An Introduction To Conformal Geometric Algebra

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Conformal geometric algebra is a model of geometry implemented in the language of geometric algebra. This document is my attempt to rigorously build the conformal model from the ground up. It is only assumed that the reader is familiar with geometric algebra. I used the books [1] and [2] to learn geometric algebra and the conformal model. I recommend them for further study.

1 Representing Geometry

We begin by defining how geometries are represented in the model. Letting $\mathbb{R}^n$ denote n-dimensional Euclidean space, we will represent geometries as subsets of this space. Having done so, we may perform unions, intersections and other operations of geometries, but we have no easy means of performing any geometric analysis. Measurements, normals, tangents, centers, shape and other things that may characterize a geometry are not so easily gleaned or inferred from a set of points. This is where geometric algebra comes in.

Letting $\mathbb{G}$ denote the geometric algebra to be used by our model of geometry, we begin by letting $p:\mathbb{R}^n\to\mathbb{G}$ be a vector-valued function of a Euclidean point, the definition of which we leave open for the moment. We then use this function in the following definition.

Definition 1.1. *For any blade $B\in\mathbb{G}$, we say that B directly represents a geometry as the set of points $G(B)=\{x\in\mathbb{R}^n|p(x)\in B\}$.*

Recall that for any vector $v\in\mathbb{G}$, we say that $v\in B$ if and only if $v\wedge B=0$. Clearly this means that v is in the vector space spanned by any vector factorization of B. Letting B^* denote a dual of B, it is not hard to show that $v \in B^*$ if and only if $v \cdot B = 0$.

Definition 1.2. *For any blade $B \in \mathbb{G}$, we say that B dually represents a geometry as the set of points $G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}$.*

Notice that by any one of these two definitions, if B represents a given geometry, then so does B^* by the other definition. That is, $G^*(B) = G(B^*)$. Furthermore, any non-zero scalar multiple of B is also representative of the same geometry. That is, for all non-zero $\lambda \in \mathbb{R}$, we have $G(B) = G(\lambda B)$.

2 Operations of Geometry

Noticing that the set of all blades in $\mathbb{G}$ is closed under the outer product operation of geometric algebra, a natural question arrises as to what geometries are represented by the results of this operation in terms of the geometries directly or dually represented by its operands. To begin to answer this, we start with a theorem.

Theorem 2.1. *For any vector $v \in \mathbb{G}$ and any two blades $A, B \in \mathbb{G}$, if $A \wedge B \neq 0$, then $v \cdot A = 0$ and $v \cdot B = 0$ if and only if $v \cdot A \wedge B = 0$.*

Proof. If $A \wedge B \neq 0$, then $A \wedge B$ represents the union of the vector sub-spaces represented by A and B. Stating the next part of the theorem another way, we can say that $v \wedge AI = 0$ and $v \wedge BI = 0$ if and only if $v \wedge (A \wedge B)I = 0$, which is also to say that $v \notin A$ and $v \notin B$ if and only if $v \notin A \wedge B$. $\square$

If $A \wedge B = 0$, then the calculation of the union of the vector sub-spaces represented by A and B is a bit more involved, but is a more general formula for what we call the join of A and B. The meet operation gives us the blade representative of the interesection of the represented vector sub-spaces. The more general meet and join operations may have their uses in the conformal model, but here we will stick to the simpler case of the join operation for now.

We can apply theorem (2.1) to get the following result.

Result 2.1. *For any two blades $A, B \in \mathbb{G}$ such that $A \wedge B \neq 0$, we have*

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Interestingly, we see here that the outer product gives the dual representation of the intersection between the two geometries dually represented by the blades taken in that product.

Theorem 2.2. *For any vector $v \in \mathbb{G}$ and any two blades $A, B \in \mathbb{G}$, if $v \wedge A = 0$ or $v \wedge B = 0$, then $v \wedge A \wedge B = 0$.*

Proof. If $A \wedge B = 0$, then we're done. If $A \wedge B \neq 0$ and $v \in A$ or $v \in B$, then $v \in A \wedge B$, which is to say that v is in the union of the vector sub-spaces represented by A and B. □

To see why the converse of theorem (2.2) does not generally hold, realize that if $v \in A \wedge B$, then the vector sub-space of $A \wedge B$ of smallest dimension containing v might non-trivially overlap both A and B, showing that $v \notin A$ and $v \notin B$.

Applying theorem (2.2), we get the following result.

Result 2.2. *For any two blades $A, B \in \mathbb{G}$, we have*

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Here we see that the outer product gives the direct representation of a geometry that is at least the union of the geometries directly represented by the blades taken in that product. Unlike the intersection result given earlier, however, here we cannot come to any certain conclusion about what is being represented, even if we know exactly what geometries are being represented by the operands of the operation. To resolve this, we'll find a relationship between the geometries generated through the use of the intersection operation and the geometries generated through the use of the union-like operation.

3 Generating Geometry

Having not yet defined the function $p(x)$ or the signature of our geometric algebra $\mathbb{G}$, what we have covered so far applies to any number of possible models of geometry based upon geometric algebra. Therefore, to start getting specific about geometry in the conformal model, we will now give an explicit formula for $p(x)$ and define $\mathbb{G}$. As part of this, we will embed $\mathbb{R}^n$ in $\mathbb{G}$. Notice, however, that this is not a requirement of the generalized model, but it is how the conformal model works. We do this by replacing $\mathbb{R}^n$ with an n-dimensional Euclidean vector space $\mathbb{V}^n$, (interpreting Euclidean vectors as Euclidean points in the usual manner), and make the geometric algebra generated by this vector space a sub-algebra of $\mathbb{G}$. Specifically, if $\{e_k\}_{k=1}^n$ is any set of n basis vectors for $\mathbb{V}^n$, then a set of basis vectors for a vector space we'll denote by $\mathbb{V}$ generating $\mathbb{G}$ is given by $\{o,\infty\}\cup\{e_k\}_{k=1}^n$, where o and ∞ are refered to as the null vectors at the origin and infinity, respectively.

Definition 3.1. *For any vector $v \in \mathbb{V}$, if $v \cdot v = 0$, we call v a null vector.*

The null vectors o and ∞ obey the relationship $\infty \cdot o = -1$. Furthermore, for all vectors $v \in \mathbb{V}^n$, we define $v \cdot o = 0$ and $v \cdot \infty = 0$.

Having precisely defined our geometric algebra $\mathbb{G}$, we define $p : \mathbb{V}^n \to \mathbb{G}$ as follows.

$$p(x) = o + x + \frac{1}{2}x^2\infty$$

It is now not hard to show that for any $x \in \mathbb{V}^n$, the vector $p(x)$ both directly and dually represents the Euclidean point x. We leave this as an exercise for the reader, as well as showing that for any scalar $r > 0$, that $p(x) - \frac{1}{2}r^2\infty$ dually represents an n-dimensional hyper-sphere at x with radius r. The reader should also convince themselves that a vector of the form $v + (x \cdot v)\infty$ dually represents an $(n-1)$-dimensional hyper-plane containing the point x and being orthogonal to the unit-normal $v \in \mathbb{V}^n$.

Now having blades that represent the spheres and planes in the highest possible dimensions of interest in n-dimensional Euclidean space, let us now apply the intersection result (2.1) to generate as many round and flat geometries as we can. Doing so, we see that we can generate hyper-spheres and hyper-planes of dimensions 0 through $n-1$ as outer products of vectors. For $n = 3$, the following table summerizes the geometries we find and the grades of the blades dually representing them.

Grade	Degenerate Dual Round	Dual Round	Dual Flat
1	Point	Sphere	Plane
2	Tangent-Point	Circle	Line
3	Tangent-Point	Point-Pair	Flat-Point

There is nothing more or less that characterizes a flat-point in comparison to a regular point, which may be thought of as a round-point, also being a degenerate sphere (a sphere of radius zero). Flat-points are called flat, because they're the first entry in the list of flat geometries in order of increasing dimension. (Flat-point, line, plane, hyper-plane, etc.)

At first glance, the point-pair may seem out-of-place, but it is simply the 1-dimensional analog of a sphere or circle. It has a center and a radius, but only two points.

The dual tangent point of grade 2 is a degenerate circle, and the dual tangent point of grade 3 is a degenerate point-pair. These occur when we intersect a plane in one point on a round, which is why they're called tangent points.

Using the intersection result (2.1) with n-dimensional spheres and $(n-1)$-dimensional planes, we come to the following result relating the grade of a blade with the geometry dually represented by that blade.

Result 3.1. *If $B \in \mathbb{G}$ is a blade dually representative of an m-dimensional hyper-sphere, then the grade of B is $n - m + 1$. If $B \in \mathbb{G}$ is a blade dually representative of an m-dimensional hyper-plane, then the grade of B is $n - m$.*

A close look at the vector forms of n-dimensional spheres and $(n-1)$-dimensional planes will reveal that we have exhausted all the types of geometries that we can represent using a vector. (Imaginary spheres will be treated in a later section.) Let us now turn our attention to the method of generating geometries using the union-like result (2.2). Interestingly, what we'll find is that we can generate the above geometries using this method. We start with a definition.

Definition 3.2. *For all $m \geq 0$, we say that the $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are co-m-hyper-planar under the following circumstances.*

For $m = 0$, the points are identical.
For $m = 1$, the points are co-linear.
For $m = 2$, the points are co-planar.
For $m = 3$, the points are co-hyper-planar.
etc.

Here, m corresponds to the dimension of the flat upon which all $m+2$ points lie.

We then need the following theorem supported by the following lemma.

Lemma 3.1. *Given any set of $m \geq 2$ points $\{x_k\}_{k=1}^{m} \subset \mathbb{V}^n$, if there exists a scalar $\lambda \in \mathbb{R}$ and a set of m scalars $\{\alpha_k\}_{k=1}^{m} \subset \mathbb{R}$, not all zero, such that*

$$\lambda\infty = \sum_{k=1}^{m} \alpha_k p(x_k),$$

then the set of m points $\{x_k\}_{k=1}^m$ are co-$(m-2)$-hyper-planar.

Proof. By equating parts, it is easy to see that

$$0 = \sum_{k=1}^m \alpha_k \text{ and } 0 = \sum_{k=1}^m \alpha_k x_k.$$

Let us first make the observation that if for all integers $k \in [1, m]$, we have $\alpha_k = 0$, then we can come to no conclusion about the points in $\{x_k\}_{k=1}^m$. Therefore, we must require that the scalars in $\{\alpha_k\}_{k=1}^m$ are not all zero.

We now make the observation that if there exists an integer $i \in [1, m]$ such that $\alpha_i \neq 0$, then there must exist an integer $j \in [1, m]$, where $i \neq j$ and $\alpha_j \neq 0$. So without loss of generality, let $i = m$ so that $1 \leq j < m$. It now follows that the sum

$$0 = \sum_{k=1}^{m-1} \alpha_k x_k - \left(\sum_{k=1}^{m-1} \alpha_k \right) x_m = \sum_{k=1}^{m-1} \alpha_k (x_k - x_m)$$

is a non-trivial linear combination of the vectors in $\{x_k - x_m\}_{k=1}^{m-1}$. It then follows that $\{x_m - x_k\}_{k=1}^{m-1}$ is a linearly dependent set of vectors. The $(m-1)$-dimensional simplex determined by the points in $\{x_k\}_{k=1}^m$, therefore, has no $(m-1)$-dimensional hyper-volume. That is,

$$0 = \frac{1}{(m-1)!} \bigwedge_{k=1}^{m-1} (x_m - x_k).$$

But this can only be if the m points are co-$(m-2)$-hyper-planar, which is what we wanted to show. □

Theorem 3.1. *For any set of $m \geq 2$ points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$, if these m points are non-co-$(m-2)$-hyper-planar, then the set of m vectors in $\{p(x_k)\}_{k=1}^m$ are linearly independent.*

Proof. This theorem is really just a corollary of lemma (3.1). By the contrapositive of lemma (3.1), there does not exist any scalar $\lambda \in \mathbb{R}$ nor set of scalars $\{\alpha_k\} \subset \mathbb{R}$, not all zero, such that $\lambda\infty = \alpha_1 p(x_1) + \cdots + \alpha_k p(x_k)$. This is therefore also true when $\lambda = 0$. □

Using this theorem, it is now not hard to show that blades directly representative of non-degenerate rounds of the conformal model have factorizations in terms of vectors representative of points. To see this, let $B \in \mathbb{G}$ be a blade directly representative of an m-dimensional round, where $m > 0$. Now convince yourself that $m+1$ points can be found on the surface of this round that are also non-co-$(m-1)$-hyper-planar. The vectors representative of these points are therefore linearly independent (by theorem (3.1)) and in the vector space represented by B (by definition (1.1)). All that remains then, to show that B is a scalar multiple of the outer product of these vectors, is that the grade of B is $m+1$. Knowing that the round in question here is m-dimensional, we see that B^* is of grade $n-m+1$ by result (3.1). The grade of B is therefore $n+2-(n-m+1) = m+1$.

For the case $m = 0$, notice that the 0-dimensional round is the degenerate n-dimensional round or point. A factorization is trivially known as a vector representative of the point.

We now see that we can build up the rounds of the conformal model using the outer product of vectors representative of points. In fact, we now see that it may be more accurate to think of this as a fitting operation instead of a union-like operation. For example, the following figure illustrates a circle fit to three points.

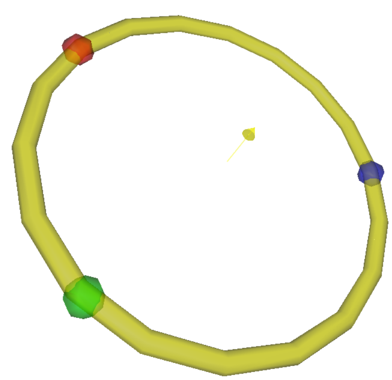

Figure 1: A circle (2-dimensional hyper-sphere) fit to three points.

Observe that the conformal model not only easily and naturally solves the problem of fitting an m-dimensional hyper-sphere to a set of $m+1$ points, but it also allows us to think of the blade directly representative of that sphere in terms of any appropriate factorization of vectors representative of points on that sphere. We can choose any $m+1$ points on the sphere to be in the outer product, provided they uniquely determine the sphere. We'll see examples of how this idea is useful when we later solve certain problems using the conformal model.

Of course, not all sets of $m+1$ points determine an m-dimensional hyper-sphere. In the cases where these points don't determine a sphere, what do we get? To answer this question, we need to start with another definition.

Definition 3.3. *For all $m \geq 0$, we say that the $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ are co-m-hyper-spherical under the following circumstances.*

For $m = 0$, the points are identical.
For $m = 1$, the points are co-point-pair.
For $m = 2$, the points are co-circular.
For $m = 3$, the points are co-spherical.
For $m = 4$, the points are co-hyper-spherical.
etc.

Here, m corresponds to the dimension of the non-degenerate round upon which all $m+2$ points lie.

For the case $m = 1$, if 3 points are non-co-point-pair, (which is to say that they are non-co-m-hyper-spherical), then no two of those points are identical. That is, they are pair-wise distinct.

With definition (3.3) in place, consider $B \in \mathbb{G}$ as a blade directly representative of an m-dimensional flat, where $m > 0$. Now convince yourself that $m+2$ points can be found on the surface of this flat that are non-co-$(m-1)$-hyper-planar and non-co-m-hyper-spherical. By the first of these two conditions, we know that there exists a subset of size $m+1$ of the $m+2$ points that determines an m-dimensional hyper-sphere in the m-dimensional hyper-plane. The second of these two conditions insures that the outer product of the blade directly representative of this m-dimensional round with the vector representative of the remaining point of the $m+2$ points is non-zero. It follows that the vectors representative of the $m+2$ points form a linearly independent set. Then since these points are on the hyper-plane, all that remains to be shown to see that the outer product of the vectors representative of these points is a scalar multiple of B is to show that the grade of B is $m+2$. Knowing that the flat in question here is m-dimensional, we see that B^* is of grade $n-m$ by result (3.1). The grade of B is therefore $n+2-(n-m) = m+2$.

For the case $m = 0$, the case of flat-points, this argument doesn't work since clearly one cannot find two unique points on a point. Fortunately, a bit of work will show that a flat point is directly represented by a 2-blade of the form $B = \lambda(i + xi \wedge \infty)I$, which simplifies to $B = \lambda(1 - x \wedge \infty)o \wedge \infty$. (Here, i is the unit psuedo-scalar of the geometric algebra genearted by V^n and I is the unit psuedo-scalar of $\mathbb{G}$.) It follows that $\infty \in B$. Then since we can clearly find a vector representative of a point that is on the flat-point, we see that B factors as a scalar multiple of the outer product of this vector and ∞. (Note that no vector representative of a point is a scalar multiple of ∞.)

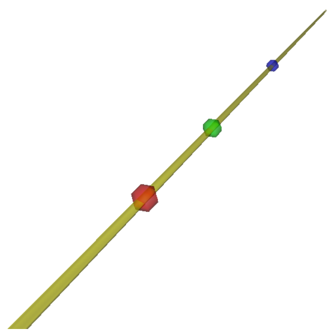

Figure 2: A line (1-dimensional hyper-plane) fit to three points.

Interestingly, what we've learned so far is that all geometries, with the exception of flat points, can be written as outer products of vectors representative of points. Our next result, however, will show that we can represent direct flat geometries in what might be considered a more convenient way.

Theorem 3.2. *If a blade $B \in \mathbb{G}$ directly represents an m-dimensional round, then $B \wedge \infty$ directly represents the m-dimensional flat containing this m-dimensional round.*

Proof. Let $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ be a set of $m+1$ points such that $B = \bigwedge_{k=1}^{m+1} p(x_k)$. Let $x_{m+2} \in \mathbb{V}^n$ be a point such that $p(x_{m+2}) \wedge B \wedge \infty = 0$. Then if x_{m+2} is not on the round directly represented by B, it follows that $p(x_{m+2}) \wedge B \neq 0$, and therefore

$$\infty = \sum_{k=1}^{m+2} \alpha_k p(x_k),$$

where the scalars in $\{\alpha_k\}_{k=1}^m \subset \mathbb{R}$ are not all zero. Our theorem now follows directly from lemma (3.1). $\square$

4 Solving for Geometry

Knowing how geometries of the conformal model factor in terms of vectors representative of points leads us to one of the reaons why the conformal model is a powerful analytical tool in geometry. Specifically, if we're given two blades $A, B \in \mathbb{G}$ that we know are both directly representative of the same non-single-point geometry, then we can easily show that A is a scalar multiple of B. Let us state this formally with a theorem.

Theorem 4.1. *For any two blades $A, B \in \mathbb{G}$, if $G(A) = G(B)$ and these are not singletons, then there exists a scalar $\lambda \in \mathbb{R}$ such that $A = \lambda B$.*

Proof. With the exception of points and flat-points, if A and B both directly represent the same geometry, then any factorization of A in terms of vectors representative of points will also be, up to scale, a factorization of B. $\square$

This is a powerful result, because the formulation of A may have been made one way, while the formulation of B, another, and now we have found a way to relate the two formulations. For example, we might formulate A as the intersection between two spheres. Our result then tells us that we can interpret A as we would write the geometry represented by A in a canonical form B. After composing A, we can decompose it as we would B. The canonical form of B, in our example here, might be the intersection of a sphere centered on a plane. Indeed, this is how the following figure was generated on a computer.

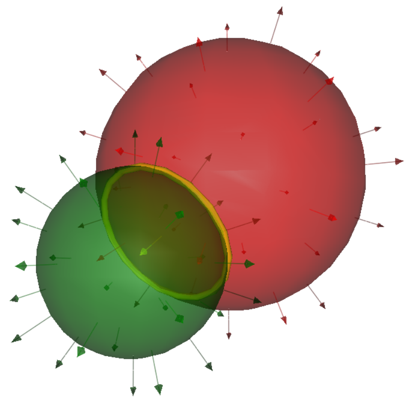

Figure 3: The intersection of two intersecting spheres.

Fascinatingly, the intersection of two non-intersecting spheres still gives us a meaningful result in the conformal model. What we get is an imaginary circle, which we can draw just the same.

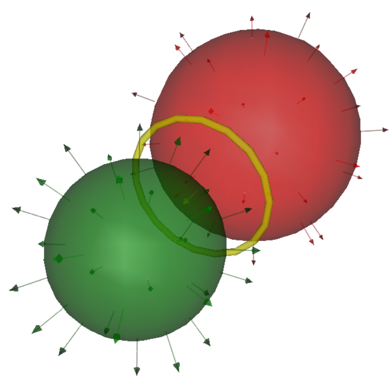

Figure 4: The intersection of two non-intersecting spheres.

Right away we can use theorem (4.1) to come up with an important result.

Theorem 4.2. *A blade $B \in \mathbb{G}$ directly represents a flat geometry if and only if $B \wedge \infty = 0$.*

Proof. By theorem (3.2), the only reason we have to believe that $B \wedge \infty \neq 0$ is in the case that B is written entirely as the outer product of vectors representative of points. But by theorem (4.1), we know that any such factorization of B can be rewritten as the outer product of a blade directly representative of a round and ∞. $\square$

Notice that this theorem can also be stated as follows. A blade $B \in \mathbb{G}$ directly represents a round geometry if and only if $B \wedge \infty \neq 0$. This is because every geometry is either round or flat.

Another useful feature of the conformal model comes from the way it lets us think about doing operations at a high level. We needed only descend to the lower levels of thinking to develop the model. Once developed, what we can do now is illustrated by the following example. Suppose we're given a dual circle A and a point B, and we want to find the dual sphere C fitting these two geometries. Well, we can think of A^* as any three points determining the circle. Combining this in the outer product with C, we then see that we get what may be four points that determine the desired sphere. Finally, we can come to the conclusion that $C = (A^* \wedge B)^* = A \cdot B$, which is a nice result! Our answer is simply the inner product of the two blades representing the geometries in question. Furthermore, the blade C gives us useful information in all situations. If $C = 0$, then B was on A. If $C \wedge \infty = 0$, then the sphere is really a plane and we may think of it as centered at infinity with radius infinity. In the remaining case, C is a finite sphere.

5 Transforming Geometry

Here we study the transformations of conformal geometries by versors. Let us therefore give a formal definition of such elements.

Definition 5.1. *Given any set of m invertible vectors $\{v_k\}_{k=1}^m \subset \mathbb{G}(\mathbb{V})$, the element*

$$V = \prod_{k=1}^{m} v_k$$

is called a versor. If m is odd, we call V an odd versor; if even, an even versor. I will refer to m as the grade of the versor V.

It follows that versors are invertible by definition. Clearly $V^{-1} = \tilde{V}/|V|$, where $|V| = \prod_{k=1}^{m} |v_k|$. It is also easy to see that versors form their own group under the geometric product. With versors as representative of transformations, this property allows us to easily concatinate transformations. As we'll come to find out, we can represent any conformal transformation with a versor. This is where the conformal model gets its name.

We will now proceed to lay some ground work for a very import result that will help us decipher the action of versors on elements representative of conformal geometries.

Lemma 5.1. *For any vector $a \in \mathbb{V}$, we have $VaV^{-1} \in \mathbb{V}$.*

Proof. Notice that we need only show that for any vector $v \in \mathbb{V}$, we have $vav^{-1} = vav/|v| \in \mathbb{V}$. By the equation

$$vav = v(a \cdot v + a \wedge v) = (a \cdot v)v + v \cdot (a \wedge v) = 2(a \cdot v)v - |v|^2 a,$$

clearly this is the case. $\square$

Lemma 5.2. *Given any two vectors $a, b \in \mathbb{V}$, we have $VaV^{-1} \cdot VbV^{-1} = a \cdot b$.*

Proof. It suffices to show that $\langle V(a \wedge b)V^{-1}\rangle_0 = 0$, since

$$VaV^{-1} \cdot VbV^{-1} = \langle VabV^{-1}\rangle_0 = a \cdot b + \langle V(a \wedge b)V^{-1}\rangle_0.$$

A proof by induction is given. Consider first $\langle v(a \wedge b)v^{-1}\rangle_0$, where $v \in \mathbb{V}$. By direct evaluation, we get

$$\langle v(a \wedge b)v^{-1}\rangle_0 = \frac{1}{|v|}\begin{vmatrix} n \cdot a & n \cdot a \\ n \cdot b & n \cdot b \end{vmatrix} = 0.$$

Suppose now that for a fixed integer k, the versor V of grade k satisfies the lemma. We must show that $\langle vV(a \wedge b)V^{-1}v^{-1}\rangle_0$ is zero. To that end, it is not hard to see that

$$\langle vV(a \wedge b)V^{-1}v^{-1}\rangle_0 = \langle v\langle V(a \wedge b)V^{-1}\rangle_0 v^{-1}\rangle_0 + \langle v\langle V(a \wedge b)V^{-1}\rangle_2 v^{-1}\rangle_0.$$

Clearly $\langle v\langle V(a \wedge b)V^{-1}\rangle_0 v^{-1}\rangle_0$ is zero by our inductive hypothesis. That $\langle v\langle V(a \wedge b)V^{-1}\rangle_2 v^{-1}\rangle_0$ is zero follows directly from the work we did to show that $\langle v(a \wedge b)v^{-1}\rangle_0$ is zero. $\square$

We're now ready to prove a theorem that we'll use to prove something very interesting about versor transformations in the conformal model.

Theorem 5.1. *Given any m-blade $B = \bigwedge_{k=1}^m b_k$, we have*

$$VBV^{-1} = \bigwedge_{k=1}^{m} Vb_iV^{-1}.$$

Proof. Our proof of this theorem will rely upon the following. If we find that $B = f(b_1, \dots, b_m)$, where f is any expression in the vector variables b_1 through b_m, then clearly

$$f(Vb_1V^{-1}, \dots, Vb_mV^{-1}) = \bigwedge_{k=1}^{m} Vb_iV^{-1}$$

by lemma (5.1). All that would remain, then, is to show that

$$VBV^{-1} = f(Vb_1V^{-1}, \dots, Vb_mV^{-1}),$$

and the theorem goes through.

We procede now by strong induction. Clearly the theorem holds for the case $m = 1$. The case $m = 2$ is not much harder to prove. Suppose now that for a fixed integer $m > 2$, that the theorem holds for each case less than m. A partial expansion of VBV^{-1} gives us

$$VBV^{-1} = V(b_1 \wedge \cdots \wedge b_{k-1})b_kV^{-1} - (-1)^{k-1}V(a_k \cdot a_1 \wedge \cdots \wedge a_{k-1})V^{-1}.$$

We will consider each part of the right-hand side seperately. For the first part, we have

$$\begin{aligned} V(b_1 \wedge \cdots \wedge b_{k-1})b_kV^{-1} &= V(b_1 \wedge \cdots \wedge b_{i-1})V^{-1}Vb_kV^{-1} \\ &= \left(\bigwedge_{i=1}^{k-1} Vb_iV^{-1}\right) Vb_kV^{-1}, \end{aligned}$$

by our inductive hypothesis. For the second part, we have

$$\begin{aligned} &(-1)^kV(a_k \cdot a_1 \wedge \cdots \wedge a_{k-1})V^{-1} \\ &= -(-1)^kV\left(\sum_{i=1}^{k-1}(-1)^i(a_k \cdot a_i) \bigwedge_{j=1, j\neq i}^{k-1} a_j\right)V^{-1} \\ &= -(-1)^k\sum_{i=1}^{k-1}(-1)^i(a_k \cdot a_i)V\left(\bigwedge_{j=1, j\neq i}^{k-1} a_j\right)V^{-1} \\ &= -(-1)^k\sum_{i=1}^{k-1}(-1)^i(Va_kV^{-1} \cdot Va_kV^{-1}) \bigwedge_{j=1, j\neq i}^{k-1} Va_jV^{-1}, \end{aligned}$$

by lemma (5.2) and our inductive hypothesis. Having now sandwitched all vectors between V and V^{-1}, our proof by induction is complete. $\square$

What theorem (5.1) tells us is that the function $f(B) = VBV^{-1}$ is an outermorphism. Applying this theorem, we get the following important result.

Result 5.1. *Given a versor V, if we understand the transformation $Vp(x)V^{-1}$ for any point $x \in \mathbb{V}^n$, then we understand the transformation VBV^{-1}, where B is any blade directly or dually representative of any geometry, with the only exception being that of flat points.*

Proof. In the previous section we showed that, with the exception of flat points, all geometries directly represented by a blade B may be written as the outer product of vectors representative of points. For all of these geometries, our result goes through directly by theorem (5.1).

To show now that the theorem goes through for blades dually representative of geometries, (with again, the exception of flat-points), we need only show the commutativity of grade k versors with the unit-psuedo scalar I of $\mathbb{G}$. For any vector $v \in \mathbb{V}$, it is not hard to show that $vI = (-1)^n Iv$. It follows that $VI = (-1)^{nk} IV$. We then see that

$$VBV^{-1} = -VBV^{-1}I^2 = -(-1)^{nk} VBIV^{-1}I,$$

which relates what we already know about the action of versors on blades directly representative of geometries with those dually representative of such geometries. □

This is a fantastic result! We now not only have a way of formulating a desired transformation, but we also know that such a formulation, once formulated, will apply to almost all geometries in the conformal model, no matter the representation! This is similar to what we know about linear transformations. A linear transformation is entirely determined by how it transforms a basis of the space. Therefore, if we know how a desired transformation transforms a basis of the space, then we have found the desired linear transformation. Analogously, if we know how a desired conformal transformation transforms a point, then we have found the desired conformal transformation. On the other hand, if we know how a given versor transforms a point, then we can predict how it will transform any conformal geometry, except flat points.

Flat points are an exception to the idea presented in result (5.1), because a given versor may not always leave ∞ invariant up to scale. We'll see an example of this in section (5.5) below.

We procede now to develop the versor transformations of the conformal model.

5.1 Reflections

Interestingly, what we find in the conformal model is that a vector not only represents a piece of geometry, but, as a versor, also represents a transformation involving that geometry. Specifically, a plane represents a reflection about that plane, and a sphere represents a reflection about that sphere!

Reflections turn out to be a very fundamental type of transformation, because so many other types of transformations, such as rotations and translations, can be described in terms of reflections about well chosen geometries. For example, two successive reflections about two well chosen planes gives us either a rotation or a translation. (Interestingly, this shows that the rotation and translation transformations are perhaps not as disimilar as we might have first thought them to be!)

Another type of reflection is performed about a sphere. That is, a point interior to a sphere is reflected out of it, while points exterior to the sphere are reflected into it. The question then comes up: if two successive reflections about two well chosen planes gives us the rotation or translation transformations, what does two successive reflections about two well chosen spheres give us? We will get to the bottom of this question later on.

5.2 Planar Reflections

Given a plane centered at $c \in \mathbb{V}^n$ with unit-length normal $v \in \mathbb{V}^n$, here we consider the action of the vector dually representative of this plane as a versor on a vector representative of a point. That is, for $\pi = v + (v \cdot c)\infty$, what do we get in the evaluation of $\pi p(x)\pi^{-1}$? Expanding this out, we get

$$\pi p(x)\pi^{-1} = -p(y),$$

where $y = x - 2(v \cdot (x - c))v$. Drawing a picture of this will convince you that y is the reflection of x about the plane!

Interestingly, here we have an example of a versor that leaves ∞ invariant but for a factor of -1. It follows that

$$\pi(p(x) \wedge \infty)\pi^{-1} = p(y) \wedge \infty,$$

showing that flat points reflect about planes the same way round points do.

Having covered the case of flat points, we can now apply result (5.1) to find that the versor π reflects any conformal geometry about the plane represented by π as we would imagine. Even a sphere partially straddling the plane will reflect as we would predict. The following figure illustrates the reflection of a circle about a plane.

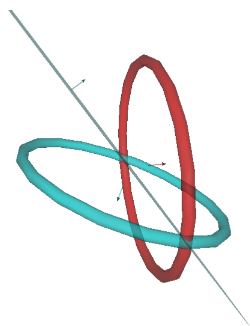

Figure 5: The reflection of a circle about a plane. The plane is shown here on edge.

5.3 Translations

It is not hard to imagine that we can perform a translation using a single planar reflection about a well chosen plane. The problem with this is that we need a different plane for every situation involving a different point, even if the translation vector is the same. Interestingly, however, what we'll find is that if we perform two successive reflections about two parallel planes, then the positions of those planes becomes arbitrary, while only the relative distance between the planes determines the amount of translation. The direction of translation is determined by the attitude of the parallel planes. It only takes a little bit of brain power to convince yourself that a point is consistently translated by the two successive planar reflections in all three cases of where a point can lie with respect to the two planes.

Let $v \in \mathbb{V}^n$ be a unit-normal shared by two parllel planes, let $c \in \mathbb{V}^n$ be the position of the first plane, and $t \in \mathbb{V}^n$ be a vector representative of the desired amount of translation. We assume that $v \wedge t = 0$. The versor representative of the translation is then given by

$$(v + (v \cdot (c + \frac{1}{2}t)\infty)(v + (v \cdot c)\infty) = 1 - \frac{1}{2}t\infty.$$

Notice that our result here is independent of the position of the first plane, but only dependent upon the relative positions of the two planes. Of course, also notice that v goes away as t is all that is needed to characterize the translation transformation.

5.4 Rotations

Suppose now that the two planes we reflect about are non-parallel. In this case we get a rotation. Let $c \in \mathbb{V}^n$ be a common point among the two planes, and let $v_0, v_1 \in \mathbb{V}^n$ be their unit-normals, respectively. Not being parallel planes, we have $v_0 \wedge v_1 \neq 0$. We can then let $a \in \mathbb{V}^n$ be a unit-length vector representing the axis of rotation, and let $\theta \in \mathbb{R}$ be the angle of rotation. Doing so, we see that $v_0 \wedge v_1 = -ai \sin \theta/2$. The versor that represents a rotation about the point c is then given by

$$(v_0 + (v_0 \cdot c)\infty)(v_1 + (v_1 \cdot c)\infty) = \cos \frac{\theta}{2} - (a + a \wedge c \wedge \infty)i \sin \frac{\theta}{2}.$$

Letting $c = 0$, we get the well-known rotor in Euclidean geometric algebras which has a polar decomposition of $\exp(-ai\theta/2)$.

5.5 Spherical Reflections

Let $c \in \mathbb{V}^n$ be the center of a sphere, and $a \in \mathbb{V}^n$ be a point exterior to the sphere. Then the point $b \in \mathbb{V}^n$ along the line from c to a given in the following figure is what we refer to as the reflection of a in the sphere.

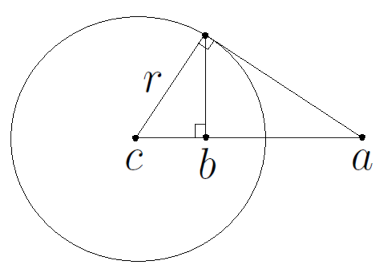

Figure 6: The reflection of a point about a sphere.

We let $r \in \mathbb{R}$ be the radius of this sphere. Using what we know about similar triangles, it is not hard to show that $b = (1 - \lambda)c + \lambda a$, where

$$\lambda = \left(\frac{r}{|c-a|}\right)^2 = \left(\frac{|c-b|}{r}\right)^2,$$

the square of a common ratio between respective sides of the similar triangles in the figure above. Interestingly, letting $\sigma = p(c) - \frac{1}{2}r^2\infty$ be the sphere in the figure, we find that

$$\sigma p(a)\sigma^{-1} = -\lambda p(b),$$

showing that vectors dually representative of spheres perform, as versors in a versor transformation, spherical reflections! As one would expect, a double spherical reflection about the same sphere leaves a point invarient. This is because we also find that

$$\sigma p(b)\sigma^{-1} = -\frac{1}{\lambda}p(a).$$

Applying result (5.1), we find that σ, when acting upon a blade directly representative of a line, gives us the blade directly representative of the circle that is the reflection of that line in the sphere. The following figure illustrates this.

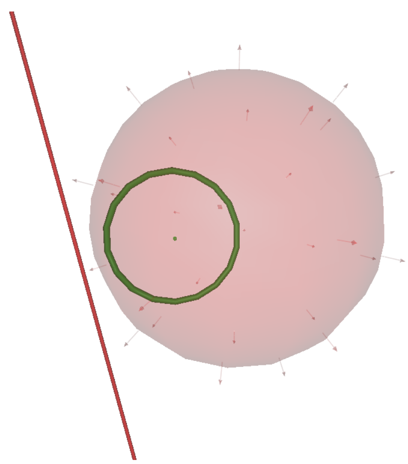

Figure 7: The reflection of a line in a sphere.

Similarly, the reflection of a plane in a sphere is a sphere.

Of course, result (5.1) doesn't tell us anything about how σ acts on a flat point. The key, as always, is to see how the versor, (in this case σ), acts on ∞. Investigating this, we find that

$$\sigma(p(a) \wedge \infty)\sigma^{-1} = \frac{2}{(c-a)^2} p(b) \wedge p(c),$$

showing that σ transforms the flat point at a into the pair of points b and c. This is because the spherical reflection of a sphere's center is, up to scale, ∞ and vice-versa. Specifically, we have

$$\sigma p(c)\sigma^{-1} = -\tfrac{1}{2}r^2\infty \text{ and } \sigma\infty\sigma^{-1} = \tfrac{-2}{r^2}p(c).$$

It is important to notice here that spherical reflections, unlike planar reflections, do not generally preserve geometric type. As we saw in figure 7, the spherical reflection of a line became a sphere, a different type of geometry than what we started with. Indeed, as we just saw, the spherical reflection of a sphere's center in itself is a multiple of ∞ which, by definition, both dually and directly represents the empty point-set geometry or the geometry of nothing.[1] The non-preservation of geometric type means that we'll need to take greater care in deciphering transformations involving spherical reflections.

So far we have only considered the spherical reflections of flat geometries, and it is not too hard to see that these are always round. Interestingly, however, the reverse is not generally true. That is, the spherical reflection of a round geometry is not always flat. To see this, first consider the reflection of any sphere in a sphere. This is always a plane. Now consider any circle sharing all points with the sphere to be spherically reflected and realize that the spherical reflection of this circle must be contained within the plane that is the spherical reflection of that sphere. What we find is that circles, under most circumstances, spherically reflect as circles, and only under a certain condition, spherically reflect as lines. Specifically, the center of the sphere to be reflected about must be in the plane determined by the circle for the circle to be spherically reflected as a line.

[1]Some authors refer to ∞ as the point at infinity. This idea may come from observing the point $\sigma p(x)\sigma^{-1}$ as x approaches c. But as the direction from which x approaches c is significant to me, and ideas like a point at infinity seem too ill-defined and poorly addressed, I prefer to think of ∞ as I have just described it. Engagine the imagination, however, I can see how the notion of a point at infinity is useful, interesting and makes intuitive sense.

Lastly, before moving on, we should not forget to consider the spherical reflection of a point in a degenerate sphere. This type of reflection simply can't be taken, because in this case σ^{-1} does not exist.

5.6 Diolations

5.7 Transversions

5.8 Other Transformations

Yet more types of transformations can be constructed in the conformal model by combining the above mentioned transformations. For example, general rigid body motions can be formulated by combining the rotation and translation versors. Questions should be answered about the set of all possible types of transformations, what groups they form under versor concatination, what their exponential forms are, and if the their logarithms can be taken, but this is beyond the scope of this paper and, admittedly, the present capabilities of the author.

In any case, it should be clear that by combining all types of reflection vector versors, (planar and spherical), that we exhaust all possible transformations by versors that we can come up with in the conformal model. We have only begun to discover what types of transformations we can do with these basic building blocks. (There are undoubtadly more combinations to consider.) But we will leave versor transformations for now, content with what we have covered thus far as an introduction to the subject.

6 Inference of Geometry

Given a blade $B \in \mathbb{G}$, is it possible for us to infer what type of geometry it represents, dually or directly? Our first clue is the grade of B. We summarize what types of geometries can be represented by the various grades in the conformal model of 3-dimensional space with the following table.

	Dual	Direct
1	Sphere, Plane, Point	Point
2	Circle, Line	Point-Pair, Flat-Point
3	Point-Pair, Flat-Point	Circle, Line
4	Point	Sphere, Plane, Point

Given a blade of any of these 4 grades, we can restrict our attention to only the grades 1 and 2 by taking the dual of blades of grades 3 and 4.

7 More Geometry

Rounds and flats are the extent of geometric primitives available in the conformal model of geometric algebra. If we, however, allow for a small variation in the way geometries are represented in the conformal model, we open up a whole new set of geometric primitives.

Definition 7.1. *Let $g : \mathbb{V}^n \to \mathbb{G}$ be any blade-valued function of a point variable. We then say that g is directly representative of a geometry as the set of all points*

$$G(g) = \{x \in \mathbb{V}^n | p(x) \in g(x)\}.$$

Similarly, we say that g is dually representative of a geometry as the set of all points

$$G^*(g) = \{x \in \mathbb{V}^n | p(x) \in g^*(x)\}.$$

Here, $g^*(x)$ refers to either of the two function $\pm g(x)I$. We refer to g as a geometric function.

From this definition it is clear that all geometric primitives of the conformal model are recovered by the set of all constant geometric functions. Realizing that all vectors representative of points are null, we see that p is a geometric function both dually and directly representative of the geometry that is all of space $\mathbb{V}^n$.

One of the motivations behind geometric functions comes from a method of derivation for the dual forms of the sphere and plane primitives of the conformal model. After writing a vector equation equated to zero, one can see how $p(x)$ factors out of this equation in terms of the inner product. This, however, cannot be done for all vector equations, such as those that represent other types of geometry, such as infinitely long cylinders, right-circular canonical surfaces and ellipsoids. But using geometric functions we can represent these new types of geometries.

Where we run into trouble with geometric functions, however, is in the performance of geometric analysis. Geometric functions that are canonical forms, while easily lending themselves to the decomposition of the represented geometry into its characteristic parts, cannot always be used to decispher the result of a geometric operation. In other words, there is no direct equivilant of theorem (4.1) for geometric functions. Indeed, for any two geometric functions g_1 and g_2, if $G(g_1) = G(g_2)$, we cannot conclude that there exists a scalar $\lambda \in \mathbb{R}$ such that for all $x \in \mathbb{V}^n$, we have $g_1(x) = \lambda g_2(x)$. This may not mean, however, that canonical forms are no longer useful to us. Suppose that g_1 takes on a canonical form while g_2 is the result of some geometric operation that is representative of the type of geometry that g_1 can represent. We then know that for some instance of g_1, we have for all $x \in \mathbb{V}^n$, $p(x) \in g_1(x)$ if and only if $p(x) \in g_2(x)$. In this sense, they are the same function, even if g_1 is not a scalar multiple of g_2. How exactly this can be useful remains to be seen.

Admittedly, the idea of geometric functions is somewhat contrary to the vision most people have about a geometric algebra. The holy grail, so to speak, of a geometric algebra is the idea that for any geometry we could think of, there exists an element in the algebra representative of that geometry, and all possible geometric operations are available and closed in the algebra. Furthermore, any question about a given geometry should be answerable through some easy calculation in the algebra. Perhaps we need a better geometric algebra and even a new type of model implemented in that algebra, but until then, let's continue with a development of geometric functions and the discovery of what we might be able to do with them.

8 Catalog of Canonical Forms

For reference, this section catalogs canonical dual representations of the geometries in the conformal model of 3-dimensional space, as well as common transformations represented by versors. In each sub-section about dual geometric representations, the blade $B \in \mathbb{G}$ is assumed to represent the geometry in question, while in sub-sections about versor transformations, the versor $V \in \mathbb{G}$ represents the transformation in question. In addition to the composition of each geometry's dual representation, a sequence of steps are also provided that show how one can decompose this representation into the variables that characterize the geometry. A similar set of composition and decomposition steps are provided for the versor transformations.

8.1 Points

Points (round points) are characterized by a Euclidean point $x \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda \left(o + x + \frac{1}{2} x^2 \infty \right)$$

We may decompose this as follows.

$$\begin{aligned} \lambda &= -\infty \cdot B \\ v &= o \wedge \infty \cdot \frac{B}{\lambda} \wedge o \wedge \infty \end{aligned}$$

8.2 Spheres

Spheres are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda \left(o + x + \frac{1}{2}(x^2 \pm r^2) \infty \right)$$

(Say something about imaginary spheres.) We may decompose this as follows.

$$\begin{aligned} \lambda &= -\infty \cdot B \\ x &= o \wedge \infty \cdot \frac{B}{\lambda} \wedge o \wedge \infty \\ r^2 &= x^2 + 2o \cdot \frac{B}{\lambda} \end{aligned}$$

Alternatively, we can find r^2 as simply the square of B^2/λ. Using what we know about spherical reflections, we can find x by reflecting ∞ into the sphere.

8.3 Planes

Planes are characterized by a Euclidean point (center, if you will) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(v + (x \cdot v)\infty)$$

If $T = 1 - \frac{1}{2}x\infty$, we may also formulate B as $Tv\tilde{T}$. We may decompose B as follows.

$$\begin{aligned} v &= o \cdot \frac{B}{\lambda} \wedge \infty \\ x &= -v\left(o \cdot \frac{B}{\lambda}\right) \end{aligned}$$

Notice here that any original weight, normal and position used in the composition of B are not recoverable in the decomposition of B. Here, x will be the point on the plane closest to the origin.

8.4 Circles

Circles are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(v + (x \cdot v)\infty) \wedge \left(o + x + \frac{1}{2}(x^2 \pm r^2)\infty\right)$$

(Say something about imaginary circles.) We may decompose this as follows.

$$\begin{aligned} v &= o \wedge \infty \cdot \frac{B}{\lambda} \wedge \infty \\ x &= v\left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge o\infty\right) \\ r^2 &= x^2 - 2v\left((x \cdot v)x - o \wedge \infty \cdot o \wedge \frac{B}{\lambda}\right) \end{aligned}$$

8.5 Lines

Lines are characterized by a Euclidean point (center, if you will) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda\left(vi - (x \cdot vi) \wedge \infty\right)$$

We may decompose this as follows.

$$\begin{aligned} v &= \left(o \cdot \frac{B}{\lambda} \wedge \infty\right) i \\ x &= -v\left(o \cdot \frac{B}{\lambda}\right) i \end{aligned}$$

8.6 Point-Pairs

Point-pairs are characterized by a Euclidean point (center) $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$, a non-zero radius $r \in \mathbb{R}$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(vi - (x \cdot vi) \wedge \infty) \wedge \left(o + x + \frac{1}{2}(x^2 \pm r^2)\infty\right)$$

(Say something about imaginary point-pairs.) We may decompose this as follows.

$$\begin{aligned}
v &= -\left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge \infty\right) i \\
x &= -v\left(o \wedge \infty \cdot \frac{B}{\lambda} \wedge o\infty\right) i \\
r^2 &= -x^2 + 2v\left((x \cdot v)v + \left(o \wedge \infty \cdot o \wedge \frac{B}{\lambda}\right) i\right)
\end{aligned}$$

8.7 Flat Points

Flat-points are characterized by a Euclidean point $x \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$.

$$B = \lambda(i + xi \wedge \infty)$$

We may decompose this as follows.

$$\begin{aligned}
\lambda &= -(B \wedge \infty)i \\
x &= \left(o \cdot \frac{B}{\lambda}\right) i
\end{aligned}$$

8.8 Tangent Points

A tangent-point is characterized by a Euclidean point $x \in \mathbb{V}^n$, a unit-normal $v \in \mathbb{V}^n$ and a non-zero scalar (weight) $\lambda \in \mathbb{R}$. Dual canonical forms of tangent points for grades 2 and 3 are given by the dual canonical forms of circles and point-pairs, respectively, with a radius r of zero. For example, given any $r > 0$, simplyfing the following equation recovers the dual form of a tangent point for grade 2.

$$B = \lambda(v + (x \cdot v)\infty) \wedge \left(o + x - rv + \frac{1}{2}((x - rv)^2 - r^2)\infty\right),$$

The reader will notice that r cancels itself out. The decomposition steps for tangent points are the same as those given for circles and point-pairs. The recovered radius will be zero in the case of tangent points.

8.9 Free Blades

Address free-blades here.

8.10 Rotate-Translate Transformations

Such a transformation is characterized by a Euclidean translation vector $t \in \mathbb{V}^n$, a unit-axis $a \in \mathbb{V}^n$, an angle $\theta \in \mathbb{R}$ and a scalar (weight) $\lambda \in \mathbb{R}$.

$$V = \lambda \left(1 - \frac{1}{2} t \infty\right) \left(\cos \frac{\theta}{2} - ai \sin \frac{\theta}{2}\right),$$

Notice that V here is not a blade. It is an even versor. If the blade $B \in \mathbb{G}$ represents a geometry, (directyl or dually), the transformation of B by V is given by VBV^{-1}, in the case that we wish to the apply the rotation first, then the translation. We may decompose this type of transformation as follows.

$$\begin{aligned}
\lambda^2 &= V\tilde{V} \\
R &= -o \cdot \frac{V}{\lambda} \wedge \infty \\
T &= \frac{V}{\lambda} \tilde{R} \\
\theta &= 2 \cos^{-1} \langle R \rangle_0 \\
a &= \frac{1}{\sin(\theta/2)} \langle R \rangle_2 \, i \\
t &= 2o \cdot (1 - T)
\end{aligned}$$

(Say something about the polar decomposition of V.)

9 Concluding Remarks

This introduction has only just begun to scratched the surface of what kinds of geometry we can do with the conformal model, and any model like it that uses a homogenous blade representation scheme, such as the model for projective geometry. It is reasonable to ask what other models of geometry there might be that use different types of geometric algebras. Instead of going in search of a model with specific features, we might as well just choose a geometric algebra, and then ask what types of geometry we can do with that algebra under various models of geometry.

References

[1] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra For Computer Science*. Morgan Kaufmann, 2007.

[2] David Hestenes. *New Foundations For Classical Machanics*. Kluwer Academic Publishing, 1987.