

# An Intro to CGA

## Conformal Geometric Algebra

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September 8, 2012

# Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Let  $\mathbb{V}^n$  denote an  $n$ -dimensional vector space.

Let  $\{b_k\}_{k=1}^m$  be a set of  $m$  vectors taken from  $\mathbb{V}^n$ .

## Definition

We say the blade  $B$ , given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero  $m$ -blade if and only if  $\{b_k\}_{k=1}^m$  is a linearly independent set of vectors.

Clearly, if  $B \neq 0$ , then we must have  $m \leq n$ .

# Visualizing Euclidean Blades

Imagine an **infinite**  $m$ -dimensional hyper-plane.

Think of  $B$  as a **finite**  $m$ -dimensional hyper-plane.

**Attitude** matters, **position** does not.

**Non-Euclidean** blades require more imagination!

Our geometric arguments will not require us to visualize the homogeneous representation space.

# Building Intuition About Euclidean Blades

Let  $v_{\parallel}$  denote the orthogonal **projection** of  $v$  down onto  $B$ .

Let  $v_{\perp} = v - v_{\parallel}$  denote the orthogonal **rejection** of  $v$  from  $B$ .

For any vector  $v \in \mathbb{V}^n$ , we have

$$v \wedge B = (v_{\parallel} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$

$$v \cdot B = (v_{\parallel} + v_{\perp}) \cdot B = v_{\parallel} \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$

$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

# Blades May Represent Vector Sub-Spaces

Recall that  $B = b_1 \wedge \cdots \wedge b_m$ .

## Definition

For any  $v \in \mathbb{V}^n$ , we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

## Definition

If  $v \notin B$ , then  $v \in B^*$ , which represents the complement  $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$ .

# Membership in Vector Spaces and Dual Vector Spaces

If  $B \neq 0$ , then  $v \in B$  if and only if  $v \wedge B = 0$ .

Proof.

The set  $\{b_k\}_{k=1}^m$  is linearly independent while the set  $\{v\} \cup \{b_k\}_{k=1}^m$  is linearly dependent. □

If  $B \neq 0$ , then  $v \in B^*$  if and only if  $v \cdot B = 0$ .

Proof.

Notice that  $0 = v \cdot B = (v \wedge B^*)^*$  if and only if  $v \wedge B^* = 0$ . □

# Blades May Represent Geometries

Let  $\mathbb{R}^n$  denote  $n$ -dimensional Euclidean space.

Let  $p : \mathbb{R}^n \rightarrow \mathbb{G}(\mathbb{V}^n)$  be a vector-valued function of a Euclidean point.

## Definition

We say that  $B$  **directly** represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n \mid p(x) \in B\}.$$

## Definition

We say that  $B$  **dually** represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n \mid p(x) \in B^*\}.$$



# We Can Combine Geometries

For any two blades  $A, B \in \mathbb{G}(\mathbb{V}^n)$  such that  $A \wedge B \neq 0$ , we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} p(x) \in A \text{ or } p(x) \in B \\ \implies p(x) \in A \wedge B \end{aligned}$$



Let  $C \subseteq A \wedge B$  represent the smallest vector sub-space such that  $p(x) \in C$ . Then we might have  $C \not\subseteq A$  and  $C \not\subseteq B$ .

# We Can Intersect Geometries

## Lemma

*For any two blades  $A, B \in \mathbb{G}(\mathbb{V}^n)$  such that  $A \wedge B \neq 0$ , we have*

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

## Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff } & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff } & p(x) \notin A \wedge B \\ \text{iff } & p(x) \in (A \wedge B)^* \end{aligned}$$



# The Homogeneous Nature Of The Model

For any non-zero scalar  $\lambda$ , we have  $G(B) = G(\lambda B)$ .

For any blade  $B$ , there is a scalar  $\lambda$  such that  $\lambda B$  is a homogenized form.

If  $B$  is the result of some geometric operations, then such a  $\lambda$  has geometric significance WRT to that operation.

# The Geometric Product

## Definition

For any vector  $v \in \mathbb{V}^n$  and any blade  $B \in \mathbb{G}(\mathbb{V}^n)$ , we define

$$vB = v \cdot B + v \wedge B.$$

Let  $\{v_k\}_{k=1}^m$  be any set of  $m$  vectors.

## Definition

We say the element  $V \in \mathbb{G}(\mathbb{V}^n)$ , given by

$$V = \prod_{k=1}^m v_k,$$

is a versor if and only if for all  $k$ , the vector  $v_k^{-1}$  exists.

# The Inverse And The Reverse Of Versors

## Definition

Given the versor  $V = v_1 \dots v_m$ , we define

$$\tilde{V} = \prod_{k=1}^m v_{m-k+1}.$$

The inverse  $V^{-1}$  of  $V$  is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

# The Versor Group

Versors form a group under the geometric product.

Proof.

**Associativity** follows from the associativity of the geometric product.

The scalar 1 is the **identity** versor.

For every versor  $V$ , there exists an **inverse**  $V^{-1}$  such that  $VV^{-1} = V^{-1}V = 1$ . □

# Properties Of Versors

Conjugation by versors is **outermorphic**!

Recall that  $B = b_1 \wedge \cdots \wedge b_m$ . We then have

$$VBV^{-1} = \bigwedge_{k=1}^m Vb_k V^{-1}.$$

Conjugation by versors is **grade preserving**!

For any vector  $v \in \mathbb{V}^n$ , we have  $VvV^{-1} \in \mathbb{V}^n$ , therefore, we have  $\text{grade}(B) = \text{grade}(VBV^{-1})$ .



# Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.

Given  $G(B)$ , it will be interesting to investigate  $G(VBV^{-1})$ .

# The Specifics Of The Conformal Model

Replace  $\mathbb{R}^n$  with  $\mathbb{V}^n$ .

Embed  $\mathbb{V}^n$  in  $\mathbb{V}^{n+2}$  as a Euclidean vector sub-space.

Let  $o, \infty \in \mathbb{V}^{n+2}$  be vectors such that  $o \cdot o = \infty \cdot \infty = 0$  and  $o \cdot \infty = \infty \cdot o = -1$  and for all  $v \in \mathbb{V}^n$ , we have  $v \cdot o = v \cdot \infty = 0$ .

## Definition

Define  $p : \mathbb{V}^n \rightarrow \mathbb{G}(\mathbb{V}^{n+2})$  as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

Having **invented** this specific model, what we are now able to **discover** about it is almost endless!

# Points in $n$ -dimensional Space

For any  $c \in \mathbb{V}^n$ , the vector  $p(c)$  both **dually** and **directly** represents the point  $c$  in space.

That is,  $G(p(c)) = G^*(p(c)) = \{c\}$ .

# $n$ -dimensional Dual Hyper-Spheres

The function  $p(x)$  factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left( p(c) - \frac{1}{2}r^2\infty \right) = 0.$$

Points are degenerate spheres, or spheres with radius zero.  
We may refer to  $p(c)$  as a **round** point.

# Generating All Dual Rounds Of CGA

Let  $\{\sigma_k\}_{k=1}^m$  be  $m$  spheres of dimension  $n$  having a **non-empty** and **non-degenerate** intersection. Then the blade  $B$ , given by

$$B = \bigwedge_{k=1}^m \sigma_k,$$

**dually** represents an  $(n - m + 1)$ -dimensional hyper-sphere.  
**Rounds** with zero radius give us **tangent** points!

# All Rounds Of CGA For 3-dimensional Space

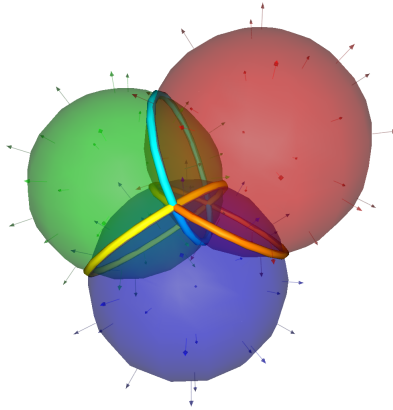


Figure : 3 Rounds, 3 Circles and 1 Point-Pair

# $(n - 1)$ -dimensional Dual Hyper-Planes

The function  $p(x)$  factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

# Generating All Dual Flats Of CGA

Let  $\{\pi_k\}_{k=1}^m$  be  $m$  planes of dimension  $n - 1$  having a **non-empty** and **non-degenerate** intersection. Then the blade  $B$ , given by

$$B = \bigwedge_{k=1}^m \pi_k,$$

**dually** represents an  $(n - m)$ -dimensional hyper-plane.  
**Flats** at infinity are **free blades**.



# All Flats Of CGA For 3-dimensional Space

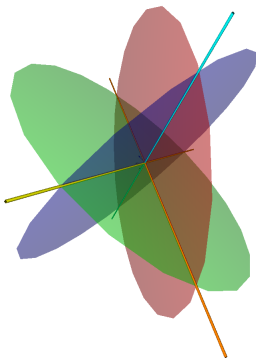


Figure : 3 Planes, 3 Lines, 1 Flat-Point

# A Generalization Of Coplanarity

## Definition

For  $m \geq 0$ , a set of  $m + 2$  points  $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$  are **co- $m$ -hyper-planar** if...

For  $m = 0$ , the points are identical,

For  $m = 1$ , the points are collinear,

For  $m = 2$ , the points are coplanar,

For  $m = 3$ , the points are co-hyper-planar,

etc...

# A Condition For Linear Independents Of Points

## Lemma

*For  $m \geq 1$ , if  $m + 1$  points  $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$  are non-co- $(m - 1)$ -hyper-planar, then  $\{p(x_k)\}_{k=1}^{m+1}$  is a linearly independent set.*

The proof is not hard, but too big for this slide.

# Generating All Direct Rounds Of CGA

Let  $m \geq 1$ . For  $m + 1$  points  $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$  on an  $m$ -dimensional hyper-sphere that are non-co- $(m - 1)$ -hyper-planar, the blade  $B$ , given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

**directly** represents the  $m$ -dimensional hyper-sphere.

**Proof.**

Let the  $(n - m + 1)$ -blade  $A$  **dually** represent the  $m$ -dimensional hyper-sphere determined by the points. If  $A$  **dually** represents this sphere, then  $A^*$  **directly** represents this sphere. Therefore, we need to show that there exists  $\lambda \in \mathbb{R}$  such that  $A^* = \lambda B$ . For all  $k$ , we have  $p(x_k) \in A^*$  and  $p(x_k) \in B$ . By our lemma,  $\{p(x_k)\}_{k=1}^m$  is a linearly independent set. Lastly,  $\text{grade}(B) = m + 1 = n + 2 - (n - m - 1) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$ . □

# A Generalization Of Cospherical

## Definition

For  $m \geq 1$ , a set of  $m + 1$  points  $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$  are **co- $m$ -hyper-planar** if...

For  $m = 1$ , the points are co-point-pair (distinct),

For  $m = 2$ , the points are co-circular,

For  $m = 3$ , the points are co-spherical,

For  $m = 4$ , the points are co-hyper-spherical,

etc...

# Generating Almost All Direct Flats Of CGA

Let  $m \geq 1$ . For  $m + 2$  points  $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$  on an  $m$ -dimensional hyper-plane that are (1) non-co- $(m - 1)$ -hyper-planar and (2) non-co- $m$ -hyper-spherical, the blade  $B$ , given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the  $m$ -dimensional hyper-plane.

Proof.

By (1), there exists the  $(n - m)$ -blade  $A$  dually representative of the  $m$ -dimensional hyper-plane. By (2),  $B \neq 0$ . Lastly,  $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$ .  $\square$

# Generating All Direct Flats Of CGA

Let  $m \geq 1$ . For  $m$  points  $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$  on an  $m$ -dimensional hyepr-plane that are non-co- $(m-1)$ -hyper-planar, the blade  $B$ , give by

$$B = \infty \wedge \bigwedge_{k=1}^{m+1} p(x_k),$$

**directly** represents the  $m$ -dimensional hyper-plane.  
The proof, again, is not hard, but can't fit here.

# Quiz Time!

**Question:** Given a **dual** line  $L$  and a point  $P$  not on  $L$ , how do I find the **dual** plane  $N$  containing  $L$  and  $P$ ?

**Answer:**  $N = (P \wedge L^*)^* = P \cdot L$ .

**Question:** Given a **dual** circle  $C$  and a point  $P$  not on  $C$  or on the plane determined by  $C$ , how do I find the **dual** sphere  $S$  containing  $C$  and  $P$ ?

**Answer:**  $S = (P \wedge C^*)^* = P \cdot C$ .

**Question:** Let  $S$  be a **dual** sphere that intersects a **dual** plane  $N$  in more than one point, and let  $P$  be a point on  $S$  but not on  $N$ . Then, if  $P'$  is the reflection of  $P$  about  $N$ , what is the **dual** sphere reflection  $S'$  of  $S$  about  $N$ ?

**Answer:**

$$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$$



# The Fun Just Doesn't Stop!

**Question:** Let  $S$  **dually** represent the planet Saturn and let the  $R$  **directly** represent one of Saturn's rings. If this ring fell out of orbit, let the **direct** circle  $F$  on the surface of  $S$  approximate the debris field. What is  $F$ ?

**Answer:**

$$F = (S \wedge (R \wedge \infty)^*)^* = S \cdot (R \wedge \infty) = (S \cdot R)\infty - (S \cdot \infty)R.$$

**Question:** Let  $\{N_k\}_{k=0}^3$  be 4 **dual** planes forming the sides of a non-degenerate tetrahedron, and let  $S$  be a **dual** sphere containing all 4 vertices of the tetrahedron. If all planes are known, how do we find  $S$ ? If  $S$  is known and all planes save one, how do we find the remaining plane?

**Answer:** Not very pretty!

# Transformations Of Direct Geometry By Versors

Recall the **outermorphic** property of versors!

Then, if the  $m$ -blade  $B$  **directly** represents any geometry, (**except a flat point**), then there exists  $m$  points  $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$  such that

$$VBV^{-1} = V \left( \bigwedge_{k=1}^m p(x_k) \right) V^{-1} = \bigwedge_{k=1}^m Vp(x_k)V^{-1}.$$

If we understand how  $V$  transforms  $p(x)$  for any  $x \in \mathbb{V}^n$ , then we can predict how  $V$  transforms any geometry!

A versor may or may not leave  $\infty$  invariant under conjugation.

# Transformations Of Dual Geometry By Versors

If the  $m$ -blade  $B$  *dually* represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

# Types Of Transformations By Versors

All conformal transformations can be represented by **versors**!  
Some of these include...

- **Translations,**
- **Rotations,**
- **Dilations,**
- **Transversions.**

# Surprise!

**Blades** representative of geometry are also **versors** representative of transformations!

- **Planar reflections** are represented by a dual planes!
- **Spherical reflections** are represented by dual spheres!
- **Translations** & **Rotations** are two successive planar reflections in well chosen planes.
- **Dilations** & **Transversions** are two successive spherical reflections in well chosen spheres.

**Corollary:** We can use versors to transform transformations!

**Note:** Points are null, (non-invertible), and therefore, planar and spherical reflections **generate** the versor group of all transformations.

# Planar Reflections

# Translations

# Rotations



# Spherical Reflections

# Dilations

# Transversions

# The End

Thank you for your time. Any questions?