Imaginary Rounds

Spencer T. Parkin

September 19, 2012

Letting B be a blade dually representative of any geometry, I believe that we can all agree that B represents an imaginary geometry if and only if the following set is empty.

$$\{x \in \mathbb{V}^n | p(x) \cdot B = 0\} \tag{1}$$

Now let $\sigma(\lambda)$ be a function defined as follows, with $r \neq 0$.

$$\sigma(\lambda) = o - \lambda \frac{1}{2} r^2 \infty$$

It is not hard to prove that $\sigma(\lambda)$ dually represents a real sphere of radius r whenever $\lambda = 1$ and an imaginary sphere of radius r whenever $\lambda = -1$. It follows that $\sigma^2(\lambda) = \lambda r^2$ may be used as a test to determine if $\sigma(\lambda)$ is real or imaginary.

$$\begin{aligned} \sigma^2(\lambda) < 0 &\implies \sigma(\lambda) \text{ is imaginary.} \\ \sigma^2(\lambda) > 0 &\implies \sigma(\lambda) \text{ is real.} \end{aligned}$$

Noticing that any origin-centered round of lower dimension may be dually represented by

$$B = \sigma(\lambda) \prod_{k=1}^{m} v_k,$$

where $\{v_k\}_{k=1}^m$ is a sequence of m pair-wise orthogonal unit-vectors taken from $\mathbb{V}^n$, how might the test be generalized? Our first task is to determine when B is imaginary by the definition established in (1). Imaginary or not, we need only consider a point $x \in \mathbb{V}^n$ on the sphere represented by B as we would draw B on paper. (All other points are clearly not on B.) It is then clear that for all integers $k \in [1, m]$, we have $p(x) \cdot v_k = 0$, showing that x is on each real dual $(n-1)$-dimensional hyper-plane v_k. It follows that $p(x) \cdot B = 0$ if and only if $p(x) \cdot \sigma(\lambda) = 0$. The test then generalizes as follows.

$$B\tilde{B} = \sigma^2(\lambda) < 0 \implies B \text{ is imaginary.}$$
$$B\tilde{B} = \sigma^2(\lambda) > 0 \implies B \text{ is real.}$$

Here, $\tilde{B}$ is the reverse of B.

Consider now the following identity, where $r > 0$ and v is a unit-length vector in $\mathbb{V}^n$.

$$\begin{aligned}
B &= \left(o - rv + \frac{1}{2}(-rv)^2\infty\right) \wedge \left(o + rv + \frac{1}{2}(rv)^2\infty\right) \\
&= o \wedge rv - rv \wedge o + \frac{1}{2}r^2 o \wedge \infty + \frac{1}{2}r^2 \infty \wedge o - \frac{1}{2}r^3 v \wedge \infty + \frac{1}{2}r^3 \infty \wedge v \\
&= 2ro \wedge v + r^3 \infty \wedge v \\
&= 2r(o + \frac{1}{2}r^2\infty) \wedge v
\end{aligned}$$

It is clear that the first part of this equation is a direct real point-pair. We will now show that the last part of this equation is a dual imaginary circle. According to our test, we calculate $B\tilde{B}$, and in doing so, get $-4r^4 < 0$.

Now, if we were to perform this test on BI, a dual of B, then our test would indicate the realness of BI as a dual point-pair. Indeed, $(BI)(IB) = -B\tilde{B} = 4r^4 > 0$.