Revisiting Time-Space Tradeoffs for Function Inversion

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Function Inversion

- Given a function

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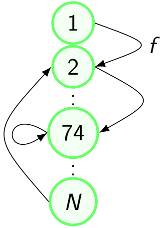

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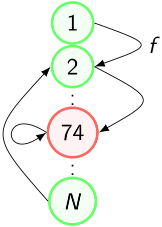

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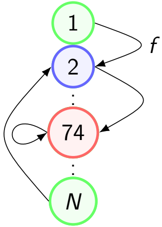

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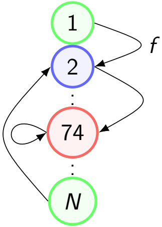

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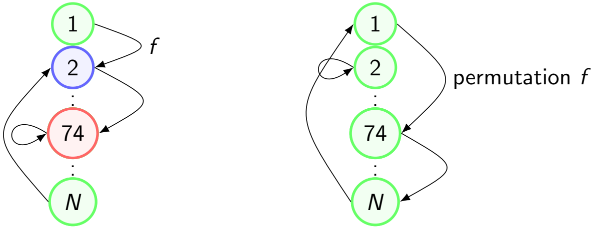

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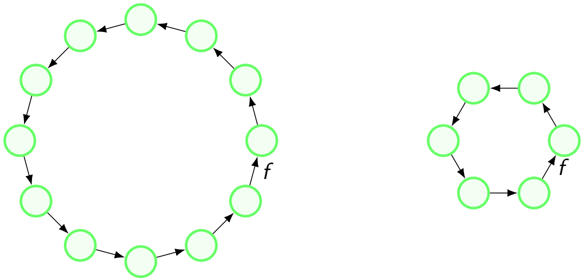

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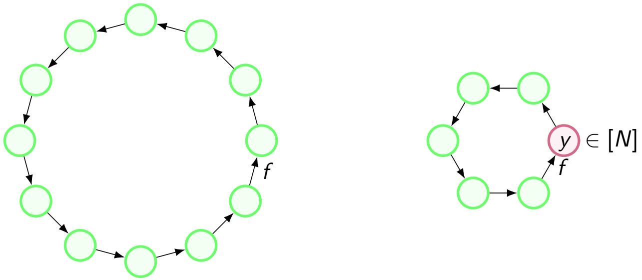

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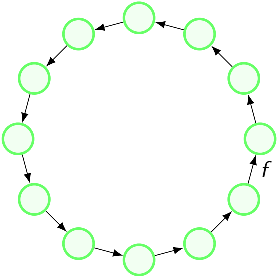

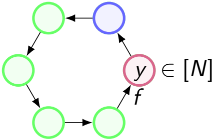

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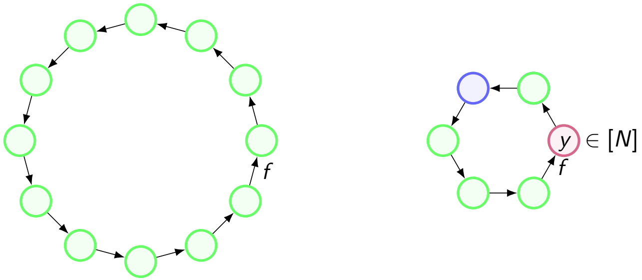

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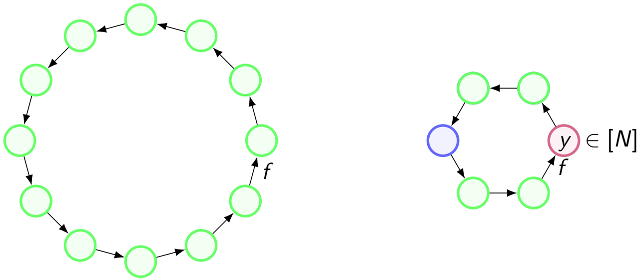

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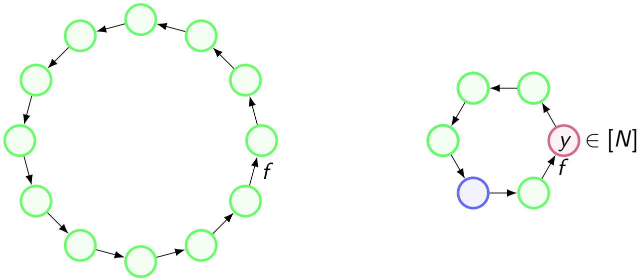

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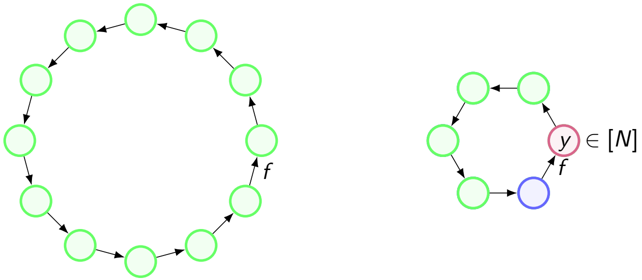

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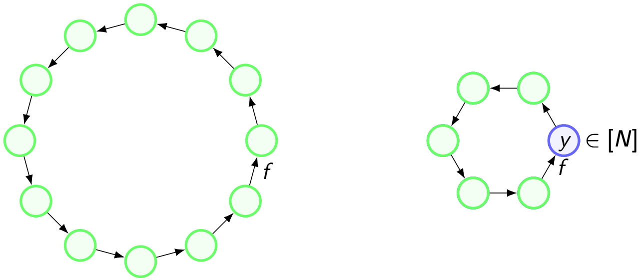

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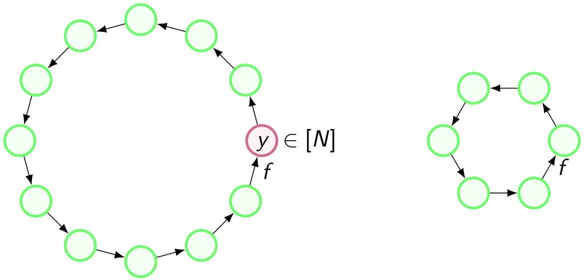

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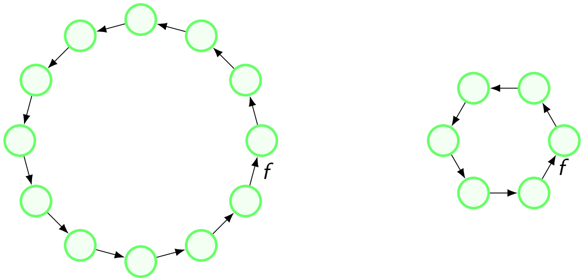

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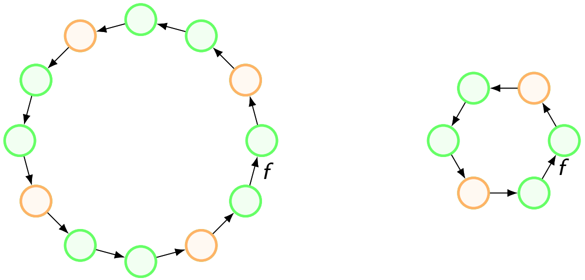

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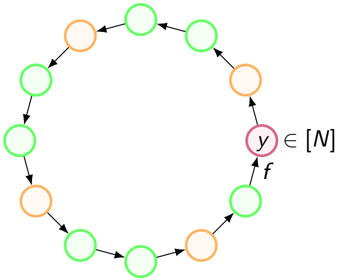

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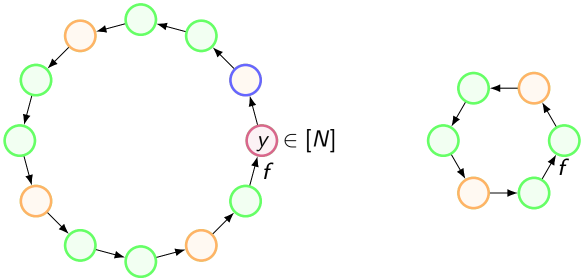

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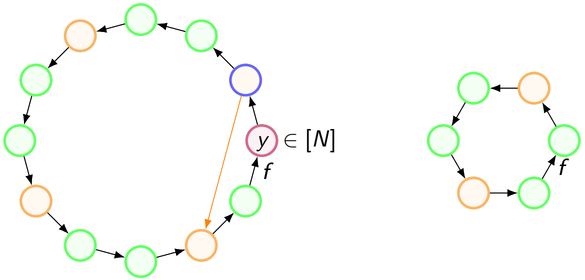

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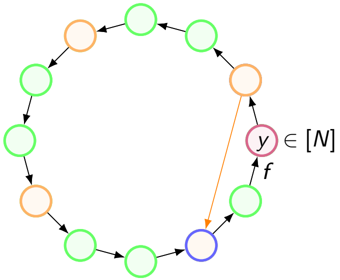

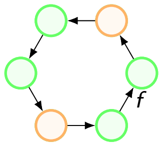

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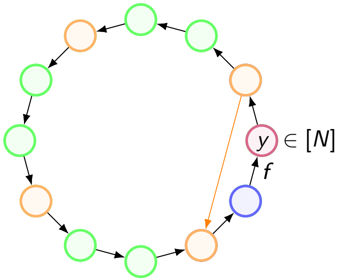

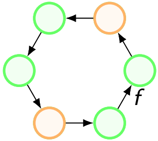

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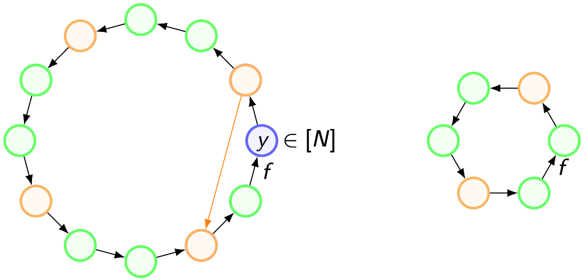

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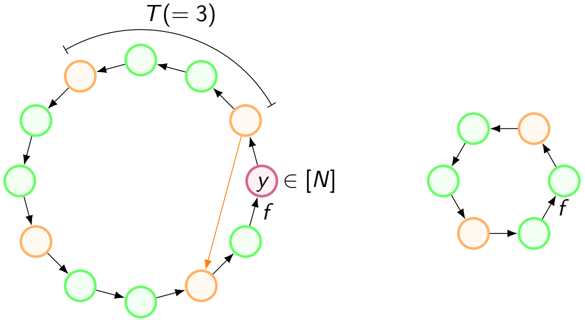

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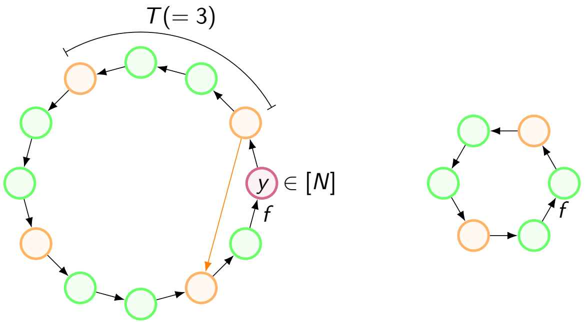

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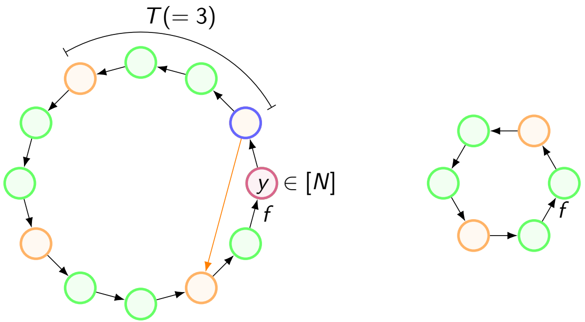

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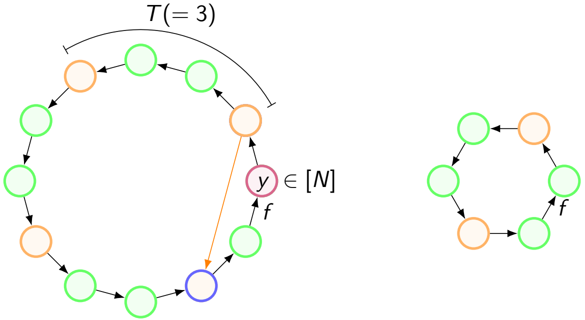

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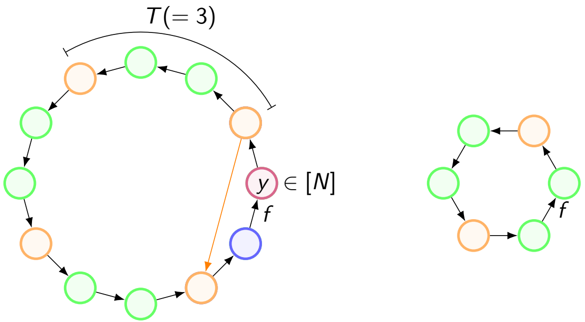

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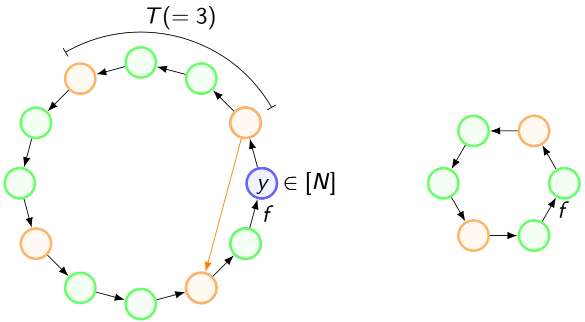

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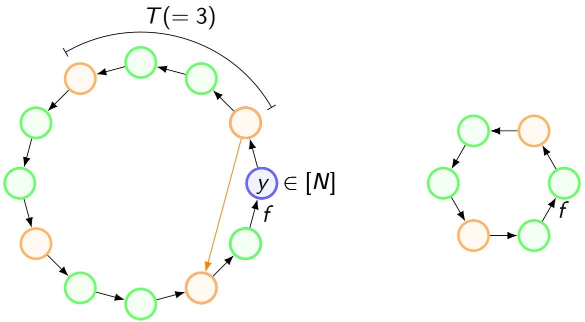

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- We need to store about N/T points total.

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- But, α must have bitlength no more than S, and $\mathcal{A}$ can make at most T evaluations of f.

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- Theoretical computer scientists want better algorithms for 3-SUM [GGH$^+$20],     , multiparty pointer jumping [CK19],   systematic substring search [CK19], ...

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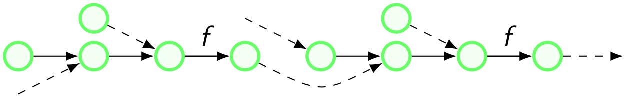

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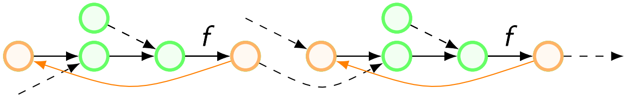

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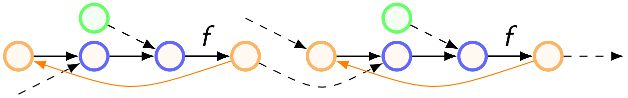

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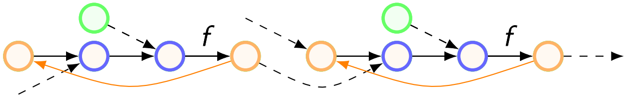

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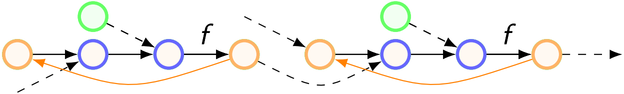

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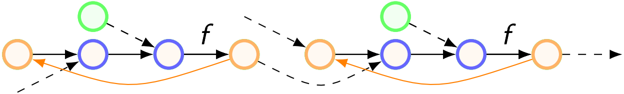

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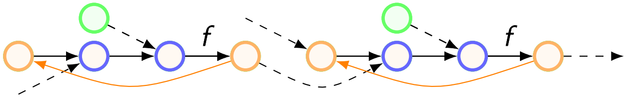

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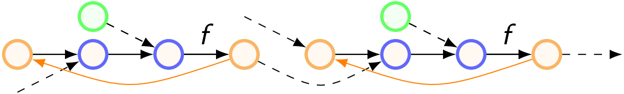

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- So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen "rerandomization" functions g_i.
- For *random* functions, Hellman showed (heuristically) this can be made to work. 

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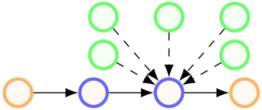

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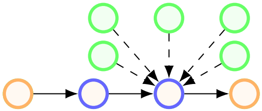

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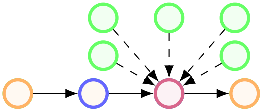

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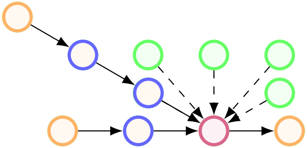

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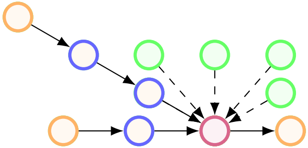

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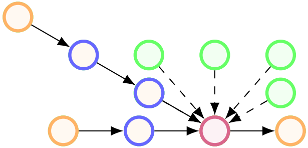

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- Intuitively, α' is the data structure for a *restriction* of f that avoids the junction points in L.
- More precisely, the "rerandomization" functions are sampled using rejection sampling so that their range is $[N] - L$.

The story so far

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Hellman 	1980	permutations	$T \lesssim N/S$	$S = T \lesssim \sqrt{N}$

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Yao 1990 	permutations	$T \gtrsim N/S$	$S = T \gtrsim \sqrt{N}$

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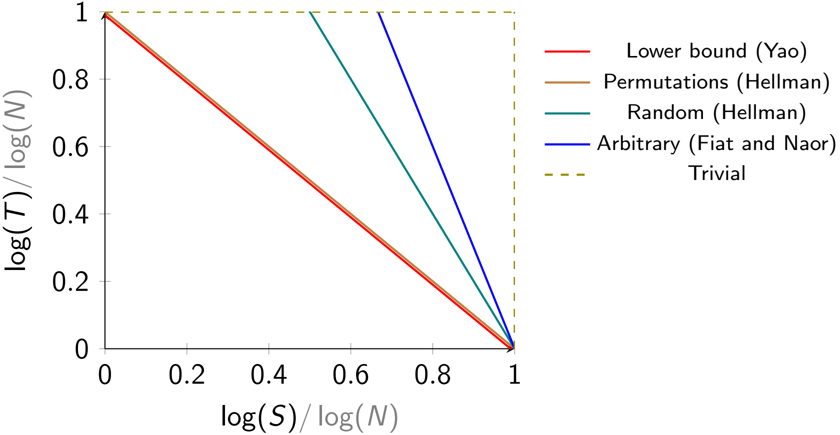

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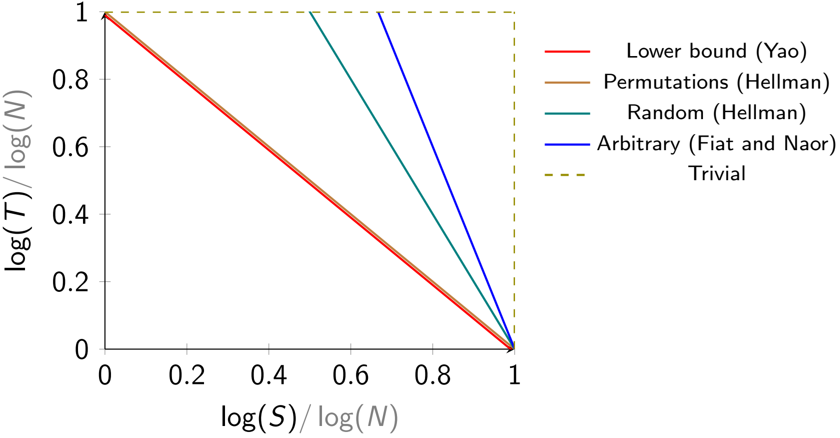

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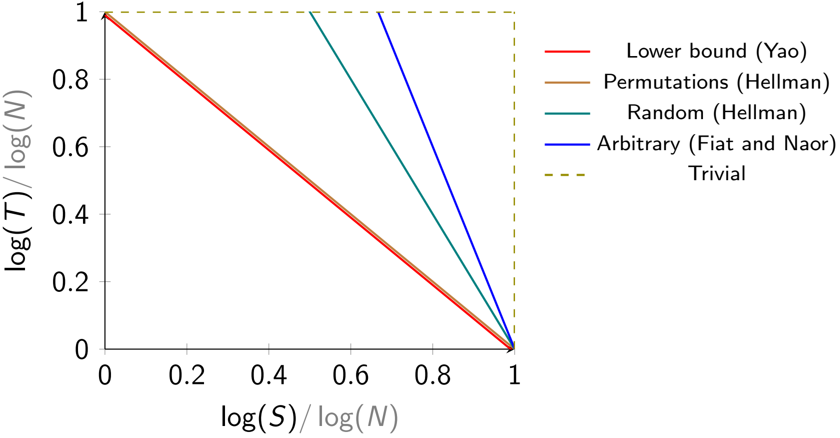

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- A: Sort of and sort of!

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- Fiat and Naor get $|L| \simeq S$, but this is the hard limit, since L needs to fit into S-bit advice α.
- Or does it?

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Sharing Randomness

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- In practice, can instantiate a random oracle.

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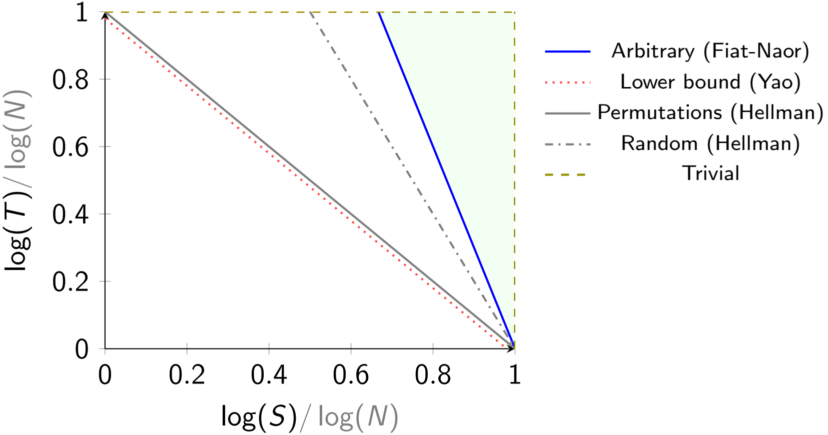

1
0.8
0.6
0.4
0.2
0
0
0.2
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$\log(T)/\log(N)$
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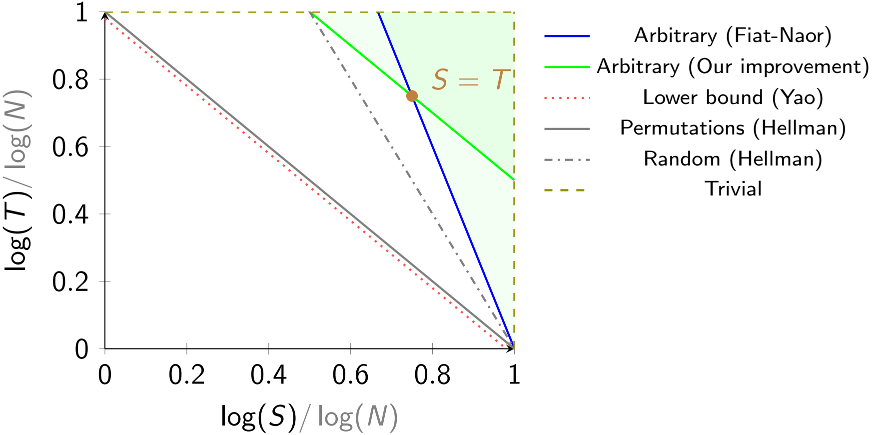

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- Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- $\implies$ preprocessing returns α maximizing

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- We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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  - Encoding is $(\alpha, i_1, \ldots, i_N)$.
  - For each y, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \ldots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

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  - A larger codomain does not make the problem harder (us and [CK19])  

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  - Better algorithms for injective functions?

Thank you!

I'm happy to take additional questions offline. You can ping me at `speters@cs.cornell.edu`.

References I

Henry Corrigan-Gibbs and Dmitry Kogan.
The function-inversion problem: Barriers and opportunities.
In *TCC*, 2019.

Anindya De, Luca Trevisan, and Madhur Tulsiani.
Time space tradeoffs for attacks against one-way functions and PRGs.
In *CRYPTO*, 2010.

Alexander Golovnev, Siyao Guo, Thibaut Horel, Sunoo Park, and Vinod Vaikuntanathan.
Data structures meet cryptography: 3SUM with preprocessing.
In *STOC*, 2020.

References II

Alexander Golovnev, Siyao Guo, Spencer Peters, and Noah Stephens-Davidowitz.
Revisiting time-space tradeoffs for function inversion.
Available at https://eccc.weizmann.ac.il/report/2022/145/, 2022.

Martin Hellman.
A cryptanalytic time-memory trade-off.
IEEE Transactions on Information Theory, 26(4):401–406, 1980.

Ilan Newman.
Private vs. common random bits in communication complexity.
Information processing letters, 39(2):67–71, 1991.