An Introduction To Conformal Geometric Algebra

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Conformal geometric algebra is a model of geometry implemented in the language of geometric algebra. This document is my attempt to rigorously build the conformal model from the ground up. It is only assumed that the reader is familiar with geometric algebra. I used the books [1] and [2] to learn geometric algebra and the conformal model. I recommend them for further study.

1 Representing Geometry

We begin by defining how geometries are represented in the model. Letting $\mathbb{R}^n$ denote n-dimensional Euclidean space, we will represent geometries as subsets of this space. Having done so, we may perform unions, intersections and other operations of geometries, but we have no easy means of performing any geometric analysis. Measurements, normals, tangents, centers, shape and other things that may characterize a geometry are not so easily gleaned or inferred from a set of points. This is where geometric algebra comes in.

Letting $\mathbb{G}$ denote the geometric algebra to be used by our model of geometry, we begin by letting $p : \mathbb{R}^n \to \mathbb{G}$ be a vector-valued function of a Euclidean point, the definition of which we leave open for the moment. We then use this function in the following definition.

Definition 1.1. *For any blade $B \in \mathbb{G}$, we say that B directly represents a geometry as the set of points $G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}$.*

Recall that for any vector $v \in \mathbb{G}$, we say that $v \in B$ if and only if $v \wedge B = 0$. Clearly this means that v is in the vector space spanned by any vector factorization of B. Letting B^* denote a dual of B, it is not hard to show that $v \in B^*$ if and only if $v \cdot B = 0$.

Definition 1.2. *For any blade $B \in \mathbb{G}$, we say that B dually represents a geometry as the set of points $G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}$.*

Notice that by any one of these two definitions, if B represents a given geometry, then so does B^* by the other definition. That is, $G^*(B) = G(B^*)$. Furthermore, any non-zero scalar multiple of B is also representative of the same geometry. That is, for all non-zero $\lambda \in \mathbb{R}$, we have $G(B) = G(\lambda B)$.

2 Operations of Geometry

Noticing that the set of all blades in $\mathbb{G}$ is closed under the inner and outer product operations of geometric algebra, a natural question arrises as to what geometries are represented by the results of these operations in terms of the geometries represented by their operands. To begin to answer this, we start with a theorem.

Theorem 2.1. *For any vector $v \in \mathbb{G}$ and any two blades $A, B \in \mathbb{G}$, if $A \wedge B \neq 0$, then $v \cdot A = 0$ and $v \cdot B = 0$ if and only if $v \cdot A \wedge B = 0$.*

Proof. This is proven in lemma (9.2). □

We can apply theorem (2.1) to get the following result.

Result 2.1. *For any two blades $A, B \in \mathbb{G}$ such that $A \wedge B \neq 0$, we have*

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Interestingly, we see here that the outer product gives the dual representation of the intersection between the two geometries dually represented by the blades taken in that product.

Theorem 2.2. *For any vector $v \in \mathbb{G}$ and any two blades $A, B \in \mathbb{G}$, if $v \wedge A = 0$ or $v \wedge B = 0$, then $v \wedge A \wedge B = 0$.*

Proof. If $A \wedge B = 0$, then we're done. If $A \wedge B \neq 0$ and $v \in A$ or $v \in B$, then $v \in A \cup B$, which is to say that v is in the union of vector sub-spaces represented by A and B. □

Applying theorem (2.2), we get the following result.

Result 2.2. *For any two blades $A, B \in \mathbb{G}$, we have*

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Here we see that the outer product gives the direct representation of a geometry that is at least the union of the geometries directly represented by the blades taken in the product. Unlike the intersection result given earlier, however, here we cannot come to any certain conclusion about what is being represented, even if we know exactly what geometries are being represented by the operands of the operation. To resolve this, we'll find a relationship between the geometries generated through the use of the intersection operation and the geometries generated through the use of the union-like operation.

3 Generating Geometry

We will now give an explicit formula for $p(x)$, but first we must embed $\mathbb{R}^n$ in $\mathbb{G}$. We do this by replacing $\mathbb{R}^n$ with an n-dimensional Euclidean vector space $\mathbb{V}^n$, and make the geometric algebra generated by this vector space a sub-algebra of $\mathbb{G}$. Specifically, if $\{e_k\}_{k=1}^n$ is any set of n basis vectors for $\mathbb{V}^n$, then the set of basis vectors for a vector space $\mathbb{V}$ generating $\mathbb{G}$ will be given by $\{o, \infty\} \cup \{e_k\}_{k=1}^n$. Here, o and ∞ are refered to as the null vectors at the origin and infinity, respectively.

Definition 3.1. *For any vector $v \in \mathbb{V}$, if $v \cdot v = 0$, we call v a null vector.*

The null vectors o and ∞ obey the relationship $\infty \cdot o = -1$. Furthermore, for all vectors $v \in \mathbb{V}^n$, we define $v \cdot o = 0$ and $v \cdot \infty = 0$.

Having precisely defined our geometric algebra $\mathbb{G}$, we define $p : \mathbb{V}^n \to \mathbb{G}$ as follows.

$$p(x) = o + x + \frac{1}{2}x^2\infty$$

It is now not hard to show that for any $x \in \mathbb{V}^n$, the vector $p(x)$ both directly and dually represents the Euclidean point x. We leave this as an exercise for the reader, as well as showing that for any scalar $r > 0$, that $p(x) - \frac{1}{2}r^2\infty$ dually represents an n-dimensional hyper-sphere at x with radius r. The reader should also convince themselves that a vector of the form $v + (x \cdot v)\infty$

dually represents an $(n-1)$-dimensional hyper-plane containing the point x and being orthogonal to the unit-normal $v \in \mathbb{V}^n$.

Now having blades that represent the spheres and planes in the highest possible dimensions of interest in n-dimensional Euclidean space, let us now apply the intersection result of the previous section to generate as many round and flat geometries as we can. Doing so, we see that we can generate hyper-spheres and hyper-planes of dimensions 0 through $n-1$ as outer products of vectors. For $n = 3$, the following table summerizes the geometries we find and the grades of the blades dually representing them.

Grade	Degenerate Dual Round	Dual Round	Dual Flat
1	Point	Sphere	Plane
2	Tangent-Point	Circle	Line
3	Tangent-Point	Point-Pair	Flat-Point

There is nothing more or less that characterizes a flat-point in comparison to a regular point, which may be thought of as a round-point, also being a degenerate sphere (a sphere of radius zero). Flat-points are called flat, because they're the first entry in the list of flat geometries in order of increasing dimension. (Flat-point, line, plane, hyper-plane, etc.)

At first glance, the point-pair may seem out-of-place, but it is simply the 1-dimensional analog of a sphere or circle. It has a center and a radius, but only two points.

The dual tangent point of grade 2 is a degenerate circle, and the dual tangent point of grade 3 is a degenerate point-pair. These occur when we intersect a plane with one point of a round, which is why they're called tangent points.

The question of what other geometric representations we may discover in the conformal model will be left open for now as we continue on, content with what we have found so far. Let us now turn our attention to the method of generating geometries using the union-like method of the previous section. What we'll find is that we can generate the above geometries using this method. We start with a definition.

Definition 3.2. *For all $m \geq 0$, we say that the $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$*

are co-m-hyper-planar under the following circumstances.

For $m = 0$, the points are identical.
For $m = 1$, the points are co-linear.
For $m = 2$, the points are co-planar.
For $m = 3$, the points are co-hyper-planar.
etc.

Here, m corresponds to the dimension of the flat upon which all $m+2$ points lie.

We then need the following theorem.

Theorem 3.1. *For any set of $m \geq 2$ points $\{x_k\}_{k=1}^{m} \subset \mathbb{V}^n$, if these m points are non-co-$(m-2)$-hyper-planar, then the set of m vectors in $\{p(x_k)\}_{k=1}^{m}$ are linearly independent.*

Proof. This is proved in lemma (9.7). $\square$

Using this theorem, it is now not hard to show that blades directly representative of non-degenerate rounds of the conformal model have factorizations in terms of vectors representative of points. To see this, let $B \in \mathbb{G}$ be a blade directly representative of an m-dimensional round, where $m > 0$. Now convince yourself that $m+1$ points can be found on the surface of this round that are also non-co-$(m-1)$-hyper-planar. The vectors representative of these points are therefore linearly independent (by theorem (3.1)) and in the vector space represented by B (by definition (1.1)). All that remains then, to show that B is a scalar multiple of the outer product of these vectors, is that the grade of B is $m+1$. Knowing that the round in question here is m-dimensional, we see that B^* is of grade $n-m+1$. (We learned this from our study of intersecting dual geometries.) The grade of B is therefore $n+2-(n-m+1) = m+1$.

For the case $m = 0$, notice that the 0-dimensional round is the degenerate n-dimensional round or point. A factorization is trivially known as a vector representative of the point.

We now see that we can build up the rounds of the conformal model using the outer product of vectors representative of points. In fact, we now see that it may be more accurate to think of this as a fitting operation instead of a union-like operation. Observe that the conformal model not only easily and naturally solves the problem of fitting an m-dimensional hyper-sphere to a set of $m+1$ points, but also allows us to think of the blade directly representative of that sphere in terms of any appropriate factorization of vectors representative of points on that sphere. We can choose any $m+1$ points on the sphere to be in the outer product, provided they uniquely determine the sphere. We'll see examples of how this idea is useful when we later solve certain problems using the conformal model.

Of course, not all sets of $m+2$ points determine an m-dimensional hyper-sphere. In the cases where these points don't determine a sphere, what do we get? To answer this question, we need to start with another definition.

Definition 3.3. *For all $m \geq 0$, we say that the $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are co-m-hyper-spherical under the following circumstances.*

For $m=0$, the points are identical.
For $m=1$, the points are co-point-pair.
For $m=2$, the points are co-circular.
For $m=3$, the points are co-spherical.
For $m=4$, the points are co-hyper-spherical.
etc.

Here, m corresponds to the dimension of the non-degenerate round upon which all $m+2$ points lie.

With this definition in place, consider $B \in \mathbb{G}$ as a blade directly representative of an m-dimensional flat, where $m > 0$. Now convince yourself that $m+2$ points can be found on the surface of this flat that are non-co-$(m-1)$-hyper-planar and non-co-m-hyper-spherical. By the first of these two conditions, we know that there exists a subset of size $m+1$ of the $m+2$ points that determines an m-dimensional hyper-sphere in the m-dimensional hyper-plane. The second of these two conditions insures that the outer product of the blade directly representative of this m-dimensional round with the vector representative of the remaining point of the $m+2$ points is non-zero. It follows that the vectors representative of the $m+2$ points form a linearly independent set. Then since these points are on the hyper-plane, all that remains to be shown to see that the outer product of the vectors representative of these points is a scalar multiple of B is to show that the grade of B is $m+2$. Knowing that the flat in question here is m-dimensional, we see that B^* is of grade $n-m$. (Again, we learned this from our study of intersecting dual geometries.) The grade of B is therefore $n+2-(n-m) = m+2$.

For the case $m = 0$, the case of flat-points, this argument doesn't work since clearly one cannot find two unique points on a point. Fortunately, a bit of work will show that a flat point is directly represented by a 2-blade of the form $B = \lambda(i + xi \wedge \infty)I$, which simplifies to $B = \lambda(1 - x \wedge \infty)o \wedge \infty$. (Here, i is the unit psuedo-scalar of the geometric algebra genearted by V^n and I is the unit psuedo-scalar of $\mathbb{G}$.) It follows that $\infty \in B$. Then since we can clearly find a vector representative of a point that is on the flat-point, we see that B factors as a scalar multiple of the outer product of this vector and ∞. (Note that no vector representative of a point is a scalar multiple of ∞.)

Interestingly, what we've learned so far is that all geometries, with the exception of flat points, can be written as outer products of vectors representative of points. Our next result, however, will show that we can represent direct flat geometries in what might be considered a more convenient way.

Theorem 3.2. *If a blade $B \in \mathbb{G}$ directly represents an m-dimensional round, then $B \wedge \infty$ directly represents the m-dimensional flat containing this m-dimensional round.*

Proof. This is proved in lemma (9.8). □

4 Solving for Geometry

Knowing how geometries of the conformal model factor in terms of vectors representative of points leads us to one of the reaons why the conformal model is a powerful analytical tool in geometry. Specifically, if we're given two blades $A, B \in \mathbb{G}$ that we know are both directly representative of the same non-point geometry, then we can easily show that A is a scalar multiple of B. Let us state this formally with a theorem.

Theorem 4.1. *For any two blades $A, B \in \mathbb{G}$, if $G(A) = G(B)$ and these are not singletons, then there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$.*

Proof. With the exception of points and flat-points, if A and B both directly represent the same geometry, then any factorization of A in terms of vectors representative of points will also be, up to scale, a factorization of B. □

This is a powerful result, because the formulation of A may have been made one way, while the formulation of B, another, and now we have found a way to relate the two formulations. For example, we might formulate A as the intersection between two spheres. Our result then tells us that we can interpret A as we would write the geometry represented by A in a canonical form B. After composing A, we can decompose it as we would B.

Right away we can use theorem (4.1) to come up with an important result.

Theorem 4.2. *A blade $B \in \mathbb{G}$ directly represents a flat geometry if and only if $B \wedge \infty = 0$.*

Proof. By the previous section and theorem (3.2), if the blade B directly represents a round, then $B \wedge \infty$ represents a flat. But also by the previous section, we know that the flat directly represented by $B \wedge \infty$ can also be directly represented by an outer product of vectors representative of points on the flat. But by theorem (4.1), these blades are scalar multiples of one another. Therefore, in any case, the outer product of ∞ and a factorization of this flat must be zero.

Prove other direction here. $\square$

Notice that this theorem can also be stated as follows. A blade $B \in \mathbb{G}$ directly represents a round geometry if and only if $B \wedge \infty \neq 0$. This is because every geometry is either round or flat.

Another useful feature of the conformal model comes from the way it lets us think about doing operations at a high level. We needed only descend to the lower levels of thinking to develop the model. Once developed, what we can do now is illustrated by the following example. Suppose we're given a dual circle A and a point B, and we want to find the dual sphere C fitting these two geometries. Well, we can think of A^* as any three points determining the circle. Combining this in the outer product with C, we then see that we get what may be four points that determine the desired sphere. Finally, we can come to the conclusion that $C = (A^* \wedge B)^* = A \cdot B$, which is a nice result! Our answer is simply the inner product of the two blades representing the geometries in question. Furthermore, the blade C gives us useful information in all situations. If $C = 0$, then B was on A. If $C \wedge \infty = 0$, then the sphere is really a plane, because its centered at infinity with radius infinity. In the remaining case, C is a finite sphere.

5 Transforming Geometry

Address transformations by versors here.

Versors transform the blades representative of geometries in the conformal model, and so versrors represent transformations.

5.1 Uniform Scaling

5.2 Rotations

5.3 Translations

5.4 Toroidal Rotations

5.5 Transversions

5.6 Hyperbolic Scaling

6 Geometries as Transformers

Interesting things happen when we use geometries as we would versors to transform other geometries. Address this here.

7 Catalog of Dual Representations

For reference, this section catalogs canonical dual representations of the geometries in the conformal model of 3-dimensional space. In each sub-section, the blade $B \in \mathbb{G}$ is assumed to represent the geometry in question. In addition to the composition of each geometry's dual representation, a sequence of steps are also provided that show how one can decompose this representation into the variables that characterize the geometry.

7.1 Points

Points (round points) are characterized by a Euclidean point $x \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda \left(o + x + \frac{1}{2}x^2\infty \right)$$

We may decompose this as follows.

$$\lambda = -\infty \cdot B$$
$$v = o \wedge \infty \cdot \frac{B}{\lambda} \wedge o \wedge \infty$$

7.2 Spheres

Spheres are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda \left(o + x + \frac{1}{2}(x^2 \pm r^2)\infty \right)$$

(Say something about imaginary spheres.) We may decompose this as follows.

$$\lambda = -\infty \cdot B$$
$$x = o \wedge \infty \cdot \frac{B}{\lambda} \wedge o \wedge \infty$$
$$r^2 = x^2 + 2o \cdot \frac{B}{\lambda}$$

7.3 Planes

Planes are characterized by a Euclidean point (center, if you will) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(v + (x \cdot v)\infty)$$

If $T = 1 - \frac{1}{2}x\infty$, we may also formulate B as $Tv\tilde{T}$. We may decompose B as follows.

$$v = o \cdot \frac{B}{\lambda} \wedge \infty$$
$$x = -v \left(o \cdot \frac{B}{\lambda} \right)$$

Notice here that any original weight, normal and position used in the composition of B are not recoverable in the decomposition of B. Here, x will be the point on the plane closest to the origin.

7.4 Circles

Circles are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(v + (x \cdot v)\infty) \wedge \left(o + x + \frac{1}{2}(x^2 \pm r^2)\infty\right)$$

(Say something about imaginary circles.) We may decompose this as follows.

$$\begin{aligned} v &= o \wedge \infty \cdot \frac{B}{\lambda} \wedge \infty \\ x &= v\left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge o\infty\right) \\ r^2 &= x^2 - 2v\left((x \cdot v)x - o \wedge \infty \cdot o \wedge \frac{B}{\lambda}\right) \end{aligned}$$

7.5 Lines

Lines are characterized by a Euclidean point (center, if you will) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda\left(vi - (x \cdot vi) \wedge \infty\right)$$

We may decompose this as follows.

$$\begin{aligned} v &= \left(o \cdot \frac{B}{\lambda} \wedge \infty\right) i \\ x &= -v\left(o \cdot \frac{B}{\lambda}\right) i \end{aligned}$$

7.6 Point-Pairs

Point-pairs are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(vi - (x \cdot vi) \wedge \infty) \wedge \left(o + x + \frac{1}{2}(x^2 \pm r^2)\infty\right)$$

(Say something about imaginary point-pairs.) We may decompose this as follows.

$$\begin{aligned}
v &= -\left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge \infty\right) i \\
x &= -v \left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge o\infty\right) i \\
r^2 &= -x^2 + 2v \left((x \cdot v)v + \left(o \wedge \infty \cdot o \wedge \frac{B}{\lambda}\right) i\right)
\end{aligned}$$

7.7 Flat Points

Flat-points are characterized by a Euclidean point $x \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(i + xi \wedge \infty)$$

We may decompose this as follows.

$$\begin{aligned}
\lambda &= -(B \wedge \infty) i \\
x &= \left(o \cdot \frac{B}{\lambda}\right) i
\end{aligned}$$

7.8 Tangent Points

A tangent-point is characterized by a Euclidean point $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$. Dual canonical forms of tangent points for grades 2 and 3 are given by the dual canonical forms of circles and point-pairs, respectively, with a radius r of zero. For example, given any $r > 0$, simplyfing the following equation recovers the dual form of a tangent point for grade 2.

$$B = \lambda(v + (x \cdot v)\infty) \wedge \left(o + x - rv + \frac{1}{2}((x - rv)^2 - r^2)\infty\right),$$

The reader will notice that r cancels itself out. The decomposition steps for tangent points are the same as those given for circles and point-pairs. The recovered radius will be zero in the case of tangent points.

7.9 Free Blades

Address free-blades here.

8 Catalog of Transformations

8.1 Scale-Rotate-Translation Transformations

Such a transformation is characterized by a Euclidean translation vector $t \in \mathbb{V}^n$, a unit-axis $a \in \mathbb{V}^n$, an angle $\theta \in \mathbb{R}$ and a scalar $\lambda \in \mathbb{R}$.

$$V = \lambda \left(1 - \frac{1}{2}t\infty\right) \left(\cos\frac{\theta}{2} - ai\sin\frac{\theta}{2}\right),$$

Notice that V here is not a blade. It is an even versor. If the blade $B \in \mathbb{G}$ represents a geometry, (directyl or dually), the transformation of B by V is given by VBV^{-1}, in the case that we wish to the apply the rotation first, then the translation. The application of the scale may be thought of as happening before or after the rotation, but not hafter the translation. We may decompose this type of transformation as follows.

$$\begin{aligned}
\lambda^2 &= VV^{\sim} \\
R &= -o \cdot \frac{V}{\lambda} \wedge \infty \\
T &= \frac{V}{\lambda}R^{\sim} \\
\theta &= 2\cos^{-1}\langle R\rangle_0 \\
a &= \frac{1}{\sin(\theta/2)}\langle R\rangle_2 i \\
t &= 2o \cdot (1 - T)
\end{aligned}$$

(Say something about the polar decomposition of V.)

9 Supporting Lemmas

So as not to detract from the main discussions of this document above, the following lemmas support the fine details needed to prove some of the main results above. (There may be a lot of mistakes here that need cleaning up.)

Lemma 9.1. *If $A \in \mathbb{G}$ is a non-zero blade and $v \in \mathbb{V}$ is a vector such that $v \cdot A = 0$, then for all vectors $w \in \mathbb{V}$ such that $w \wedge A = 0$, we have $v \cdot w = 0$.*

Proof. We give a proof by induction. The case when A is a vector is easy to see. Suppose the lemma holds for blades A of grade $m-1$. We will then show that the lemma holds for blades A of grade m. Let $\{a_k\}_{k=1}^m$ be a set of m vectors such that $A = \bigwedge_{k=1}^m a_k$. Let w be a vector such that $w \wedge A = 0$. It follows that w is a linear combination of the vectors in $\{a_k\}_{k=1}^m$. Let v be a vector such that $v \cdot A = 0$.

$$0 = v \cdot A = v \cdot \bigwedge_{k=1}^{m} a_k = \left(v \cdot \bigwedge_{k=1}^{m-1} a_k \right) \wedge b_m + (-1)^m (v \cdot b_m) \bigwedge_{k=1}^{m-1} b_k$$

From this we see that if $v \cdot A = 0$, then $v \cdot \bigwedge_{k=1}^{m-1} a_k = 0$ and $v \cdot a_m = 0$. Now, by inductive hypothesis, it follows that for all integers $k \in [1, m-1]$, we have $v \cdot a_k = 0$. Now since w is a linear combination of the vectors in $\{a_k\}_{k=1}^m$, we see that $v \cdot w = 0$. □

Lemma 9.2. *Given any $v \in \mathbb{V}$ and any two blades $A, B \in \mathbb{G}$ such that $A \wedge B \neq 0$, it follows that*

$$v \cdot A = 0 \text{ and } v \cdot B = 0 \iff v \cdot (A \wedge B) = 0.$$

Proof. Let $\{a_k\}_{k=1}^i$ be a set of i vectors such that $A = \bigwedge_{k=1}^i a_k$. Similarly, let $\{b_k\}_{k=1}^i$ be a set of j vectors such that $B = \bigwedge_{k=1}^j b_k$. Then, using lemma (9.1) in both directions of the proof, we have $v \cdot A = 0$ and $v \cdot B = 0$ if and only if for all integers $k \in [1, i]$, we have $v \cdot a_k = 0$, and for all integers $k \in [1, j]$, we have $v \cdot b_k = 0$, which, in turn, is true if and only if $v \cdot (A \wedge B) = 0$. Notice that one direction of this proof fails if $A \wedge B = 0$. □

Lemma 9.3. *For any integer $m > 2$, if a set of m position vectors $\{v_k\}_{k=1}^m$ taken from $\mathbb{V}^n$ is co-$(m-3)$-hyper-planer, then this set of position vectors is also co-$(m-2)$-hyper-planar.*

Proof. Clearly for any $(m-3)$-dimensional hyper-plane there exists an $(m-2)$-dimensional hyper-plane containing it. □

Lemma 9.4. *If a set of position vectors $\{v_k\}_{k=1}^m$ are non-co-$(m-2)$-hyper-planar, then for any integer $w \in [1, m]$, the set of $m-1$ vectors $\{v_w - v_k\}_{k=1, k \neq w}^m$ is a linearly independent set of vectors.*

Proof. We will prove the contrapositive. If the set of $m-1$ vectors $\{v_w - v_k\}_{k=1,k\neq w}^m$ are linearly dependent, then the $(m-1)$-dimensional hyper-volume of the $(m-1)$-dimensional parallel-piped determined by the $m-1$ vectors in $\{v_w - v_k\}_{k=1,k\neq w}^m$ is zero. Then, at most, this flattened paralle-piped has $(m-1)$-dimensional hyper-area, which is $(m-2)$-dimensional hyper-volume. We can now see that if all directional vectors in $\{v_w - v_k\}_{k=1,k\neq w}^m$ are confined to an $(m-2)$-dimensional hyper-plane, then, as position vectors, the vectors in $\{v_k\}_{k=1}^m$ must all be co-$(m-2)$-hyper-planar. Notice that if our flattened parallel-piped has only volume in lower dimensions, or no volume in any dimension, then our lemma here still goes through by lemma (9.3). $\square$

Lemma 9.5. *Let m be an integer such that $m \geq 2$. Then, if $\{\alpha_k\}_{k=1}^m$ is a set of m scalars, not all zero, taken from $\mathbb{R}$, and $\{v_k\}_{k=1}^m$ is a set of m vectors taken from $\mathbb{V}^n$, such that*

$$0 = \textstyle\sum_{k=1}^m \alpha_k \text{ and } 0 = \sum_{k=1}^m \alpha_k v_k,$$

then the set of m position vectors $\{v_k\}_{k=1}^m$ are co-$(m-2)$-hyper-planar.

Proof. Let us first make the observation that if for all integers $k \in [1, m]$, we have $\alpha_k = 0$, then we can come to no conclusion about the vectors in $\{v_k\}_{k=1}^m$. Therefore, we must require that the scalars in $\{\alpha_k\}_{k=1}^m$ are not all zero.

We now make the observation that if there exists an integer $i \in [1, m]$ such that $\alpha_i \neq 0$, then there must exist an integer $j \in [1, m]$, where $i \neq j$, such that $\alpha_j \neq 0$ also. So without loss of generality, let $i = m$ so that $1 \leq j < m$. It now follows that the sum

$$0 = \sum_{k=1}^{m-1} \alpha_k v_k - \left(\sum_{k=1}^{m-1} \alpha_k\right) v_m = \sum_{k=1}^{m-1} \alpha_k (v_k - v_m)$$

is a non-trivial linear combination of the vectors in $\{v_k - v_m\}_{k=1}^{m-1}$. It then follows that $\{v_m - v_k\}_{k=1}^{m-1}$ is a linearly dependent set of vectors. Therefore, by the contrapositive of lemma (9.4), we see the set of m position vectors $\{v_k\}_{k=1}^m$ are co-$(m-2)$-hyper-planar, which is what we wanted to show. $\square$

Lemma 9.6. *Given a set of m position vectors $\{v_k\}_{k=1}^m$ taken from $\mathbb{V}^n$, if there exists a scalar $\lambda \in \mathbb{R}$ and a set of m scalars $\{\alpha_k\}_{k=1}^m$, not all zero, taken from $\mathbb{R}$ such that*

$$\lambda\infty = \sum_{k=1}^m \alpha_k p(v_k),$$

then the set of m position vectors in $\{v_k\}_{k=1}^m$ are co-$(m-2)$-hyper-planar.

Proof. By equating parts, it is easy to see that

$$0 = \sum_{k=1}^m \alpha_k \text{ and } 0 = \sum_{k=1}^m \alpha_k v_k.$$

Our theorem now follows directly from lemma (9.5). □

Lemma 9.7. *If a given set of m position vectors $\{v_k\}_{k=1}^m$ taken from $\mathbb{V}^n$ are non-co-$(m-2)$-hyper-planar, then the set of vectors $\{p(v_k)\}_{k=1}^m$ are linearly independent.*

Proof. By the contrapositive of lemma (9.6), there does not exist any scalar $\lambda \in \mathbb{R}$ nor set of scalars $\{\alpha_k\}_{k=1}^m$, not all zero, such that

$$\lambda\infty = \sum_{k=1}^{m} \alpha_k p(v_k).$$

This is therefore also true when $\lambda = 0$. □

Lemma 9.8. *Let $A \in \mathbb{G}$ be an m-blade that is directly representive of an $(m-1)$-dimensional hyper-sphere of radius $r > 0$. Then the $(m+1)$-blade $A \wedge \infty$ is directly representative of the $(m-1)$-dimensional hyper-plane containing the hyper-sphere represented by A.*

Proof. By result (**??**), there exists a scalar $\lambda \in \mathbb{R}$ and a set of m vectors $\{p(v_k)\}_{k=1}^m$, such that $A = \lambda \bigwedge_{k=1}^m p(v_k)$. Now since the position vectors $\{v_k\}_{k=1}^m$ are non-co-$(m-2)$-hyper-planar, we see that ∞ is not any non-trivial linear combination of the m vectors $\{p(v_k)\}_{k=1}^m$ by lemma (9.6), and therefore $A \wedge \infty \neq 0$.

Now consider the set of all vectors $v \in \mathbb{V}^n$ such that $p(v) \wedge A \wedge \infty = 0$. This implies that if $p(v) \wedge A \neq 0$, then there exists a set of $m+1$ scalars $\{\alpha_k\}_{k=1}^{m+1}$, not all zero, such that

$$\infty = \sum_{k=1}^{m} \alpha_k p(v_k) + \alpha_{m+1} p(v).$$

If such a non-trivial linear combination does not exist, then $p(v) \wedge A = 0$, and so v is on the hyper-sphere, which is in the desired hyper-plane. Therefore, what we need to show is that the remainder of the hyper-plane is given by the set of all position vectors $v \in \mathbb{V}^n$ for which this non-trivial linear combination does exist. This follows directly from lemma (9.6). □

References

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