



Revisiting Time-Space Tradeoffs for Function Inversion

Spencer Peters

Noah S.D.



Siyao Guo



Sasha Golovnev



Function Inversion

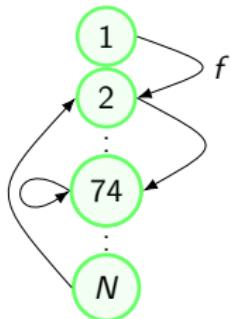
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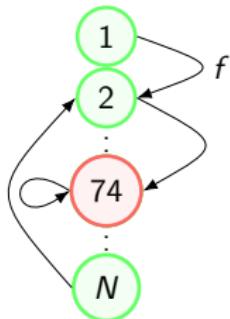
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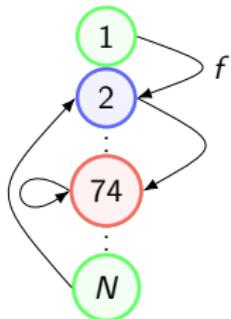
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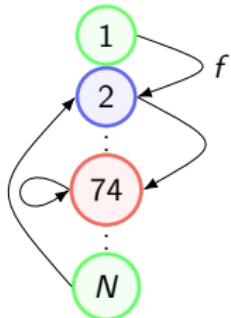
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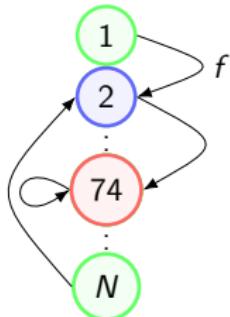
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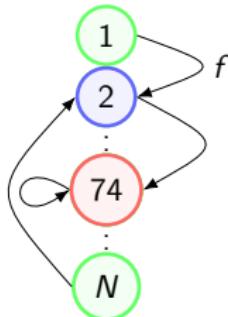
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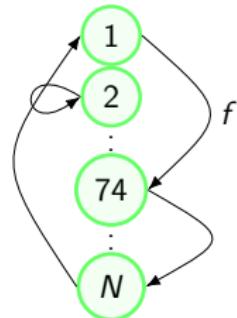
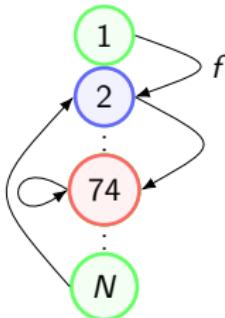
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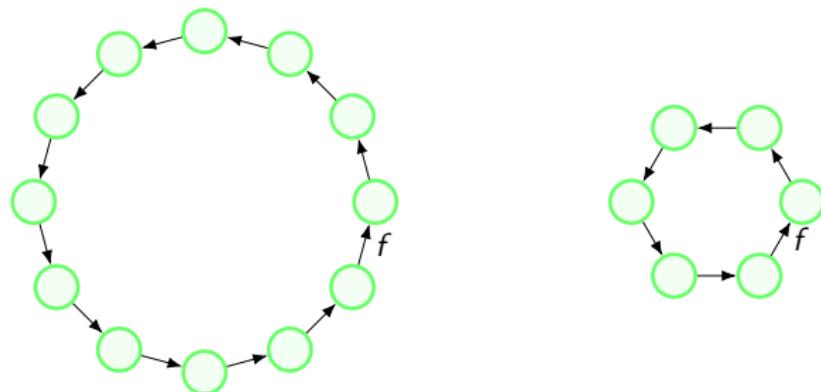
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- If f is a permutation, its *graph* is a disjoint union of cycles.

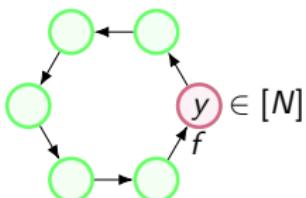
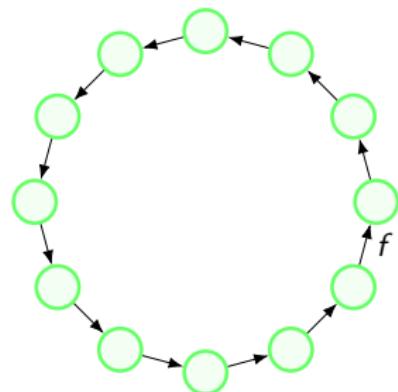
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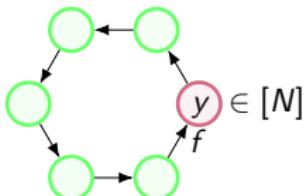
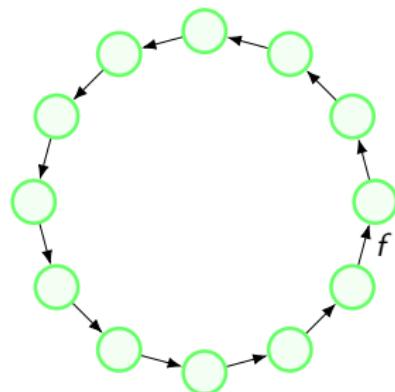
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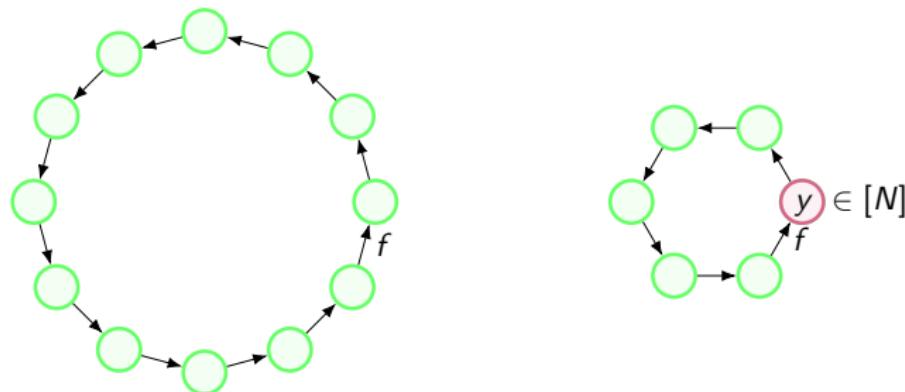
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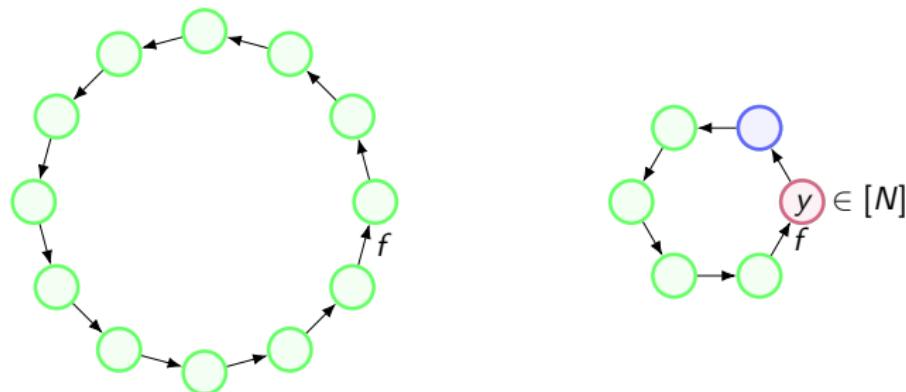
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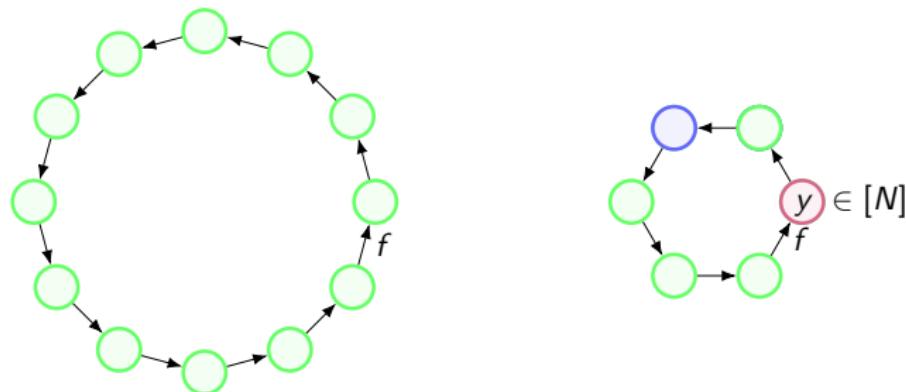
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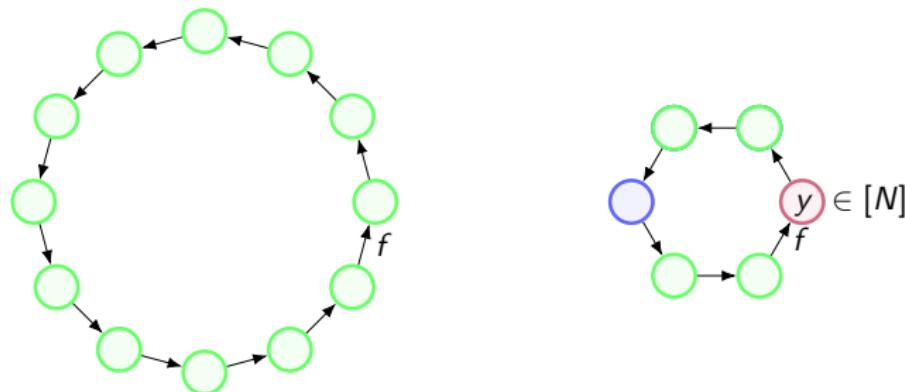
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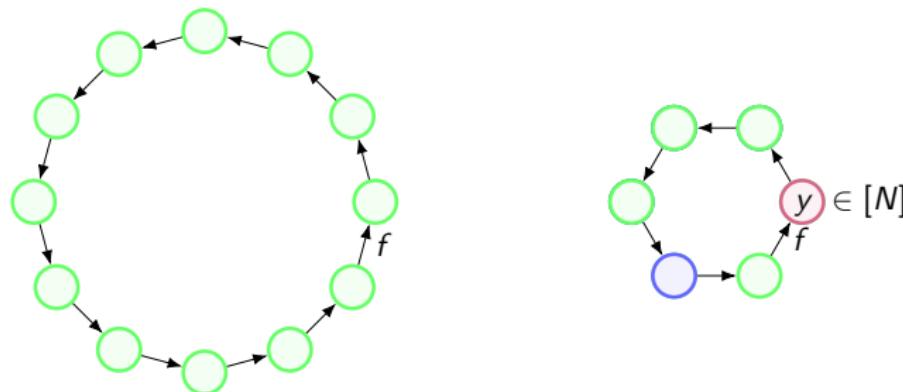
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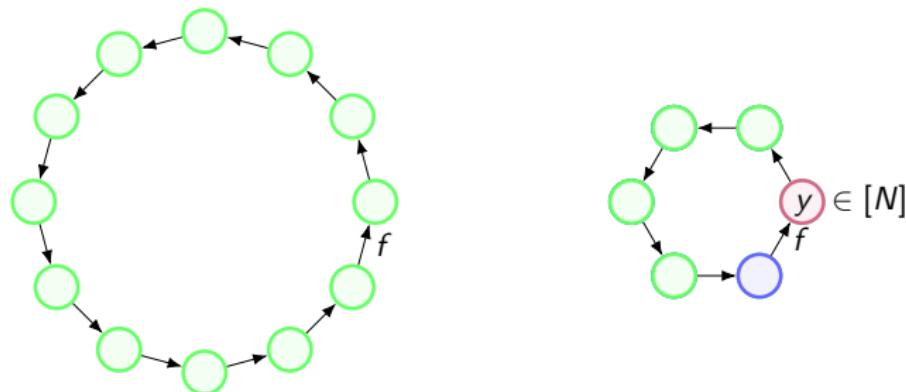
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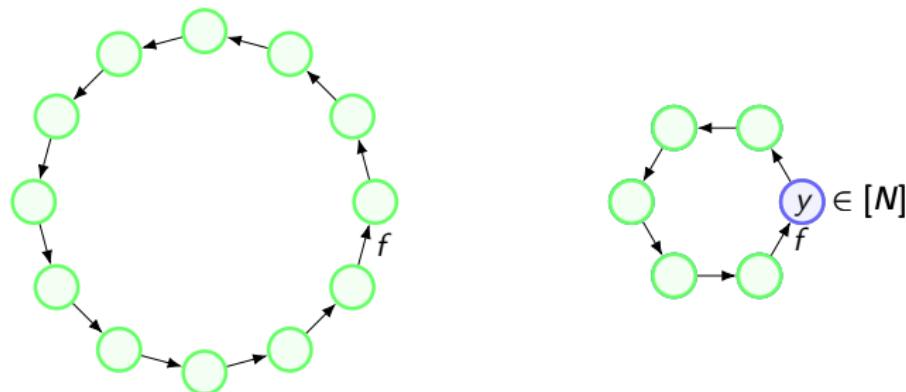
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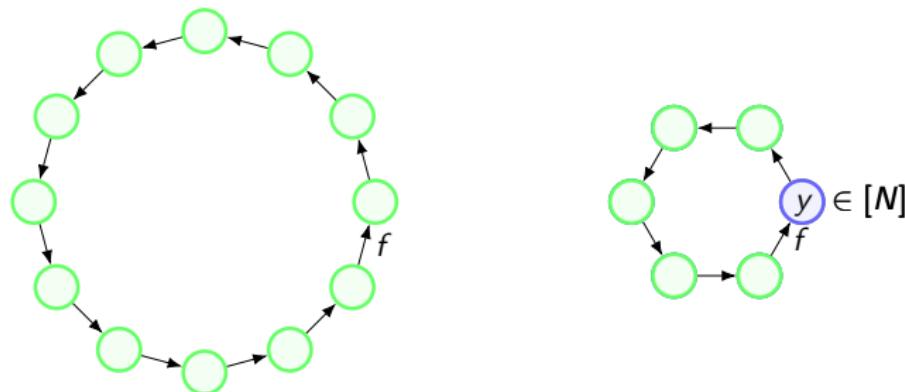
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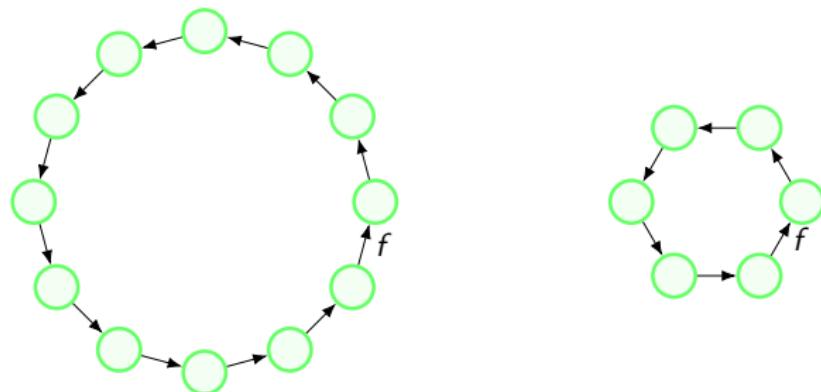
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- ▶ Compute $f(y)$, $f(f(y))$, and so on until you reach y again.
- ▶ Of course, y might not be on a small cycle...

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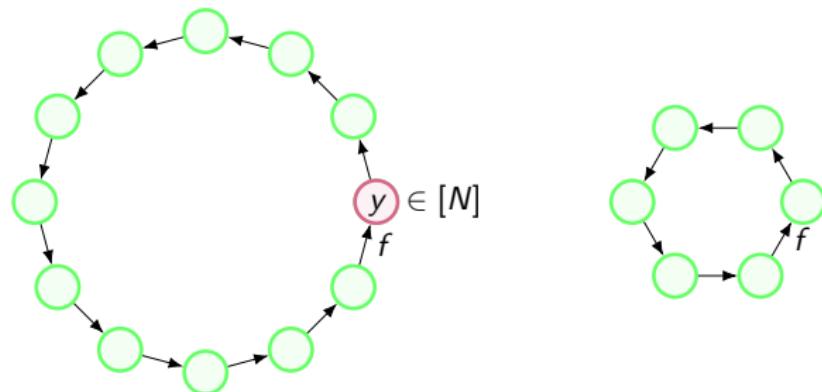
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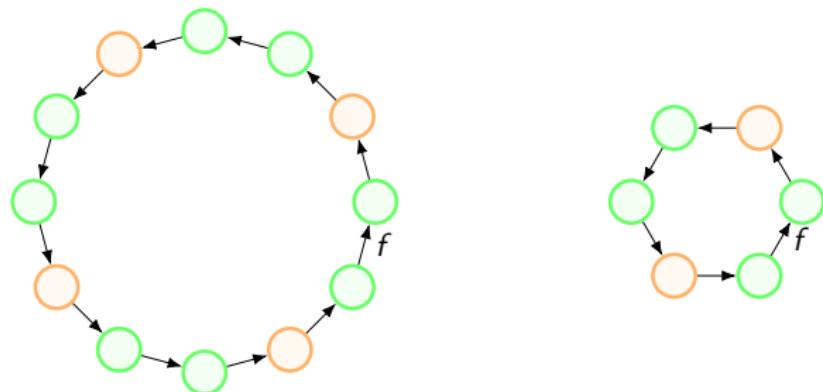
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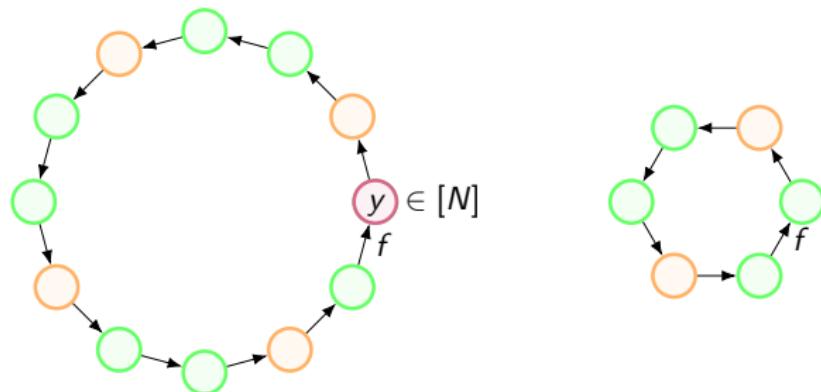
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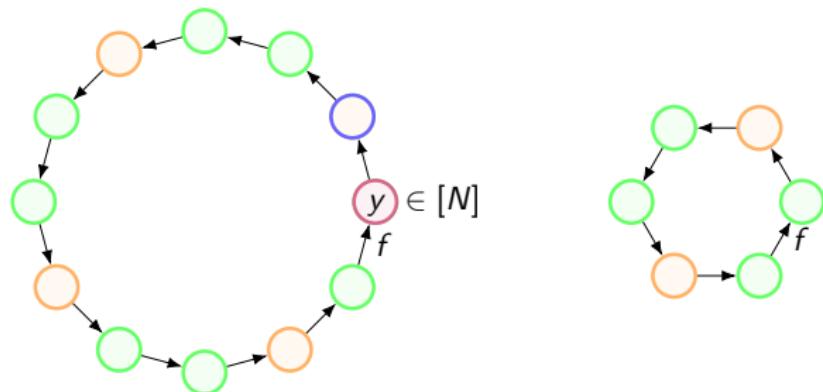
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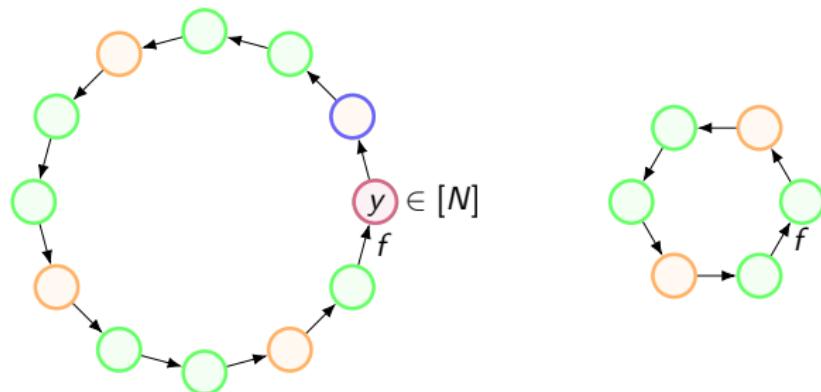
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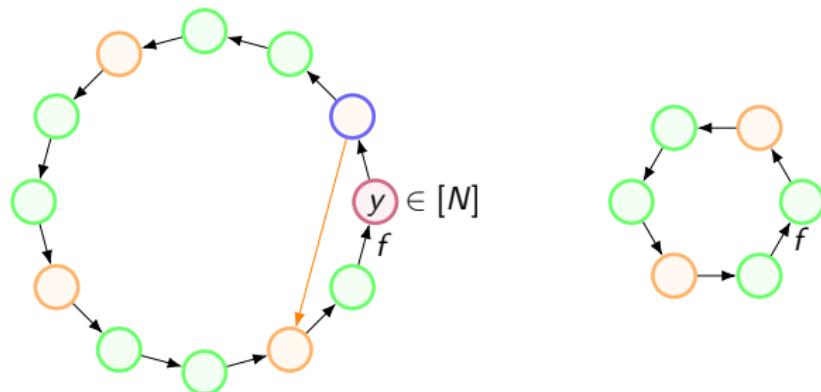
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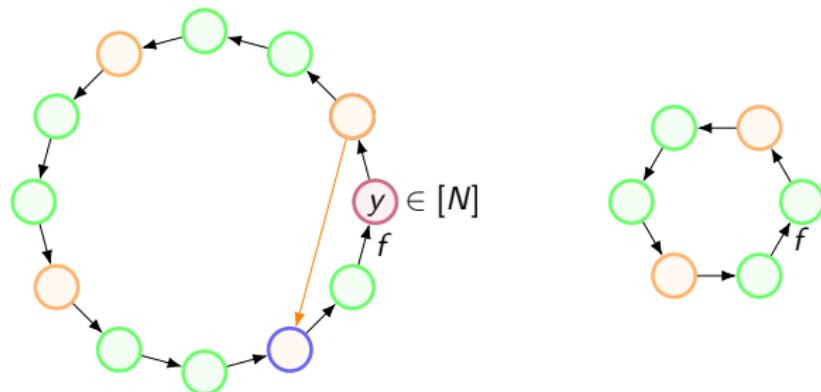
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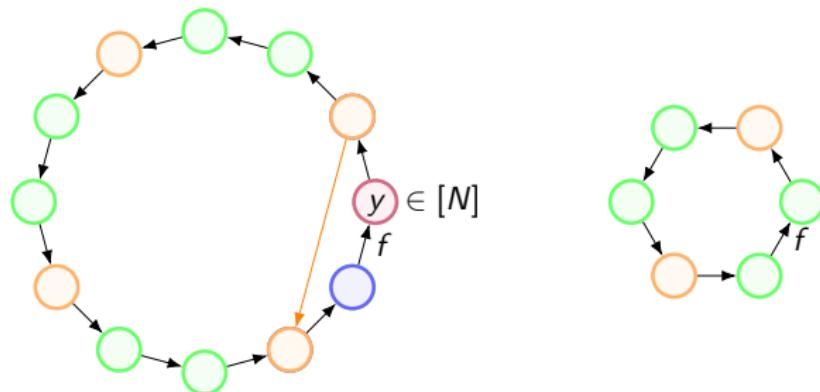
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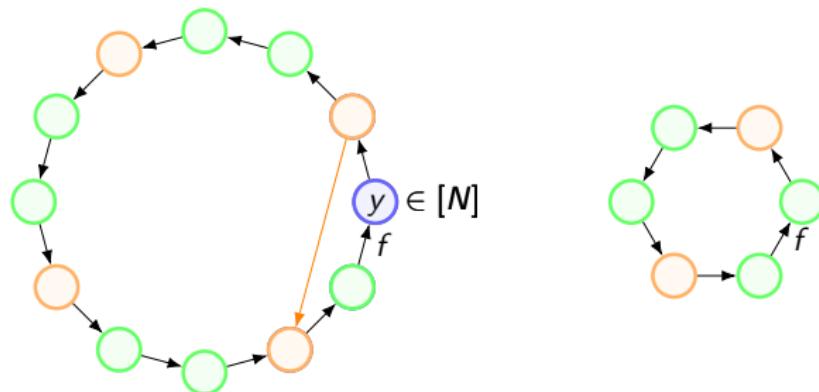
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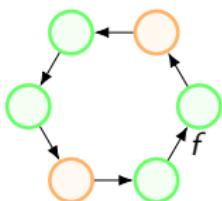
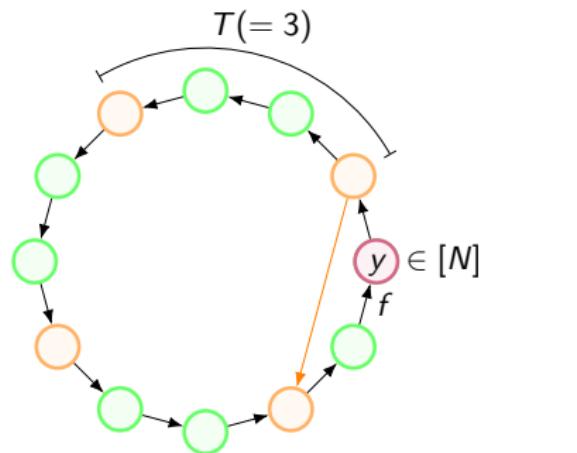
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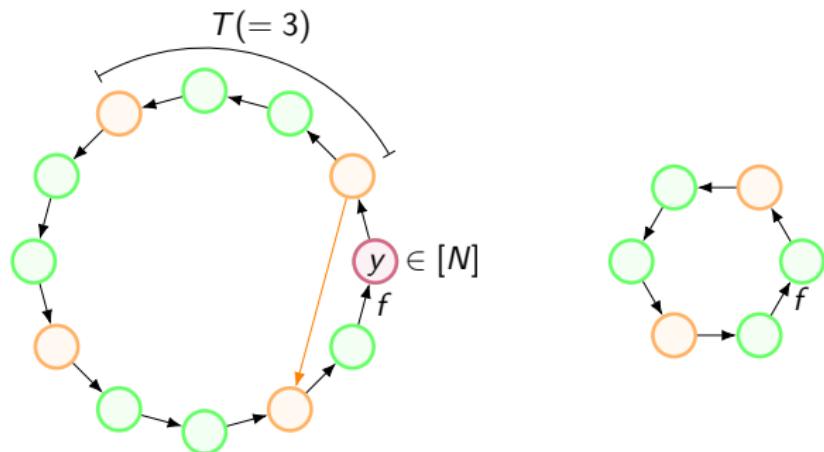
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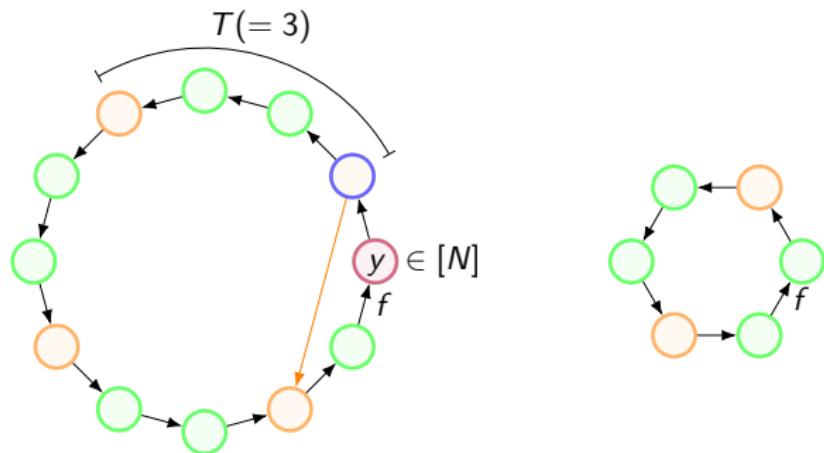
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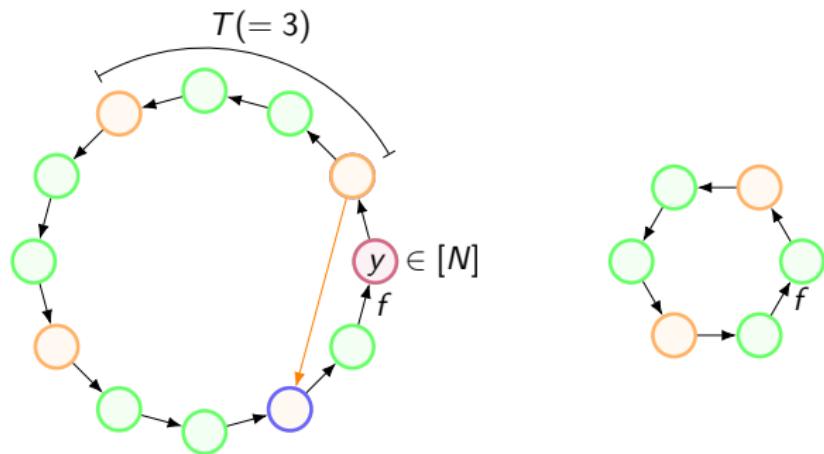
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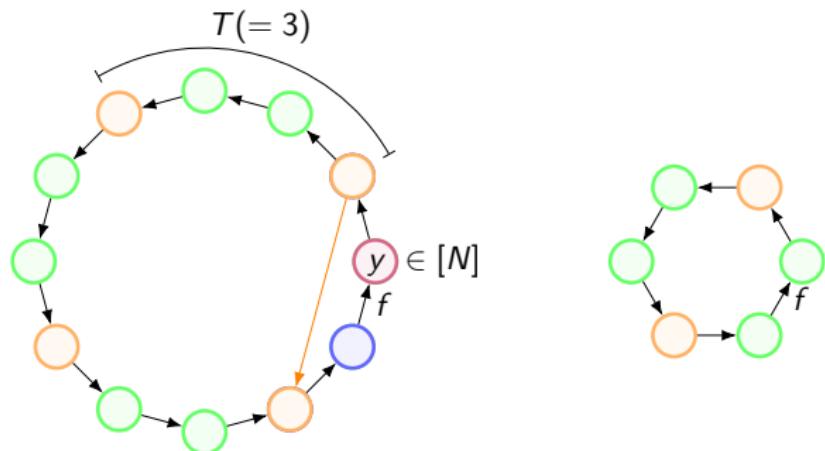
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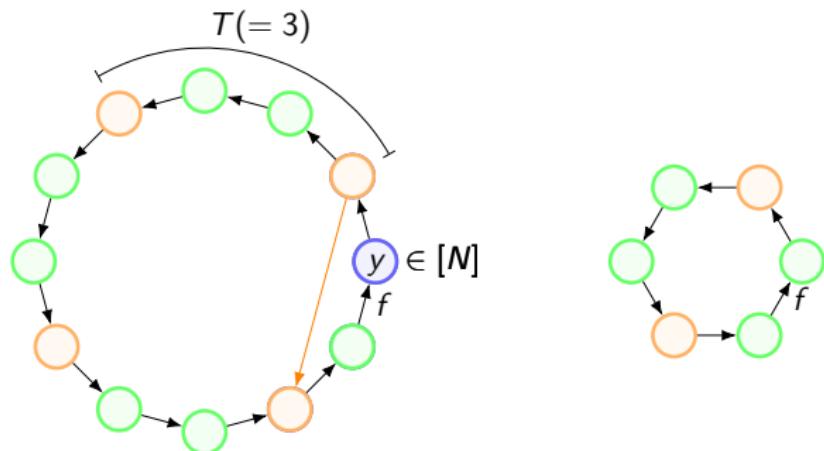
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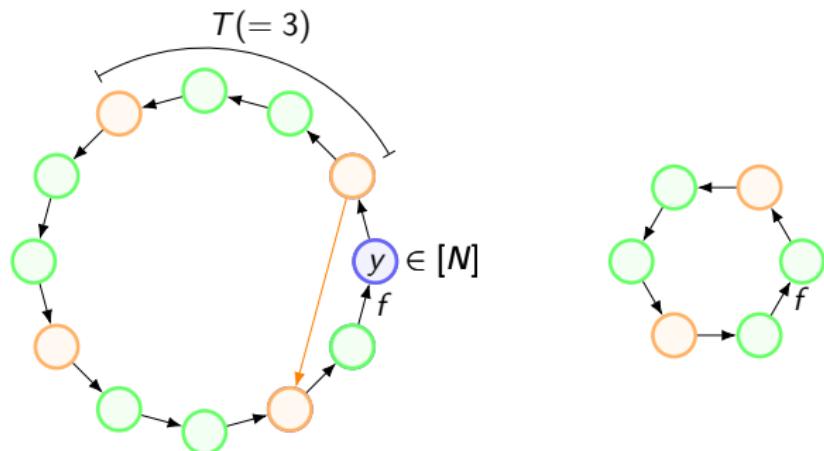
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- ▶ We need to store about N/T points total.

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- ▶ Need $S = O((N \log N)/T)$.

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- ▶ Formally, for all $f : [N] \rightarrow [N]$ and $y \in f([N])$, the algorithms $(\mathcal{P}, \mathcal{A})$ should satisfy

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- ▶ But, α has bitlength at most S , and \mathcal{A} can make at most T evaluations of f .

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- ▶ Hackers want to recover passwords from a stolen database of password hashes
- ▶ Theoretical computer scientists want better algorithms for 3-SUM [GGH⁺20],



,

multiparty pointer jumping [CK19],



systematic substring search [CK19], ...

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- ▶ So we'd love for L to be longer, but it needs to fit in $\alpha \in \{0, 1\}^S$.

Result #1: A Simple Improvement to Fiat-Naor

- ▶ Recall that for f arbitrary, Hellman's algorithm is tripped up by points with many inverses ("junction points").
- ▶ Fiat and Naor's idea is to (implicitly) work with a *restriction* of f that leaves out the junction points.
- ▶ Preprocessing conveys the restriction to online via a list L of junction points (along with their inverses).
- ▶ We observe that the tradeoff $T \lesssim N^3/S^3$ comes from

$$T \lesssim \frac{1}{|L|} \cdot \frac{N^3}{S^2}.$$

- ▶ So we'd love for L to be longer, but it needs to fit in $\alpha \in \{0, 1\}^S$.
- ▶ Or does it?

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- ▶ But I've cheated here...
- ▶ How do \mathcal{A} and \mathcal{P} agree on the same list of random values x_i ?

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- ▶ If the number $\Sigma = O(N)$ of fixed strings is large enough, this will be almost as good as fully random.
- ▶ This approach leans fairly heavily on non-uniformity. But in practice, can just instantiate a random oracle.



In Context

Result		Applies To	Tradeoff	Key Point
Hellman 	1980	permutations	$T \leq O((N \log N)/S)$	$S = T \lesssim \sqrt{N}$
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- ▶ Thus non-adaptive algorithms are not the barren, lifeless desert previously expected...

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- ▶ \implies preprocessing returns α maximizing

$$|\{y \in [N] : \alpha = \alpha_y\}|.$$

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- ▶ Balls and bins! With high probability, $\Omega(\log N / \log \log N)$ balls in the bin with most balls.
- ▶ $\implies \max_{\alpha} |\{y \in [N] : \alpha_y = \alpha\}| \geq \Omega(\log N / \log \log N)$.

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- ▶ That is, it can be thought of returning T candidate inverses x_1, \dots, x_T , *without querying f at all*.
- ▶ We show that all *guess-and-check* algorithms satisfy the matching lower bound $S = \Omega(N \log(N/T))$.

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 - ▶ Encoding is $(\alpha, i_1, \dots, i_N)$.
 - ▶ For each y , decoder again runs $\mathcal{A}(\alpha, y)$ and receives x_1, \dots, x_T . It sets $f^{-1}(y) = x_{i_y}$.

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- ▶ Improving Fiat-Naor: a

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 - ▶ The size of the codomain doesn't matter (us and [CK19])



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- ▶ $S^2 T = N^2$ algorithm for worst-case function inversion?

Thank you!

I'm happy to take additional questions offline. You can ping me at sp2473@cornell.edu.

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