A Model For Quadric Surfaces Using Geometric Algebra

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Abstract. Inspired by the conformal model of geometric algebra, a similar model of geometry is developed for the set of all quadric surfaces in n-dimensional space. Bivectors of the geometric algebra are found to be representative of quadric surfaces. Coordinate free canonical forms of such bivectors are found for common quadric surfaces. The model is investigated for usefulness and compared to the conformal model.

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1. The Construction Of The Model

The stage for this model of n-dimensional quadric surfaces is set in the geometric algebra we'll denote by $\mathbb{G}$ that is generated by a Euclidean vector space $\mathbb{W}$ of dimension $2(n+1)$. Letting $\{e_i\}_{i=0}^{2n+1}$ be an orthonormal set of basis vectors generating $\mathbb{W}$, we let $\{e_i\}_{i=0}^{n}$ be such a set of vectors generating the $(n+1)$-dimensional vector sub-space $\mathbb{V}$ of $\mathbb{W}$ upon which we'll impose the usual interpretation of $(n+1)$-dimensional homogeneous space. Specifically, a vector $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ represents the point given by[1]

$$\frac{e_0 \cdot e_0 \wedge v}{e_0 \cdot v} \tag{1.1}$$

in n-dimensional Euclidean space, imposing the usual correlation between n-dimensional vectors and n-dimensional points.[2] We will take the liberty of letting vectors $v \in \mathbb{V}$ with $v \cdot e_0 = 0$ represent points under the same

[1] Throughout this paper we let the outer product take precedence over the inner product, and the geometric product take precedence over both the inner and outer products.

[2] The correlation between vectors and points spoken of here is that of having a vector represent the point at its tip when its tail is placed at the origin.

interpretation of which has just been spoken, as well as pure directions with magnitude. The intended interpretation will be made clear in the context of our usage. We will refer to all vectors $v \in \mathbb{V}$ with $v \cdot e_0 \neq 0$ as affine points, and such vectors with $v \cdot e_0 = 0$ as Euclidean points or sometimes directions. See section 2.1.1 of [1] for a great introduction to homogeneous coordinates.

We now introduce a function defined on $\mathbb{G}$ having the outermorphic property. This means that it is a linear function and that it preserves the outer product. We will use over-bar notation to denote the use of this function. Doing so, for any element $E \in \mathbb{G}$, we define $\overline{E}$ as

$$\overline{E} = SES^{-1}, \tag{1.2}$$

where the rotor S is given by

$$S = \prod_{i=0}^{n} (1 - e_i e_{i+n+1}). \tag{1.3}$$

As the reader can check, for any integer $i \in [0, n]$, we have $\overline{e_i} = e_{i+n+1}$, and similarly, $\overline{e_{i+n+1}} = -e_i$. The rotor S simply rotates any element taken from the geometric algebra generated by $\mathbb{V}$, (which we'll denote by $\mathbb{G}(\mathbb{V})$), into the identical geometric algebra generated by the vector space we'll denote by $\overline{\mathbb{V}}$ that is complement to $\mathbb{V}$ with respect to $\mathbb{W}$. The over-bar function is an isomorphism between the geometric algebras $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$. We will find the over-bar function convenient when performing algebraic manipulations in our model to the extent that in many cases we can forget about the versor S, letting the over-bar notation be nothing more than a device used to distinguish between elements of $\mathbb{G}(\mathbb{V})$ and $\mathbb{G}(\overline{\mathbb{V}})$.

The geometric algebra $\mathbb{G}$ that we have constructed here is similar to "the mother algebra" in [2], except that while $\mathbb{G}(\mathbb{V})$ is Euclidean, so is the geometric algebra $\mathbb{G}(\overline{\mathbb{V}})$. We do not make use of an anti-Euclidean geometric algebra. Although doing so might prove beneficial, it is worth forgoing for now in the realization that $\mathbb{G}$, as it stands, and as we'll see, sufficiently fulfills at least the minimum requirements of a model for quadric surfaces.

We are now ready to give the definition by which we will interpret bivectors in $\mathbb{G}$ as n-dimensional quadric surfaces. It is as follows. For any element $E \in \mathbb{G}$, we say that E is representative of the n-dimensional quadric surface generated by the set of all affine points $p \in \mathbb{V}$ such that

$$0 = p \wedge \overline{p} \cdot E. \tag{1.4}$$

Notice that when grade$(E) > 1$, there is no ambiguity, despite the non-associativity of the inner product, in rewriting equation (1.4) as

$$0 = p \cdot E \cdot \overline{p}, \tag{1.5}$$

which resembles a sort of conjugation of E by p. This may perhaps be a more familiar form for readers familiar with the study of quadric surfaces in projective geometry. For example, see equation (1) of the chapter on quadric surfaces in [4]. Also notice that we have not required that E be a bivector in definition (1.4), because we may find this condition useful and meaningful for any element of $\mathbb{G}$. For now, however, we will restrict our attention to the case when E is a bivector.

To see why definition (1.4) works, simply notice that when E is a bivector, we have

$$p \wedge \overline{p} \cdot E = \sum_{i=0}^{n} \sum_{j=i}^{n} \lambda_{ij} (p \cdot e_i)(p \cdot e_j), \tag{1.6}$$

which we can recognize as a homogeneous polynomial of degree 2 in the vector components of p. The scalars λ_{ij}, with $0 \leq i \leq j \leq n$, may be formulated in terms of E by

$$\lambda_{ij} = \begin{cases} e_i \overline{e_j} \cdot E & \text{if } i = j, \\ (e_i \overline{e_j} + e_j \overline{e_i}) \cdot E & \text{if } i \neq j. \end{cases} \tag{1.7}$$

It should be noted that bivectors do not uniquely represent quadric surfaces, not even up to scale. This is apparent from equation (1.7) when we see that for $i \neq j$, we can freely choose certain components of the bivector without changing the represented quadric so long as their sum is still λ_{ij}. The problem this may pose in our model comes from a very important result in the conformal model. In the conformal model, if two blades are known to represent the same non-trivial geometry in the same way, then it can be shown that the two blades are equal, up to scale. In our present model, it may take more than just multiplying by a non-zero scalar factor to get a bivector known to represent a certain geometry in a known canonical form. To account for this during the performance of algebraic manipulations on paper, it is convenient to use the following notation. We say that quadrics E_a and E_b are equivalent, writing $E_a \equiv E_b$, whenever E_a and E_b represent the same quadric under definition (1.4). For example, for any two vectors $u, v \in \mathbb{V}$, we have

$$u \wedge \overline{v} \equiv -2u \wedge \overline{v} \equiv u \wedge \overline{v} + v \wedge \overline{u} \equiv (u + \overline{v}) \wedge (u - \overline{v}). \tag{1.8}$$

Be aware that if $E = E_a + E_b$ and $E_a \equiv E_c$, then this does not imply that $E \equiv E_b + E_c$ unless it can be shown that for all affine points $p \in \mathbb{V}$, we have

$$p \wedge \overline{p} \cdot E_a = p \wedge \overline{p} \cdot E_c. \tag{1.9}$$

This condition is weaker than $E_a = E_c$ yet stronger than $E_a \equiv E_c$.

Another important difference to point out here between our present model and the conformal model is that, unlike what we can analogously expect from the point-definition of the conformal model, here the 2-blade form $a \wedge \overline{a}$ found in definition (1.4), for any affine point $a \in \mathbb{V}$ not at origin, does not represent the affine point a under definition (1.4). In homogenized form, the affine point represented by $a \wedge \overline{a}$ is given by

$$e_0 - \left(\frac{e_0 \cdot e_0 \wedge a}{e_0 \cdot a} \right)^{-1}, \tag{1.10}$$

which is the reflection about the origin of the spherical inversion of the affine point a about the unit-sphere centered at the origin. The affine point e_0 at the origin simply represents the empty point-set geometry, or the geometry of nothing. It is also easy to see that $a \wedge \overline{a}$ does not represent the affine point

a, because there are no null blades in our purely Euclidean geometric algebra $\mathbb{G}$.

2. The Construction Of Quadric Surfaces In The Model

Having constructed our model, we are now ready to find canonical forms of bivectors representing a variety of well-known quadric surfaces. Our approach here will be similar to that taken in section 3 of [5].

Let us begin with the spheroid, (a special case of ellipsoid), the circular cylinder, and the circular hyperboloid of one sheet. We will find that all of these surfaces share the same canonical form, because they may all be characterized as the Euclidean point solution set of the equation

$$0 = -r^2 + (x-c)^2 + \lambda((x-c)\cdot v)^2 \tag{2.1}$$

in the Euclidean point $x \in \mathbb{V}$, where $c \in \mathbb{V}$ is a Euclidean point denoting the center of the surface, $v \in \mathbb{V}$ is a unit-length direction vector, $r \in \mathbb{R}$ is the radius of the geometry about the axis v at c, and $\lambda \in \mathbb{R}$ is a scalar indicating the type and extremity of the surface. Specifically, if $\lambda < -1$, we get a circular hyperboloid of one sheet; if $\lambda = -1$, we get a circular cylinder; if $-1 < \lambda < 0$, we get a stretched sphere; if $\lambda = 0$, a sphere; and if $\lambda > 0$, a squished sphere. Interestingly, when $r = 0$ and $\lambda < -1$, we get circular conical surfaces; a right-circular conical surface if $\lambda = -2$.

Expanding equation (2.1), we get

$$0 = x^2 + \lambda(x\cdot v)^2 - 2x\cdot(c+\lambda(c\cdot v)v) + c^2 + \lambda(c\cdot v)^2 - r^2, \tag{2.2}$$

from which it is possible to factor out $-p \wedge \overline{p}$ in terms of the inner product, where $p = e_0 + x$ is a homogenized affine point. Doing so, we see that the bivector E given by

$$E = \Omega + \lambda v \wedge \overline{v} - 2(c+\lambda(c\cdot v)v)\wedge\overline{e_0} + (c^2 + \lambda(c\cdot v)^2 - r^2)A, \tag{2.3}$$

is representative of the three surface types by definition (1.4), where the constant Ω is defined as

$$\Omega = \sum_{i=1}^{n} e_i\overline{e_i}, \tag{2.4}$$

and A is the constant defined as $A = e_0\overline{e_0}$. We will find each of these useful as frequently recurring constants in our calculations.

Such forms as that in equation (2.3) are useful, not only for composition, but especially decomposition in the cases where we have formulated what may, for example, be a spheroid by some means other than composition. This gives the model power as an analytical tool. If we can solve a problem whose solution is a bivector known to represent a spheroid, then we can use this canonical form to answer questions about that spheroid. Where is its center? What is its axis? What is its radius about that axis? As is often the case in mathematics, however, decomposition is harder than composition. A general sequence of decomposition steps for the form (2.3) is not obvious, if it exists, but we will proceed now to give such a sequence for the case when E is known to be a cylinder. That is, when $\lambda = -1$.

The first thing to notice is that the canonical form E in equation (2.3) is in a homogenized form, because the coeficient of Ω is 1. Looking at any canonical form, if there exists a term in that form with a consistent magnitude, (a magnitude that does not change with any instantiation of that form with a given set of parameters), then we can usually find a way to homogenize that form – the process by which we transform any non-homogenized element E' known to represent the same quadric as that of a homogenized and canonical form E into E. For the canonical form (2.3) with $\lambda = -1$, a common[3] non-homogenized form is given by

$$E' = \omega(\Omega - v \wedge \overline{v} - 2u \wedge \overline{e_0} + (u^2 - r^2)A), \tag{2.5}$$

where $u = c - (c \cdot v)v$, $\omega \neq 1$ and $\omega \neq 0$. To find the homogenized form $E = E'/\omega$, it is not hard to show that

$$\omega = -\frac{\Omega \cdot E'}{n-1}. \tag{2.6}$$

We can then proceed to decompose the canonical form E as follows.

We start by recovering the unit-length direction vector v. This can be done as

$$v = \sum_{i=1}^{n} \left(e_i - \overline{e_i \wedge e_0 \cdot e_0 \wedge E}\right). \tag{2.7}$$

It is unfortunate that we had to refer to a basis to obtain v; nevertheless, it is done. The rest of the decomposition will proceed with greater satisfaction.

There is no way to recover c for cylinders, which is quite obvious. The choice for the point c, the center of the cylinder, may be arbitrarily chosen as any point along its spine. This information is lost in composition, so we may therefore arbitrarily choose

$$c = -\frac{1}{2}A \cdot (E \wedge e_0) \tag{2.8}$$

as the cylinder's center, which, incidentally, will also be the point on the spine of the cylinder closest to the origin.

Lastly, we may find the radius of the cylinder from the simple equation

$$r^2 = c^2 + A \cdot E. \tag{2.9}$$

A generalization of equation (2.1) should be mentioned before moving on. It is given by

$$0 = -r^2 + (x - c)^2 + \sum_{i=1}^{k} \lambda_i((x - c) \cdot v_i)^2, \tag{2.10}$$

which would probably give us the general set of ellipsoids, provided the set of k direction vectors in $\{v_i\}_{i=1}^k$ are linearly independent.

[3]Recall that it may take more than multiplying by a simple scalar factor to homogenize a bivector in $\mathbb{G}$ as discussed in Section 1.

In any case, specific canonical forms for circles, lines, planes, plain-pairs, and so on, can be found using the more general form (2.3). One can then add and subtract these forms and then identify the result as an instance of some other form. For example, the difference of two spheres is a plane, and the sum of two points is an imaginary sphere.

3. Intersecting Quadric Surfaces

In this section we briefly consider the ability to intersect quadric surfaces in the model. A desire to do such a thing is motivated by an ability to do it in the conformal model, and by its applications to drawing solid objects composed of quadric surfaces. (See [3], [6] and [5].) After finding vectors dually representative of planes and spheres, the ability to intersect such geometries is one method of the conformal model for generating all geometries of that model. It can be seen how the model at present is capable of representing planes and spheres, so likewise we would wish to intersect them.

Noticing that any trivector $T \in \mathbb{G}$ can be written in the form

$$T = \sum_{i=1}^{k} v_i \wedge B_i, \tag{3.1}$$

where $\{v_i\}_{i=1}^k \subset \mathbb{W}$ is a set of k vectors and $\{B_i\}_{i=1}^k \subset \mathbb{G}$ is a set of k bivectors, we may proceed to apply definition (1.4) to find that

$$0 = p \wedge \overline{p} \cdot T = p \cdot \sum_{i=1}^{k} (\overline{p} \cdot v_i) B_i - \overline{p} \cdot \sum_{i=1}^{k} (p \cdot v_i) B_i + \sum_{i=1}^{k} (p \wedge \overline{p} \cdot B_i) v_i, \tag{3.2}$$

showing that T is representative of the quadric surface that is the intersection of all quadrics in $\{B_i\}_{i=1}^k$, provided that $\{v_i\}_{i=1}^k$ is a linearly independent set of vectors, and that for all affine points $p \in \mathbb{V}$, and for any integer $i \in [1, k]$, we have $p \cdot v_i = \overline{p} \cdot v_i = 0$. This last condition, unfortunately, cannot be satisfied in $\mathbb{G}$ as we have defined it, nor does it seem practical or reasonable to extend $\mathbb{G}$ in a way that makes the condition possible, mainly because the resulting trivector would not appear capable of ever directly characterizing the intersection, but only indirectly as the intersection of the quadrics in $\{B_i\}_{i=1}^k$.

A 3-way intersection, however, (which was given a great deal of consideration in [7]), is quite possible in the present model. For any given non-zero 3-blade $T \in \mathbb{G}$, given by $T = a \wedge b \wedge c$, the geometry represented by this 3-blade under definition (1.4) is the intersection of the 3 quadrics $a \wedge b$, $a \wedge c$ and $b \wedge c$. The proof of this follows directly from the following identity, which the reader can easily verify.

$$p \wedge \overline{p} \cdot a \wedge b \wedge c = (p \wedge \overline{p} \cdot a \wedge b)c - (p \wedge \overline{p} \cdot a \wedge c)b + (p \wedge \overline{p} \cdot b \wedge c)a \tag{3.3}$$

Now realize that since $a \wedge b \wedge c \neq 0$, $\{a, b, c\}$ is a linearly independent set, and therefore, $0 = p \wedge \overline{p} \cdot T$ if and only if p is on $a \wedge b$, $a \wedge c$ and $b \wedge c$.

This ability to intersect three quadrics in this way, however, is restrictive in at least two ways. First, all three quadrics must be blades, and secondly, the three quadrics must pair-wise share a common vector in their respective factorizations.

4. Transformations In The Model

In this section we show that bivector quadrics can be transformed by versors in a meaningful way. Specifically, we can rotate any quadric about any axis through the origin using a carefully formulated rotor.

We begin by observing that for any Euclidean point $v \in \mathbb{V}$, we can easily rotate this point as $Rv\tilde{R}$, where R is given by

$$R = \cos\frac{\theta}{2} - aI\sin\frac{\theta}{2}, \tag{4.1}$$

where the axis $a \in \mathbb{V}$ is a unit-length direction vector, and $I = \prod_{i=1}^{n} e_i$. (The element $e_0 I$ is the unit psuedo-scalar of $\mathbb{G}(\mathbb{V})$.) Furthermore, for any Euclidean point $v \in \mathbb{V}$, notice that

$$\overline{v} = R\overline{v}\tilde{R}, \tag{4.2}$$

showing that the counter-part $\overline{v}$ of v in $\overline{\mathbb{V}}$ remains invarient under this rotation. (The proof of this is similar to the proof we'll give shortly that R leaves e_0 invariant.) Of course, we can formulate an equivilant of R that will rotate $\overline{v}$, and it is simply $\overline{R}$. Then, seeing that $\overline{R}$ leaves v invariant, it follows that

$$V = R\overline{R} \tag{4.3}$$

is a rotor that will rotate the 2-blade $v \wedge \overline{v}$ in a desired way. Specifically, we have

$$V(v \wedge \overline{v})\tilde{V} = Rv\tilde{R} \wedge \overline{Rv\tilde{R}}. \tag{4.4}$$

Now, for all quadrics that are sums of blades of the form $a \wedge \overline{b}$, with $a, b \in \mathbb{V}$, and each of a and b being a Euclidean position or direction related to the quadric, we see that for such quadrics $E \in \mathbb{G}$, the rotation E' of this quadric about an axis $a \in \mathbb{V}$ by an angle θ, is given by

$$E' = VE\tilde{V}. \tag{4.5}$$

Interestingly, this formula applies to all quadrics, because it can be shown that V leaves Ω and A invariant under versor conjugation. Indeed, a spheroid in the form of equation (2.3) can be rotated as illustrated in Figure 1.

To see that V leaves A invariant, notice that

$$VA\tilde{V} = Re_0\tilde{R} \wedge \overline{Re_0\tilde{R}}. \tag{4.6}$$

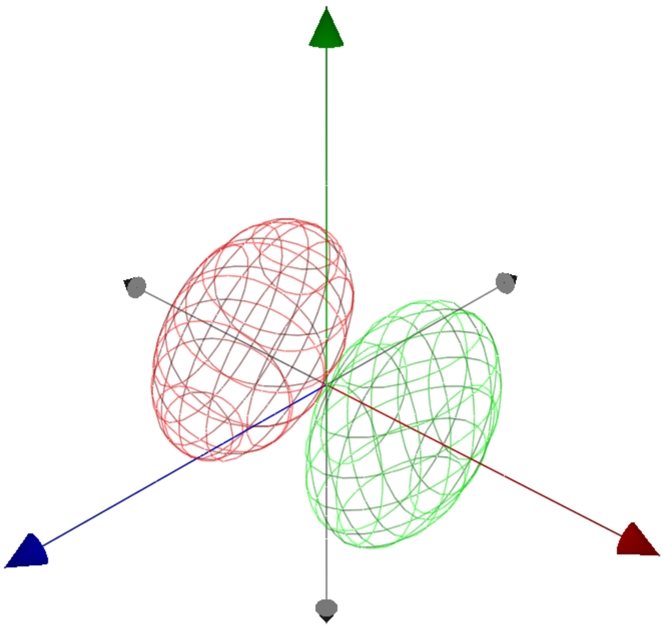

FIGURE 1. The rotation of a spheroid about the axis $(e_1 + e_2 + e_3)/\sqrt{3}$ by π radians.

We need only show now that R leaves e_0 invariant. To that end, we see that

$$R e_0 \tilde{R} = \cos^2\frac{\theta}{2} e_0 + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(e_0 aI - aIe_0) - \sin^2\frac{\theta}{2} aIe_0aI \tag{4.7}$$
$$= \cos^2\frac{\theta}{2} e_0 - \sin^2\frac{\theta}{2}(aI)^2 e_0 \tag{4.8}$$
$$= \left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) e_0 = e_0, \tag{4.9}$$

since $|a| = 1$. Seeing that V leaves Ω invariant is a bit trickier. We first observe that

$$V\Omega\tilde{V} = \sum_{i=1}^{n} Re_i\tilde{R} \wedge \overline{Re_i\tilde{R}}. \tag{4.10}$$

It is important to realize at this point that for all integers $i \in [1, n]$, that $e_i \neq Re_i\tilde{R}$, yet V really does leave Ω invariant. To see why, we will rewrite e_i in equation (4.10) as

$$e_i = \sum_{j=1}^{n} (e_i \cdot e_j)e_j. \tag{4.11}$$

Now realize that

$$Re_i\tilde{R} = \sum_{j=1}^{n}(Re_i\tilde{R}\cdot e_j)e_j. \tag{4.12}$$

It then follows that for any integer $i \in [1, n]$, we have

$$-e_i \wedge \overline{e_i} \cdot V\Omega\tilde{V} = \sum_{j=1}^{n}(Re_i\tilde{R}\cdot e_j)^2 = (Re_i\tilde{R})^2 = 1, \tag{4.13}$$

showing that the coefficient of $e_i \wedge \overline{e_i}$ in $V\Omega\tilde{V}$ is 1. Realize that the application of a rotor leaves the magnitude of a vector unchanged. To finish the proof, we observe that for all integers $i \neq j$ in $[1, n]$, we have

$$-e_i \wedge \overline{e_j} \cdot V\Omega\tilde{V} = \sum_{k=1}^{n}(Re_i\tilde{R}\cdot e_k)(Re_j\tilde{R}\cdot e_k) = (Re_i\tilde{R})\cdot(Re_j\tilde{R}) = 0, \tag{4.14}$$

showing that the coefficient of $e_i \wedge \overline{e}_j$ in $V\Omega\tilde{V}$ is 0. Realize that the action of a rotor taken with two orthogonal vectors does not change their orthogonal relationship. It now follows that $V\Omega\tilde{V} = \Omega$.

Rotations and translations in the conformal model can be developed together in a very nice uniform way by first developing the ability to reflect any conformal geometry about an arbitrary plane, which is really only about as hard as showing that any conformal point can be reflected about such a plane. It is easy to see that planes in our present model are also versors. Unlike the conformal model, however, applying such versors to quadrics in our model does not produce a reflection about the plane. But this doesn't mean that we can't find some way to translate quadrics in our model anyway.

Given a direction vector $t \in \mathbb{V}$ and a quadric $E \in \mathbb{G}$, it is not at all hard to show that the quadric E', given by

$$E' = E + (t\cdot E)\wedge e_0 + (\overline{t}\cdot E)\wedge\overline{e_0} - (t\wedge\overline{t}\cdot E)A, \tag{4.15}$$

represents the quadric E translated by the direction vector t. To see this, simply expand the equation

$$0 = (p-t)\wedge\overline{(p-t)}\cdot E \tag{4.16}$$

and then factor out $p \wedge \overline{p}$. (It helps to realize that $\overline{t}\cdot E \in \mathbb{V}$ and $t \cdot E \in \overline{\mathbb{V}}$.) That E' is not the result of applying some versor to E is unfortunate, and leaves us to conclude that only in a better model do the rigid body motions of quadrics exist as versor transformations.

5. Concluding Remarks

While it has been shown that elements of $\mathbb{G}$ do indeed, under a given definition, represent quadric surfaces, there really is nothing more or less interesting about adding and subtracting these elements than adding and subtracting vector equations whose solution sets represent the quadric surfaces. There might not be any great advantage in using the elemental form over the functional form. There is some wonder, however, whether the model can be helpful in studying what is referred to in [5] and [7] as the pencil of two quadrics, which is also essential to the study of intersecting quadrics in [3].

That $\mathbb{G}$ was not something fancy like a Minkowski space or some other type of non-Euclidean geometric algebra was perhaps our first clue from the beginning that the potential for great things coming out of this model was, let's say, less than likely. On the other hand, it is very hard to see all ends, and so perhaps there are deep results to be found or new insights to be had using this method of studying quadric surfaces. In any case, geometric algebra has proven to be a fundamental, versatile and unifying language that perhaps most naturally extends mathematics beyond the real number line. Perhaps there is a much better way to use geometric algebra to study quadric surfaces. It is possible that as the model for projective geometry using geometric algebra may be inferior to the conformal model, so the model of this paper may be inferior to a conformal-like model for quadric surfaces.

References

1. S. Birchfield, *An introduction to projective geometry*, `http://vision.stanford.edu/~birch/projective/projective.pdf`.
2. C. Doran and D. Hestenes, *Lie groups as spin groups*, J. Math. Phys. **34** (1993), 8.
3. J. Levin, *A parametric algorithm for drawing pictures of solid objects composed of quadric surfaces*, Commun. ACM **19, 10**.
4. K. Mehlhorn and C. Yap, *Robust geometric computation (tentative title)*, `http://cs.nyu.edu/yap/bks/egc/09/21Surfaces.pdf`, 2004.
5. J. Miller, *Geometric approaches to nonplanar quadric surface intersection curves*, ACM Transactions on Graphics **Vol. 6, No. 4**, pp. 274–307.
6. W. Wang, R. Goldman, and C. Tu, *Enhancing levin's method for computing quadric-surface intersections*, Computer Aided Geometric Design **20** (2003), 401–422.
7. Z. Xu, X. Wang, X. Chen, and J. Sun, *A robust algorithm for finding the real intersections of three quadric surfaces*, Computer Aided Geometric Design **22, Issue 6, 1** (2005), 515–530.

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