



Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.



Siyao Guo

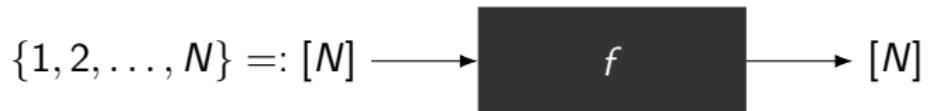


Sasha Golovnev



Function Inversion

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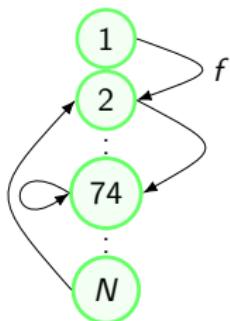
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$$\{1, 2, \dots, N\} =: [N] \ni x \rightarrow \boxed{f} \rightarrow y \in [N]$$

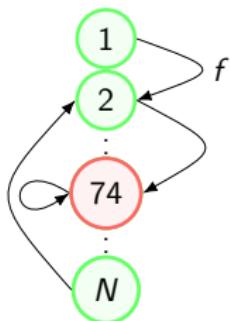
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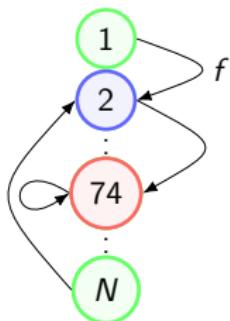
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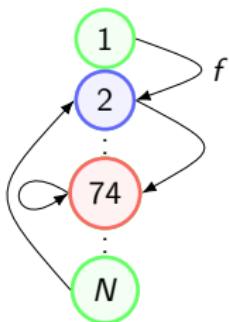
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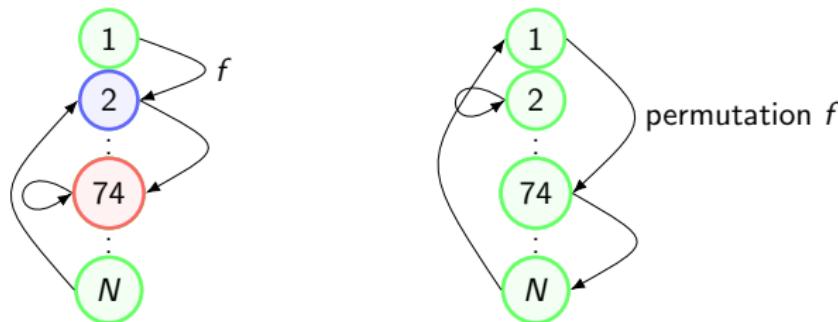
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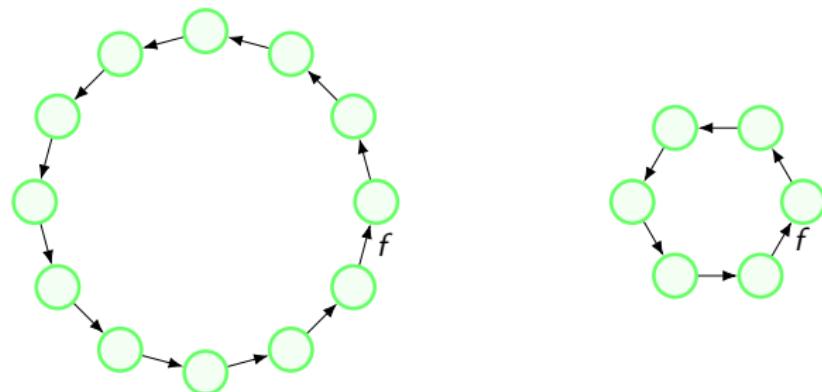
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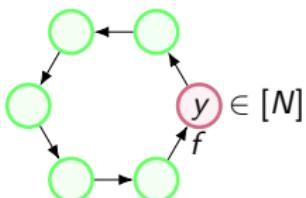
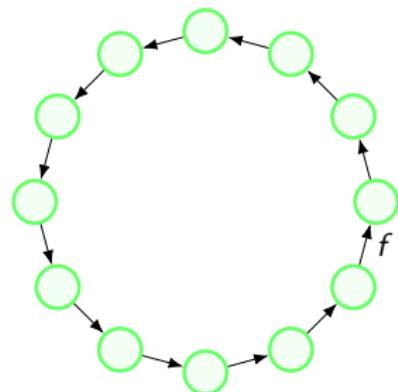
Hellman's algorithm

- If f is a permutation, its *graph* is a disjoint union of cycles.



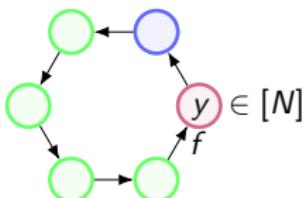
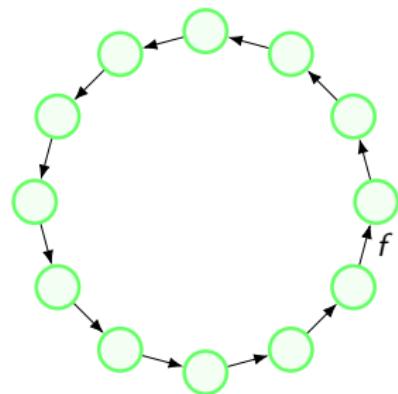
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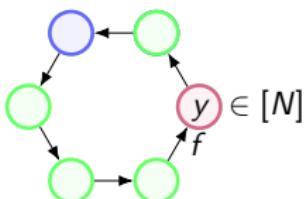
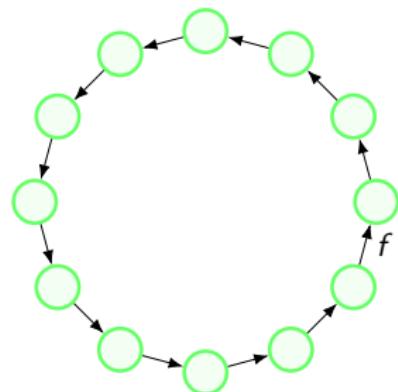
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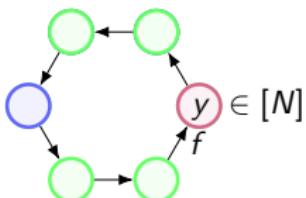
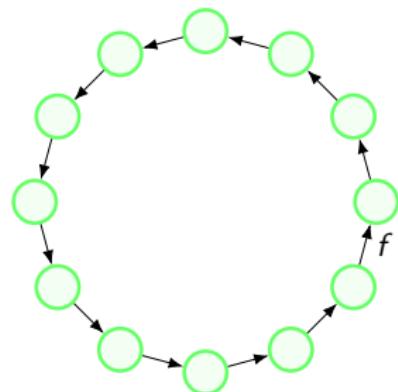
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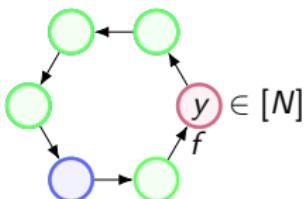
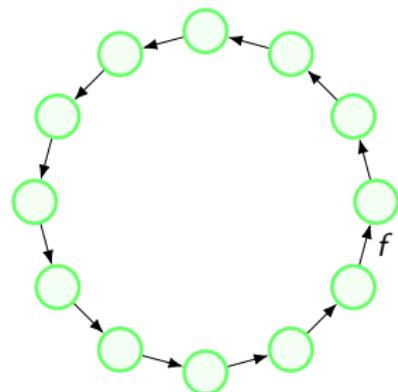
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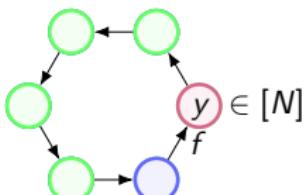
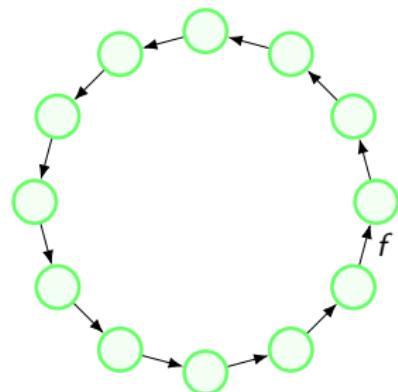
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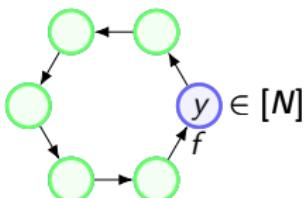
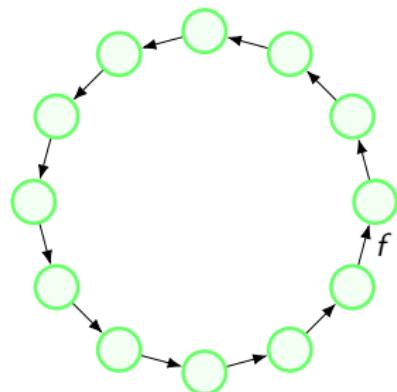
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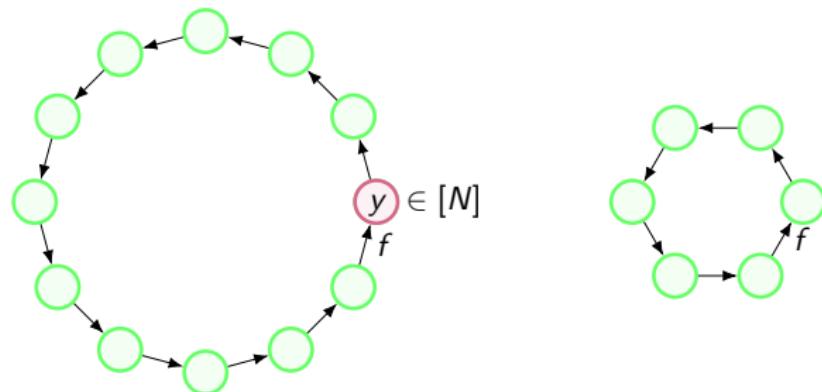


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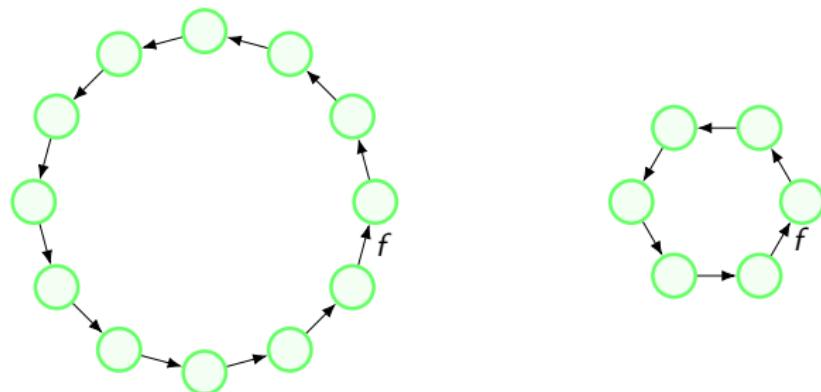
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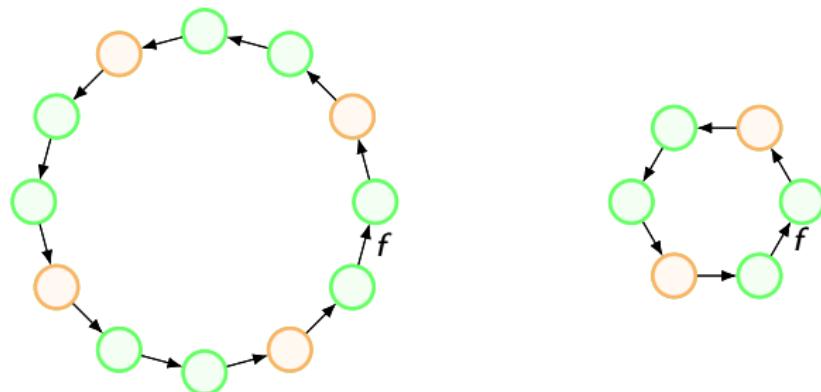
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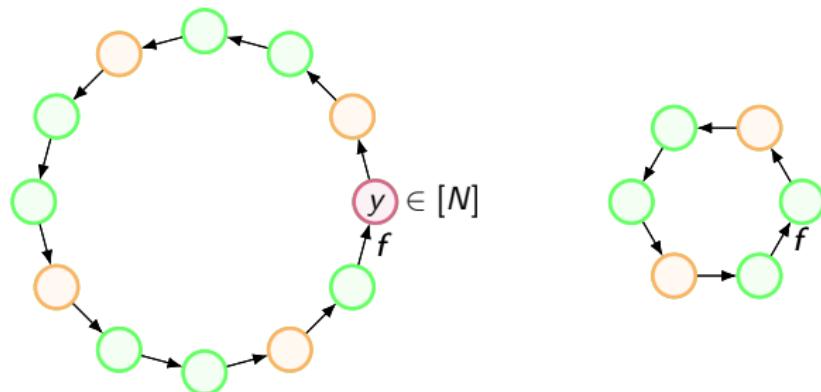
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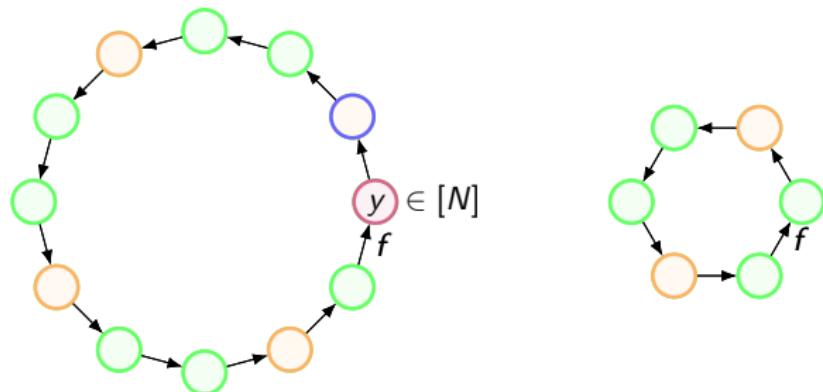
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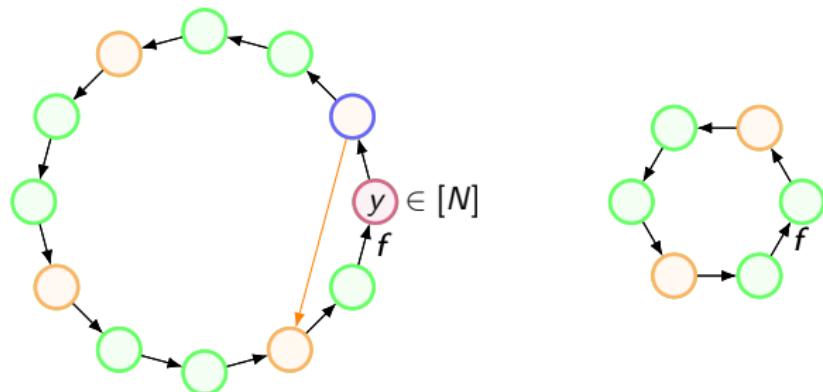
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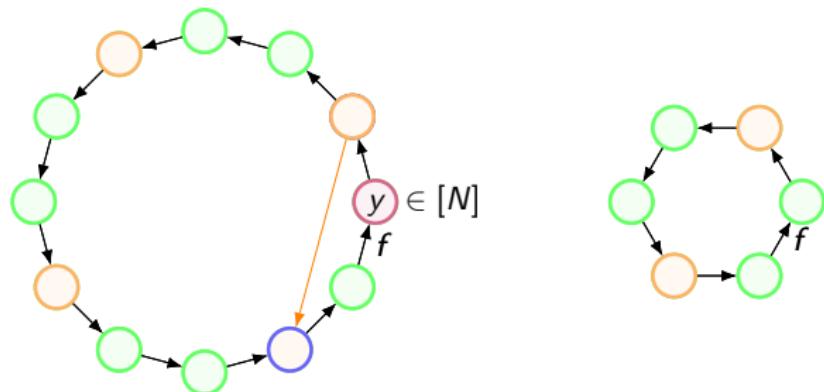
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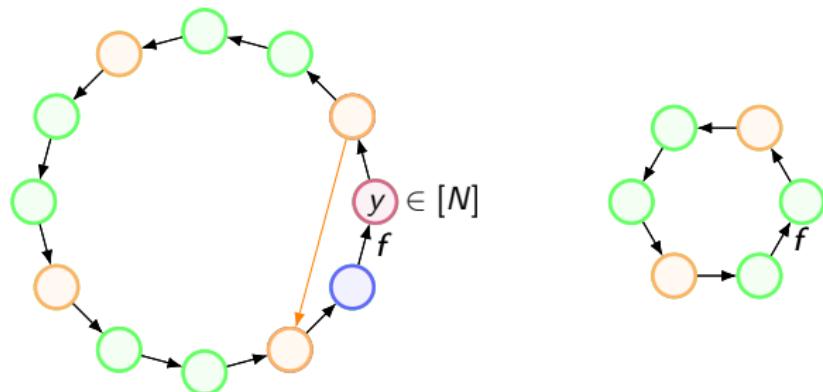
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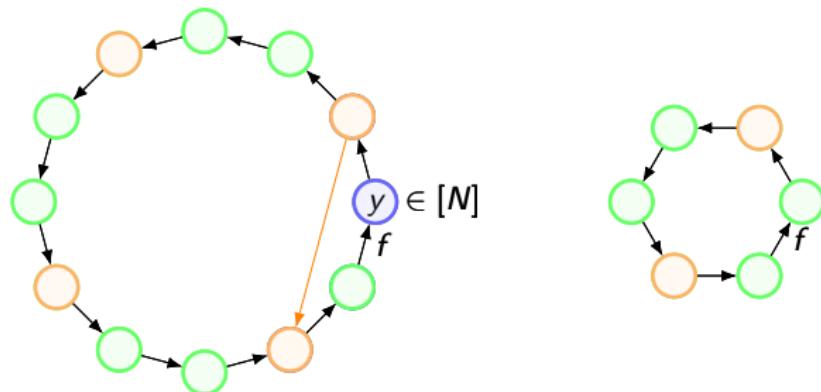
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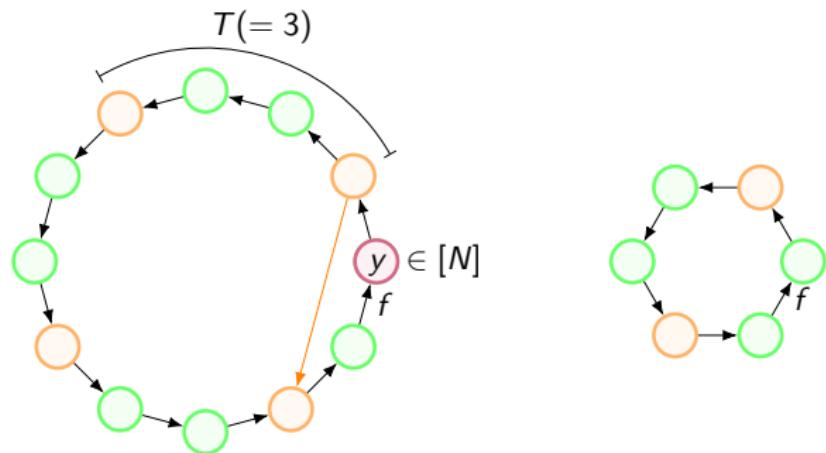
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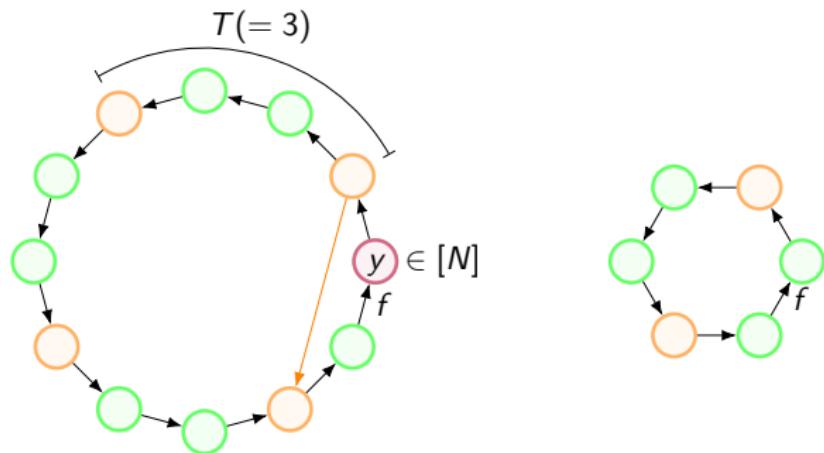
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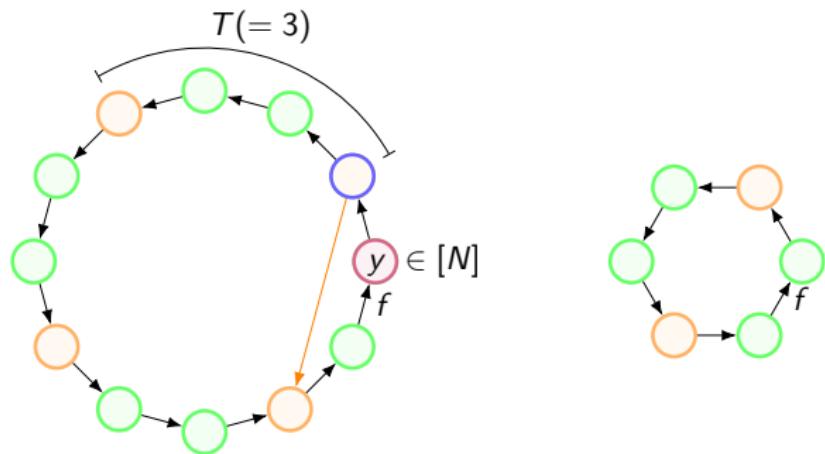
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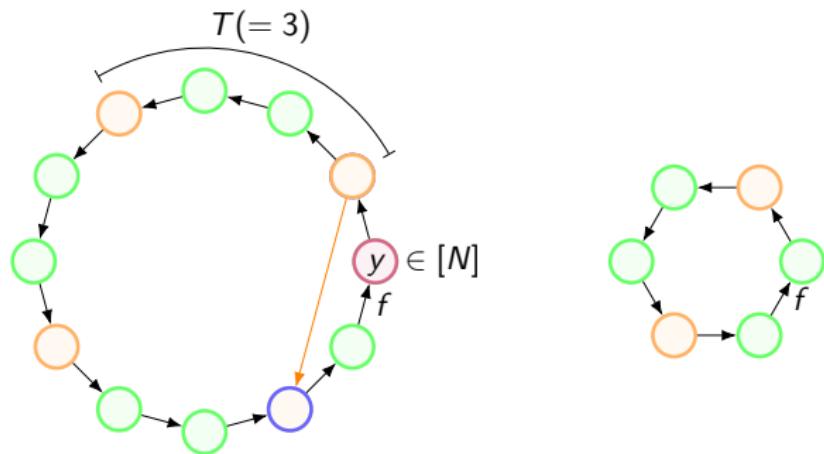
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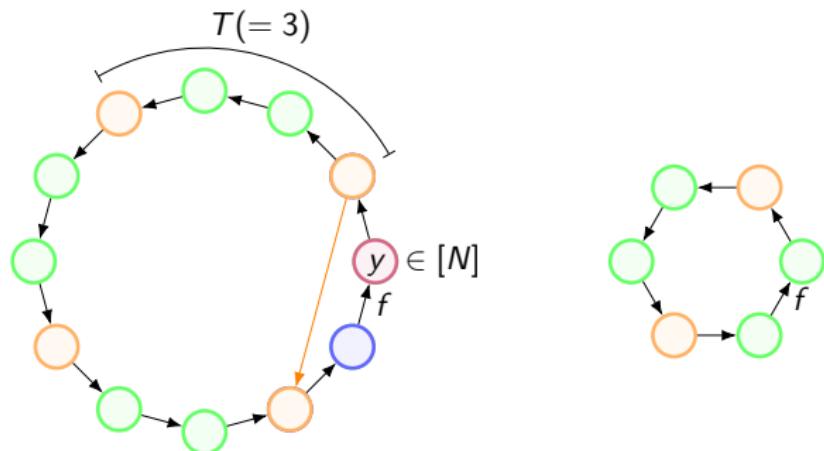
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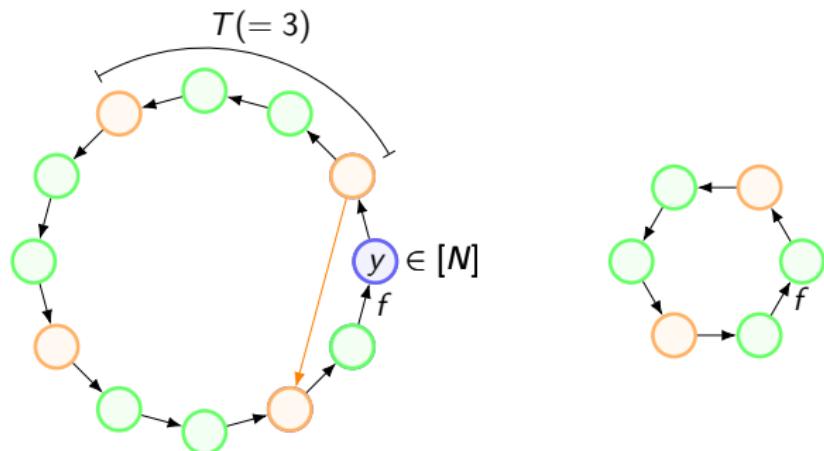
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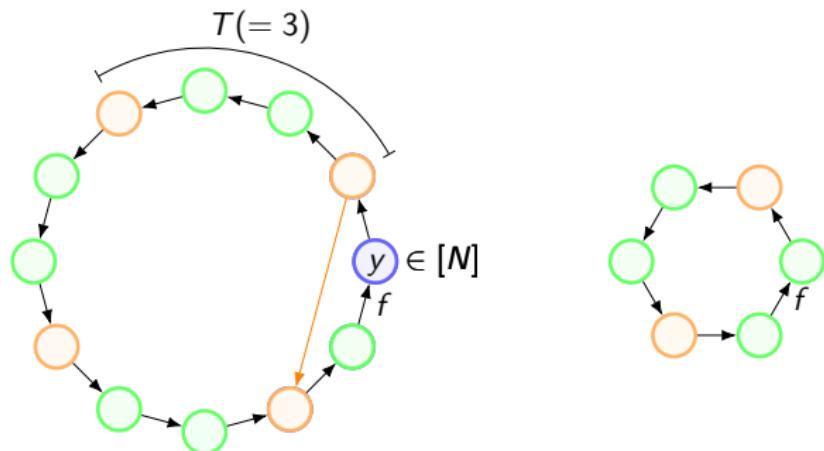
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- ▶ We need to store about N/T points total.

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- ▶ But, α must have bitlength no more than S , and \mathcal{A} can make at most T evaluations of f .

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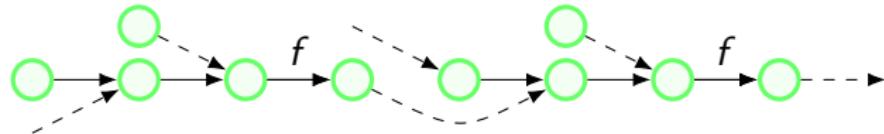
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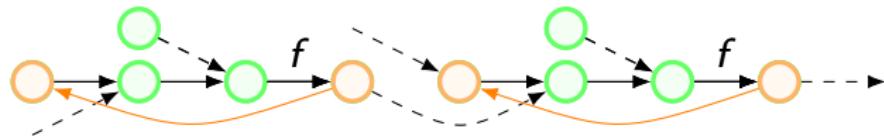
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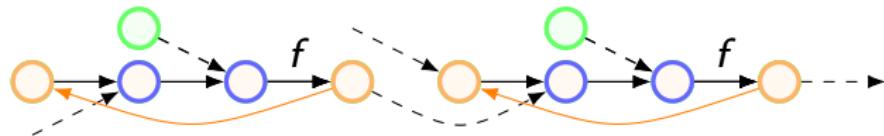
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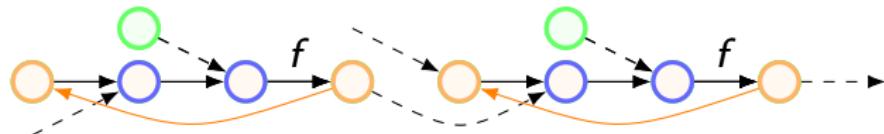
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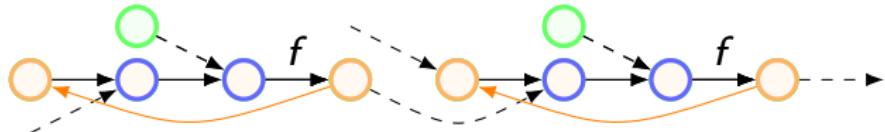
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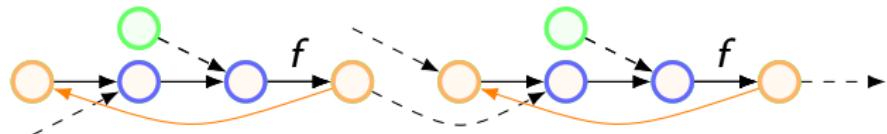
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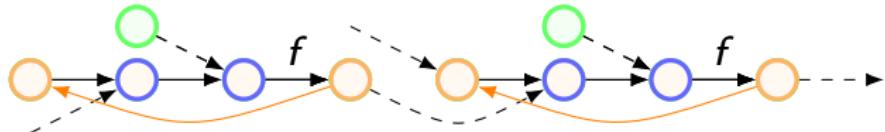
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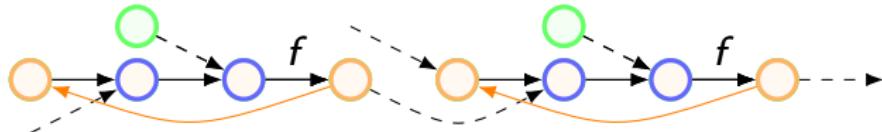
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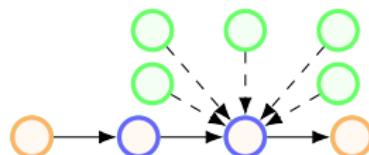


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- ▶ So, can repeatedly apply the basic scheme to many compositions $g_i \circ f$, for suitably chosen “rerandomization” functions g_i .
- ▶ For *random* functions, Hellman showed (heuristically) this can be made to work.



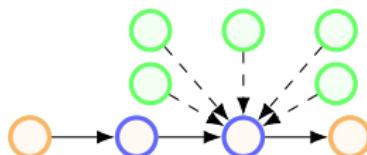
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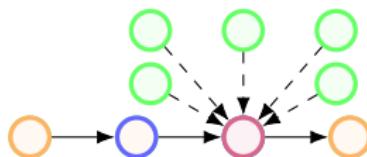
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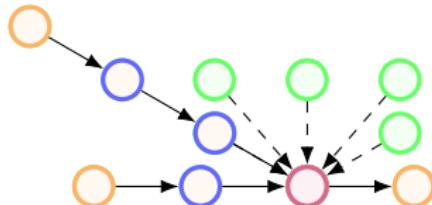
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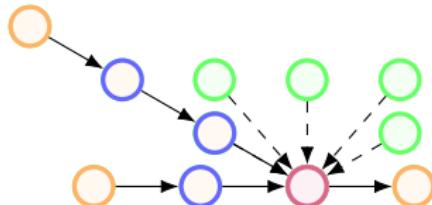
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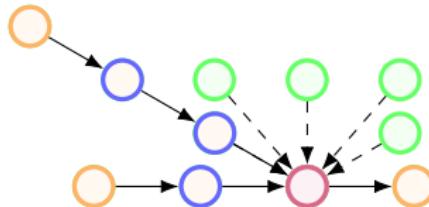
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- ▶ Fiat and Naor deal with this by storing $\alpha = (\alpha', L)$, where L contains junction points along with their inverses.
- ▶ Intuitively, α' is the data structure for a *restriction* of f that avoids the junction points in L .
- ▶ More precisely, the “rerandomization” functions are sampled using rejection sampling so that their range is $[N] - L$.

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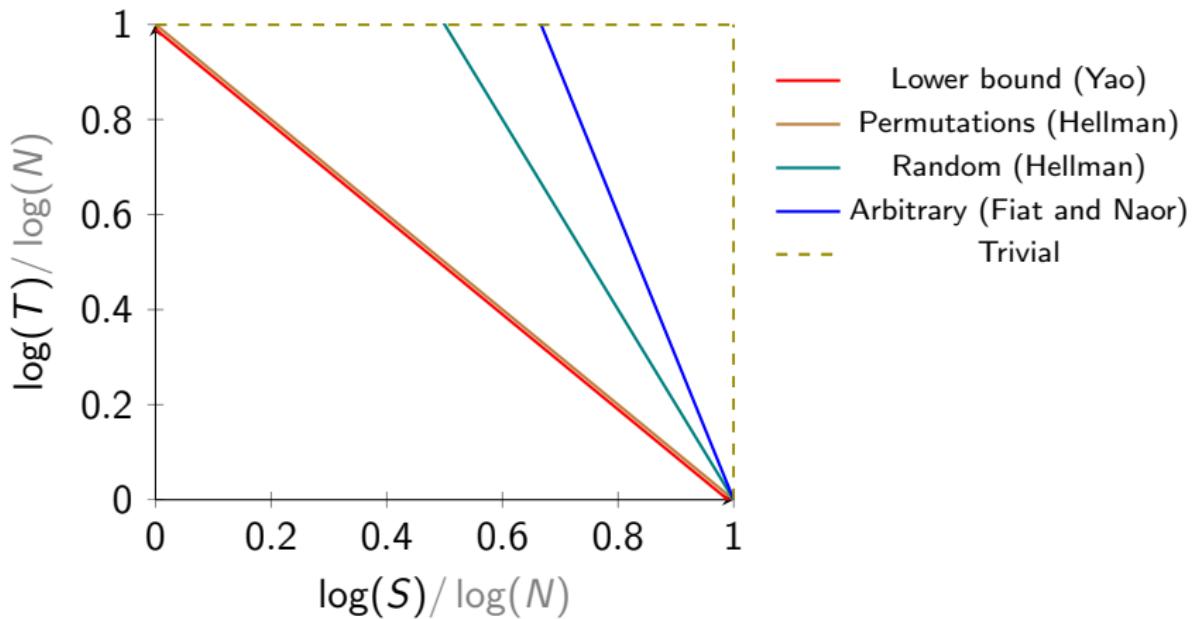
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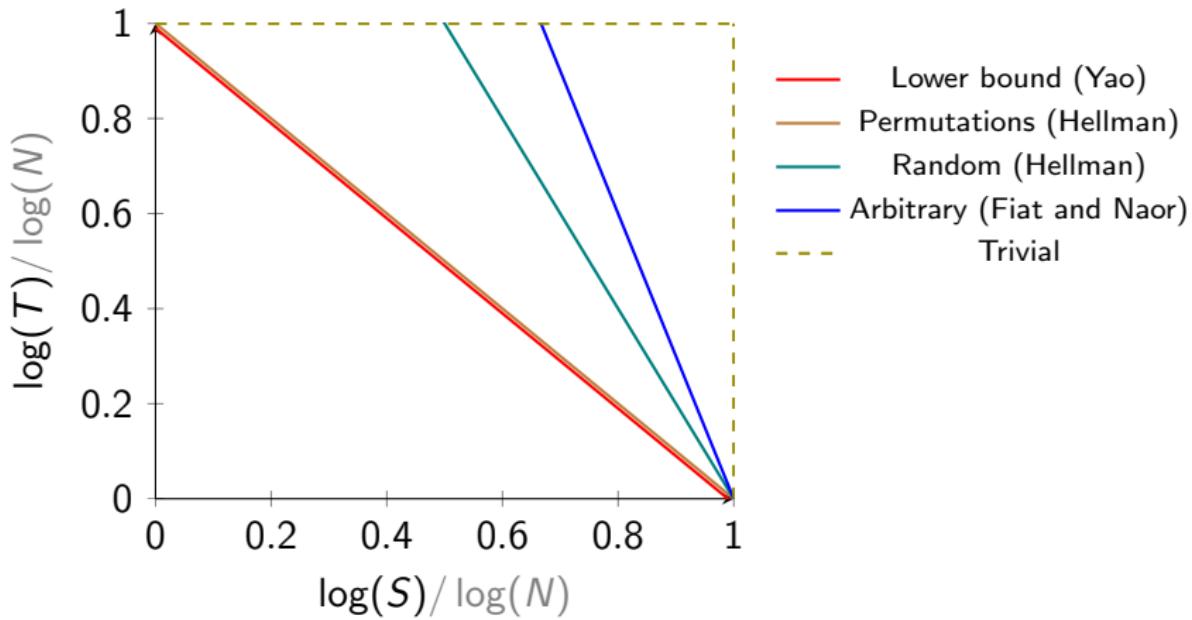
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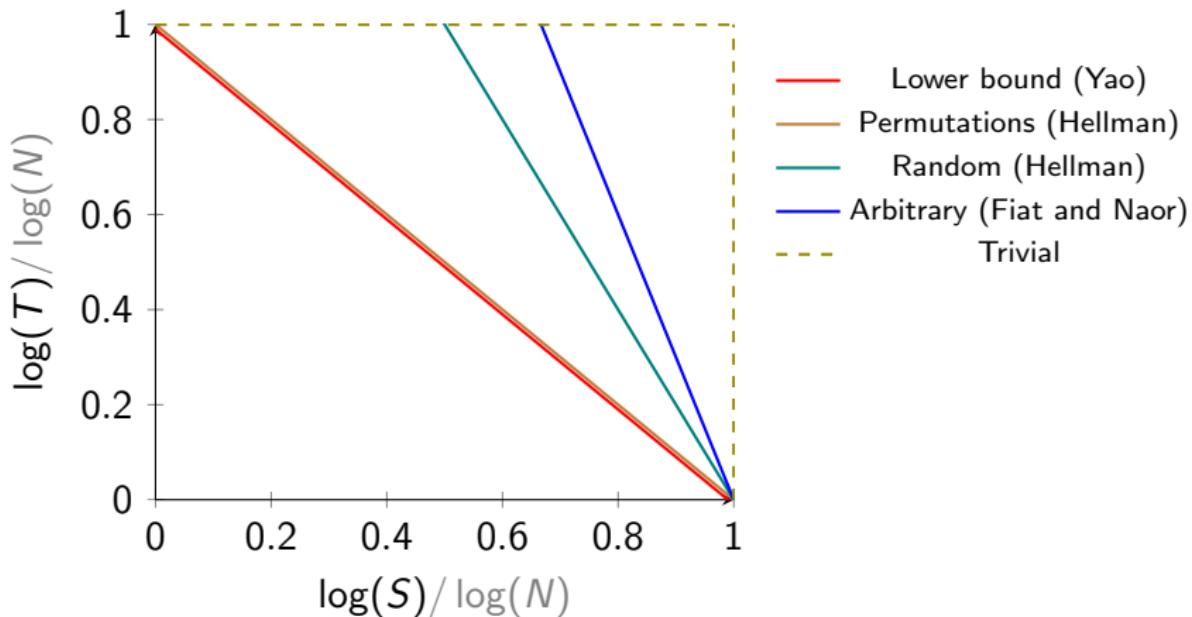


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- ▶ A: Sort of and sort of!

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- ▶ Fiat and Naor get $|L| \simeq S$, but this is the hard limit, since L needs to fit into S -bit advice α .
- ▶ Or does it?

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- ▶ In practice, can instantiate a random oracle.



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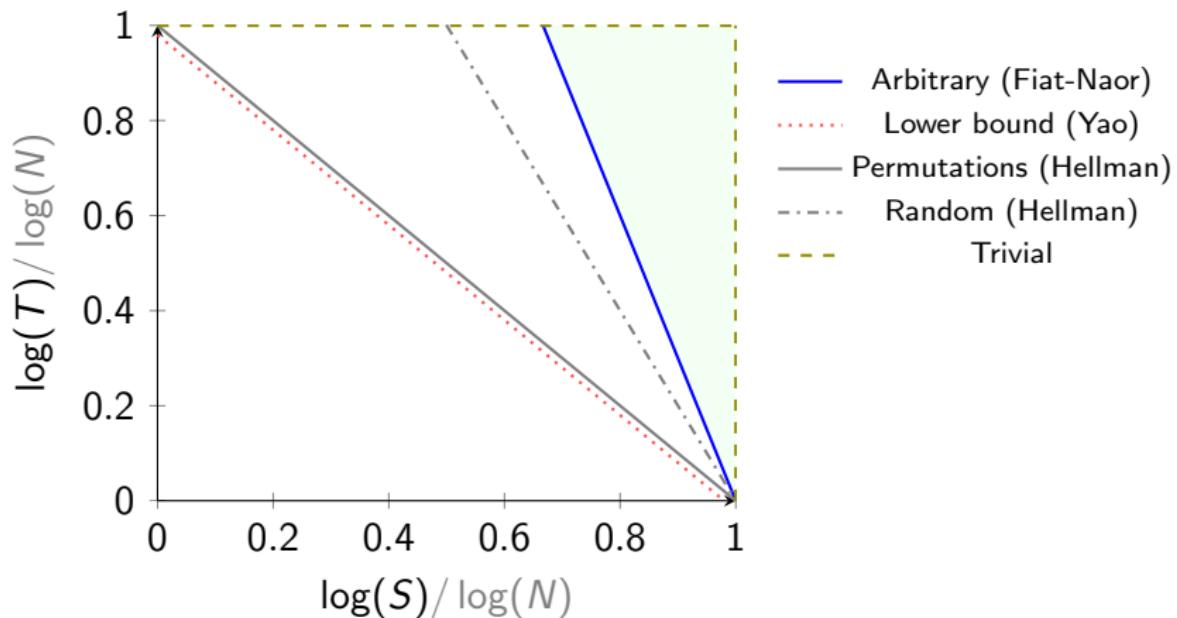
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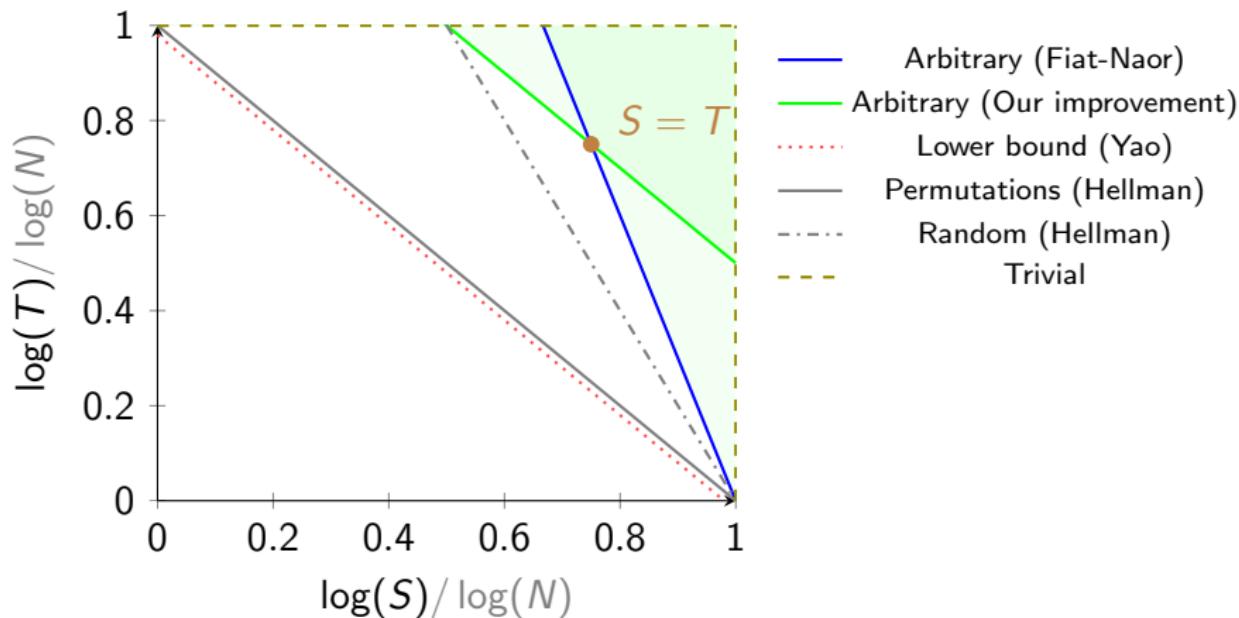
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- ▶ Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- ▶ \implies preprocessing returns α maximizing

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- ▶ We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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 - ▶ For each y , decoder again runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It sets $f^{-1}(y) = x_{i_y}$.

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 - ▶ A larger codomain does not make the problem harder (us and [CK19])



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- ▶ Possibly more tractable:
 - ▶ Better algorithms for inverting a small fraction of the range?
That is, improving on De, Trevisan and Tulsiani [DTT10]?

Open Problems

- ▶ Only one real open problem—**close the gap!**
- ▶ Moonshots:
 - ▶ Improve Yao's lower bound against (general) non-adaptive algorithms?
 - ▶ $S^2 T = N^2$ algorithm for worst-case function inversion?
- ▶ Possibly more tractable:
 - ▶ Better algorithms for inverting a small fraction of the range?
That is, improving on De, Trevisan and Tulsiani [DTT10]?
 - ▶ Better algorithms for injective functions?

Thank you!

I'm happy to take additional questions offline. You can ping me at speters@cs.cornell.edu.

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