Planar Physics with Geometric Algebra

Spencer T. Parkin

December 7, 2023

The goal of this paper is to develop the math and algorithms necessary to perform basic rigid-body dynamics in the plane using geometric algebra. As the reader will see, the equations fall out quite nicely under this alternative mathematical framework. As we go along we can make comparisons with traditional methods involving matrices and the cross product. Geometric algebra (or GA for short) generalizes the inner product by first developing the notion of the outer product. An introduction to GA is first given before diving into the main material. Readers already familiar with GA can just skip over that part.

1 Intro to GA

A GA is constructed using a vector space $\mathbb{V}$, a set of scalars $\mathbb{R}$, addition, and the outer product. The scalars can be any field, such as the complex numbers, but since complex numbers arise naturally as a sub-structure of the GA, it makes sense to just stick with $\mathbb{R}$.

Given a set of n vectors $\{v_i\}_{i=1}^n$ taken from $\mathbb{V}$, we define

$$V = \bigwedge_{i=1}^{n} v_i \tag{1}$$

to be an n-blade if the vectors form a linearly independent set, or zero otherwise. We call n the grade of the blade, and it follows immediately that the dimension of $\mathbb{V}$ is an upper-bound on the highest non-zero grade blade we can make.

But how do we imagine a blade? Well, in the case $n = 1$, these are already familiar, because they're just vectors. In the case $n = 2$, you can think of them as pieces of area. Similar to vectors, they need not be rooted in any particular location, and they have an orientation and magnitude. Unlike vectors, however, they have a non-unique factorization. Often, the ability to choose how we factor a blade can make it easier to algebraically manipulate them in equations in order to work them out. In the case $n = 3$, you can think of these as pieces of volume, also factorable in many ways, and so on.

Now enters the inner product, and we can understand this product in terms of the outer product in the sense that it simply does the opposite thing that the outer product does. That is, while the outer product builds blades up, the inner product breaks them down. You're probably already familiar with the inner product between two vectors (or 1-blades) as a scalar value (or 0-blade), and you already know the geometric interpretation of this operation. We are now going to generalize this with higher-grade blades.

First, imagine the outer product as something that extrudes a point (0-blade) through space to make a vector (1-blade). Now use the outer product again to extrude a vector (1-blade) through space to make a 2-blade, and think of it in the shape of a parallelagram. This parallelagram, though, is not the only one representative of the 2-blade, remember, because the factorization in terms of the outer product is not unique. (Just bare that in mind for now.) Now extruding the 2-blade (or parallelagram) again, to get a 3-blade, or peice of volume, and think of it in the shape parallelepiped, or even just a box.

Okay, now let's break it down using the inner product. With each extrusion, a vector was taken in the outer product with what we had to build it into a higher-grade blade. We'll now take what we have in the inner product with a vector to break it down into a lower-grade blade. As we do this, only the component of the vector parallel to (or in the space of) the blade will contribute to the operation, while the component of the vector perpendicular to the blade will play no part. That is, a vector taken in the inner product with a blade is zero if that vector is perpendicular to the blade. If we're working in 3 dimensions, then a vector will always collapse a 3-blade into a 2-blade perpendicular to that vector. Similarly, another vector can then collapse the 2-blade into a vector orthogonal to it, provided it wasn't orthogonal to the 2-blade.

This has all been a bit hand-wavey so far, so let's be more concrete about it with some algebra. Given a 2-blade B, and a 1-blade v, let's consider the inner product $v \cdot B$. We begin by writing

$$v \cdot B = (v_\perp + v_\parallel) \cdot B = v_\perp \cdot B + v_\parallel \cdot B = v_\parallel \cdot B. \tag{2}$$

This equation illustrates the fact that only the component $v_\parallel$ of v parallel to B will contribute to the operation, since $v_\perp \cdot B = 0$. Now, since factorizations of blades are not unique, we can write $B = v_\parallel \wedge b$ for some appropriate vector b such that $v_\parallel \cdot b = 0$. This is because $v_\parallel$ is in the space of the blade, which is also to say that $v_\parallel \wedge B = 0$. We can then define

$$v_\parallel \cdot B = v_\parallel \cdot (v_\parallel \wedge b) \equiv |v_\parallel|^2 b. \tag{3}$$

And there we have it! How should we interpret this? Well, I like to think of it as a collapsing of B into the space of b along the $v_\parallel$ dimension; the opposite of an extrusion, but it shouldn't be confused with a projection. It's not quite that. We can actually calculate the projection p of v onto B as

$$p = B \cdot v \cdot B^{-1}. \tag{4}$$

Note that the inner product is not necessarily associatve in all cases. Also, what are we to make of B^{-1}? Neither the inner nor the outer product supports a multiplicative inverse. Well, that's where the geometric product comes in. Given a vector $a \in \mathbb{V}$, and blade B, we define

$$aB \equiv a \cdot B + a \wedge B \tag{5}$$

to be the geometric product of a and B. To show that this is invertible, realize that the product is associative and commutative, and then write

$$\begin{aligned} aaB &= a(a \cdot B) + a(a \wedge B) && (6)\\ &= a \cdot (a \cdot B) + a \wedge (a \cdot B) + a \cdot (a \wedge B) + a \wedge (a \wedge B). && (7) \end{aligned}$$

We can now apply some logic. Clearly, $a \wedge a \wedge B = 0$, because a vector repeated in a list never gives a linearly independent set of vectors. Also, we must have $a \cdot (a \cdot B) = 0$, because a will be orthogonal to the collapse $a \cdot B$ of B by a. This leaves us

$$aaB = a \wedge (a \cdot B) + a \cdot (a \wedge B). \tag{8}$$

Again, applying what we know about the inner and outer products, we see here that in one case, B is collapsed by a, then extruded by a; and in another case, B is extruded by a, then collpased by a. In either case, the net result is just some scalar multiple of B. What scalar multiple? Well, we can also write

$$aaB = (a \cdot a + a \wedge a)B = |a|^2 B. \tag{9}$$

All in all, we've shown that $a^{-1} = a/|a|^2$, but what about B^{-1} more generally? A trick for determining this is to write B in terms of an orthogonal set of basis vectors $\{b_i\}_{i=1}^n$. (We then have, for any two distinct integers $i, j \in [1, n]$, the property $b_i \cdot b_j = 0$.) Doing so, we have

$$B = \bigwedge_{i=1}^{n} b_i = \prod_{i=1}^{n} b_i. \tag{10}$$

Realizing that the outer product is anti-commutative, we also have

$$B = \bigwedge_{i=1}^{n} b_i = (-1)^k \bigwedge_{i=1}^{n} b_{n-i+1} = (-1)^k \prod_{i=1}^{n} b_{n-i+1}, \tag{11}$$

where $k = n(n-1)/2$ is a triangle number. Thus,

$$B^2 = (-1)^k \left(\prod_{i=1}^{n} b_{n-1+1} \right) \left(\prod_{i=1}^{n} b_i \right) = \prod_{i=1}^{n} |b_i|^2. \tag{12}$$

Finally, we see that

$$B^{-1} = \frac{(-1)^k B}{\prod_{i=1}^{n} |b_i|^2}. \tag{13}$$

So far we've covered blades, the inner and outer products, and the geometric product. The last thing to cover to give one a general overview of GA is simply a linear combination of blades, which is what we call a multivector. Realize that while all vectors are closed under addition, this is not true of blades in general. That is, the sum of two blades is not necessarily a blade, even if they have the same grade. If they do have the same grade, then the result is what we call a k-vector. Every k-blade is a k-vector, but not every k-vector is a k-blade. In the case $k = 1$, we call them vectors; $k = 2$, bivectors; $k = 3$ trivectors, and so on. It's not hard to see why some k-vectors are not k-blades. Just add two 2-blades from partially disjoint vector spaces.

There are many applications of GA; most notably are those of projective and conformal geometry. In each of these, a vector space of dimension greater than 3 is used to generate a geometric algebra, and then the projections into 3-dimensional space of the elements of that algebra are studied. Operations in the higher-dimensional algebra produce geometrically meaningful results in terms of their "shadows" in 3-dimensional space. Some are quite fascinating. For example, in conformal geometric algebra,[1] the outer product of 4 vectors, each representative of a point in 3-dimensional space, gives an element (a 4-blade) representative of the 3-dimensional sphere fitting those 4 points, if the blade is non-zero, of course. Pretty cool stuff! Can you think of calculating the sphere fitting 4 given points without such an algebra?

But I believe I've digressed too far. Let's get back to physics in the plane as our chosen application of GA.

2 Rigid Body Model

[1]CGA is patented, by the way, which I think is silly, but whatever.