An Introduction To Projective Geometry Using Geometric Algebra

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This paper is my attempt to build up the subject of projective geometry using geometric algebra in my own words. I do not claim originality to any result in this paper. If nothing else, this paper simply represents a formal compilation of my notes on the subject. I have mainly used [3], [1] and [2] for research in the preparation of this paper.

I began my study of projective geometry using geometric algebra after having already put much effort into understanding the conformal model of geometric algebra. I may therefore make reference to concepts in the conformal model as we go along, but it is not a prerequisite that the reader be familiar at all with the conformal model.

I don't know for certain, but I believe that the conformal model might be superior to projective geometry, though to-date the latter has been studied far more than the former, it having been around a lot longer.

1 Representing Geometry

Like the conformal model, we may think of geometries as subsets of the set of all points in some n-dimensional Euclidean space, which we'll denote by $\mathbb{V}^n$. This also denotes an n-dimensional Euclidean vector space as we adopt here the standard correlation between vectors in such a vector space with points in an n-dimensional Euclidean space.

Points sets, of course, do not lend themselves easily to goemetric analysis. So, like the conformal model, we represent them using blades in a geometric algebra. Why we use blades will become apparent after we define how a blade represents a point set, because then it will become clear how the meet and join operations of blades will allow us to do some interesting geometric operations, just as we can in the conformal model.

For n-dimensional projective geometry, we use a geometric algebra generated by an $(n+1)$-dimesnional Euclidean vector space. If $\{e_k\}_{k=0}^{n-1}$ is any set of orthonormal basis vectors spanning $\mathbb{V}^n$, let $\{e_k\}_{k=0}^{n}$ be a set of orthonormal basis vectors spanning $\mathbb{V}^{n+1}$, which we'll use to generate our geometric algebra $\mathbb{G}(\mathbb{V}^{n+1})$.

In projective geometry we can represent points, lines, planes, hyperplanes, and so on to higher dimensions. Certainly results in geometry involving all of these types of geometric primitives can be found by simply using $\mathbb{V}^n$ alone, but what we'll see is that the extra dimension in $\mathbb{V}^{n+1}$ will facilitate some amazingly useful constructions in $\mathbb{G}(\mathbb{V}^{n+1})$ that make the finding of such results much easier than it would be otherwise. Indeed, in [3], it is shown how geometric algebra easily and naturally explains many fundamental theorems in projective geometry. It is my guess that interpretations of how these constructions work based on $(n+1)$-dimensional projections into n-dimensional space are at least partially to blame for the title of the subject being projective geometry.

Without further delay, we begin with a function $p : \mathbb{V}^n \to \mathbb{V}^{n+1}$ that defines a mapping from points in our Euclidean space with vectors in our geometric algebra. We then say that a blade $A \in \mathbb{G}(\mathbb{V}^{n+1})$ represents a piece of geometry as the set of all points $x \in \mathbb{V}^n$ such that $p(x) \wedge A = 0$. From this it is clear that all non-zero scalar multiples A also represent the same piece of geometry. This is the nature of a homogeneous representation model.

We define p simply as

$$p(x) = x + e_3.$$

Having done so, it is easy to see that for any vector $b \in \mathbb{V}^n$, that $p(b)$ represents the point b. Now here's where it gets interesting. Let $\{b_k\}_{k=0}^{m-1}$ be any set of m points taken from $\mathbb{V}^n$ such that they are non-co-$(m-1)$-hyper-planar. That is, if $m = 2$, the 2 points are distinct; if $m = 3$, the 3 points are non-co-linear; if $m = 4$, the 4 points are non-co-planar; if $m = 5$, the points are non-co-hyper-planar, and so on. We will now show that if $m \leq n+1$, then the blade $B = \bigwedge_{k=0}^{m-1} p(b_k)$ represents an $(m-1)$-dimensional hyper-plane.

The case $m = n + 1$ is trivial. In that case, B is a psuedo-scalar of $\mathbb{G}(\mathbb{V}^{n+1})$, and so for all points $x \in \mathbb{V}^n$, we have $p(x) \wedge B = 0$, showing that B represents an n-dimensional hyper-plane. Let us now consider all $1 < m < n + 1$. We start by putting B in a form that exposes its two primary characteristics.

$$B = p(b_0) \wedge A = b_0 \wedge A + e_n \wedge A \tag{1}$$

Here we have arbitrarily chosen a_0 to represent what we'll call the center of the hyper-plane, if you will. (Any point on a plane may be thought of as its center. Let's just choose one.) The blade A here represents the attitude of the hyper-plane, and is given by

$$A = \bigwedge_{k=1}^{m-1} (b_k - b_0).$$

Examining now the equation $p(x) \wedge B = 0$, we see that this reduces to

$$x \wedge a_0 \wedge A + e_n \wedge (x - a_0) \wedge A = 0,$$

showing that for x to be part of the geometry represented by A, there are two conditions that need to be met. The first is that $x \wedge a_0 \wedge A = 0$, which is a necessary yet insufficient condition that x be on the hyper-plane. The second, however, that $(x - a_0) \wedge A = 0$, is both a necessary and sufficient condition that x be on the $(m - 1)$-dimensional hyper-plane. And there we have it!

Looking back at the canonical form of a hyper-plane in this model, equation (1) shows how we might decompose a given hyper-plane into its constituent characteristics. Of course, given a blade B, we couldn't hope to recover the original center and attitude used to compose it, but we can find a center and the original attitude up to scale. An attidue can be found as $A = e_n \cdot B$. Then, interestingly, we can find the center on the plane closest to the origin by requiring $b_0 \cdot A = 0$. It follows that

$$b_0 = (e_n \cdot (e_n \wedge B))\frac{\tilde{A}}{|A|^2}.$$

We have now exhausted the geometric primitives of the projective geometry model of geometric algebra. What remains to is to explore what types of operations we can perform on these geometries, what results we can derive in this model, and however else we can find the model useful in the subject of geometry.

2 Intersecting Geometry

Explore that here...

References

[1] Stan Birchfield. An introduction to projective geometry (for computer vision), 1998.

[2] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra For Computer Science*. Morgan Kaufmann, 2007.

[3] David Hestenes. Projective geometry with clifford algebra. *Acta Applicandae Mathematicae*, 1991.