

The Inverse And The Reverse Of Versors

Definition

Given the versor $V = v_1 \dots v_m$, we define

$$\tilde{V} = \prod_{k=1}^m v_{m-k+1}.$$

The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

The Versor Group

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.

The scalar 1 is the **identity** versor.

For every versor V , there exists an **inverse** V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

Conjugation by versors is **outermorphic**!

Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^m Vb_k V^{-1}.$$

Conjugation by versors is **grade preserving**!

For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.

Given $G(B)$, it will be interesting to investigate $G(VBV^{-1})$.

The Specifics Of The Conformal Model

Replace \mathbb{R}^n with \mathbb{V}^n .

Embed \mathbb{V}^n in \mathbb{V}^{n+2} as a Euclidean vector sub-space.

Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \rightarrow \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Having **invented** this specific model, what we are now able to **discover** about it is almost endless!

Points in n -dimensional Space

For any $c \in \mathbb{V}^n$, the vector $p(c)$ both **dually** and **directly** represents the point c in space.

That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

n -dimensional Dual Hyper-Spheres

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2}r^2\infty \right) = 0.$$

Points are degenerate spheres, or spheres with radius zero.
We may refer to $p(c)$ as a **round** point.

Generating All Dual Rounds Of CGA

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a **non-empty** and **non-degenerate** intersection. Then the blade B , given by

$$B = \bigwedge_{k=1}^m \sigma_k,$$

dually represents an $(n - m + 1)$ -dimensional hyper-sphere.
Rounds with zero radius give us **tangent** points!

All Rounds Of CGA For 3-dimensional Space

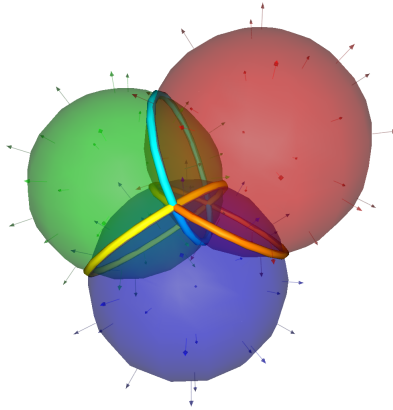


Figure : 3 Rounds, 3 Circles and 1 Point-Pair

$(n - 1)$ -dimensional Dual Hyper-Planes

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

Generating All Dual Flats Of CGA

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n - 1$ having a **non-empty** and **non-degenerate** intersection. Then the blade B , given by

$$B = \bigwedge_{k=1}^m \pi_k,$$

dually represents an $(n - m)$ -dimensional hyper-plane.
Flats at infinity are **free blades**.

All Flats Of CGA For 3-dimensional Space

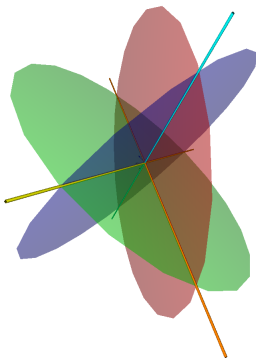


Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are **co- m -hyper-planar** if...

For $m = 0$, the points are identical,

For $m = 1$, the points are collinear,

For $m = 2$, the points are coplanar,

For $m = 3$, the points are co-hyper-planar,

etc...

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co- $(m - 1)$ -hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Generating All Direct Rounds Of CGA

Let $m \geq 1$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m -dimensional hyper-sphere that are non-co- $(m - 1)$ -hyper-planar, the blade B , given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m -dimensional hyper-sphere.

Proof.

Let the $(n - m + 1)$ -blade A **dually** represent the m -dimensional hyper-sphere determined by the points. If A **dually** represents this sphere, then A^* **directly** represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k , we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^m$ is a linearly independent set. Lastly, $\text{grade}(B) = m + 1 = n + 2 - (n - m - 1) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are **co- m -hyper-planar** if...

For $m = 1$, the points are co-point-pair (distinct),

For $m = 2$, the points are co-circular,

For $m = 3$, the points are co-spherical,

For $m = 4$, the points are co-hyper-spherical,

etc...

Generating Almost All Direct Flats Of CGA

Let $m \geq 1$. For $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m -dimensional hyper-plane that are (1) non-co- $(m - 1)$ -hyper-planar and (2) non-co- m -hyper-spherical, the blade B , given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m -dimensional hyper-plane.

Proof.

By (1), there exists the $(n - m)$ -blade A dually representative of the m -dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. \square

Generating All Direct Flats Of CGA

Let $m \geq 1$. For m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ on an m -dimensional hyepr-plane that are non-co- $(m-1)$ -hyper-planar, the blade B , give by

$$B = \infty \wedge \bigwedge_{k=1}^{m+1} p(x_k),$$

directly represents the m -dimensional hyper-plane.
The proof, again, is not hard, but can't fit here.

Quiz Time!

Question: Given a **dual** line L and a point P not on L , how do I find the **dual** plane N containing L and P ?

Answer: $N = (P \wedge L^*)^* = P \cdot L$.

Question: Given a **dual** circle C and a point P not on C or on the plane determined by C , how do I find the **dual** sphere S containing C and P ?

Answer: $S = (P \wedge C^*)^* = P \cdot C$.

Question: Let S be a **dual** sphere that intersects a **dual** plane N in more than one point, and let P be a point on S but not on N . Then, if P' is the reflection of P about N , what is the **dual** sphere reflection S' of S about N ?

Answer:

$$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$$

The Fun Just Doesn't Stop!

Question: Let S **dually** represent the planet Saturn and let the R **directly** represent one of Saturn's rings. If this ring fell out of orbit, let the **direct** circle F on the surface of S approximate the debris field. What is F ?

Answer:

$$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$$

Question: Let $\{N_k\}_{k=0}^3$ be 4 **dual** planes forming the sides of a non-degenerate tetrahedron, and let S be a **dual** sphere containing all 4 vertices of the tetrahedron. If all planes are known, how do we find S ? If S is known and all planes save one, how do we find the remaining plane?

Transformations Of Direct Geometry By Versors

Recall the **outermorphic** property of versors!

Then, if the m -blade B **directly** represents any geometry, (**except a flat point**), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V \left(\bigwedge_{k=1}^m p(x_k) \right) V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

If we understand how V transforms $p(x)$ for any $x \in \mathbb{V}^n$, then we can predict how V transforms any geometry!

A versor may or may not leave ∞ invariant under conjugation.

Transformations Of Dual Geometry By Versors

If the m -blade B *dually* represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

Types Of Transformations By Versors

All conformal transformations can be represented by **versors**!
Some of these include...

- **Translations,**
- **Rotations,**
- **Dilations,**
- **Transversions.**

Surprise!

Blades representative of geometry are also **versors** representative of transformations!

- **Planar reflections** are represented by a dual planes!
- **Spherical reflections** are represented by dual spheres!
- **Translations** & **Rotations** are two successive planar reflections in well chosen planes.
- **Dilations** & **Transversions** are two successive spherical reflections in well chosen spheres.

Corollary: We can use versors to transform transformations!

Planar Reflections

Translations

Rotations

Spherical Reflections

Dilations

Transversions

The End

Thank you for your time. Any questions?