The Quadratic Form In Geometric Algebra

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Abstract. Blah.

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1. Finding The Quadratic Form

Let $\mathbb{V}^n$ be an n-dimensional Euclidean vector space, and identify vectors in this space with points in n-dimensional Euclidean space. That is, for any vector $v \in \mathbb{V}^n$, identify this vector with the point at its tip when its tail is placed at origin. Letting any subset of $\mathbb{V}^n$ be what we refer to as a geometry, the goal of this paper is to use geometric algebra in the study of all such geometries that occur as the zero set of one or more quadratic forms.[1] A quadratic form $q : \mathbb{V}^n \to \mathbb{R}$ is a quadratic polynomial in the vector components of any vector $v \in \mathbb{V}^n$. Specifically, we have

$$q(v) = C + \sum_{i=1}^{n} C_i(v \cdot e_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}(v \cdot e_i)(v \cdot e_j), \tag{1.1}$$

where C, each of C_i and each of C_{ij} are scalars in $\mathbb{R}$. The coeficients C, C_i and C_{ij} collectively determine the geometry that is the zero set of q. Adding a Euclidean vector e_0 representative of the origin to $\mathbb{V}^n$ to obtain the $(n+1)$-dimensional Euclidean vector space $\mathbb{V}^{n+1}$, we see that the quadratic form q is determined by a symmetric billinear form $B : \mathbb{V}^{n+1} \times \mathbb{V}^{n+1} \to \mathbb{R}$ as

$$q(v) = B(e_0 + v, e_0 + v) \tag{1.2}$$

$$= B(e_0, e_0) + 2\sum_{i=1}^{n} B(e_0, e_i)(v \cdot e_i) + \sum_{i=1}^{n}\sum_{j=1}^{n} B(e_i, e_j)(v \cdot e_i)(v \cdot e_j), \tag{1.3}$$

[1] In algebraic geometry, the zero set of one or more polynomials is called an affine variety.

if we let $B(e_0, e_0) = C$, each of $B(e_0, e_i) = B(e_i, e_0) = \frac{1}{2}C_i$ and each of $B(e_i, e_j) = C_{ij}$. In turn, we see that the symmetric billinear form B is determined entirely by how it maps a basis of $\mathbb{V}^{n+1}$.

To find the quadratic form q in geometric algebra, it is clear now that one approach is to go about looking for the symmetric bilinear form B. Two instances of this form are found and detailed in the following two sections.

2. The Quadratic Form In $\mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$

Here we let $\overline{\mathbb{V}}^{n+1}$ be an $(n+1)$-dimensional Euclidean vector space isomorphic to $\mathbb{V}^{n+1}$, and then define the over-bar notation on elements of the geometric algebra $\mathbb{G}(\mathbb{W})$, with $\mathbb{W} = V^{n+1} \oplus \overline{\mathbb{V}}^{n+1}$, as an outermorphic isomorphism between $\mathbb{G}(\mathbb{V}^{n+1})$ and $\mathbb{G}(\overline{\mathbb{V}}^{n+1})$. That is, for any element $E \in \mathbb{G}(\mathbb{W})$, we have

$$\overline{E} = QEQ^{-1}, \tag{2.1}$$

where the rotor Q is given by

$$Q = \prod_{i=0}^{n}(1 - e_i\overline{e}_i). \tag{2.2}$$

Having defined $\mathbb{G}(\mathbb{W})$, we will introduce the function $S : \mathbb{V}^{n+1} \to \mathbb{G}(\mathbb{W})$, given by

$$S(p) = p\overline{p}, \tag{2.3}$$

and then see that for all vectors $p \in \mathbb{V}^{n+1}$, the symmetric bilinear form B occurs in $\mathbb{G}(\mathbb{W})$ as

$$B(p,p) = -S(p) \cdot \sum_{i=0}^{n}\sum_{j=0}^{n} B(e_i, e_j)e_i\overline{e}_j, \tag{2.4}$$

showing that the bivectors $E \in \mathbb{G}(\mathbb{W})$ are representative of n-dimensional quadric surfaces as the set of all vectors $v \in \mathbb{V}^n$ such that $q(v) = 0$, where

$$q(v) = S(e_0 + v) \cdot E. \tag{2.5}$$

This approach is especially advantageous in the realization that for any versor $V \in \mathbb{G}(\mathbb{V}^{n+1})$, we have

$$S(V^{-1}pV) \cdot E = S(p) \cdot V\overline{V}E(V\overline{V})^{-1}, \tag{2.6}$$

by the fact that S has the property

$$S(V^{-1}pV) = (V\overline{V})^{-1}S(p)V\overline{V}, \tag{2.7}$$

which shows that if we understand how V transforms homogeneous points $p \in \mathbb{V}^{n+1}$ as $V^{-1}pV$, then we also understand how V transforms quadric surfaces $E \in \mathbb{G}(\mathbb{W})$ as $(V\overline{V})E(V\overline{V})^{-1}$. In a variation of this approach that uses the geometric algebra $\mathbb{G}(\mathbb{V}^{n+1,1} \oplus \mathbb{V}^{n+1,1})$, the versors of the conformal model of geometric algebra may be used to transform quadric surfaces.

A down-side to this approach, however, is in the fact that we're not using blades to represent quadric surfaces in the same way that blades are representive of geometries in the conformal model of geometric algebra. Consequently, we cannot similarly benefit from the meet and join operations. We will attempt to remedy this problem in the next section.

3. The Quadratic Form In $\mathbb{G}(\mathbb{V}^{(n+1)^2})$

Notice that in the previous method, the Euclidean space $\mathbb{V}^n$ was embedded in the representation space $\mathbb{G}(\mathbb{W})$. For the method to follow, we show that this need not be the case. Specifically, we do not let $\mathbb{V}^n$ be a vector sub-space of the $(n+1)^2$-dimensional anti-Euclidean vector space $\mathbb{V}^{(n+1)^2}$.[2] We will, however, continue to let $\mathbb{V}^n$ be a proper vector sub-space of $\mathbb{V}^{n+1}$.

Letting $\{e_{ij}\}$ be a set of orthonormal basis vectors spanning $\mathbb{V}^{(n+1)^2}$, we reintroduce the function $S : \mathbb{V}^{n+1} \to \mathbb{V}^{(n+1)^2}$ as

$$S(p) = p \otimes p, \tag{3.1}$$

where $\otimes : \mathbb{V}^{n+1} \times \mathbb{V}^{n+1} \to \mathbb{V}^{(n+1)^2}$ is a commutative bilinear and binary operator, defined as

$$x \otimes y = \sum_{i=0}^{n} \sum_{j=0}^{n} (x \cdot e_i)(y \cdot e_j) e_{ij}, \tag{3.2}$$

and then find that for all vectors $p \in \mathbb{V}^{n+1}$, the symmetric billinear form B in $\mathbb{G}(\mathbb{V}^{(n+1)^2})$ is found as

$$B(p, p) = -S(p) \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} B(e_i, e_j) e_{ij}, \tag{3.3}$$

showing that the vectors $E \in \mathbb{G}(\mathbb{V}^{(n+1)^2})$ are representative of n-dimensional quadric surfaces as the set of all vectors $v \in \mathbb{V}^n$ such that $q(v) = 0$, where q is again given by equation (2.5).

Immediately we see that the advantage to this approach is that a non-zero blade $E \in \mathbb{G}(\mathbb{V}^{(n+1)^2})$ of grade k is representative of an $(n + 1 - k)$-dimensional quadric surface. To see this, let $E = E_1 \wedge \cdots \wedge E_k$, and realize that

$$0 = S(p) \cdot \bigwedge_{i=1}^{k} E_i = \sum_{i=1}^{k} (S(p) \cdot E_i) \bigwedge_{j=1, j \neq i}^{k} E_j \tag{3.4}$$

if and only if for all integers $i \in [1, k]$, we have $S(p) \cdot E_i = 0$. In other words, E represents the affine variety generated by the set of all quadradic polynomials determined by each E_i.

We will refer to E as a dual quadric if we are interpreting it as being representative of a quadric surface in terms of the equation

$$0 = S(e_0 + v) \cdot E. \tag{3.5}$$

[2] To be anti-Euclidean means that the inner product square of any two vectors is a non-positive scalar.

Similarly, we will refer to E as a direct quadric if we are interpreting it as being representative of such a surface in terms of the equation

$$0 = S(e_0 + v) \wedge E. \tag{3.6}$$

To see that this is also the previously mentioned affine variety, simply realize that

$$0 = S(e_0 + v) \wedge E \text{ if and only if } 0 = S(e_0 + v) \cdot EI, \tag{3.7}$$

where I is the unit-psuedo scalar of $\mathbb{G}(\mathbb{V}^{(n+1)^2})$.

Notice that any single blade $E \in \mathbb{G}(\mathbb{V}^{(n+1)^2})$ is simultaneously representative of both a dual and direct quadric, which are not necessarily the same peice of the geometry.[3] It is sometimes useful to reinterpret a dual quadric as a direct quadric, or vice versa. For example, if the dual intersection of two dual quadrics is imaginary, the imaginary intersection may be a real quadric in direct form.

4. Point Fitting Quadrics In $\mathbb{G}(\mathbb{V}^{(n+1)^2})$

Suppose $E \in \mathbb{G}(\mathbb{V}^{(n+1)^2})$ is a direct quadric of grade k, and that $\{p_i\}_{i=1}^k$ is a set of k homogeneous points taken from $\mathbb{V}^{n+1}$ such that for all integers $i \in [1, k]$, we have $S(p_i) \wedge E = 0$. Then, if $\{S(p_i)\}_{i=1}^k$ is a linearly independent set, it follows that there exists a scalar $\lambda \in \mathbb{R}$ such that

$$\bigwedge_{i=1}^{k} S(p_i) = \lambda E. \tag{4.1}$$

What this shows is that, given a set of k points $\{p_i\}_{i=1}^k$, we can find a quadric E that fits the k points, provided the set $\{S(p_i)\}_{i=1}^k$ is linearly independent. Two questions arise from this. First, under what circumstances do the k points generate a linearly independent set $\{S(p_i)\}_{i=1}^k$; and secondly, under those circumstances, what quadric surface do we get? (Address this somehow, preferably by finding the answers.)

From what we have thus far gathered, an $(n+1-k)$-dimensional quadric surface would be fit to $(n+1)^2 - k$ points if it were at all possible to find such a set of points generating a linearly independent set. Possible or not, it is easy to show that this is certainly not the least upper bound on the number of points needed to determine such a surface. To see why, define $S_0 : \mathbb{V}^n \to \mathbb{V}^m$ as

$$S_0(p) = e_{00} + \sum_{i=1}^{n} (p \cdot e_i) e_{0i} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (p \cdot e_i)(p \cdot e_j) e_{ij} \tag{4.2}$$

[3] A dual quadric is directly represented by its dual, and a direct quadric is dually represented by its dual. As a given blade simultaneously represents two geometries, (one dually, the other directly), a single given geometry is simultaneously represented by two blades, (which are duals of one another).

where m is given by

$$m = \binom{n}{0} + \binom{n}{1} + \binom{n}{2}, \tag{4.3}$$

and $\mathbb{V}^m$ is a proper vector sub-space of $\mathbb{V}^{(n+1)^2}$ and spanned by the vectors in $\{e_{00}\} \cup \{e_{0i}\}_{i=1}^n \cup \{e_{ij}\}_{i<j}$. Using now I_0, what we'll use to denote the unit psuedo-scalar of $\mathbb{G}(\mathbb{V}^m)$, to transition between dual and direct quadrics, we see that an $(n+1-k)$-dimensional quadric surface may be fit to $m-k$ points, from which it is more likely that we'll generate a linearly independent set.

5. Switching Between $\mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$ And $\mathbb{G}(\mathbb{V}^{(n+1)^2})$

If you found the choice of an anti-Euclidean vector space in section (**??**) odd, the reason for this will now come to light. To gain the advantages of working in both $\mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$ and $\mathbb{G}(\mathbb{V}^{(n+1)^2})$, it may not be unreasonable to switch between the two algebras when needed. To do this, we simply use the linear function $f : \mathbb{V}^{(n+1)^2} \to \mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$, defined in terms of how it maps the basis vectors of $\mathbb{V}^{(n+1)^2}$ onto the basis bivectors of the linear sub-space of bivectors in $\mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$ as follows. For all pairs of integers $(i,j) \in [0,n] \times [0,n]$, we define

$$f(e_{ij}) = e_i \bar{e}_j. \tag{5.1}$$

We now see that for any vector $E \in \mathbb{V}^{(n+1)^2}$ representative of a quadric surface through the use of equation(3.3), the bivector $f(E) \in \mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$ is representative of the same quadric through the use of equation (2.4).

This gives us the ability to transform any intersection of one or more quadrics in $\mathbb{G}(\mathbb{V}^{(n+1)^2})$ as we would a single quadric in $\mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$. For a given blade $E \in \mathbb{G}(\mathbb{V}^{(n+1)^2})$, we need only find a factorization of the blade E as $E_i \wedge \cdots \wedge E_k$, then formulate the transformation E' of E by a versor $V \in \mathbb{G}(\mathbb{V}^{n+1} \oplus \overline{\mathbb{V}}^{n+1})$ as

$$E' = \bigwedge_{i=1}^{k} f^{-1}\left(V\overline{V}f(E_i)(V\overline{V})^{-1}\right). \tag{5.2}$$

The problem of blade factorization has been given a great deal of treatment in [].

6. Closing Remarks

References

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