An Intro to CGA
Conformal Geometric Algebra

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What is CGA?

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Invented by David Hestens in his paper, "Old Wine in New Bottles: A new algebraic framework for computational geometry."

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Presentation Outline

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- Find forms for all geometric primitives of the CGA model,

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- Introduce concepts from GA only as necessary,
- Introduce the generalized homogeneous model of geometry over GA,
- Define the specific conformal model of GA,
- Find forms for all geometric primitives of the CGA model,
- Discuss the fundamental transformations of CGA.

Blades

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Definition

We say the blade B, given by

$$B = \bigwedge_{k=1}^m b_k = b_1 \wedge \cdots \wedge b_m,$$

is a non-zero *m*-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

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is a non-zero *m*-blade if and only if $\{b_k\}_{k=1}^m$ is a linearly independent set of vectors.

Clearly, if $B \neq 0$, then we must have $\text{grade}(B) = m \leq n$.

Imagine an infinite m-dimensional hyper-plane.

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Our geometric arguments will not require us to visualize the homogeneous representation space.

Recall that $B = b_1 \wedge \cdots \wedge b_m$.

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Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \in B \text{ if and only if } v \in \text{span}\{b_k\}_{k=1}^m.$$

Blades May Represent Vector Sub-Spaces

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Definition

For any $v \in \mathbb{V}^n$, we say that

$$v \notin B \text{ if and only if } v \in B^*,$$

where B^* represents the complement $(\mathbb{V}^n - \text{span}\{b_k\}_{k=1}^m) \cup \{0\}$.

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For any vector $v \in \mathbb{V}^n$, we have

$$v \wedge B = (v_\parallel + v_\perp) \wedge B = v_\perp \wedge B,$$
$$v \cdot B = (v_\parallel + v_\perp) \cdot B = v_\parallel \cdot B.$$

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$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

Building Intuition About Euclidean Blades

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$$\text{grade}(v \wedge B) = \text{grade}(B) + 1$$
$$\text{grade}(v \cdot B) = \text{grade}(B) - 1$$

We may also imagine $v \cdot B = (v \wedge B^*)^*$, where B^* is the complement of B with respect to $\mathbb{V}^n$.

If $B \neq 0$, then $v \in B$ if and only if $v \wedge B = 0$.

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Proof.

The set $\{b_k\}_{k=1}^m$ is linearly independent while the set $\{v\} \cup \{b_k\}_{k=1}^m$ is linearly dependent. $\qquad\square$

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If $B \neq 0$, then $v \in B^*$ if and only if $v \cdot B = 0$.

Proof.

Notice that $0 = v \cdot B = (v \wedge B^*)^*$ if and only if $v \wedge B^* = 0$. $\square$

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Let $\mathbb{R}^n$ denote n-dimensional Euclidean space.
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We say that B directly represents a geometry as the set of all points

$$G(B) = \{x \in \mathbb{R}^n | p(x) \in B\}.$$

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Definition

We say that B **dually** represents a geometry as the set of all points

$$G^*(B) = \{x \in \mathbb{R}^n | p(x) \in B^*\}.$$

Note that $G(B) = G^*(B^*)$ and $G^*(B) = G(B^*)$.

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

We Can Combine Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G(A) \cup G(B) \subseteq G(A \wedge B).$$

Proof.

$$p(x) \in A \text{ or } p(x) \in B$$
$$\implies p(x) \in A \wedge B$$

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Why doesn't the converse hold?

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

We Can Intersect Geometries

For any two blades $A, B \in \mathbb{G}(\mathbb{V}^n)$ such that $A \wedge B \neq 0$, we have

$$G^*(A) \cap G^*(B) = G^*(A \wedge B).$$

Proof.

$$p(x) \in A^* \text{ and } p(x) \in B^*$$
$$\text{iff } p(x) \notin A \text{ and } p(x) \notin B$$
$$\text{iff } p(x) \notin A \wedge B$$
$$\text{iff } p(x) \in (A \wedge B)^*$$

□

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The Homogeneous Nature Of The Model

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For any blade B, there is a scalar λ such that λB is a homogenized form.

If B is the result of some geometric operation, then such a λ has geometric signficance WRT to that operation.

The Geometric Product

Definition

For any vector $v \in \mathbb{V}^n$ and any blade $B \in \mathbb{G}(\mathbb{V}^n)$, we define

$$vB = v \cdot B + v \wedge B.$$

Let $\{v_k\}_{k=1}^m$ be any set of m vectors.

Versors

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Definition

We say the element $V \in \mathbb{G}(\mathbb{V}^n)$, given by

$$V = \prod_{k=1}^m v_k,$$

is a versor if and only if for all k, the vector v_k^{-1} exists.

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Notice that $v^{-1} = v/v^2$.

The Inverse And The Reverse Of Versors

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The inverse V^{-1} of V is therefore given by

$$V^{-1} = \frac{\tilde{V}}{V\tilde{V}}.$$

Versors form a group under the geometric product.

The Versor Group

Versors form a group under the geometric product.

Proof.

Associativity follows from the associativity of the geometric product.

The scalar 1 is the identity versor.

For every versor V, there exists an inverse V^{-1} such that $VV^{-1} = V^{-1}V = 1$. □

Properties Of Versors

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Conjugation by versors is grade preserving!
For any vector $v \in \mathbb{V}^n$, we have $VvV^{-1} \in \mathbb{V}^n$, therefore, we have $\text{grade}(B) = \text{grade}(VBV^{-1})$.

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Let $o, \infty \in \mathbb{V}^{n+2}$ be vectors such that $o \cdot o = \infty \cdot \infty = 0$ and $o \cdot \infty = \infty \cdot o = -1$ and for all $v \in \mathbb{V}^n$, we have $v \cdot o = v \cdot \infty = 0$.

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Definition

Define $p : \mathbb{V}^n \to \mathbb{G}(\mathbb{V}^{n+2})$ as

$$p(x) = o + x + \frac{1}{2}x^2\infty.$$

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Having invented this specific model, what we are now able to discover about it is almost endless!

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That is, $G(p(c)) = G^*(p(c)) = \{c\}$.

The function $p(x)$ factors out of the equation

$$(x - c)^2 - r^2 = 0$$

as the alternative equation

$$p(x) \cdot \left(p(c) - \frac{1}{2} r^2 \infty \right) = 0.$$

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Points are degenerate spheres, or spheres with radius zero. We may refer to $p(c)$ as a round point.

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Rounds with zero radius give us tangent points!
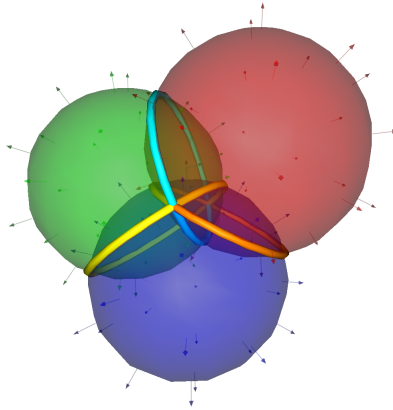
Figure : 3 Rounds, 3 Circles and 1 Point-Pair

The function $p(x)$ factors out of the equation

$$(x - c) \cdot v = 0$$

as the alternative equation

$$p(x) \cdot (v + (c \cdot v)\infty) = 0.$$

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Flats at infinity are free blades.

0-dimensional flats are called flat points.
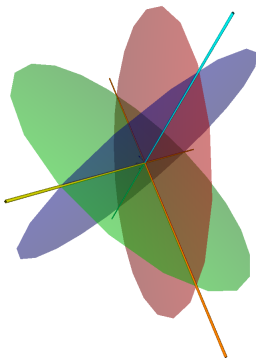
Figure : 3 Planes, 3 Lines, 1 Flat-Point

A Generalization Of Coplanarity

Definition

For $m \geq 0$, a set of points are co-m-hyper-planar if...

> For $m = 0$, the points are identical,
> For $m = 1$, the points are collinear,
> For $m = 2$, the points are coplanar,
> For $m = 3$, the points are co-hyper-planar,
> etc...

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m-1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

Lemma

For $m \geq 1$, if $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ are non-co-$(m - 1)$-hyper-planar, then $\{p(x_k)\}_{k=1}^{m+1}$ is a linearly independent set.

The proof is not hard, but too big for this slide.

Let $m \geq 1$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyper-sphere that are non-co-$(m-1)$-hyper-planar, the blade B, given by

$$B = \bigwedge_{k=1}^{m+1} p(x_k)$$

directly represents the m-dimensional hyper-sphere.

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Proof.

Let the $(n - m + 1)$-blade A dually represent the m-dimensional hyper-sphere determined by the points. If A dually represents this sphere, then A^* directly represents this sphere. Therefore, we need to show that there exists $\lambda \in \mathbb{R}$ such that $A^* = \lambda B$. For all k, we have $p(x_k) \in A^*$ and $p(x_k) \in B$. By our lemma, $\{p(x_k)\}_{k=1}^m$ is a linearly independent set. Lastly, $\text{grade}(B) = m + 1 = n + 2 - (n - m + 1) = n + 2 - \text{grade}(A) = \text{grade}(A^*)$. $\qquad \square$

A Generalization Of Cospherical

Definition

For $m \geq 1$, a set of points are co-m-hyper-spherical if...

> For $m = 1$, the points are co-point-pair,
> For $m = 2$, the points are co-circular,
> For $m = 3$, the points are co-spherical,
> For $m = 4$, the points are co-hyper-spherical,
> etc...

Let $m \geq 1$. For $m + 2$ points $\{x_k\}_{k=1}^{m+2} \subset \mathbb{V}^n$ on an m-dimensional hyper-plane that are (1) non-co-$(m-1)$-hyper-planar and (2) non-co-m-hyper-spherical, the blade B, given by

$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

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$$B = \bigwedge_{k=1}^{m+2} p(x_k),$$

directly represents the m-dimensional hyper-plane.

Proof.

By (1), there exists the $(n-m)$-blade A dually representative of the m-dimensional hyper-plane. By (2), $B \neq 0$. Lastly, $\operatorname{grade}(B) = m + 2 = n + 2 - (n - m) = n + 2 - \operatorname{grade}(A) = \operatorname{grade}(A^*)$. $\quad\square$

Let $m \geq 0$. For $m + 1$ points $\{x_k\}_{k=1}^{m+1} \subset \mathbb{V}^n$ on an m-dimensional hyepr-plane that are non-co-$(m-1)$-hyper-planar, the blade B, give by

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The proof, again, is not hard, but can't fit here.

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Answer:
$$S' = (P' \wedge (S \wedge N)^*)^* = P' \cdot (S \wedge N) = (P' \cdot S)N - (P' \cdot N)S.$$

Question: Let S dually represent the planet Saturn and let the R directly represent one of Saturn's rings. If this ring fell out of orbit, let the direct circle F on the surface of S approximate the debris field. What is F?

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Answer:

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The Fun Just Doesn't Stop!

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Answer: Not very pretty!

Pictures Of Quiz Answers

Figure : blah
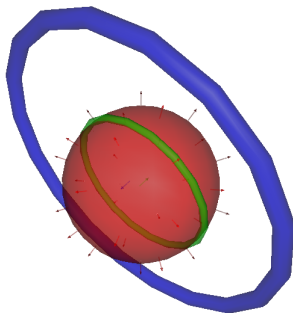
Figure : Saturn's Fallen Ring

Recall the outermorphic property of versors!

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Then, if the m-blade B directly represents any geometry, (except a flat point), then there exists m points $\{x_k\}_{k=1}^m \subset \mathbb{V}^n$ such that

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A versor may or may not leave ∞ invariant under conjugation.

If the m-blade B dually represents any geometry, then we can write

$$VBV^{-1} = V(B^*)^*V^{-1} = (VB^*V^{-1})^*,$$

relating this to what we know about the transformation of *directly* represented geometries.

All conformal transformations can be represented by versors! Some of these include...

- Translations,
- Rotations,
- Dilations,
- Transversions.

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Note: Points are null, (non-invertible), and therefore, planar and spherical reflections generate the versor group of all transformations.
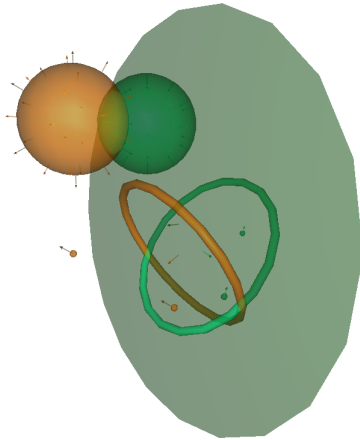
Figure : Planar reflections of a sphere, circle and point-pair.

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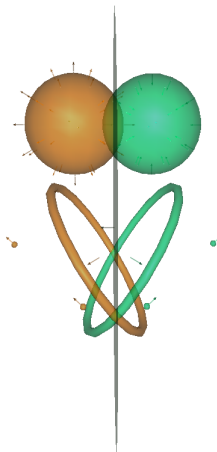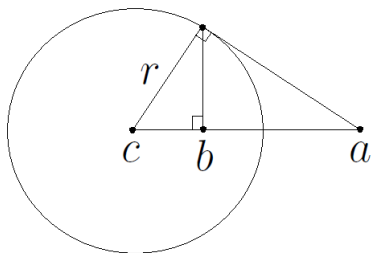
Figure : How we define spherical reflections.
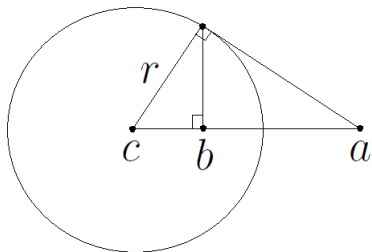
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It can be shown that $b = c + \lambda(b - a)$, where $\lambda = r^2/(c - a)^2$.
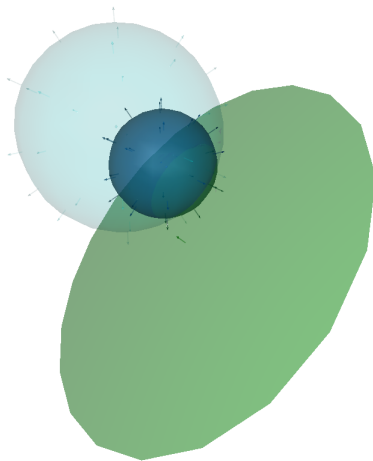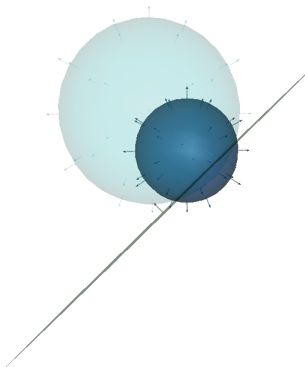
Figure : The spherical reflection of a plane about a sphere is a sphere.

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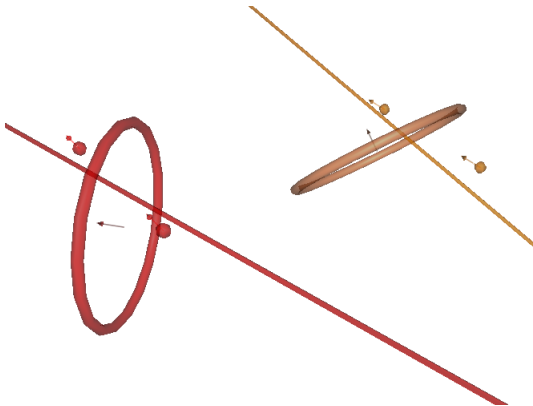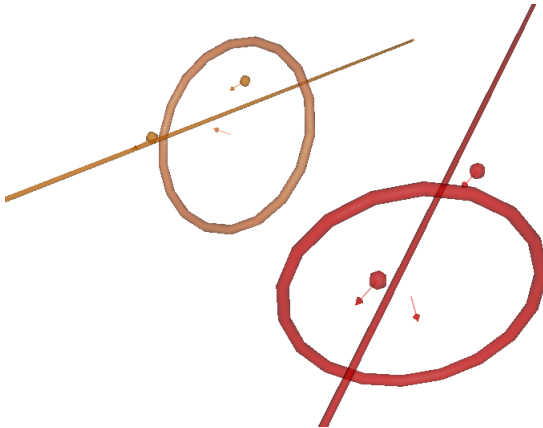
Figure : A line, circle and point-pair rotated/translated.
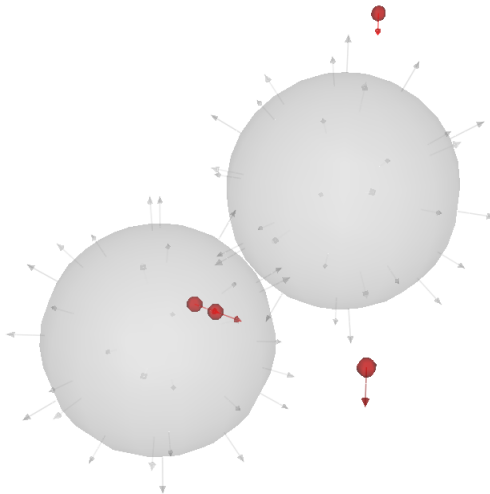
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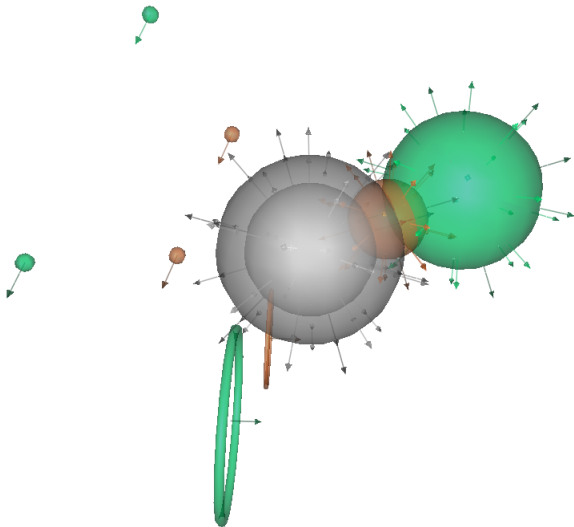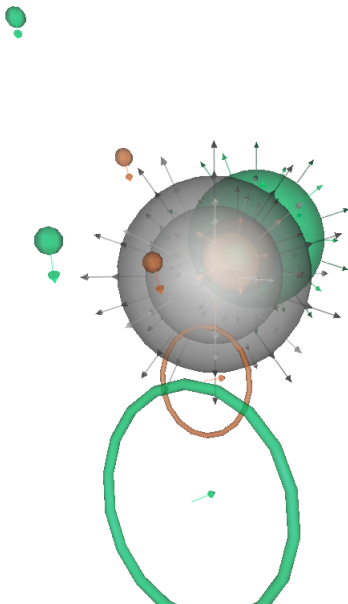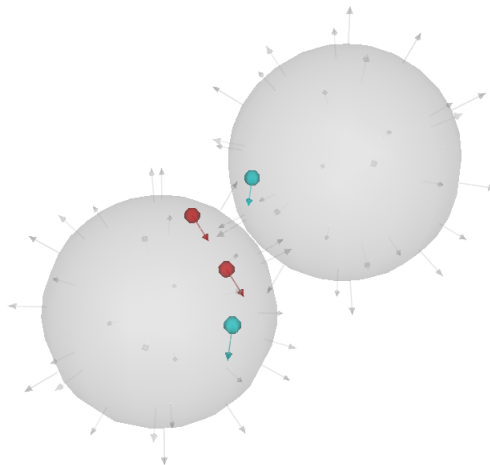
Figure : The transversion of a point-pair.

Figure : The transversion of some other point-pair.

The End

Thank you for your time. Any questions?