



Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.



Siyao Guo

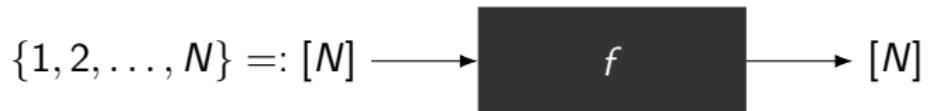


Sasha Golovnev



Function Inversion

- ▶ Given a function



Function Inversion

- ▶ Given a function, and a point y in its range,



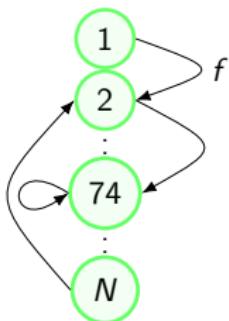
Function Inversion

- Given a function, and a point y in its range, find x with $f(x) = y$.

$$\{1, 2, \dots, N\} =: [N] \ni x \rightarrow \boxed{f} \rightarrow y \in [N]$$

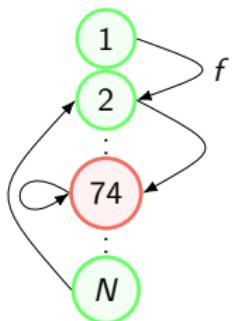
Function Inversion

- Given a function, and a point y in its range, find x with $f(x) = y$.



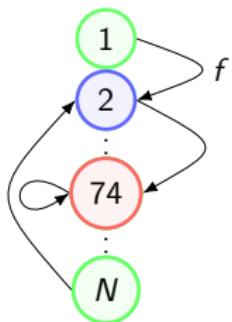
Function Inversion

- ▶ Given a function, and a point y in its range, find x with $f(x) = y$.



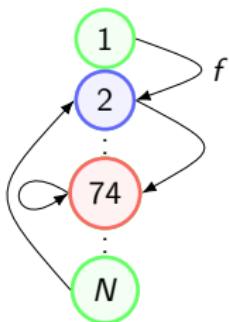
Function Inversion

- Given a function, and a point y in its range, find x with $f(x) = y$.



Function Inversion

- ▶ Given a function, and a point y in its range, find x with $f(x) = y$.



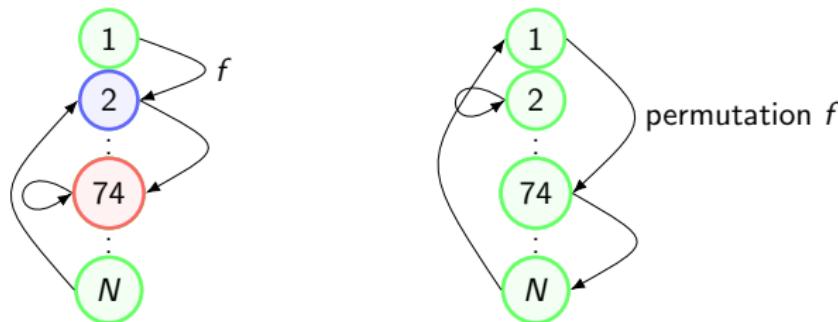
- ▶ The study of this *black-box* function inversion problem was initiated by Martin Hellman



in 1980 [Hel80].

Function Inversion

- Given a function, and a point y in its range, find x with $f(x) = y$.

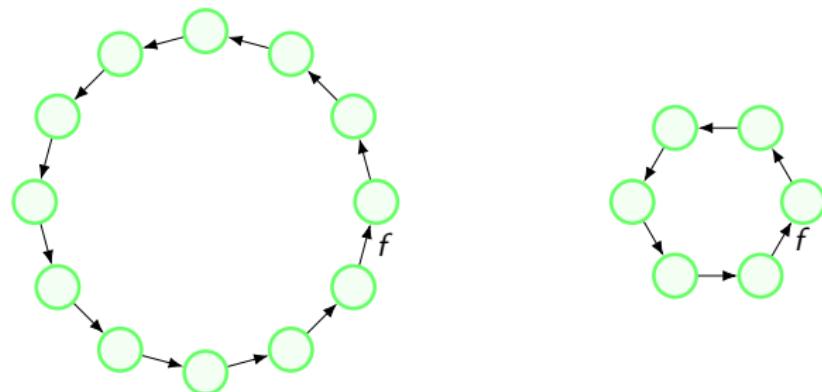


- The study of this *black-box* function inversion problem was initiated by Martin Hellman in 1980 [Hel80].



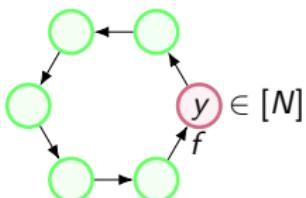
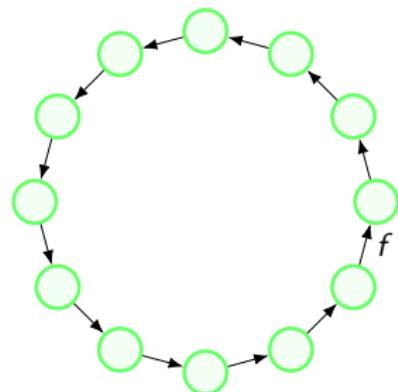
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



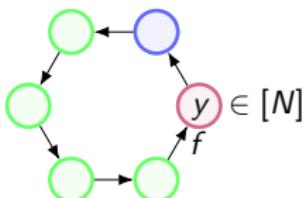
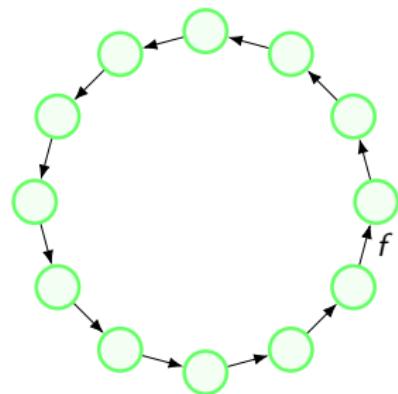
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



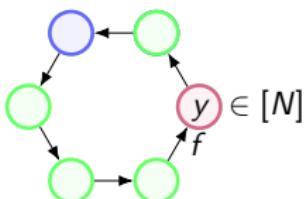
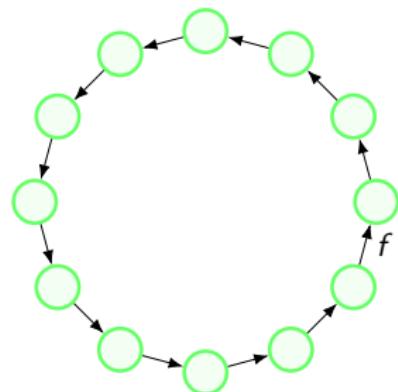
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



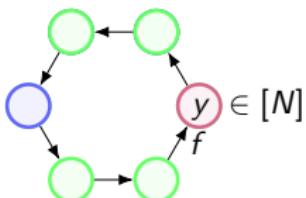
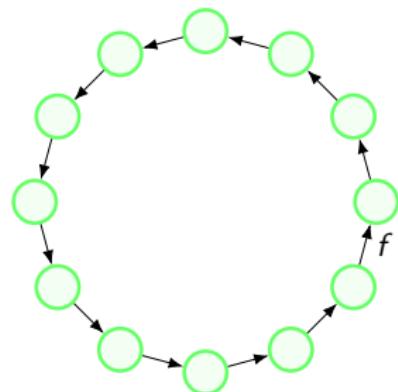
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



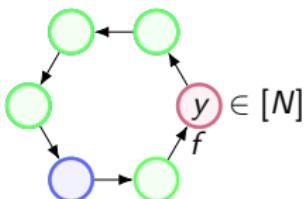
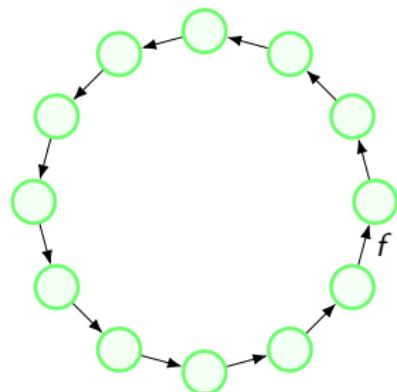
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



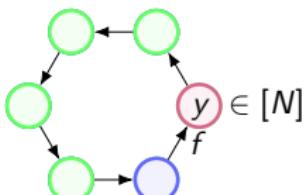
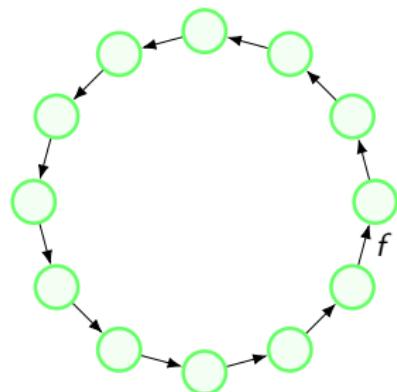
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



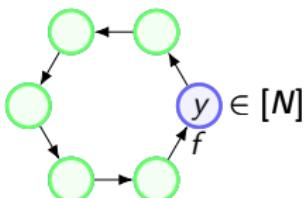
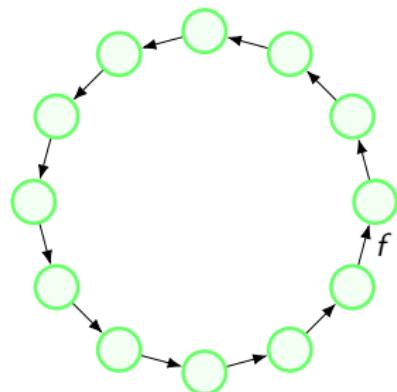
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.

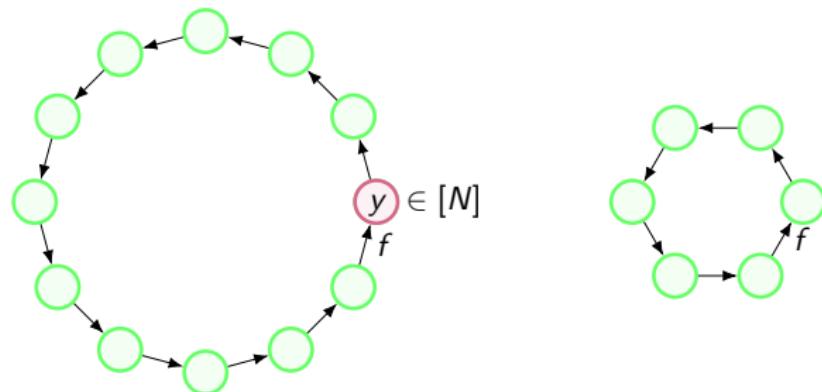


Hellman's algorithm

- ▶ What if y is on a large cycle?

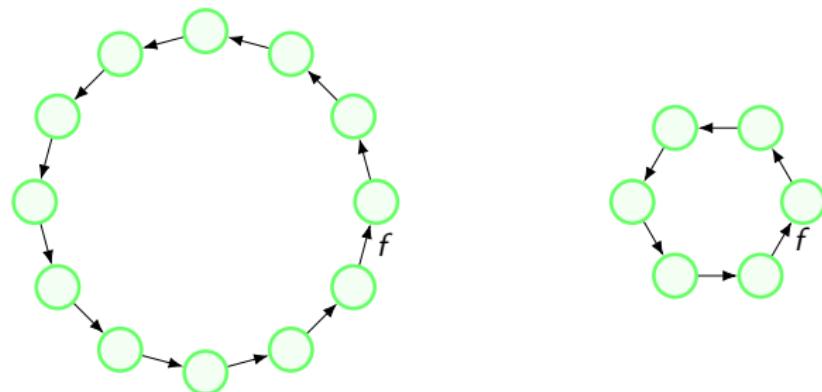
Hellman's algorithm

- ▶ What if y is on a large cycle?



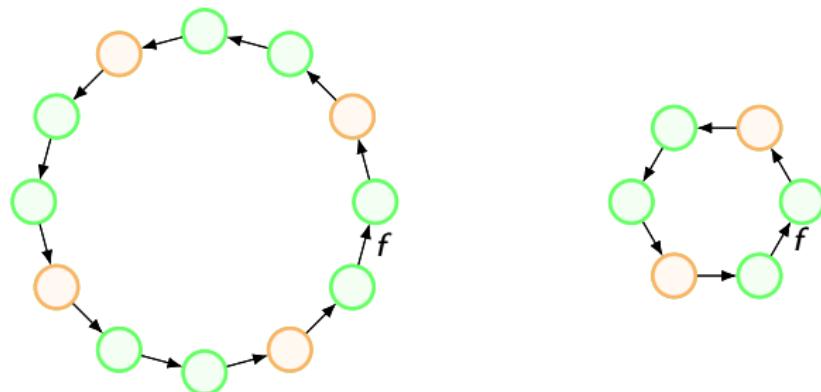
Hellman's algorithm

- ▶ What if y is on a large cycle?



Hellman's algorithm

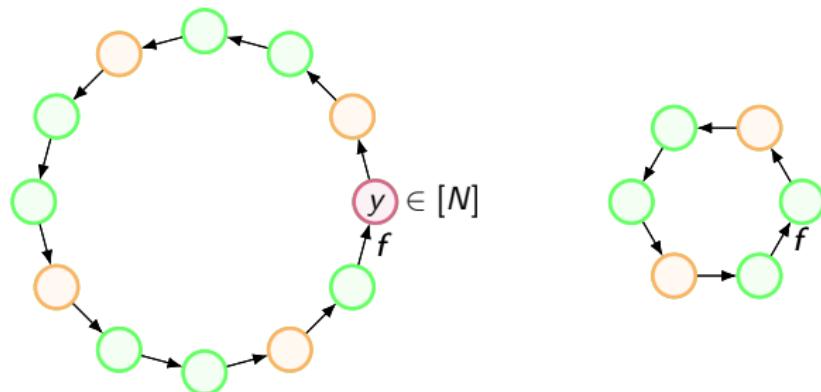
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

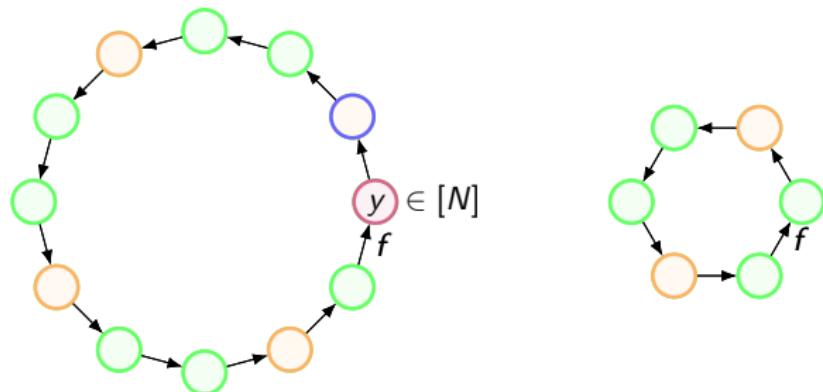
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

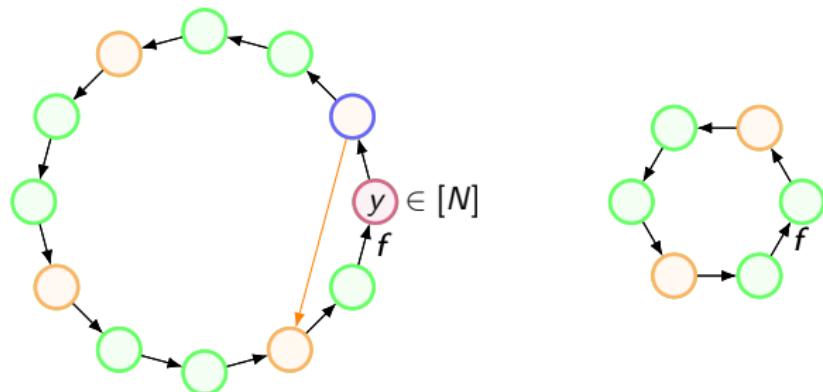
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

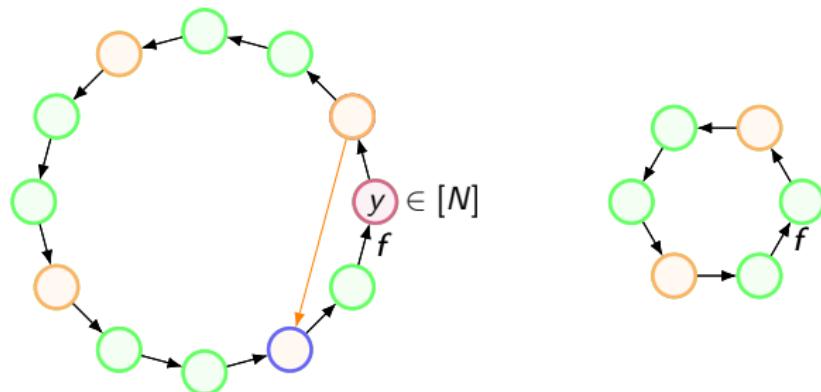
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

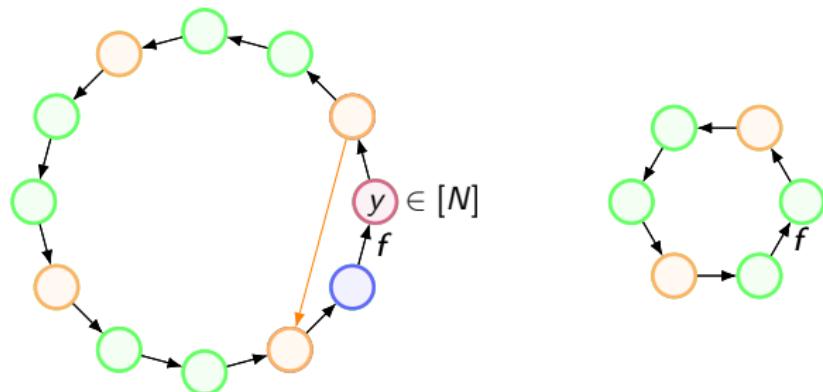
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

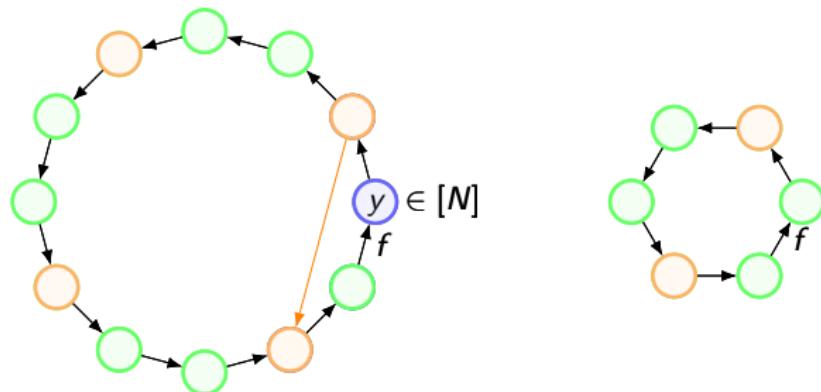
- ▶ What if y is on a large cycle?



- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

- ▶ What if y is on a large cycle?



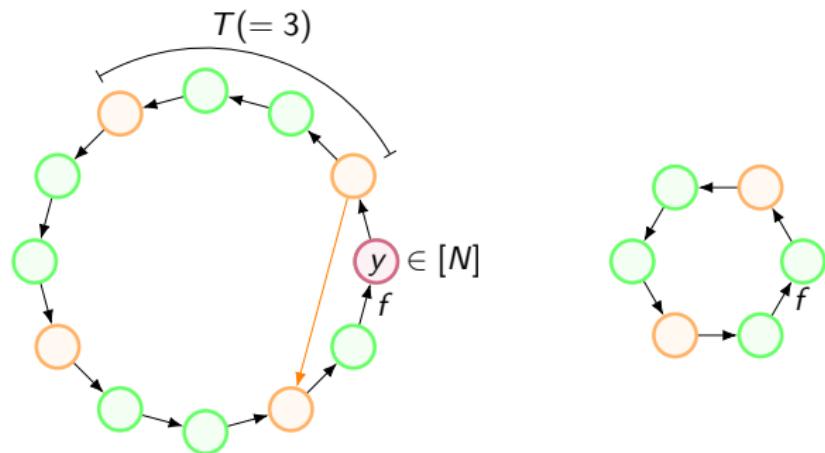
- ▶ In a preprocessing step, store points uniformly spaced around each cycle.

Analysis

- ▶ If we space the stored points T hops apart:

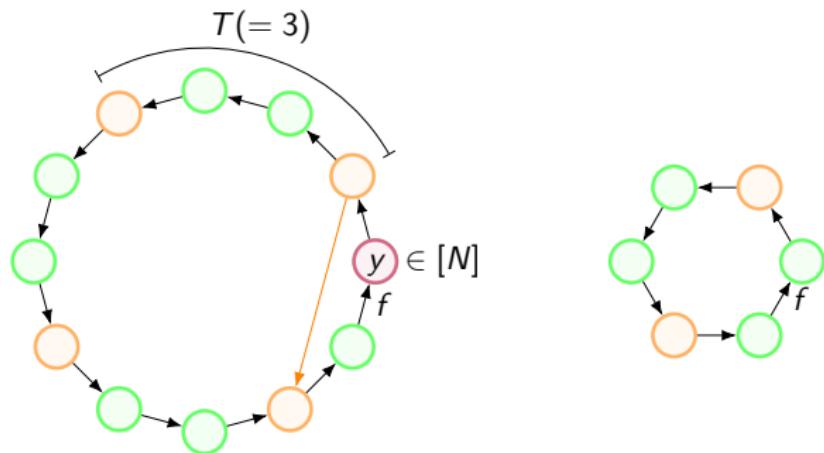
Analysis

- If we space the stored points T hops apart:



Analysis

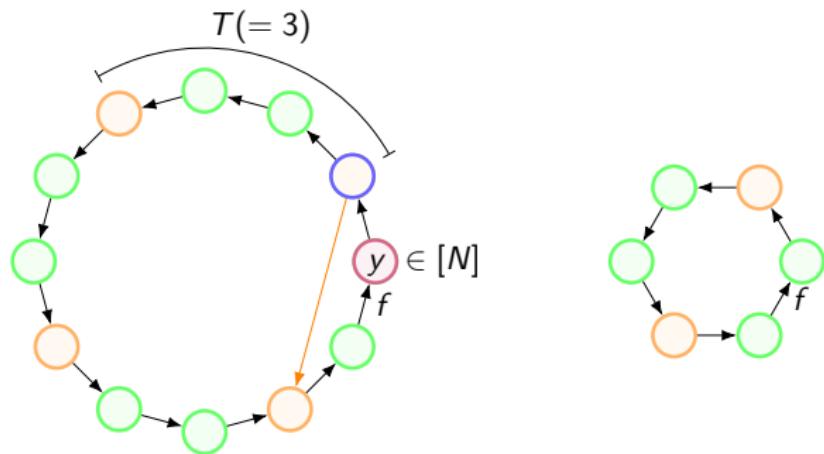
- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .

Analysis

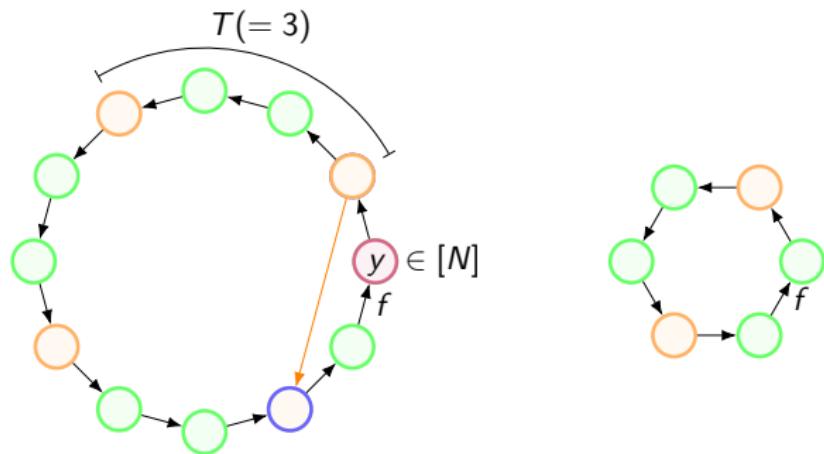
- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .

Analysis

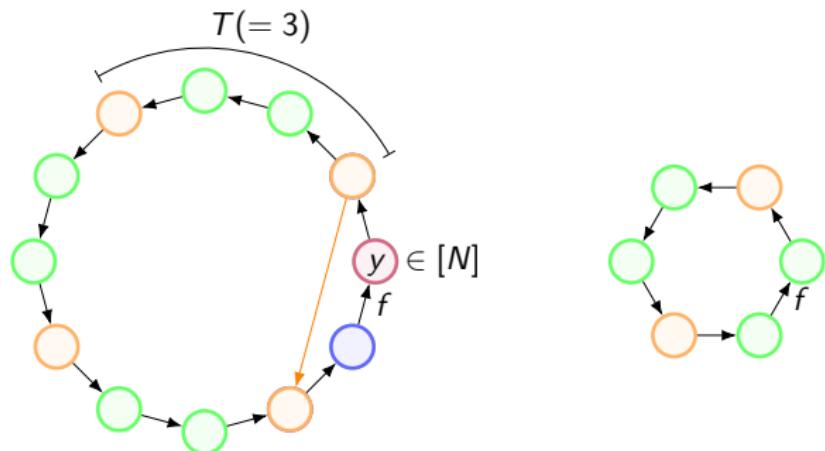
- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .

Analysis

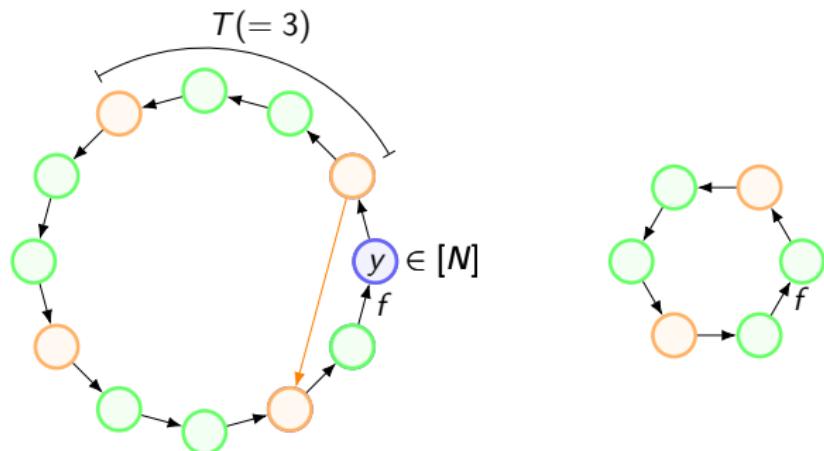
- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .

Analysis

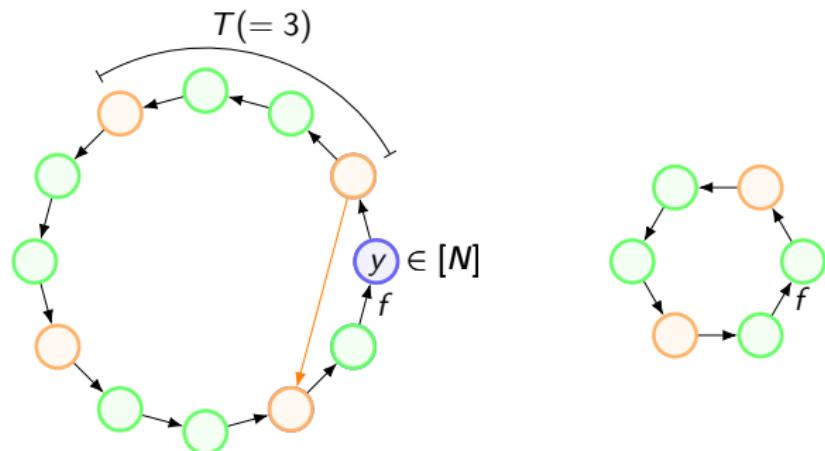
- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .

Analysis

- ▶ If we space the stored points T hops apart:



- ▶ We need T evaluations of f to invert y .
- ▶ We need to store about N/T points total.

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.
- ▶ $\mathcal{A}^f(\alpha, y)$ uses α to invert y . It treats f as a *black box* and evaluates it $\leq T$ times.

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.
- ▶ $\mathcal{A}^f(\alpha, y)$ uses α to invert y . It treats f as a *black box* and evaluates it $\leq T$ times.
- ▶ Hellman's algorithm uses $S = O((N \log N)/T)$.

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.
- ▶ $\mathcal{A}^f(\alpha, y)$ uses α to invert y . It treats f as a *black box* and evaluates it $\leq T$ times.
- ▶ Hellman's algorithm uses $S = O((N \log N)/T)$.
- ▶ More generally, **the goal** is to design a preprocessing algorithm $(\mathcal{P}, \mathcal{A})$ such that for all f and all $y \in f([N])$,

$$\Pr[\alpha \leftarrow \mathcal{P}(f); x \leftarrow \mathcal{A}^f(\alpha, y); f(x) = y] \geq 9/10.$$

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.
- ▶ $\mathcal{A}^f(\alpha, y)$ uses α to invert y . It treats f as a *black box* and evaluates it $\leq T$ times.
- ▶ Hellman's algorithm uses $S = O((N \log N)/T)$.
- ▶ More generally, **the goal** is to design a preprocessing algorithm $(\mathcal{P}, \mathcal{A})$ such that for all f and all $y \in f([N])$,

$$\Pr[\alpha \leftarrow \mathcal{P}(f); x \leftarrow \mathcal{A}^f(\alpha, y); f(x) = y] \geq 9/10.$$

- ▶ In this model, \mathcal{P} and \mathcal{A} have **unbounded** computational power,

What did we just do?

- ▶ We have a preprocessing algorithm \mathcal{P} and an online algorithm \mathcal{A} . Preprocessing makes a S -bit data structure $\alpha = \mathcal{P}(f)$.
- ▶ $\mathcal{A}^f(\alpha, y)$ uses α to invert y . It treats f as a *black box* and evaluates it $\leq T$ times.
- ▶ Hellman's algorithm uses $S = O((N \log N)/T)$.
- ▶ More generally, **the goal** is to design a preprocessing algorithm $(\mathcal{P}, \mathcal{A})$ such that for all f and all $y \in f([N])$,

$$\Pr[\alpha \leftarrow \mathcal{P}(f); x \leftarrow \mathcal{A}^f(\alpha, y); f(x) = y] \geq 9/10.$$

- ▶ In this model, \mathcal{P} and \mathcal{A} have **unbounded** computational power,
- ▶ But, α has bitlength at most S , and \mathcal{A} can make at most T evaluations of f .

Applications

Scenarios

- ▶ The NSA wants to break a widely used cryptographic function, such as AES-128

Applications

Scenarios

- ▶ The NSA wants to break a widely used cryptographic function, such as AES-128
- ▶ Hackers want to recover passwords from a stolen database of password hashes (Rainbow Tables)

Applications

Scenarios

- ▶ The NSA wants to break a widely used cryptographic function, such as AES-128
- ▶ Hackers want to recover passwords from a stolen database of password hashes (Rainbow Tables)
- ▶ Theoretical computer scientists want better algorithms for 3-SUM [GGH⁺20],



multiparty pointer jumping [CK19],



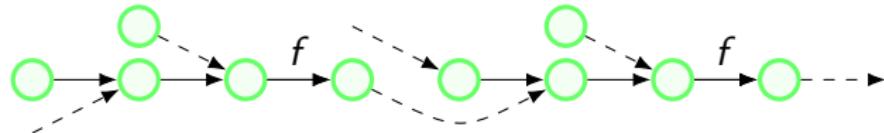
systematic substring search [CK19], ...

Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.

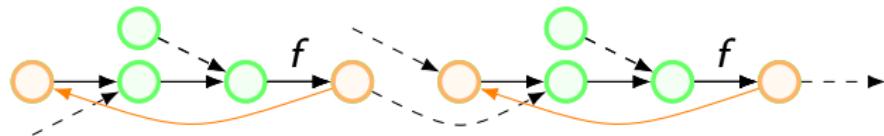
Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.



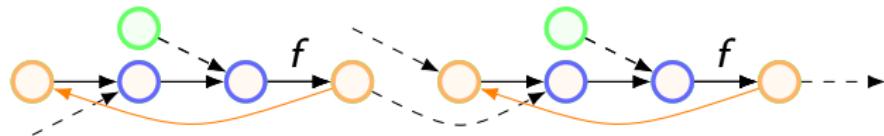
Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.



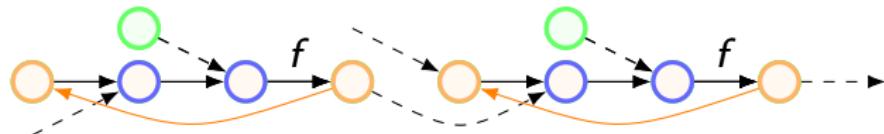
Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.



Beyond permutations

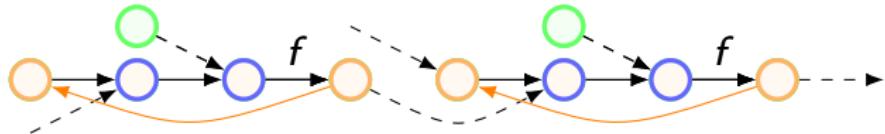
- ▶ Preprocessing stores the endpoints of disjoint paths.



- ▶ No longer possible to cover the entire range.

Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.

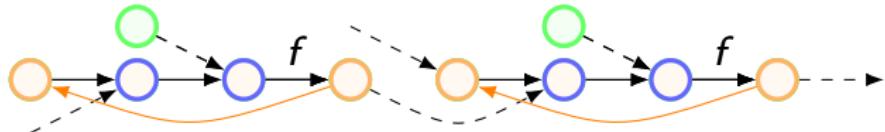


- ▶ No longer possible to cover the entire range.
- ▶ But, one can cover a small fraction with disjoint paths, then compose with a “random” function and repeat.



Beyond permutations

- ▶ Preprocessing stores the endpoints of disjoint paths.

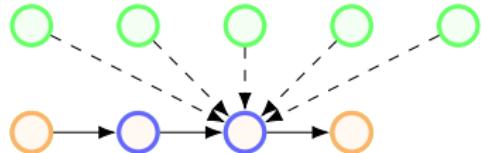


- ▶ No longer possible to cover the entire range.
- ▶ But, one can cover a small fraction with disjoint paths, then compose with a “random” function and repeat.
- ▶ For *random* functions, Hellman showed (heuristically) this can be made to work.



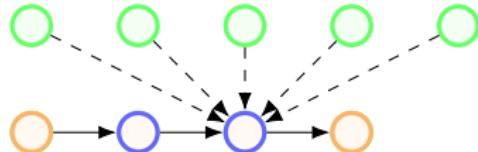
Beyond permutations

- ▶ Hellman's argument fails for arbitrary functions.



Beyond permutations

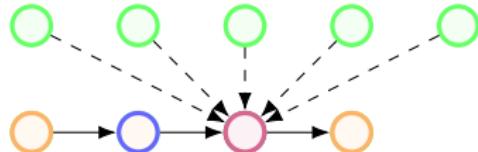
- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses (“junction points”), which cause paths to overlap quickly.

Beyond permutations

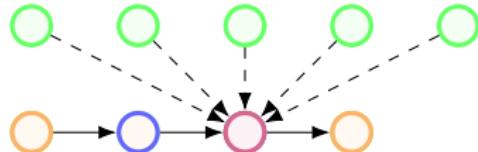
- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses (“junction points”), which cause paths to overlap quickly.

Beyond permutations

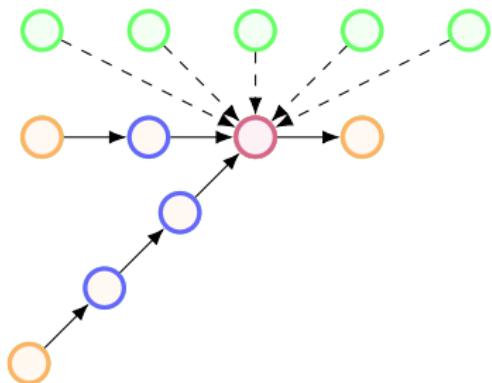
- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses (“junction points”), which cause paths to overlap quickly.

Beyond permutations

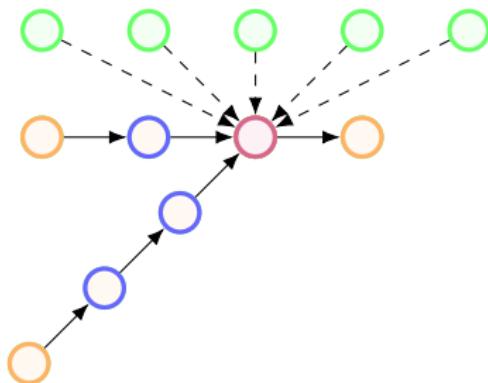
- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses ("junction points"), which cause paths to overlap quickly.

Beyond permutations

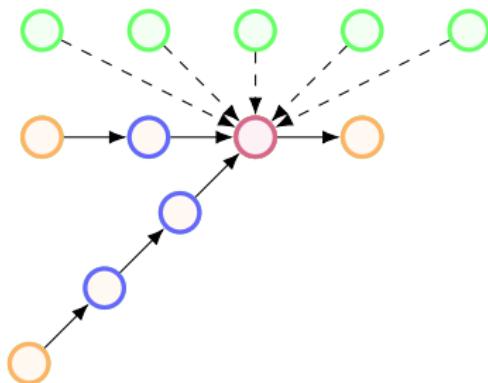
- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses (“junction points”), which cause paths to overlap quickly.
- ▶ Fiat and Naor deal with this by storing $\alpha = (\alpha', L)$, where L contains junction points along with their inverses.

Beyond permutations

- ▶ Hellman's argument fails for arbitrary functions.



- ▶ There may be points with very many inverses (“junction points”), which cause paths to overlap quickly.
- ▶ Fiat and Naor deal with this by storing $\alpha = (\alpha', L)$, where L contains junction points along with their inverses.
- ▶ Intuitively, α' is the data structure for a *restriction* of f that avoids the junction points in L .

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 1980 	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 1980 	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 1991 	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 1980 	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 1991 	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$

- ▶ Q: Can we improve Fiat-Naor? Can we improve Yao's lower bound?

The story so far

Result	Applies To	Tradeoff	Key Point
Hellman 1980 	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 1980 	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 1991 	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$

- ▶ Q: Can we improve Fiat-Naor? Can we improve Yao's lower bound?
- ▶ A: Sort of and sort of!

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.
- ▶ Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.
- ▶ Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).
- ▶ We observe that the tradeoff $T \lesssim N^3/S^3$ comes from

$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.
- ▶ Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).
- ▶ We observe that the tradeoff $T \lesssim N^3/S^3$ comes from

$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$

- ▶ So we'd love for L to be longer, but it needs to fit in $\alpha \in \{0, 1\}^S$.

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.
- ▶ Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).
- ▶ We observe that the tradeoff $T \lesssim N^3/S^3$ comes from

$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$

- ▶ So we'd love for L to be longer, but it needs to fit in $\alpha \in \{0, 1\}^S$.
- ▶ Or does it?

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Fiat and Naor's list L actually consists of images $f(x_i)$ of random points $x_i \sim [N]$.

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Fiat and Naor's list L actually consists of images $f(x_i)$ of random points $x_i \sim [N]$.
- ▶ Our idea: Instead of reading L from α , \mathcal{A} recovers L by evaluating f on the random points x_i .

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Fiat and Naor's list L actually consists of images $f(x_i)$ of random points $x_i \sim [N]$.
- ▶ Our idea: Instead of reading L from α , \mathcal{A} recovers L by evaluating f on the random points x_i .
- ▶ This allows $|L| \simeq T$, so we can get $T \lesssim N^3/(S^2 T)$, or

$$T \lesssim N^{3/2}/S.$$

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Fiat and Naor's list L actually consists of images $f(x_i)$ of random points $x_i \sim [N]$.
- ▶ Our idea: Instead of reading L from α , \mathcal{A} recovers L by evaluating f on the random points x_i .
- ▶ This allows $|L| \simeq T$, so we can get $T \lesssim N^3/(S^2 T)$, or

$$T \lesssim N^{3/2}/S.$$

- ▶ But I've cheated here...

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Fiat and Naor's list L actually consists of images $f(x_i)$ of random points $x_i \sim [N]$.
- ▶ Our idea: Instead of reading L from α , \mathcal{A} recovers L by evaluating f on the random points x_i .
- ▶ This allows $|L| \simeq T$, so we can get $T \lesssim N^3/(S^2 T)$, or

$$T \lesssim N^{3/2}/S.$$

- ▶ But I've cheated here...
- ▶ How do \mathcal{A} and \mathcal{P} agree on the same list of random values x_i ?

Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .

Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .
- ▶ More precisely, shared randomness costs an additive $O(\log N)$ in S and a factor 2 increase in failure probability.

Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .
- ▶ More precisely, shared randomness costs an additive $O(\log N)$ in S and a factor 2 increase in failure probability.
- ▶ The proof adapts *Newman's lemma* [New91] from communication complexity.



Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .
- ▶ More precisely, shared randomness costs an additive $O(\log N)$ in S and a factor 2 increase in failure probability.
- ▶ The proof adapts *Newman's lemma* [New91] from communication complexity.
- ▶ The idea: rather than storing the random bits in full, store a random *index* into a fixed list of strings.



Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .
- ▶ More precisely, shared randomness costs an additive $O(\log N)$ in S and a factor 2 increase in failure probability.
- ▶ The proof adapts *Newman's lemma* [New91] from communication complexity.
- ▶ The idea: rather than storing the random bits in full, store a random *index* into a fixed list of strings.
- ▶ If the number $\Sigma = O(N)$ of fixed strings is large enough, this will be almost as good as fully random.



Sharing Randomness

- ▶ We show that we can assume *shared randomness* without loss of generality. Here, that means shared random x_i .
- ▶ More precisely, shared randomness costs an additive $O(\log N)$ in S and a factor 2 increase in failure probability.
- ▶ The proof adapts *Newman's lemma* [New91] from communication complexity.
- ▶ The idea: rather than storing the random bits in full, store a random *index* into a fixed list of strings.
- ▶ If the number $\Sigma = O(N)$ of fixed strings is large enough, this will be almost as good as fully random.
- ▶ This approach leans fairly heavily on non-uniformity. But in practice, can just instantiate a random oracle.



In Context

Result		Applies To	Tradeoff	Key Point
Hellman 	1980	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 		permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 	1980	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 	1991	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$

In Context

Result		Applies To	Tradeoff	Key Point
Hellman 	1980	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 		permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 	1980	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 	1991	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$
This work		all functions	$T \leq \tilde{O}(N^{3/2}/S)$	$S = T \lesssim N^{3/4}$

In Context

Result	Applies To	Tradeoff	Key Point
Hellman 	1980 permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$
Hellman 	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
Fiat-Naor 1991 	all functions	$T \leq \tilde{O}(N^3/S^3)$	$S = T \lesssim N^{3/4}$
This work	all functions	$T \leq \tilde{O}(N^{3/2}/S)$ $T \lesssim N^3/(S^2 T)$	$S = T \lesssim N^{3/4}$

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.
- ▶ Corrigan-Gibbs and Kogan   [CK19]: any small improvement \implies new lower bounds in circuit complexity.

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.
- ▶ Corrigan-Gibbs and Kogan   [CK19]: any small improvement \implies new lower bounds in circuit complexity.
- ▶ Even improving Yao's bound just for *non-adaptive* algorithms would do it!

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.
- ▶ Corrigan-Gibbs and Kogan   [CK19]: any small improvement \implies new lower bounds in circuit complexity.
- ▶ Even improving Yao's bound just for *non-adaptive* algorithms would do it!
- ▶ \mathcal{A} is non-adaptive if its evaluation points x_1, \dots, x_T are chosen up front, before any evaluations of f are seen.

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.
- ▶ Corrigan-Gibbs and Kogan   [CK19]: any small improvement \implies new lower bounds in circuit complexity.
- ▶ Even improving Yao's bound just for *non-adaptive* algorithms would do it!
- ▶ \mathcal{A} is non-adaptive if its evaluation points x_1, \dots, x_T are chosen up front, before any evaluations of f are seen.
- ▶ Non-adaptive algorithms seem very weak. Hellman's algorithm is *very* adaptive.

Non-Adaptive Algorithms

- ▶ Recall that Yao's lower bound (for inverting *arbitrary* functions) hasn't been improved in 30+ years.
- ▶ Corrigan-Gibbs and Kogan   [CK19]: any small improvement \implies new lower bounds in circuit complexity.
- ▶ Even improving Yao's bound just for *non-adaptive* algorithms would do it!
- ▶ \mathcal{A} is non-adaptive if its evaluation points x_1, \dots, x_T are chosen up front, before any evaluations of f are seen.
- ▶ Non-adaptive algorithms seem very weak. Hellman's algorithm is *very* adaptive.
- ▶ Corrigan-Gibbs and Kogan speculated that there is no non-adaptive algorithm with

$$S = o(N \log N) \text{ and } T = o(N).$$

Result #2: A Non-Adaptive Algorithm

- ▶ Corrigan-Gibbs and Kogan speculated that there is no non-adaptive algorithm with
$$S = o(N \log N) \text{ and } T = o(N).$$
- ▶ We show that there IS such an algorithm, that (barely!) outperforms the trivial inverter.

Result #2: A Non-Adaptive Algorithm

- ▶ Corrigan-Gibbs and Kogan speculated that there is no non-adaptive algorithm with

$$S = o(N \log N) \text{ and } T = o(N).$$

- ▶ We show that there IS such an algorithm, that (barely!) outperforms the trivial inverter.
- ▶ We achieve, for example,

$$S = O(N \log \log N), T = O(N / \log^c N)$$

for any constant c .

Result #2: A Non-Adaptive Algorithm

- ▶ Corrigan-Gibbs and Kogan speculated that there is no non-adaptive algorithm with

$$S = o(N \log N) \text{ and } T = o(N).$$

- ▶ We show that there IS such an algorithm, that (barely!) outperforms the trivial inverter.
- ▶ We achieve, for example,

$$S = O(N \log \log N), T = O(N / \log^c N)$$

for any constant c .

- ▶ Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

The Algorithm

- ▶ Try $T = 0$, for starters.

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?
- ▶ How many y 's can we invert? (What's trivial here?)

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?
- ▶ How many y 's can we invert? (What's trivial here?)
- ▶ $\mathcal{A}(\alpha, y)$ is just some (possibly random) function...

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?
- ▶ How many y 's can we invert? (What's trivial here?)
- ▶ $\mathcal{A}(\alpha, y)$ is just some (possibly random) function...
- ▶ Try $\mathcal{A}(\alpha, y) = \alpha + g(y) \pmod{N}$ where $g : [N] \rightarrow [N]$ is random.

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?
- ▶ How many y 's can we invert? (What's trivial here?)
- ▶ $\mathcal{A}(\alpha, y)$ is just some (possibly random) function...
- ▶ Try $\mathcal{A}(\alpha, y) = \alpha + g(y) \pmod{N}$ where $g : [N] \rightarrow [N]$ is random.
- ▶ For simplicity, assume f is a permutation. Then

$$\begin{aligned}y \text{ inverted} &\Leftrightarrow \alpha + g(y) = f^{-1}(y) \\&\Leftrightarrow \alpha = \alpha_y := f^{-1}(y) - g(y)\end{aligned}$$

The Algorithm

- ▶ Try $T = 0$, for starters.
- ▶ Why not $S = \log N$, so that $\alpha \in [N]$?
- ▶ How many y 's can we invert? (What's trivial here?)
- ▶ $\mathcal{A}(\alpha, y)$ is just some (possibly random) function...
- ▶ Try $\mathcal{A}(\alpha, y) = \alpha + g(y) \pmod{N}$ where $g : [N] \rightarrow [N]$ is random.
- ▶ For simplicity, assume f is a permutation. Then

$$\begin{aligned}y \text{ inverted} &\Leftrightarrow \alpha + g(y) = f^{-1}(y) \\&\Leftrightarrow \alpha = \alpha_y := f^{-1}(y) - g(y)\end{aligned}$$

- ▶ \implies preprocessing returns α maximizing

$$|\{y \in [N] : \alpha = \alpha_y\}|.$$

The Algorithm

- ▶ Recall $\alpha \in [N]$ inverts all $y \in [N]$ with

$$\alpha = \alpha_y := f^{-1}(y) - g(y).$$

The Algorithm

- ▶ Recall $\alpha \in [N]$ inverts all $y \in [N]$ with

$$\alpha = \alpha_y := f^{-1}(y) - g(y).$$

- ▶ Key point: g is random $\implies \alpha_y$'s are uniform and independent.

The Algorithm

- ▶ Recall $\alpha \in [N]$ inverts all $y \in [N]$ with

$$\alpha = \alpha_y := f^{-1}(y) - g(y).$$

- ▶ Key point: g is random $\implies \alpha_y$'s are uniform and independent.
- ▶ Balls and bins! With high probability, $\Omega(\log N / \log \log N)$ balls in the bin with most balls.

The Algorithm

- ▶ Recall $\alpha \in [N]$ inverts all $y \in [N]$ with

$$\alpha = \alpha_y := f^{-1}(y) - g(y).$$

- ▶ Key point: g is random $\implies \alpha_y$'s are uniform and independent.
- ▶ Balls and bins! With high probability, $\Omega(\log N / \log \log N)$ balls in the bin with most balls.
- ▶ $\implies \max_{\alpha} |\{y \in [N] : \alpha_y = \alpha\}| \geq \Omega(\log N / \log \log N)$.

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.
- ▶ Toss the remaining balls into the bins again, using a fresh random function g' , and store new bin α' with most balls.

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.
- ▶ Toss the remaining balls into the bins again, using a fresh random function g' , and store new bin α' with most balls.
- ▶ Online computes the two candidates $x = g(y) + \alpha$, $x' = g'(y) + \alpha'$.

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.
- ▶ Toss the remaining balls into the bins again, using a fresh random function g' , and store new bin α' with most balls.
- ▶ Online computes the two candidates $x = g(y) + \alpha$, $x' = g'(y) + \alpha'$.
- ▶ It evaluates f on each to check if $f(x) = y$ or $f(x') = y$, and outputs whichever passes the check (if any). Non-adaptive!

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.
- ▶ Toss the remaining balls into the bins again, using a fresh random function g' , and store new bin α' with most balls.
- ▶ Online computes the two candidates $x = g(y) + \alpha$, $x' = g'(y) + \alpha'$.
- ▶ It evaluates f on each to check if $f(x) = y$ or $f(x') = y$, and outputs whichever passes the check (if any). Non-adaptive!
- ▶ For the full algorithm, instead of repeating twice, we repeat $r = O(N/(\log N / \log \log N))$ times.

The Algorithm

- ▶ What if we allow ourselves to store another element $\alpha' \in [N]$?
- ▶ First remove the “balls” $\{y \in [N] : \alpha_y = \alpha\}$ from consideration.
- ▶ Toss the remaining balls into the bins again, using a fresh random function g' , and store new bin α' with most balls.
- ▶ Online computes the two candidates $x = g(y) + \alpha$, $x' = g'(y) + \alpha'$.
- ▶ It evaluates f on each to check if $f(x) = y$ or $f(x') = y$, and outputs whichever passes the check (if any). Non-adaptive!
- ▶ For the full algorithm, instead of repeating twice, we repeat $r = O(N / (\log N / \log \log N))$ times.
- ▶ $S = O(r \log N) = O(N \log \log N)$,
 $T = O(r) = O(N \log \log N / \log N)$.

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.
- ▶ By partitioning the range and letting each α specify multiple candidates, we can get $S = O(N \log(N/T))$.

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.
- ▶ By partitioning the range and letting each α specify multiple candidates, we can get $S = O(N \log(N/T))$.
- ▶ Is this tradeoff the “right answer” ?

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.
- ▶ By partitioning the range and letting each α specify multiple candidates, we can get $S = O(N \log(N/T))$.
- ▶ Is this tradeoff the “right answer”?
- ▶ Our algorithm is a *guess-and-check algorithm*—it’s non-adaptive, and it returns one of the points x_1, \dots, x_T that it queries.

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.
- ▶ By partitioning the range and letting each α specify multiple candidates, we can get $S = O(N \log(N/T))$.
- ▶ Is this tradeoff the “right answer”?
- ▶ Our algorithm is a *guess-and-check algorithm*—it’s non-adaptive, and it returns one of the points x_1, \dots, x_T that it queries.
- ▶ That is, it can be thought of returning T candidate inverses x_1, \dots, x_T , *without querying f at all*.

The Algorithm

- ▶ Generalizing to arbitrary functions is easy: just assign a unique inverse x_y to each $y \in f([N])$ and define $\alpha_y = x_y - g(y)$.
- ▶ By partitioning the range and letting each α specify multiple candidates, we can get $S = O(N \log(N/T))$.
- ▶ Is this tradeoff the “right answer”?
- ▶ Our algorithm is a *guess-and-check algorithm*—it’s non-adaptive, and it returns one of the points x_1, \dots, x_T that it queries.
- ▶ That is, it can be thought of returning T candidate inverses x_1, \dots, x_T , *without querying f at all*.
- ▶ We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

$$\implies S + N \log T = \Omega(N \log N)$$

$$\implies S = \Omega(N \log(N/T)).$$

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

$$\begin{aligned}\implies S + N \log T &= \Omega(N \log N) \\ \implies S &= \Omega(N \log(N/T)).\end{aligned}$$

- ▶ Proof:
 - ▶ Encoder computes $\alpha \leftarrow \mathcal{P}(f)$.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

$$\begin{aligned}\implies S + N \log T &= \Omega(N \log N) \\ \implies S &= \Omega(N \log(N/T)).\end{aligned}$$

- ▶ Proof:
 - ▶ Encoder computes $\alpha \leftarrow \mathcal{P}(f)$.
 - ▶ For each $y \in [N]$, encoder runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It writes down the $i_y \in [T]$ that satisfies $f(x_{i_y}) = y$.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

$$\begin{aligned}\implies S + N \log T &= \Omega(N \log N) \\ \implies S &= \Omega(N \log(N/T)).\end{aligned}$$

- ▶ Proof:
 - ▶ Encoder computes $\alpha \leftarrow \mathcal{P}(f)$.
 - ▶ For each $y \in [N]$, encoder runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It writes down the $i_y \in [T]$ that satisfies $f(x_{i_y}) = y$.
 - ▶ Encoding is $(\alpha, i_1, \dots, i_N)$.

Lower Bound

- ▶ For simplicity, assume a guess-and-check algorithm that always succeeds, with parameters S, T . We use a compression argument.
- ▶ Theorem: Can encode any permutation f using $S + N \log T$ bits.

$$\begin{aligned}\implies S + N \log T &= \Omega(N \log N) \\ \implies S &= \Omega(N \log(N/T)).\end{aligned}$$

- ▶ Proof:
 - ▶ Encoder computes $\alpha \leftarrow \mathcal{P}(f)$.
 - ▶ For each $y \in [N]$, encoder runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It writes down the $i_y \in [T]$ that satisfies $f(x_{i_y}) = y$.
 - ▶ Encoding is $(\alpha, i_1, \dots, i_N)$.
 - ▶ For each y , decoder again runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It sets $f^{-1}(y) = x_{i_y}$.

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

- ▶ The first nontrivial non-adaptive function inversion algorithm.

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

- ▶ The first nontrivial non-adaptive function inversion algorithm.
 - ▶ Best possible among *guess-and-check* algorithms.

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

- ▶ The first nontrivial non-adaptive function inversion algorithm.
 - ▶ Best possible among *guess-and-check* algorithms.
- ▶ Shared randomness is essentially free

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

- ▶ The first nontrivial non-adaptive function inversion algorithm.
 - ▶ Best possible among *guess-and-check* algorithms.
- ▶ Shared randomness is essentially free
- ▶ Not in this talk:
 - ▶ Search-to-decision reductions

Function Inversion Revisited: Our Results

- ▶ Improving Fiat-Naor: a

$$T \lesssim N^{3/2}/S$$

algorithm for inverting arbitrary functions.

- ▶ The first nontrivial non-adaptive function inversion algorithm.
 - ▶ Best possible among *guess-and-check* algorithms.
- ▶ Shared randomness is essentially free
- ▶ Not in this talk:
 - ▶ Search-to-decision reductions
 - ▶ A larger codomain does not make the problem harder (us and [CK19])



Open Problems

- ▶ Only one real open problem—

Open Problems

- ▶ Only one real open problem—**close the gap!**

Open Problems

- ▶ Only one real open problem—**close the gap!**
- ▶ Improve Yao's lower bound against (general) non-adaptive algorithms?

Open Problems

- ▶ Only one real open problem—**close the gap!**
- ▶ Improve Yao's lower bound against (general) non-adaptive algorithms?
- ▶ $S^2 T = N^2$ algorithm for worst-case function inversion?

Open Problems

- ▶ Only one real open problem—**close the gap!**
- ▶ Improve Yao's lower bound against (general) non-adaptive algorithms?
- ▶ $S^2 T = N^2$ algorithm for worst-case function inversion?
- ▶ Improved algorithm for injective functions?

Thank you!

I'm happy to take additional questions offline. You can ping me at sp2473@cornell.edu.

References I

-  Henry Corrigan-Gibbs and Dmitry Kogan.
The function-inversion problem: Barriers and opportunities.
In *TCC*, 2019.
-  Alexander Golovnev, Siyao Guo, Thibaut Horel, Sunoo Park,
and Vinod Vaikuntanathan.
Data structures meet cryptography: 3SUM with preprocessing.

In *STOC*, 2020.
-  Alexander Golovnev, Siyao Guo, Spencer Peters, and Noah
Stephens-Davidowitz.
Revisiting time-space tradeoffs for function inversion.
Available at
<https://eccc.weizmann.ac.il/report/2022/145/>, 2022.

References II

-  Martin Hellman.
A cryptanalytic time-memory trade-off.
IEEE Transactions on Information Theory, 26(4):401–406,
1980.
-  Ilan Newman.
Private vs. common random bits in communication complexity.
Information processing letters, 39(2):67–71, 1991.