Examination of Definition 5.2

I assume that a function F defined on open set in $\mathbb{R}^n$ that is multi-vector valued has the property that for any $x \in \mathbb{R}^n$, we have $F(x) \in \mathbb{G}(\mathbb{R}^n)$, where $\mathbb{G}(\mathbb{R}^n)$ is the geometric algebra generated by $\mathbb{R}^n$ when thought of as a vector space. Is this right? We have to be able to relate the input space and the output space of F for definition 5.2 to make sense, because we are combining each e_i in some product with elements in the image of $\partial_i F(x)$.

For any integer $i \in [1, n]$, to understand definition 5.2, we must understand the meaning of $e_i \partial_i F(x)$, where $x = \sum_{i=1}^n x_i e_i$ and each $x_i \in \mathbb{R}$. If I understand correctly, we have

$$\partial_i F(x) = \lim_{\epsilon \to 0} \frac{F(x + \epsilon e_i) - F(x)}{\epsilon},$$

which, like F, is also a multivector-valued function defined on an open set in $\mathbb{R}^n$. It follows that $\nabla F(x)$, as defined in definition 5.2, may be written as

$$\begin{aligned}
\nabla F(x) &= \sum_{i=1}^n e_i \partial_i F(x) \\
&= \sum_{i=1}^n (e_i \cdot \partial_i F(x) + e_i \wedge \partial_i F(x)) \\
&= \sum_{i=1}^n e_i \cdot \partial_i F(x) + \sum_{i=1}^n e_i \wedge \partial_i F(x),
\end{aligned}$$

if I am not mistaken in my understanding that the product between each e_i and $\partial_i F(x)$ in definition 5.2 is the geometric product. If we then define, (and we need a definition here because I do not believe it follows from anything we have done thus far), $\nabla \cdot F(x) = \sum_{i=1}^n e_i \cdot \partial_i F(x)$ and $\nabla \wedge F(x) = \sum_{i=1}^n e_i \wedge \partial_i F(x)$, then we may write

$$\nabla F(x) = \nabla \cdot F(x) + \nabla \wedge F(x),$$

which is my first experience seeing the operator ∇ behave as a vector. (Does it behave generally like a vector? That remains to be seen.) But I do not really know how $\nabla \cdot F(x)$ and $\nabla \wedge F(x)$ are defined, or if they can be inferred from more fundamental definitions and results. I would think not, since the combining of an operator with a function in one of the inner or outer products is something that must be defined first to make any sense.

I can see that for all integers $i \in [1, n]$, if $\partial_i F(x)$ is a scalar, then $\nabla F(x)$ needs not be thought of as behaving like a vector, because it is a vector! But this is not always the case.

If I'm not off my rocker yet, the linearity of the ∇ operator follows from the linearity of the ∂_i operator and the distributivity of the geometric product over addition.

Examination of Exercise 5.3a

What does Alan mean by $\nabla x_i = e_i$? If we let $x = \sum_{i=1}^n x_i e_i$, then perhaps he means $\nabla F(x)$, where

$$F(x) = \sum_{i=1}^{n} x \cdot e_i = \sum_{i=1}^{n} x_i.$$

Applying definition 5.2, we then have

$$\nabla F(x) = \sum_{i=1}^{n} e_i \partial_i F(x) = \sum_{i=1}^{n} e_i \partial_i \sum_{j=1}^{n} x_j = \sum_{i=1}^{n} e_i.$$

If, on the other hand, Alan means $\nabla F(x)$, where $F(x) = x \cdot e_i = x_i$ for some fixed integer $i \in [1, n]$, then

$$\nabla F(x) = \sum_{j=1}^{n} e_j \partial_j x_i = e_i.$$

I believe that either interpretation is okay, since he's using an implicit summation notation.

Examination of Exercise 5.3b

What in the world does Alan mean by $e_i \cdot \nabla$?! How do we take a vector and an operator in the inner product? This just can't make sense, because the inner product is not defined over a space that includes operators. It does make sense, however, to write

$$e_i \cdot \nabla F(x) = \sum_{j=1}^{n} e_i \cdot \left(e_j \partial_j F(x)\right).$$

Now, if for all integers $j \in [1, n]$, we have $\partial_j F(x) \in \mathbb{R}$, then it is easy to see that

$$e_i \cdot \nabla F(x) = \sum_{j=1}^{n} (e_i \cdot e_j) \partial_j F(x) = \partial_i F(x).$$

But again, in general, we do not have $\partial_j F(x)$ as a scalar-valued function, so we would have to be very careful, I believe, to lazily rewrite $e_i \cdot \nabla F(x) = \partial_i F(x)$ as $e_i \cdot \nabla = \partial_i$. For example, let's suppose that $\partial_j F(x)$ was always a vector valued function. We then have

$$\begin{aligned}
e_i \cdot \nabla F(x) &= \sum_{j=1}^{n} e_i \cdot (e_j \cdot \partial_j F(x) + e_j \wedge \partial_j F(x)) \\
&= \sum_{j=1}^{n} (e_j \cdot \partial_j F(x)) e_i + \sum_{j=1}^{n} (e_i \cdot e_j) \partial_j F(x) - \sum_{j=1}^{n} (e_i \cdot \partial_j F(x)) e_j \\
&= \partial_i F(x) + \sum_{j=1}^{n} ((e_j \cdot \partial_j F(x)) e_i - (e_i \cdot \partial_j F(x)) e_j) \\
&= \partial_i F(x) + \sum_{j=1}^{n} \partial_j F(x) \cdot (e_j \wedge e_i).
\end{aligned}$$

In this case we can only claim that $e_i \cdot \nabla F(x) = \partial_i F(x)$ when for all $i \neq j$, we have $\partial_j F(x)$ as zero or as some vector orthogonal to both e_i and e_j. What does this say about F? I'm not sure without further analysis. In any case, I am not convinced that an identity such as $e_i \cdot \nabla = \partial_i$, if we can write it that way, holds generally.

Misc. Stuff

Let's suppose $F(x) = x = \sum_{i=1}^{n} x_i e_i$ is the identity function. What is ∇F in this case?

Well, let's start by showing that

$$\partial_i F(x) = \lim_{\epsilon \to 0} \frac{(x_i + \epsilon) e_i - x_i e_i}{\epsilon} = e_i,$$

which, admittedly, can be done much faster using rules of differentiation. We then have

$$\nabla F(x) = \sum_{i=1}^{n} e_i \partial_i F(x) = \sum_{i=1}^{n} e_i e_i = \sum_{i=1}^{n} 1 = n.$$