An Intro to CGA
Conformal Geometric Algebra

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- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

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We say the blade B, given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero *m*-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

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Clearly, if $B \neq 0$, then we must have $\text{grade}(B) = m \leq n$.

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Our geometric arguments will not require us to visualize the homogeneous representation space.

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Let $v_\perp = v - v_\parallel$ denote the orthogonal rejection of v from B.

Building Intuition About Euclidean Blades

Let $v_{\parallel}$ denote the orthogonal projection of v down onto B.
Let $v_{\perp} = v - v_{\parallel}$ denote the orthogonal rejection of v from B.
For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{\parallel} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
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We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the complement of B with respect to $\mathbb{V}^n$.

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Blades May Represent Vector Sub-Spaces

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Definition

If $v \notin B$, then $v \in B^*$, which represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

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Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\square$

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If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. $\qquad\square$

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We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

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Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

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$$\implies p(x) \in A \wedge B$$

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Let $C \subseteq A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subseteq A$ and $C \not\subseteq B$.

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

We Can Intersect Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$p(x) \in A^* \text{ and } p(x) \in B^*$$
$$\text{iff } p(x) \notin A \text{ and } p(x) \notin B$$
$$\text{iff } p(x) \notin A \wedge B$$
$$\text{iff } p(x) \in (A \wedge B)^*$$

$\square$

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For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operation, then such a λ has geometric signficance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Versors

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Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

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Notice that $v^{-1} = v/v^2$.

The Inverse And The Reverse Of Versors

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Given the versor $V = v_1 \ldots v_m$, we define

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The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

Versors form a group under the geometric product.

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Proof.

Associativity follows from the associativity of the geometric product.

The scalar 1 is the identity versor.

For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

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Properties Of Versors

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Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

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Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\operatorname{grade}(B) = \operatorname{grade}(VBV^{-1})$.

Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.

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Given $G(B)$, it will be interesting to investigate $G(VBV^{-1})$.

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Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

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Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2 \infty.$$

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Having invented this specific model, what we are now able to discover about it is almost endless!

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That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

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Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

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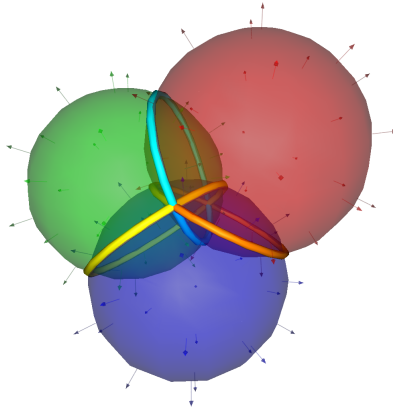Rounds with zero radius give us tangent points!

Figure : 3 Rounds, 3 Circles and 1 Point-Pair

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

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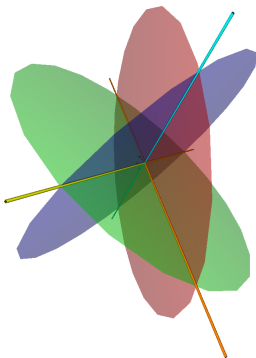
Flats at infinity are free blades.

0-dimensional flats are called flat points.

Figure : 3 Planes, 3 Lines, 1 Flat-Point

Definition

For $m \geq 0$, a set of $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are co-m-hyper-planar if...

> For $m = 0$, the points are identical,
> For $m = 1$, the points are collinear,
> For $m = 2$, the points are coplanar,
> For $m = 3$, the points are co-hyper-planar,
> etc...

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m - 1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Let $m \geq 1$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

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Proof.

Let the $(n-m+1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^m$ is a linearly independent set. Lastly, grade$(B) = m+1 = n+2-(n-m+1) = n+2-$grade$(A) = $grade$(A^*)$. □

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are co-m-hyper-planar if...

> For $m = 1$, the points are co-point-pair (distinct),
> For $m = 2$, the points are co-circular,
> For $m = 3$, the points are co-spherical,
> For $m = 4$, the points are co-hyper-spherical,
> etc...

Let $m \geq 1$. For $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

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$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

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Proof.

By (1), there exists the $(n-m)$-blade A dually representative of the m-dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\mathrm{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \mathrm{grade}(A) = \mathrm{grade}(A^*)$. $\square$

Let $m \geq 0$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

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The proof, again, is not hard, but can't fit here.

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Answer:

$$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$$

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Answer:

$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R$.

Question: Let $\{N_k\}_{k=0}^3$ be 4 dual planes forming the sides of a non-degenerate tetrahedron, and let S be a dual sphere containing all 4 vertices of the tetrahedron. If all planes are known, how do we find S? If S is known and all planes save one, how do we find the remaining plane?

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

Answer:

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Answer: Not very pretty!

Recall the outermorphic property of versors!

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Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

$$VBV^{-1} = V \left(\bigwedge_{k=1}^m p(x_k) \right) V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

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A versor may or may not leave ∞ invariant under conjugation.

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^* V^{-1} = (VB^* V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

All conformal transformations can be represented by versors!
Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

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Corollary: We can use versors to transform transformations!

Surprise!

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Corollary: We can use versors to transform transformations!
Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.

Planar Reflections

Spencer T. Parkin   An Intro to CGA

The End

Thank you for your time. Any questions?