An Extension Of The Quadric Model Of Geometric Algebra

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Abstract. What can be seen as an extension of the model set forth in [2], a new model of n-dimensional quadric surfaces is found in which all conformal transformations are supported in the form of versors.

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1. The Geometric Algebra

An explanation of any model of geometry based upon geometric algebra must begin with a description of the structure of the geometric algebra upon which the model is imposed. The model set forth in this paper is set in a geometric algebra containing the following nested vector space.

Notation	Basis
$\mathbb{V}^e$	$\{e_i\}_{i=1}^n$
$\mathbb{V}^h$	$\{e_0\} \cup \{e_i\}_{i=1}^n$
$\mathbb{V}^o$	$\{o, e_0\} \cup \{e_i\}_{i=1}^n$
$\mathbb{V}$	$\{o, e_0\} \cup \{e_i\}_{i=1}^n \cup \{\infty\}$

(1.1)

The set of vectors $\{e_i\}_{i=1}^n$ forms an orthonormal set of basis vectors for the n-dimensional Euclidean vector space $\mathbb{V}^e$, which we'll use to represent n-dimensional Euclidean space. The basis vector e_0 will be a unit-length Euclidean vector orthogonal to all vectors in $\{e_i\}_{i=1}^n$. The vectors o and ∞ are familiar null-vectors at origin and infinity taken from the conformal model of geometric algebra. An inner-product table for these basis vectors is given as follows.

$$\begin{array}{c|cccc} \cdot & o & e_0 & e_i & \infty \\ \hline o & 0 & 0 & 0 & -1 \\ e_0 & 0 & 1 & 0 & 0 \\ e_i & 0 & 0 & 1 & 0 \\ \infty & -1 & 0 & 0 & 0 \end{array} \tag{1.2}$$

We will now let $\mathbb{G}(\mathbb{V})$ denote the Minkowski geometric algebra generated by $\mathbb{V}$. For each vectors space in table (1.1), we will let an over-bar above this vector space denote an identical copy of that vector space. The vector space $\mathbb{W}$ will denote the smallest vector space containing each $\mathbb{V}$ and $\overline{\mathbb{V}}$ as vector subspaces. In symbols, one may write

$$\mathbb{G}(\mathbb{W}) = \mathbb{G}(\mathbb{V}) \oplus \mathbb{G}(\overline{\mathbb{V}}) \tag{1.3}$$

to illustrate the structure of $\mathbb{G}(\mathbb{W})$ in terms of its two isomorphic Minkowski geometric sub-algebras $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$.

We will use over-bar notation to distinguish between vectors taken from $\mathbb{V}$ with vectors taken from $\overline{\mathbb{V}}$. Though not necessary, we can work exclusively in $\mathbb{G}(\mathbb{V})$ by defining the over-bar notation as an outermorphic ismorphism between $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. Doing so, we see that for any element $E \in \mathbb{G}(\mathbb{V})$, we may define $\overline{E} \in \mathbb{G}(\overline{\mathbb{V}})$ as

$$\overline{E} = SE\tilde{S}, \tag{1.4}$$

where S is the versor given by

$$S = 2^{-n/2}(1 + e_-\overline{e}_-)(1 - e_+\overline{e}_+)\prod_{i=0}^{n}(1 - e_i\overline{e}_i). \tag{1.5}$$

This definition is non-recursive if we let the over-bars in equation (1.5) be purely notation. The vectors e_- and e_+, taken from [1], are defined as

$$e_- = \frac{1}{2}\infty + o \tag{1.6}$$

$$e_+ = \frac{1}{2}\infty - o. \tag{1.7}$$

The vectors $\overline{e}_-$ and $\overline{e}_+$ are defined similarly in terms of $\overline{o}$ and $\overline{\infty}$. Defined this way, it is important to realize that, unlike the over-bar function defined in [2], here we do not have the property that for any vector $w \in \mathbb{W}$, we have $\overline{\overline{w}} = w$. This is because $\overline{\overline{o}} = -o$ and $\overline{\overline{\infty}} = -\infty$.

2. The Form Of Quadric Surfaces In $\mathbb{G}(\mathbb{W})$

Let $\mathbb{V}^p$ denote the vector space having $\{o^*\} \cup \{e_i\}_{i=1}^n$ as a basis, where

$$o^* = e_0 + o. \tag{2.1}$$

We will also define

$$\infty^* = \frac{1}{2}(e_0 - \infty). \tag{2.2}$$

We now give a formal definition under which elements $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces.

Definition 2.1. Referring to an element $E \in \mathbb{G}(\mathbb{W})$ as a quadric surface, it is representative of such an n-dimensional surface as the set of all points $p \in \mathbb{V}^p$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{2.3}$$

From this definition it can be seen that the general form of a quadric $E \in \mathbb{G}(\mathbb{W})$ is given by

$$E = \sum_{i=1}^{k} a_i \wedge \overline{b}_i + a_0 \wedge \overline{\infty} + \infty \wedge \overline{b}_0 + \lambda \infty \wedge \overline{\infty}, \tag{2.4}$$

where $\lambda \in \mathbb{R}$, and each of $\{a_i\}_{i=1}^k$ and $\{b_i\}_{i=1}^k$ are sequences of vectors taken from $\mathbb{V}^h$. This is because an element of the form (2.4), when substituted into equation (2.3), produces a homogeneous polynomial of degree 2 in the vector components of $p - (\infty^* \cdot p)o$. Prove that here...

3. Transformations Of The Extended Quadric Model

Here we begin with the following lemma to be used in the main result of this paper, which shows that all conformal transformations exist as versors applicable to quadrics in the extended model.

Lemma 3.1. *For any versor $V \in \mathbb{G}(\mathbb{W})$, and any four vectors $a, b, c, d \in \mathbb{V}$, we have*

$$V^{-1}aV \wedge \overline{V^{-1}bV} \cdot c \wedge \overline{d} = a \wedge \overline{b} \cdot V\overline{V}(c \wedge \overline{d})(V\overline{V})^{-1}. \tag{3.1}$$

Proof. Give proof here... □

The main result, as promised, can now be given as follows.

Theorem 3.2. *Let $E \in \mathbb{G}(\mathbb{W})$ be a bivector of the form (2.4). Then, for any versor $V \in \mathbb{G}(\mathbb{V})$ of the conformal model having the property that*

$$e_0 = V^{-1}e_0V, \tag{3.2}$$

the element E', given by

$$E' = V\overline{V}E(V\overline{V})^{-1}, \tag{3.3}$$

is representative of the n-dimensional quadric surface by Definition 2.1 that is the transformation of E by V.

Proof. Let $P : \mathbb{V}^e \to \mathbb{V}$ be the conformal mapping defined by

$$P(v) = o + v + \frac{1}{2}v^2\infty. \tag{3.4}$$

We then make the observation that if $p' \in \mathbb{V}^o$ is representative of the point that is the transformation of $p \in \mathbb{V}^o$ by the transformation of V^{-1}, then the transformation of E by V is described as the set of all points $p \in \mathbb{V}^o$ such that $0 = p' \wedge \overline{p}' \cdot E$. We then see that

$$
\begin{aligned}
&\frac{p' \wedge \overline{p}'}{(\infty^* \cdot p')^2} \cdot E && (3.5)\\
&= P\left(\frac{p'}{\infty^* \cdot p'} - o\right) \wedge \overline{P}\left(\frac{p'}{\infty^* \cdot p'} - o\right) \cdot E && (3.6)\\
&= V^{-1}P\left(\frac{p}{\infty^* \cdot p} - o\right) V \wedge \overline{V^{-1}P}\left(\frac{p}{\infty^* \cdot p} - o\right) \overline{V} \cdot E && (3.7)\\
&= P\left(\frac{p}{\infty^* \cdot p} - o\right) \wedge \overline{P}\left(\frac{p}{\infty^* \cdot p} - o\right) \cdot E' && (3.8)\\
&= \frac{p \wedge \overline{p}}{(\infty^* \cdot p)^2} \cdot E', && (3.9)
\end{aligned}
$$

showing that E' is representative of this very same set of points under Definition 2.1. Lemma 3.1 was employed in the step taken from (3.7) to (3.8). □

4. Computational Verification

In this section we present the result of implementing the above model with a computer program.

References

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