An Intro to CGA

Conformal Geometric Algebra

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Presentation Outline

In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find the forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

Let $\mathbb{V}^n$ denote an n-dimensional vector space.
Let $\{b_k\}_{k=1}^m$ be a set of m vectors taken from $\mathbb{V}^n$.

Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^{m} b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Visualizing Euclidean Blades

Imagine an infinite m-dimensional hyper-plane.
Think of B as a finite m-dimensional hyper-plane.
Non-Euclidean blades require more imagination!

Building Intuition About Blades

For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{\|} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
$$v \cdot B = (v_{\|} + v_{\perp}) \cdot B = v_{\|} \cdot B.$$

$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Definition

If $v \notin B$, then $v \in B^*$, which represents the complement $\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m$.

Membership in Vector Spaces and Dual Vector Spaces

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. □

If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

Blades May Represent Geometries

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. Let $p : \mathbb{R}^n \to \mathbb{G}(\mathbb{V}^n)$ be a vector-valued function of a Euclidean point.

Definition

We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

Definition

We say that B dually represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \implies & p(x) \in A \wedge B \end{aligned}$$

$\square$

Let $C \subseteq A \wedge B$ represent the smallest vector sub-space such that $p(x) \in C$. Then we might have $C \not\subseteq A$ and $C \not\subseteq B$.

We Can Intersect Geometries

Lemma

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ *such that* $A \wedge B \neq 0$*, we have*

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff} \quad & p(x) \notin A \wedge B \\ \text{iff} \quad & p(x) \in (A \wedge B)^* \end{aligned}$$

□

The Homogeneous Nature Of The Model

For any non-zero scalar λ, we have $G(B) = G(\lambda B)$.

For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operations, then such a λ has geometric signficance WRT to that operation.

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^{m} v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

Conjugation by versors is outermorphic!
Recall that $B = b_1 \wedge \cdots \wedge b_m$. We then have

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_k V^{-1}.$$

Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

Versors May Represent Transformations

It follows that versors may be used to represent transformations of geometry as versors conjugated with blades representative of geometry.

Replace $\mathbb{R}^n$ with $\mathbb{V}^n$.
Embed $\mathbb{V}^n$ in $\mathbb{V}^{n+2}$ as a Euclidean vector sub-space.
Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$