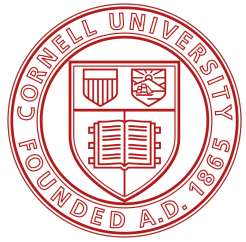

Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.

Siyao Guo

Sasha Golovnev

Function Inversion

- Given a function $f : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\}$ and a point y in its image, find x with $f(x) = y$.

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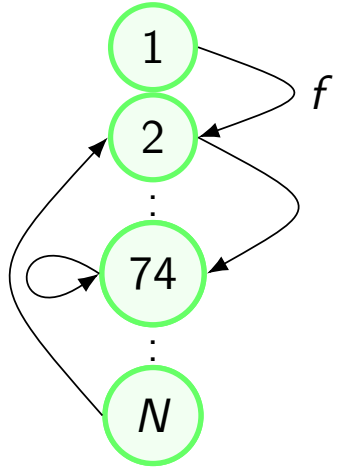

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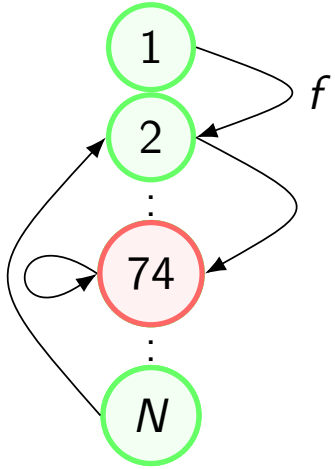

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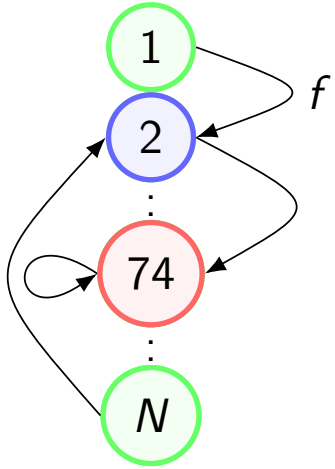

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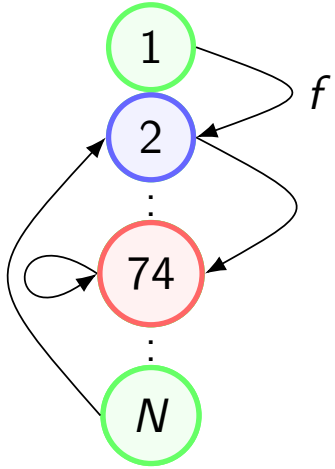

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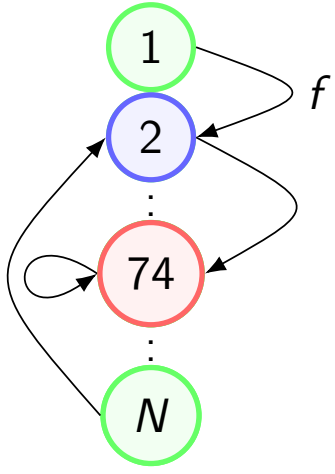

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- The study of this *black-box* function inversion problem was initiated by Martin Hellman  in 1980 [Hel80].

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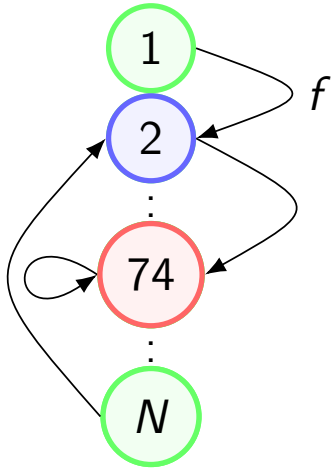

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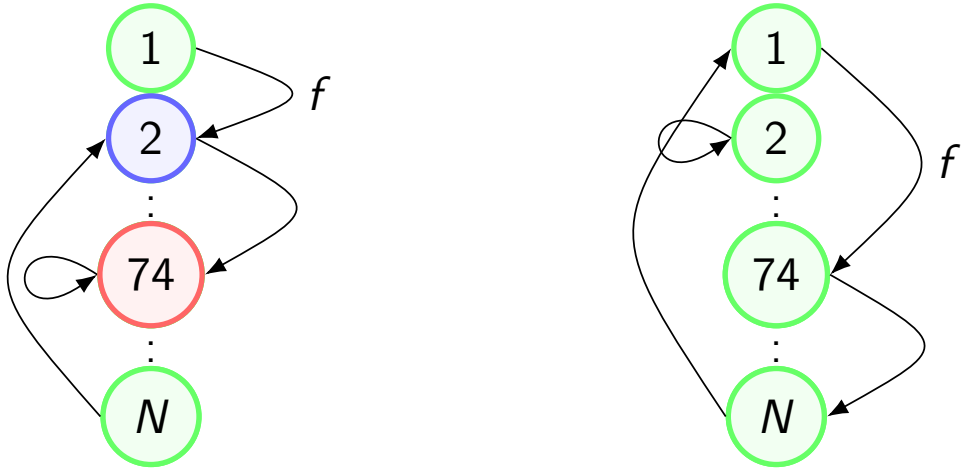

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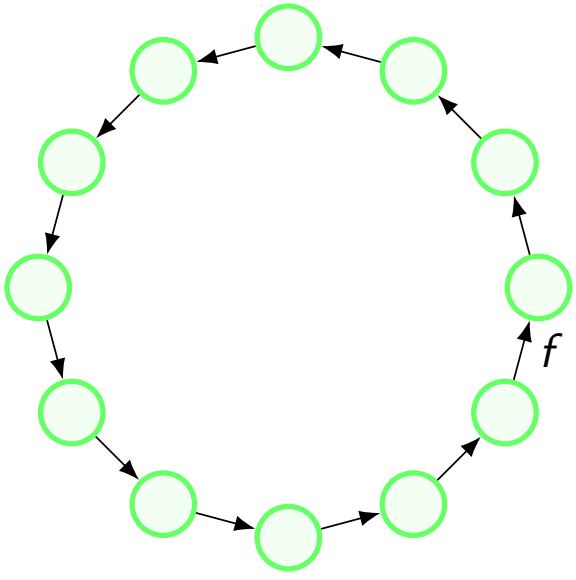

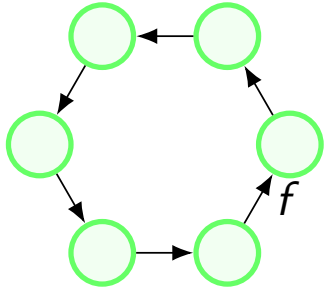

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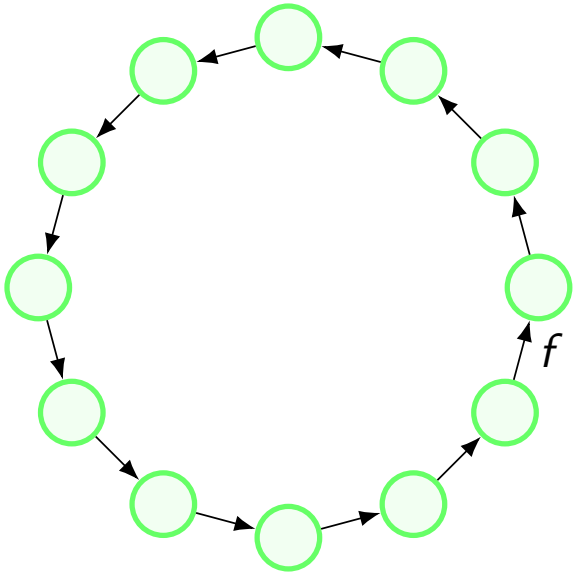

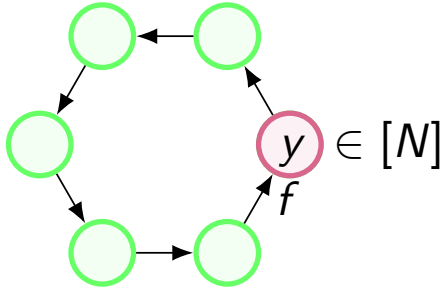

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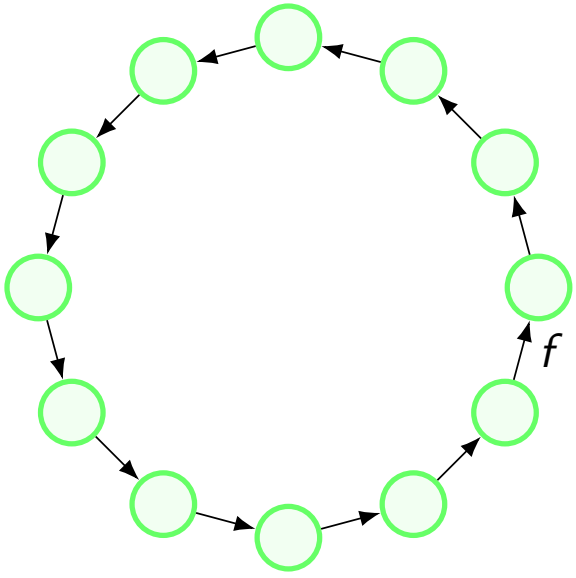

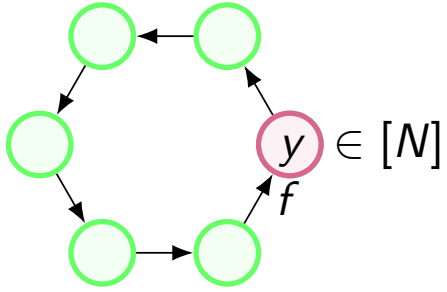

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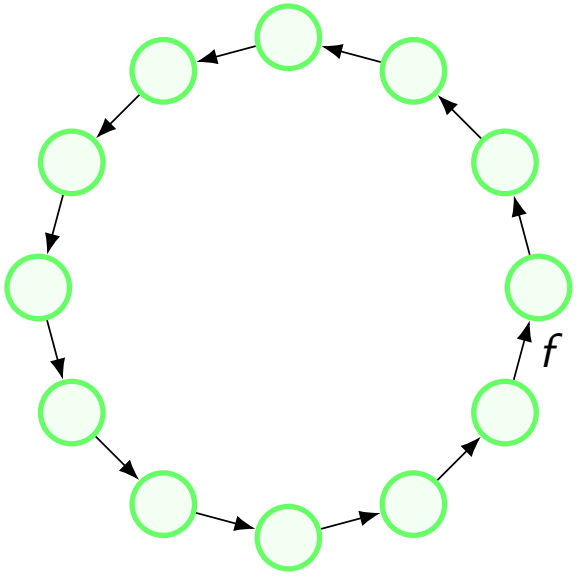

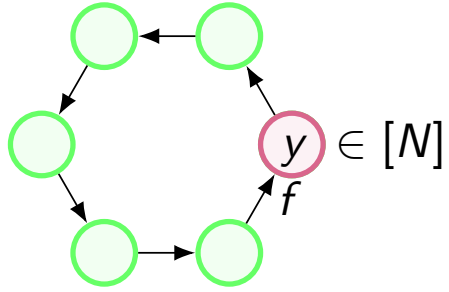

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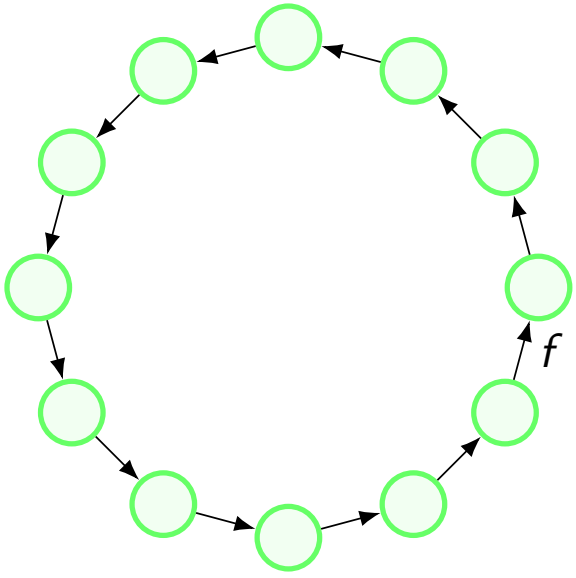

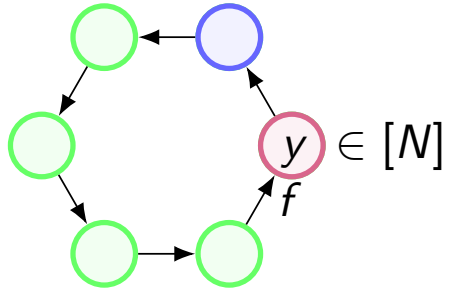

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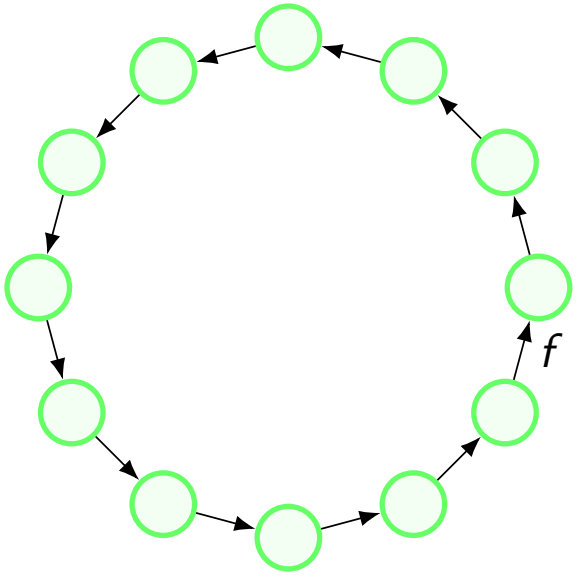

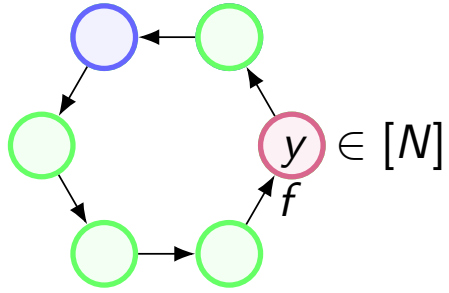

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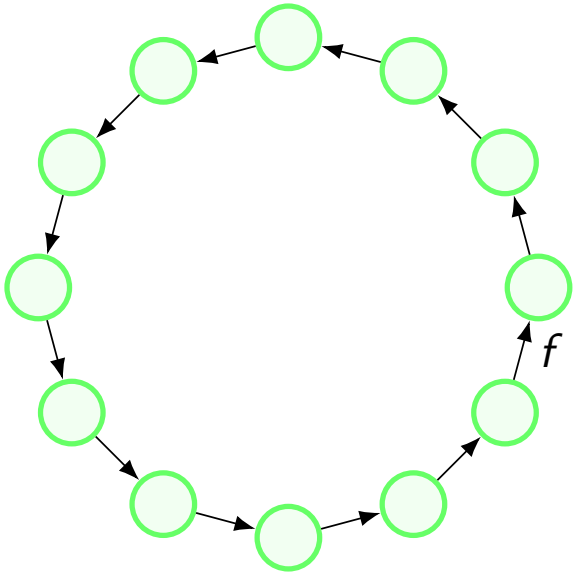

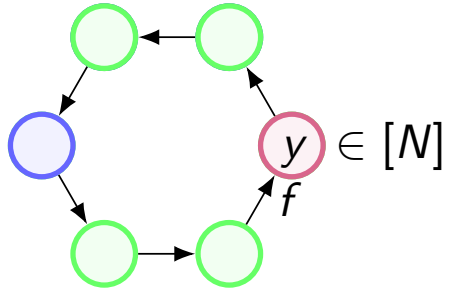

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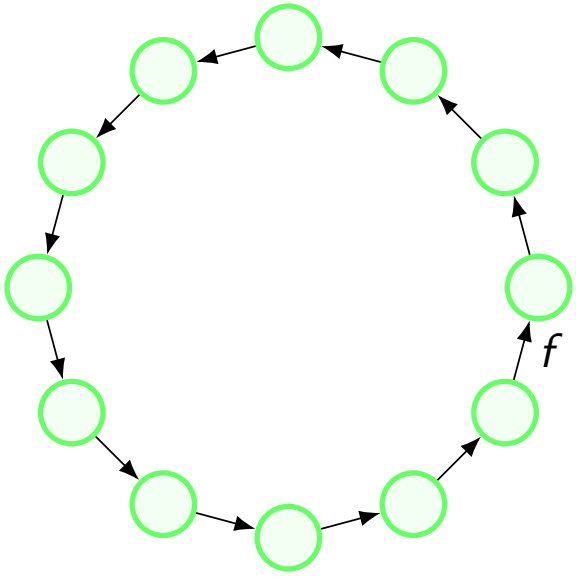

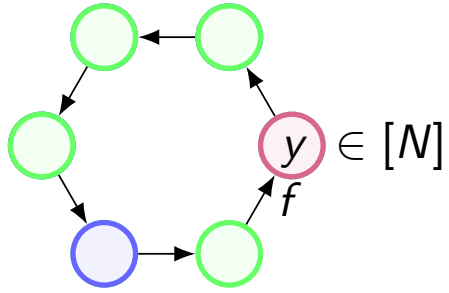

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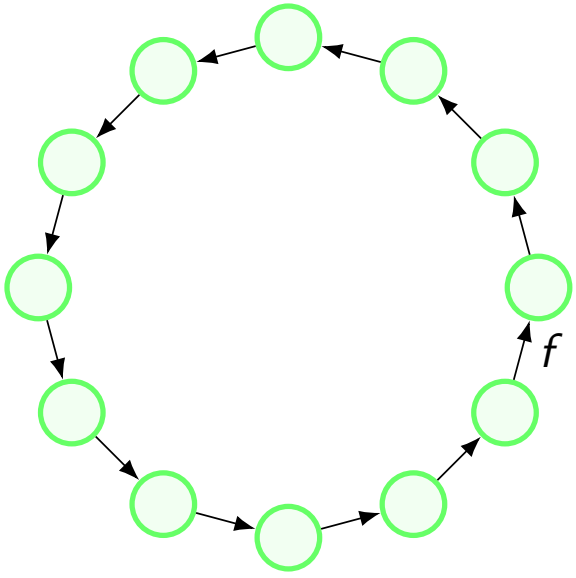

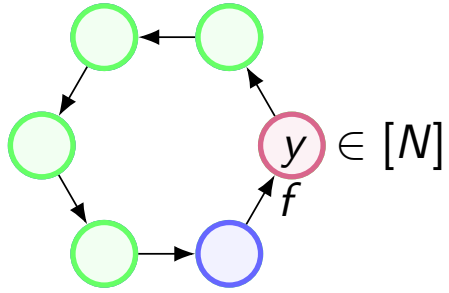

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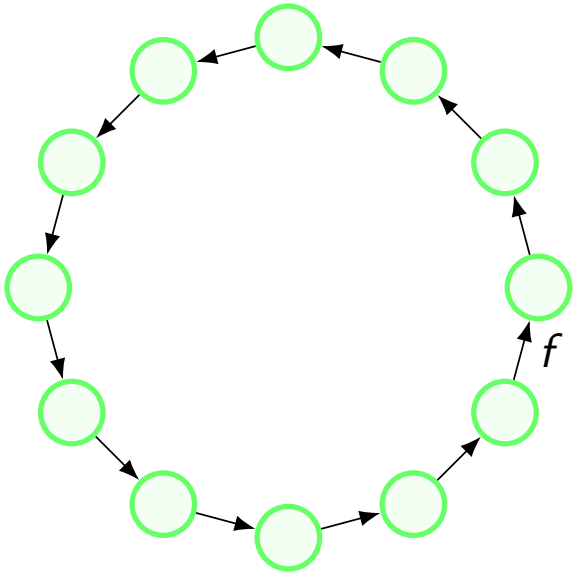

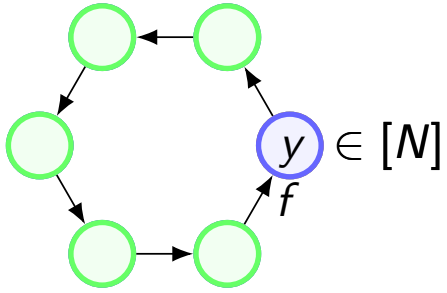

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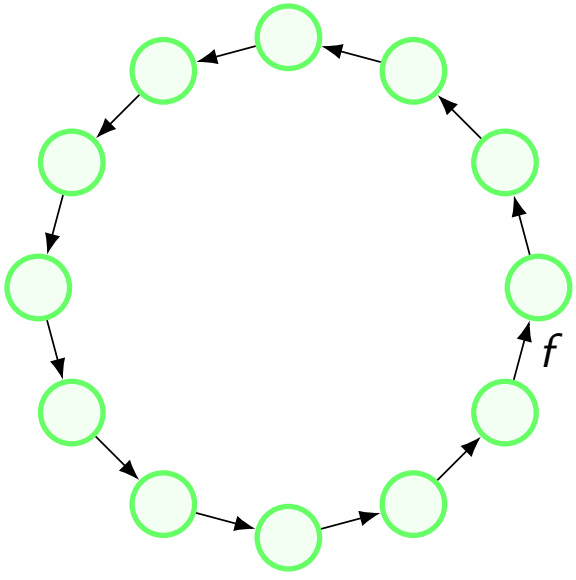

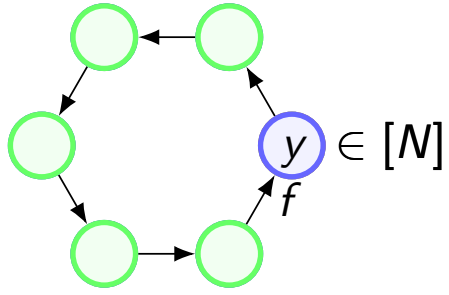

- Starting point: Hope y is on a small cycle.
- Compute $f(y)$, $f(f(y))$, and so on until you reach y again.
- Of course, y might not be on a small cycle...

Hellman's algorithm

- What if y is on a large cycle?

Hellman's algorithm

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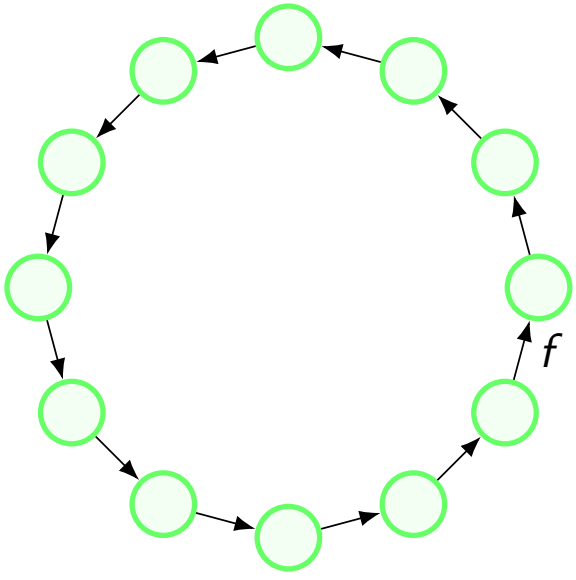

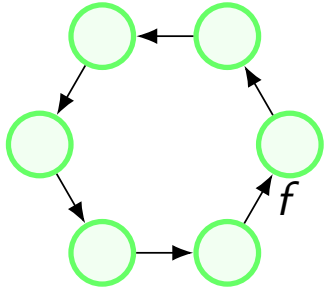

Hellman's algorithm

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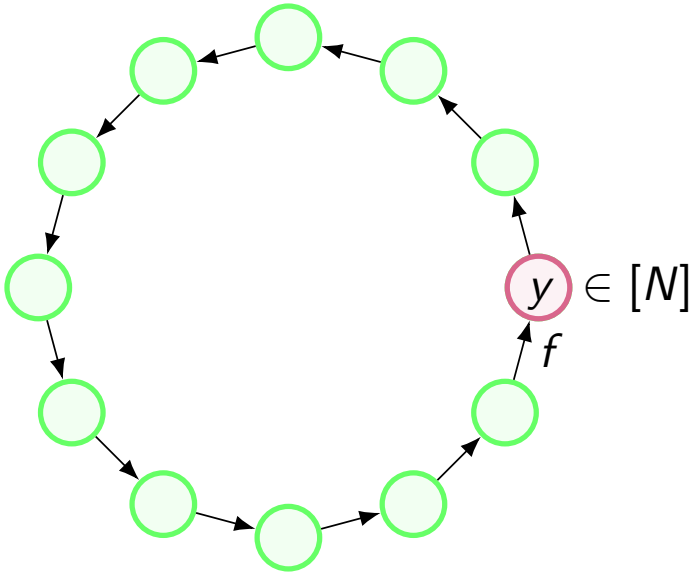

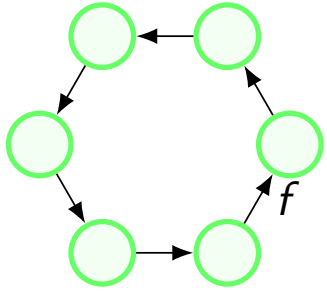

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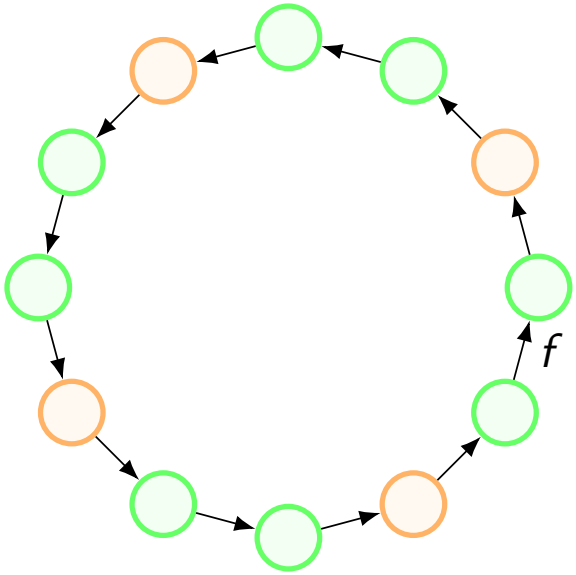

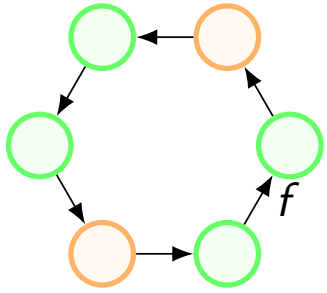

- In a preprocessing step, store points uniformly spaced around each cycle.

Hellman's algorithm

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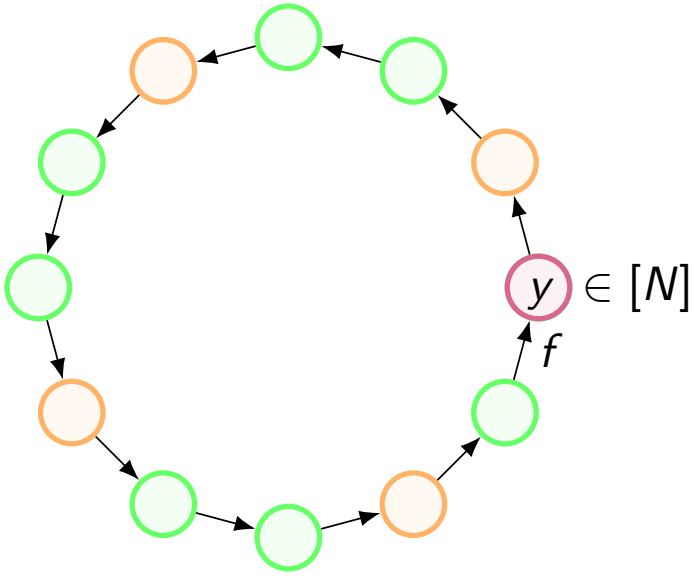

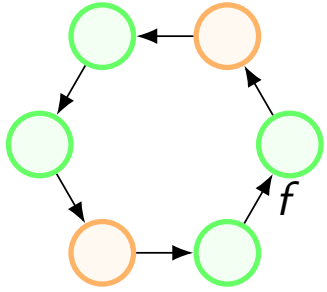

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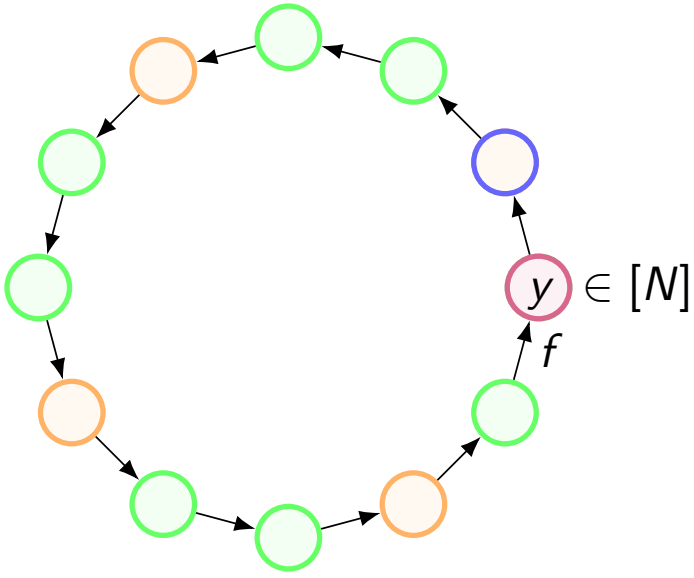

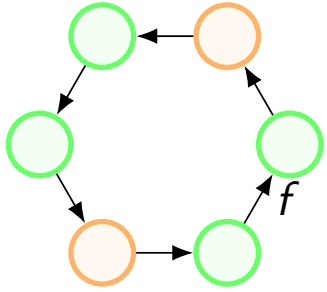

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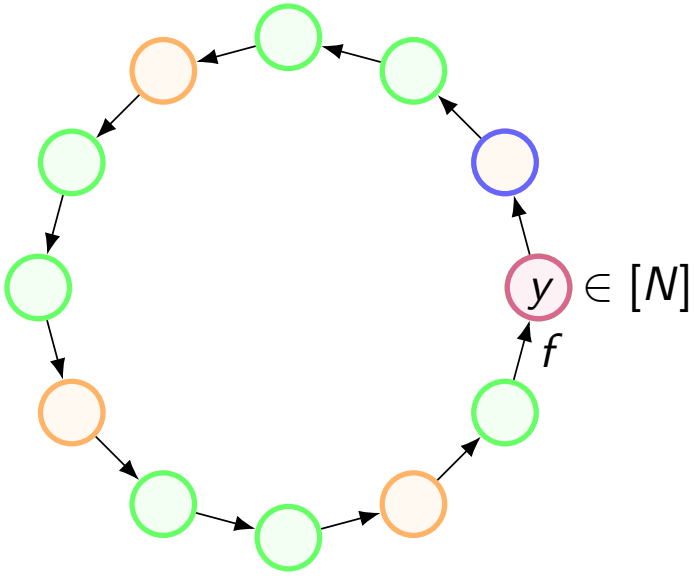

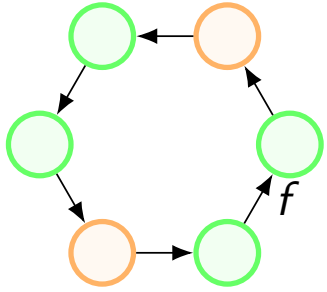

- In a preprocessing step, store points uniformly spaced around each cycle.
- If you hit a stored point, jump to the previous one!

Hellman's algorithm

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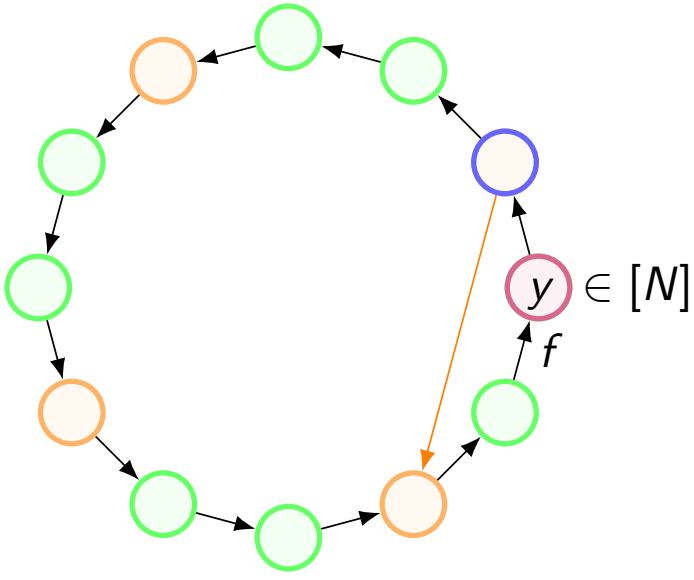

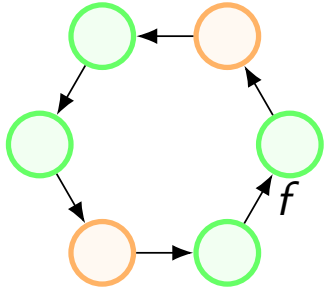

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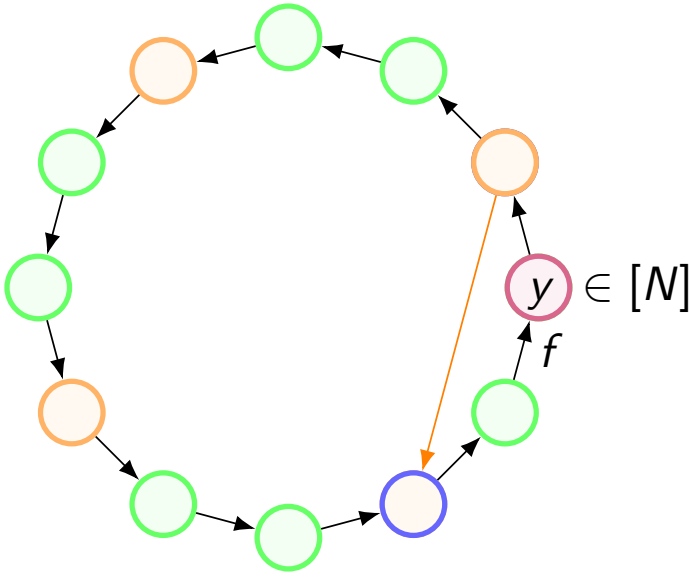

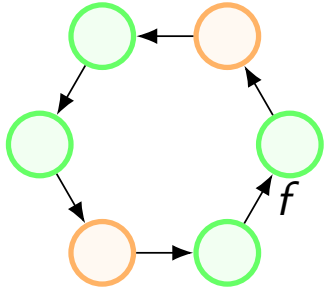

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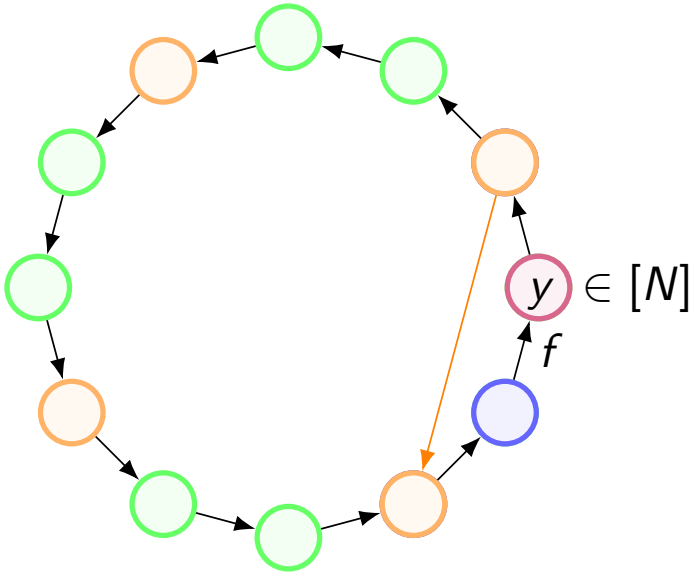

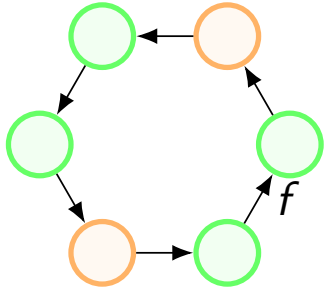

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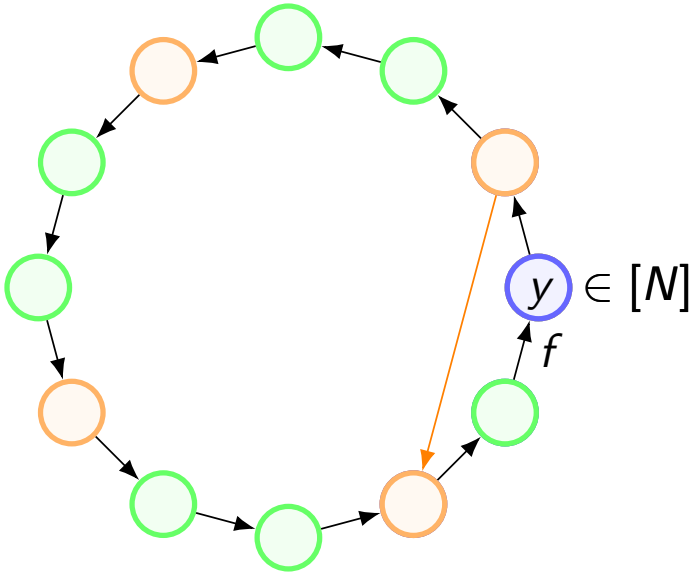

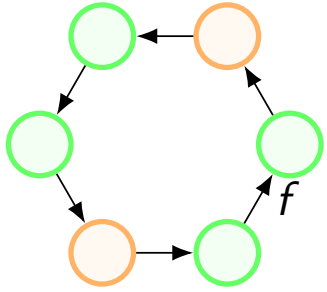

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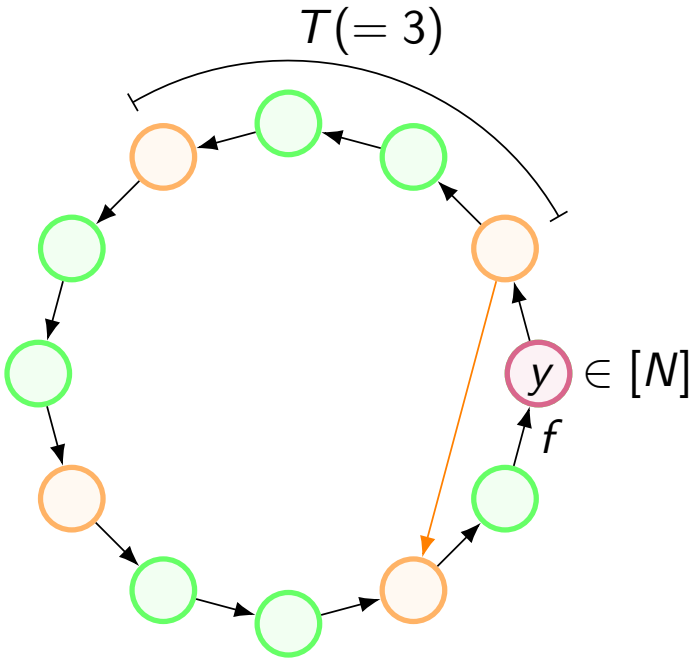

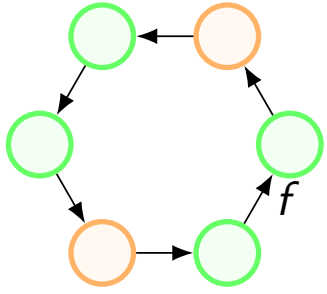

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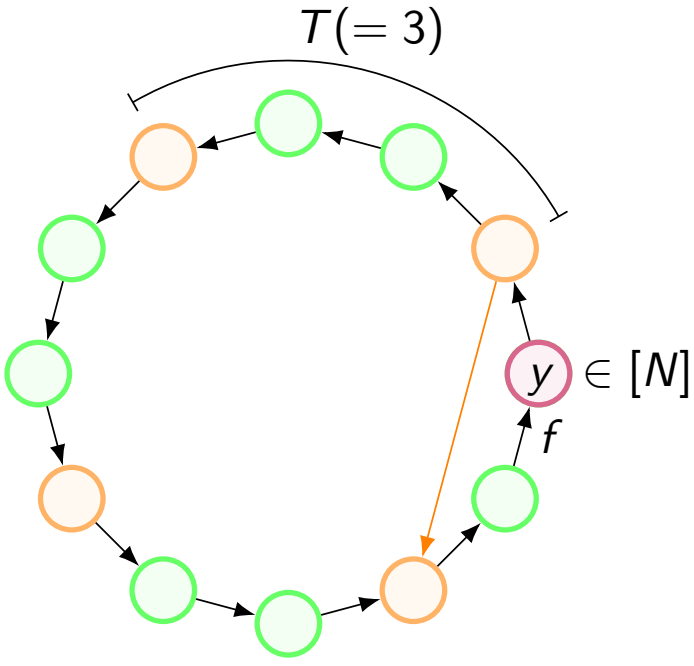

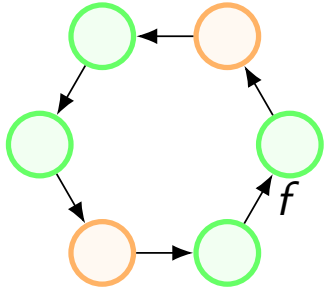

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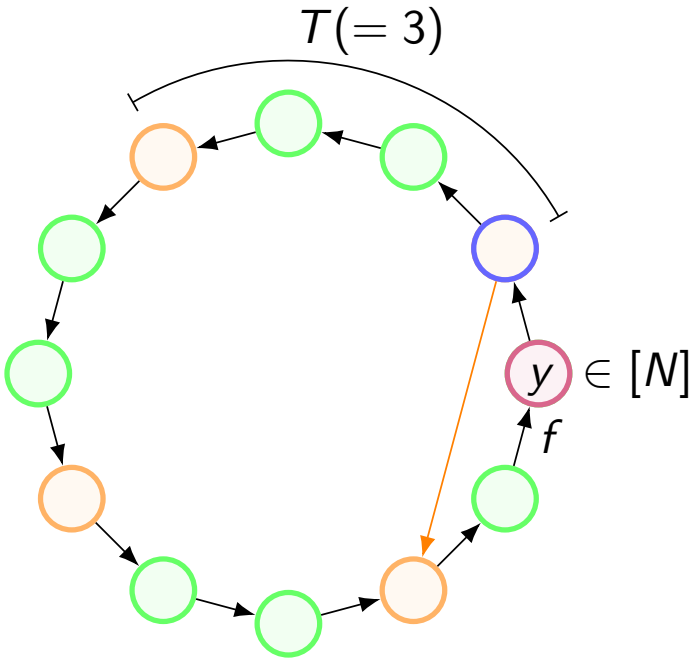

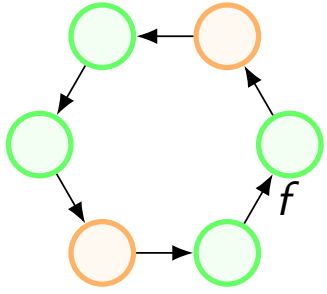

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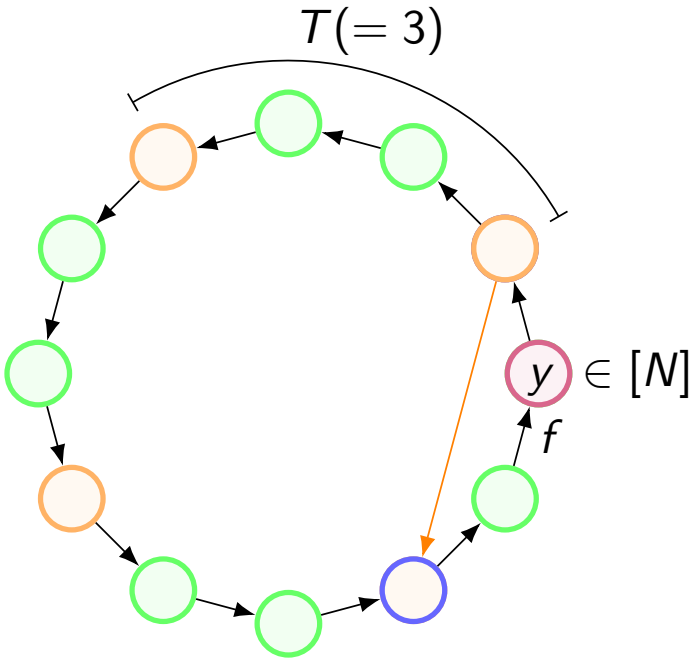

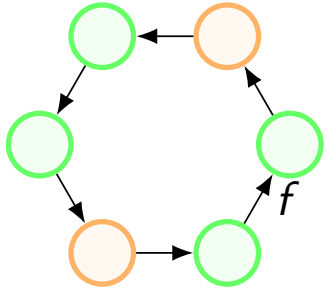

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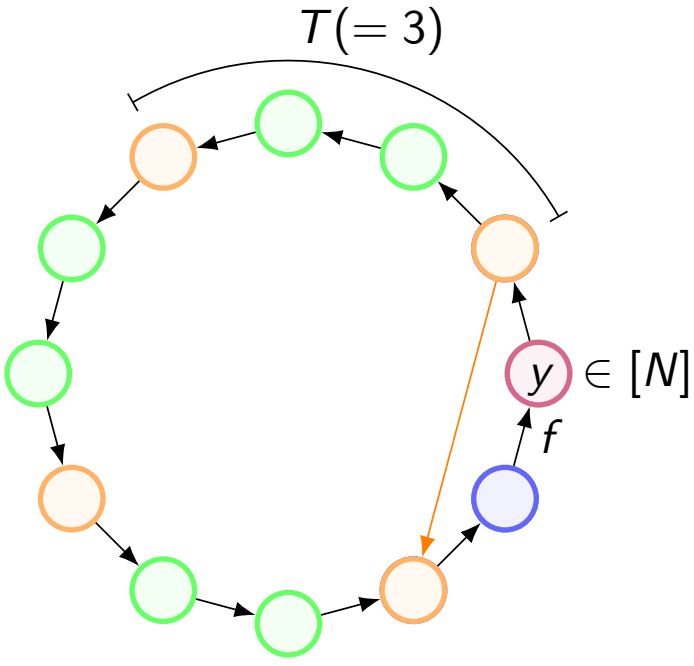

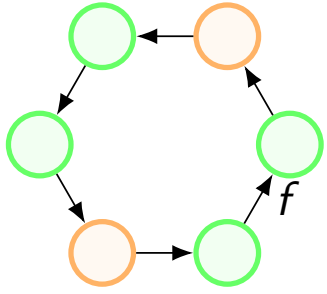

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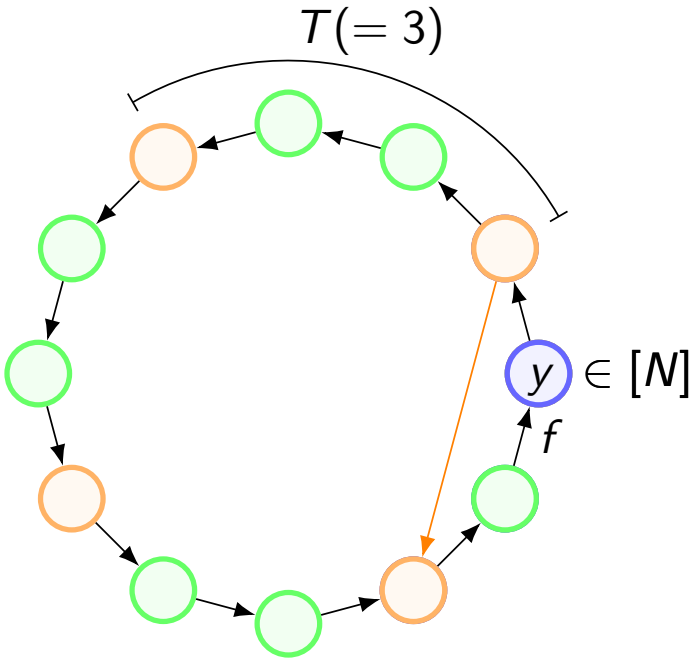

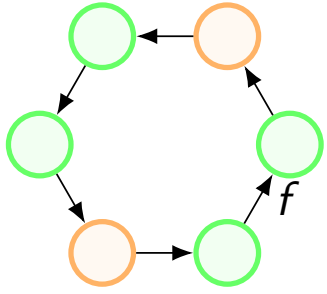

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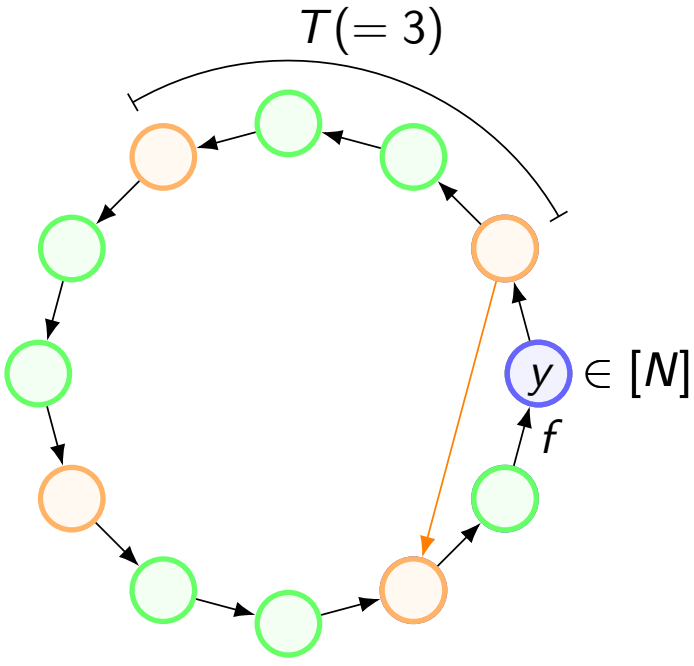

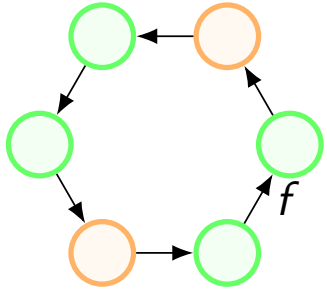

- We need T evaluations of f to invert y.
- We need to store about N/T points total.

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- Need $S = O((N \log N)/T)$.

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- Formally, for all $f : [N] \to [N]$ and $y \in f([N])$, the algorithms $(\mathcal{P}, \mathcal{A})$ should satisfy

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Scenarios

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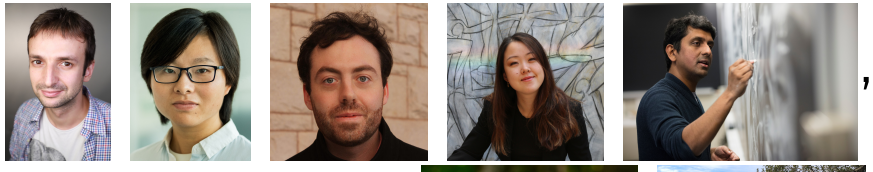
- Theoretical computer scientists want better algorithms for 3-SUM [GGH+20],  , multiparty pointer jumping [CK19],  , systematic substring search [CK19], ...

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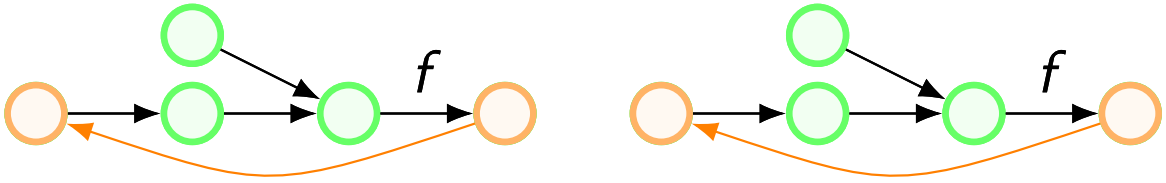

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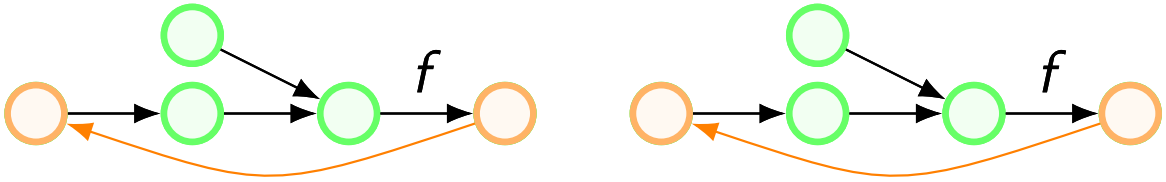

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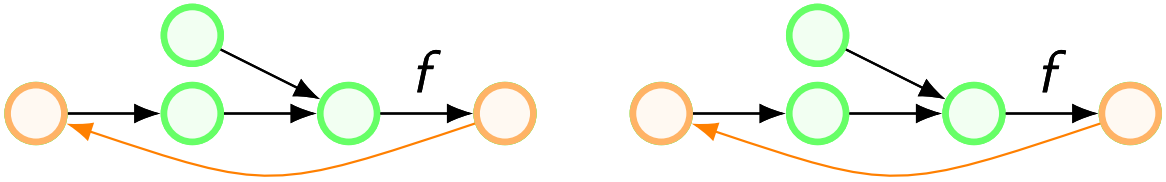

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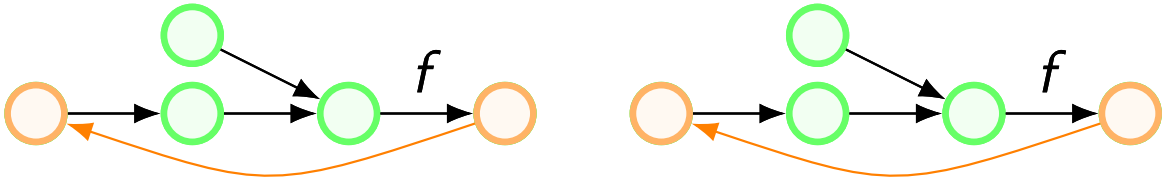

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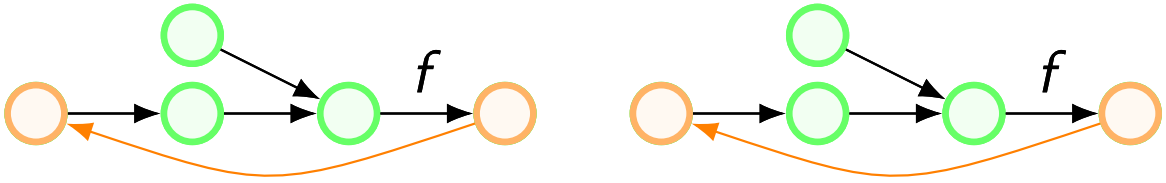

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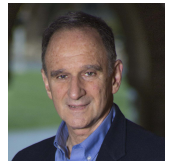

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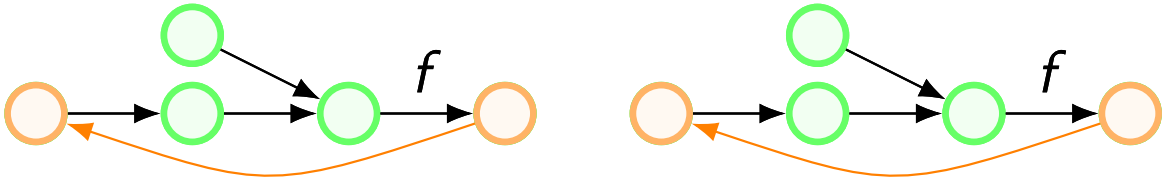

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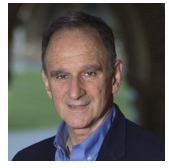

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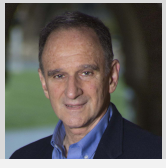
Hellman 	1980	random f	$T \leq \tilde{O}(N^2/S^2)$	$S = T \lesssim N^{2/3}$
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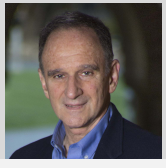

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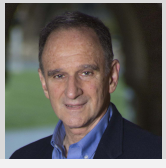

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- Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).
- We observe that the tradeoff $T \lesssim N^3/S^3$ comes from

$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$

- So we'd love for L to be longer, but it needs to fit in $\alpha \in \{0,1\}^S$.
- Or does it?

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- But I've cheated here...
- How do $\mathcal{A}$ and $\mathcal{P}$ agree on the same list of random values x_i?

Sharing Randomness

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- If the number $\Sigma = O(N)$ of fixed strings is large enough, this will be almost as good as fully random.
- This approach leans fairly heavily on non-uniformity. But in practice, can just instantiate a random oracle.

In Context

Result		Applies To	Tradeoff	Key Point
Hellman 	1980	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
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- Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- $\Longrightarrow$ preprocessing returns α maximizing

$$|\{y \in [N] : \alpha = \alpha_y\}|.$$

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- Balls and bins! With high probability, $\Omega(\log N / \log\log N)$ balls in the bin with most balls.
- $\implies$ $\max_\alpha |\{y \in [N] : \alpha_y = \alpha\}| \geq \Omega(\log N / \log\log N)$.

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- That is, it can be thought of returning T candidate inverses $x_1, \ldots, x_T$, *without querying f at all*.
- We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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  - Encoding is $(\alpha, i_1, \ldots, i_N)$.
  - For each y, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \ldots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

Function Inversion Revisited: Our Results

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  - The size of the codomain doesn't matter (us and [CK19])


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- $S^2 T = N^2$ algorithm for worst-case function inversion?

Thank you!

I'm happy to take additional questions offline. You can ping me at sp2473@cornell.edu.

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