Revisiting Time-Space Tradeoffs for Function Inversion

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Noah S.D.

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Function Inversion

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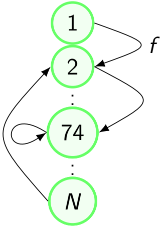

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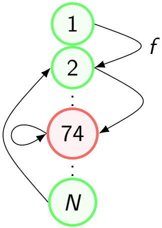

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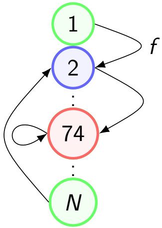

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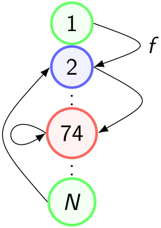

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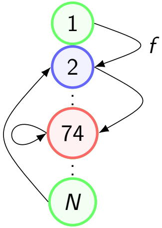

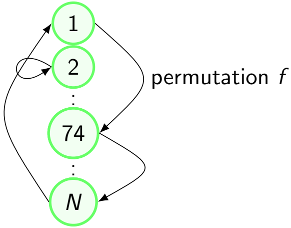

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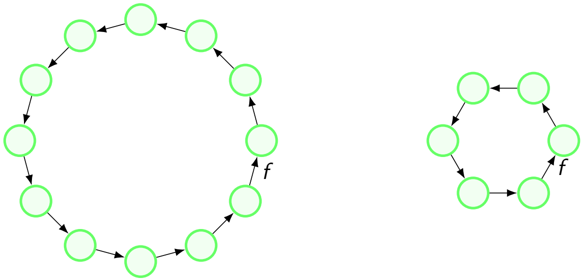

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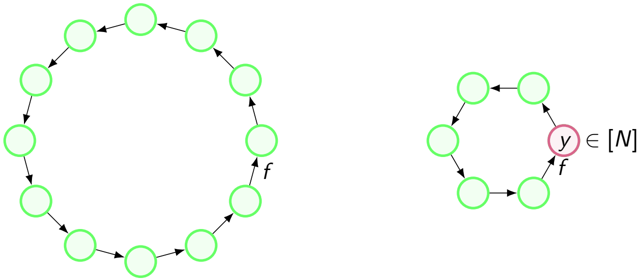

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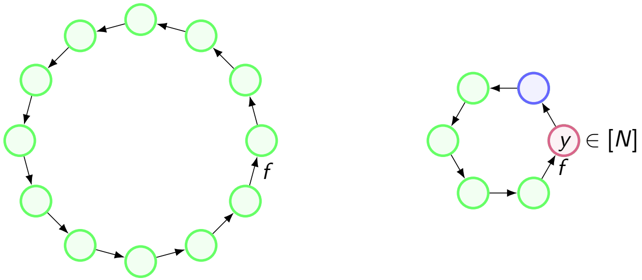

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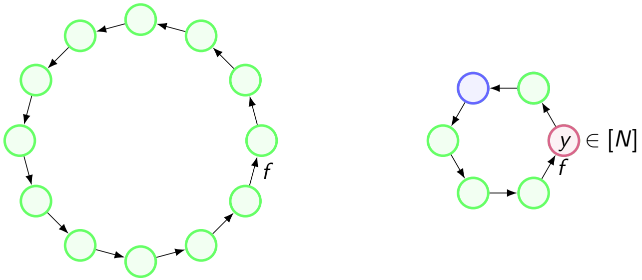

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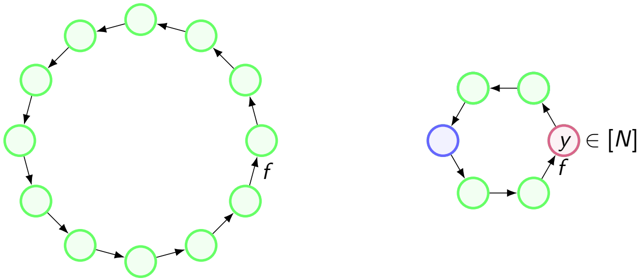

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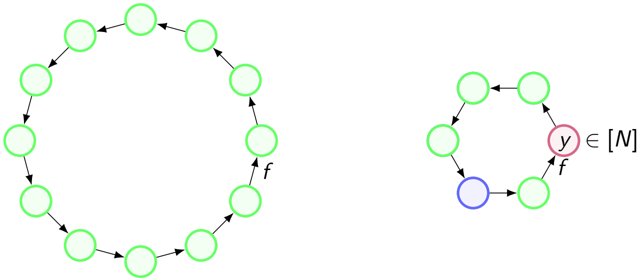

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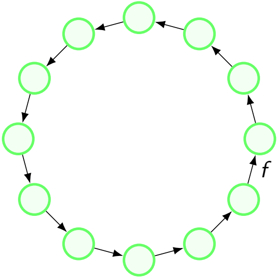

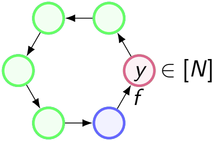

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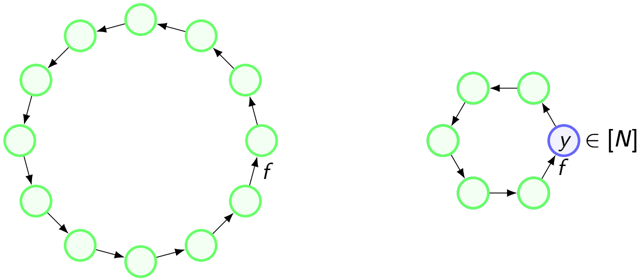

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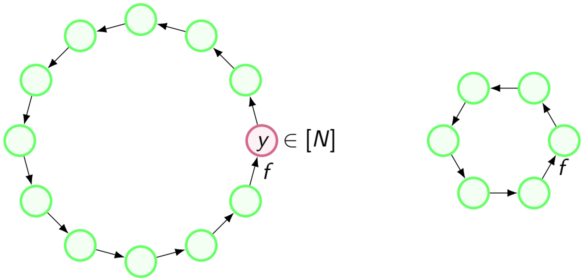

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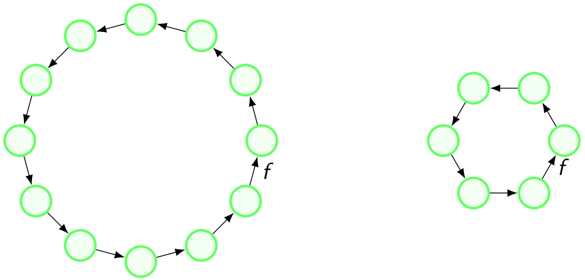

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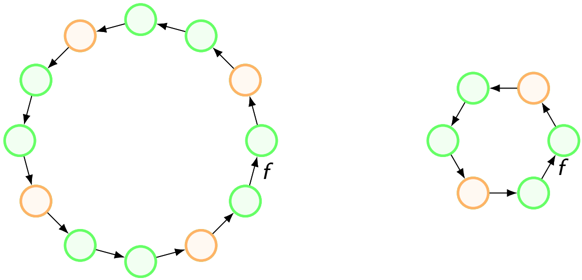

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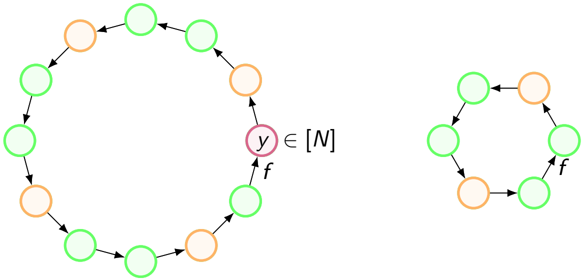

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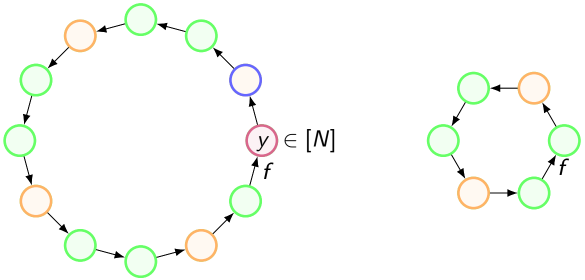

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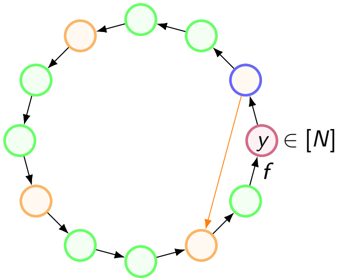

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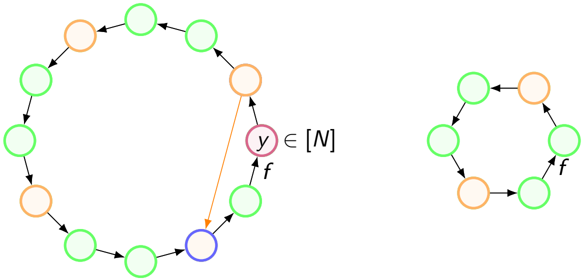

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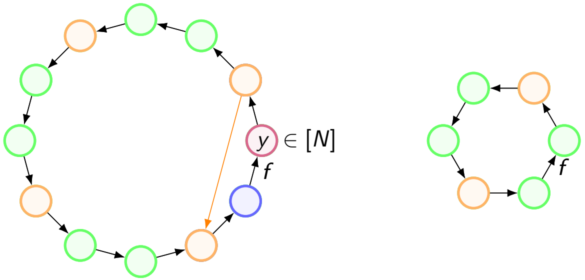

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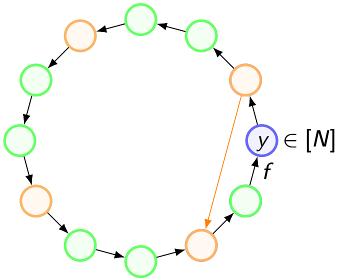

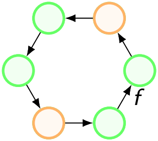

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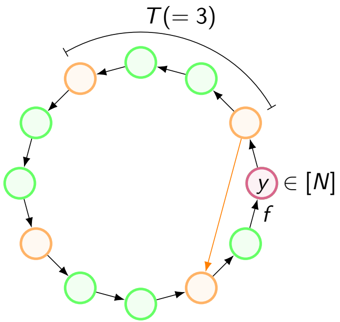

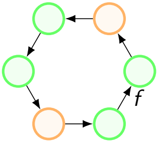

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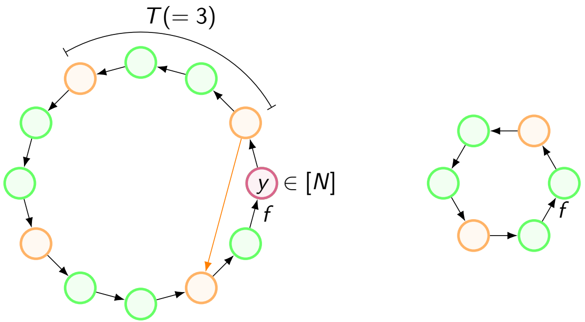

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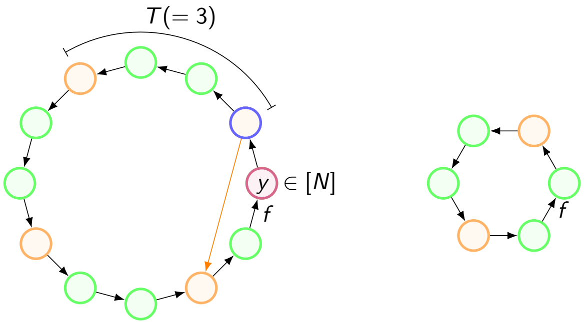

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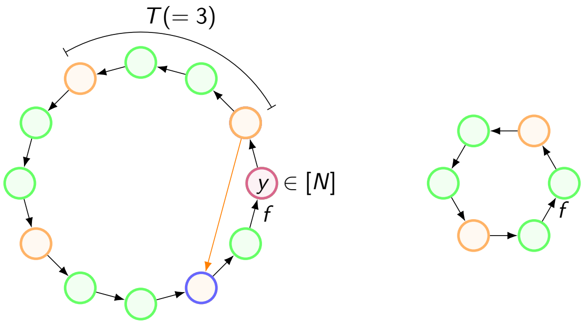

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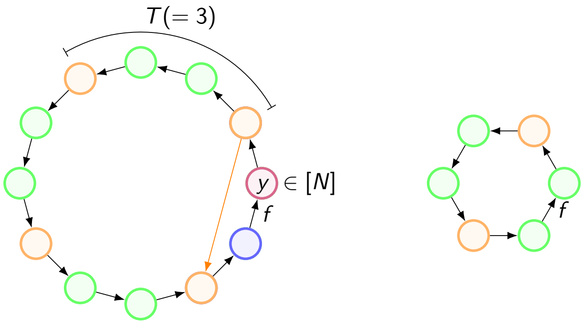

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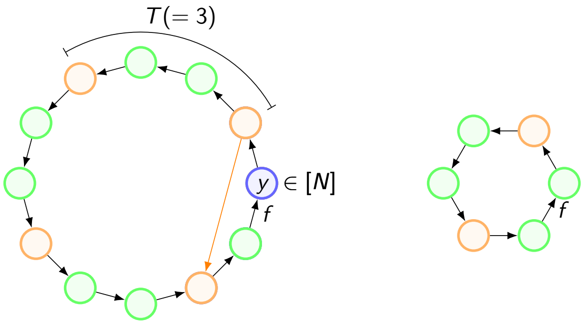

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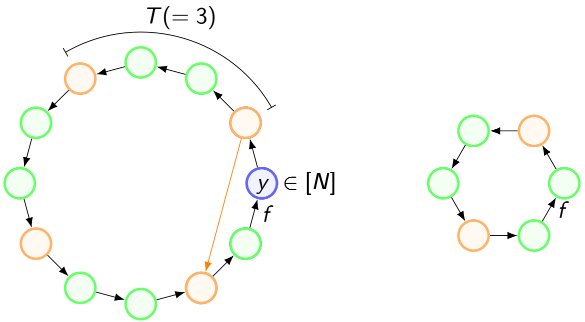

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- We need to store about N/T points total.

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- We aim to **minimize** the bitlength S of α, and the number of queries T that $\mathcal{A}$ makes to f.

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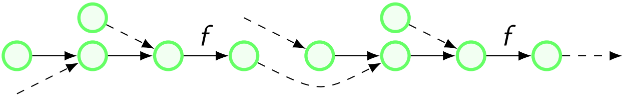

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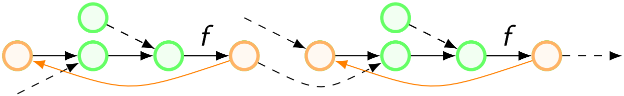

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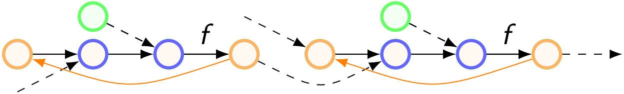

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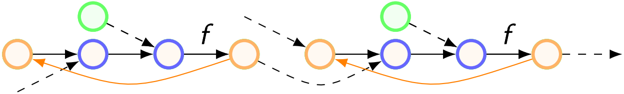

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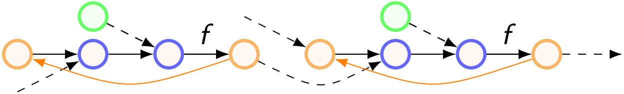

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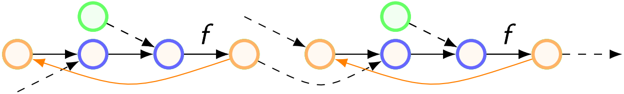

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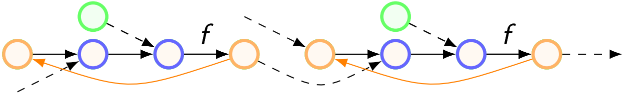

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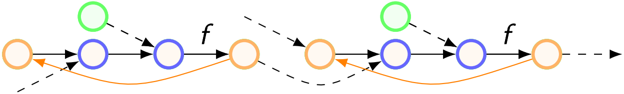

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- For *random* functions, Hellman showed (heuristically) this can be made to work. 

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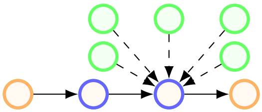

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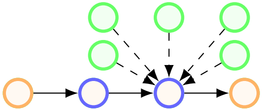

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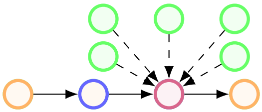

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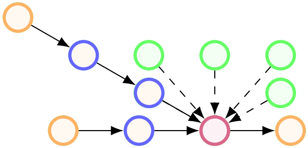

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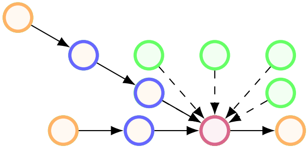

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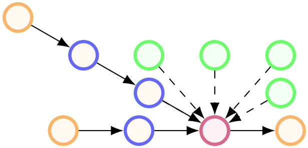

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- Intuitively, α' is the data structure for a *restriction* of f that avoids the junction points in L.
- More precisely, the "rerandomization" functions are sampled using rejection sampling so that their range is $[N] - L$.

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Fiat-Naor 1991  	all functions	$T \lesssim N^3/S^3$	$S = T \lesssim N^{3/4}$

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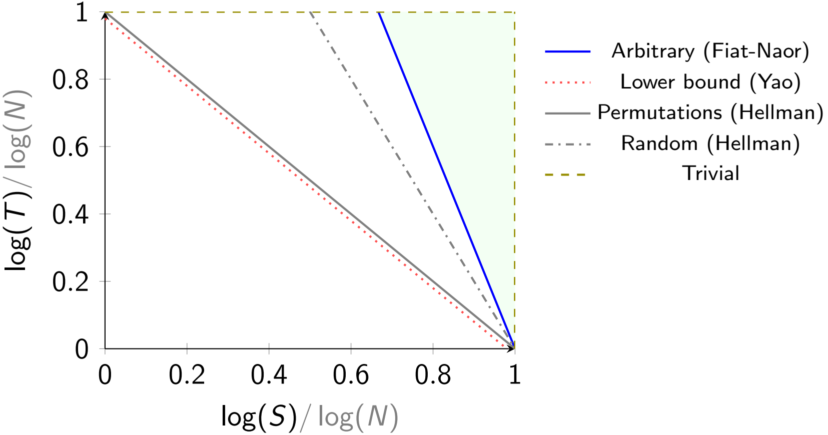

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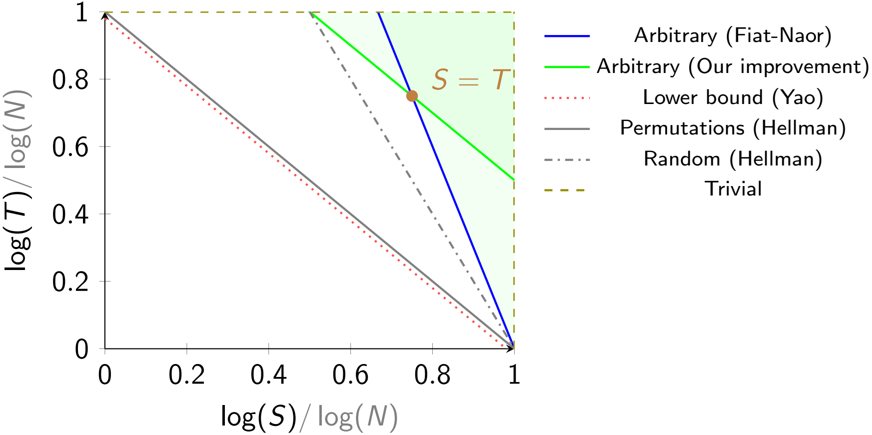

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- Fiat and Naor get $|L| \simeq S$, but this is the hard limit, since L needs to fit into S-bit advice α.
- Or does it?

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- How do $\mathcal{A}$ and $\mathcal{P}$ agree on the *same* list of random values x_i?

Sharing Randomness

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- In practice, can instantiate a random oracle.

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  - Encoding is $(\alpha, i_1, \ldots, i_N)$.
  - For each y, decoder again runs $\mathcal{A}(\alpha, y)$ and receives $x_1, \ldots, x_T$. It sets $f^{-1}(y) = x_{i_y}$.

Function Inversion Revisited: Our Results

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  - Better algorithms for injective functions?

Thank you!

I'm happy to take additional questions offline. You can ping me at `speters@cs.cornell.edu`.

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