

Notes On Geometric Calculus

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This paper is a formal account of everything I have or have tried to learn about geometric calculus. Take it for what it's worth, if anything.

I will compile a list of references later. Most of this comes from Hestenes' stuff. I go into more detail to prove his statements so that I can understand what-the-hell he's saying.

1 Outermorphism

Let $f : \mathbb{A} \rightarrow \mathbb{B}$ be a linear transformation from the vector space \mathbb{A} to the vector space \mathbb{B} . In the context of geometric algebra, if f preserves the outer product, (or if we extend f to do so), then f is what we call an outermorphism. A linear transformation f that is also an outermorphism will be denoted by \underline{f} .

It follows that since \underline{f} preserves the outer product that it also preserves grade. That is, for any k -blade $A \in \mathbb{G}(\mathbb{A})$, the blade $\underline{f}(A) \in \mathbb{G}(\mathbb{B})$ is also a k -blade. Letting $A = \bigwedge_{i=1}^k a_i$ with each $a_i \in \mathbb{A}$, we may write

$$\underline{f}(A) = \underline{f} \left(\bigwedge_{i=1}^k a_i \right) = \bigwedge_{i=1}^k \underline{f}(a_i),$$

which is clearly a k -blade. Of particular interest is how \underline{f} maps the unit psuedo-scalar of $\mathbb{G}(\mathbb{A})$, which we'll denote by $I_{\mathbb{A}}$. Clearly this will be some scalar multiple of the unit psuedo-scalar of $\mathbb{G}(\mathbb{B})$, which we'll denote by $I_{\mathbb{B}}$. We define this scalar multiple as the determinant of \underline{f} and write

$$\underline{f}(I_{\mathbb{A}}) = (\det \underline{f}) I_{\mathbb{B}}.$$

Associated with every outermorphism is a function \overline{f} denoting what we call the adjoint of \underline{f} . We define $\overline{f} : \mathbb{A} \rightarrow \mathbb{B}$ as an outermorphism with the property that for any pair of vectors $a, b \in \mathbb{A}$, we have

$$a \cdot \underline{f}(b) = \overline{f}(a) \cdot b.$$

Using the k -blade A given earlier, this leads to the following result.

$$\begin{aligned} a \cdot \underline{f}(A) &= - \sum_{i=1}^k (-1)^i (a \cdot \underline{f}(a_i)) \bigwedge_{j=1, j \neq i}^k \underline{f}(a_j) \\ &= - \sum_{i=1}^k (-1)^i (\overline{f}(a) \cdot a_i) \bigwedge_{j=1, j \neq i}^k \underline{f}(a_j) \\ &= \underline{f} \left(- \sum_{i=1}^k (-1)^i (\overline{f}(a) \cdot a_i) \bigwedge_{j=1, j \neq i}^k a_j \right) \\ &= \underline{f}(\overline{f}(a) \cdot A) \end{aligned}$$

Now since \overline{f} is an outermorphism, we may interchange underbars and overbars in the above equation.

Letting $A, B \in \mathbb{G}(\mathbb{A})$ be i and j -blades, respectively, recall that if $i \leq j$, we have the identity

$$A \cdot B = \left(\bigwedge_{k=1}^{i-1} a_k \right) \cdot (a_i \cdot B),$$

which is not hard to show. Recursively applying this identity, we get

$$A \cdot B = a_1 \cdot \dots \cdot a_i \cdot B,$$

where here, right to left associativity of the inner product is understood. It then follows that

$$A \cdot \underline{f}(B) = \underline{f}(\overline{f}(a_1) \cdot \dots \cdot \overline{f}(a_i) \cdot B) = \underline{f}(\overline{f}(A) \cdot B), \quad (1)$$

where here again we recursively applied the identity above.

We can now use the result in equation (1) to show that \underline{f} and \overline{f} have the same determinant in the case that $\mathbb{A} = \mathbb{B}$. In the case that $\mathbb{A} = \mathbb{B}$, let

I denote the unit psuedo-scalar of $\mathbb{G}(\mathbb{A}) = \mathbb{G}(\mathbb{B})$. Recalling that for any k -blade A , we have $\bar{A} = (-1)^{k(k-1)/2}A$, it follows that $I^{-1} = \lambda I$, where $\lambda = \pm 1$, depending on the dimension of \mathbb{A} . We then see that

$$\begin{aligned}
\det \underline{f} &= I^{-1} \cdot \underline{f}(I) \\
&= \underline{f}(\bar{f}(I^{-1}) \cdot I) \\
&= \underline{f}(\bar{f}(\lambda I) \cdot I) \\
&= \underline{f}(\bar{f}(I) \cdot \lambda I) \\
&= \underline{f}(\bar{f}(I) \cdot I^{-1}) \\
&= \underline{f}(\det \bar{f}) \\
&= \det \bar{f}.
\end{aligned}$$

We also have enough at this point to find a formula for the inverse of the outermorphism \underline{f} . Letting A be a blade in $\mathbb{G}(\mathbb{A})$, we have

$$(\det \underline{f})A \cdot I_{\mathbb{B}} = A \cdot \underline{f}(I_{\mathbb{A}}) = \underline{f}(\bar{f}(A) \cdot I_{\mathbb{A}}).$$

From this we get

$$(\det \underline{f})\underline{f}^{-1}(A \cdot I_{\mathbb{B}}) = \bar{f}(A) \cdot I_{\mathbb{A}}.$$

We can then make the substitution $B = A \cdot I_{\mathbb{B}}$ to get

$$\underline{f}^{-1}(B) = \frac{\bar{f}(B \cdot I_{\mathbb{B}}^{-1}) \cdot I_{\mathbb{A}}}{\det \underline{f}}. \quad (2)$$

We'll now show that \underline{f}^{-1} is an outermorphism. Do that here...

Using some calculus, we can find a formula for the adjoint \bar{f} in terms of \underline{f} . Do that here...