THE QUADRIC MODEL OF GEOMETRIC ALGEBRA

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ABSTRACT. Inspired by the conformal model of geometric algebra, a model
of geometry is developed in this paper capable of representing any quadric
surface.

1. FINDING THE QUADRIC EQUATION

Taking our cue from [1], the n-dimensional quadric surfaces may be characterized
as the set of all projective points in an $(n + 1)$-dimensional homogeneous space
satisfying a matrix equation involving a symmetric matrix. We will let $\mathbb{V}^{n+1}$ be an
$(n + 1)$-dimensional vector space and identify vectors in this space with projective
points of n-dimensional space in the usual manner. That is, letting $\{e_i\}_{i=0}^n$ be an
orthonormal basis for $\mathbb{V}^{n+1}$, we identify the n-dimensional point represented by any
$p \in \mathbb{V}^{n+1}$ as the point $p/(p \cdot e_0)$ in the $e_0 = 1$ plane, provided $p \cdot e_0 \neq 0$.

Letting $\{\alpha_{ij}\} \subset \mathbb{R}$ with $0 \leq i \leq j \leq n$ be the scalar elements of a symmetric
matrix M, an n-dimensional quadric surface is the projective solution set to the
matrix equation

$$0 = pMp^{\mathsf{T}}, \tag{1.1}$$

where here we have abused notation by interpreting the vector p taken from $\mathbb{V}^{n+1}$
as a row-vector with p^{T} as the corresponding column-vector, and where M is the
$(n + 1) \times (n + 1)$ symmetric matrix given by

$$\begin{bmatrix} \alpha_{00} & \alpha_{01} & \dots & \alpha_{0n} \\ \alpha_{01} & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{0n} & \alpha_{1n} & \dots & \alpha_{nn} \end{bmatrix}. \tag{1.2}$$

Written another way without abusing notation, we have

$$0 = \sum_{i=0}^{n} \sum_{j=i}^{n} \sigma_{ij} \alpha_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.3}$$

where σ_{ij} is defined as

$$\sigma_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } i \neq j. \end{cases} \tag{1.4}$$

The form (1.1) lends itself to the study of quadrics using matrix algebra, while
the form (1.3) may be thought of as a low-level form of the equation in geometric
algebra. What we might think of as a high-level form in geometric algebra, coming

from a framework of computation, may provide a better means of studying quadrics using geometric algebra. We proceed now to develop such a form.

Let $\mathbb{W}^{2(n+1)}$ denote a $2(n+1)$-dimensional vector space having $\{e_i\}_{i=0}^{2n+1}$ as a set of orthonormal basis vectors generating it. The vector space $\mathbb{V}^{n+1}$ is therefore a vector sub-space of $\mathbb{W}^{2(n+1)}$ and we will let $\overline{\mathbb{V}}^{n+1}$ denote the $(n+1)$-dimensional vector sub-space of $\mathbb{W}^{2(n+1)}$ that is complement to $\mathbb{V}^{n+1}$. It is then helpful to introduce the notation $\bar{p}$ as the vector in $\overline{\mathbb{V}}^{n+1}$ related to the vector $p \in \mathbb{V}^{n+1}$ by the equation

$$(1.5) \qquad\qquad\qquad\qquad \bar{p} = Rp\tilde{R},$$

where R is a rotor defined as

$$(1.6) \qquad\qquad\qquad R = 2^{-n/2} \prod_{i=0}^{n} \left(1 - e_i e_{i+n+1}\right).$$

This idea comes from [2], and it is easy to see that for any integer $i \in [0, n]$, we have $\bar{e}_i = e_{i+n+1}$ and $\bar{e}_{i+n+1} = e_i$. Notice that the over-bar operator is also an outermorphic function that we may apply to any element of the geometric algebra $\mathbb{G}(\mathbb{W}^{2(n+1)})$.

We are now ready to give the high-level form of equation (1.3) as[1]

$$(1.7) \qquad\qquad\qquad\qquad 0 = p \wedge \bar{p} \cdot B,$$

where $B \in \mathbb{G}(\mathbb{W}^{2(n+1)})$ is a bivector of the form

$$(1.8) \qquad\qquad B = -\frac{1}{2} \sum_{i=0}^{n} \sum_{j=i}^{n} \alpha_{ij}(e_i \bar{e}_j + (-1)^{\sigma_{ij}} \bar{e}_i e_j).$$

Here, as in the form (1.1) where we may think of the symmetric matrix as representative of the quadric, the bivector B may also be thought of as a quadric representative.

Realizing that we need to be careful, because the inner product is not associative, it is interesting to write equation (1.7) in a form similar to that of equation (1.1). Doing so, we get

$$(1.9) \qquad\qquad\qquad\qquad 0 = p \cdot B \cdot \bar{p}.$$

We can get away with this, because the choice of associativity here only changes the sign of the right-hand side, and the sign of the left-hand side clearly doesn't matter. Considering $\bar{p}$ a type of conjugate to p, we may refer to equation (1.9) as the inner product conjugation of B by p.

2. Using The Quadric Equation

Having developed the quadric equation (1.7) in geometric algebra, we can now benefit from the language of geometric algebra in using it to answer questions about quadric geometry.

Notice that in our model we can make a distinction between members of $\mathbb{V}^{n+1}$ that are representative of points and those representative of directions. Specifically, a vector $v \in \mathbb{V}^{n+1}$ is a direction if and only if $v \cdot e_0 = 0$. Although we will use an

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[1]Here and throughout this paper, we assume that the outer product takes precedence over the inner product. We also assume that the geometric product takes precedence over the inner and outer products.

arrow accent to distinguish between direction vectors and position vectors, there should be no confusion on the form of a vector and what we intend it to represent when we refer to it as a direction or a position. Similarly, we will take the liberty of referring to bivectors taken from $\mathbb{G}(\mathbb{V}^{2(n+1)})$ as quadrics. This helps eliminate phrases that would otherwise sound a bit too pedantic.

2.1. **Characterizing Flat Quadrics.** Letting $f : \mathbb{V}^{n+1} \to \mathbb{R}$ be the function defined as

$$(2.1) \qquad f(x) = x \wedge \overline{x} \cdot B,$$

we arrive at our first result.

Lemma 2.1. *Given any quadric B, if for all pairs of homogenized points $p, x \in \mathbb{V}^{n+1}$ on B we have $f(x - p) = 0$, then B is a linear (flat) quadric.*

Proof. For any pair of homogenized points $p, x \in \mathbb{V}^{n+1}$, there is a direction $\vec{x} \in \mathbb{V}^{n+1}$ such that $x = p + \vec{x}$. We then find that

$$(2.2) \qquad f(x) = f(p + \vec{x}) = f(p) + \nabla_{\vec{x}} f(p) + f(x - p) = \nabla_{\vec{x}} f(p),$$

in the case that p is on B, where $\nabla_{\vec{x}} f(p)$ is the directional derivative of f at p in the direction of $\vec{x}$. It follows that the tangent space of any point on the quadric is also in the quadric. The quadric is therefore flat at any point upon its surface. To see this, consider all such pairs of points p, x where both are on B. Both are on B and both are in the tangent space of B at p. $\qquad \square$

Recalling that for any $x \in \mathbb{V}^{n+1}$, the definition of $\nabla f(x)$ is given by

$$(2.3) \qquad \nabla f(x) = \sum_{i=0}^{n} e_i \nabla_{e_i} f(x),$$

it is not hard to show that for any vector $y \in \mathbb{V}^{n+1}$, we have $y \cdot \nabla f(x) = \nabla_y f(x)$. Seeing that $\vec{x} \cdot \nabla f(p) = \nabla_{\vec{x}} f(p)$ in the light of Lemma 2.1, it follows that the direction $e_0 \cdot e_0 \wedge \nabla f(p)$ is normal to the surface of the quadric B at p.

2.2. **Quadric 2-Blades.** All quadrics are bivectors, but not all bivectors are 2-blades. Here we study the class of quadrics that are 2-blades. For any four points $a, b, c, d \in \mathbb{V}^{n+1}$, such a quadric B has the form

$$(2.4) \qquad B = (a + \overline{b}) \wedge (c + \overline{d})$$

$$(2.5) \qquad = a \wedge c + a \wedge \overline{d} + \overline{b} \wedge c + \overline{b \wedge d}.$$

It is curious to think what geometric significance the quadric B has in relation to these four points. Whatever the case may be, it is clear from equation (2.5) that the quadric B contains the intersection, if any, of the four quadrics appearing in the sum. Considering the three forms of 2-blades found in the expansion of equation (2.4) to be more fundamental, (namely, $a \wedge c$, $\overline{b \wedge d}$ and the identical forms $a \wedge \overline{d}$ and $-c \wedge \overline{b}$), we'll start with a treatment of each of these forms.

We first notice that the quadrics of the form $a \wedge c$ and $\overline{b \wedge d}$ trivially represent the quadric of all space. They therefore contribute nothing to the shape of B. The remaining form $a \wedge \overline{d}$, therefore, deserves our full attention. We break this form into two cases, the first being the case when $a = d$, and the second when $a \neq d$.

In the first case, a quick application of equation (1.7) reveals the type of quadric represented by $a \wedge \overline{a}$. Doing so, we see that it represents the set of all projective points $p \in \mathbb{V}^{n+1}$ such that

$$(2.6) \qquad\qquad 0 = p \wedge \overline{p} \cdot a \wedge \overline{a} = -(p \cdot a)^2,$$

which holds if and only if $p \cdot a = 0$. Letting $p = e_0 + \vec{p}$ and $a = e_0 + \vec{a}$, our equation becomes

$$(2.7) \qquad\qquad -1 = \vec{p} \cdot \vec{a} = \langle \vec{p}\vec{a} \rangle_0,$$

where it is now clear that $a \wedge \overline{a}$ is the solitary point $e_0 - \vec{a}^{-1}$ in homogenized form. What is perhaps unfortunate is that this also means that the element $p \wedge \overline{p}$ does not represent the projective point p as we might like to think of it in the context of equation (1.7). That is, while $0 = p \wedge \overline{p} \cdot B$ implies that the projective point p is on B, it does not imply that the point $p \wedge \overline{p}$ is on B under the definition implied by equation (1.7) of what $p \wedge \overline{p}$ represents. We therefore have to be careful with our manner of speech when we refer to any element as representing any particular geometry. The definition under which that element is representative of a geometry must always be known.

In the second case, our use of equation (1.7) barrows from what we have just learned in the first case.

$$(2.8) \qquad\qquad 0 = p \wedge \overline{p} \cdot a \wedge \overline{d} = -(p \cdot a)(p \cdot d)$$

Clearly $a \wedge \overline{d}$ represents the pair of points $e_0 - \vec{a}^{-1}$ and $e_0 - \vec{d}^{-1}$ if we let $a = e_0 + \vec{a}$ and $d = e_0 + \vec{d}$.

Returning to the general quadric 2-blade (2.4), to understand this general form, we must come to some understanding of...

2.3. **Exploiting The Entire Algebra.** It should be observed that a bivector is not the sole form of an element of our geometric algebra that is capable of representing a quadric. Indeed, any element of the algebra can represent a quadric, because all of the necessary mixing of coordinates is done in the $p \wedge \overline{p}$ calculation, while the quadric representative B, not necessarily a bivector, contains nothing more than the coefficients characterizing the quadric. A formal definition of the desired algebraic variety is now warranted.

Definition 2.2. For any element $E \in \mathbb{G}(\mathbb{V}^{2(n+1)})$, this element is representative of the geometry that is the set of all projective points $p \in \mathbb{V}^{n+1}$ such that

$$(2.9) \qquad\qquad 0 = p \wedge \overline{p} \cdot E.$$

From the beginning of this paper we have been using the version of Definition 2.2 where E is strictly a bivector. We will now consider some cases where E is not a bivector.

The first consideration is naturally that of vectors. Letting $a, b \in \mathbb{V}^{n+1}$ be any two vectors, we see that

$$(2.10) \qquad\qquad p \wedge \overline{p} \cdot (a + \overline{b}) = (p \cdot b)p - (p \cdot a)\overline{p}.$$

Recalling that if p is a projective point, then $p \neq 0$, it is clear that p is on $a + \bar{b}$ if and only if $p \cdot a = 0$ and $p \cdot b = 0$, which is impossible unless $a = b$. Therefore, if $a = b$, we can say that $a + \bar{b}$ is a point.[2]

General trivectors are as formidable an object of study as bivectors themselves. Let us therefore consider cases of trivectors that we can relate back to what we already know about bivectors.

3. Concluding Remarks

Admittedly, more questions have been brought up than answered in this paper. It also appears that while the equation (1.7) works as an alternative to the matrix equation (1.1), it may prove to be nothing more than a curious mathematical construction. If there are any new insights to be gained about quadrics using geometric algebra, I have no idea if this model can be helpful in that regard.

References

1. *Quadric*, `http://en.wikipedia.org/wiki/Quadric`.
2. C. Doran and D. Hestenes, *Lie groups as spin groups*, J. Math. Phys. **34** (1993), 8.

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[2]Notice that we must be careful here in referring to a vector as any geometry. For example, it is common for us to think of a vector $p \in \mathbb{V}^{n+1}$ as a projective point, but under Definition 2.2, p represents the projective point $e_0 - (e_0 \cdot (e_0 \wedge p))^{-1}$.