A Transformational Result Of The Quadric Model

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Abstract. An important feature of the conformal model is found to be possessed by the quadric model. Specifically, it is shown in this paper that the action of a versor on a point reveals the action of this versor on any geometry of the model. This leads to a possible direction in which we might look for a better model of quadric surfaces.

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1. Introduction

It is well known that, with the exceptoin of flat points, all geometries of the conformal model may be expressed directly as the outer product of one or more vectors representative of points. It then immediately follows by the outermorphic property of versor conjugation, (see equation (2.1) below), that the action of a versor on a conformal point reveals the action of this versor on any point on the surface of a conformal geometry, and therefore the conformal geometry itself. This feature of the conformal model facilitates the search for versors performing desired actions, and the analysis of what action a given versor performs. In this paper we will find that the quadric model possess its own version of this very feature. We'll also find that this feature may help point us in a direction of where we might look for a better model of quadric geometry.

The definitions and results of [2] will be assumed for the remainder of this paper so that no background will need be given before we can dive into the new material.

2. The Versor Identities Of Geometric Algebra

There are two well known identities in geometric algebra involving versors. For any two vectors $p, b \in \mathbb{W}$ and any versor $V \in \mathbb{G}$, we have

$$V(p \wedge b)V^{-1} = VpV^{-1} \wedge VbV^{-1}, \tag{2.1}$$

as well as

$$p \cdot b = VpV^{-1} \cdot VbV^{-1}. \tag{2.2}$$

Proofs of these identities may be found in [1]. A perhaps less known identity, however, is the following.

$$VpV^{-1} \cdot b = p \cdot V^{-1}bV \tag{2.3}$$

Let us give a proof of it now. We will proceed by induction. Letting $v \in \mathbb{W}$ be a vector, it is easy to see that

$$vpv^{-1} \cdot b = \frac{2(v \cdot p)(v \cdot b)}{v^2} - p \cdot b = p \cdot v^{-1}bv. \tag{2.4}$$

Assuming now that the identity (2.3) holds for a versor composed as the geometric product of some fixed number of vectors, the proof of identity (2.3) follows by induction with

$$\begin{aligned}
&vVp(vV)^{-1} \cdot b && && (2.5)\\
&= vVpV^{-1}v^{-1} \cdot b && && (2.6)\\
&= VpV^{-1} \cdot v^{-1}bv && \text{by equation (2.4),} && (2.7)\\
&= p \cdot V^{-1}v^{-1}bvV && \text{by our inductive hypothesis,} && (2.8)\\
&= p \cdot (vV)^{-1}bvV. && && (2.9)
\end{aligned}$$

In the next section we'll make use of identity (2.3) as well as (2.1) to prove the main result. The identity (2.2) has many use cases while working in $\mathbb{G}$ of [2], but will not be needed to prove the main result. Looking back, however, it is not hard to see that (2.2) implies (2.3) in an easier proof than what has just been given.

3. Relating The Action Of Versors On Quadrics To That Of Points

It was established in [2] that quadrics $E \in \mathbb{G}$ are bivectors of the form

$$E = \sum_{i=1}^{k} a_i \wedge \overline{b_i} \tag{3.1}$$

where for each integer $i \in [1, k]$, each of a_i and b_i are taken from $\mathbb{V}$. Being a quadric, the set of all projective points $p \in \mathbb{V}$ on E is given by the set of all projective points $p \in \mathbb{V}$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{3.2}$$

Clearly now, if we can visualize the quadric E, and if we can understand the action of a versor V on a projective point p, then our imaginations are likely able to visualize the geometry that is the set of all projective points $p \in \mathbb{V}$ such that

$$0 = VpV^{-1} \wedge \overline{VpV^{-1}} \cdot E. \tag{3.3}$$

For example, if V translates p by a direction vector t, then E must be translated by the direction vector $-t$. Similarly, if V rotates p on an axis a by an angle θ, then E must be rotated by an angle $-\theta$ about the axis a. Of course, no claim is being made here that either of such versor exists. (A translation versor is not known to exist, but it has been shown in [2] that the rotation versor does exist.) The idea, however, that the action of V on p translates into the inverse action of V on E, should be well understood.

We will now proceed to show that the geometry represented in equation (3.3) is the very geometry represented by the quadric $(V\overline{V})^{-1}EV\overline{V}$, provided that V has the property that for all vectors $v \in \mathbb{V}$, we have

$$\begin{aligned} &v = \overline{V}v\overline{V^{-1}}, \\ &\overline{v} = V\overline{v}V^{-1}, \quad \text{(which follows from } v = \overline{V}v\overline{V^{-1}}), \end{aligned} \tag{3.4}$$

as well as

$$\begin{aligned} &VvV^{-1} \in \mathbb{V}, \\ &\overline{V}\overline{v}\overline{V^{-1}} \in \overline{\mathbb{V}}, \quad \text{(which follows from } VvV^{-1} \in \mathbb{V}), \end{aligned} \tag{3.5}$$

which is to say that V leaves vectors in $\overline{\mathbb{V}}$ invariant under versor conjugation as $\overline{V}$ leaves vectors in $\mathbb{V}$ invariant under versor conjugation, as well as that conjugation of a vector in $\mathbb{V}$ by the versor V is an operation closed in $\mathbb{V}$. It will then immediately follow that if we understand the action of V^{-1} on a projective point p, then we understand the action of V on E as

$$V\overline{V}E(V\overline{V})^{-1}. \tag{3.6}$$

The proof is straight forward as it follows from the equality of (3.7) with (3.14).

$$VpV^{-1} \wedge \overline{VpV^{-1}} \cdot E \tag{3.7}$$

$$= \sum_{i=1}^{k} VpV^{-1} \wedge \overline{VpV^{1}} \cdot a_i \wedge \overline{b_i} \tag{3.8}$$

$$= -\sum_{i=1}^{k} (VpV^{-1} \cdot a_i)(\overline{VpV^{-1}} \cdot \overline{b_i}) \qquad \text{by property (3.5),} \tag{3.9}$$

$$= -\sum_{i=1}^{k} (p \cdot V^{-1}a_iV)(\overline{p} \cdot \overline{V^{-1}b_iV}) \qquad \text{by identity (2.3),} \tag{3.10}$$

$$= \sum_{i=1}^{k} p \wedge \overline{p} \cdot V^{-1}a_iV \wedge \overline{V^{-1}b_iV} \qquad \text{by property (3.5),} \tag{3.11}$$

$$= \sum_{i=1}^{k} p \wedge \overline{p} \cdot (V\overline{V})^{-1} a_i V\overline{V} \wedge (V\overline{V})^{-1} \overline{b_i} V\overline{V} \quad \text{by property (3.4)}, \tag{3.12}$$

$$= \sum_{i=1}^{k} p \wedge \overline{p} \cdot (V\overline{V})^{-1} (a_i \wedge \overline{b_i}) V\overline{V} \quad \text{by identity (2.1)}, \tag{3.13}$$

$$= p \wedge \overline{p} \cdot (V\overline{V})^{-1} E V\overline{V}. \tag{3.14}$$

Of course, this is just one of perhaps many algebraic routes one could take to prove the identity that (3.7) is (3.14). In fact, it is not hard to see that a shorter route can be found from (3.8) to (3.12) using only (2.2) and (3.4). Nevertheless, the route shown above illustrates algebraic techniques that are useful as their need is frequently encountered.

The property (3.4) is not unreasonable at all since a versor providing any action on a projective point $p \in \mathbb{V}$ must come from $\mathbb{G}(\mathbb{V})$ anyway, and by so doing, naturally leaves vectors in $\overline{\mathbb{V}}$ untouched, up to scale. In fact, the condition of (3.4) may be relaxed to allow for a sign change, as such a change leaves the geometry represented by a bivector invariant.

4. The Search For A Better Model

The main result now given, we have a tool that we can use to discover or derive the action of a versor upon an element of the form (3.1) representative of some piece of geometry by equation (3.2). As noted earlier, [2] shows that such elements may be representative of quadric surfaces; but that there appears to be a limited number of versors in $\mathbb{G}$ for desired transformations suggests, among other evidences, that there must be a better model based upon geometric algebra, or perhaps some other algebra, that is capable of both representing the quadrics and offering a wider array of operations we can perform on such geometries.

Where we might start looking for a better model is in a re-examination of how we're using vectors to represent points.[1] Experiences to date suggest that there is not much of interest we can do with versors acting upon vectors representing points in classical homogenous space. That is, a projective point $p \in \mathbb{V}$ of the form

$$p = \lambda(e_0 + \vec{p}), \tag{4.1}$$

where $\lambda \in \mathbb{R}$ is a non-zero scalar, and $\vec{p} \in \mathbb{V}$ is a non-projective point or direction vector. Reflections and therefore rotations about the origin are one

[1]Note that the way spoken of here by which such vectors are representative of points is different than how elements of the form (3.1) represent quadrics by the definition (3.2). One must not confuse the two interpretations. Note that there are elements of the form (3.1) representative of points, but, unlike the conformal model, these are not used in definition (3.2). That the points of the conformal model appear in the definition of any conformal geometry, (which includes conformal points themselves), is one of many features contributing to how interesting and pleasing the results of the conformal model are.

immediately observable type of transformation of interest that we can perform on such points of the form (4.1) using a versor. If there is a versor that reflects such points about an arbitrary plane, which would be highly desirable as it would lead to versors representing the rigid body motions, it is not obvious.

We'll therefore close with the following question with some consideration attached. Assuming that a better model for quadric surfaces still makes use of the form (3.1) and the definition (3.2) in some geometric algebra other than $\mathbb{G}$ in [2], how might we maintain the ability of this form and definition, collectively, to allow for the representation of all quadrics while altering the form (4.1), if possible, to lend itself to a wider variety of transformations by versor conjugation? As we've shown in the main result, if a better form than (4.1) can be found to allow for more transformations, then the quadrics of the form (3.1) will become endowed with the ability to be transformed by this larger set of transformations.

To give an example, it is possible to extend $\mathbb{G}$ to a geometric algebra having two isomorphic sub-algebras that are Minkowski algebras. There is more than one way to do it. The motivation for our choice here will be seen in the outcome.

We start by changing the signature of e_0 and e_{n+1} to be anti-Euclidean. That is, let $-1 = e_0 \cdot e_0 = e_{n+1} \cdot e_{n+1}$. We then must change S as it is defined in [2] to

$$S = 2^{-(n+1)/2}(1 + e_0 e_{n+1}) \prod_{i=1}^{n} (1 - e_i e_{i+n+1}). \tag{4.2}$$

We have now preserved the property that for all integers $i \in [0, n]$, we have $\overline{e_i} = e_{i+n+1}$ and $\overline{e_{i+n+1}} = e_i$. We have also maintained the integrity of equation (3.2) in that it still expands as a homogeneous polynomial of degree 2 in the vector components of p with the coeficients being found in E.

For any integer $i \in [0, n]$, we now introduce the null-vectors u_i and v_i having the relationship $-1 = u_i \cdot v_i$. Then, for any integer $i \in [0, n]$, we have

$$e_i = \begin{cases} 2^{-1/2}(u_i - u_i) & \text{if } e_i \text{ is Euclidean,} \\ -2^{-1/2}(u_i + u_i) & \text{if } e_i \text{ is anti-Euclidean.} \end{cases} \tag{4.3}$$

The form (4.1) can now be changed to

$$p = \lambda(u_0 - \sqrt{2}v + v_0). \tag{4.4}$$

References

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