A Class of Nonharmonic Fourier Series
Author(s): R. J. Duffin and A. C. Schaeffer
Reviewed work(s):
Source: *Transactions of the American Mathematical Society*, Vol. 72, No. 2 (Mar., 1952), pp. 341-366
Published by: American Mathematical Society
Stable URL: http://www.jstor.org/stable/1990760
Accessed: 19/02/2013 17:25

A CLASS OF NONHARMONIC FOURIER SERIES[1]

BY

R. J. DUFFIN AND A. C. SCHAEFFER

1. **Introduction.** A sequence $\{\lambda_n\}$, $n=0, \pm 1, \pm 2, \cdots$, of real or complex numbers we shall say has *uniform density* 1 if there are constants L and δ such that $|\lambda_n - n| \leqq L$ and $|\lambda_n - \lambda_m| \geqq \delta > 0$ for $n \neq m$. This is a more restrictive notion than density, for, considering only those λ_n for which $n>0$, it is clear that a sequence of uniform density 1 has a density as defined by Pólya equal to 1, but the converse is not true. Sequences of uniform density d are defined in a later part of the present paper for any $d>0$. If $f(z)$ is an entire function of exponential type γ, $0 \leqq \gamma < \pi$, that is,

$$f(z) = O(e^{\gamma |z|})$$

uniformly in all directions as $|z| \to \infty$, then $f(z)$ is completely determined by its values at any sequence of uniform density 1. Some properties of entire functions of exponential type extend in a natural way from a sequence of uniform density to all points of the real axis or of a strip parallel to the real axis. For example, the authors have shown [6] that if an entire function of exponential type γ, $0 \leqq \gamma < \pi$, is uniformly bounded at a sequence of uniform density 1, then it is uniformly bounded on the entire real axis. It also has a bound in every strip parallel to the real axis. This result was applied to questions concerning the coefficients of power series.

In the present paper a further property of sequences of uniform density is proved. It is shown that if $f(z)$ is an entire function of exponential type γ, $0 \leqq \gamma < \pi$, belonging to $L_2(-\infty, \infty)$ on the real axis and $\{\lambda_n\}$ is a sequence of uniform density 1, then the ratio $\{\sum_n |f(\lambda_n)|^2\} / \int_{-\infty}^{\infty} |f(x)|^2 dx$ has positive upper and lower bounds independent of the function. An essentially equivalent statement is that if $g(t) \in L_2(-\gamma, \gamma)$ where $0 < \gamma < \pi$, and $\{\lambda_n\}$ is a sequence of uniform density 1, then there are positive constants A and B independent of the function $g(t)$ such that

$$A \leqq \frac{\frac{1}{2\pi} \sum_n \left| \int_{-\gamma}^{\gamma} g(t) e^{i\lambda_n t} dt \right|^2}{\int_{-\gamma}^{\gamma} |g(t)|^2 dt} \leqq B. \tag{1}$$

Presented to the International Congress of Mathematicians, August 31, 1950; received by the editors August 1, 1951.

(1) Part of the work on this paper was done while the authors were at Purdue University, and another part while they were under contracts with the Office of Naval Research, the Office of Ordnance Research, and the Flight Research Laboratory.

There are other sequences for which this inequality is true; we shall say that a sequence of functions $\{\exp(i\lambda_n t)\}$ is a *frame* over the interval $(-\gamma, \gamma)$ if there are positive constants A and B such that (1) is true for all $g(t) \in L_2(-\gamma, \gamma)$. If $\gamma=\pi$ and $\lambda_n = n$, then $A=B=1$ is Parseval's relation. The proof that a constant B exists is quite direct, the proof of the existence of A in case $\{\lambda_n\}$ is a sequence of uniform density 1 and $0<\gamma<\pi$ is one of the central results of the present paper.

It was shown by Paley and Wiener [9], who initiated much of the work in nonharmonic Fourier series, that if λ_n is real and $|\lambda_n - n| < \pi^{-2}$, in which case the condition $|\lambda_n - \lambda_m| \geqq \delta > 0$ for $n \neq m$ is automatically satisfied, then $\{\exp(i\lambda_n x)\}$ is closed over $(-\pi, \pi)$. Boas [1] pointed out that their results imply that (1) is true in this case with $\gamma = \pi$. Duffin and Eachus [5] showed that this inequality is true in the case $\gamma=\pi$ if λ_n are real or complex numbers satisfying $|\lambda_n - n| \leqq 0.22 \cdots$. In the present paper it is shown that (1) is true in the case $\gamma = \pi$ if $\{\lambda_n\}$ is a sequence such that $|\operatorname{Re}(\lambda_n) - n| \leqq 0.22 \cdots$ and the imaginary part of λ_n is uniformly bounded. Boas [3] has considered problems analogous to some of these for L_p spaces.

If relation (1) is true, then the sequence of functions $\{\exp(i\lambda_n t)\}$ is clearly complete over $(-\gamma, \gamma)$; that is, if $g(t) \in L_2(-\gamma, \gamma)$, then the set of relations

$$\int_{-\gamma}^{\gamma} g(t) e^{i\lambda_n t} dt = 0$$

imply that $g(t)$ vanishes almost everywhere in $(-\gamma, \gamma)$. A proof of completeness in a more general case than any mentioned above was given by Levinson [8]. Levinson's result, without stating the most general form, shows that if the points lie in a strip parallel to the real axis and if $|\operatorname{Re}(\lambda_n) - n| \leqq \alpha < 1/4$, then the set of functions $\{\exp(i\lambda_n t)\}$ are complete in $(-\pi, \pi)$. On the other hand, completeness is a less strong conclusion than the frame condition. We shall show, for example, that if $\{\lambda_n\}$ is a sequence of uniform density 1, then the sequence of functions $\{\exp(i\lambda_n t)\}$ for which $n>0$ form a complete set in any interval $(-\gamma, \gamma)$ where $0<\gamma<\pi$, but they do not constitute a frame in any such interval.

Relation (1) gives the set of functions $\{\exp(i\lambda_n t)\}$ properties quite similar to an orthonormal set such as $\{\exp(int)\}$ in Hilbert space. However, the situation is more complicated because the set $\{\exp(i\lambda_n t)\}$ is highly overcomplete on an interval of length less than 2π. Most of the previous study of nonharmonic Fourier series has been for the exactly complete case; that is, the case in which the sequence of functions is complete but becomes incomplete by the omission of any one of them. It has therefore seemed worthwhile to give in detail some of the elementary relationships between moment sequences, expansion coefficients, etc. It is shown that these relations are consequences of well known properties of positive definite transformations in Hilbert space.

These considerations give information only about mean convergence of the series. However, combining properties of mean convergence with properties of the Dirichlet kernel gives conditions for pointwise convergence. It results that nonharmonic Fourier series have to a large extent the same convergence and summability properties as ordinary Fourier series.

2. **Fourier frames.** We begin the proofs by making the following more precise definitions.

DEFINITION. A sequence $\{\lambda_n\}$ of real or complex numbers has *uniform density* d, $d>0$, if there are constants L and δ such that

$$\left|\lambda_n - \frac{n}{d}\right| \leqq L, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{2}$$

$$|\lambda_n - \lambda_m| \geqq \delta > 0, \qquad n \neq m. \tag{3}$$

DEFINITION. A set of functions $\{\exp(i\lambda_n t)\}$ is a *frame* over an interval $(-\gamma, \gamma)$ if there exist positive constants A and B which depend exclusively on γ and the set of functions $\{\exp(i\lambda_n t)\}$ such that

$$A \leqq \frac{\frac{1}{2\pi}\sum_n \left|\int_{-\gamma}^{\gamma} g(t)e^{i\lambda_n t}dt\right|^2}{\int_{-\gamma}^{\gamma} |g(t)|^2 dt} \leqq B \tag{4}$$

for every function $g(t) \in L_2(-\gamma, \gamma)$.

In the case of a frame, we shall suppose in part 2, except where the contrary is stated, that the index n runs through all positive and negative integers and zero; however, we do not suppose that the λ_n are distinct. The following theorem is to be proved.

THEOREM I. *If* $\{\lambda_n\}$ *is a sequence of uniform density* d, *then the set of functions* $\{\exp(i\lambda_n t)\}$ *is a frame over the interval* $(-\gamma, \gamma)$ *where* $0<\gamma<\pi d$.

The following equivalent theorem is also to be proved.

THEOREM I′. *Let* $\{\lambda_n\}$ *be a sequence of uniform density* d *and let* $0<\gamma<\pi d$. *If* $f(z)$ *is an entire function of exponential type* γ *such that* $f(x) \in L_2(-\infty, \infty)$, *then*

$$A \leqq \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \leqq B. \tag{5}$$

Here A *and* B *are positive constants which depend exclusively on* γ *and* $\{\lambda_n\}$.

It will be shown that either of these theorems implies the other, and that any set of positive constants A and B which suffices in (4) or (5) also suffices in the other inequality. It will be clear from the sequel that the constant B exists under conditions much milder than are necessary to infer the existence of A. The proof of these theorems depends on several lemmas. The following lemma is a now classical result of Paley and Wiener [9].

LEMMA I. *If $f(z)$ is an entire function of exponential type γ and if $f(x) \in L_2(-\infty, \infty)$, then there is a function $g(t) \in L_2(-\gamma, \gamma)$ such that*

$$f(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} g(t) e^{izt} dt. \tag{6}$$

It is clear in this lemma that $g(t)$ is the Fourier transform of $f(x)$, that is, the Fourier transform of $f(x)$ vanishes almost everywhere outside $(-\gamma, \gamma)$, and so Plancherel's theorem states that

$$\int_{-\infty}^{\infty} | f(x) |^2 dx = \int_{-\gamma}^{\gamma} | g(t) |^2 dt. \tag{7}$$

From these relations it follows that if $f(z)$ is an entire function of exponential type γ such that $f(x) \in L_2(-\infty, \infty)$, then

$$| f(x + iy) | \leqq \left(\frac{\gamma}{\pi}\right)^{1/2} e^{\gamma |y|} \left\{ \int_{-\infty}^{\infty} | f(x) |^2 dx \right\}^{1/2}. \tag{8}$$

Differentiating (6) k times we also have

$$f^{(k)}(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} g(t)(it)^k e^{izt} dt,$$

so, by Plancherel's theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} | f^{(k)}(x) |^2 dx &= \int_{-\gamma}^{\gamma} | g(t) |^2 t^{2k}\, dt \\ &\leqq \gamma^{2k} \int_{-\gamma}^{\gamma} | g(t) |^2 dt. \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} | f^{(k)}(x) |^2 dx \leqq \gamma^{2k} \int_{-\infty}^{\infty} | f(x) |^2 dx \tag{9}$$

for every entire function $f(z)$ of exponential type γ.

From Lemma I it clearly follows that Theorem I and Theorem I′ are equivalent. More generally, a set of functions $\{\exp (i\lambda_n t)\}$ is a frame over $(-\gamma, \gamma)$ if and only if there exist positive constants A and B such that

$$A \leqq \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \leqq B \tag{10}$$

for every entire function $f(z)$ of exponential type γ satisfying $f(x) \in L_2(-\infty, \infty)$.

Results similar to Lemma II were obtained by Plancherel and Pólya [10] and by Boas [2] under different conditions.

LEMMA II. *Let $\{\exp(i\lambda_n t)\}$ be a frame over the interval $(-\gamma, \gamma)$. If M is any constant and $\{\mu_n\}$ is a sequence satisfying $|\mu_n - \lambda_n| \leqq M$, then there is a number $C = C(M, \gamma, \{\lambda_n\})$ such that*

$$\frac{\sum_n |f(\mu_n)|^2}{\sum_n |f(\lambda_n)|^2} \leqq C$$

for every entire function $f(z)$ of exponential type γ.

Proof. It is clearly sufficient to prove this lemma under the additional hypothesis that $f(x) \in L_2(-\infty, \infty)$. Let $\{\exp(i\lambda_n t)\}$ be a frame over $(-\gamma, \gamma)$ such that inequality (10) is satisfied, and, if M is a given positive number, let $\{\mu_n\}$ be a sequence such that $|\lambda_n - \mu_n| \leqq M$. It is to be shown that if $f(z)$ is an entire function of exponential type γ belonging to $L_2(-\infty, \infty)$ on the real axis and ρ is a given positive number, then

$$\sum_n |f(\mu_n) - f(\lambda_n)|^2 \leqq T \sum_n |f(\lambda_n)|^2 \tag{11}$$

where

$$T = \frac{B}{A}\left(e^{\gamma^2/\rho^2} - 1\right)\left(e^{M^2\rho^2} - 1\right). \tag{12}$$

If $f(z)$ satisfies these conditions, then Taylor's theorem shows that

$$f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{k!} (\mu_n - \lambda_n)^k.$$

Multiplying and dividing the last series termwise by ρ^k, we have from Cauchy's inequality

$$|f(\mu_n) - f(\lambda_n)|^2 \leqq \left\{ \sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{\rho^{2k} k!} \right\} \left\{ \sum_{k=1}^{\infty} \frac{(M\rho)^{2k}}{k!} \right\}.$$

The second sum on the right side of this inequality is $\exp(M^2\rho^2) - 1$. The function $f^{(k)}(z)$ is an entire function of exponential type γ, and, according to inequality (9), it belongs to class $L_2(-\infty, \infty)$ on the real axis. Then the function $f^{(k)}(z)$ satisfies inequality (10), so

$$\sum_n |f^{(k)}(\lambda_n)|^2 \leqq B\int_{-\infty}^{\infty} |f^{(k)}(x)|^2 dx \leqq B\gamma^{2k}\int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Since the function $f(z)$ also satisfies (10), it is clear that the last expression is equal to or less than $\{B\gamma^{2k}/A\}\sum|f(\lambda_n)|^2$, and inequality (11) follows. Then Minkowski's inequality shows that

$$\left(\sum_n |f(\mu_n)|^2\right)^{1/2} \leqq \left(\sum_n |f(\lambda_n)|^2\right)^{1/2} + \left(T\sum_n |f(\lambda_n)|^2\right)^{1/2},$$

so the lemma follows with $C=(1+T^{1/2})^2$.

Lemma II shows that the constant B of Theorems I, I′ exists. For in the case $d=1$ it is well known that the sequence of functions e^{int} is a frame over $(-\gamma, \gamma)$ where $0<\gamma\leqq\pi$. Since $|\lambda_n-n|\leqq L$, the existence of B follows from Lemma II. The case $d\neq 1$ may be reduced to the case $d=1$ by a change of variables.

The following result shows that the set of points $\{\lambda_n\}$ such that $\{\exp(i\lambda_n t)\}$ is a frame over a fixed interval is in a sense an open set.

LEMMA III. *Let $\{\exp(i\lambda_n t)\}$ be a frame over $(-\gamma, \gamma)$. There is a $\delta_1>0$ such that $\{\exp(i\mu_n t)\}$ is a frame over the same interval whenever $|\mu_n-\lambda_n|\leqq\delta_1$.*

Proof. Let $f(z)$ be an entire function of exponential type γ such that $f(x)\in L_2(-\infty, \infty)$, and let $\{\exp(i\lambda_n t)\}$ be a frame over $(-\gamma, \gamma)$. Then $f(z)$ satisfies inequality (11) where $|\mu_n-\lambda_n|\leqq M$, and T is defined by (12) with ρ and M any positive numbers. From (11) and Minkowski's inequality we see that

$$\left(\sum_n |f(\lambda_n)|^2\right)^{1/2} \leqq \left(\sum_n |f(\mu_n)|^2\right)^{1/2} + \left(T\sum_n |f(\lambda_n)|^2\right)^{1/2}. \tag{13}$$

Now let $\delta_1=M=1/\rho$ and choose ρ so large that $T<1/4$. Then inequalities (10) and (13) show that

$$\frac{A}{4}\int_{-\infty}^{\infty} |f(x)|^2 dx \leqq \sum_n |f(\mu_n)|^2.$$

Lemma II completes the proof.

The above method is similar to that used by Duffin and Eachus [5] to show that $\{\exp(i\mu_n t)\}$ is a frame over $(-\pi, \pi)$ if $|\mu_n-n|\leqq M<(\log 2)/\pi$. To obtain this result let $\lambda_n=n$, $\gamma=\pi$, $\rho=(\gamma/M)^{1/2}$. Then since in this case $B=A$, we see from (12) that $T=(e^{\gamma M}-1)^2$, so $T<1$ if $M<(\log 2)/\pi$. From (10) and (13) it is seen that $\{\exp(i\mu_n t)\}$ is a frame over $(-\pi, \pi)$ if $|\mu_n-n|\leqq M<(\log 2)/\pi$. Theorem II will serve to strengthen this result by showing that $\{\exp(i\mu_n t)\}$ is a frame over $(-\pi, \pi)$ if there are constants β and M such that $|I(\mu_n)| \leqq \beta$, $|\mathrm{Re}(\mu_n)-n| \leqq M<(\log 2)/\pi$.

The proof of the existence of the positive constant A in Theorem I depends on several lemmas. The following result has previously been proved by the authors [6].

LEMMA IV. *If $f(z)$ is an entire function of exponential type γ and $\{\lambda_n\}$ is a sequence of uniform density d, $d>\gamma/\pi$, then*

$$|f(x+iy)| \leqq M e^{\gamma|y|} \sup |f(\lambda_n)|$$

where the constant M is independent of $f(z)$.

The following lemma is closely related to results of Bourgin [4] and Ibragimov [7]; however, these authors were not concerned with sequences of uniform density. It is stated in a more general form than needed because of its intrinsic interest.

LEMMA V. *Let A_2 be the closed subspace of $L_2(-\pi, \pi)$ generated by the set of functions $1, e^{i\theta}, e^{2i\theta}, \cdots$. If $f(z)$ is an entire function of exponential type γ such that $f^{(n)}(0)\neq 0$, $n=0, 1, 2, \cdots$, and the sequence $\{\lambda_n\}$ has uniform density d, $d>\gamma/\pi$, then the set of functions $f(\lambda_n e^{i\theta})$, $n=0, \pm 1, \pm 2, \cdots$, is complete in A_2.*

Proof. Let

$$F(z) = \int_{-\pi}^{\pi} f(ze^{i\theta})g^*(\theta)d\theta \tag{14}$$

where $g(\theta)\in A_2$, that is,

$$g(\theta) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{0}^{N} c_\nu e^{i\nu\theta}$$

with $\sum|c_\nu|^2<\infty$. Here and elsewhere the * represents the complex conjugate. Since $f(z)$ is an entire function of exponential type γ, it is clear that $F(z)$ is also. Under the hypothesis

$$F(\lambda_n) = \int_{-\pi}^{\pi} f(\lambda_n e^{i\theta})g^*(\theta)d\theta = 0, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

it is to be shown that $g(\theta)=0$ almost everywhere. Since $F(\lambda_n)=0$, Lemma IV shows that $F(z)$ vanishes identically. Then differentiating (14) we have

$$F^{(k)}(0) = \int_{-\pi}^{\pi} f^{(k)}(0)e^{ik\theta}g^*(\theta)d\theta$$

for $k=0, 1, 2, \cdots$. Since $f^{(k)}(0)\neq 0$, the lemma follows.

LEMMA VI. *Given R satisfying $0<R<\pi$, let $p(z)$ be regular in the circle*

$|z| \leqq R$ and let $\{\lambda_n\}$ be a sequence of uniform density 1. For each positive number h there is an integer N and a finite set of numbers $a_{-N}, a_{-N+1}, \cdots, a_0, \cdots, a_N$ such that

$$p(z) - \sum_{-N}^{N} a_n e^{i\lambda_n z} = \sum_{k=0}^{\infty} b_k z^k \tag{15}$$

and

$$| b_k | \leqq \frac{h}{R^k}, \qquad | a_n | \leqq N. \tag{16}$$

Moreover, given h, R, $p(z)$, L, δ, the same N suffices for all sequences satisfying

$$| \lambda_n - n | \leqq L, \qquad | \lambda_n - \lambda_m | \geqq \delta > 0 \quad \text{for} \quad n \neq m. \tag{17}$$

Proof. the function $f(z) = e^{iRz}$ satisfies the conditions of Lemma V with $\gamma = R < \pi$, so the set of functions $\{\exp(i\lambda_n Re^{i\theta})\}$ is complete in A_2. It is well known that closure and completeness are equivalent in A_2 so there is a finite set of numbers $a_{-M}, a_{-M+1}, \cdots, a_M$ such that if

$$\tau(Re^{i\theta}) = p(Re^{i\theta}) - \sum_{-M}^{M} a_n e^{i\lambda_n Re^{i\theta}}$$

then

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} | \tau(Re^{i\theta}) |^2 d\theta\right\}^{1/2} \leqq \frac{h}{2} \cdot \tag{18}$$

The Taylor's series of $\tau(z)$ about the origin

$$\tau(z) = p(z) - \sum_{-M}^{M} a_n e^{i\lambda_n z} = \sum_{k=0}^{\infty} b_k z^k$$

converges in $|z| < R$. Then making use of Cauchy's integral representation of b_k we have from (18) and Schwarz's inequality

$$| b_k | = \left| \frac{1}{2\pi i}\int_{|z|=R} \tau(z) \frac{dz}{z^{k+1}} \right| \leqq \frac{h}{2R^k} \cdot$$

Let $a_n = 0$ for $|n| > M$. Then choosing a sufficiently large integer N, the first part of the lemma follows.

The proof of the second part of the lemma is by contradiction. Let h, R, $p(z)$, L, δ be fixed and suppose there is no N which suffices for all sequences $\{\lambda_n\}$ satisfying (17). Then there are sequences $\{\lambda_n^{(1)}\}, \{\lambda_n^{(2)}\}, \cdots, \{\lambda_n^{(j)}\}, \cdots$ such that the least integer $N = N(j)$ for which (15) and (16) are true for some $a_n^{(j)}, b_k^{(j)}$ satisfies

$$N(j) > j.$$

From relation (17) it is clear that there is a subset of the sequences $\{\lambda_n^{(j)}\}$, which by a renumbering we suppose is the entire set, which converges to a limit sequence $\{\lambda_n^{(0)}\}$,

$$\lim_{j\to\infty} \lambda_n^{(j)} = \lambda_n^{(0)}, \qquad n = 0 \pm 1, \pm 2, \cdots.$$

The sequence $\{\lambda_n^{(0)}\}$ satisfies (17) and so has uniform density 1. Then, by what has been shown, there is an N_0 and a set of numbers $a_n^{(0)}$ such that

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|p(Re^{i\theta}) - \sum_{-N_0}^{N_0} a_n^{(0)} \exp\left(i\lambda_n^{(0)} Re^{i\theta}\right)\right|^2 d\theta\right\}^{1/2} \leqq \frac{h}{2},$$

$$\left| a_n^{(0)} \right| \leqq N_0, \qquad n = 0, \pm 1, \pm 2, \cdots, \pm N_0.$$

The finite sum

$$\sum_{-N_0}^{N_0} a_n^{(0)} \exp\left(i\lambda_n Re^{i\theta}\right)$$

is a continuous function of the $2N_0+1$ variables λ_n, so for sufficiently large j

$$\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|p(Re^{i\theta}) - \sum_{-N_0}^{N_0} a_n^{(0)} \exp\left(i\lambda_n^{(j)} Re^{i\theta}\right)\right|^2 d\theta\right\}^{1/2} \leqq h.$$

This gives a contradiction since $N = N_0$ suffices in (15) and (16) for all large j.

Proof of Theorem I. It has been shown that the constant B of inequality (4) exists. To show the existence of the positive constant A, it is sufficient to consider the case $d=1$, $0<\gamma<\pi$. Given the sequence $\{\lambda_n\}$ of uniform density 1, let

$$\lambda_n^{(\nu)} = \lambda_{n+\nu} - \nu. \tag{19}$$

Then for each positive and negative integer ν the sequence $\{\lambda_n^{(\nu)}\}$ is of uniform density 1 with the same bounds L, δ as the given sequence,

$$\left| \lambda_n^{(\nu)} - n \right| \leqq L, \qquad \left| \lambda_n^{(\nu)} - \lambda_m^{(\nu)} \right| \geqq \delta > 0 \qquad \text{for } n \neq m.$$

In Lemma VI define $p(z)$, R, h by

$$p(z) = 1, \qquad R = \frac{1}{2}(\gamma + \pi), \qquad h = \frac{1}{2R}(R - \gamma).$$

Then the lemma asserts that for suitable $a_n^{(\nu)}$, $b_k^{(\nu)}$, N we have

$$1 - \sum_{n=-N}^{N} a_n^{(\nu)} \exp\left(i\lambda_n^{(\nu)} x\right) = \sum_{k=0}^{\infty} b_k^{(\nu)} x^k \tag{20}$$

where

$$|b_k^{(\nu)}| \leqq \frac{h}{R^k}, \qquad |a_n^{(\nu)}| \leqq N; \tag{21}$$

and the same constant N suffices for $\nu=0, \pm 1, \pm 2, \cdots$. If we define

$$\psi_\nu(x) = e^{i\nu x} - e^{i\nu x}\sum_{k=0}^{\infty} b_k^{(\nu)} x^k, \tag{22}$$

it follows from (19) and (20) that

$$\psi_\nu(x) = \sum_{n=-N}^{N} a_n^{(\nu)} \exp(i\lambda_{n+\nu}x). \tag{23}$$

Writing

$$\zeta^{(\nu)}(x) = e^{i\nu x}\sum_{k=0}^{\infty} b_k^{(\nu)} x^k, \tag{24}$$

we have

$$\psi_\nu(x) = e^{i\nu x} - \zeta^{(\nu)}(x).$$

It is to be shown that $\zeta^{(\nu)}(x)$ has in some sense a small average value, and it will follow that $\psi_\nu(x)$ has some average behavior similar to $e^{i\nu x}$. In the remainder of the proof of Theorem I we shall use the notation

$$(\phi, g) = \frac{1}{2\pi}\int_{-\gamma}^{\gamma} \phi(x)g(x)dx, \qquad \|g\| = \left\{\frac{1}{2\pi}\int_{-\gamma}^{\gamma} |g(x)|^2 dx\right\}^{1/2}.$$

Given a function $g(x) \in L_2(-\gamma, \gamma)$, define $g(x)=0$ in the part of $(-\pi, \pi)$ that lies outside $(-\gamma, \gamma)$. Then Parseval's relation takes the form

$$\|g\|^2 = \sum_{-\infty}^{\infty} |(e^{i\nu x}, g)|^2.$$

Since $R>\gamma$, inequality (21) shows that the series (24) converges uniformly in $-\gamma \leqq x \leqq \gamma$; hence after multiplying (24) by $g(x)$ and integrating termwise we obtain

$$(\zeta^{(\nu)}, g) = \sum_{k=0}^{\infty} b_k^{(\nu)}(e^{i\nu x}, x^k g).$$

If this series is multiplied and divided termwise by $(R\gamma)^{k/2}$, then Cauchy's inequality and (21) show that

$$|(\zeta^{(\nu)}, g)|^2 \leqq h^2\left\{\sum_{k=0}^{\infty}\left(\frac{\gamma}{R}\right)^k\right\} \sum_{k=0}^{\infty} |(e^{i\nu x}, x^k g)|^2 R^{-k}\gamma^{-k}.$$

Then making use of Parseval's relation, we have

$$\sum_{\nu=-\infty}^{\infty} |(\zeta^{(\nu)}, g)|^2 \leqq \frac{h^2 R}{R-\gamma} \sum_{k=0}^{\infty} \sum_{\nu=-\infty}^{\infty} |(e^{i\nu x}, x^k g)|^2 R^{-k}\gamma^{-k}$$
$$= \frac{h^2 R}{R-\gamma} \sum_{k=0}^{\infty} \frac{\|x^k g\|^2}{R^k \gamma^k}.$$

In the last series the integration defining $\|x^k g\|$ need only be carried over $(-\gamma, \gamma)$, so $\|x^k g\| \leqq \gamma^k \|g\|$. Thus, recalling the definition of h, we have

$$(25) \qquad \sum_{\nu=-\infty}^{\infty} |(\zeta^{(\nu)}, g)|^2 \leqq \|g\|^2/4.$$

Since $e^{i\nu x} = \psi_\nu(x) + \zeta^{(\nu)}(x)$, we have $(e^{i\nu x}, g) = (\psi_\nu, g) + (\zeta^{(\nu)}, g)$, and then Minkowski's inequality shows that

$$\left(\sum_{-\infty}^{\infty} |(e^{i\nu x}, g)|^2 \right)^{1/2} \leqq \left(\sum_{-\infty}^{\infty} |(\psi_\nu, g)|^2 \right)^{1/2} + \left(\sum_{-\infty}^{\infty} |(\zeta^{(\nu)}, g)|^2 \right)^{1/2}.$$

The left side of this inequality is equal to $\|g\|$ by Parseval's theorem, and the second of the two terms on the right we have shown in (25) is dominated by $\|g\|/2$. Thus

$$(26) \qquad \sum_{-\infty}^{\infty} |(\psi_\nu, g)|^2 \geqq \|g\|^2/4.$$

Now substituting in (23), we have

$$(\psi_\nu, g) = \sum_{n=-N}^{N} a_n^{(\nu)} (e^{\lambda_{n+\nu} x}, g),$$

so Cauchy's inequality and the estimate of $a_n^{(\nu)}$ given in (21) shows that

$$|(\psi_\nu, g)|^2 \leqq (2N+1)N^2 \sum_{n=-N}^{N} |(e^{i\lambda_{n+\nu} x}, g)|^2.$$

If the last inequality is summed on ν from $-\infty$ to ∞, then in the double summation the sum $n+\nu$ runs through each integer precisely $2N+1$ times. Thus

$$(27) \qquad \sum_{-\infty}^{\infty} |(\psi_\nu, g)|^2 \leqq (2N+1)^2 N^2 \sum_{k=-\infty}^{\infty} |(e^{i\lambda_k x}, g)|^2.$$

If we combine (26) and (27, Theorem I follows; and with $d=1$ we have $A \geqq 1/\{4N^2(2N+1)^2\}$. The magnitude of N is not determined by the previous argument so this is not an estimate of A.

As previously remarked, Theorem I is equivalent to Theorem I′. However, the conditions of Theorem I′ are unnecessarily restrictive because it is not necessary to suppose that $f(x)\in L_2(-\infty, \infty)$. If $\{\lambda_n\}$, γ, d satisfy the conditions of Theorem I′, then every entire function $f(z)$ of exponential type γ satisfies inequality (5). If either of $\int_{-\infty}^{\infty}|f(x)|^2dx$, $\sum|f(\lambda_n)|^2$ is finite, then the other is also and (5) holds. For if $\int_{-\infty}^{\infty}|f(x)|^2dx<\infty$, then inequality (5) follows from Theorem I′, so suppose that $\sum|f(\lambda_n)|^2<\infty$. Then $f(z)$ is bounded at the points $z=\lambda_n$, so Lemma IV shows that $f(z)$ is bounded on the entire real axis. The function

$$F_\epsilon(z) = f(z)\frac{\sin \epsilon z}{\epsilon z}$$

where $0<\epsilon<(\pi d-\gamma)/2$ is an entire function of exponential type $(\pi d+\gamma)/2$ and $F_\epsilon(x)\in L_2(-\infty, \infty)$. By Theorem I′ it follows that there are positive constants $A^\Delta=A((\gamma+\pi d)/2)$, $B^\Delta=B((\gamma+\pi d)/2)$ such that

$$A^\Delta \leqq \frac{\sum|F_\epsilon(\lambda_n)|^2}{\int_{-\infty}^{\infty}|F_\epsilon(x)|^2dx} \leqq B^\Delta.$$

Letting ϵ approach zero, the constants A^Δ, B^Δ may be supposed fixed since $F_\epsilon(z)$ remains of type $(\pi d+\gamma)/2$. It then follows that the limit $f(z)$ of $F_\epsilon(z)$ also satisfies this inequality. Thus $f(x)\in L_2(-\infty, \infty)$, so Theorem I′ shows that $f(z)$ satisfies inequality (5).

This remark leads to a strengthening of Theorem I. Without attempting a complete analogy to the stronger form of Theorem I′, let $\{\lambda_n\}$, γ satisfy the conditions of Theorem I with $d=1$. Thus $0<\gamma<\pi$ and $\{\lambda_n\}$ has uniform density 1. Let $G(t)$ be a function of bounded variation over $(-\gamma, \gamma)$. If

$$\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left|\int_{-\gamma}^{\gamma}e^{i\lambda_n t}dG(t)\right|^2<\infty,$$

then $G(t)$ is essentially the indefinite integral of a function of class $L_2(-\infty, \infty)$, and

$$A \leqq \frac{\frac{1}{2\pi}\sum_{-\infty}^{\infty}\left|\int_{-\gamma}^{\gamma}e^{i\lambda_n t}dG(t)\right|^2}{\int_{-\gamma}^{\gamma}|G'(t)|^2dt} \leqq B.$$

This follows from Lemma I and the stronger form of Theorem I′ since

$$f(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\gamma}^{\gamma}e^{izt}dG(t)$$

is an entire function of exponential type γ.

Not every frame over an interval $(-\gamma, \gamma)$ can be strengthened in the same manner to Stieltjes integrals. This is shown by the example in which $\gamma=\pi$, $\lambda_n=n$, and $G(t)=0$ for $-\pi<t<\pi$, $G(-\pi)=1$, $G(\pi)=1$. For in this case, although $\{e^{int}\}$ is a frame over $(-\pi, \pi)$, we see that $G(t)$ is not absolutely continuous but

$$\int_{-\pi}^{\pi} e^{int}dG(t) = 0, \qquad n = 0, \pm 1, \pm 2, \cdots .$$

Let $\{\lambda_n\}$ be a sequence of uniform density 1, and let $0<\gamma<\pi$. Over the interval $(-\gamma, \gamma)$, the set of functions $\{\exp (i\lambda_n t)\}$, $n=0, \pm 1, \pm 2, \cdots$, is a frame, and is therefore complete. The subset of functions $\{\exp (i\lambda_n t)\}$ for which $n>0$ is complete over $(-\gamma, \gamma)$ according to a result of Levinson [8, p. 3], but it is not a frame over this interval. For let $g(t)=e^{i\alpha t}$ where $\alpha>0$. Then

$$f(z) = \frac{1}{(2\pi)^{1/2}}\int_{-\gamma}^{\gamma} e^{i\alpha t}e^{izt}dt = \left(\frac{2}{\pi}\right)^{1/2} \frac{\sin \gamma(\alpha+z)}{\alpha+z}. \tag{28}$$

Now $\int_{-\gamma}^{\gamma}|g(t)|^2dt$ is a positive constant independent of α, but

$$\sum_{n=1}^{\infty}|f(\lambda_n)|^2 = \frac{2}{\pi}\sum_{n=1}^{\infty}\left|\frac{\sin \gamma(\alpha+\lambda_n)}{\alpha+\lambda_n}\right|^2 \leqq \frac{2}{\pi}e^{2\gamma L}\sum_{n=1}^{\infty}\frac{1}{|\alpha+\lambda_n|^2}$$

and this approaches zero as $\alpha\to\infty$. Thus there is no positive constant A such that (4) is satisfied.

We shall say that a set of functions $\{\exp (i\lambda_n t)\}$ is an exact frame over an interval I if it is a frame over I but fails to be a frame over I by the removal of any function of the set. This use of exact is analogous to that of Paley and Wiener in the case of exactly complete sets. If $\{\exp (i\lambda_n t)\}$ is a frame over I but is not an exact frame over I, then we say it is an overcomplete frame over I.

If $\{\lambda_n\}$ is a set of uniform density 1 and $0<\gamma<\pi$, then, according to Theorem I, the set of functions $\{\exp (i\lambda_n t)\}$ is a frame over $(-\gamma, \gamma)$. This set of functions is an overcomplete frame over $(-\gamma, \gamma)$, indeed if any finite number of the functions are deleted the remaining functions are a frame over $(-\gamma, \gamma)$. This follows from the fact that if a finite number of the λ_n are deleted, the remaining λ_n may be reindexed so as to satisfy inequalities (2), (3) with the bound L replaced by a larger number.

LEMMA VII. *If $\{\exp (i\lambda_n t)\}$ is a frame over $(-\gamma, \gamma)$ but fails to be a frame over this interval by the removal of some function of the set, then it fails to be a frame over $(-\gamma, \gamma)$ by the removal of any function of the set.*

Proof. Suppose that there are positive constants A, B such that for all $g(t) \in L_2(-\gamma, \gamma)$,

$$A \leqq \frac{\frac{1}{2\pi}\sum_n \left| \int_{-\gamma}^{\gamma} e^{i\lambda_n t} g(t)dt \right|^2}{\int_{-\gamma}^{\gamma} | g(t) |^2 dt} \leqq B$$

but that if λ_m is omitted there are no such A, B. The failure must be in A rather than B. Then there is a sequence of functions $g_k(t)$, $k=1, 2, 3, \cdots$, whose Fourier transforms we write $f_k(z)$,

$$f_k(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} g_k(t) e^{izt} dt, \tag{29}$$

which are normalized by the relation

$$\int_{-\infty}^{\infty} | f_k(x) |^2 dx = \int_{-\gamma}^{\gamma} | g_k(t) |^2 dt = 1$$

and such that

$$\sum_n | f_k(\lambda_n) |^2 \geqq A, \qquad \sum_{n \neq m} | f_k(\lambda_n) |^2 \leqq \frac{1}{k} \cdot$$

Clearly $f_k(\lambda_n)$ tends to zero as $k \to \infty$ for each $n \neq m$, but $|f_k(\lambda_m)|$ has a lower bound equal to or greater than $A^{1/2} > 0$ as $k \to \infty$. Now $f_k(z)$ is an entire function of exponential type γ, and, according to (8), it satisfies

$$| f_k(x + iy) | \leqq (\gamma/\pi)^{1/2} e^{\gamma |y|}, \qquad k = 1, 2, 3, \cdots.$$

The sequence of entire functions being uniformly dominated, there is a subsequence which converges to a limit $f(z)$, and uniformly in every bounded domain. Thus $f(z)$ is an entire function of exponential type γ and

$$| f(x + iy) | \leqq \left(\frac{\gamma}{\pi}\right)^{1/2} e^{\gamma |y|}, \qquad \int_{-\infty}^{\infty} | f(x) |^2 dx \leqq 1.$$

Now $f(\lambda_n) = 0$ for $n \neq m$, and $f(\lambda_m) \neq 0$, in particular, $f(z)$ does not vanish identically. Therefore, if j is any integer and

$$F(z) = \frac{z - \lambda_m}{z - \lambda_j} f(z),$$

then $F(z)$ is a not identically vanishing entire function of exponential type γ. It is readily shown that $F(x) \in L_2(-\infty, \infty)$ and $F(\lambda_n) = 0$ for $n \neq j$. According to Lemma I there is a function $G(t) \in L_2(-\gamma, \gamma)$ such that

$$F(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} G(t) e^{izt} dt.$$

$G(t)$ is not equivalent to zero in $(-\gamma, \gamma)$, but

$$\frac{1}{(2\pi)^{1/2}}\int_{-\gamma}^{\gamma} G(t)e^{i\lambda_n t}dt = 0, \qquad n \neq j,$$

so the lemma follows.

THEOREM II. *Let* $\lambda_n = \alpha_n + i\beta_n$ *where* α_n *and* β_n *are real and* $|\beta_n| \leqq \beta$ *for some constant* β. *If the set of functions* $\{\exp(i\alpha_n t)\}$ *is a frame over an interval* $(-\gamma, \gamma)$, *then* $\{\exp(i\lambda_n t)\}$ *is a frame over the same interval.*

Proof. Let $f(z)$ be an entire function of exponential type γ such that $f(x) \in L_2(-\infty, \infty)$. According to Hadamard's factorization theorem it may be written in the form

$$f(z) = z^k e^{az+b}\prod_{\nu=1}^{\infty}\left(1 - \frac{z}{z_\nu}\right)e^{z/z_\nu} \tag{30}$$

where $z_1, z_2, z_3, \cdots$ are the zeros of $f(z)$ other than at the origin. Now $f(z)$ satisfies inequality (8), so Carleman's formula [11] written for the upper half-plane and for the lower half-plane in turn shows that $\sum \operatorname{Im}(1/z_\nu)$ is an absolutely convergent series. If

$$1/z_\nu = p_\nu + iq_\nu$$

where p_ν and q_ν are real, Hadamard's formula may be written in the form

$$f(z) = z^k e^{(c+id)z+b}\prod_{\nu=1}^{\infty}\left(1 - \frac{z}{z_\nu}\right)e^{p_\nu z} \tag{31}$$

where c and d are real,

$$c = \operatorname{Re}(a), \qquad d = \operatorname{Im}(a) + \sum_{\nu=1}^{\infty} q_\nu.$$

This product converges uniformly in every bounded domain since (30) does and $\sum q_\nu$ is an absolutely convergent series.

We are going to define a sequence $\{\lambda_n^{(1)}\}$ and an entire function $f_1(z)$ of exponential type γ belonging to $L_2(-\infty, \infty)$ on the real axis such that

$$\lambda_n^{(1)} = \alpha_n + i\beta_n^{(1)}, \qquad |\beta_n^{(1)}| \leqq \beta/2, \tag{32}$$

and

$$\frac{\sum_n |f_1(\lambda_n^{(1)})|^2}{\int_{-\infty}^{\infty}|f_1(x)|^2dx} \leqq e^{\beta\gamma}\frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx}. \tag{33}$$

There are two cases to consider, $d \geqq 0$ and $d < 0$. Suppose first that $d \geqq 0$. Now form from the given function $f(z)$ another entire function whose zeros are obtained by reflecting in the real axis those zeros of $f(z)$ that lie in the lower half-plane. Let

$$\overset{\Delta}{z_\nu} = \begin{cases} z_\nu & \text{if} \quad \operatorname{Im}(z_\nu) \geqq 0, \\ z_\nu^* & \text{if} \quad \operatorname{Im}(z_\nu) < 0 \end{cases}$$

and define

$$f^\Delta(z) = z^k e^{(c+id)z+b} \prod_{\nu=1}^{\infty} \left(1 - \frac{z}{z_\nu^\Delta}\right) e^{p_\nu z}. \tag{34}$$

Then $f^\Delta(z)$ is an entire function of exponential type satisfying

$$\left| f^\Delta(x) \right| = \left| f(x) \right|. \tag{35}$$

To show this we note that the zeros of $f(z)$ that lie in the lower half-plane may be reflected in the real axis one at a time. Thus, if m is any positive integer, the function

$$\overset{\Delta}{f_m}(z) = z^k e^{(c+id)z+b} \prod_{\nu=1}^{m} \left(1 - \frac{z}{z_\nu^\Delta}\right) e^{p_\nu z} \prod_{\nu=m+1}^{\infty} \left(1 - \frac{z}{z_\nu}\right) e^{p_\nu z}$$

has the same modulus on the real axis as $f(x)$, and it is an entire function of exponential type γ since the ratio $f_m^\Delta(z)/f(z)$ tends to 1 as z tends to infinity in any direction. Thus, $f_m^\Delta(z)$ is dominated by the right side of (8) for $m = 1, 2, 3, \cdots$, and hence its limit $f^\Delta(z)$ is dominated by the right side of (8). The product (34) defining $f^\Delta(z)$ converges uniformly in every bounded domain. If we compare (31) and (34), it is clear that

$$\left| f^\Delta(x + iy) \right| \leqq \left| f(x + iy) \right|, \qquad y \geqq 0.$$

Since $d \geqq 0$, we see also that

$$\left| f^\Delta(x + iy) \right| \leqq \left| f(x - iy) \right|, \qquad y \geqq 0.$$

Now define a new sequence $\{\lambda_n^\Delta\}$ by reflecting in the real axis those points of $\{\lambda_n\}$ that lie in the lower half-plane. Let

$$\overset{\Delta}{\lambda_n} = \begin{cases} \lambda_n & \text{if} \quad \operatorname{Im}(\lambda_n) \geqq 0, \\ \lambda_n^* & \text{if} \quad \operatorname{Im}(\lambda_n) < 0. \end{cases}$$

Then, according to the inequalities stated in the preceding paragraph,

$$\left| \overset{\Delta}{f}(\overset{\Delta}{\lambda_n}) \right| \leqq \left| f(\lambda_n) \right|.$$

The points λ_n were assumed to lie in a strip of width 2β which is symmetric about the real axis. The points λ_n^Δ lie in a strip of width β which lies in the upper half-plane. We now translate the points λ_n^Δ to obtain a new sequence which lies in a strip of width β symmetric about the real axis. Let

$$\lambda_n^{(1)} = \lambda_n^\Delta - i\beta/2, \qquad f_1(z) = f^\Delta(z + i\beta/2).$$

Then the points $\lambda_n^{(1)}$ lie in the strip $|y| \leqq \beta/2$ and

$$|f_1(\lambda_n^{(1)})| = |f^\Delta(\lambda_n^\Delta)| \leqq |f(\lambda_n)|.$$

Also, according to Lemma I, there is a function $g^\Delta(t) \in L_2(-\gamma, \gamma)$ such that

$$f^\Delta(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} g^\Delta(t) e^{izt} dt$$

and, by (35),

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |f^\Delta(x)|^2 dx = \int_{-\gamma}^{\gamma} |g^\Delta(t)|^2 dt.$$

Thus, writing $g_1(t) = g^\Delta(t) e^{-\beta t/2}$, we see that

$$f_1(z) = \frac{1}{(2\pi)^{1/2}} \int_{-\gamma}^{\gamma} g_1(t) e^{izt} dt.$$

Hence

$$\int_{-\infty}^{\infty} |f_1(x)|^2 dx = \int_{-\gamma}^{\gamma} |g^\Delta(t)|^2 e^{-\beta t} dt \geqq e^{-\beta\gamma} \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

The sequence $\lambda_n^{(1)}$ and the function $f_1(z)$ thus defined satisfy (32) and (33).

If $d < 0$, we reflect in the real axis the zeros of $f(z)$ and the points that lie in the upper half-plane. Then after a translation we obtain a sequence $\{\lambda_n^{(1)}\}$ and an entire function $f_1(z)$ satisfying (32) and (33).

The above process may be iterated. If $f(z)$ is an entire function of exponential type γ which belongs to $L_2(-\infty, \infty)$ on the real axis, then for $k = 1, 2, 3, \cdots$ there is a sequence $\{\lambda_n^{(k)}\}$ and an entire function $f_k(z)$ of exponential type γ which belongs to $L_2(-\infty, \infty)$ on the real axis such that

$$\lambda_n^{(k)} = \alpha_n + \beta_n^{(k)}, \qquad |\beta_n^{(k)}| \leqq \beta/2^k, \tag{36}$$

and

$$\frac{\sum_n |f_k(\lambda_n^{(k)})|^2}{\int_{-\infty}^{\infty} |f_k(x)|^2 dx} \leqq e^{2\beta\gamma} \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx}. \tag{37}$$

The constant $\exp(2\beta\gamma)$ is here written in place of $\exp(\gamma\beta+\gamma\beta/2+\cdots+\gamma\beta/2^{k-1})$.

Suppose now that $\{\exp(i\alpha_n t)\}$ is a frame over $(-\gamma, \gamma)$. According to Lemma III there is a $\delta_1>0$ such that the set of functions $\{\exp(i\mu_n t)\}$ is a frame whenever $|\mu_n-\alpha_n|\leqq\delta_1$. Let k be so large that $\beta/2^k<\delta_1$. Then $\{\exp(i\lambda_n^{(k)}t)\}$ is a frame over $(-\gamma, \gamma)$, and the left side of (37) therefore has a positive lower bound A'. Thus

$$A'e^{-2\beta\gamma} \leqq \frac{\sum_n |f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx}.$$

Since $|\lambda_n-\alpha_n|\leqq\beta$, Lemma II asserts that the right side of (37) has a finite upper bound. It follows that the set of functions $\{\exp(i\lambda_n t)\}$ is a frame over $(-\gamma, \gamma)$.

3. **Abstract frames.** For two vectors u and v of Hilbert space, let $(u, v) = (v, u)^*$ be the complex scalar product which defines the norm $\|u\| = (u, u)^{1/2}$. We define a *frame* to be an infinite sequence of nonzero vectors $\phi_1, \phi_2, \phi_3, \cdots$ such that for an arbitrary vector v,

$$A\|v\|^2 \leqq \sum_n |(v, \phi_n)|^2 \leqq B\|v\|^2. \tag{38}$$

Here A and B are positive constants called *bounds* of the frame. The numbers $\alpha_n=(v, \phi_n)$, $n=1, 2, \cdots$, are called the moment sequence of the vector v relative to the frame. Since a frame is clearly a complete set, the finite linear combinations of the ϕ_n are everywhere dense. For the space $L_2(0, 1)$ the scalar product is defined as $(u, v)=\int_0^1 u(x)v^*(x)dx$.

If $\{\phi_n\}$ is a frame and $\{c_n\}$ is a sequence of numbers such that $\sum|c_n|^2<\infty$, then $\sum c_n\phi_n$ converges and

$$\left\|\sum_1^\infty c_n\phi_n\right\|^2 \leqq B\sum_1^\infty |c_n|^2. \tag{39}$$

For if

$$v_k = \sum_{n=1}^k c_n\phi_n, \qquad k \geqq 1,\ v_0 = 0,$$

then for $k\geqq j$ we have from Schwarz' inequality and the frame condition,

$$\|v_k - v_j\|^2 = \sum_{n=j+1}^k c_n(\phi_n, v_k - v_j)$$
$$\leqq \left\{\sum_{n=j+1}^k |c_n|^2\right\}^{1/2}\left\{B\|v_k - v_j\|^2\right\}^{1/2}.$$

Hence

$$\|v_k - v_j\|^2 \leqq B \sum_{n=j+1}^{k} |c_n|^2,$$

and (39) follows.

This shows that a linear transformation S is defined by the relation

$$Sv = \sum_{n=1}^{\infty} (v, \phi_n)\phi_n. \tag{40}$$

The transformation S is self-adjoint, and, if we make use of (38), it follows that

$$A\|v\|^2 \leqq (Sv, v) \leqq B\|v\|^2. \tag{41}$$

This states that S is positive definite with positive upper and lower bounds. Hence the inverse S^{-1} exists as a self-adjoint transformation, and

$$B^{-1}\|v\|^2 \leqq (S^{-1}v, v) \leqq A^{-1}\|v\|^2. \tag{42}$$

LEMMA VIII. *Let $\{\phi_n\}$ be a frame and let v be an arbitrary vector. Then there exists a moment sequence $\{\beta_n\}$ such that*

$$v = \sum_{1}^{\infty} \beta_n\phi_n \tag{43}$$

and

$$B^{-1}\|v\|^2 \leqq \sum_{1}^{\infty} |\beta_n|^2 \leqq A^{-1}\|v\|^2. \tag{44}$$

If $\{b_n\}$ is any other sequence such that $v = \sum_1^\infty b_n\phi_n$, then $\{b_n\}$ is not the moment sequence of any vector, and

$$\sum_{1}^{\infty} |b_n|^2 = \sum_{1}^{\infty} |\beta_n|^2 + \sum_{1}^{\infty} |b_n - \beta_n|^2. \tag{45}$$

Proof. The first part of the lemma follows from the preceding discussion when we write $v = Su$, $\beta_n = (u, \phi_n)$. To prove (45) there is no loss of generality in supposing $\sum |b_n|^2 < \infty$. Then $0 = v - v = \sum(b_n - \beta_n)\phi_n$, so, with $v = Su$, $\beta_n = (u, \phi_n)$, we have $0 = \sum(b_n - \beta_n)\beta_n^*$ and (45) follows. The uniqueness of the transformation S^{-1} shows that $\{b_n\}$ is not a moment sequence.

Now define a new sequence $\{\phi_n'\}$ by $\phi_n' = S^{-1}\phi_n$. Then $\{\phi_n'\}$ is a frame with the positive bounds A^{-1}, B^{-1}. For if v is any vector and $u = S^{-1}v$, then

$$\sum_n |(v, \phi_n')|^2 = \sum_n |(Su, \phi_n')|^2 = \sum_n |(u, \phi_n)|^2 = (Su, u) = (S^{-1}v, v)$$

so the result follows from (42). If v is any vector, then it may be expanded by *conjugate frames* in the form

$$v = \sum_n (v, \phi_n')\phi_n = \sum_n (v, \phi_n)\phi_n'. \tag{46}$$

The following result also gives a method of finding the expansion coefficients β_n by a rapidly converging process of successive approximations.

THEOREM III. *Let $\{\phi_n\}$ be a frame, and let $\rho = 2/(A+B)$. If v is an arbitrary vector, define*

$$v^{(1)} = v - \rho \sum_n (v, \phi_n)\phi_n,$$

$$v^{(k+1)} = v^{(k)} - \rho \sum_n (v^{(k)}, \phi_n)\phi_n, \qquad k \geqq 1.$$

Let

$$\beta_n^{(k)} = \rho(v + v^{(1)} + v^{(2)} + \cdots + v^{(k-1)}, \phi_n),$$

$$v_k = \sum_{n=1}^{\infty} \beta_n^{(k)} \phi_n.$$

Then

$$\|v - v_k\| \leqq \left(\frac{B-A}{B+A}\right)^k \|v\|. \tag{47}$$

Proof. The transformation $T = I - \rho S$ satisfies

$$|(Tv, v)| \leqq \theta \|v\|^2, \qquad \theta = \frac{B-A}{B+A}.$$

Since T is self-adjoint, $\|Tv\| \leqq \theta\|v\|$. Thus $\|v^{(1)}\| \leqq \theta\|v\|$, and in general, $\|v^{(k)}\| \leqq \theta^k\|v\|$. Adding the relations $v^{(k+1)} - v^{(k)} = -\rho S v^{(k)}$ for $k = 0, 1, \cdots, m-1$, we see that $v^{(m)} - v = -v_m$. Thus $\|v_m - v\| = \|v^{(m)}\| \leqq \theta^m\|v\|$.

A frame which fails to be a frame on the removal of any one of its vectors is termed an *exact frame*. It is not difficult to show by an example that the abstract analogue of Lemma VII is false.

LEMMA IX. *The removal of a vector from a frame leaves either a frame or an incomplete set.*

Proof. Suppose that ϕ_m is removed from the frame $\{\phi_n\}$. As a special case of (43) we may write

$$\phi_m = \sum \beta_n \phi_n \tag{48}$$

where $\phi_m' = S^{-1}\phi_m$ and $\beta_n = (\phi_m', \phi_n)$.

If $\beta_m \neq 1$, then

$$\phi_m = (1 - \beta_m)^{-1} \sum{}' \beta_n \phi_n$$

where $\sum{}'$ indicates the omission of the mth term. Thus if v is an arbitrary vector, then

$$| (v, \phi_m) |^2 \leqq | 1 - \beta_m |^{-2} \{ \sum{}' | \beta_n |^2 \} \{ \sum{}' | (\phi_n, v) |^2 \}$$

so

$$\sum | (v, \phi_n) |^2 \leqq \{ 1 + | 1 - \beta_m |^{-2} \sum{}' | \beta_n |^2 \} \sum{}' | (v, \phi_n) |^2.$$

It follows that the subset of $\{\phi_n\}$ with $n \neq m$ is a frame, for in place of (38) we have $A' \|v\|^2 \leqq \sum{}' | (v, \phi_n) |^2 \leqq B \|v\|^2$ where

$$A' = \frac{A}{1 + | 1 - \beta_m |^{-2} \sum{}' | \beta_n |^2}.$$

Now suppose that in (48) we have $\beta_m = 1$. We show in this case that $\{\phi_n\}$, $n \neq m$, is incomplete. Since $\beta_m = 1$, we obtain

$$0 = \sum{}' \beta_n \phi_n$$

and this may be written

$$0 = \sum \beta_n' \phi_n$$

where $\beta_n' = \beta_n$ if $n \neq m$, $\beta_m' = 0$. But $0 = \sum 0 \cdot \phi_n$ so relation (45) with $b_n = 0$ shows that

$$2 \sum | \beta_n' |^2 = 0.$$

Thus $\beta_n = 0$ for $n \neq m$. Now $\beta_n = (\phi_m', \phi_n)$ where $\phi_m' = S^{-1} \phi_m$, and the vector ϕ_m' is therefore orthogonal to all vectors of the set $\{\phi_n\}$, $n \neq m$. Thus

$$(\phi_m', \phi_n) = \delta_{mn}. \tag{49}$$

LEMMA X. *If $\{\phi_n\}$ is an exact frame, then $\{\phi_n\}$ and $\{\phi_n'\}$, where $\phi_n' = S^{-1}\phi_n$, are biorthogonal. Any sequence of numbers $\{c_n\}$ such that $\sum | c_n |^2 < \infty$ is the moment sequence of some vector with respect to $\{\phi_n\}$, and*

$$A \sum | c_n |^2 \leqq \| \sum c_n \phi_n \| \leqq B \sum | c_n |^2. \tag{50}$$

Proof. If $\{\phi_n\}$ is an exact frame, then (49) is true for all m as well as for all n so $\{\phi_n\}$ and $\{\phi_n'\}$ are biorthogonal. Given the sequence $\{c_n\}$, according to (39) the vector v, $v = \sum c_n \phi_n$, has finite norm. Then

$$c_n = (v, \phi_n') = (u, \phi_n)$$

where $u = S^{-1} v$, so $\{c_n\}$ is the moment sequence of the vector u. Then (50) follows from (44).

Paley and Wiener have also given a Hilbert space development of their theory. It is of interest to show the precise relationship with the present theory. Their theory concerns an infinite sequence of vectors $f_1, f_2, \cdots$ which is close to a complete orthonormal sequence $\psi_1, \psi_2, \cdots$. By "close" is meant that for any sequence of complex numbers $\{c_n\}$,

$$\left\| \sum c_n(f_n - \psi_n)\right\|^2 \leqq \theta^2 \sum |c_n|^2 \tag{51}$$

where θ, $0<\theta<1$, is a constant independent of $\{c_n\}$. An arbitrary vector v may be represented as $v=\sum c_n\psi_n$ and it may be shown that (51) implies that the sequence $\{f_n\}$ satisfies the frame condition (38). This was first pointed out by Boas [1]. Applying the triangle inequality to (51) gives

$$(1-\theta)^2\sum |c_n|^2 \leqq \left\|\sum c_nf_n\right\|^2 \leqq (1+\theta)^2\sum |c_n|^2. \tag{52}$$

Hence $\{f_n\}$ fails to be complete on the removal of any one of the f_n. Thus $\{f_n\}$ is an exact frame.

Conversely suppose that $\{\phi_n\}$ is an exact frame. Let $f_n = 2(B^{1/2}+A^{1/2})^{-1}\phi_n$ where A and B are the upper and lower bounds of the sequence $\{\phi_n\}$. Then (50) may be expressed in the form (52) with $\theta=(B^{1/2}-A^{1/2})/(B^{1/2}+A^{1/2})$. It was shown by Duffin and Eachus [5] that relation (52) for a complete set $\{f_n\}$ implies the existence of a complete orthonormal set $\{\psi_n\}$ satisfying (51). *Thus the theory of Paley and Wiener and the theory of exact frames are equivalent.*

The following theorem gives a new example of an exact frame. The proof is omitted. *Let $\{\lambda_n\}$ be a sequence of uniform density d such that for some positive constant τ, $\{\lambda_n\}$ and $\{\lambda_n+\tau\}$ are the same set of points. Then the set of functions $\{\exp(i\lambda_n t)\}$ is an exact frame over $(-\pi d, \pi d)$.*

4. **Pointwise convergence.** Let $\{\lambda_n\}$ be a sequence of uniform density 1 and suppose that $g(x)\in L_2(-\pi,\pi)$. Then Theorem I and Lemma VIII together imply that corresponding to each positive constant γ, $0<\gamma<\pi$, there exist expansion coefficients c_n such that $\sum |c_n|^2<\infty$ and

$$g(x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{-N}^{N} c_n e^{i\lambda_n x}, \qquad -\gamma \leqq x \leqq \gamma.$$

We shall show that indeed the limit in the mean $\sum_{-N}^{N} c_n \exp(i\lambda_n x)$ exists over the larger interval $(-\pi,\pi)$, defining a function $g(x)$ in the latter interval. It is also to be shown that the nonharmonic Fourier series actually converges to $g(x)$ at a given point of $(-\pi,\pi)$ if and only if the ordinary Fourier series of $g(x)$ converges to $g(x)$ at this point. A similar statement could be made for summability. Among all sequences $\{c_n\}$ corresponding to the same $g(x)$ in $(-\gamma,\gamma)$ there exists a unique sequence which minimizes $\sum |c_n|^2$, according to Lemma VIII. Whether or not $\{c_n\}$ is this minimizing sequence does not effect the convergence of the nonharmonic Fourier series.

Paley and Wiener have also obtained convergence properties for their class of nonharmonic Fourier series. Theorem IV is a generalization of their result and has a sharper conclusion. The proof is along different lines than that of these authors. The convergence theory of Paley and Wiener has been generalized along different lines by Levinson [8].

In order to investigate the convergence of the nonharmonic Fourier series, we write

$$e^{i\lambda_n x} = e^{inx}e^{i(\lambda_n - n)x} = e^{inx}\sum_{k=0}^{\infty} \frac{i^k(\lambda_n - n)^k x^k}{k!}$$

and we recall that since $\{\lambda_n\}$ is a sequence of uniform density 1, $|\lambda_n - n|$ is bounded. Thus, in the notation of Theorem IV below,

$$g_N(x) = \sum_{-N}^{N} c_n e^{\lambda_n x} = \sum_{-N}^{N} c_n \zeta_n(x)$$

are the partial sums of the nonharmonic Fourier series of $g(x)$.

THEOREM IV. *Let $\{b_k\}$ be a sequence of positive constants such that $\sum_0^\infty b_k \pi^k < \infty$, and write*

$$\zeta_n(x) = e^{inx}\sum_{k=0}^{\infty} b_{nk}x^k \tag{53}$$

where $|b_{nk}| \leqq b_k$, $n = 0, \pm 1, \pm 2, \cdots$. If $\{c_n\}$ is any set of complex numbers such that $\sum |c_n|^2 < \infty$, then

$$g_N(x) = \sum_{n=-N}^{N} c_n \zeta_n(x) \tag{54}$$

converges in mean to a function $g(x)$ of class $L_2(-\pi, \pi)$. If $\{a_n\}$ are the Fourier coefficients of $g(x)$ over $(-\pi, \pi)$, then

$$\phi_N(x) = (\pi^2 - x^2)\sum_{-N}^{N} (c_n\zeta_n(x) - a_n e^{inx}) \tag{55}$$

converges uniformly to zero for $-\pi \leqq x \leqq \pi$.

Proof. We have

$$g_N(x) = \sum_{n=-N}^{N} c_n\zeta_n(x) = \sum_{k=0}^{\infty} \psi_{Nk}(x)x^k$$

where

$$\psi_{Nk}(x) = \sum_{n=-N}^{N} c_n b_{nk} e^{inx}. \tag{56}$$

Then

$$\|\psi_{Nk}(x)\|^2 = \sum_{n=-N}^{N} |c_n b_{nk}|^2 \leqq c^2 b_k^2$$

where $c^2 = \sum_{-\infty}^{\infty} |c_n|^2$. It is clear that

$$\psi_k(x) = \underset{N\to\infty}{\text{l.i.m.}}\, \psi_{Nk}(x) \tag{57}$$

exists and that

$$\|\psi_k(x)\| \leqq cb_k, \qquad \psi_k(x) \sim \sum_{-\infty}^{\infty} c_n b_{nk} e^{inx}.$$

Then

$$\left\| \sum_{k=\mu}^{\nu} x^k \psi_k(x) \right\| \leqq \sum_{k=\mu}^{\nu} \|x^k \psi_k(x)\| \leqq \sum_{k=\mu}^{\nu} \pi^k c b_k$$

so it follows that $\sum_0^\nu x^k \psi_k(x)$ converges in mean to some function of class $L_2(-\pi, \pi)$. We define this function as $g(x)$ and show that $g(x)$ has the properties given in the statement of the theorem. Thus

$$g(x) = \underset{\nu\to\infty}{\text{l.i.m.}} \sum_{k=0}^{\nu} x^k \psi_k(x).$$

Then

$$\begin{aligned} \|g(x) - g_N(x)\| &\leqq \sum_{k=0}^{\infty} \|x^k(\psi_{Nk}(x) - \psi_k(x))\| \\ &\leqq \sum_{k=0}^{\infty} \pi^k \|\psi_{Nk}(x) - \psi_k(x)\| \leqq \sum_{k=0}^{\infty} \pi^k c b_k. \end{aligned}$$

But $\|\psi_{Nk}(x) - \psi_k(x)\| \to 0$ as $N \to \infty$ for each fixed k so it follows that

$$g(x) = \underset{N\to\infty}{\text{l.i.m.}}\, g_N(x).$$

To investigate the question of convergence we write

$$D_N(u) = \frac{\sin (N + 1/2)u}{\sin (u/2)}$$

for the Dirichlet kernel. Now (56), (57) show that

$$g_N(x) = \sum_{k=0}^{\infty} x^k (\psi_k(t), D_N(x - t)).$$

Now define $f_N(x)$ by the relation

$$f_N(x) = (g(t), D_N(x - t)) = \sum_{k=0}^{\infty} (t^k \psi_k(t), D_N(x - t)).$$

Then $f_N(x)$ is the Nth partial sum of the Fourier series of $g(x)$ over $(-\pi, \pi)$ so

$$\begin{aligned}\phi_N(x) &= (\pi^2 - x^2)(g_N(x) - f_N(x)) \\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_{-\pi}^{\pi}(\pi^2 - x^2)(x^k - t^k)\psi_k(t)D_N(x - t)dt \\ &= (\pi^2 - x^2)\sum_{k=0}^{\infty}(\psi_k(t), (x^k - t^k)D_N(x - t)).\end{aligned} \tag{58}$$

We now show that there is a constant A independent of k such that

$$\left|\frac{(\pi^2 - x^2)(x^k - t^k)}{\sin((x - t)/2)}\right| \leqq A\pi^k, \qquad |x| \leqq \pi, \ |t| \leqq \pi. \tag{59}$$

By a change of variables, this is equivalent to the relation

$$\left|\frac{(1 - x^2)(x^k - t^k)}{\sin(\pi(x - t)/2)}\right| \leqq A\pi^{-2}, \qquad |x| \leqq 1, \ |t| \leqq 1.$$

Clearly it is sufficient to take $x \geqq 0$, in which case $(x-t)/2$ lies in the interval $(-1/2, 1)$. But $u(u-1)/\sin \pi u$ is bounded when u lies in this interval, so it is sufficient to show that

$$h = \left|\frac{(1 - x)(x^k - t^k)}{(x - t)(2 + t - x)}\right|$$

is bounded by 4 for $-1 \leqq t \leqq 1$, $0 \leqq x \leqq 1$. If $-1 \leqq t \leqq -1/2$, then $h \leqq 4(1-x)/(2+t-x) \leqq 4$. If $-1/2 < t \leqq 1$, then $h \leqq 2(1-x)(x^k - t^k)/(x-t) = 2(1-x)(x^{k-1} + tx^{k-2} + \cdots + t^{k-1}) \leqq 2$.

From (59) we see that

$$|(\psi_k(t), (\pi^2 - x^2)(x^k - t^k)D_N(x - t))| \leqq A\pi^k\|\psi_k(t)\|$$

so

$$\sum_{k=M}^{\infty}|(\psi_k(t), (\pi^2 - x^2)(x^k - t^k)D_N(x - t))| \leqq Ac\sum_{k=M}^{\infty}\pi^k b_k. \tag{60}$$

Given ϵ, $\epsilon > 0$, choose M so large that

$$Ac\sum_{k=M}^{\infty}\pi^k b_k < \frac{\epsilon}{3}\cdot$$

It is clear that

$$(\pi^2 - x^2)\sum_{k=0}^{M-1}(\psi_k(t), (x^k - t^k)D_N(x - t))$$

tends to zero as $x \to \pm\pi$, so there is a positive number $\delta = \delta(\epsilon)$ such that this sum is bounded by $\epsilon/3$ in the intervals $(-\pi, -\pi+\delta)$ and $(\pi - \delta, \pi)$. If x lies in the interval $-\pi+\delta \leqq x \leqq \pi-\delta$ and $k=0, 1, 2, \cdots, M-1$, then

$$H_k(x, t) = \frac{x^k - t^k}{\sin((x-t)/2)}$$

is a continuous function of x and t, $-\pi \leqq t \leqq \pi$. Then

$$(\psi_k(t), (x^k - t^k)D_N(x-t)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_k(t) H_k(x, t) \sin\left(N + \frac{1}{2}\right)(x-t)dt,$$

and a simple extension of the Riemann-Lebesgue lemma shows that this tends to zero uniformly in x, $|x| \leqq x-\delta$, as N tends toward infinity. Thus choose N so large that

$$\left| (\pi^2 - x^2) \sum_{k=0}^{M-1} (\psi_k(t), (x^k - t^k)D_N(x-t)) \right| < \frac{\epsilon}{3}$$

for $|x| \leqq \pi - \delta$. Then

$$|\phi_N(x)| < \epsilon, \qquad -\pi \leqq x \leqq \pi,$$

and the theorem follows.

References

1. R. P. Boas, Jr., *A trigonometric moment problem*, J. London Math. Soc. vol. 14 (1939) pp. 242–244.

2. ———, *Entire functions bounded on a line*, Duke Math J. vol. 6 (1940) pp. 148–149.

3. ———, *A general moment problem*, Amer. J. Math. vol. 63 (1941) pp. 361–370.

4. D. G. Bourgin, *A class of sequences of functions*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 478–518.

5. R. J. Duffin and J. J. Eachus, *Some notes on an expansion theorem of Paley and Wiener*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 850–855.

6. R. J. Duffin and A. C. Schaeffer, *Power series with bounded coefficients*, Amer. J. Math. vol. 67 (1945) pp. 141–154.

7. I. Ibragimov, *On the completeness of systems of analytic functions*, Bull. Acad. Sci. URSS. Sér. Math. vol. 13 (1949) pp. 45–54.

8. N. Levinson, *Gap and density theorems*, Amer. Math. Soc. Colloquium Publications, vol. 26, 1940, chaps. I and IV.

9. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 19, 1934, chaps. I and VII.

10. M. Plancherel and G. Pólya, *Functions entières et intégrales de Fourier multiples*, Comment. Math. Helv. vol. 10 (1937–1938) pp. 110–163.

11. E. C. Titchmarsh, *The theory of functions*, Oxford, 1932, chap. III.

Carnegie Institute of Technology,
Pittsburgh, Pa.
University of Wisconsin,
Madison, Wis.