# Revisiting Time-Space Tradeoffs for Function Inversion

**Spencer Peters**

Noah S.D.

Siyao Guo

Sasha Golovnev

# Function Inversion

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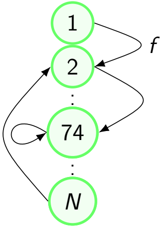

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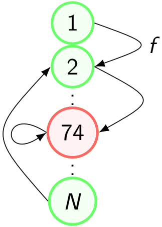

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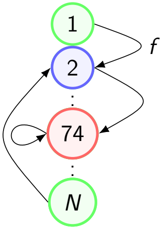

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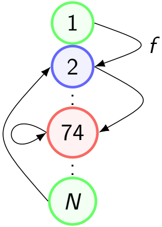


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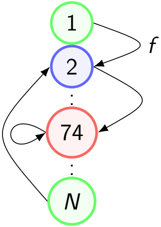


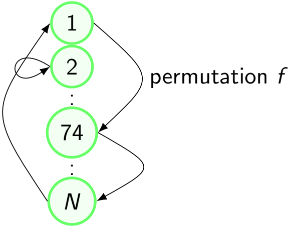


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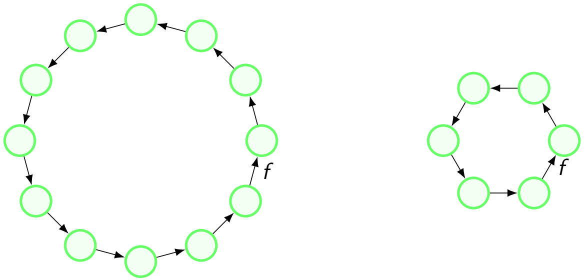

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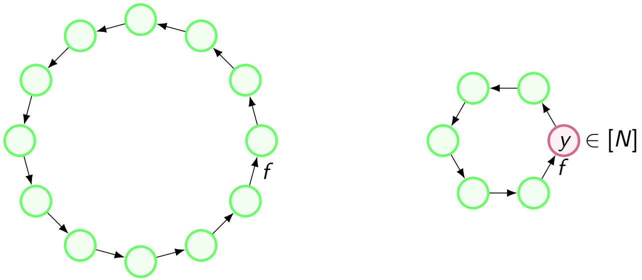

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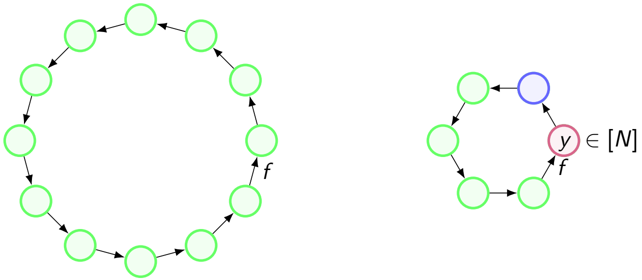

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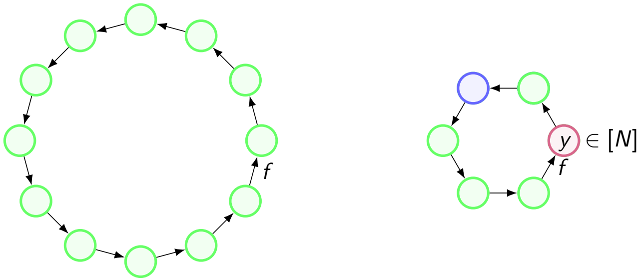

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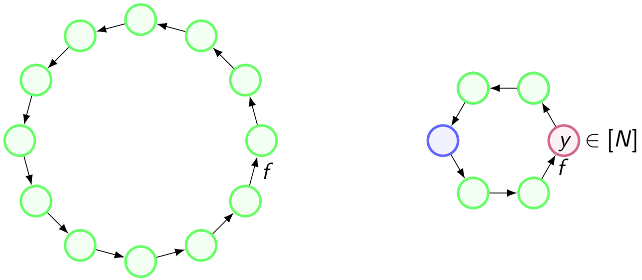

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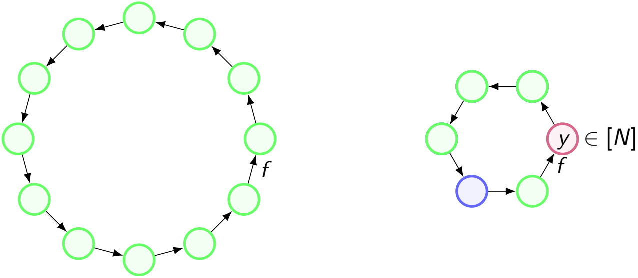

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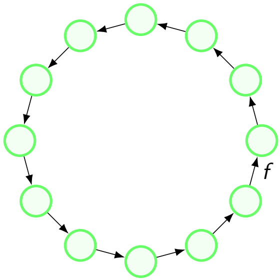


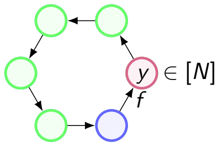

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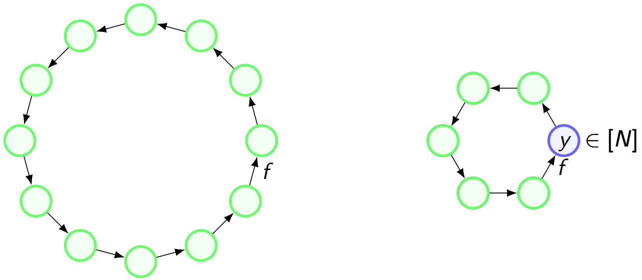

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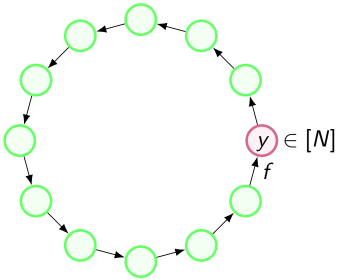


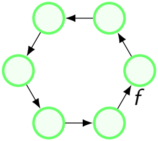

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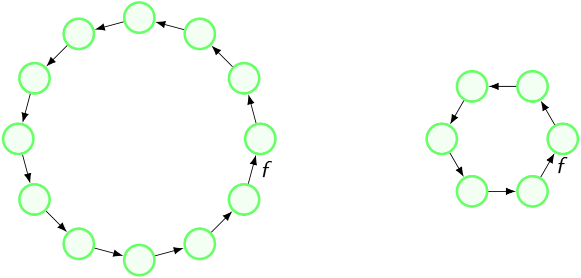

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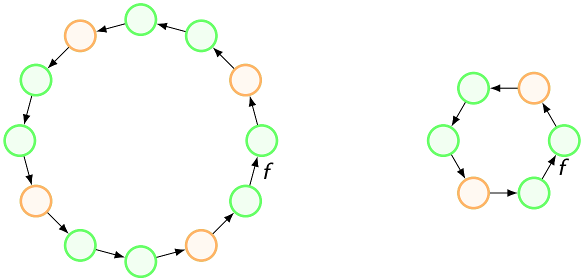


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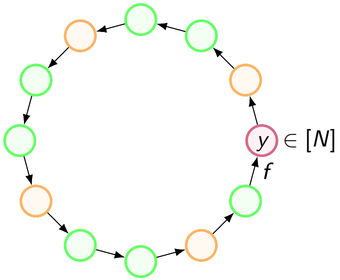


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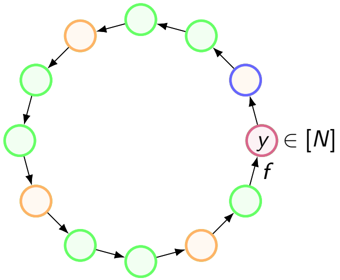


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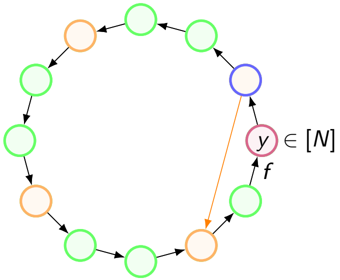


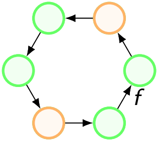


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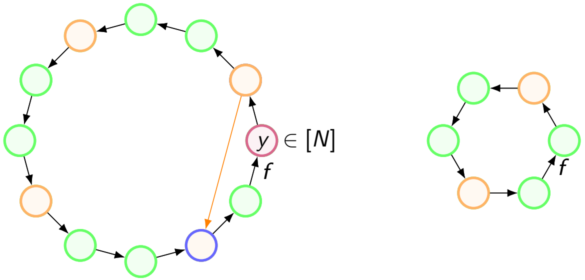


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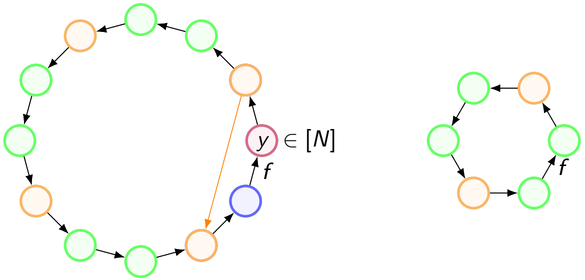


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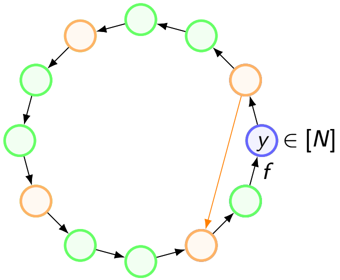


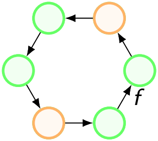


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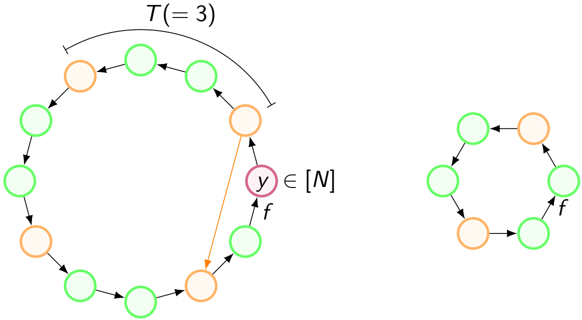

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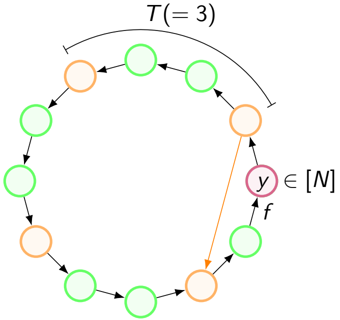


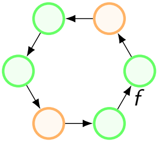


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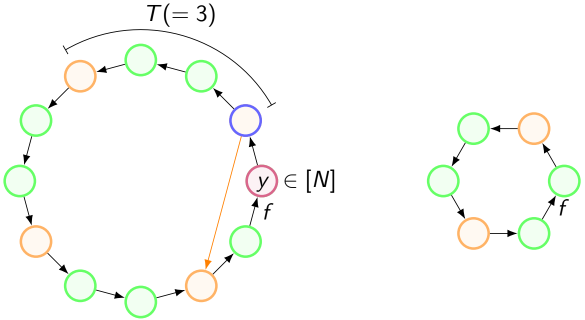


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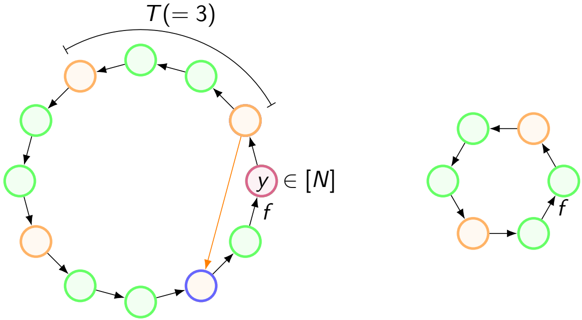


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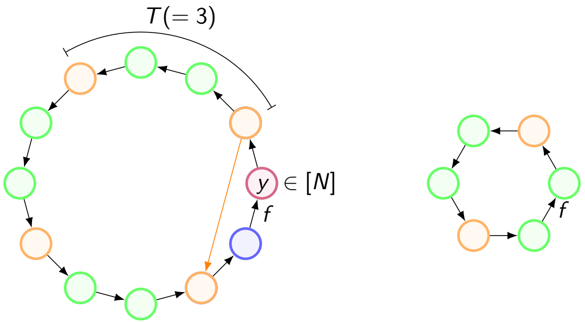


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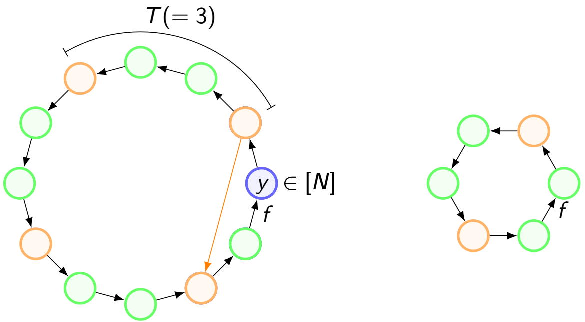


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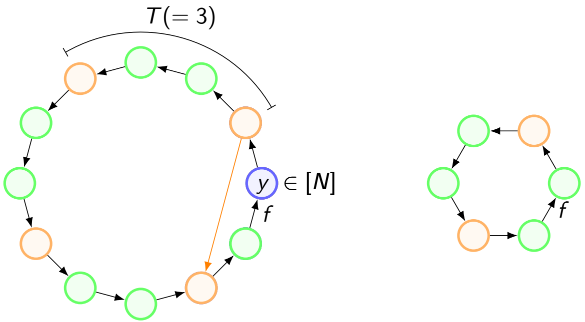


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- We need to store about $N/T$ points total.

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- But, $\alpha$ has bitlength at most $S$, and $\mathcal{A}$ can make at most $T$ evaluations of $f$.

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- Theoretical computer scientists want better algorithms for 3-SUM [GGH+20],     , multiparty pointer jumping [CK19],   systematic substring search [CK19], ...

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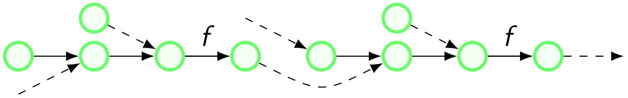

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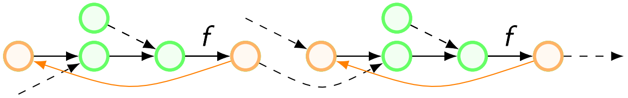

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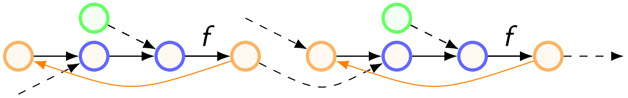

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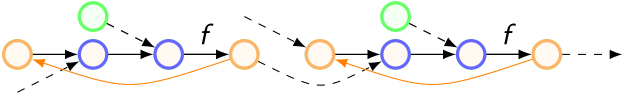


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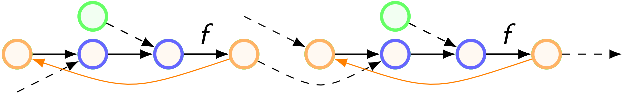


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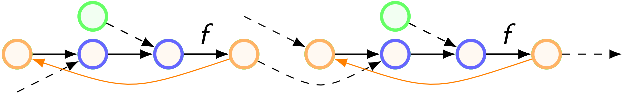


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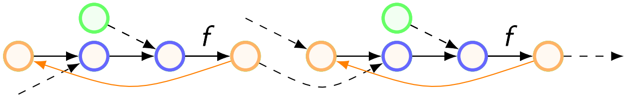


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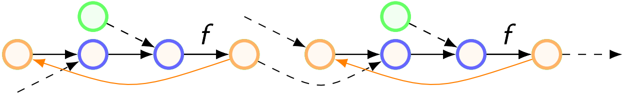


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- So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen “rerandomization” functions $g_i$.
- For *random* functions, Hellman showed (heuristically) this can be made to work. 

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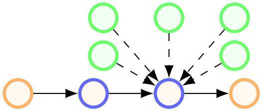

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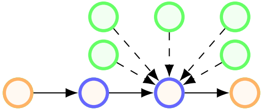

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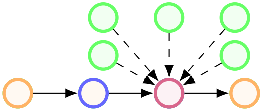

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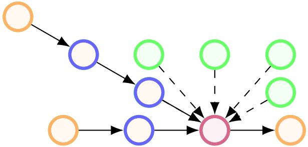

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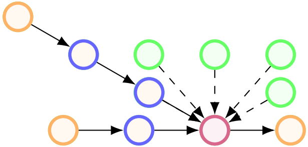

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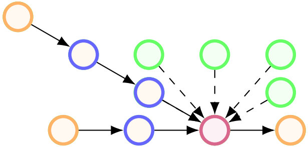

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- Among other things, overlapping paths are redundant, causing massive inefficiency!
- Fiat and Naor deal with this by storing $\alpha = (\alpha', L)$, where $L$ contains junction points along with their inverses.
- Intuitively, $\alpha'$ is the data structure for a *restriction* of $f$ that avoids the junction points in $L$.
- More precisely, the "rerandomization" functions are sampled using rejection sampling so that their range is $[N] - L$.

# The story so far

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- A: Sort of and sort of!

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- How do $\mathcal{A}$ and $\mathcal{P}$ agree on the same list of random values $x_i$?

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- This approach leans fairly heavily on non-uniformity. But in practice, can just instantiate a random oracle.

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- Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- $\implies$ preprocessing returns $\alpha$ maximizing

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- $\implies \max_\alpha |\{y \in [N] : \alpha_y = \alpha\}| \geq \Omega(\log N / \log\log N)$.

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- That is, it can be thought of returning $T$ candidate inverses $x_1, \ldots, x_T$, *without querying $f$ at all*.
- We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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  - For each $y$, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \ldots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

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  - A larger codomain does not make the problem harder (us and [CK19])  

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- $S^2 T = N^2$ algorithm for worst-case function inversion?
- Improved algorithm for injective functions?

# Thank you!

I'm happy to take additional questions offline. You can ping me at `sp2473@cornell.edu`.

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