An Intro to CGA

Conformal Geometric Algebra

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What is CGA?

$$\left\{ \begin{array}{c} \text{Synthetic} \\ \text{geometry} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Computational} \\ \text{geometry} \end{array} \right\}$$

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Invented by David Hestens in his paper, "Old Wine in New Bottles: A new algebraic framework for computational geometry."

Presentation Outline

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In this presentation, we will...

- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

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Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^{m} b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero m-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

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Clearly, if $B \neq 0$, then we must have $\text{grade}(B) = m \leq n$.

Imagine an infinite m-dimensional hyper-plane.

Visualizing Euclidean Blades

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Think of B as a finite m-dimensional hyper-plane.

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Non-Euclidean blades require more imagination!
Our geometric arguments will not require us to visualize the homogeneous representation space.

Blades May Represent Vector Sub-Spaces

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

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For any $v \in \mathbb{V}^n$, we say that

$$v \notin B \text{ if and only if } v \in B^*,$$

where B^* represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

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Notice that $v_{||} \in B$, but $v_{\perp} \notin B$.
For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_{||} + v_{\perp}) \wedge B = v_{\perp} \wedge B,$$
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We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the complement of B with respect to $\mathbb{V}^n$.

Membership in Vector Spaces and Dual Vector Spaces

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

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Proof.

The set $\{b_k\}_{k=1}^{m}$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^{m}$ is linearly dependent. $\square$

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Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. □

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Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

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$$\begin{aligned} & p(x) \in A \text{ or } p(x) \in B \\ \implies & p(x) \in A \wedge B \end{aligned}$$

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Why doesn't the converse hold?

We Can Intersect Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

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Proof.

$$\begin{aligned} & p(x) \in A^* \text{ and } p(x) \in B^* \\ \text{iff} \quad & p(x) \notin A \text{ and } p(x) \notin B \\ \text{iff} \quad & p(x) \notin A \wedge B \\ \text{iff} \quad & p(x) \in (A \wedge B)^* \end{aligned}$$

$\square$

Notice that $G^*(A \wedge B) = \emptyset$ if and only if $G(A \wedge B) \neq \emptyset$.

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Main point: We can't hope to cover all combinations of outer products between all various types and representations of geometry, but you must explore all of these to get full use out of the model!

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A direct real round is also a dual imaginary round having complementary dimension.
Main point: We can't hope to cover all combinations of outer products between all various types and representations of geometry, but you must explore all of these to get full use out of the model!
Application of imaginary intersection: Finding the contour of a sphere from a point.

Visualization Of Example

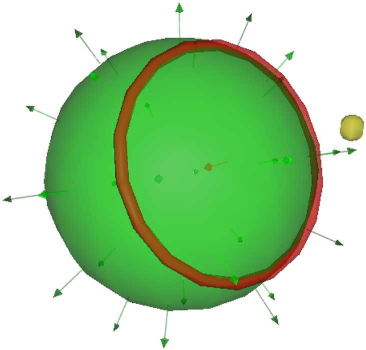

Figure : The contour of a circle from a point.

The Homogeneous Nature Of The Model

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For any blade B, there is a scalar λ such that λB is a homogenized form.
If B is the result of some geometric operation, then such a λ has geometric signficance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

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Notice that $v^{-1} = v/v^2$.

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The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

Versors form a group under the geometric product.

The Versor Group

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.
The scalar 1 is the identity versor.
For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

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Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

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Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

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Having invented this specific model, what we are now able to discover about it is almost endless!

Points in n-dimensional Space

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That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

The function $p(x)$ factors out of the equation

$$(x-c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

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Points are degenerate spheres, or spheres with radius zero.

n-dimensional Dual Hyper-Spheres

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Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

Let $\{\sigma_k\}_{k=1}^{m}$ be m spheres of dimension n having a non-empty and non-degenerate intersection.

Generating All Dual Rounds Of CGA

Let $\{\sigma_k\}_{k=1}^m$ be m spheres of dimension n having a non-empty and non-degenerate intersection.
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dually represents an $(n - m + 1)$-dimensional hyper-sphere.

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Rounds with zero radius give us tangent points!

All Rounds Of CGA For 3-dimensional Space

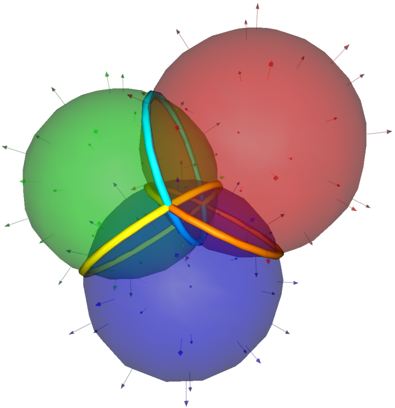

Figure : 3 Rounds, 3 Circles and 1 Point-Pair

$(n-1)$-dimensional Dual Hyper-Planes

The function $p(x)$ factors out of the equation

$$(x-c)\cdot v = 0$$

as the alternative equation

$$p(x)\cdot(v+(c\cdot v)\infty) = 0.$$

Let $\{\pi_k\}_{k=1}^m$ be m planes of dimension $n-1$ having a non-empty and non-degenerate intersection.

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Flats at infinity are free blades.
0-dimensional flats are called flat points.

All Flats Of CGA For 3-dimensional Space

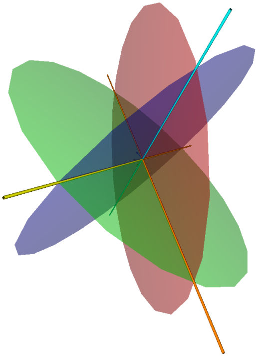

Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of points are co-m-hyper-planar if...

For $m = 0$, the points are identical,
For $m = 1$, the points are collinear,
For $m = 2$, the points are coplanar,
For $m = 3$, the points are co-hyper-planar,
etc...

A Condition For Linear Independents Of Points

Lemma

For $m \geq 1$, if $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

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The proof is not hard, but too big for this slide.

Generating All Direct Rounds Of CGA

Let $m \geq 1$. For $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

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Proof.

Let the $(n-m+1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^{m}$ is a linearly independent set. Lastly, $\text{grade}(B) = m+1 = n+2-(n-m+1) = n+2-\text{grade}(A) = \text{grade}(A^*)$. □

A Generalization Of Cospherical

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For $m \geq 1$, a set of points are co-m-hyper-spherical if...

For $m = 1$, the points are co-point-pair,
For $m = 2$, the points are co-circular,
For $m = 3$, the points are co-spherical,
For $m = 4$, the points are co-hyper-spherical,
etc...

Generating Almost All Direct Flats Of CGA

Let $m \geq 1$. For $m+2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

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Proof.

By (1), there exists the $(n-m)$-blade A dually representative of the m-dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\text{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. □

Generating All Direct Flats Of CGA

Let $m \geq 0$. For $m+1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

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The proof, again, is not hard, but can't fit here.

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$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$

The Fun Just Doesn't Stop!

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Answer: Not very pretty!

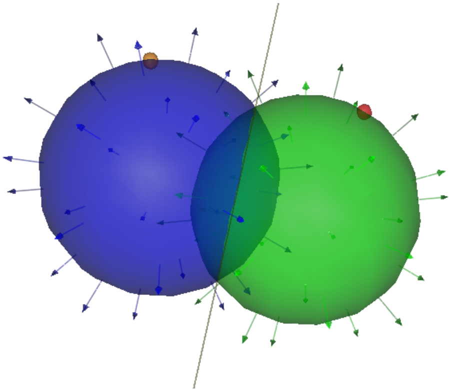

Figure : Reflection of sphere with aide of point.

Pictures Of Quiz Answers (Continued)

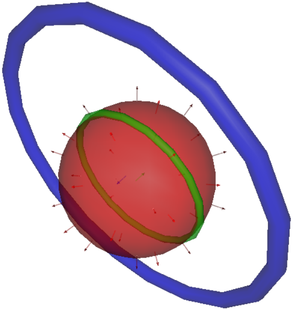

Figure : Saturn's Fallen Ring

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Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

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A versor may or may not leave ∞ invariant under conjugation.

Transformations Of Dual Geometry By Versors

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

All conformal transformations can be represented by versors!
Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

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Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.

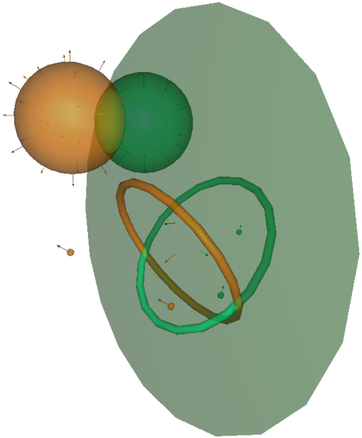

Figure : Planar reflections of a sphere, circle and point-pair.

Planar Reflections (Again)

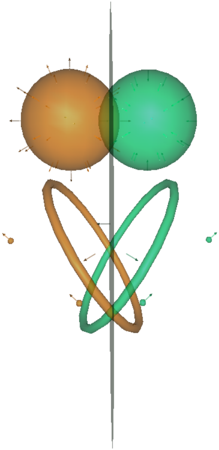

Figure : Planar reflections of a sphere, circle and point-pair.

Spherical Reflections

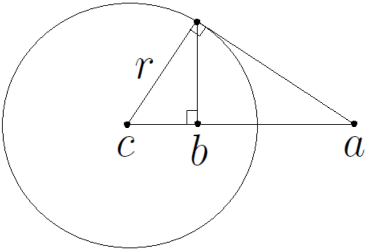

Figure : How we define spherical reflections.

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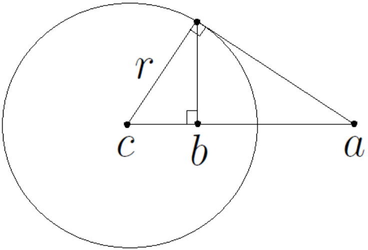


Figure : How we define spherical reflections.

It can be shown that $b = c + \lambda(b - a)$, where $\lambda = r^2/(c - a)^2$.

Spherical Reflections (Again)

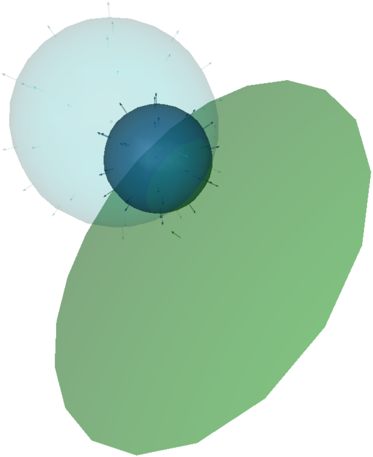
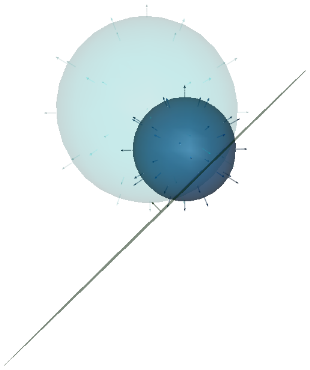


Figure : The spherical reflection of a plane about a sphere is a sphere.

Spherical Reflections (Yet Again)



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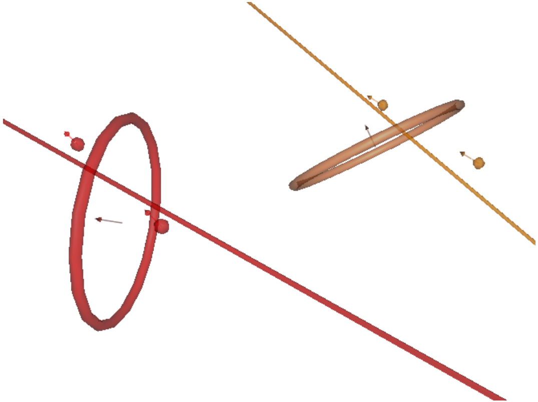
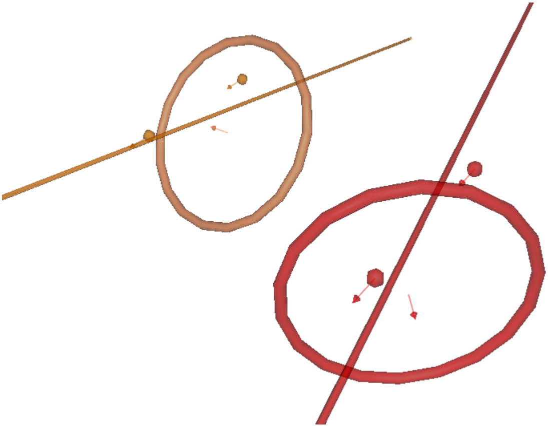


Figure : A line, circle and point-pair rotated/translated.

Rigid Body Motions (Again)



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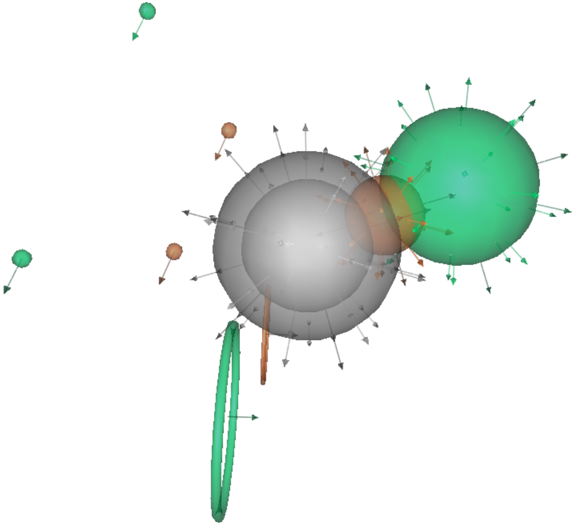
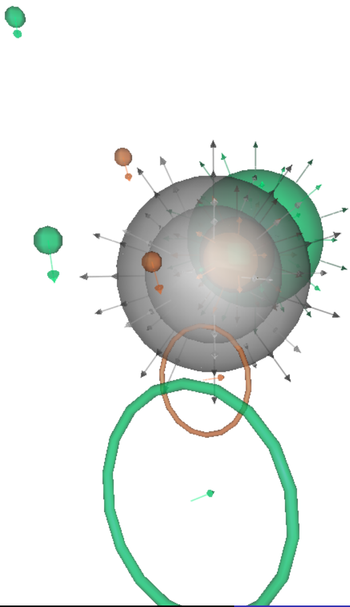
Dilations



Dilations (Again)



Transversions

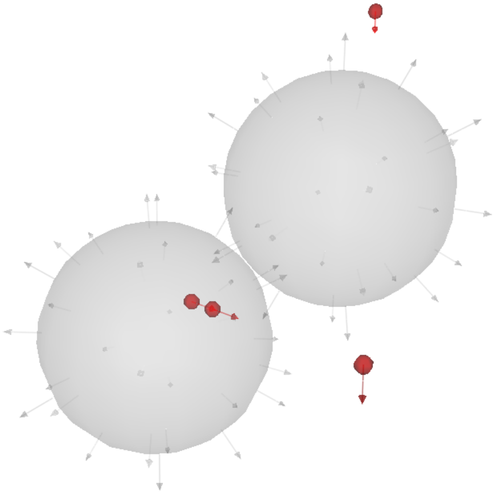


Figure : The transversion of a point-pair.

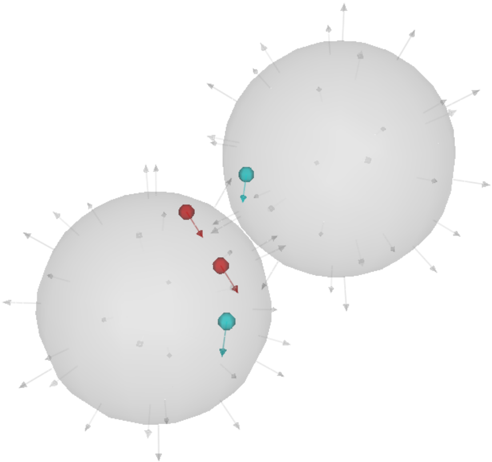


Figure : The transversion of some other point-pair.

The End

Thank you for your time. Any questions?