Reasoning About Causal Models With Infinitely Many Variables

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Abstract

Generalized structural equations models (GSEMs) (Anonymous 2021), are, as the name suggests, a generalization of structural equations models (SEMs). They can deal with (among other things) infinitely many variables with infinite ranges, which is critical for capturing dynamical systems. We provide a sound and complete axiomatization of causal reasoning in GSEMs that is an extension of the sound and complete axiomatization provided by Halpern [2000] for SEMs. Considering GSEMs helps clarify what properties Halpern's axioms capture.

1 Introduction

Systems that evolve in continuous time are ubiquitous in all areas of science and engineering. A number of approaches have been used to model causality in such systems, ranging from dynamical systems involving differential equations to *rule-based models* (Laurent *et al.* 2018) for capturing complex interactions in molecular biology and *hybrid automata* (Alur *et al.* 1992) for describing mixed discrete-continuous systems. The standard approach to modeling causality, *structural-equations models* (SEMs), introduced by Pearl [2000], cannot handle such systems, since it allows only finitely many variables, which each have finite ranges. But continuous systems typically have real-valued variables indexed by time, which ranges over the reals (e.g., the temperature at time t).

An extension of SEMs, *generalized structural-equations models (GSEMs)*, which can capture such systems, was recently proposed (Anonymous 2021).[1] The goal of this paper is to provide a sound and complete axiomatization of GSEMs, in the spirit of that provided for SEMs by Halpern [2000]. There are a number of features of GSEMs that make reasoning about them subtle. We briefly discuss some of them here.

Like SEMs, GSEMs are defined with respect to a signature $\mathcal{S}$ that describes the variables in the model, their possible values, and the allowed interventions. The language $\mathcal{L}(\mathcal{S})$ of causal formulas that we consider (like that of Halpern [2000]) is parameterized by $\mathcal{S}$. A GSEM is a mapping that, given an intervention, produces a set of assignments to the variables given in $\mathcal{S}$, called *outcomes*; intuitively, they correspond to possible outcomes after the intervention is performed. If the signature $\mathcal{S}$ is finite (i.e., there are finitely many variables, each of which can take on only finitely many values), there can be only finitely many outcomes. This is, in particular, the case with SEMs. But in general, in a GSEM, there may be infinitely many outcomes. This complicates reasoning about them, as we shall see.

Another complication involves *allowed interventions*. In SEMs, all possible interventions are allowed; that is, we can intervene by setting any subset of the variables to any of the values in their ranges. In GSEMs, we have more expressive power: we can specify which interventions are allowed. The idea of limiting the set of interventions has already appeared in earlier work (Beckers and Halpern 2019; Rubenstein *et al.* 2017). Intuitively, allowed interventions are the ones that are feasible or meaningful. The set of allowed interventions is part of the signature; it also has an impact on the language. In the language $\mathcal{L}(\mathcal{S})$, we allow a formula of the form $[\vec{X} \leftarrow \vec{x}]\varphi$ (which can be read "after intervening by setting the variables in $\vec{X}$ to $\vec{x}$, φ holds") only if $\vec{X} \leftarrow \vec{x}$ is an allowed intervention: if an intervention is not allowed, we cannot talk about it in the language. As shown in (Anonymous 2021), restricting to allowed interventions is useful even when the signature is finite; we can describe interesting situations that are inconsistent with all interventions being allowed (see Example C.1).

Besides creating the possibility for infinitely many outcomes, the infinitary signatures required for continuous-time systems pose certain technical problems. If, for example, we have variables ranging over the reals, and we can refer to all possible real numbers in the language, then the language must be uncountable. Although we believe that all our results continue to hold for uncountable languages, having uncountably many formulas makes soundness and completeness arguments much more complicated. We thus restrict the language so that it can refer explicitly to only countably many values, and so that only countably many interventions are allowed. This still leaves us with an extremely rich language, which easily suffices to characterize systems that occur in practice.

It is shown in (Anonymous 2021, Theorem 2.1) that $\mathcal{L}(\mathcal{S})$

[1](Anonymous 2021) is currently under submission to another conference. We have included it in the supplementary material for the reviewer's benefit.

is rich enough to completely characterize SEMs, as well as GSEMs over infinitary signatures where there are only finitely many outcomes to each intervention (so, in particular, GSEMs with finite signatures); specifically, it is shown that if each of $\mathcal{M}$ and $\mathcal{M}'$ is either a SEM or a GSEM for which there are only finitely many outcomes to each intervention, then $\mathcal{M}$ and $\mathcal{M}'$ are $\mathcal{L}(\mathcal{S})$-*equivalent*, that is, they agree on all formulas in $\mathcal{L}(\mathcal{S})$, iff they are *equivalent*, that is, iff they have the same outcomes under all allowed interventions. (We remark that this is no longer the case if we consider GSEMs for which there may be infinitely many outcomes for a given context and intervention, something that can certainly be the case in some dynamical systems; see Example 4.3.)

Halpern [2000] provided axiom systems $AX^+(\mathcal{S})$ and $AX^+_{rec}(\mathcal{S})$ that he showed were sound and complete for general SEMs and acyclic SEMs, respectively. In this paper, we extend $AX^+(\mathcal{S})$ to arbitrary GSEMs, and several interesting subclasses of GSEMs (such as GSEMs with unique outcomes). First, we show that $AX^+(\mathcal{S})$ is sound and complete for the class of GSEMs satisfying $AX^+(\mathcal{S})$, if $\mathcal{S}$ is finite and $\mathcal{S}$ is *universal*; that is, if all interventions are allowed. This is an easy corollary of (Anonymous 2021, Theorem 3.4), which states that if $\mathcal{S}$ is finite and universal, then every SEM with signature $\mathcal{S}$ is equivalent to a GSEM satisfying $AX^+(\mathcal{S})$, and vice versa. The assumption that $\mathcal{S}$ is universal is critical here. Example 3.6 in (Anonymous 2021) gives a GSEM over a finite signature $\mathcal{S}$ that satisfies all the axioms of $AX^+(\mathcal{S})$ but is not equivalent to any SEM. This implies that $AX^+(\mathcal{S})$ is no longer complete for SEMs when $\mathcal{S}$ is not universal (Theorem 5.6).

We then show that a subsystem of $AX^+(\mathcal{S})$ that we call $AX^+_{basic}(\mathcal{S})$ is sound and complete for arbitrary GSEMs over a finite signature $\mathcal{S}$. We also show that as in SEMs, extending $AX^+(\mathcal{S})$ with one more axiom gives a sound and complete system for finite *acyclic* GSEMs; we conjecture a similar result holds for general acyclic GSEMs. Extending these results to arbitrary (possibly infinite) signatures $\mathcal{S}$ is nontrivial, because one of the axioms of $AX^+_{basic}(\mathcal{S})$ is no longer in the language $\mathcal{L}(\mathcal{S})$ when $\mathcal{S}$ is infinite. We show that this axiom can be replaced with a new inference rule that gives an equivalent system when $\mathcal{S}$ is finite. Moreover, the resulting axiom system, $AX^*_{basic}(\mathcal{S})$, is sound and complete for arbitrary GSEMs (Theorem 6.2). We further show that several properties of SEMs (such as having unique outcomes for all interventions in all contexts) can be recovered from arbitrary GSEMs by adding axioms from $AX^+(\mathcal{S})$ to $AX^*_{basic}(\mathcal{S})$ (Theorem 6.2). Doing so helps clarify what properties the added axioms capture.

2 SEMs: a review

Formally, a *structural-equations model* M is a pair $(\mathcal{S}, \mathcal{F})$, where $\mathcal{S}$ is a *signature*, which explicitly lists the endogenous and exogenous variables and characterizes their possible values, and $\mathcal{F}$ defines a set of *modifiable structural equations*, relating the values of the variables. We extend the signature to include a set of *allowed interventions*, as was done in earlier work (Beckers and Halpern 2019; Rubenstein *et al.* 2017). Intuitively, allowed interventions are the ones that are feasible or meaningful. A signature $\mathcal{S}$ is a tuple $(\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{I})$, where $\mathcal{U}$ is a set of exogenous variables, $\mathcal{V}$ is a set of endogenous variables, and $\mathcal{R}$ associates with every variable $Y \in \mathcal{U} \cup \mathcal{V}$ a nonempty, finite set $\mathcal{R}(Y)$ of possible values for Y (i.e., the set of values over which Y *ranges*). We assume (as is typical for SEMs) that $\mathcal{U}$ and $\mathcal{V}$ are finite sets, and adopt the convention that for $\vec{Y} \subseteq \mathcal{U} \cup \mathcal{V}$, $\mathcal{R}(\vec{Y})$ denotes the product of the ranges of the variables appearing in $\vec{Y}$; that is, $\mathcal{R}(\vec{Y}) := \times_{Y \in \vec{Y}} \mathcal{R}(Y)$. Finally, an intervention $I \in \mathcal{I}$ is a set of pairs (X, x), where $X \in \mathcal{V}$ and $x \in \mathcal{R}(X)$. We abbreviate an intervention I by $\vec{X} \leftarrow \vec{x}$, where $\vec{X} \subseteq \mathcal{V}$. Although this notation makes most sense if $\vec{X}$ is nonempty, we allow $\vec{X}$ to be empty (which amounts to not intervening at all).

$\mathcal{F}$ associates with each endogenous variable $X \in \mathcal{V}$ a function denoted F_X such that $F_X : \mathcal{R}(\mathcal{U} \cup \mathcal{V} - \{X\}) \rightarrow \mathcal{R}(X)$. This mathematical notation just makes precise the fact that F_X determines the value of X, given the values of all the other variables in $\mathcal{U} \cup \mathcal{V}$. If there is one exogenous variable U and three endogenous variables, X, Y, and Z, then F_X defines the values of X in terms of the values of Y, Z, and U. For example, we might have $F_X(u, y, z) = u + y$, which is usually written as $X = U + Y$. Thus, if $Y = 3$ and $U = 2$, then $X = 5$, regardless of how Z is set.

The structural equations define what happens in the presence of external interventions. Setting the value of some variable X to x in a SEM $M = (\mathcal{S}, \mathcal{F})$ results in a new SEM, denoted $M_{X \leftarrow x}$, which is identical to M, except that the equation for X in $\mathcal{F}$ is replaced by $X = x$. Interventions on subsets $\vec{X}$ of $\mathcal{V}$ are defined similarly. Notice that $M_{\vec{X} \leftarrow \vec{x}}$ is always well defined, even if $(\vec{X} \leftarrow \vec{x}) \notin \mathcal{I}$. In earlier work, the reason that the model included allowed interventions was that, for example, relationships between two models were required to hold only for allowed interventions (i.e., the interventions that were meaningful). As we shall see, here, the set of allowed interventions plays a different role, influencing the language (what we are allowed to talk about).

Given context $\mathbf{u} \in \mathcal{R}(\mathcal{U})$, the *outcomes* of a SEM M under intervention $\vec{X} \leftarrow \vec{x}$ are all assignments of values $\mathbf{v} \in \mathcal{R}(\mathcal{V})$ such that the assignments $\mathbf{u}$ and $\mathbf{v}$ together satisfy the structural equations of $M_{\vec{X} \leftarrow \vec{x}}$. This set of outcomes is denoted $M(\mathbf{u}, \vec{X} \leftarrow \vec{x})$. Given an outcome $\mathbf{v}$, we denote by $\mathbf{v}[X]$ and $\mathbf{v}[\vec{X}]$ the value that $\mathbf{v}$ assigns to X and the restriction of $\mathbf{v}$ to $\mathcal{R}(\vec{X})$ respectively. An important special case of SEMs are acyclic (or recursive) SEMs. Formally, an acyclic SEM is one for which, for every context $\mathbf{u} \in \mathcal{R}(U)$, there is some total ordering $\prec_{\mathbf{u}}$ of the endogenous variables (the ones in $\mathcal{V}$) such that if $X \prec_{\mathbf{u}} Y$, then X is independent of Y, that is, $F_X(\mathbf{u}, \ldots, y, \ldots) = F_X(\mathbf{u}, \ldots, y', \ldots)$ for all $y, y' \in \mathcal{R}(Y)$.

3 Axiomatizing SEMs

In order to talk about SEMs and the information they represent more precisely, we use the formal language $\mathcal{L}(\mathcal{S})$ for

SEMs having signature $\mathcal{S}$, introduced by Halpern [2000].

We restrict the language used by Halpern [2000] to formulas containing only allowed interventions. Fix a signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{I})$. A *primitive event* (over signature $\mathcal{S}$) has the form $X = x$, where $X \in \mathcal{V}$ and $x \in \mathcal{R}(\mathcal{V})$. An *event* is a Boolean combination of primitive events.

An *atomic formula* (over $\mathcal{S}$) has the form $[\vec{Y} \leftarrow \vec{y}]\varphi$, where $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}$ (i.e., it is an allowed intervention), and φ is an event. A *causal formula* (over $\mathcal{S}$) is a Boolean combination of atomic formulas. The language $\mathcal{L}(\mathcal{S})$ consists of all causal formulas over $\mathcal{S}$. There are a number of minor differences between the language considered here and that considered by Halpern [2000]. First, since Halpern implicitly assumed that all interventions were allowed, he did not have the restriction to allowed interventions. Second, Halpern considered a slightly richer language, where the context $\mathbf{u}$ was part of the formula, not on the left-hand side of the $\models$. Specifically, a primitive event had the form $X(\mathbf{u}) = x$. It has become standard not to include the context $\mathbf{u}$ in the formula (see, e.g., (Halpern and Pearl 2005; Halpern 2016)).

Next we define the semantics of $\mathcal{L}(\mathcal{S})$. An assignment $\mathbf{v} \in \mathcal{R}(\mathcal{V})$ *satisfies* the primitive event $X = x$, written $\mathbf{v} \models (X = x)$ if $\mathbf{v}[X] = x$. We extend this definition to Boolean combinations of primitive events by structural induction in the obvious way, that is, say that $\mathbf{v} \models e_1 \wedge e_2$ iff $\mathbf{v} \models e_1$ and $\mathbf{v} \models e_2$, and similarly for the other Boolean connectives $\vee$ and $\neg$. Fix a SEM M with signature $\mathcal{S}$. Given a context $\mathbf{u} \in \mathcal{R}(U)$, we say that M satisfies the atomic formula $[\vec{Y} \leftarrow \vec{y}]\varphi$ in context $\mathbf{u}$, written $(M, \mathbf{u}) \models [\vec{Y} \leftarrow \vec{y}]\varphi$, if all outcomes $\mathbf{v} \in M(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ satisfy φ. Finally, we extend this definition to causal formulas by structural induction as above. That is, $M, \mathbf{u} \models [\vec{Y} \leftarrow \vec{y}]\varphi \wedge [\vec{Z} \leftarrow \vec{z}]\psi$ iff $(M, \mathbf{u}) \models [\vec{Y} \leftarrow \vec{y}]\varphi$ and $(M, \mathbf{u}) \models [\vec{Z} \leftarrow \vec{z}]\psi$, and similarly for $\vee$ and $\neg$. As usual, $\langle \vec{Y} \leftarrow \vec{y}\rangle\varphi$ is taken to be an abbreviation for $\neg[\vec{Y} \leftarrow \vec{y}](\neg\varphi)$. It is easy to check that $(M, \mathbf{u}) \models \langle \vec{Y} \leftarrow \vec{y}\rangle\varphi$ iff φ is true of at least one outcome $\mathbf{v} \in M(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$. The causal formula ψ is *valid in* M, written $M \models \psi$, if $M, \mathbf{u} \models \psi$ for all $\mathbf{u} \in \mathcal{R}(\mathcal{U})$; ψ is *satisfied in* M if $(M, \mathbf{u}) \models \psi$ for some context $\mathbf{u} \in \mathcal{R}(\mathcal{U})$.

We now review Halpern's axiomatization of SEMs where all interventions are allowed (which is based on that of Galles and Pearl [1998]). To axiomatize acyclic SEMs, following Halpern, we define $Y \rightsquigarrow Z$, read "Y affects Z", as an abbreviation for the formula

$$\vee_{\vec{X} \subseteq \mathcal{V}, \vec{x} \in \mathcal{R}(\vec{X}), y \in \mathcal{R}(y), z \neq z' \in \mathcal{R}(Z)} ([\vec{X} \leftarrow \vec{x}](Z = z) \wedge [\vec{X} \leftarrow \vec{x}, Y \leftarrow y](Z = z'));$$

that is, Y affects Z if there is some setting of some endogenous variables $\vec{X}$ for which changing the value of Y changes the value of Z. This definition is used in axiom D6 below, which characterizes acyclicity.

Consider the following axioms:

D0. All instances of propositional tautologies.

D1. $[\vec{Y} \leftarrow \vec{y}](X = x \Rightarrow X \neq x')$ if $x, x' \in \mathcal{R}(X)$, $x \neq x'$ (functionality)

D2. $[\vec{Y} \leftarrow \vec{y}](\bigvee_{x \in \mathcal{R}(X)} X = x)$ (definiteness)

D3. $\langle \vec{X} \leftarrow \vec{x}\rangle(W = w \wedge \varphi) \Rightarrow \langle \vec{X} \leftarrow \vec{x}; W \leftarrow w\rangle(\varphi)$ (composition)

D4. $[\vec{X} \leftarrow \vec{x}](\vec{X} = \vec{x})$ (effectiveness)

D5. $(\langle \vec{X} \leftarrow \vec{x}; Y \leftarrow y\rangle(W = w \wedge \vec{Z} = \vec{z}) \wedge \langle \vec{X} \leftarrow \vec{x}; W \leftarrow w\rangle(Y = y \wedge \vec{Z} = \vec{z}))$
$\Rightarrow \langle \vec{X} \leftarrow \vec{x}\rangle(W = w \wedge Y = y \wedge \vec{Z} = \vec{z})$, if $\vec{Z} = \mathcal{V} - (\vec{X} \cup \{W, Y\})$ (reversibility)

D6. $(X_0 \rightsquigarrow X_1 \wedge \ldots \wedge X_{k-1} \rightsquigarrow X_k) \Rightarrow \neg(X_k \rightsquigarrow X_0)$ (recursiveness)

D7. $([\vec{X} \leftarrow \vec{x}]\varphi \wedge [\vec{X} \leftarrow \vec{x}](\varphi \Rightarrow \psi)) \Rightarrow [\vec{X} \leftarrow \vec{x}]\psi$ (distribution)

D8. $[\vec{X} \leftarrow \vec{x}]\varphi$ if φ is a propositional tautology (generalization)

D9. $\langle \vec{Y} \leftarrow \vec{y}\rangle true \wedge (\langle \vec{Y} \leftarrow \vec{y}\rangle\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi)$ if $\vec{Y} = \mathcal{V} - \{X\}$ (unique outcomes for $\mathcal{V} - \{X\}$)

D10(a). $\langle \vec{Y} \leftarrow \vec{y}\rangle true$ (at least one outcome)

D10(b). $\langle \vec{Y} \leftarrow \vec{y}\rangle\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi$ (at most one outcome)

MP. From φ and $\varphi \Rightarrow \psi$, infer ψ (modus ponens)

Let AX^+ consist of axiom schema D0-D5 and D7-D9, and inference rule MP; let AX^+_{rec} be the result of adding D6 and D10 to AX^+.

These are not quite the same axioms that Halpern [2000] used, although they are equivalent for SEMs. In more detail:

- Instead of an arbitrary formula φ in D3, Halpern had just formulas of the form $\vec{Y} = \vec{y}$. But since in the case of SEMs, every propositional formula φ is equivalent to a disjunction of formulas of the form $\vec{Y} = \vec{y}$, and $\langle \vec{X} \leftarrow \vec{x}\rangle(\varphi \vee \psi) \Rightarrow \langle \vec{X} \leftarrow \vec{x}\rangle\varphi \vee \langle \vec{X} \leftarrow \vec{x}\rangle\psi$ is provable from the axioms (cf. Lemma A.2), our version of D3 is easily seen to be equivalent to the original version for SEMs, but is stronger in the case of GSEMs.
- D5 follows from D2, D3, D6, D7, D8, D10, and MP, so Halpern did not include it in his axiomatization. (This was already essentially observed by Galles and Pearl [1998].) Indeed, as we show below (Proposition B.2), in the presence of these other axioms, D5 holds even without the requirement that $\vec{Z} = \mathcal{V} - \{\vec{X}, \vec{Y}\}$.
- Halpern's version of D4 said $[\mathbf{W} \leftarrow \mathbf{w}; X \leftarrow x](X = x)$. Using D0, D7, and D8 (and some standard modal logic reasoning), it is easy to see that the two versions are equivalent.
- Halpern had slightly different versions of D9 and D10. Specifically, the second conjunct is Halpern's version of D9 is $\vee_{x \in \mathcal{R}(X)}[\vec{Y} \leftarrow \vec{y}](X = x)$. For finite signatures, our version of D9 is equivalent to Halpern's in the presence of the other axioms, as we prove in the supplementary material (see Theorem B.1).
- Finally, Halpern also had an additional axiom D11; we discuss this below.

As mentioned before, the language $\mathcal{L}(\mathcal{S})$ considered here differs from the language considered by Halpern [2000], which we denote $\mathcal{L}_H(\mathcal{S})$, in two ways. First, Halpern implicitly assumed that all interventions were allowed, so he did not have the restriction to allowed interventions. That is, all formulas of the form $[\vec{Y} \leftarrow \vec{y}]\varphi$ were included in $\mathcal{L}_H(\mathcal{S})$, where $\vec{Y} \subseteq \mathcal{V}$ and $\vec{y} \subseteq \mathcal{R}(\vec{Y})$. Second, the causal formulas in $\mathcal{L}_H(\mathcal{S})$ were built from atomic events of the form $X(\mathbf{u}) = x$ as opposed to the form $X = x$. Halpern [2000] gave semantics to formulas with respect to models M, not with respect to pairs $(M, \mathbf{u})$. The semantics of atomic formulas in $\mathcal{L}_H(\mathcal{S})$ is given by $M \models [\vec{Y} \leftarrow \vec{y}](X(\mathbf{u}) = x)$ if all outcomes $\mathbf{v} \in M(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ satisfy $X = x$. It is easy to see that $M \models [\vec{Y} \leftarrow \vec{y}](X(\mathbf{u}) = x)$ iff, in the semantics we are using for this paper, $(M, \mathbf{u}) \models [\vec{Y} \leftarrow \vec{y}](X = x)$. To deal with the richer language, Halpern [2000] had an additional axiom:

D11. $\langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi_1(\mathbf{u}_1) \wedge \ldots \wedge \varphi_k(\mathbf{u}_k)) \Leftrightarrow (\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_1(\mathbf{u}_1) \wedge \ldots \wedge \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_k(\mathbf{u}_k)$, if $\varphi_i(\mathbf{u}_i)$ is a Boolean combination of formulas of the form $X(\mathbf{u}_i) = x$ and $\mathbf{u}_i \neq \mathbf{u}_j$ for $i \neq j$.

D11 is used in Halpern's completeness proof only to reduce consideration from formulas that mention multiple contexts to formulas that mention only one context, which are easily seen to be equivalent to formulas in $\mathcal{L}(\mathcal{S})$. We can show that the axioms without D11 are sound and complete for the language $\mathcal{L}(\mathcal{S})$ using exactly the same proof as used by Halpern to show that the axioms with D11 are sound and complete for $\mathcal{L}_H(\mathcal{S})$, just skipping the step that uses D11 to reduce to formulas involving just one context. This is formalized in the following theorem, where a signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{I})$ is *universal* if $\mathcal{I} = \mathcal{I}_{univ}$, the set of all interventions.

Theorem 3.1: (Halpern 2000) *If $\mathcal{S}$ is a universal signature, then AX^+ (resp., AX^+_{rec}) is a sound and complete axiomatization for the language $\mathcal{L}(\mathcal{S})$ for SEMs (resp., acyclic SEMs) with a universal signature $\mathcal{S}$.*

As we shall see (Theorem 5.6), the assumption that $\mathcal{S}$ is universal is critical here; Theorem 3.1 is not true in general without it.

4 GSEMs

In this section, we briefly review the definition of GSEMs (Anonymous 2021); we encourage the reader to consult (Anonymous 2021) for more details and intuition. We also prove some results regarding the extent to which the language $\mathcal{L}(\mathcal{S})$ characterizes GSEMs, and introduce the class of *acyclic* GSEMs.

The main purpose of causal modeling is to reason about a system's behavior under intervention. A SEM can be viewed as a function that takes a context $\mathbf{u}$ and an intervention $\vec{Y} \leftarrow \vec{y}$ and returns a set of assignments to the endogenous variables (i.e., a set of outcomes), namely, the set of all solutions to the structural equations after replacing the equations for the variables in $\vec{Y}$ with $\vec{Y} = \vec{y}$. Viewed in this way, generalized structural-equations models (GSEMs) are a generalization of SEMs. In a GSEM, there is a function that takes a context $\mathbf{u}$ and an intervention $\vec{Y} \leftarrow \vec{y}$ and returns a set of outcomes. However, the outcomes need not be obtained by solving a set of suitably modified equations as they are for SEMs. This relaxation gives GSEMs the ability to concisely represent dynamical systems and other systems with infinitely many variables, and the flexibility to handle situations involving finitely many variables that cannot be modeled by SEMs.

Because GSEMs don't have the structure that SEMs have by virtue of being defined in terms of structural equations, we may want to rule out certain unintuitive possibilities. In particular, we require that after intervening to set $\vec{X} \leftarrow \vec{x}$, all outcomes satisfy $\vec{X} = \vec{x}$. This is in fact the axiom D4 (effectiveness) given above.

Formally, a *generalized structural-equations model (GSEM)* M is a pair $(\mathcal{S}, \mathbf{F})$, where $\mathcal{S}$ is a signature, and $\mathbf{F}$ is a mapping from contexts and interventions to sets of outcomes. As before, a signature $\mathcal{S}$ is a quadruple $(\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{I})$, except that we no longer require $\mathcal{U}$ and $\mathcal{V}$ to be finite, nor $\mathcal{R}(Y)$ to be finite for all $Y \in \mathcal{U} \cup \mathcal{V}$. The big difference is that $\mathbf{F}$ is a function $\mathbf{F} : \mathcal{I} \times \mathcal{R}(\mathcal{U}) \rightarrow \mathcal{P}(\mathcal{R}(\mathcal{V}))$, where $\mathcal{P}$ denotes the powerset operation. That is, it maps a context $\mathbf{u} \in \mathcal{R}(U)$ and an allowed intervention $I \in \mathcal{I}$ to a set of *outcomes* $\mathbf{F}(\mathbf{u}, I) \subseteq \mathcal{P}(\mathcal{R}(\mathcal{V}))$. As with SEMs, we denote these outcomes by $M(\mathbf{u}, I)$. As we said above, we require that each outcome $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x})$ satisfy $\mathbf{v}[\vec{X}] = \vec{x}$. Since the semantics of $\models$ as we have given it is defined in terms of $M(\mathbf{u}, I)$, we can define $\models$ for GSEMs in the identical way.

It is shown in (Anonymous 2021) that GSEMs generalize SEMs in the following sense: Two causal models M and M', which may either be SEMs or GSEMs, are *equivalent*, denoted $M \equiv M'$, if they have the same signature, and they have the same outcomes; that is, if for all $\vec{X} \subseteq \mathcal{V}$, all values $\vec{x} \in \mathcal{R}(\vec{X})$ such that $\vec{X} \leftarrow \vec{x} \in \mathcal{I}$, and all contexts $\mathbf{u} \in \mathcal{R}(\mathcal{U})$, we have $M(\mathbf{u}, \vec{X} \leftarrow \vec{x}) = M'(\mathbf{u}, \vec{X} \leftarrow \vec{x})$.

Theorem 4.1: *(Anonymous 2021, Theorem 3.1) For all SEMs M, there is a GSEM M' such that $M \equiv M'$.*

Recall that in the introduction we defined two models with signature $\mathcal{S}$ to be *$\mathcal{L}(\mathcal{S})$-equivalent* if they agree on all formulas in $\mathcal{L}(\mathcal{S})$. Call a GSEM M *finitary* if, for all contexts and interventions, the set of outcomes is finite. Of course, a GSEM with a finite signature (a finite GSEM) is bound to be finitary, but even infinite GSEMs may be finitary. It is shown in (Anonymous 2021) that equivalence and $\mathcal{L}(\mathcal{S})$-equivalence coincide in SEMs and finitary GSEMs.

Theorem 4.2: *(Anonymous 2021, Theorem 2.1) If M and M' are finitary causal models over the same signature $\mathcal{S}$, then $M \equiv M'$ iff M and M' are $\mathcal{L}(\mathcal{S})$-equivalent.*

As the following example shows, the assumption that M and M' are finitary is critical.

Example 4.3: Consider two GSEMs M, M' with the same signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{R}, \mathcal{I})$. $\mathcal{V}$ consists of countably many binary endogenous variables, that is, $\mathcal{V} = \{X_1, X_2, \ldots\}$ and $\mathcal{R}(X_i) = \{0, 1\}$ for all i. The models have only one context $\mathbf{u}$ (i.e., $\mathcal{U}$ consists of one exogenous variable with a single value). There is only one allowed intervention, the null intervention $\emptyset$. The outcomes of this intervention are as follows. $\mathbf{F}(\mathbf{u}, \emptyset)$ consists of all assignments to the variables X_i where only finitely many of the X_i take the value 0. $\mathbf{F}'(\mathbf{u}, \emptyset)$ consists of all assignments to the X_i where only finitely many X_i take the value 1. Note that for a finite subset of the variables, restricting the outcomes of either model to that subset yields all assignments to that subset. Hence, a formula of the form $\langle\emptyset\rangle\varphi$ is false in both models if $\neg\varphi$ is valid, and true in both models otherwise, because φ is a finite formula and as such can depend only on finitely many variables. Hence, the distinct models M and M' satisfy the same set of causal formulas over $\mathcal{L}(\mathcal{S})$. ∎

4.1 Acyclic GSEMs

In this subsection, we introduce a class of GSEMs analogous to acyclic SEMs. Just as many SEMs used in practice are acyclic, we expect that many GSEMs of practical interest will also be acyclic. For example, the GSEMs constructed in (Anonymous 2021) to model dynamical systems are acyclic according to our definition.

In SEMs, acyclicity is defined using the notion of *independence*. Recall from Section 2 that given a SEM M and endogenous variables X and Y, we say that Y *is independent of* X (in context $\mathbf{u}$) if the structural equation $F_Y(\mathbf{u}, \dots)$ for Y does not depend on X. An acyclic SEM is a SEM whose endogenous variables $\mathcal{V}$ can be totally ordered (for all contexts $\mathbf{u}$) such that if $X \preceq_{\mathbf{u}} Y$, then X is independent of Y in context $\mathbf{u}$.

In acyclic SEMs, there is always a unique outcome, and you cannot change the outcome for variables preceding X by intervening on X. More precisely, let $\mathcal{V}_{\prec_{\mathbf{u}} X} = \{Y \in \mathcal{V} : Y \prec_{\mathbf{u}} X\}$. Then the outcome $\mathbf{v}$ of doing $I; X \leftarrow x$ in context $\mathbf{u}$ and the outcome $\mathbf{v}'$ of doing $I; X \leftarrow x'$ in context $\mathbf{u}$ agree on $\mathcal{V}_{\prec_{\mathbf{u}} X}$ ($\mathbf{v}[\mathcal{V}_{\prec_{\mathbf{u}} X}] = \mathbf{v}'[\mathcal{V}_{\prec_{\mathbf{u}} X}]$). In words, in acyclic SEMs, changing variables later in the ordering does not affect variables earlier in the ordering. In fact, a SEM is acyclic if and only if there are orderings $\preceq_{\mathbf{u}}$ such that this condition holds.

This gives a natural way to extend the definition of acyclicity to GSEMs. The definition of acyclicity for SEMs does not extend to GSEMs, because GSEMs do not have structural equations F_X. A GSEM M only has outcomes $M(\mathbf{u}, I) = \mathbf{F}(\mathbf{u}, I)$. But the condition $\mathbf{v}[\mathcal{V}_{\prec_{\mathbf{u}} X}] = \mathbf{v}'[\mathcal{V}_{\prec_{\mathbf{u}} X}]$ is a condition on outcomes, so it makes sense for GSEMs. Acyclic GSEMs may have multiple solutions, so we need to strengthen the condition slightly. Given a set S of outcomes and a subset $\vec{Y}$ of $\mathcal{V}$, define the *restriction of* S *to* $\vec{Y}$, denoted $S[\vec{Y}]$, as $S[\vec{Y}] = \{\mathbf{v}[\vec{Y}] \mid \mathbf{v} \in S\}$.

Definition 4.4: A GSEM M is acyclic if, for all contexts $\mathbf{u}$, there is a total ordering $\prec_{\mathbf{u}}$ of $\mathcal{V}$ such that the following holds.

Acyc1. For all $X \in \mathcal{V}$, for all $x, x' \in \mathcal{R}(X)$, for all $I \in \mathcal{I}$, we have $M(\mathbf{u}, I; X \leftarrow x)[\mathcal{V}_{\prec_{\mathbf{u}} X}] = M(\mathbf{u}, I, X \leftarrow x')[\mathcal{V}_{\prec_{\mathbf{u}} X}]$.

It is natural to wonder whether this condition needs to involve all variables preceding X. After all, in SEMs, acyclicity is defined in terms of independence, and independence is defined pairwise. Indeed, the pairwise version of this condition is sufficient for SEMs; a SEM M is acyclic if and only if for all contexts $\mathbf{u}$, there is a total ordering $\prec_{\mathbf{u}}$ such that the following holds.

Acyc2. If $Y \prec_{\mathbf{u}} X$, then for all I, x, x', we have $M(\mathbf{u}, I; X \leftarrow x)[Y] = M(\mathbf{u}, I; X \leftarrow x')[Y]$.

Clearly Acyc1 implies Acyc2. In SEMs, they are equivalent.

Proposition 4.5: *If M is a SEM, then M satisfies Acyc1 iff M satisfies Acyc2 (for a fixed context $\mathbf{u}$).*

However, in GSEMs, the two conditions are not equivalent; we claim that the stronger condition Acyc1 is more appropriate for characterizing acyclicity. The following example illustrates why.

Example 4.6: Define a GSEM M with binary variables A, B, C, a single context $\mathbf{u}$, allowed interventions $\mathcal{I} = \{A \leftarrow 0, A \leftarrow 1, B \leftarrow 0, B \leftarrow 1, C \leftarrow 0, C \leftarrow 1\}$, and the outcomes

$$M(\mathbf{u}, C \leftarrow 0) = \{(0,0,0), (1,1,0)\} \text{ and}$$
$$M(\mathbf{u}, C \leftarrow 1) = \{(0,1,1), (1,0,1)\}$$

where (a, b, c) is short for $(A = a, B = b, C = c)$. The outcomes for $A \leftarrow a$ and $B \leftarrow b$ are symmetric.

M is not acyclic when acyclicity is defined using Acyc2. To see this, fix an ordering of the variables; since the model is symmetric, we take the ordering A, B, C without loss of generality. Then intervening on C, the last variable in the ordering, changes the outcomes for the other two; $M(\mathbf{u}, C \leftarrow 0)[\{A, B\}] = \{(0,0), (1,1)\}$, but $M(\mathbf{u}, C \leftarrow 1)[\{A, B\}] = \{(0,1), (1,0)\}$, violating Acyc1. This seems to us the correct classification: M should not be acyclic. The fact that intervening on C changes the possible values for (A, B), but both A and B precede C in $\prec_{\mathbf{u}}$, cannot occur in acyclic SEMs. However, M is acyclic when acyclicity is defined using Acyc2. This is because intervening on C does not affect the possible values for A ($A = 0$ and $A = 1$ in the two outcomes for each intervention) or for B ($B = 0$ and $B = 1$ in the two outcomes for each intervention). ∎

All the GSEMs introduced in (Anonymous 2021) for modeling of dynamical systems, namely, ODE GSEMs (Section 4), GSEMs for hybrid automata (Appendix D.1), and GSEMs for rule-based models (Appendix D.2) are acyclic. The order $\prec_{\mathbf{u}}$ in each case corresponds to time; intervening on variables at a given time cannot affect variables earlier in time (or at the same time).

Next we discuss our approach to axiomatizing this important class of GSEMs. In SEMs, acyclicity corresponds to the axiom D6, which captures the weaker Acyc2. To get an axiom for acyclicity for GSEMs, we need a modification of D6 that captures the stronger Acyc1. For $X \in \mathcal{V}$, $x, x' \in \mathcal{R}'(X)$, $\vec{Y}$ a finite subset of $\mathcal{V}$, $\vec{y} \in \mathcal{R}'(\vec{Y})$, and $I \in \mathcal{I}'$, we write $X \twoheadrightarrow_{x,x',\vec{w},I} \vec{Y}$ as an abbreviation for the formula $[I; X \leftarrow x](\vec{Y} \neq \vec{y}) \wedge \langle I; X \leftarrow x'\rangle(\vec{Y} = \vec{y})$. Consider the following axiom:

D6$^+$. $\bigvee_{i=1}^{k} \neg \bigvee_{x,x' \in U_i, \vec{y} \in S_i, I \in T} X_i \twoheadrightarrow_{x,x',\vec{y},I} \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\}$, where T is a finite subset of $\mathcal{I}'$, U_i is a finite subset of $\mathcal{R}(X)$ for $1 \leq i \leq k$, and $S_i = U_1 \times \cdots \times U_{i-1} \times U_{i+1} \times \cdots \times U_k$.

Over finite signatures, it is straightforward to verify that D6$^+$ is equivalent to

$$\bigvee_{i=1}^{k} \neg(X_i \twoheadrightarrow \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\}) \qquad (1)$$

where $X \twoheadrightarrow \vec{Y}$ abbreviates the formula $\bigvee_{x,x' \in \mathcal{R}(X), \vec{y} \in \mathcal{R}(\vec{Y}), I \in \mathcal{I}} X \twoheadrightarrow_{x,x',\vec{y},I} \vec{Y}$. Over finite signatures, D6$^+$ states that every (finite) subset of the variables has an element that does not affect the others, in the sense that intervening on that element does not change the outcomes restricted to the others. Over infinite signatures, D6$^+$ is weaker: it makes the same statement, but only for interventions of the form $I; X_i \leftarrow x$ and $I; X_i \leftarrow x'$ where $I \in T$ and $x, x' \in U_i$, and only if the changed value is in S_i. D6$^+$ is clearly sound in acyclic GSEMs, since in an acyclic GSEM, every finite subset of variables has a maximum element under $\preceq_{\mathbf{u}}$, and this element does not affect the others. Moreover, in SEMs, D6$^+$ implies D6, which Halpern used to axiomatize acyclic SEMs. This is because D6$^+$ holds iff there is an ordering $\prec_{\mathbf{u}}$ of the endogenous variables such that Acyc1 holds, and D6 holds iff there is an ordering such that the weaker condition Acyc2 holds.

In Section 5 we show that an axiom system containing D6$^+$ is in fact complete for finite acyclic GSEMs. We believe this is also true for infinite GSEMs, but we have not yet finished working out the details.

5 Axiomatizing finite GSEMs

Our goal is to provide a sound and complete axiomatization of GSEMs. We start with finite GSEMs, that is, GSEMs over a finite signature; in the next section, we consider arbitrary GSEMs. Note that the language $\mathcal{L}(\mathcal{S})$ given above for SEMs makes perfectly good sense for finite GSEMs; the semantics of the language for GSEMs is identical to the semantics for SEMs. Because GSEMs are more flexible than SEMs, they do not satisfy all the axioms in AX^+. As we now show, a strict subset of AX^+ provides a sound and complete axiomatization of finite GSEMs.

Definition 5.1: AX^+_{basic} consists of axiom schema D0, D1, D2, D4, D7, D8, and inference rule MP.

Theorem 5.2: *AX^+_{basic} is sound and complete for finite GSEMs.*

The proof of this and all other results not in the main text can be found in the supplementary material. Let $AX^+_{basic,rec}$ consist of the axioms in AX^+_{basic} along with axiom schema D6$^+$. Then

Theorem 5.3: *$AX^+_{basic,rec}$ is sound and complete for acyclic finite GSEMs.*

As shown in (Anonymous 2021), if $\mathcal{S}$ is a universal signature, then SEMs over $\mathcal{S}$ are equivalent in expressive power to finite GSEMs where all the axioms in AX^+ are valid.

Theorem 5.4: *(Anonymous 2021) If $\mathcal{S}$ is a universal signature, then for every finite GSEM over $\mathcal{S}$ that satisfies the axioms of $AX^+(\mathcal{S})$ there is an equivalent SEM over $\mathcal{S}$, and for every SEM over $\mathcal{S}$ there is an equivalent finite GSEM over $\mathcal{S}$ that satisfies the axioms of $AX^+(\mathcal{S})$.*

Since equivalence is the same as $\mathcal{L}(\mathcal{S})$-equivalence, this immediately implies the following.

Corollary 5.5: *If $\mathcal{S}$ is a universal signature, then $AX^+(\mathcal{S})$ is a sound and complete axiomatization for $\mathcal{L}(\mathcal{S})$ for finite GSEMs satisfying $AX^+(\mathcal{S})$.*

Despite its apparently trivial nature, Theorem 5.5 does not hold in general for non-universal signatures, as (Anonymous 2021, Example 3.6) shows. This example, included in the supplementary material for completeness C.1), is a GSEM over a finite signature $\mathcal{S}$ that satisfies the axioms of $AX^+(S)$ but is not equivalent to a SEM.

The existence of a finite GSEM satisfying AX^+ that is not equivalent to an SEM has a significant implication.

Theorem 5.6: *$AX^+(\mathcal{S})$ is not complete for SEMs of signature $\mathcal{S}$ when some interventions are not allowed in $\mathcal{S}$, although it is sound.*

6 Axiomatizing infinite GSEMs

Things change significantly in infinite GSEMs. To see just one of the problems, note that if X is a variable with infinite range, then instances of D2 corresponding to X, namely $[\vec{Y} \leftarrow \vec{y}](\bigvee_{x \in \mathcal{R}(X)} X = x)$, are no longer in the language, since the disjunction is infinitary. Moreover, if $\mathcal{R}(X)$ is uncountable and the language includes all formulas of the form $X = x$ for $x \in \mathcal{R}(X)$, then the language will be uncountable. While there is no difficulty giving semantics to this uncountable language, there seem to be nontrivial technical problems when it comes to axiomatizations.

On the other hand, suppose that, for example, the range of X is the real numbers. In practice, we do not want to make statements like $X = \pi^3 - e$. It should certainly suffice in practice to be able to mention explicitly only countably many real numbers. (Indeed, we expect that, in practice, it will suffice to talk explicitly about only finitely many real numbers.) Similarly, we expect that it will suffice to talk explicitly about only countably many variables and interventions. To get a countable language, we thus proceed as follows.

Given a signature $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathcal{F}, \mathcal{I})$ or $\mathcal{S} = (\mathcal{U}, \mathcal{V}, \mathbf{F}, \mathcal{I})$, let $\vec{W}$ be a countable subset of $\mathcal{V}$; we call the elements of $\vec{W}$ *named variables*. For each named variable X, let $\mathcal{R}'(X)$ be a countable subset of $\mathcal{R}(X)$, except that we require that (a) if $\mathcal{R}(X)$ is finite, then $\mathcal{R}'(X) = \mathcal{R}(X)$ and (b) if $\mathcal{R}(X)$ is infinite, then so is $\mathcal{R}'(X)$. The elements of $\mathcal{R}'(X)$ are called *named values*. Finally, let $\mathcal{I}'$ be an arbitrary countable subset of $\mathcal{I}$, except that we require that if $\vec{X} \leftarrow \vec{x} \in \mathcal{I}'$, then $\vec{X} \subseteq \vec{W}$ and $\vec{x} \subseteq \mathcal{R}'(X)$ and we assume that $\mathcal{I}'$ *is closed under finite differences with* $\mathcal{I}$, so that if $I_1 \in \mathcal{I}'$,

$I_2 \in \mathcal{I}$, $(I_1 - I_2) \cup (I_2 - I_1)$ is finite, and $I_2 = \vec{X} \leftarrow \vec{x}$, where $\vec{X} \subseteq \vec{W}$ and $\vec{x} \in \mathcal{R}'(\vec{X})$, then $I_2 \in \mathcal{I}'$. That is, if we are willing to talk about the intervention I_1, and I_2 is an allowable intervention that differs from I_1 only in how it sets a finite number of variables, all of which we are willing to talk about, as well as the values that they are set to, then we should be willing to talk about I_2 as well. The language $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ consists of Boolean combinations of basic causal formulas $[\vec{Y} \leftarrow \vec{y}]\varphi$ where $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$ and φ is a Boolean combination of events of the form $X = x$, where $X \in \vec{W}$ and $x \in \mathcal{R}'(X)$. $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ is clearly a sublanguage of $\mathcal{L}(\mathcal{S})$. Intuitively, it consists only of entities (variables, values, and interventions) that can be named. Since there are only countably many entities that can be named, it easily follows that $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ is countable.

This sublanguage, although not as expressive as the full language, is still extraordinarily expressive. For example, if the exogenous variables are X_t for t ranging over the real numbers, we could choose $\vec{W}$ to be the subset of $\{X_t \mid t \in \mathbb{R}\}$ for which t is rational. Likewise, if each variable X_t ranges over the real numbers, we could choose $\mathcal{R}(X_t)$ to be the rationals.

We are interested in axiomatizing classes of GSEMs essentially using subsets of the axioms in AX^+, but it seems that we need one new inference rule. While we keep axiom D2, it applies only to variables X such that $\mathcal{R}(X)$ is finite. However, even if $\mathcal{R}(X)$ is infinite, we still want to be able to conclude something like $[\vec{Y} \leftarrow \vec{y}](\exists x(X = x))$: after setting $\vec{Y}$ to $\vec{y}$, X takes on *some* value. Of course, we cannot say this, since we have no existential quantification in the language. Although it is far from obvious, the following rule of inference plays the same role as D2 for variables X with infinite ranges.

D2$^+$. Suppose that $S \subseteq \mathcal{R}'(X)$ is a finite subset of values of X that contains all the values of X mentioned in the formula $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\psi$, and some value in $\mathcal{R}'(X)$ not in the formula if there is such a value. Then from $\varphi \Rightarrow \wedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x))$ infer $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\psi$.

Note that if $\mathcal{R}(X)$ is infinite, since we have assumed that $\mathcal{R}'(X)$ is infinite if $\mathcal{R}(X)$ is, there will always be an element in $\mathcal{R}'(X)$ that is not mentioned in φ or ψ.

While D2$^+$ may not look anything like D2, we can show that in the case of variables X with finite range, it is equivalent to D2 in the sense made precise in the following result.

Proposition 6.1: *If AX^*_{basic} is the result of replacing D2 with D2$^+$ in AX^+_{basic}, then we can derive D2 for variables with finite ranges in AX^*_{basic}. Moreover, D2$^+$ is derivable in AX^+_{basic} for variables X with finite range, in the sense that if $AX^+_{basic} \vdash \varphi \Rightarrow \wedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x))$ and $\mathcal{R}(X)$ is finite, then $AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\psi$.*

While D2$^+$ is unnecessary for finite GSEMs, it is necessary for infinite GSEMs. Let $AX^*_{basic}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ consist of all the axioms and inference rules in $AX^+_{basic}(\mathcal{S})$ together with D2$^+$, restricted to formulas in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$. Then $AX^*_{basic}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ is sound and complete for GSEMs over $\mathcal{S}$ (this is a special case of Theorem 6.2 below).

Considering GSEMs also helps explain the role of some of the other axioms. A GSEM $\mathcal{S}$ is *coherent* if for all interventions $\vec{X} \leftarrow \vec{x}$ and $\vec{X} \leftarrow \vec{x}; \vec{Y} \leftarrow \vec{y}$ in $\mathcal{I}$, if $\vec{Y}$ is finite, $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x})$, and $\mathbf{v}[\vec{Y}] = \vec{y}$, then $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x}; \vec{Y} \leftarrow \vec{y})$. The intuition for coherence is straightforward: if we think of the assignments in $\mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x})$ as representing the "closest" assignments to $\mathbf{u}$ for which $\vec{X} = \vec{x}$ holds, and $\vec{Y} = \vec{y}$ already holds in some assignment $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x})$, then $\mathbf{v}$ is one of the assignments closest to $\mathbf{u}$ where both $\vec{X} = \vec{x}$ and $\vec{Y} = \vec{y}$ hold, so $\mathbf{v}$ should be in $\mathbf{F}(\mathbf{u}, \vec{X} \leftarrow \vec{x}; \vec{Y} \leftarrow \vec{y})$. Note that this intuition applies equally to the case where $\vec{Y}$ is infinite ("strong coherence"). However, this seems to be harder to axiomatize, because the intervention $\vec{X} \leftarrow \vec{x}; \vec{Y} \leftarrow \vec{y}$ is not guaranteed to be in $\mathcal{I}'$. We leave the problem of axiomatizing "strong coherence" to future work. As we show, D3 correponds to coherence. As their names suggest, D10(a) corresponds to each intervention having at least one outcome (in any given context), and D10(b) corresponds to each intervention having at most one outcome, so D10 (i.e., the combination of 10(a) and 10(b)) corresponds to each intervention having a unique outcome. This is made precise in Theorem 6.2 below.

On the other hand, D5 and D9 do not seem meaningful in GSEMs. D5 and D9 do not have analogues if we have infinitely many variables, since we cannot express $\vec{Z} = \vec{z}$, and there are uncountably many complete interventions (interventions of the form $\vec{Y} \leftarrow \vec{y}$ for $\vec{Y} = \mathcal{V} - \{X\}$). They also do not seem to represent particularly interesting properties, since their scope is very limited (D5 always mentions all variables, whereas D9 applies only to complete interventions).

For finite GSEMs, we showed that D6$^+$ corresponds to acyclicity. We believe this is also true for infinite GSEMs, but we have not yet finished working out the details.

Let $\mathcal{G}^{\emptyset}(\mathcal{S})$ denote the class of GSEMs over $\mathcal{S}$. Let $\mathcal{G}^{\geq 1}(\mathcal{S})$, $\mathcal{G}^{\leq 1}(\mathcal{S})$, and $\mathcal{G}^{=1}(\mathcal{S})$ denote the class of GSEMs over $\mathcal{S}$ where each intervention has at least one, at most one, and exactly one outcome; let $\mathcal{G}^{coh}$ denote the class of coherent GSEMs over $\mathcal{S}$. Given a subset A of {D3, D10(a), D10(b)}, let $\mathcal{A}$ be the corresponding subset of $\{coh, \geq 1, \leq 1\}$. Let $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ be the axiom system consisting of the axioms and rules of inference of AX^*_{basic} together with the axioms in A, restricted to the language $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$. Let $\mathcal{G}^{\mathcal{A}}$ be the class of GSEMs satisfying all of the properties in $\mathcal{A}$; that is, $\mathcal{G}^{\mathcal{A}} = \bigcap_{P \in \mathcal{A}} \mathcal{G}^P$. Then

Theorem 6.2: *$AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ is sound and complete for the class $\mathcal{G}^{\mathcal{A}}$ of GSEMs with signature $\mathcal{S}$ over language $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$.*

For reasons of space, we defer the full proof of Theorem 6.2 to the supplementary material. However, since the completeness proof requires several nontrival ideas beyond what is needed for the analogous results for SEMs (and other soundness and completeness results in modal logic), we present a sketch of it here.

Proof: (Sketch)

To prove completeness, we proceed in the same spirit as for finite GSEMs (Theorem 5.2). The proof of that theorem (in the supplementary material) uses essentially the same technique as that used by Halpern [2000]. We briefly sketch the details here: Suppose that $\varphi \in \mathcal{L}(\mathcal{S})$ is *consistent with* $AX^+_{basic}(\mathcal{S})$ (i.e., we cannot prove $\neg\varphi$ in $AX^+_{basic}(\mathcal{S})$). Then φ can be extended to a maximal consistent set C of formulas; that is, $\varphi \in C$, every finite subset C' of C is consistent with $AX^+_{basic}(\mathcal{S})$ (i.e., the conjunction of the formulas in C' is consistent with $AX^+_{basic}(\mathcal{S})$), and no strict superset C^* of C has the property that every finite subset of C^* is consistent with $AX^+_{basic}(\mathcal{S})$. Standard arguments show that, for every formula $\psi \in \mathcal{L}(\mathcal{S})$, either ψ or $\neg\psi$ must be in C; moreover, every instance of the axioms in $AX^+_{basic}(\mathcal{S})$ is in C. Define a finite GSEM M^C with signature $\mathcal{S}$ as follows. For all contexts $\mathbf{u}$ and allowed interventions $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}$, define $\mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y}) = \{\mathbf{v} \in \mathcal{R}(\mathcal{V}) \mid \langle\vec{Y} \leftarrow \vec{y}\rangle(\mathcal{V} = \mathbf{v}) \in C\}$. Thus, the formulas in C tell us how interventions work in M^C. To finish the proof, we need to show that M^C is a GSEM, and that it models φ. That is, given a fixed context $\mathbf{u}$, $(M^C, \mathbf{u}) \models \varphi$.

However, in the case of infinite signatures, there are a number of new subtleties. We can no longer define the function $\mathbf{F}$ in M by taking $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ if $\langle\vec{Y} \leftarrow \vec{y}\rangle(\mathcal{V} = \mathbf{v}) \in C$, because $\mathcal{V} = \mathbf{v}$ is an infinitary formula, so is not in the language. We deal with this by saying that the outcome $\mathbf{v}$ is in $\mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ if $\langle\vec{Y} \leftarrow \vec{y}\rangle(\vec{X} = \mathbf{v}[\vec{X}]) \in C$ for all finite $\vec{X} \subseteq \vec{W}$.

But there is a more serious problem. In the proof of Theorem 5.2, we showed that (for the model M constructed from the maximal consistent set C), we have that $(M, \mathbf{u}) \models \varphi$ iff $\varphi \in C$. But suppose that X is a variable such that $\mathcal{R}(X) = \{x_1, x_2, x_3, \ldots\}$ is countable, $\mathcal{R}'(X) = \mathcal{R}(X)$, and $\mathcal{Y} \leftarrow y \in \mathcal{I}' \cap \mathcal{I}$. It is not hard to see that the set $C' = \{\langle Y \leftarrow y\rangle true, [Y \leftarrow y](X \neq x_1), [Y \leftarrow y](X \neq x_2), \ldots\}$ is consistent with AX^*_{basic} (since every finite subset of this set is obviously consistent). Hence C' can be extended to a maximal consistent set C. But there is no model M such that $(M, u) \models \varphi$ for all formulas $\varphi \in C'$. Thus, we have to restrict the set of maximal consistent sets that we consider.

Definition 6.3: A *conjunctive formula* is a conjunction of formulas of the form $X = x$ and $X \neq x$. (The formula $true$ is viewed as conjunctive, since it is an empty conjunction.) A set C of formulas in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ is *acceptable for* $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi$ *with respect to* $Z \in \vec{W}$, where φ is a conjunctive formula, if there is some $z \in \mathcal{R}'(Z)$ and a conjunctive formula ψ such that every conjunct of φ is a conjunct of ψ, $Z = z$ is a conjunct of ψ for some $z \in \mathcal{R}'(Z)$, and $\langle\vec{Y} \leftarrow \vec{y}\rangle\psi \in C$. C is *acceptable* if C is acceptable for every formula of the form $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi \in C$ such that φ is conjunctive formula and $Z \in \vec{W}$. ∎

The set C' above is not acceptable for $\langle Y \leftarrow y\rangle true$ and X, and cannot be extended to an acceptable consistent set. Our proof technique involves constructing a model from an acceptable maximal consistent set. So we must show that every consistent formula is included in an acceptable maximal consistent set. The next lemma gives the key step for doing this. Its proof can be found in the supplementary material.

Lemma 6.4: *If C is a finite subset of $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi \in C$, and $X \in \vec{W}$, then there exists a formula $\psi \in \mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S}) - C$ such that $C \cup \{\psi\}$ is consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ and acceptable with respect to $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi$ and X.*

We can now prove completeness. Given a formula φ consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, we construct a maximal acceptable set C consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ containing φ as follows. Let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an enumeration of the formulas in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ such that $\sigma_0 = \varphi$ and let $X_0, X_1, X_2, \ldots$ be an enumeration of the variables in $\vec{W}$. It is well known that there is a bijection b from $I\!N$ to $I\!N \times I\!N$ such that if $b(n) = (n_1, n_2)$, then $n_1 \leq n$. We construct a sequence of sets $C_0, C_1, C_2, \ldots$ such that $C_0 = \{\varphi\}$, $C_k \subseteq C_{k+1}$, and (a) either $\sigma_k \in C_k$ or $C_k \cup \{\sigma_k\}$ is inconsistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, (b) C_k is consistent, (c) if $b(k) = (k_1, k_2)$ and σ_{k_1} has the form $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi$, then C_{k+1} is acceptable with respect to σ_{k_1} and X_{k_2}. We construct the sequence inductively. Given C_k, then we add σ_{k+1} to C_k if $C_k \cup \{\sigma_{k+1}\}$ is consistent. In addition, if σ_{k_1} has the form $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi$, then we apply Lemma 6.4 to add a formula if necessary to make C_{k+1} acceptable with respect to σ_{k_1} and X_{k_2}.

Let $C = \cup_{k=0}^{\infty} C_k$. Clearly C contains φ. It is consistent, since if not, some finite subset of C must be inconsistent. But this finite subset must be contained in C_k for some k, and C_k is consistent, by construction. Finally, C is acceptable. For suppose that $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi \in C$ and $X \in \vec{W}$. There must exist k_1 and k_2 such that $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi = \sigma_{k_1}$ and $X = X_{k_2}$. Let $k = b^{-1}(k_1, k_2)$. Since $\sigma_{k_1} = \langle\vec{Y} \leftarrow \vec{y}\rangle\varphi \in C$, it must already be in C_{k_1} (since it would be added in the construction of C_{k_1} if it was not already in C_{k_1-1}). By the choice of b, $k_1 \leq k$, so $\sigma_{k_1} \in C_k$. By construction, C_{k+1} is acceptable with respect to $\langle\vec{Y} \leftarrow \vec{y}\rangle\varphi$ and X_{k_2}, hence so is C.

We now construct a model M^C with signature $\mathcal{S}$. For interventions $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$, we take $\mathbf{F}(\vec{Y} \leftarrow \vec{y}, \mathbf{u}) = \{\mathbf{v} \mid$ for all finite subsets $\vec{X} \subseteq \vec{W}$, $\langle\vec{Y} \leftarrow \vec{y}\rangle(\vec{X} = \mathbf{v}[\vec{X}]) \in C\}$. Now we still need to define $\mathbf{F}$ on interventions in $\mathcal{I} - \mathcal{I}'$. Let $\mathbf{v}^*$ be a fixed assignment. For $I \in \mathcal{I} - \mathcal{I}'$, define $\mathbf{F}(I, \mathbf{u}) = \{\mathbf{v}^*\}$. We claim that $M^C \in \mathcal{G}^{\mathcal{A}}$ and that $\psi \in C$ iff, for all contexts $\mathbf{u}$, we have $(M^C, \mathbf{u}) \models \psi$. If these claims hold, then we have produced a model M^C of φ that satisfies the properties in $\mathcal{A}$, completing the consistency proof. We refer the interested reader to the supplementary material for the proofs of these claims. Both proofs use the definition of M^C to relate the formulas in C to the properties of M^C. We show the first claim separately for each property in $\mathcal{A}$ given its counterpart axiom in A. The second claim is proved by structural induction on ψ. ■

7 Conclusion

GSEMs (Anonymous 2021) are a flexible, powerful generalization of SEMs that allow us to reason about systems with infinitely many variables, such as dynamical systems. In this paper, building on the axiomatization of SEMs given by Halpern [2000], we give sound and complete axiomatizations for GSEMs, and for several interesting subclasses of GSEMs. To prove the completeness results, we develop several ideas that may be generally useful in developing other completeness results in in infinitary modal logics. Specifically, we (1) define a restricted language for causal reasoning that is countable but still highly expressive, and (2) develop a notion of *acceptable* maximal consistent sets. Our results also clarify the properties that Halpern's axioms capture in SEMs.

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A Proofs

Proposition 4.5: *If M is a SEM, then M satisfies Acyc1 iff M satisfies Acyc2 (for a fixed context $\mathbf{u}$).*

Proof: We need only prove that Acyc2 implies Acyc1 in SEMs, since the other direction is obvious. Fix a context $\mathbf{u}$. We first claim that if M satisfies Acyc2, then M has unique solutions. Fix an intervention I; it suffices to show that for all variables X there is x such that for all solutions $\mathbf{v} \in M(\mathbf{u}, I)$, $\mathbf{v}[X] = x$. We show this by induction on $\prec_{\mathbf{u}}$. For the base case, let X be the minimum variable with respect to $\preceq_{\mathbf{u}}$. Acyc2 implies that the structural equation for X does not depend on any variables; that is, it is a constant $\mathcal{F}_X = x$. Hence $\mathbf{v}[X] = x$ for all solutions $\mathbf{v} \in \mathcal{M}(\mathbf{u}, I)$. For the inductive step, Acyc2 implies that the structural equation for Z does not depend on variables $Y \succeq_{\mathbf{u}} Z$. Hence $\mathcal{F}_Z$ only depends on variables preceding Z. But by inductive hypothesis, those variables have the same values in all solutions. Plugging those values into $\mathcal{F}_Z$, we find that $\mathcal{F}_Z$ is also a constant. Now that we have shown that M has unique solutions, we can finish the proof by observing that Acyc2 states that the unique outcome of $I; X \leftarrow x$ and the unique outcome of $I; X \leftarrow x'$ agree on all variables $Y \prec_{\mathbf{u}} X$. But this implies Acyc1. ∎

Theorem 5.2: AX^+_{basic} *is sound and complete for finite GSEMs.*

Proof: First we prove soundness. Fix a finite GSEM M and context $\mathbf{u}$. D0 is trivially sound. D1 is sound since for all $\mathbf{v} \in M(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$, we have that $\mathbf{v}[X] = x \Rightarrow \mathbf{v}[X] \neq x'$ for $x \neq x'$. D2 is sound since for all $\mathbf{v} \in M(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$, we have that $\mathbf{v}[X] \in \mathcal{R}(X)$. D4 is sound by definition of GSEMs. D7, D8 and MP are trivially sound.

Next we prove completeness. Fix a finite signature $\mathcal{S}$. It suffices to show that for any causal formula $\varphi \in \mathcal{L}(\mathcal{S})$ that is consistent with $AX^+_{basic}(\mathcal{S})$, there is a finite GSEM M and context $\mathbf{u}$ such that $(M, \mathbf{u}) \models \varphi$.

To prove this, we use essentially the same technique as that used by Halpern [2000]. We briefly sketch the details here: Suppose that $\varphi \in \mathcal{L}(\mathcal{S})$ is *consistent with* $AX^+_{basic}(\mathcal{S})$ (i.e., we cannot prove $\neg\varphi$ in $AX^+_{basic}(\mathcal{S})$). Then φ can be extended to a maximal consistent set C of formulas; that is, $\varphi \in C$, every finite subset C' of C is consistent with $AX^+_{basic}(\mathcal{S})$ (i.e., the conjunction of the formulas in C' is consistent with $AX^+_{basic}(\mathcal{S})$), and no strict superset C^* of C has the property that every finite subset of C^* is consistent with $AX^+_{basic}(\mathcal{S})$. Standard arguments show that, for every formula $\psi \in \mathcal{L}(\mathcal{S})$, either ψ or $\neg\psi$ must be in C; moreover, every instance of the axioms in $AX^+_{basic}(\mathcal{S})$ is in C. Define a finite GSEM M with signature $\mathcal{S}$ as follows. For all contexts $\mathbf{u}$ and allowed interventions $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}$, define $\mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y}) = \{\mathbf{v} \in \mathcal{R}(\mathcal{V}) \mid \langle \vec{Y} \leftarrow \vec{y} \rangle (\mathcal{V} = \mathbf{v}) \in C\}$. Thus, the formulas in C tell us how interventions work in M. Fix a context $\mathbf{u}$. To finish the proof, we need to show two properties of M: first, that M is in fact a GSEM (that is, it satisfies effectiveness) and second, that $(M, \mathbf{u}) \models \varphi$. These both follow from a standard *truth lemma*:

Lemma A.1: *For all $\psi \in \mathcal{L}(\mathcal{S})$,*

$$\psi \in C \textit{ iff } (M, \mathbf{u}) \models \psi. \qquad (2)$$

That is, C contains exactly the formulas true of M. Effectiveness follows from Lemma A.1 because C contains all instances of D4. The desired result $(M, \mathbf{u}) \models \varphi$) also follows easily from Lemma A.1 using the fact $\varphi \in C$.

To prove Lemma A.1, the key step is to prove (2) for the basic causal formulas $[\vec{Y} \leftarrow \vec{y}]\rho$. Since C is maximal, and causal formulas are, by definition, Boolean combinations of basic causal formulas, the result then follows by a straightforward structural induction on ψ.

It suffices to check (2) for basic causal formulas of the form $[\vec{Y} \leftarrow \vec{y}](\bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$, since for all events ρ,

$$AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}]\rho \Leftrightarrow [\vec{Y} \leftarrow \vec{y}] \bigwedge_{\mathbf{v} \models \neg\rho} \neg(\mathcal{V} = \mathbf{v}), \qquad (3)$$

where, as usual, we write $AX \vdash \varphi$ to denote that φ is provable in the axiom system AX. Viewing the formula ρ as a predicate on outcomes, (3) expresses the intuitively obvious claim that ρ is equivalent to the predicate that checks that its input is not equal to any outcome not satisfying ρ.

To prove (3), we need one lemma, which we state here without proof, but which follows from just D0, D7 and D8 using standard modal logic arguments.

Lemma A.2:

(a) $AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}]\varphi_1 \wedge [\vec{Y} \leftarrow \vec{y}]\varphi_2 \Leftrightarrow [\vec{Y} \leftarrow \vec{y}](\varphi_1 \wedge \varphi_2)$

(b) $AX^+_{basic} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi_1 \vee \varphi_2) \Leftrightarrow_{AX^+_{basic}} \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_1 \vee \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_2$.

Let $\varphi_{D1} = (\bigwedge_{X \in \mathcal{V}, x \in \mathcal{R}(X), x' \neq x} (X = x \Rightarrow X \neq x')$ and $\varphi_{D2} = (\bigvee_{X \in \mathcal{V}, x \in \mathcal{R}(X)} (X = x))$. These formulas are the conjunctions of the formulas after the $[\vec{Y} \leftarrow \vec{y}]$ in D1 and D2 respectively. Let $\varphi_{EQ} = \varphi_{D1} \wedge \varphi_{D2}$. φ_{EQ} says that, for each variable $X \in \mathcal{V}$, there is exactly one value $x \in \mathcal{R}(X)$ such that $X = x$.

It follows from D1, D2, and Lemma A.2 that $AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}]\varphi_{EQ}$. It is straightforward to show that $\varphi_{EQ} \Rightarrow (\rho \Leftrightarrow \bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$ is a propositional tautology. By D8, we have that $AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}](\varphi_{EQ} \Rightarrow (\rho \Leftrightarrow \bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v}))$. Hence, by D7, $AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}](\rho \Leftrightarrow \bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$. Using D7 again, it follows that $AX^+_{basic} \vdash [\vec{Y} \leftarrow \vec{y}]\rho \Leftrightarrow [\vec{Y} \leftarrow \vec{y}] \bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$, which is exactly (3).

It remains to prove (2) for formulas of the form $[\vec{Y} \leftarrow \vec{y}](\bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$. Since C is maximal and consistent with AX^+_{basic},

it follows from Lemma A.2 that $[\vec{Y} \leftarrow \vec{y}](\bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v}) \in C$ if and only if for all $\mathbf{v} \models \neg\rho$, $[\vec{Y} \leftarrow \vec{y}](\mathcal{V} \neq \mathbf{v}) \in C$. Clearly, we also have that $(M, \mathbf{u}) \models [\vec{Y} \leftarrow \vec{y}](\bigwedge_{\mathbf{v} \models \neg\rho} \mathcal{V} \neq \mathbf{v})$ if and only if for all $\mathbf{v} \models \neg\rho$, $(M, \mathbf{u}) \models [\vec{Y} \leftarrow \vec{y}](\mathcal{V} \neq \mathbf{v})$. Finally, by definition, $\neg[\vec{Y} \leftarrow \vec{y}](\mathcal{V} \neq \mathbf{v})$ is $\langle \vec{Y} \leftarrow \vec{y} \rangle(\mathcal{V} = \mathbf{v})$. Therefore, it suffices to prove (2) for formulas ψ of the form $\langle \vec{Y} \leftarrow \vec{y} \rangle(\mathcal{V} = \mathbf{v})$.

Suppose that $\psi \in C$. Then, by definition of M, we have that $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$, so $(M, \mathbf{u}) \models \psi$. Conversely, suppose that $(M, \mathbf{u}) \models \psi$. Then $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$. Again, by definition of M, this implies that $\psi \in C$. ∎

Theorem 5.3: *$AX^+_{basic,rec}$ is sound and complete for acyclic finite GSEMs.*

Proof: Consider (1), which is equivalent to D6$^+$ over finite signatures. Soundness is clear; given an acyclic GSEM M and context $\mathbf{u}$, for all instances of D6$^+$, the definition of acyclicity implies that the disjunct corresponding to the largest element of $\{X_1, \ldots, X_k\}$ with respect to $\prec_\mathbf{u}$ is satisfied by M. For completeness, we build off the proof of Theorem 5.2. It suffices to show that the model M constructed in that proof is acyclic. M has only one context $\mathbf{u}$, so it suffices to show the acyclicity condition Acyc1 holds for that $\mathbf{u}$. Lemma A.1 implies that M satisfies all instances of D6$^+$ in context $\mathbf{u}$. Hence, for every instance φ of D6$^+$, M satisfies at least one disjunct $1 \leq i \leq k$ of φ (in context $\mathbf{u}$). We use this to construct an order $\prec_\mathbf{u}$ over $\mathcal{V}$ as follows. Consider the unique instance φ_k of D6$^+$ where $\{X_1, \ldots, X_k\} = \mathcal{V}$. Let i be any disjunct of φ_k satisfied by M, and define the maximum element of $\prec_\mathbf{u}$ to be X_i. Next consider the unique instance φ_{k-1} of D6$^+$ where $\{X_1, \ldots, X_k\} = \mathcal{V} - \{X_i\}$. Let j be any disjunct of φ_{k-1} satisfied by M, and define the second largest element of $\prec_\mathbf{u}$ to be X_j. Repeating this process gives an order $\prec_\mathbf{u}$ of all the variables, such that for all $X \in \mathcal{V}$, M satisfies $\neg(X \twoheadrightarrow \mathcal{V}_{\prec_\mathbf{u} X})$. But this implies that M satisfies Acyc1. ∎

Corollary 5.5: *If $\mathcal{S}$ is a universal signature, then $AX^+(\mathcal{S})$ is a sound and complete axiomatization for $\mathcal{L}(\mathcal{S})$ for finite GSEMs satisfying $AX^+(\mathcal{S})$.*

Proof: Since for every finite GSEM satisfying $AX^+(\mathcal{S})$ there is an $\mathcal{S}$-equivalent SEM and vice versa, the same set of causal formulas is valid in each class of models. Since, as shown by Halpern [2000], $AX^+(\mathcal{S})$ is a sound and complete axiomatization of SEMs, the result follows. ∎

Theorem 5.6: *$AX^+(\mathcal{S})$ is not complete for SEMs of signature $\mathcal{S}$ when some interventions are not allowed in $\mathcal{S}$, although it is sound.*

Proof: For incompleteness, consider the causal formula φ that characterizes the shell game and its outcomes, that is, $\varphi = [S_1 \leftarrow 1](S_1 = 1 \wedge S_2 = 1 \wedge Z = 1) \wedge [S_2 \leftarrow 1](S_1 = 1 \wedge S_2 = 1 \wedge Z = 0)$. This formula is false in all SEMs with signature $\mathcal{S}_{shell}$ (the signature of M_{shell}). Hence, $\neg\varphi$ is true in all SEMs with signature $\mathcal{S}_{shell}$. However, $\neg\varphi$ is not provable from $AX^+(\mathcal{S}_{shell})$, because M_{shell} satisfies all axioms of $AX^+(\mathcal{S}_{shell})$, but φ is true in M_{shell}. For soundness, as we observed, all the axioms of $AX^+(\mathcal{S}_{shell})$ are valid in M_{shell}. ∎

Proposition 6.1: *If AX^*_{basic} is the result of replacing D2 with D2$^+$ in AX^+_{basic}, then we can derive D2 for variables with finite domains in AX^*_{basic}. Moreover, D2$^+$ is derivable in AX^+_{basic} for variables X with finite range, in the sense that if $AX^+_{basic} \vdash \varphi \Rightarrow \wedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x))$ and $\mathcal{R}(X)$ is finite, then $AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\psi$.*

Proof: To prove the first claim, suppose that $\mathcal{R}(X)$ is finite. Define $S = \mathcal{R}(X)$, $\varphi = true$, and $\psi = \bigwedge_{x \in S} X \neq x$. Clearly, $AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow \bigwedge_{x \in S} X \neq x)$, so $AX^*_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}](\neg\psi)$. But $\neg\psi$ is equivalent to $\bigvee_{x \in \mathcal{R}(X)} X = x$, so using D7 and D8, we can conclude that $AX^*_{basic} \vdash [\vec{Y} \leftarrow \vec{y}](\bigvee_{x \in \mathcal{R}(X)} X = x)$, as desired.

For the second claim, suppose that $AX^+_{basic} \vdash \varphi \Rightarrow \wedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x))$, where $S \subseteq \mathcal{R}'(X)$ and S contains some value in $\mathcal{R}'(X)$ not in $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\psi$ if there is one. First we consider the case $S = \mathcal{R}(X)$. Applying Lemma A.2, we have $AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}](\bigwedge_{x \in \mathcal{R}(X)} \psi \Rightarrow (X \neq x))$. It is easy to see that $\bigwedge_{x \in \mathcal{R}(X)}(\psi \Rightarrow (X \neq x))$ is logically equivalent to $\psi \Rightarrow \bigwedge_{x \in \mathcal{R}(X)}(X \neq x)$, which is logically equivalent to $\vee_{x \in \mathcal{R}(X)}(X = x) \Rightarrow \neg\psi$. Thus, using D7 and D8, we get

$$AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}](\vee_{x \in \mathcal{R}(X)}(X = x) \Rightarrow \neg\psi).$$

Now using D2, D7, and D8, we easily get

$$AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\psi,$$

as desired.

Next we consider the case $S \neq \mathcal{R}(X)$. Since we assumed that $\mathcal{R}(X)$ is finite, $\mathcal{R}'(X) = \mathcal{R}(X)$. Hence there is a value in $\mathcal{R}'(X)$ that is not mentioned in $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\psi$, so there must be such an element in S, say x. By assumption, $AX^+_{basic} \vdash \varphi \Rightarrow \bigwedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow (X \neq x))$. Thus, there exists a derivation of $\varphi \Rightarrow \bigwedge_{x \in S}[\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow (X \neq x))$ in AX^+_{basic}. For all $x' \notin S$, we can replace x with x' everywhere in this derivation. The result is a derivation of $\varphi \Rightarrow \bigwedge_{x \in S \cup \{x'\} \setminus \{x\}}[\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow (X \neq x))$ in AX^+_{basic}. (Notice that φ and ψ remain unchanged, since they do not mention x.) Thus, $AX^+_{basic} \vdash \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow (X \neq x'))$. Using the assumption that $\mathcal{R}(X)$ is finite, it then follows that $AX^+_{basic} \vdash \varphi \Rightarrow \wedge_{x \in \mathcal{R}(X)}[\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x))$. The result now follows from the argument given for the case $S = \mathcal{R}(X)$. ∎

Theorem 6.2:

If $A \subseteq \{$D3, D10(a),D10(b)$\}$ and $\mathcal{A}$ is the corresponding subset of $\{coh, \geq 1, \leq 1\}$, then $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ is sound and complete for the class $\mathcal{G}^\mathcal{A}$ of GSEMs with signature $\mathcal{S}$ over language $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$.

Proof: The soundness of all the axioms and inference rules with respect to the appropriate class of structures is straightforward except for D2$^+$. To see that D2$^+$ is sound in $\mathcal{G}^\mathcal{A}(\mathcal{S})$, suppose that $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\psi$ is a formula in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ that mentions only $x_1, \ldots, x_k \in \mathcal{R}'(X)$, and there is a value

$x_{k+1} \in \mathcal{R}'(X) - \{x_1, \ldots, x_k\}$. Suppose further that the formula

$$\theta = \quad \varphi \Rightarrow \wedge_{i=1}^{k+1} [\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x_i))$$

is valid in $\mathcal{G}^{\mathcal{A}}(\mathcal{S})$. We want to show that $\varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\psi$ is valid in $\mathcal{G}^{\mathcal{A}}$. Suppose not. Then there is a GSEM $M \in \mathcal{G}^A$ and context $\mathbf{u}$ such that $(M, \mathbf{u}) \models \varphi \wedge \langle \vec{Y} \leftarrow \vec{y} \rangle \psi$. Thus, there must be some outcome $\mathbf{v}^* \in \mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ such that $\mathbf{v}^*$ satisfies ψ. Let $x^* := \mathbf{v}^*[X]$. Since $(M, \mathbf{u}) \models \wedge_{i=1}^{k+1} [\vec{Y} \leftarrow \vec{y}](\psi \Rightarrow (X \neq x_i))$, it cannot be the case that $x^* \in \{x_1, \ldots, x_{k+1}\}$.

Now define a model $M' = (\mathcal{S}, \mathbf{F}')$ with the same signature $\mathcal{S}$ as M, but a modified mapping $\mathbf{F}'$, defined as follows. Let $\mathbf{v}[x/X]$ denote the outcome such that $\mathbf{v}[x/X][X] = x$ and for all $Y \neq X$, $\mathbf{v}[x/X][Y] = \mathbf{v}[Y]$. Define the outcome $\mathbf{v}^r$ (the r indicates "replacing" x^* by x_{k+1}) as $\mathbf{v}^r = \mathbf{v}[x_{k+1}/X]$ if $\mathbf{v}[X] = x^*$, otherwise $\mathbf{v}^r = \mathbf{v}$.

Suppose $X \notin \vec{W}$. Define

$$\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w}) = \{\mathbf{v}^r \mid \mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})\}.$$

For $x \neq x_{k+1}$, define

$$\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w}; X \leftarrow x) = \mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w}; X \leftarrow x).$$

Finally, define

$$\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w}; X \leftarrow x_{k+1}) = \{\mathbf{v}[x_{k+1}/X] \mid \mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})\}.$$

We claim that M' is a counterexample to the supposed validity of θ in $\mathcal{G}^A(\mathcal{S})$. This claim consists of two subclaims: first, that M' does not satisfy θ, and second, that $M' \in \mathcal{G}^A(\mathcal{S})$.

First we argue M' does not satisfy θ. It suffices to show that a formula φ that does not mention the values x, x_{k+1} cannot distinguish M and M'; that is, $M \models \varphi$ iff $M' \models \varphi$. To show this, it suffices to consider atomic causal formulas, since all other causal formulas are Boolean combinations of these. Let $\varphi = [\vec{W} \leftarrow \vec{w}]\gamma$ be an atomic causal formula in $\mathcal{I}$ that does not mention x or x_{k+1}. The claim that $M \models \varphi \Leftrightarrow M' \models \varphi$ follows from a straightforward structural induction on γ in the first two cases defining $\mathbf{F}'$, namely $X \notin \vec{W}$ and $X \in \vec{W}, \vec{w}[X] \neq x_{k+1}$. The third case ($X \in \vec{W}, \vec{w}[X] = x_{k+1}$) is not possible since, by assumption, $[\vec{W} \leftarrow \vec{w}](\gamma)$ does not mention x_{k+1}.

Next we argue that $M' \in \mathcal{G}^A(\mathcal{S})$. M' was defined to have signature $\mathcal{S}$, so it suffices to show that M' satisfies the axioms of A. It is clear from the definition of $\mathbf{F}'$ that $|\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w})| \leq |\mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})|$, where $|\cdot|$ denotes set cardinality. To see this, note that in the second case defining $\mathbf{F}'$, the two outcome sets are identical, whereas in the first and third cases, $\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w})$ is the image of $\mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})$ under the maps $\mathbf{v} \mapsto \mathbf{v}^r$ and $\mathbf{v} \mapsto \mathbf{v}[x_{k+1}/X]$, respectively. Suppose that A includes the property ≤ 1, so M satisfies ≤ 1. Then $|\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w})| \leq |\mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})| \leq 1$, so M' satisfies ≤ 1. It is also clear that if $|\mathbf{F}(\mathbf{u}, \vec{W} \leftarrow \vec{w})| \geq 1$, then $|\mathbf{F}'(\mathbf{u}, \vec{W} \leftarrow \vec{w})| \geq 1$, since the image of a nonempty set is a nonempty set. Hence, if $\geq 1 \in A$, M' satisfies ≥ 1.

It remains to show that if $coh \in A$, then M' satisfies coh. That is, for all interventions $\vec{Z} \leftarrow \vec{z}$, $\vec{T} \leftarrow \vec{t}$, all contexts $\mathbf{u}$, and all assignments $\mathbf{v}'$,

$$\begin{aligned} &\text{if } \mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}) \text{ and } \mathbf{v}'[\vec{T}] = \vec{t}, \\ &\text{then } \mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t}). \end{aligned} \tag{4}$$

This turns out to be the most involved part of the soundness proof.

It suffices to prove (4) for two cases: (a) $\vec{T} \leftarrow \vec{t} = X \leftarrow x$ and (b) $X \notin \vec{T}$. This follows from two facts. First, if (4) holds for $\vec{T_1} \leftarrow \vec{t_1}$ and $\vec{T_2} \leftarrow \vec{t_2}$, it also holds for $\vec{T_1} \leftarrow \vec{t_1}; \vec{T_2} \leftarrow \vec{t_2}$. To see this, suppose that $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$ and $\mathbf{v}[\vec{T_1}; \vec{T_2}] = \vec{t_1}; \vec{t_2}$. Applying (4) to $\vec{T_1} \leftarrow \vec{t_1}$ gives $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T_1} \leftarrow \vec{t_1})$, then applying it to $\vec{T_2} \leftarrow \vec{t_2}$ gives $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T_1} \leftarrow \vec{t_1}; \vec{T_2} \leftarrow \vec{t_2})$. Second, any intervention $\vec{T} \leftarrow \vec{t}$ can be written as $\vec{T_1} \leftarrow \vec{t_1}; \vec{T_2} \leftarrow \vec{t_2}$, where $X \notin \vec{T_1}$, and either $\vec{T_2} = \emptyset$, if $X \notin \vec{T}$; or $\vec{T_2} = \vec{t_2} = X \leftarrow x$, if $X \in \vec{T}$.

We now prove (4) for these two cases. Suppose that $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$ and $\mathbf{v}'[\vec{T}] = \vec{t}$. We want to show that $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$.

First suppose that $X \notin \vec{Z}$. Then, by definition of $\mathbf{F}'$, there is some $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$ with $\mathbf{v}' = \mathbf{v}^r$. For case (a), if $x \neq x_{k+1}$, then $\mathbf{v}[X] \neq x^*$; otherwise $\mathbf{v}'[X] = \mathbf{v}^r[X] = x_{k+1}$, but $\mathbf{v}'[X] = x \neq x_{k+1}$, a contradiction. It follows $\mathbf{v}' = \mathbf{v}$, since $\mathbf{v}^r[X] = \mathbf{v}[X]$ when $\mathbf{v}[X] \neq x^*$. Thus $\mathbf{v}[X] = x$, and by the coherence of M, $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x)$. But by definition of $\mathbf{F}'$, $\mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x) = \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x)$. Thus $\mathbf{v}' = \mathbf{v} \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x)$, the desired conclusion. And if $x = x_{k+1}$, then $\mathbf{v}' = \mathbf{v}^r = \mathbf{v} = \mathbf{v}[x_{k+1}/X]$. But by definition of $\mathbf{F}'$, $\mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x_{k+1}) = \{\mathbf{v}_1[x_{k+1}/X] \mid \mathbf{v}_1 \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z})\}$. In particular, $\mathbf{v}' = \mathbf{v}[x_{k+1}/X] \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; X \leftarrow x_{k+1})$ as desired.

For case (b), if $X \notin \vec{T}$, then $\mathbf{v}[\vec{T}] = \mathbf{v}^r[\vec{T}] = \vec{t}$, so by the coherence of M, $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$. Thus by definition of $\mathbf{F}'$, $\mathbf{v}^r \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$. But $\mathbf{v}' = \mathbf{v}^r$, so $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$, as desired.

Now suppose that $X \in \vec{Z}$ and $x_{\vec{z}} = \vec{z}[X]$. For case (a), we have $\mathbf{v}'[X] = x_{\vec{z}}$, by effectiveness, and $\mathbf{v}'[X] = x$ by assumption, so $x_{\vec{z}} = x$. Hence, $\vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t} = \vec{Z} \leftarrow \vec{z}; X \leftarrow x = \vec{Z} \leftarrow \vec{z}$. The desired conclusion $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$ trivially follows from the assumption $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$.

For case (b), write $\vec{Z} \leftarrow \vec{z} = \vec{Z}' \leftarrow \vec{z}'; X \leftarrow x$, where $X \notin \vec{Z}'$. If $x \neq x_{k+1}$, then by definition of $\mathbf{F}'$, we have that $\mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}) = \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$. Hence $\mathbf{v}' \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z})$, and by the coherence of M, $\mathbf{v}' \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t}) = \mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; X \leftarrow x; \vec{T} \leftarrow \vec{t}) = \mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t}; X \leftarrow x)$. But again, by definition of $\mathbf{F}'$, $\mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t}; X \leftarrow x) = \mathbf{F}'(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t}; X \leftarrow x) = \mathbf{F}'(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; X \leftarrow x; \vec{T} \leftarrow \vec{t}) = \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$.

The desired conclusion $\mathbf{v}' \in \mathbf{F}(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$ follows. On the other hand, if $x = x_{k+1}$, then by definition of $\mathbf{F}'$, $\mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}) = \{\mathbf{v}[x_{k+1}/X] \mid \mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}')\}$. Thus, there is some $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}')$ with $\mathbf{v}' = \mathbf{v}[x_{k+1}/X]$. Further, $\mathbf{v}[T] = \mathbf{v}'[T] = \vec{t}$, since $\mathbf{v}[x_{k+1}/X]$ differs from $\mathbf{v}'$ only at X, if at all, and $X \notin \vec{T}$. So, by the coherence of M, $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t})$. Thus again by definition of $\mathbf{F}'$, $\mathbf{v}' = \mathbf{v}[x_{k+1}/X] \in \mathbf{F}'(\mathbf{u}, \vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t}; X \leftarrow x_{k+1})$. But $\vec{Z}' \leftarrow \vec{z}'; \vec{T} \leftarrow \vec{t}; X \leftarrow x_{k+1} = \vec{Z}' \leftarrow \vec{z}'; X \leftarrow x_{k+1}; \vec{T} \leftarrow \vec{t} = \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t}$, so we have $\mathbf{v}' \in \mathbf{F}'(\mathbf{u}, \vec{Z} \leftarrow \vec{z}; \vec{T} \leftarrow \vec{t})$, as desired.

For completeness, we proceed in the same spirit as the proof of Theorem 5.2. But there are a number of new subtleties. In Theorem 5.2, a maximal consistent set C was used to construct a causal model M. We defined the function $\mathbf{F}$ in M by taking $\mathbf{v} \in \mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ if $\langle \vec{Y} \leftarrow \vec{y} \rangle (\mathcal{V} = \mathbf{v}) \in C$. But if there are infinitely many variables, $\mathcal{V} = \mathbf{v}$ is an infinitary formula, so is not in the language. We deal with this by saying that the outcome $\mathbf{v}$ is in $\mathbf{F}(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ if $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{X} = \mathbf{v}[\vec{X}]) \in C$ for all finite $\vec{X} \subseteq \vec{W}$.

But there is a more serious problem. In the proof of Theorem 5.2, we showed that (for the model M constructed from the maximal consistent set C), we have that $(M, \mathbf{u}) \models \varphi$ iff $\varphi \in C$. But suppose that X is a variable such that $\mathcal{R}(X) = \{x_1, x_2, x_3, \ldots\}$ is countable, $\mathcal{R}'(X) = \mathcal{R}(X)$, and $\mathcal{Y} \leftarrow y \in \mathcal{I}' \cap \mathcal{I}$. It is not hard to see that the set $C' = \{\langle Y \leftarrow y \rangle true, [Y \leftarrow y](X \neq x_1), [Y \leftarrow y](X \neq x_2), \ldots\}$ is consistent with AX^*_{basic} (since every finite subset of this set is obviously consistent). Hence C' can be extended to a maximal consistent set C. But there is no model M such that $(M, u) \models \varphi$ for all formulas $\varphi \in C'$. Thus, we have to restrict the set of maximal consistent sets that we consider.

Definition 6.3: *A* conjunctive formula *is a conjunction of formulas of the form $X = x$ and $X \neq x$. (The formula $true$ is viewed as conjunctive, since it is an empty conjunction.) A set C of formulas in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ is* acceptable for $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ with respect to $Z \in \vec{W}$, *where φ is a conjunctive formula, if there is some $z \in \mathcal{R}'(Z)$ and a conjunctive formula ψ such that every conjunct of φ is a conjunct of ψ, $Z = z$ is a conjunct of ψ for some $z \in \mathcal{R}'(Z)$, and $\langle \vec{Y} \leftarrow \vec{y} \rangle \psi \in C$. C is* acceptable *if C is acceptable for every formula of the form $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$ such that φ is conjunctive formula and $Z \in \vec{W}$.*

The set C' above is not acceptable for $\langle Y \leftarrow y \rangle true$ and X, and cannot be extended to an acceptable consistent set. Our proof technique involves constructing a model from an acceptable maximal consistent set. So we must show that every consistent formula is included in an acceptable maximal consistent set. The next lemma gives the key step for doing this.

Lemma 6.4: *If C is a finite subset of $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$, and $X \in \vec{W}$, then there exists a formula $\psi \in \mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S}) - C$ such that $C \cup \{\psi\}$ is consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ and acceptable with respect to $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ and X.*

Proof: Let C be a finite subset of $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$. Let θ be the conjunction of the formulas in C, and let $\{x_1, \ldots, x_k\}$ be the values in $\mathcal{R}'(X)$ mentioned in θ. There are two cases. If there exists a value $x_{k+1} \in \mathcal{R}'(X) - \{x_1, \ldots, x_k\}$, then we claim that one of the formulas $\theta \wedge \langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi \wedge X = x_i)$, $i = 1, \ldots, k+1$ must be consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$. For if not, then $\theta \Rightarrow [\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow X \neq x_i)$ must be provable in $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ for $i = 1, \ldots, k+1$. By D2$^+$, so is $\theta \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\varphi$. But since, by assumption, $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ is one of the conjuncts of θ, $\theta \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ is also provable, so θ is not consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, contradicting the consistency of C.

If there does not exist a value $x_{k+1} \in \mathcal{R}'(X) - \{x_1, \ldots, x_k\}$, then $\mathcal{R}'(X) = \{x_1, \ldots, x_k\}$, so by D2$^+$, $[\vec{Y} \leftarrow \vec{y}](X = x_1 \vee \ldots \vee X = x_k)$ is provable in $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$. Thus, so is $\theta \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi \wedge X = x_1 \vee \ldots \vee X = x_k)$. It easily follows that $\theta \wedge \langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi \wedge X = x_i)$ must be consistent for some $i = 1, \ldots, k$. In either case, we can add a formula of the form $\langle \vec{Y} \leftarrow \vec{y} \rangle (\varphi \wedge X = x)$ to C while maintaining its consistency. ∎

We can now prove completeness. Given a formula φ consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, we construct a maximal acceptable set C consistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$ containing φ as follows. Let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an enumeration of the formulas in $\mathcal{L}_{\vec{W},\mathcal{R}',\mathcal{I}'}(\mathcal{S})$ such that $\sigma_0 = \varphi$ and let $X_0, X_1, X_2, \ldots$ be an enumeration of the variables in $\vec{W}$. It is well known that there is a bijection b from $I\!N$ to $I\!N \times I\!N$ such that if $b(n) = (n_1, n_2)$, then $n_1 \leq n$. We construct a sequence of sets $C_0, C_1, C_2, \ldots$ such that $C_0 = \{\varphi\}$, $C_k \subseteq C_{k+1}$, and (a) either $\sigma_k \in C_k$ or $C_k \cup \{\sigma_k\}$ is inconsistent with $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}')$, (b) C_k is consistent, (c) if $b(k) = (k_1, k_2)$ and σ_{k_1} has the form $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$, then C_{k+1} is acceptable with respect to σ_{k_1} and X_{k_2}. We construct the sequence inductively. Given C_k, then we add σ_{k+1} to C_k if $C_k \cup \{\sigma_{k+1}\}$ is consistent. In addition, if σ_{k_1} has the form $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$, then we apply Lemma 6.4 to add a formula if necessary to make C_{k+1} acceptable with respect to σ_{k_1} and X_{k_2}.

Let $C = \cup_{k=0}^{\infty} C_k$. Clearly C contains φ. It is consistent, since if not, some finite subset of C must be inconsistent. But this finite subset must be contained in C_k for some k, and C_k is consistent, by construction. Finally, C is acceptable. For suppose that $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$ and $X \in \vec{W}$. There must exist k_1 and k_2 such that $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi = \sigma_{k_1}$ and $X = X_{k_2}$. Let $k = b^{-1}(k_1, k_2)$. Since $\sigma_{k_1} = \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$, it must already be in C_{k_1} (since it would be added in the construction of C_{k_1} if it was not already in $C_{k_1 - 1}$). By the choice of b, $k_1 \leq k$, so $\sigma_{k_1} \in C_k$. By construction, C_{k+1} is acceptable with respect to $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ and X_{k_2}, hence so is C.

We now construct a model M^C with signature $\mathcal{S}$. For interventions $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$, we take $\mathbf{F}(\vec{Y} \leftarrow \vec{y}, \mathbf{u}) = \{\mathbf{v} \mid$ for all finite subsets $\vec{X} \subseteq \vec{W}$, $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{X} = \mathbf{v}[\vec{X}]) \in C\}$. Now we still need to define $\mathbf{F}$ on interventions in $\mathcal{I} - \mathcal{I}'$. Let $\mathbf{v}^*$ be a fixed assignment. For $I \in \mathcal{I} - \mathcal{I}'$, define $\mathbf{F}(I, \mathbf{u}) = \{\mathbf{v}^*\}$. We claim that $M^C \in \mathcal{G}^{\mathcal{A}}$ and that $\psi \in C$ iff, for all contexts $\mathbf{u}$, we have $(M^C, \mathbf{u}) \models \psi$. If these claims hold, then we have produced a model M^C of φ that satisfies the properties in $\mathcal{A}$, completing the consistency proof.

First we show that for all contexts $\mathbf{u}$ and formulas $\varphi \in \mathcal{L}_{\vec{W}, \mathcal{R}', \mathcal{I}'}(\mathcal{S})$, we have

$$\varphi \in C \Leftrightarrow (M^C, \mathbf{u}) \models \varphi. \tag{5}$$

Fix a context $\mathbf{u} \in \mathcal{U}$. It suffices to show (5) for the dual basic causal formulas $\varphi = \langle \vec{Y} \leftarrow \vec{y} \rangle \rho$. This is because it is easy to see that every causal formula is a Boolean combination of dual basic causal formulas. The result follows for all causal formulas by a straightforward structural induction.

Now ρ is a Boolean combination of primitive events of the form $X = x$. We can write ρ in DNF. That is, there exists $k \geq 0$ and conjunctive events d_i for $1 \leq i \leq k$ such that $\rho \Leftrightarrow \bigvee_{1 \leq i \leq k} d_i$ is a propositional tautology.

By Lemma A.2(b), we have $AX_{basic} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \rho \Leftrightarrow \bigvee_{1 \leq i \leq k} \langle \vec{Y} \leftarrow \vec{y} \rangle d_i$. So, again by structural induction, it suffices to show (5) for formulas φ of the form $\varphi = \langle \vec{Y} \leftarrow \vec{y} \rangle \gamma$, where γ is a conjunctive formula.

We now do this. For the "only if" direction, suppose that $(M^C, \mathbf{u}) \models \langle \vec{Y} \leftarrow \vec{y} \rangle \gamma$. Then there is some assignment $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ such that $\mathbf{v} \models \gamma$. By definition of M^C, for all finite sets $\vec{X} \subseteq \vec{W}$, we have $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{X} = \mathbf{v}[\vec{X}]) \in C$. Let $\vec{Z}$ be the (finite) set of variables mentioned in γ; then, in particular, $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{Z} = \mathbf{v}[\vec{Z}]) \in C$. Using D0, D1, D7, and D8, we get that $AX^*_{basic,A}(\mathcal{S}, \vec{W}, \mathcal{R}', \mathcal{I}') \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{Z} = \mathbf{v}[\vec{Z}]) \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$. Since C is a maximal consistent set, we must have $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{Z} = \mathbf{v}[\vec{Z}]) \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$. The fact that C is a maximal consistent set implies that it is closed under implication, so since $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{Z} = \mathbf{v}[\vec{Z}]) \in C$, we must have $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \in C$, as desired.

Conversely, suppose that $\langle \vec{Y} \leftarrow \vec{y} \rangle \gamma \in C$. Fix an enumeration $X_i, i \geq 1$, of the set of named variables $\vec{W}$. For $i \geq 1$, let $\vec{Z}_i = \{X_1, \ldots, X_i\}$. Let $\varphi_0 = \gamma$. Since C is acceptable, we can find formulas $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_1, \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_2, \ldots \in C$ such that for $i \geq 1$, every conjunct of φ_{i-1} is a conjunct of φ_i, and for some $x_i \in \mathcal{R}(X_i)$, $X_i = x_i$ is a conjunct of φ_i. Let $\overrightarrow{w}$ be any assignment to the variables $\mathcal{V} - \vec{W}$, and define $\mathbf{v} = \{(X_i, x_i) \mid i \geq 1\} \cup ((\mathcal{V} - \vec{W}) \leftarrow \overrightarrow{w})$. Since every conjunct of γ is a conjunct of each φ_i, we have $\mathbf{v} \models \gamma$. But every finite subset S of $\vec{W}$ is contained in $\vec{Z}_i$ for some i. Thus, $AX^*_{basic} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi_i \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle (S = \mathbf{v}[S])$ for all i sufficiently large. It follows from the maximality of C that for all finite subsets S of $\vec{W}$, $\langle \vec{Y} \leftarrow \vec{y} \rangle (S = \mathbf{v}[S]) \in C$. By definition of M^C, it follows $\mathbf{v} \in M^C(\vec{Y} \leftarrow \vec{y}, \mathbf{u})$. But then by definition of $\models$, we have $(M^C, \mathbf{u}) \models \gamma$, as desired. It remains to show that $M^C \in \mathcal{G}^{\mathcal{A}}$, that is, that M^C satisfies all the properties of $\mathcal{A}$. We proceed one property at a time. Suppose that $\geq 1 \in \mathcal{A}$. Then A contains D10(a). Fix an intervention $\vec{Y} \leftarrow \vec{y}$ and context $\mathbf{u} \in \mathcal{R}(\mathcal{U})$. If $\vec{Y} \leftarrow \vec{y} \notin \mathcal{I}'$, then $M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}) = \{v^*\}$, so the intervention $\vec{Y} \leftarrow \vec{y}$ has exactly one outcome. If $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$, then by D10(a), we have $\langle \vec{Y} \leftarrow \vec{y} \rangle true \in C$. Since C is acceptable, we can use the construction in the proof of the truth lemma to find $\mathbf{v}$ such that $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{X} = \mathbf{v}[\vec{X}]) \in C$ for all finite subsets $\vec{X}$ of $\vec{W}$. Then, by the definition of M^C, $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$, so the intervention $\vec{Y} \leftarrow \vec{y}$ has at least one outcome.

Next suppose that $\leq 1 \in \mathcal{A}$. Then A contains D10(b). Again, the case $\vec{Y} \leftarrow \vec{y} \notin \mathcal{I}'$ is immediate. Suppose that $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$. If $\mathbf{v}, \mathbf{v}' \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$. then by definition of M^C, we must have $\langle \vec{Y} \leftarrow \vec{y} \rangle (X = \mathbf{v}[X]) \in C$ for all $X \in \mathcal{V}$. Suppose for contradiction that $\mathbf{v} \neq \mathbf{v}'$. Then there is a variable Z such that $\mathbf{v}[Z] \neq \mathbf{v}'[Z]$. Since $\mathbf{v}' \in M^C(\vec{Y} \leftarrow \vec{y}, \mathbf{u})$, we must have $\langle \vec{Y} \leftarrow \vec{y} \rangle (Z = \mathbf{v}'[Z]) \in C$. By D10(b), for all $\psi \in \mathcal{L}_{\vec{W}, \mathcal{R}', \mathcal{I}'}(\mathcal{S})$, we have $\langle \vec{Y} \leftarrow \vec{y} \rangle \psi \Rightarrow [\vec{Y} \leftarrow \vec{y}] \psi \in C$. Thus, $[\vec{Y} \leftarrow \vec{y}](Z = \mathbf{v}'[Z]) \in C$. Then by D2, we have $[\vec{Y} \leftarrow \vec{y}](Z \neq \mathbf{v}[Z]) \in C$, and so $[\vec{Y} \leftarrow \vec{y}](Z = \mathbf{v}[Z]) \wedge [\vec{Y} \leftarrow \vec{y}](Z = \mathbf{v}'[Z]) \in C$. It follows from Lemma A.2 that $[\vec{Y} \leftarrow \vec{y}](Z = \mathbf{v}[Z] \wedge Z = \mathbf{v}'[Z]) \in C$, so $[\vec{Y} \leftarrow \vec{y}] false \in C$. But since $\langle \vec{Y} \leftarrow \vec{y} \rangle true \in C$, and $\langle \vec{Y} \leftarrow \vec{y} \rangle true$ is just an abbreviation for $\neg [\vec{Y} \leftarrow \vec{y}] false$, C is inconsistent, a contradiction. Hence $\mathbf{v} = \mathbf{v}'$, and so the intervention $\vec{Y} \leftarrow \vec{y}$ has at most one outcome.

Finally, suppose that $coh \in \mathcal{A}$. Then A contains D3. Fix an intervention $\vec{Y} \leftarrow \vec{y}$ and context $\mathbf{u} \in \mathcal{R}(U)$. Suppose that $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$ and that $\mathbf{v}[W] = w$. We want to show that $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}; W \leftarrow w)$. First suppose that $\vec{Y} \leftarrow \vec{y} \notin \mathcal{I}'$. Then (because $\mathcal{I}'$ is closed under finite differences with $\mathcal{I}$), we have $\vec{Y} \leftarrow \vec{y}; W \leftarrow w \notin \mathcal{I}'$. By the definition of M^C, we have $M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}) = M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}; W \leftarrow w) = \{\mathbf{v}^*\}$. Thus, if $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y})$, then $\mathbf{v} = \mathbf{v}^*$ so $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}; W \leftarrow w)$, as desired. Next, suppose that $\vec{Y} \leftarrow \vec{y} \in \mathcal{I}'$. Then by the definition of M^C, we have $\langle \vec{Y} \leftarrow \vec{y} \rangle (\vec{X} = \mathbf{v}[\vec{X}] \wedge W = w) \in C$ for all finite subsets $\vec{X}$ of $\vec{W}$. By D3, it follows that $\langle \vec{Y} \leftarrow \vec{y}; W \leftarrow w \rangle (\vec{X} = \mathbf{v}[\vec{X}]) \in C$ for all finite subsets $\vec{X}$ of $\vec{W}$. But, by the definition of M^C, this implies that $\mathbf{v} \in M^C(\mathbf{u}, \vec{Y} \leftarrow \vec{y}; W \leftarrow w)$, as desired. ∎

B Additional results

We want to make precise and prove the claim in the main text that Halpern's version of D9,

$$\langle \vec{Y} \leftarrow \vec{y} \rangle true \wedge (\bigvee_{x \in \mathcal{R}(X)} [\vec{Y} \leftarrow \vec{y}](X = x)),$$

which we call HD9, is equivalent to ours,

$$\langle \vec{Y} \leftarrow \vec{y} \rangle true \wedge (\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi),$$

in the presence of the other axioms. It turns out that the only other axioms we need are D0, D1, D2, D7, and D8. So let AX_{EQ} be the axiomatization consisting of these axioms and MP, let $AX_{\text{HD9}} = AX_{EQ} \cup \{\text{HD9}\}$, and let $AX_{\text{D9}} = AX_{EQ} \cup \{\text{D9}\}$.

Theorem B.1: *For a finite signature $\mathcal{S}$, AX_{HD9} is equivalent to AX_{D9}.*

Proof: Since $\mathcal{S}$ is finite, without loss of generality we can assume that $\mathcal{V} = \{X_1, X_2, \ldots, X_n\}$ for some n. It clearly suffices to show that each instance of D9 is provable from AX_{HD9} and that each instance of HD9 is provable from AX_{D9}. For the first claim, we show that

$$AX_{\text{HD9}} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle true \wedge (\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi).$$

Since one conjunct of HD9 is the formula $\langle \vec{Y} \leftarrow \vec{y} \rangle true$, it follows that

$$AX_{\text{HD9}} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle true.$$

(Formally, we are using D0, MP, and the observation that $a \wedge b \Rightarrow a$ is a propositional tautology; in the sequel, we omit the details of such obvious propositional reasoning.) Thus, it suffices to show

$$AX_{\text{HD9}} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi.$$

Using an argument almost identical to that given to prove (3), we can show that

$$AX_{EQ} \vdash [\vec{Y} \leftarrow \vec{y}](\varphi \Leftrightarrow \bigvee_{\mathbf{v} \models \varphi} \mathcal{V} = \mathbf{v}); \tag{6}$$

we leave details to the reader. (Note that our argument for (3) used only the axioms of AX_{EQ}.)

From D0, D7, D8, and MP, using standard modal logic arguments, it follows

$$AX_{EQ} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \wedge [\vec{Y} \leftarrow \vec{y}](\varphi \Rightarrow \psi) \Rightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle \psi. \tag{7}$$

Combining (6) and (7), it follows that $AX_{EQ} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Leftrightarrow \langle \vec{Y} \leftarrow \vec{y} \rangle (\bigvee_{\mathbf{v} \models \varphi} \mathcal{V} = \mathbf{v})$. Applying Lemma A.2, it follows that

$$AX_{EQ} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Leftrightarrow \bigvee_{\mathbf{v} \models \varphi} \langle \vec{Y} \leftarrow \vec{y} \rangle (\mathcal{V} = \mathbf{v}). \tag{8}$$

Taking the conjunction over $\mathcal{V}$ of the second conjunct of HD9 gives

$$AX_{HD9} \vdash \bigwedge_{X \in \mathcal{V}} \bigvee_{x \in \mathcal{R}(X)} [\vec{Y} \leftarrow \vec{y}](X = x).$$

Then using D0 to distribute and over or and then applying Lemma A.2 gives

$$AX_{HD9} \vdash \bigvee_{\mathbf{v} \in \mathcal{R}(\mathcal{V})} [\vec{Y} \leftarrow \vec{y}](\mathcal{V} = \mathbf{v}).$$

If $\mathbf{v} \models \neg\varphi$, then $\mathcal{V} = \mathbf{v} \Rightarrow \neg\varphi$ is a propositional tautology. Thus, using D7 and D8, we have that

$$AX_{HD9} \vdash [\vec{Y} \leftarrow \vec{y}](\mathcal{V} = \mathbf{v}) \Rightarrow [\vec{Y} \leftarrow \vec{y}]\neg\varphi.$$

Taking the contrapositive and using the fact that $\langle \vec{Y} \leftarrow \vec{y} \rangle \varphi$ is an abbreviation for $\neg[\vec{Y} \leftarrow \vec{y}]\neg\varphi$, we get that

$$AX_{HD9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow \neg[\vec{Y} \leftarrow \vec{y}](\mathcal{V} = \mathbf{v}).$$

Since this is true for all $\mathbf{v}$ such that $\mathbf{v} \models \neg\varphi$, it follows from (8) that

$$AX_{HD9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow \bigvee_{\mathbf{v} \models \varphi} [\vec{Y} \leftarrow \vec{y}](\mathcal{V} = \mathbf{v}).$$

If $\mathbf{v} \models \varphi$, then standard arguments using D7 and D8 show that

$$AX_{HD9} \vdash [\vec{Y} \leftarrow \vec{y}](\mathcal{V} = \mathbf{v}) \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi.$$

It follows

$$AX_{HD9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Rightarrow [\vec{Y} \leftarrow \vec{y}]\varphi,$$

as desired.

For the second claim, we show that $AX_{D9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle true \wedge (\bigvee_{x \in \mathcal{R}(X)} [\vec{Y} \leftarrow \vec{y}](X = x))$. As before, $AX_{D9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle true$, so it suffices to show $AX_{D9} \vdash \bigvee_{x \in \mathcal{R}(X)} [\vec{Y} \leftarrow \vec{y}](X = x)$. Since AX_{D9} includes D2, $AX_{D9} \vdash [\vec{Y} \leftarrow \vec{y}](\bigvee_{x \in \mathcal{R}(X)} (X = x))$. Combining this with $AX_{D9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle true$ using (7) gives $AX_{D9} \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle (\bigvee_{x \in \mathcal{R}(X)} (X = x))$. Applying Lemma A.2, it follows $AX_{D9} \vdash \bigvee_{x \in \mathcal{R}(X)} \langle \vec{Y} \leftarrow \vec{y} \rangle (X = x)$. But then by D9, $AX_{D9} \vdash \bigvee_{x \in \mathcal{R}(X)} [\vec{Y} \leftarrow \vec{y}](X = x)$, as desired. ∎

Theorem B.2: *Let AX^- consist of D0, D2, D3, D6, D7, D8, D10, and MP. Then AX^- implies D5. That is,*

$$\begin{aligned} AX^- \vdash (&\langle \vec{X} \leftarrow \vec{x}; Y \leftarrow y \rangle (W = w \wedge \vec{Z} = \vec{z}) \\ &\wedge \langle \vec{X} \leftarrow \vec{x}; W \leftarrow w \rangle (Y = y \wedge \vec{Z} = \vec{z})) \\ &\Rightarrow \langle \vec{X} \leftarrow \vec{x} \rangle (W = w \wedge Y = y \wedge \vec{Z} = \vec{z}. \end{aligned} \tag{9}$$

Proof: Using D10(a) and D10(b), we get that $AX^- \vdash \langle \vec{Y} \leftarrow \vec{y} \rangle \varphi \Leftrightarrow [\vec{Y} \leftarrow \vec{y}]\varphi$. Hence, by D0, it suffices to prove the propositionally equivalent statement

$$\begin{aligned} AX^- \vdash (&[\vec{X} \leftarrow \vec{x}; Y \leftarrow y](W = w \wedge \vec{Z} = \vec{z}) \\ &\wedge [\vec{X} \leftarrow \vec{x}; W \leftarrow w](Y = y \wedge \vec{Z} = \vec{z})) \\ &\Rightarrow [\vec{X} \leftarrow \vec{x}](W = w \wedge Y = y \wedge \vec{Z} = \vec{z}). \end{aligned} \tag{10}$$

Assume $[\vec{X} \leftarrow \vec{x}; Y \leftarrow y](W = w \wedge \vec{Z} = \vec{z}) \wedge [\vec{X} \leftarrow \vec{x}; W \leftarrow w](Y = y \wedge \vec{Z} = \vec{z})$. First observe that D6 implies $\neg(Y \rightsquigarrow W) \vee \neg(W \rightsquigarrow Y))$. Assume without loss of generality that $\neg(Y \rightsquigarrow W)$. We will use Lemma A.2 frequently in this proof, to justify treating $[\vec{T} \leftarrow \vec{t}](\varphi \wedge \psi)$ as $[\vec{T} \leftarrow \vec{t}]\varphi \wedge [\vec{T} \leftarrow \vec{t}]\psi$ and vice versa. Thus from $[\vec{X} \leftarrow \vec{x}; Y \leftarrow$

$y](W \leftarrow w \wedge \vec{Z} = \vec{z})$ we get $[\vec{X} \leftarrow \vec{x}; Y \leftarrow y](W \leftarrow w)$. Then $\neg(Y \rightsquigarrow W)$ implies $[\vec{X} \leftarrow \vec{x}](W \leftarrow w)$.

Second, D2 implies that $\vec{Z}$ must take on some value in the outcome resulting from the intervention $\vec{X} \leftarrow \vec{x}$, that is, $\bigvee_{\vec{z} \in \mathcal{R}(\vec{Z})} \vec{Z} = \vec{z}$. To see this, take the conjunction of instances of D2 over $Z \in \vec{Z}$, that is, $\bigwedge_{Z \in \vec{Z}} [\vec{X} \leftarrow \vec{x}](\bigvee_{z \in \mathcal{R}(Z)} Z = z)$. Then applying Lemma A.2(a) and distributing and over or, we have $[\vec{X} \leftarrow \vec{x}](\bigvee_{\vec{z}' \in \mathcal{R}(\vec{Z})} \vec{Z} = \vec{z}')$, or equivalently by D10, $\langle \vec{X} \leftarrow \vec{x} \rangle (\bigvee_{\vec{z}' \in \mathcal{R}(\vec{Z})} \vec{Z} = \vec{z}')$. Then applying Lemma A.2(b) yields $\bigvee_{\vec{z}' \in \mathcal{R}(\vec{Z})} \langle \vec{X} \leftarrow \vec{x} \rangle (\vec{Z} = \vec{z}')$.

Suppose without loss of generality that $\langle \vec{X} \leftarrow \vec{x} \rangle (\vec{Z} = \vec{z}^*)$, or equivalently by D10, $[\vec{X} \leftarrow \vec{x}](\vec{Z} = \vec{z}^*)$. But then $[\vec{X} \leftarrow \vec{x}](\vec{Z} = \vec{z}^* \wedge W = w)$. By D3, this implies $[\vec{X} \leftarrow \vec{x}; W \leftarrow w](\vec{Z} = \vec{z}^*)$. However, we assumed $[\vec{X} \leftarrow \vec{x}; W \leftarrow w](Y = y \wedge \vec{Z} = \vec{z})$. Thus $\vec{z}^* = \vec{z}$, and so $[\vec{X} \leftarrow \vec{x}](\vec{Z} = \vec{z} \wedge W = w)$.

Finally it remains to show $[\vec{X} \leftarrow \vec{x}](Y = y)$. Using D2 again, there must be y^* such that $[\vec{X} \leftarrow \vec{x}](W = w \wedge \vec{Z} = \vec{z} \wedge Y = y*)$. Again, by D3, this implies $[\vec{X} \leftarrow \vec{x}; W \leftarrow w](\vec{Z} \leftarrow \vec{z} \wedge Y \leftarrow y^*)$. But we assumed $[\vec{X} \leftarrow \vec{x}; W \leftarrow w](Y = y \wedge \vec{Z} = \vec{z})$. It follows $y^* = y$. Thus $[\vec{X} \leftarrow \vec{x}](W = w \wedge Y = y \wedge \vec{Z} = \vec{z})$ as desired. ∎

C Shell game example

As mentioned in Section 5, the following example, which is Example 3.6 of (Anonymous 2021), is a GSEM over a finite signature $\mathcal{S}$ which satisfies the axioms of $AX^+(\mathcal{S})$, but is not equivalent to a SEM.

Example C.1: Suppose that Suzy is playing a shell game with two shells. One of the shells conceals a dollar; the other shell is empty. Suzy can choose to flip over a shell. If she does, the house flips over the other shell. If Suzy picks shell 1, which hides the dollar, she wins the dollar; otherwise she wins nothing. This example can be modeled by a GSEM $M_{shell} = (\mathcal{S}_{shell}, \mathbf{F}_{shell})$ with two binary endogenous variables S_1, S_2 describing whether shell 1 is flipped over and shell 2 is flipped over, respectively, and a binary endogenous variable Z describing the change in Suzy's winnings. (The GSEM also has a trivial exogenous variable whose range has size 1, so that there is only one context $\mathbf{u}$.) That defines $\mathcal{U}_{shell}, \mathcal{V}_{shell}$, and $\mathcal{R}_{shell}$; we set $\mathcal{I}_{shell} = \{S_1 \leftarrow 1, \mathcal{S}_2 \leftarrow 1\}$; finally, $\mathbf{F}_{shell}$ is defined as follows:

$$\begin{aligned}
\mathbf{F}(\mathbf{u}, S_1 \leftarrow 1) &= M_{shell}(\mathbf{u}, S_1 \leftarrow 1) \\
&= \{(S_1 = 1, S_2 = 1, Z = 1)\} \\
\mathbf{F}(\mathbf{u}, S_2 \leftarrow 1) &= M_{shell}(\mathbf{u}, S_2 \leftarrow 1) \\
&= \{(S_1 = 1, S_2 = 1, Z = 0)\}.
\end{aligned}$$

As shown in (Anonymous 2021), M_{shell} is a GSEM where all the axioms in $AX^+(\mathcal{S}_{shell})$ are valid (note that in the case of D9, validity is vacuous, since none of the relevant interventions is in $\mathcal{L}(\mathcal{S}_{shell})$. However, M_{shell} is not equivalent to a SEM; no SEM M' over $\mathcal{S}_{shell}$ can have the outcomes $M_{shell}(\mathbf{u}, S_1 \leftarrow 1) = \{(S_1 = 1, S_2 = 1, Z = 1)\}$ and $M_{shell}(\mathbf{u}, S_1 \leftarrow 2) = \{(S_1 = 1, S_2 = 1, Z = 0)\}$. This is because in a SEM, the value of Z would be specified by a structural equation $Z = \mathcal{F}_Z(\mathcal{U}, S_1, S_2)$. This cannot be the case here, since there are two outcomes having $S_1 = S_2 = 1$, but with different values of Z. ∎