Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.

Siyao Guo

Sasha Golovnev

Function Inversion

- Given a function

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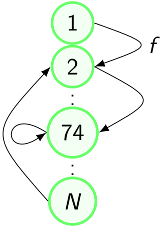

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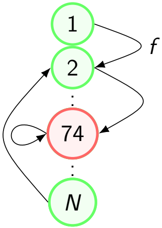

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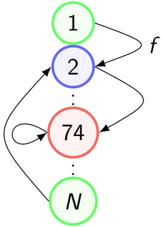

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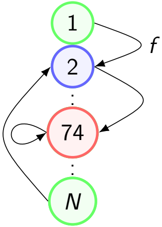

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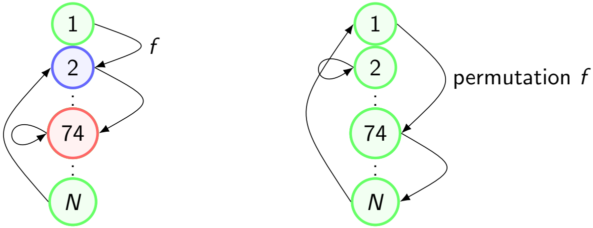

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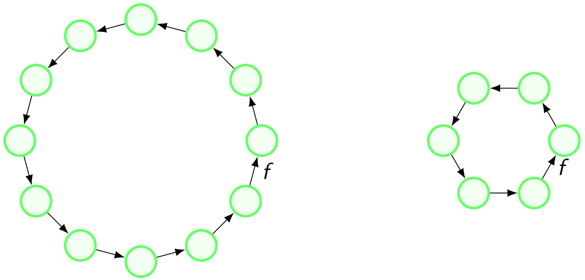

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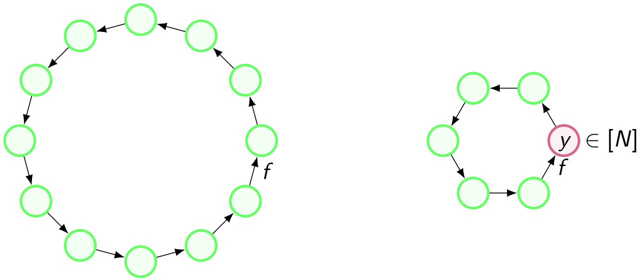

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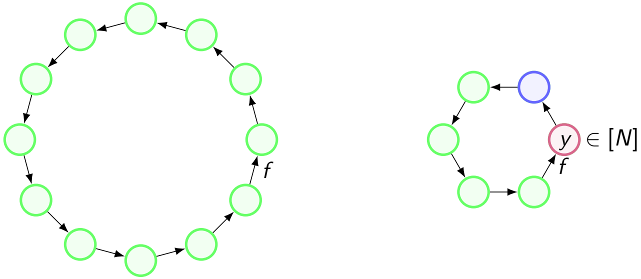

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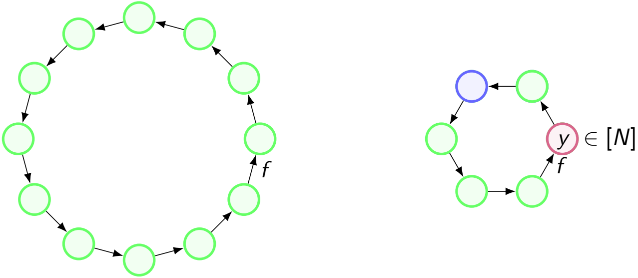

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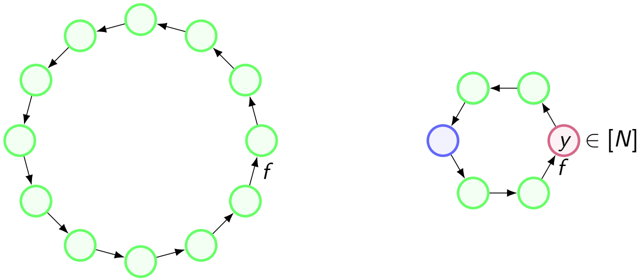

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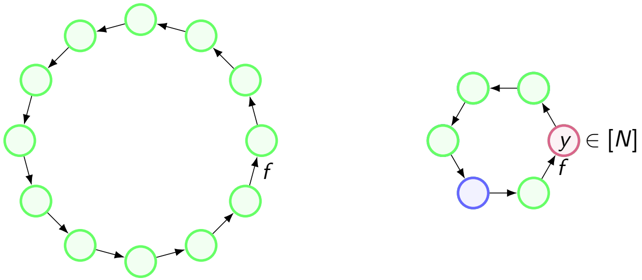

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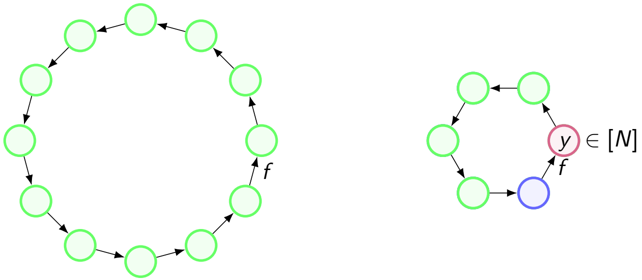

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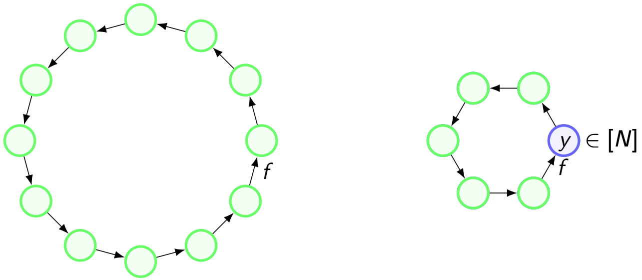

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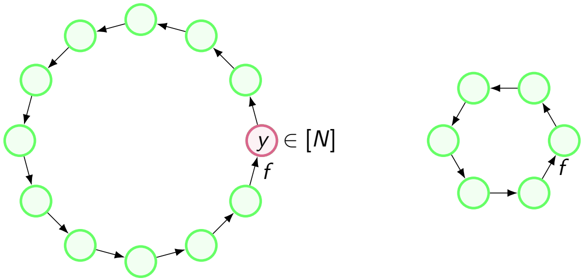

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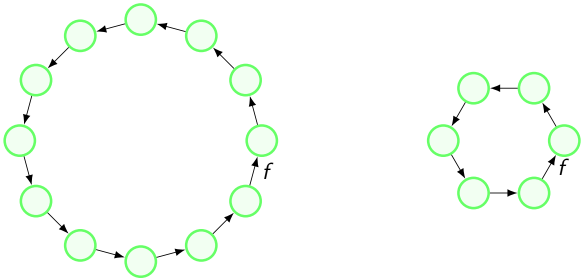

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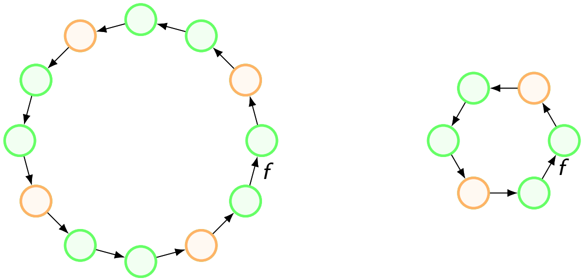

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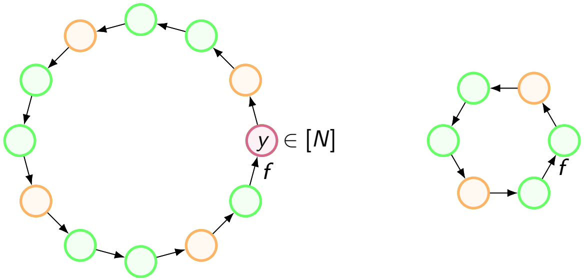

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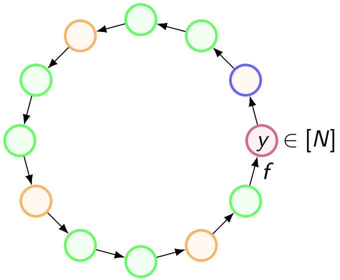

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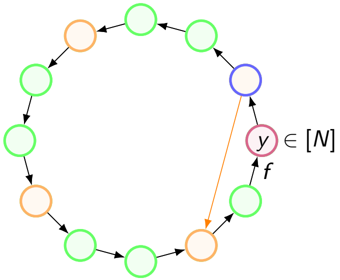

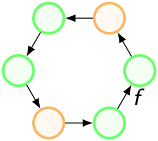

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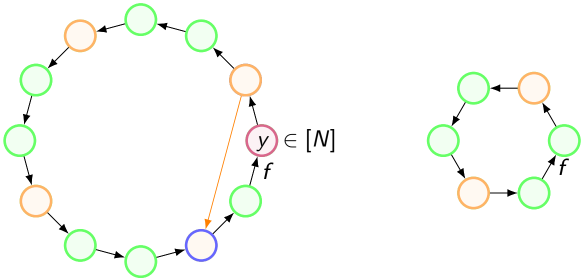

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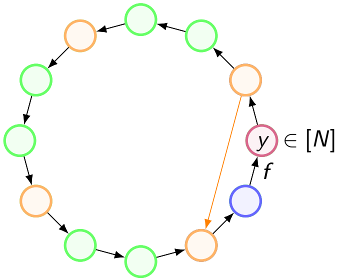

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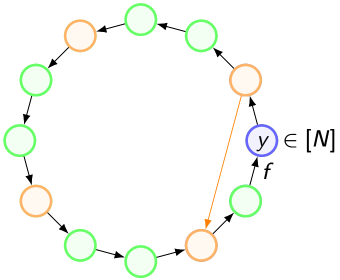

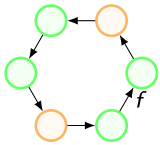

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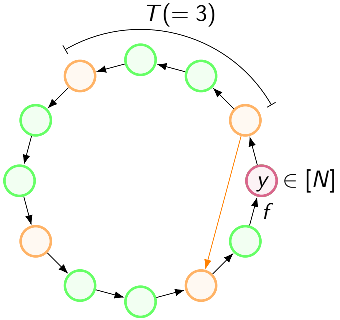

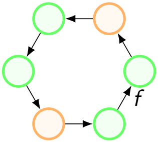

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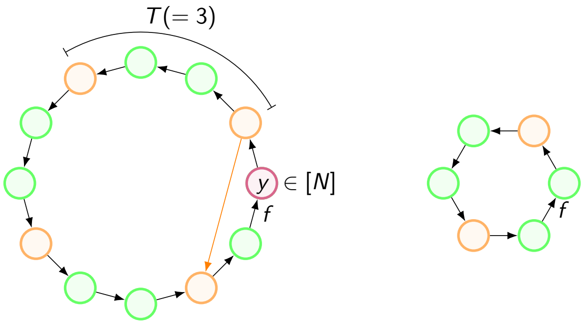

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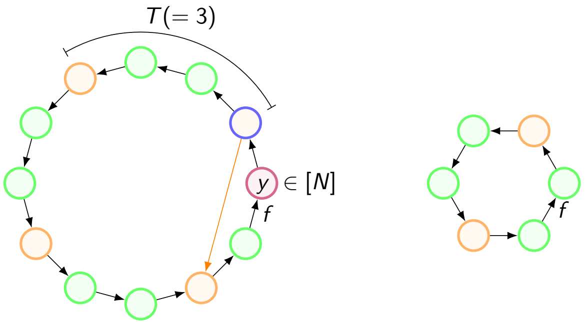

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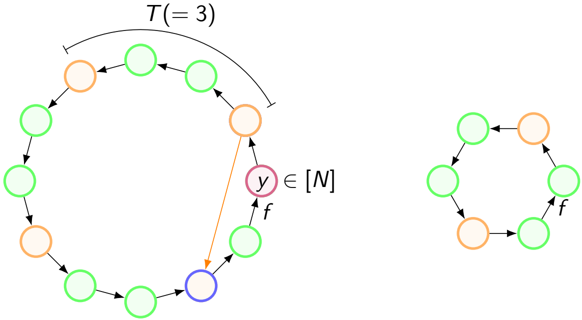

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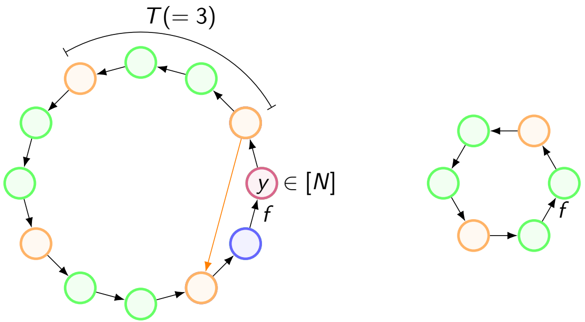

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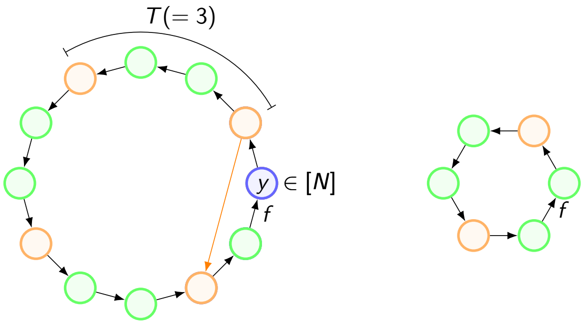

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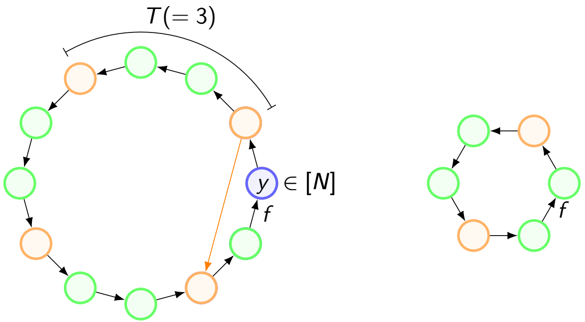

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- We need to store about N/T points total.

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- But, α has bitlength at most S, and $\mathcal{A}$ can make at most T evaluations of f.

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- Theoretical computer scientists want better algorithms for 3-SUM [GGH$^+$20],     , multiparty pointer jumping [CK19],   systematic substring search [CK19], ...

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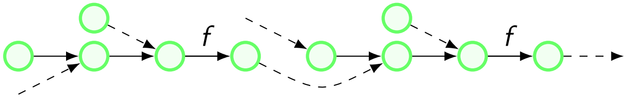

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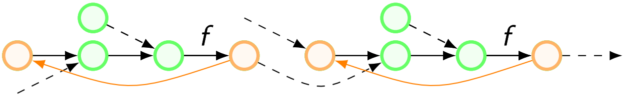

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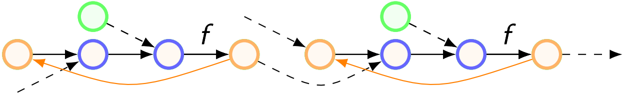

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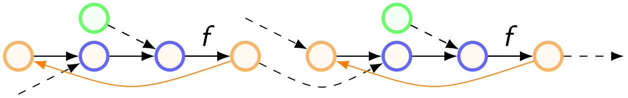

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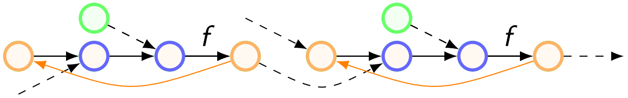

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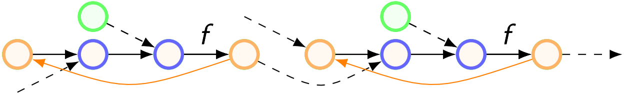

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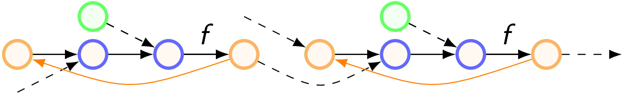

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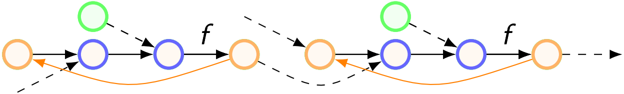

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- So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen "rerandomization" functions g_i.
- For *random* functions, Hellman showed (heuristically) this can be made to work. 

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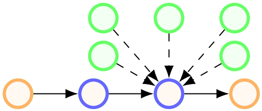

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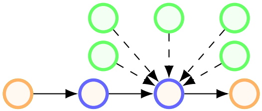

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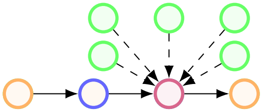

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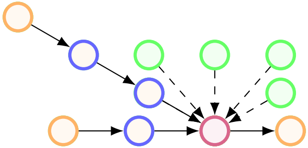

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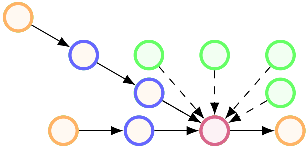

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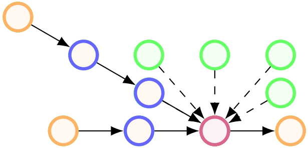

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- Intuitively, α' is the data structure for a *restriction* of f that avoids the junction points in L.
- More precisely, the "rerandomization" functions are sampled using rejection sampling so that their range is $[N] - L$.

The story so far

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Yao 1990 	permutations	$T \geq \Omega((N \log N)/S)$	$S = T \gtrsim \sqrt{N}$

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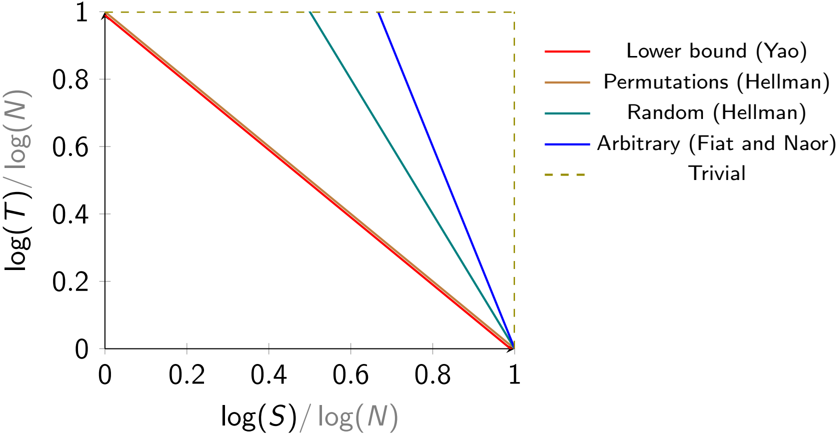

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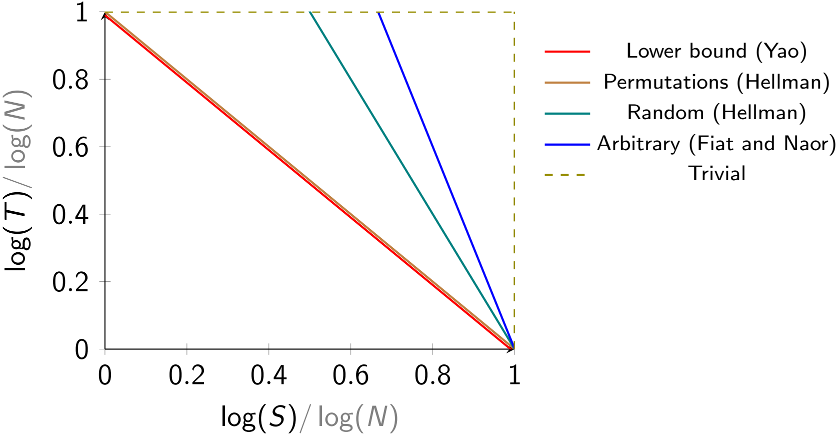

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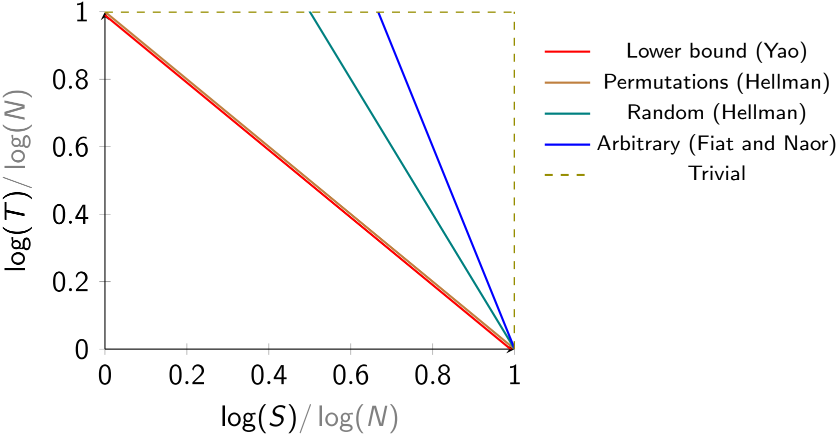

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$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$
- Fiat and Naor get $|L| \simeq S$, but this is the hard limit, since L needs to fit into S-bit advice α.
- Or does it?

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- How do $\mathcal{A}$ and $\mathcal{P}$ agree on the *same* list of random values x_i?

Sharing Randomness

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- In practice, can instantiate a random oracle.

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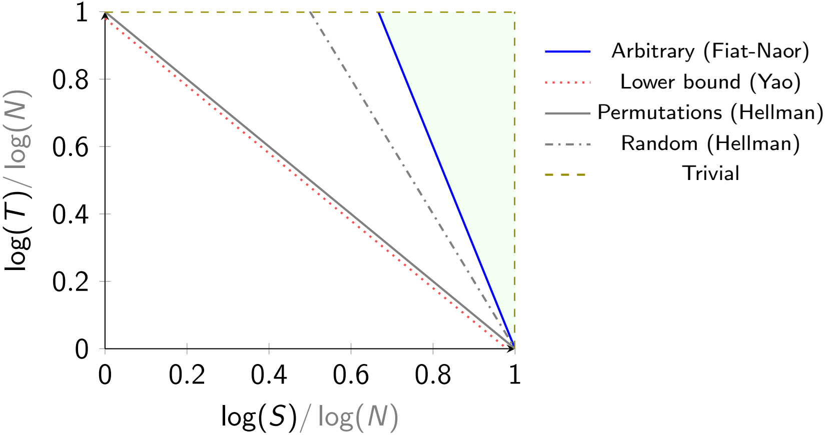

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0
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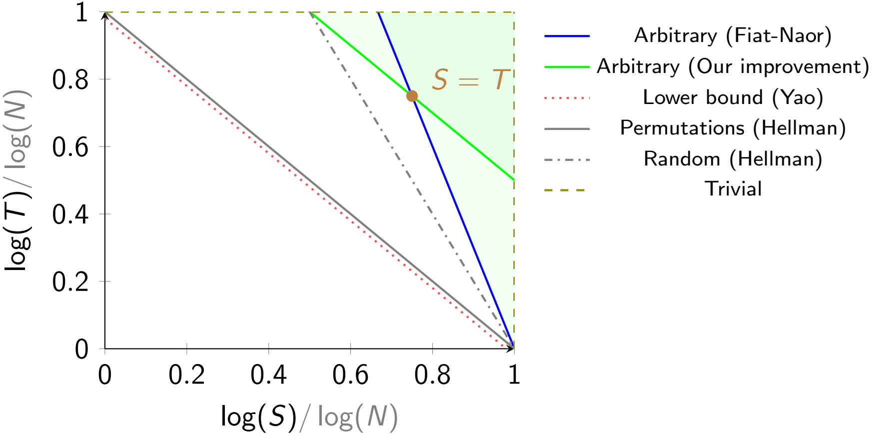

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- Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- $\implies$ preprocessing returns α maximizing

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- We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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  - Encoding is $(\alpha, i_1, \ldots, i_N)$.
  - For each y, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \ldots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

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  algorithm for inverting arbitrary functions.
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  - Best possible among *guess-and-check* algorithms.
- Shared randomness is essentially free
- Not in this talk:
  - Search-to-decision reductions
  - A larger codomain does not make the problem harder (us and [CK19])  

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  - Better algorithms for inverting a small fraction of the range? That is, improving on De, Trevisan and Tulsiani [DTT10]?

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- Moonshots:
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- Possibly more tractable:
  - Better algorithms for inverting a small fraction of the range? That is, improving on De, Trevisan and Tulsiani [DTT10]?
  - Better algorithms for injective functions?

Thank you!

I'm happy to take additional questions offline. You can ping me at `sp2473@cornell.edu`.

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