New twists on an old standard

Small but important phrases are sometimes tucked into individual standards statements. These may indicate shifts in the traditional approach to the topic in question. Such cases might be good starting points for professional development activities—simple ways to get into the document by looking at a single detail that quickly opens up to larger issues.

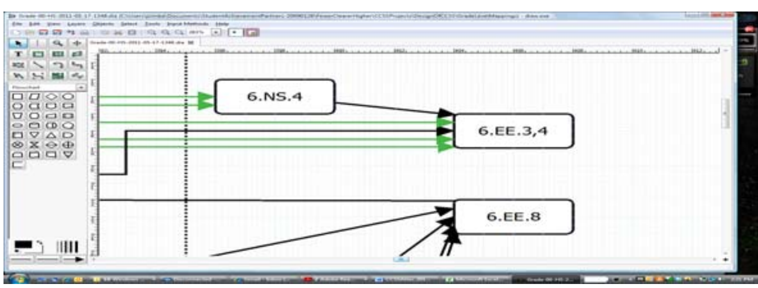
There are any number of cases in which the specific wording of individual standards suggests a shift in approach. Here are just a couple of perhaps less obvious examples:

6.NS.4

This standard refers to greatest common factor—a staple of in-class worksheets. But note the second sentence and the example given:

> Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. **Use the distributive property to express a sum of two whole numbers 1–100 with a common factor as a multiple of a sum of two whole numbers with no common factor. For example, express 36 + 8 as 4 (9 + 2).**

The example given is to express 36 + 8, a sum of two numbers with a common factor, as 4(9 + 2), a multiple of a sum of two numbers with no common factor.



This example sends a signal about how “greatest common factor” can be piped into the main flow of ideas leading to algebra. For example, consider what we do when we express $ax + a$ as $a(x + 1)$. We are using the distributive property to express a sum of terms with a common factor as a multiple of a sum of terms with no common factor (6.EE.3,4). Standard 6.NS.4 is a kind of rehearsal for this, with numbers instead of variables.

3.OA.1,2,6

> 3.OA.1. Interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7.
>
> 3.OA.2. Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.
>
> 3.OA.6. Understand division as an unknown-factor problem. For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.

These three standards are notable for what they don’t say. For example, nowhere in the Standards is there an explicit expectation that students will interpret products as repeated addition, jumps on a number line, skip counting, or the like. The Grade 3 standards only explicitly require students to interpret a product in terms of equal groups, and to interpret a quotient as a number of groups, a number in each group, or an unknown factor.

Of course, students come into Grade 3 knowing only addition and subtraction, so at first they will have to use addition in order to find products. But that makes repeated addition a strategy for finding a product, not what the product itself fundamentally means. (And likewise for repeated subtraction as a strategy for finding a quotient, not what a quotient fundamentally means.) The distinction between the meaning of a quotient vs. the steps one takes to compute that quotient is of course an important one. Consider that fractions and decimals have distinct algorithms for computing quotients, yet the operation and its meaning are the same in either case.

Multiplicative reasoning differs in kind from additive reasoning. This difference becomes acute by Grade 5.NF, and it connects to proportional reasoning in Grades 6 and 7. While proportional reasoning problems can sometimes be solved by skip-counting, that is certainly not the goal. Moreover, repeated addition is pretty hopeless as a way to understand a Grade 7 problem such as $(^8/_3)\, x = {}^3/_4$.

6.EE.3

> Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression $3\,(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6\,(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.

Normally, the three examples given in this standard would be called, respectively, distributing, factoring, and collecting like terms. But in the standard itself, none of those terms are used. The implication is that instead of thinking of these techniques as a to-do list of disconnected items, they are all applications of a few fundamental and familiar principles. This is how the Standards aim to approach both arithmetic and algebra altogether; this is coherence in a nutshell.

We end this section on shifts by calling attention to a couple of “eloquent silences” in the Standards. Granting that eloquence is probably an overstatement, one can at least say that certain silences were meant to be audible. Two of these are:

Fractions in lowest terms. There is no explicit requirement in the Standards about simplifying fractions or putting fractions into lowest terms. What there is instead is an important progression of concepts and skills relating to fraction equivalence (e.g., 4.NF.4). Observe that putting a fraction into lowest terms is a special case of generating equivalent fractions.

(Of course, generating equivalent fractions can go "either way" – starting from $^{4}/_{12}$, we might generate an equivalent fraction $^{1}/_{3}$, or we might generate an equivalent fraction $^{40}/_{120}$.)

While the standards don't make an explicit demand that answers to fraction problems be put in lowest terms, teachers are of course free to impose that requirement if they wish. If students have a good understanding of fraction equivalence, and fluency with multiplication and division and knowledge of the times table, then putting fractions into lowest terms is presumably not too much of a problem. But in any case, $^{4}/_{12}$ and $^{1}/_{3}$ are equally correct ways to express $4 \times {}^{1}/_{12}$ as a fraction.

"Simplifying." Putting fractions into lowest terms is related to the issue of simplifying expressions in general. Apart from a single instance in the Mathematical Practices, the word "simplify" does not appear in the Standards. One reason for this is that "simple" is sometimes in the eye of the beholder. Is $x^2 + x$ simpler than $x(x + 1)$? (Is adding to a product simpler than multiplying by a sum? Perhaps the edge goes to the latter expression, because one of its atomic expressions is a number and not a variable? Perhaps the edge goes to the former expression, because all of its atomic expressions are the same? Or cosmetically because it has no parentheses?)

Sometimes it can also be important to "complexify" an expression. Depending on the context, writing $1.05P$ in the less(?) simple form $P + 0.05P$ could add insight.

Even in the case of numerical fractions, putting into lowest terms might not always be best—and might not always be particularly simple either. I think for example about my father, who was a screw machine operator at the end of his career. In the shirt pocket of his work uniform, he carried a metal machinist's scale marked in thousandths of an inch. For him, eight thousandths was a perfectly useful measure. In lowest terms, this is

$^1/_{125}$—an observation which is pretty academic given that no such number was etched onto his scale. I'm not sure if he even could have located $^1/_{125}$ on a thousandths scale; but if he could, it would be because of a good understanding of fraction equivalence.

Of course, none of this is meant to suggest that simplification is never straightforward, or never useful. And mathematics itself is a subject with a very strong aesthetic favoring elegance and simplicity. $^{(xy^2 + y^3)}/_{y^5}$ is by no means a sensible way to express $^x/_{y^3} + {}^1/_{y^2}$ as a fraction. But a student well-trained in the habit of seeing structure in expressions (A-SSE.2) would quickly see an opportunity to rewrite the numerator of the former expression as $y^2(x + y)$ en route to rewriting finally as $^{(x + y)}/_{y^3}$. Anybody would agree that this is a simplification. The point is that simplification, least common denominators, and putting into lowest terms shouldn't be little tin gods that mock the fundamental mathematical ideas involved in generating equivalent fractions and generating equivalent expressions.