A Model For Quadric Surfaces Using Geometric Algebra

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Abstract. Inspired by the conformal model of geometric algebra, a similar model of geometry is developed for the set of all quadric surfaces in n-dimensional space. Bivectors of the geometric algebra are found to be representative of quadric surfaces. Coordinate free canonical forms of such bivectors are found for common quadric surfaces. The model is investigated for usefulness and compared to the conformal model.

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1. The Construction Of The Model

The stage for this model of n-dimensional quadric surfaces is set in the geometric algebra we'll denote by $\mathbb{G}$ that is generated by a Euclidean vector space $\mathbb{W}$ of dimension $2(n+1)$. Letting $\{e_i\}_{i=0}^{2n+1}$ be an orthonormal set of basis vectors generating $\mathbb{W}$, we let $\{e_i\}_{i=0}^{n}$ be such a set of vectors generating the $(n+1)$-dimensional vector sub-space $\mathbb{V}$ of $\mathbb{W}$ upon which we'll impose the usual interpretation of $(n+1)$-dimensional homogeneous space. Specifically, a vector $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ represents the point given by[1]

$$\frac{e_0 \cdot e_0 \wedge v}{e_0 \cdot v} \tag{1.1}$$

in n-dimensional Euclidean space, imposing the usual correlation between n-dimensional vectors and n-dimensional points.[2] We will take the liberty of letting vectors $v \in \mathbb{V}$ with $v \cdot e_0 = 0$ represent points under the same

[1]Throughout this paper we let the outer product take precedence over the inner product, and the geometric product take precedence over both the inner and outer products.
[2]The correlation between vectors and points spoken of here is that of having a vector represent the point at its tip when its tail is placed at the origin.

interpretation of which has just been spoken, as well as pure directions with magnitude. The intended interpretation will be made clear in the context of our usage. We will refer to all vectors $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ as affine points, and such vectors with $v \cdot e_0 = 0$ as Euclidean points or sometimes directions. See section 2.1.1 of [1] for a great introduction to homogeneous coordinates.

We now introduce a function defined on $\mathbb{G}$ having the outermorphic property. This means that it is a linear function and that it preserves the outer product. We will use over-bar notation to denote the use of this function. Doing so, for any element $E \in \mathbb{G}$, we define $\overline{E}$ as

$$\overline{E} = SE\tilde{S}, \tag{1.2}$$

where the rotor S is given by

$$S = 2^{-(n+1)/2} \prod_{i=0}^{n} (1 - e_i e_{i+n+1}), \tag{1.3}$$

and $\tilde{S}$ denotes the reverse of S. As the reader can check, for any integer $i \in [0, n]$, we have $\overline{e_i} = e_{i+n+1}$, and similarly, $\overline{e_{i+n+1}} = -e_i$. The rotor S simply rotates any element taken from the geometric algebra generated by $\mathbb{V}$, (which we'll denote by $\mathbb{G}(\mathbb{V})$), into the identical geometric algebra generated by the vector space we'll denote by $\overline{\mathbb{V}}$ that is complement to $\mathbb{V}$ with respect to $\mathbb{W}$. The over-bar function is an isomorphism between the geometric algebras $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. We will find the over-bar function convenient when performing algebraic manipulations in our model to the extent that we can really forget about the versor S, letting the over-bar notation be nothing more than a device used to distinguish between elements of $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$.

The geometric algebra $\mathbb{G}$ that we have constructed here is similar to "the mother algebra" in [2], except that while $\mathbb{G}(\mathbb{V})$ is Euclidean, so is the geometric algebra $\mathbb{G}(\overline{\mathbb{V}})$. We do not make use of an anti-Euclidean geometric algebra. Although doing so might prove beneficial, it is worth forgoing for now in the realization that $\mathbb{G}$, as it stands, and as we'll see, sufficiently fulfills at least the minimum requirements of a model for quadric surfaces.

We are now ready to give the definition by which we will interpret bivectors in $\mathbb{G}$ as n-dimensional quadric surfaces. It is as follows. For any element $E \in \mathbb{G}$, we say that E is representative of the n-dimensional quadric surface generated by the set of all affine points $p \in \mathbb{V}$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{1.4}$$

Notice that when $\text{grade}(E) > 1$, there is no ambiguity, despite the non-associativity of the inner product, in rewriting equation (1.4) as

$$0 = p \cdot E \cdot \overline{p}, \tag{1.5}$$

which resembles a sort of conjugation of E by p. This may perhaps be a more familiar form for readers familiar with the study of quadric surfaces in projective geometry. Also notice that we have not required that E be a bivector in definition (1.4), because we may find this condition useful and meaningful for any element of $\mathbb{G}$. For now, however, we will restrict our attention to the case when E is a bivector.

To see why definition (1.4) works, simply notice that when E is a bivector, we have

$$p \wedge \overline{p} \cdot E = \sum_{i=0}^{n} \sum_{j=i}^{n} \lambda_{ij}(p \cdot e_i)(p \cdot e_j), \tag{1.6}$$

which we can recognize as a homogeneous polynomial of degree 2 in the vector components of p. The scalars λ_{ij}, with $0 \leq i \leq j \leq n$, may be formulated in terms of E by

$$\lambda_{ij} = \begin{cases} e_i\overline{e_j} \cdot E & \text{if } i = j, \\ (e_i\overline{e_j} + e_j\overline{e_i}) \cdot E & \text{if } i \neq j. \end{cases} \tag{1.7}$$

It should be noted that bivectors do not uniquely represent quadric surfaces, not even up to scale. This is apparent from equation (1.7) when we see that for $i \neq j$, we can freely choose certain components of the bivector without changing the represented quadric so long as their sum is still λ_{ij}. The problem this may pose in our model comes from a very important result in the conformal model. In the conformal model, if two blades are known to represent the same non-trivial geometry in the same way, then it can be shown that the two blades are equal, up to scale. In our present model, it may take more than just multiplying by a non-zero scalar factor to get a bivector known to represent a certain geometry in a known canonical form. To account for this during the performance of algebraic manipulations, we will introduce the following notation. We will say that quadrics E_a and E_b are equivalent, writing $E_a \equiv E_b$, whenever E_a and E_b represent the same quadric under definition (1.4). For example, for any two vectors $u, v \in \mathbb{V}$, we have

$$u \wedge \overline{v} \equiv -2u \wedge \overline{v} \equiv u \wedge \overline{v} + v \wedge \overline{u} \equiv (u + \overline{v}) \wedge (u - \overline{v}). \tag{1.8}$$

Be aware that if $E = E_a + E_b$ and $E_a \equiv E_c$, then this does not imply that $E \equiv E_b + E_c$ unless it can be shown that for all affine points $p \in \mathbb{V}$, we have

$$p \wedge \overline{p} \cdot E_a = p \wedge \overline{p} \cdot E_c. \tag{1.9}$$

This condition is weaker than $E_a = E_c$ yet stronger than $E_a \equiv E_c$.

Another important difference to point out here between our present model and the conformal model is that, unlike what we can analogously expect from the point-definition of the conformal model, here the 2-blade form $a \wedge \overline{a}$ found in definition (1.4), for any affine point $a \in \mathbb{V}$ not at origin, does not represent the affine point a under definition (1.4). In homogenized form, the affine point represented by $a \wedge \overline{a}$ is given by

$$e_0 - \left(\frac{e_0 \cdot e_0 \wedge a}{e_0 \cdot a}\right)^{-1}, \tag{1.10}$$

which is the reflection about the origin of the spherical inversion of the affine point a about the unit-sphere centered at the origin. The affine point e_0 at the origin simply represents the empty point-set geometry, or the geometry of nothing. It is also easy to see that $a \wedge \overline{a}$ cannot represent itself, because there are no null blades in our purely Euclidean geometric algebra $\mathbb{G}$.

2. The Construction Of Quadric Surfaces In The Model

Having constructed our model, we are now ready to find canonical forms of bivectors representing a variety of well-known quadric surfaces. Our approach here will be similar to that taken in Section 3 of [3].

Let us begin with the spheroid, (a special case of ellipsoid), the circular cylinder, and the circular hyperboloid of one sheet. We will find that all of these surfaces share the same canonical form, because they may all be characterized as the Euclidean point solution set of the equation

$$0 = -r^2 + (x-c)^2 + \lambda((x-c)\cdot v)^2 \tag{2.1}$$

in the Euclidean point $x \in \mathbb{V}$, where $c \in \mathbb{V}$ is a Euclidean point denoting the center of the surface, $v \in \mathbb{V}$ is a unit-length direction vector, $r \in \mathbb{R}$ is the radius of the geometry about the axis v at c, and $\lambda \in \mathbb{R}$ is a scalar indicating the type and extremity of the surface. Specifically, if $\lambda < -1$, we get a circular hyperboloid of one sheet; if $\lambda = -1$, we get a circular cylinder; if $-1 < \lambda < 0$, we get a stretched sphere; if $\lambda = 0$, a sphere; and if $\lambda > 0$, a squished sphere. Interestingly, when $r = 0$ and $\lambda < -1$, we get circular conical surfaces; a right-circular conical surface if $\lambda = -2$.

Expanding equation (2.1), we get

$$0 = x^2 + \lambda(x\cdot v)^2 - 2x\cdot(c+\lambda(c\cdot v)v) + c^2 + \lambda(c\cdot v)^2 - r^2, \tag{2.2}$$

from which it is possible to factor out $-p\wedge\overline{p}$ in terms of the inner product, where $p = e_0 + x$ is a homogenized affine point. Doing so, we see that the bivector E given by

$$E = \Omega + \lambda v\wedge\overline{v} - 2(c+\lambda(c\cdot v)v)\wedge\overline{e_0} + (c^2 + \lambda(c\cdot v)^2 - r^2)A, \tag{2.3}$$

is representative of the three surface types by definition (1.4), where the constant Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i\overline{e_i}, \tag{2.4}$$

and A is the constant defined as $A = e_0\overline{e_0}$. We will find each of these useful as frequently recurring constants in our calculations.

Such forms as that in equation (2.3) are useful, not only for composition, but especially decomposition in the cases where we have formulated what may, for example, be a spheroid by some means other than composition. This gives the model power as an analytical tool. If we can solve a problem whose solution is a bivector known to represent a spheroid, then we can use this canonical form to answer questions about that spheroid. Where is its center? What is its axis? What is its radius about that axis? As is often the case in mathematics, however, decomposition is harder than composition. A general sequence of decomposition steps for the form (2.3) is not obvious, if it exists, but we will proceed now to give such a sequence for the case when E is known to be a cylinder. That is, when $\lambda = -1$.

The first thing to notice is that the canonical form E in equation (2.3) is in a homogenized form, because the coeficient of Ω is 1. Looking at any canonical form, if there exists a term in that form with a consistent magnitude, (a magnitude that does not change with any instantiation of that form with a given set of parameters), then we can usually find a way to homogenize that form – the process by which we transform any non-homogenized element E' known to represent the same quadric as that of a homogenized and canonical form E into E. For the canonical form (2.3) with $\lambda = -1$, a common[3] non-homogenized form is given by

$$E' = \omega(\Omega - v \wedge \overline{v} - 2u \wedge \overline{e_0} + (u^2 - r^2)A), \tag{2.5}$$

where $u = c - (c \cdot v)v$, $\omega \neq 1$ and $\omega \neq 0$. To find the homogenized form $E = E'/\omega$, it is not hard to show that

$$\omega = -\frac{\Omega \cdot E'}{n-1}. \tag{2.6}$$

We can then proceed to decompose the canonical form E as follows.

We start by recovering the unit-length direction vector v. This can be done as

$$v = \sum_{i=1}^{n} \left(e_i - \overline{e_i \wedge e_0 \cdot e_0 \wedge E}\right). \tag{2.7}$$

It is unfortunate that we had to refer to a basis to obtain v; nevertheless, it is done. The rest of the decomposition will proceed with greater satisfaction.

There is no way to recover c for cylinders, which is quite obvious. The choice for the point c, the center of the cylinder, may be arbitrarily chosen as any point along its spine. This information is lost in composition, so we may therefore arbitrarily choose

$$c = -\frac{1}{2}A \cdot (E \wedge e_0) \tag{2.8}$$

as the cylinder's center, which, incidentally, will also be the point on the spine of the cylinder closest to the origin.

Lastly, we may find the radius of the cylinder from the simple equation

$$r^2 = c^2 + A \cdot E. \tag{2.9}$$

A generalization of equation (2.1) should be mentioned before moving on. It is given by

$$0 = -r^2 + (x - c)^2 + \sum_{i=1}^{k} \lambda_i((x - c) \cdot v_i)^2, \tag{2.10}$$

which would probably give us the general set of ellipsoids, provided the set of k direction vectors in $\{v_i\}_{i=1}^k$ are linearly independent.

[3]Recall that it may take more than multiplying by a simple scalar factor to homogenize a bivector in $\mathbb{G}$ as discussed in Section 1.

The following table summarizes a few additional canonical forms.

Geometry	Canonical/Homogenized Form
Plane	$v \wedge \overline{e_0} - (c \cdot v)A,\ \lvert v\rvert = 1$
Sphere	$\Omega - 2c \wedge \overline{e_0} + (c^2 - r^2)A$
Point	$\Omega - 2c \wedge \overline{e_0} + c^2 A$
Line	$\Omega - v \wedge \overline{v} - 2u \wedge \overline{e_0} + u^2 A,\ u = c - (c \cdot v)v$
Plane-Pair	$((c_a \cdot v_a)e_0 - v_a) \wedge ((c_b \cdot v_b)\overline{e_0} - \overline{v_b})$

(2.11)

The plain-pair form is discussed below with equation (5.4). Here, in table (2.11), $c_a, c_b \in \mathbb{V}$ are Euclidean points on the two planes, and $v_a, v_b \in \mathbb{V}$ are direction vectors normal to each of the two planes.

3. Making Use Of The Model

Admittedly, there is really nothing interesting about this model unless we can prove that it has some utility. The conformal model, for example, has at least two great features. The first is the utility of the wedge product in generating intersections between geometries in dual form, or point-fitting between geometries in direct form. It is even possible to make use of dual imaginary intersections by reinterpreting them as real geometries in direct form. The second great feature of the conformal model is the surprising fact that all geometries in the conformal model are, as versors, also conformal transformations with geometric significance relative to the simultaneously represented geometry. Then, realizing that all conformal geometries, (with the exception of flat points), have a factorization in direct form as an outer product of points, the outermorphic property of versor conjugation allows us to predict the action of any versor transformation on almost any conformal geometry.

These are great features! But what can the model at present do for us? Well, the first observation we must make is that the set of all known quadrics is represented by the set of all bivectors in $\mathbb{G}$, under-which the inner and outer products are obviously not closed. Only addition and subtraction are closed in this set, and so we're left to wonder what we might be able to prove about the addition and subtraction of n-dimensional quadric surfaces. Letting $B_a, B_b \in \mathbb{G}$ be bivectors, it is not hard to see that $B_a \pm B_b$, under definition (1.4), must represent at least the intersection, if any, of the quadric surfaces B_a and B_b, but this is not an exact answer to the question of what surface $B_a \pm B_b$ represents.

Let's try an example. Suppose B_a and B_b are both homogenized spheres with a real intersection and having Euclidean centers $c_a, c_b \in \mathbb{V}$, respectively. Let $r_a, r_b \in \mathbb{R}$ be the respective radii of B_a and B_b. It then follows from table (2.11) that $B_a - B_b$, in homogenized form, is given by

$$\frac{v}{|v|} \wedge \overline{e_0} - \left(\frac{v}{|v|} \cdot \frac{c_a + c_b + (r_b^2 - r_a^2)v^{-1}}{2} \right) A, \tag{3.1}$$

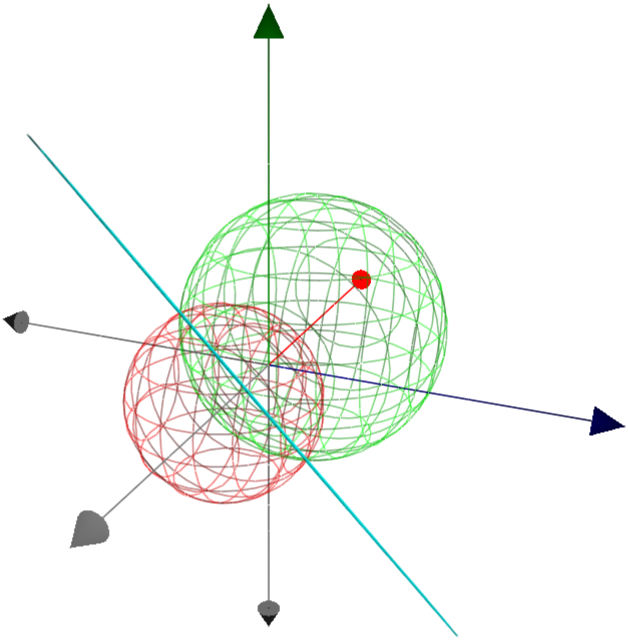

FIGURE 1. The difference of two homogenized spheres gives the plane, shown here on edge, containing their intersection. The spheres were rendered as a number of traces in various parallel planes.

where v is the vector $c_a - c_b$, which, again by table (2.11), tells us that this is a plane with normal v. A point on the plane is also apparent from (3.1), namely $(c_a + c_b + (r_b^2 - r_a^2)v^{-1})/2$. Then, knowing that $B_a - B_b$ must contain the intersection of the two spheres, we can conclude that this point must be in the plane containing the circle that is the intersection of the two spheres, because $B_a - B_b$ must be the said plane. Notice that even if the spheres don't intersect, we still get a meaningful result. A picture of $B_a - B_b$ is given in Figure 1.

At first sight, the sum of a sphere and a plane may not seem that interesting. However, the sum of a homogenized sphere and a non-homogenized plane is interesting, because the result is always a sphere in homogenized form. The scalar amount at which the plane is non-homogenized simply indicates twice the length along the normal of the plane that the center of the original sphere is displaced in the opposite direction of that normal to find a sphere intersecting the plane in the same circle as that of the original sphere.

Interestingly, the difference of spheres generalizes to the idea of subtracting spheroids. A picture of this is given in Figure 2. Of course, there is

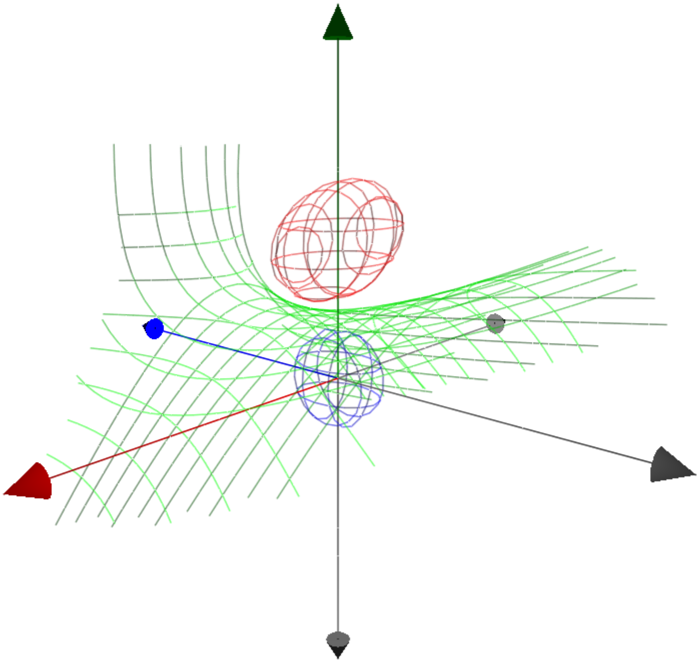

FIGURE 2. The difference of two homogenized spheroids gives a hyperbolic paraboloid. In this case, the two paraboloids have no real intersection.

undoubtedly a geometric significance in the difference between any two homogenized quadric surfaces containing Ω. It would be interesting to find out exactly what that is. The sum of such elements is also interesting. For example, in the case of two homogenized points, it can be shown that their sum is the imaginary sphere having the two points as opposite poles. Specifically, we have

$$\left(\Omega - 2a \wedge \overline{e_0} + a^2 A\right) + \left(\Omega - 2b \wedge \overline{e_0} + b^2 A\right) \tag{3.2}$$

$$\equiv \Omega - 2\left(\frac{a+b}{2}\right) \wedge \overline{e_0} + \left(\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right) A, \tag{3.3}$$

where $a, b \in \mathbb{V}$ are Euclidean points, which we can recognize from table (2.11) as an imginary sphere.

4. The Consideration Of Trivector Quadrics

Notice that any trivector $T \in \mathbb{G}$ can be written in the form

$$T = \sum_{i=1}^{k} v_i \wedge B_i, \tag{4.1}$$

where $\{v_i\}_{i=1}^k \subset \mathbb{W}$ is a set of k vectors and $\{B_i\}_{i=1}^k \subset \mathbb{G}$ is a set of k bivectors. Applying definition (1.4), we get the equation

$$0 = p \wedge \overline{p} \cdot T = p \cdot \sum_{i=1}^{k} (\overline{p} \cdot v_i) B_i - \overline{p} \cdot \sum_{i=1}^{k} (p \cdot v_i) B_i + \sum_{i=1}^{k} (p \wedge \overline{p} \cdot B_i) v_i. \tag{4.2}$$

Now, if $\{v_i\}_{i=1}^k$ was a linearly independent set, and for all affine points $p \in \mathbb{V}$, we have $p \cdot v_i = \overline{p} \cdot v_i = 0$ for any integer $i \in [1, k]$, then it is clear from equation (4.2) that T represents the intersection of all quadrics in $\{B_i\}_{i=1}^k$. Unfortunately, it is obviously not possible to satisfy this condition in $\mathbb{G}$ without expanding it. Doing so, we might introduce two new basis vectors b_1 and b_2, thereby finding the trivector $T = b_1 \wedge B_1 + b_2 \wedge B_2$ as representative of the intersection of the quadrics B_1 and B_2. This, however, may be undesirable, because T cannot directly characterize the intersection in this form, but only indirectly as the characterization of the quadrics B_1 and B_2 taken in the intersection operation. Such indirect characterizations should not be so easily dismissed, however, because a common theme in the process of performing geometry in a model based in geometric algebra is the idea of simply transforming one characterization or interpretation of a given geometry into another. If we were able to formulate T through some means other than that of the intersection of B_1 and B_2, then the original characterization, whatever that may have been, may be transformed into this one, thereby providing a new interpretation of the geometry represented by T as the intersection of the two quadrics B_1 and B_2.

In any case, we are not going to expand $\mathbb{G}$, because it is already complicated enough as it is, and we are far from discovering everything possible in the present model imposed upon it. Let's take a step back for a moment, then, and narrow our scope to that of 3-blades. Doing so, we see that what might motivate us to investigate the set of all quadrics that are 2-blades, (or to find a better model where all quadrics are 2-blades), is the following result.

For any given non-zero 3-blade $T \in \mathbb{G}$, given by $T = a \wedge b \wedge c$, the geometry represented by this 3-blade under definition (1.4) is the intersection of the 3 quadrics $a \wedge b$, $a \wedge c$ and $b \wedge c$. The proof of this follows directly from the following identity, which the reader can easily verify.

$$p \wedge \overline{p} \cdot a \wedge b \wedge c = (p \wedge \overline{p} \cdot a \wedge b)c - (p \wedge \overline{p} \cdot a \wedge c)b + (p \wedge \overline{p} \cdot b \wedge c)a \tag{4.3}$$

Now realize that since $a \wedge b \wedge c \neq 0$, $\{a, b, c\}$ is a linearly independent set, and therefore, $0 = p \wedge \overline{p} \cdot T$ if and only if p is on $a \wedge b$, $a \wedge c$ and $b \wedge c$.

This, of course, would generalize to blades of higher grade. The 3-way intersection of quadrics was given a great deal of consideration in [4]. This ability to intersect three quadrics, however, is restrictive in at least two ways.

First, all three quadrics must be blades. And secondly, the three quadrics must pair-wise share a common vector in their respective factorizations. In any case, the result leads us to a consideration of quadrics that are 2-blades. This deserves its own section.

5. The Consideration Of 2-Blade Quadrics

Letting $B \in \mathbb{G}$ be a 2-blade of the form

$$B = (a + \overline{b}) \wedge (c + \overline{d}) \tag{5.1}$$

$$= a \wedge c + a \wedge \overline{d} - c \wedge \overline{b} + \overline{b \wedge d} \tag{5.2}$$

$$\equiv a \wedge \overline{d} - c \wedge \overline{b}, \tag{5.3}$$

where $a, b, c, d \in \mathbb{V}$, our first observation is that $a \wedge c$ and $\overline{b \wedge d}$ contribute nothing to the shape of the quadric, because they represent the geometry of all space under definition (1.4), which is easily verified. What remains is the difference between two quadrics of the form $u \wedge \overline{v}$, where $u, v \in \mathbb{V}$. It is easy to show that a quadric of this form is a plane-pair.

$$0 = p \wedge \overline{p} \cdot u \wedge \overline{v} = -(p \cdot v)(p \cdot u) \tag{5.4}$$

The affine point solution set of equation (5.4) is clearly the union of such a set for the equation $0 = p \cdot v$ and $0 = p \cdot u$, both of which are planes. The plane for v has normal $e_0 \cdot e_0 \wedge v$, and the point $e_0 - (v \cdot e_0)(e_0 \cdot e_0 \wedge v)^{-1}$ as the point on the plane closest to the origin.

Returning to (5.1), it is clear now that B represents the affine point solution set to the equation

$$0 = \begin{vmatrix} p \cdot a & p \cdot b \\ p \cdot c & p \cdot d \end{vmatrix}. \tag{5.5}$$

It is not at all immediately obvious as to what type of surface this may be. What we do know, however, is that it must contain the intersection, if any, of the two pairs of planes $a \wedge \overline{d}$ and $c \wedge \overline{d}$. In most cases this is a pair of lines, many instances of which can be seen to fit the ruled surface given in Figure 3.

One immediate observation about equation (5.5) is that if all points lie in a plane through the origin, then the equation remains invariant when p is replaced with $p + v$, where $v \in \mathbb{V}$ is a direction vector orthogonal to that plane. This shows that in this case, B must be the extrusion of a conic section through a dimension parallel to the norm of the common plane of the four points. It is not clear whether all conic sections can be represented this way, though, because certainly not all quadrics can be represented with a 2-blade. It may, howerver, be possible to represent all extruded conic sections with bivectors that are sums of blades having the property just mentioned.

Lastly, it should be noted that, unlike equation (5.4), equation (5.5) is sensative to which of the four points are homogenized and which are not. This is something we would need to carefully consider in any analysis of this equation.

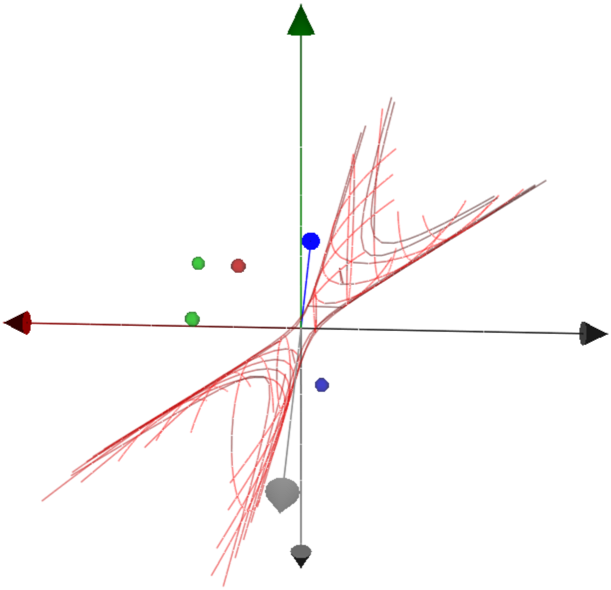

Figure 3. A quadric formed by four affine points composing a 2-blade. Two of the points are green, one is blue and the other red.

6. Transformations In The Model

In this section we finally show that this model has merit with at least one interesting feature, which is that bivector quadrics can be transformed by versors in a meaningful way. Specifically, we can rotate any quadric about any axis through the origin using a carefully formulated rotor.

We begin by observing that for any Euclidean point $v \in \mathbb{V}$, we can easily rotate this point as $Rv\tilde{R}$, where R is given by

$$R = \cos\frac{\theta}{2} - aI\sin\frac{\theta}{2}, \tag{6.1}$$

where the axis $a \in \mathbb{V}$ is a unit-length direction vector, and $I = \prod_{i=1}^{n} e_i$. (The element $e_0 I$ is the unit psuedo-scalar of $\mathbb{G}(\mathbb{V})$.) Furthermore, for any Euclidean point $v \in \mathbb{V}$, notice that

$$\overline{v} = R\overline{v}\tilde{R}, \tag{6.2}$$

showing that the counter-part $\overline{v}$ of v in $\overline{\mathbb{V}}$ remains invarient under this rotation. (The proof of this is similar to the proof we'll give shortly that R leaves e_0 invariant.) Of course, we can formulate an equivilant of R that will rotate $\overline{v}$, and it is simply $\overline{R}$. Then, seeing that $\overline{R}$ leaves v invariant, it follows that

$$V = R\overline{R} \tag{6.3}$$

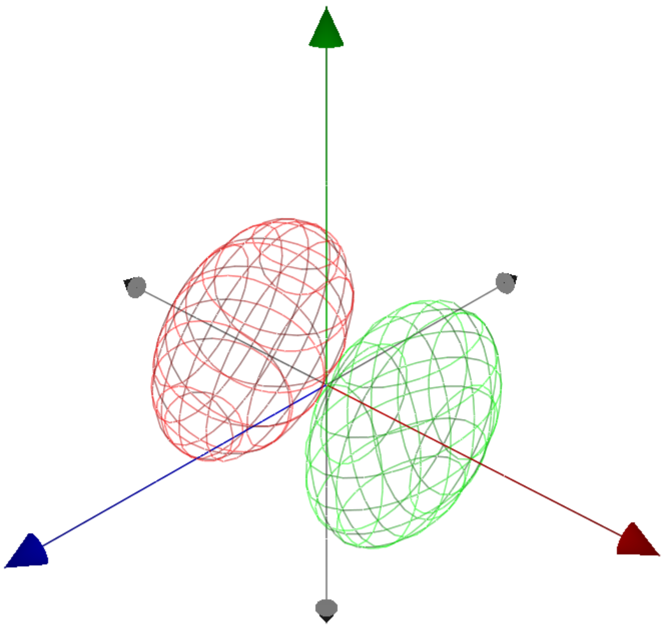

FIGURE 4. The rotation of a spheroid about the axis $(e_1 + e_2 + e_3)/\sqrt{3}$ by π radians.

is a rotor that will rotate the 2-blade $v \wedge \overline{v}$ in a desired way. Specifically, we have

$$V(v \wedge \overline{v})\tilde{V} = Rv\tilde{R} \wedge \overline{Rv\tilde{R}}. \tag{6.4}$$

Now, for all quadrics that are sums of blades of the form $a \wedge \overline{b}$, with $a, b \in \mathbb{V}$, and each of a and b being a Euclidean position or direction related to the quadric, we see that for such quadrics $E \in \mathbb{G}$, the rotation E' of this quadric about an axis $a \in \mathbb{V}$ by an angle θ, is given by

$$E' = VE\tilde{V}. \tag{6.5}$$

Interestingly, this formula applies to all quadrics, because it can be shown that V leaves Ω and A invariant under versor conjugation. Indeed, a spheroid in the form of equation (2.3) can be rotated as illustrated in Figure 4.

To see that V leaves A invariant, notice that

$$VA\tilde{V} = Re_0\tilde{R} \wedge \overline{Re_0\tilde{R}}. \tag{6.6}$$

We need only show now that R leaves e_0 invariant. To that end, we see that

$$R e_0 \tilde{R} = \cos^2\frac{\theta}{2} e_0 + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(e_0 aI - aIe_0) - \sin^2\frac{\theta}{2} aIe_0 aI \tag{6.7}$$

$$= \cos^2\frac{\theta}{2} e_0 - \sin^2\frac{\theta}{2}(aI)^2 e_0 \tag{6.8}$$

$$= \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) e_0 = e_0, \tag{6.9}$$

since $|a| = 1$. Seeing that V leaves Ω invariant is a bit trickier. We first observe that

$$V\Omega\tilde{V} = \sum_{i=1}^{n} Re_i\tilde{R} \wedge \overline{Re_i\tilde{R}}. \tag{6.10}$$

It is important to realize at this point that for all integers $i \in [1, n]$, that $e_i \neq Re_i\tilde{R}$, yet V really does leave Ω invariant. To see why, we will rewrite e_i in equation (6.10) as

$$e_i = \sum_{j=1}^{n} (e_i \cdot e_j)e_j. \tag{6.11}$$

Now realize that

$$Re_i\tilde{R} = \sum_{j=1}^{n} (Re_i\tilde{R} \cdot e_j)e_j. \tag{6.12}$$

It then follows that for any integer $i \in [1, n]$, we have

$$-e_i \wedge \overline{e_i} \cdot V\Omega\tilde{V} = \sum_{j=1}^{n} (Re_i\tilde{R} \cdot e_j)^2 = (Re_i\tilde{R})^2 = 1, \tag{6.13}$$

showing that the coefficient of $e_i \wedge \overline{e_i}$ in $V\Omega\tilde{V}$ is 1. Realize that the application of a rotor leaves the magnitude of a vector unchanged. To finish the proof, we observe that for all integers $i \neq j$ in $[1, n]$, we have

$$-e_i \wedge \overline{e_j} \cdot V\Omega\tilde{V} = \sum_{k=1}^{n} (Re_i\tilde{R} \cdot e_k)(Re_j\tilde{R} \cdot e_k) = (Re_i\tilde{R}) \cdot (Re_j\tilde{R}) = 0, \tag{6.14}$$

showing that the coefficient of $e_i \wedge \overline{e}_j$ in $V\Omega\tilde{V}$ is 0. Realize that the action of a rotor taken with two orthogonal vectors does not change their orthogonal relationship. It now follows that $V\Omega\tilde{V} = \Omega$.

If there is a versor that translates quadrics in our model, it is not at all obvious. Rotations and translations in the conformal model can be developed together in a very nice uniform way by first developing the ability to reflect any conformal geometry about an arbitrary plane, which is really only about as hard as showing that any conformal point can be reflected about such a plane. It is easy to see that planes in our present model, as shown in table (2.11), are also versors. Unlike the conformal model, unfortunately, applying such versors to quadrics in our model does not produce a reflection about the plane. This doesn't mean, however, that we can't find some way to translate quadrics in our model.

Given a direction vector $t \in \mathbb{V}$ and a quadric $E \in \mathbb{G}$, it is not at all hard to show that the quadric E', given by

$$E' = E + (t \cdot E) \wedge e_0 + (\bar{t} \cdot E) \wedge \overline{e_0} - (t \wedge \bar{t} \cdot E)A, \tag{6.15}$$

represents the quadric E translated by the direction vector t. To see this, simply expand the equation

$$0 = (p - t) \wedge \overline{(p - t)} \cdot E \tag{6.16}$$

and then factor out $p \wedge \overline{p}$. (It helps to realize that $\bar{t} \cdot E \in \mathbb{V}$ and $t \cdot E \in \overline{\mathbb{V}}$.) Then, recalling that $A = e_0 \wedge \overline{e_0}$, the form E' in equation (6.15) begins to resemble what might be the result of a transformation of E by some versor. Exactly what versor this might be, however, if it even exists, remains to be seen. If we could find a translation versor, we could combine such versors with those of the form (6.3) to get the rigid body motions, even despite our limitation in being able to rotate only about axes going through the origin. It may not be possible to do this, in which case we must conclude that there must be a better model for quadric surfaces.

7. Flat Quadrics

In the conformal model there is a distinction made between flat and round geometries. Interestingly, it can be seen that flats of the conformal model are really just rounds taken to an infinite extreme. For example, a plane is simply a sphere centered at infinity with infinite radius. Such geometries of the conformal model have the property that ∞ is in the vector sub-space of the blade representing the geometry. We give here a condition that we might use in the quadric model to test a given quadric for its flatness.

For any given quadric $E \in \mathbb{G}$, let $f : \mathbb{V} \to \mathbb{R}$ be the function

$$f(x) = x \wedge \overline{x} \cdot E. \tag{7.1}$$

Then, if for all pairs of homogenized affine points $a, b \in \mathbb{V}$ on E we have

$$0 = f(a - b), \tag{7.2}$$

then B is a flat quadric. We really should be mathematically precise here about what we mean when we say that a given quadric is flat. The reasoning behind this condition, however, may suffice as adequate meaning.

Let $x \in \mathbb{V}$ be the direction vector $a - b$, and see that

$$0 = f(a) = f(b + x) = f(b) + \nabla_x f(b) + f(a - b) = \nabla_x f(b), \tag{7.3}$$

where $\nabla_x f(b)$ is the directional derivative of f at b. It follows that for any b on E, the directional derivative of f at b in every direction from b to any other point a on E is zero. Then, seeing that $\nabla_x f(b) = x \cdot \nabla f(b)$, and that $\nabla f(b)$, (the gradient of f at b), is a direction vector orthogonal to the surface of the quadric at b, it is clear that all other points a on E must be in the tangent space of E at b. It follows that E is flat in any neighborhood of a point upon its surface.

Notice that under this criterion, lines, as they are considered flat in the conformal model, are also considered flat under our present model. To see this, notice that $f(w) = 0$ when E is the line in table (2.11) and w is any non-zero direction vector parallel to the unit-length direction vector v. Notice that cylinders are similarly flat in such directions w, but clearly not in any other direction.

8. Concluding Remarks

While it has been shown that elements of $\mathbb{G}$ do indeed, under a given definition, represent quadric surfaces, there really is nothing more or less interesting about adding and subtracting these elements than adding and subtracting vector equations whose solution sets represent the quadric surfaces. There might not be any great advantage in using the elemental form over the functional form. There is some wonder, however, whether the model can be helpful in studying what is referred to in [3] and [4] as the pencil of two quadrics. It would be particularly interesting if our model could provide a nice proof that a ruled quadric must exist in the pencil of any two quadrics. To begin to answer such a question, we would first need to know how to classify ruled quadrics in $\mathbb{G}$. These would appear to the quadrics $E \in \mathbb{G}$ that are flat in at least one direction on every point of the surface.

That $\mathbb{G}$ was not something fancy like a Minkowski space or some other type of non-Euclidean geometric algebra was perhaps our first clue from the beginning that the potential for great things coming out of this model was, let's say, less than likely. On the other hand, it is very hard to see all ends, and so perhaps there are deep results to be found or new insights to be had using this method of studying quadric surfaces. In any case, geometric algebra has proven to be a fundamental, versatile and unifying language that perhaps most naturally extends mathematics beyond the real number line. Perhaps there is a much better way to use geometric algebra to study quadric surfaces. It is possible that as the model for projective geometry using geometric algebra may be inferior to the conformal model, so the model of this paper may be inferior to a conformal-like model for quadric surfaces.

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