An Introduction To Projective Geometry Using Geometric Algebra

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This paper is my attempt to build up the subject of projective geometry using geometric algebra in my own words. I do not claim originality to any result in this paper. If nothing else, this paper simply represents a formal compilation of my notes on the subject. I have mainly used [2] and [1] for research in the preparation of this paper.

I began my study of projective geometry using geometric algebra after having already put much effort into understanding the conformal model of geometric algebra. I may therefore make reference to concepts in the conformal model as we go along, but it is not a prerequisite that the reader be familiar at all with the conformal model.

I don't know for certain, but I believe that the conformal model might be superior to projective geometry, though to-date the latter has been studied far more than the former, it having been around a lot longer.

1 Representing Geometry

Like the conformal model, we may think of geometries as subsets of the set of all points in some n-dimensional Euclidean space, which we'll denote by $\mathbb{V}^n$. This also denotes an n-dimensional Euclidean vector space as we adopt here the standard correlation between vectors in such a vector space with points in an n-dimensional Euclidean space.

Points sets, of course, do not lend themselves easily to goemetric analysis. So, like the conformal model, we represent them using blades in a geometric algebra. Why we use blades will become apparent after we define how a blade represents a point set, because then it will become clear how the meet and join operations of blades will allow us to do some interesting geometric operations, just as we can in the conformal model.

For n-dimensional projective geometry, we use a geometric algebra generated by an $(n+1)$-dimesnional Euclidean vector space. If $\{e_k\}_{k=0}^{n-1}$ is any set of orthonormal basis vectors spanning $\mathbb{V}^n$, let $\{e_k\}_{k=0}^{n}$ be a set of orthonormal basis vectors spanning $\mathbb{V}^{n+1}$, which we'll use to generate our geometric algebra $\mathbb{G}(\mathbb{V}^{n+1})$.

In projective geometry we can represent points, lines, planes, hyperplanes, and so on to higher dimensions. Certainly results in geometry involving all of these types of geometric primitives can be found by simply using $\mathbb{V}^n$ alone, but what we'll see is that the extra dimension in $\mathbb{V}^{n+1}$ will facilitate some amazingly useful constructions in $\mathbb{G}(\mathbb{V}^{n+1})$ that make the finding of such results much easier than it would be otherwise. Indeed, in [2], it is shown how geometric algebra easily and naturally explains many fundamental theorems in projective geometry. It is my guess that interpretations of how these constructions work based on $(n+1)$-dimensional projections into n-dimensional space are at least partially to blame for the title of the subject being projective geometry.

Without further delay, we begin with a function $p : \mathbb{V}^n \to \mathbb{V}^{n+1}$ that defines a mapping from points in our Euclidean space with vectors in our geometric algebra. We then say that a blade $A \in \mathbb{G}(\mathbb{V}^{n+1})$ represents a piece of geometry as the set of all points $x \in \mathbb{V}^n$ such that $p(x) \wedge A = 0$. From this it is clear that all non-zero scalar multiples A also represent the same piece of geometry. This is the nature of a homogeneous representation model.

We define p simply as

$$p(x) = x + e_3.$$

Having done so, it is easy to see that for any vector $a \in \mathbb{V}^n$, that $p(a)$ represents the point a. Now here's where it gets interesting. Let $\{a_k\}_{k=0}^{m-1}$ be any set of m points taken from $\mathbb{V}^n$ such that they are non-co-$(m-1)$-hyper-planar. That is, if $m = 2$, the 2 points are distinct; if $m = 3$, the 3 points are non-co-linear; if $m = 4$, the 4 points are non-co-planar; if $m = 5$, the points are non-co-hyper-planar, and so on. We will now show that if $m \leq n+1$, then the blade $A = \bigwedge_{k=0}^{m-1} p(a_k)$ represents an $(m-1)$-dimensional hyper-plane.

The case $m = n + 1$ is trivial. In that case, A is a psuedo-scalar of $\mathbb{G}(\mathbb{V}^{n+1})$, and so for all points $x \in \mathbb{V}^n$, we have $p(x) \wedge A = 0$, showing that A represents an n-dimensional hyper-plane. Let us now consider all $1 < m < n + 1$. It is not hard to show that we can rewrite the equation $p(x) \wedge A = 0$ as

$$x \wedge a_0 \wedge \bigwedge_{k=1}^{m-1} (a_k - a_0) + (x - a_0) \wedge \left(\bigwedge_{k=1}^{m-1} (a_k - a_0) \right) \wedge e_n = 0 \tag{1}$$

From this we see two conditions that must be satisifed in that the first and second terms must be zero. The first term being zero is a necessary yet insufficient condition that x be on the $(m-1)$-dimensional hyper-plane. The second term being zero, however, is a necessary and sufficient condition that x be on the $(m - 1)$-dimensional hyper-plane.

In equation (1), we can think of a_0 as being the center of the hyper-plane, if you will. For each integer $k \in [1, m-1]$, we can think of the vectors $a_k - a_0$ as vectors in the hyper-plane. Clearly for x to be on the hyper-plane, we need the vector $x - a_0$ to be in the hyper-plane.

A way to write A such that we can decompose it into its characteristic parts would be

$$A = a_0 \wedge \bigwedge_{k=1}^{m-1} (a_k - a_0) + \left(\bigwedge_{k=1}^{m-1} (a_k - a_0) \right) \wedge e_n.$$

From this we see the composition of A in terms of two characteristics. The first is a center, if you will, a_0, and the second an $(m-1)$-blade $\bigwedge_{k=1}^{m-1}(a_k - a_0)$ determining the attitude of the $(m - 1)$-dimensional hyper-plane.

References

[1] Leo Dorst, Daniel Fontijne, and Stephen Mann. *Geometric Algebra For Computer Science*. Morgan Kaufmann, 2007.

[2] David Hestenes. Projective geometry with clifford algebra. *Acta Applicandae Mathematicae*, 1991.