

# **SDS 383D: Exercises 4 – Hierarchical Models**

April 24, 2017

*Professor Scott*

**Spencer Woody**

## Problem 1

### Math Tests

We have a model where  $y_{ij}$  is the test score of the  $j$ th student in school  $i$ , with indices  $i = 1, 2, \dots, I$  and  $j = 1, 2, \dots, N_i$ , so  $N_i$  is the sample size for school  $i$  and there are  $N = \sum_{i=1}^I$  total test scores. Let  $\lambda = 1/\sigma^2$  and  $\gamma = 1/\tau^2$  be the precision parameters. Further, let  $y_i = [y_{i1}, y_{i2}, \dots, y_{iN_i}]^T$  and  $y = [y_1^T, y_2^T, \dots, y_I^T]^T$  and  $\theta = [\theta_1, \theta_2, \dots, \theta_I]^T$ . As we can see in Figure 1, schools with smaller sample sizes tend to have more extreme average test scores.

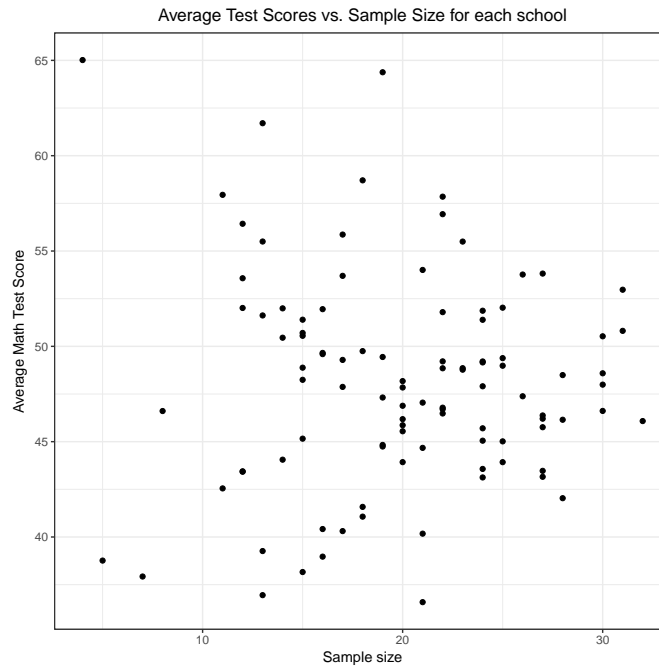


Figure 1: Scatter plot of sample size and average test scores

The hierarchical model for these data is

$$\begin{aligned} (y_{ij}|\theta_i, \lambda) &\sim \mathcal{N}(\theta_i, \lambda^{-1}) \\ (\theta_i|\mu, \lambda, \gamma) &\sim \mathcal{N}(\mu, (\lambda\gamma)^{-1}). \end{aligned}$$

We set the priors

$$\begin{aligned} \pi(\mu) &\propto 1, \quad -\infty < \mu < \infty \\ \pi(\lambda) &\propto \lambda^{-1}, \quad \lambda > 0 \\ \pi(\gamma) &\propto 1, \quad \gamma > 0, \end{aligned}$$

that is to say, .... In order to implement the Gibbs sampler, we need the posterior full conditionals for each  $\theta_i$ ,  $\mu$ ,  $\lambda$ , and  $\gamma$ .

- For each  $\theta_i$ ,

$$\begin{aligned} f(\theta_i|y_i, \mu, \lambda, \gamma) &\propto f(y_i|\theta_i, \lambda) \cdot f(\theta_i|\mu, \lambda, \gamma) \\ &\sim \mathcal{N}\left((N_i\lambda + \lambda\gamma)^{-1} \cdot (N_i\lambda\bar{y}_i + \lambda\gamma\mu), (N_i\lambda + \lambda\gamma)^{-1}\right), \end{aligned}$$

which we know from the normal-normal conjugacy derived in Exercises 1.

- For  $\mu$ ,

$$\begin{aligned}
 \pi(\mu|y, \theta, \lambda, \gamma) &\propto f(\theta|\lambda, \gamma, \mu) \cdot \pi(\mu) \\
 &\propto \left( \prod_{i=1}^I \exp \left[ -\frac{1}{2} \lambda \gamma (\theta_i - \mu)^2 \right] \right) \cdot 1 \\
 &= \exp \left[ -\frac{1}{2} \lambda \gamma \sum_{i=1}^I (\theta_i - \mu)^2 \right] \\
 &= \exp \left[ -\frac{1}{2} \lambda \gamma \sum_{i=1}^I (\theta_i^2 - 2\theta_i \mu + \mu^2) \right] \\
 &\propto \exp \left[ -\frac{1}{2} \lambda \gamma (I\mu^2 - 2I\bar{\theta}\mu) \right] \\
 &\sim \mathcal{N}(\bar{\theta}, (I\lambda\gamma)^{-1}).
 \end{aligned}$$

- For  $\lambda$ ,

$$\begin{aligned}
 \pi(\lambda|y, \mu, \gamma, \theta) &\propto f(y|\lambda, \theta) \cdot f(\theta|\lambda, \gamma, \mu) \cdot \pi(\lambda) \\
 &\propto \left( \prod_{i=1}^I \prod_{j=1}^{N_i} \lambda^{1/2} \exp \left[ -\frac{1}{2} (y_{ij} - \theta_i)^2 \right] \right) \cdot \left( \prod_{i=1}^I \lambda^{1/2} \exp \left[ -\frac{1}{2} \lambda \gamma (\theta_i - \mu)^2 \right] \right) \cdot \lambda^{-1} \\
 &= \lambda^{(N+I)/2-1} \exp \left[ -\frac{1}{2} \left( \sum_{i=1}^I \sum_{j=1}^{N_i} (y_{ij} - \theta_i)^2 + \gamma \sum_{i=1}^I (\theta_i - \mu)^2 \right) \lambda \right] \\
 &\sim \text{Gamma} \left( \frac{N+I}{2}, \frac{1}{2} \left[ \sum_{i=1}^I \sum_{j=1}^{N_i} (y_{ij} - \theta_i)^2 + \gamma \sum_{i=1}^I (\theta_i - \mu)^2 \right] \right).
 \end{aligned}$$

- For  $\gamma$ ,

$$\begin{aligned}
 \pi(\gamma|y, \mu, \lambda, \theta) &\propto f(\theta|\lambda, \gamma, \mu) \cdot \pi(\gamma) \\
 &\propto \left( \prod_{i=1}^I \gamma^{1/2} \exp \left[ -\frac{1}{2} \lambda \gamma (\theta_i - \mu)^2 \right] \right) \cdot 1 \\
 &= \gamma^{I/2} \exp \left[ -\frac{1}{2} \lambda \sum_{i=1}^I (\theta_i - \mu)^2 \cdot \gamma \right] \\
 &\sim \text{Gamma} \left( \frac{I}{2} + 1, \frac{1}{2} \lambda \sum_{i=1}^I (\theta_i - \mu)^2 \right).
 \end{aligned}$$

Table 1: 95% posterior credible intervals

	2.5%	50%	97.5%
$\mu$	47.03	48.10	49.18
$\lambda$	0.0111	0.0118	0.0126
$\gamma$	2.43	3.49	5.03

Given the posterior mean  $\hat{\theta}_i$  as an estimate of  $\theta_i$ , define the shrinkage coefficient

$$\kappa_i = \frac{\bar{y}_i - \hat{\theta}_i}{\bar{y}_i},$$

which is a measure incomplete pooling. Figure 2 shows the absolute shrinkage coefficient for each school as a function of sample size. As sample size increases, the shrinkage decreases because we are gaining precision in estimating the school-level mean  $\theta_i$ .

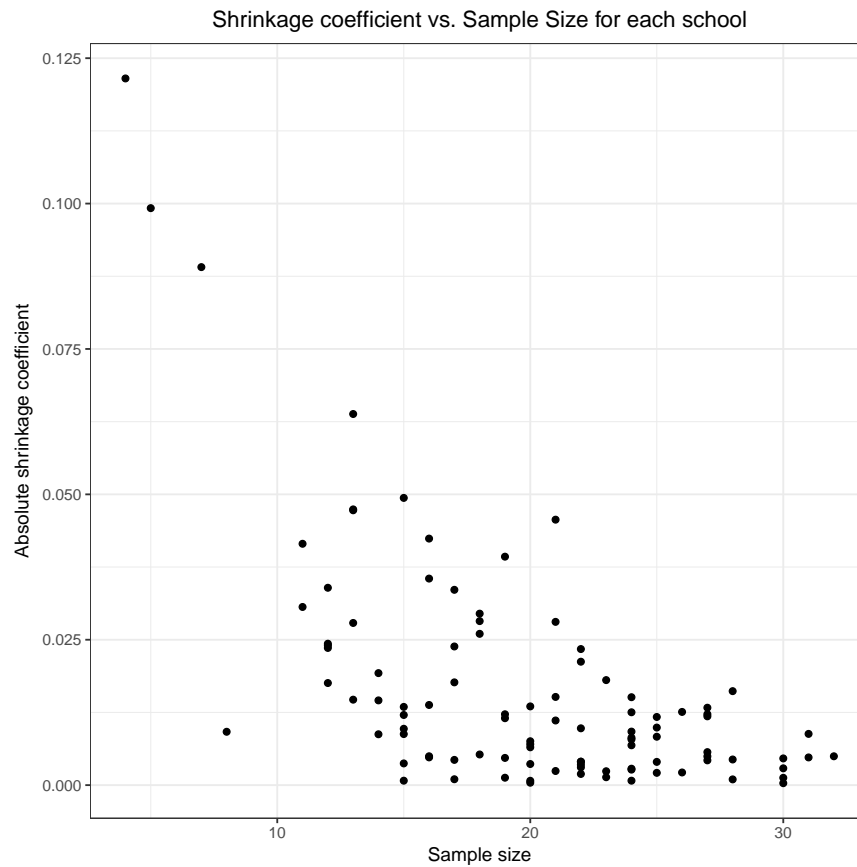


Figure 2: Absolute shrinkage coefficient as a function of sample size

## Problem 2

### Price elasticity of demand

Here we model the demand curve for cheese, which is given by

$$Q = \alpha P^\beta,$$

where  $Q$  is the quantity of cheese demanded,  $P$  is price,  $\beta$  is a parameter for the *price elasticity of demand* and  $\alpha$  is a (rather unremarkable) scaling parameter. Note that if we take a logarithmic transform of the equation in our demand model, we obtain the linear relationship

$$\log Q = \log \alpha + \beta \log P.$$

Figure 3 shows all the data with a fitted OLS line, and Figure 4 shows the data on a store-by-store level with the same OLS line from all data on each panel. The fact that the OLS line performs poorly on any given individual store's data suggests that a hierarchical approach would be beneficial. The hierarchical linear model for the quantity of cheese sold for the  $t$ th observation at store  $i$  is

$$y_{it} = \alpha_i + \beta_i x_{it} + \gamma_i z_{it} + \theta_i z_{it} x_{it} + \epsilon_{it},$$

where  $x_{it}$  is the log-price of cheese and  $z_{it}$  is an indicator variable taking on a value of 1 when the display is shown, and 0 otherwise.

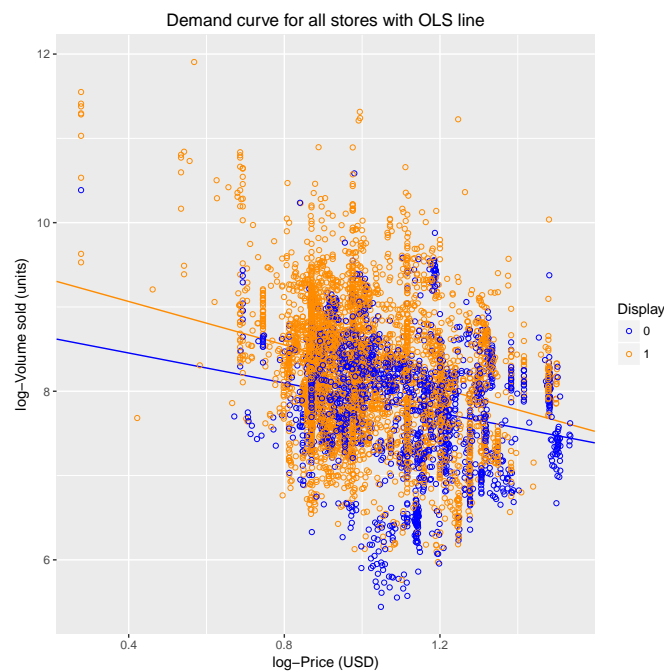


Figure 3: Scatterplot for data from all stores with OLS line

Using frequentist REML to build this model we obtain these results,

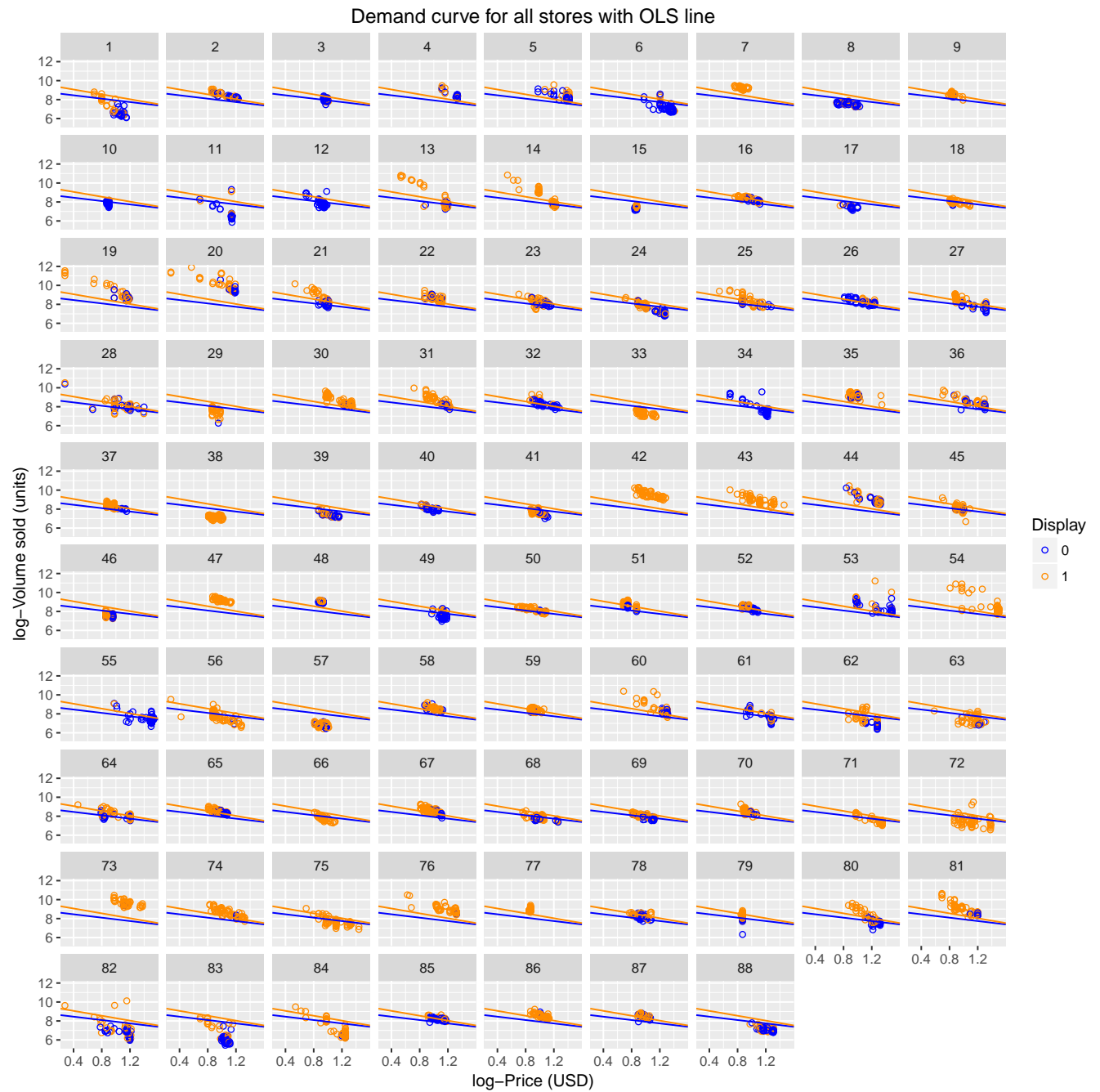


Figure 4: Scatterplot for data from all stores with OLS line

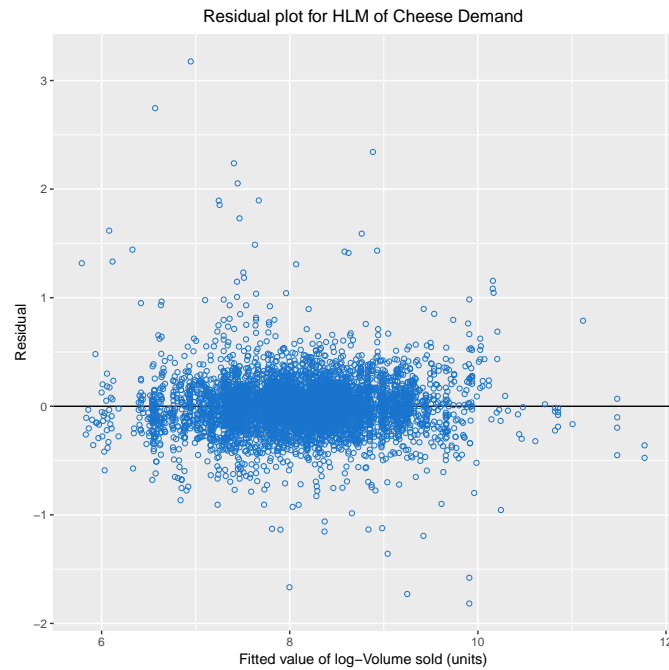


Figure 5: Residual plot using HLM and REML method

*Full Bayesian***Model specification**

Here we specify a general Bayesian hierarchical linear model. Let  $y_i$  be a  $n_i$ -length vector representing the responses of group  $i$ . There are  $N = \sum_i n_i$  total responses.  $X_i$  is the  $n_i \times p$  design matrix for the observations in group  $i$ , and  $Z_i$  is a  $n_i \times q$ ,  $q \leq p$  matrix whose columns are a subset of the columns of  $X_i$ , and this represents the subject-level effects, sometimes called “random effects.”. Then the responses  $y_i$  are distributed as:

$$y_i | \beta, b_i, \lambda \sim \mathcal{N}_{n_i}(X_i \beta + Z_i b_i, \lambda^{-1} \mathcal{I}_{n_i})$$

$$b_i | D \stackrel{\text{iid}}{\sim} \mathcal{N}_q(0, D)$$

Note that the responses  $y_{it}$  for subject  $i$  are therefore assumed to iid, and also note two results of this model,

$$E(y_i | b_i) = X_i \beta + Z_i b_i$$

$$E(y_i) = E(E(y_i | b_i)) = X_i \beta,$$

or in other words, The priors are

$$\pi(\lambda) \propto \lambda^{-1}$$

$$\pi(\beta) \propto 1$$

$$\pi(D) \sim \text{IW}(\nu, \Psi).$$

To implement a Gibbs sampler, we need the full conditional posterior distributions for  $b_i$ ,  $\lambda$ ,  $\beta$ , and  $D$ .

- For each  $b_i$ , first define  $v_i := y_i - X_i\beta$ ,

$$\begin{aligned}
 p(b_i|y_i, \lambda, \beta, D) &\propto p(y_i|\beta, b_i, \lambda)p(b_i|D) \\
 &\propto \exp\left[-\frac{1}{2}\lambda(y_i - X_i\beta - Z_i b_i)^T(y_i - X_i\beta - Z_i b_i)\right] \cdot \exp\left[-\frac{1}{2}b_i^T D^{-1} b_i\right] \\
 &= \exp\left[-\frac{1}{2}\lambda(Z_i b_i - v_i)^T(Z_i b_i - v_i)\right] \cdot \exp\left[-\frac{1}{2}b_i^T D^{-1} b_i\right] \\
 &\propto \exp\left[-\frac{1}{2}b_i^T (\lambda Z_i^T Z_i + D^{-1}) b_i - 2b_i^T \lambda Z_i^T v_i\right] \\
 &\propto \exp\left[-\frac{1}{2}\left(b_i - [\lambda Z_i^T Z_i + D^{-1}]^{-1} \lambda Z_i^T v_i\right)^T (\lambda Z_i^T Z_i + D^{-1}) \left(b_i - [\lambda Z_i^T Z_i + D^{-1}]^{-1} \lambda Z_i^T v_i\right)\right] \\
 &\sim \mathcal{N}\left([\lambda Z_i^T Z_i + D^{-1}]^{-1} \lambda Z_i^T v_i, [\lambda Z_i^T Z_i + D^{-1}]^{-1}\right) \\
 &\sim \mathcal{N}\left([\lambda Z_i^T Z_i + D^{-1}]^{-1} \lambda Z_i^T (y_i - X_i\beta), [\lambda Z_i^T Z_i + D^{-1}]^{-1}\right).
 \end{aligned}$$

- For  $\lambda$ ,

$$\begin{aligned}
 \pi(\lambda|y, \beta, b) &\propto p(y|\lambda, \beta) \cdot \pi(\lambda) \\
 &= \left(\prod_{i=1}^I \lambda^{n_i/2} \exp\left[-\frac{1}{2}\lambda(y_i - X_i\beta - Z_i b_i)^T(y_i - X_i\beta - Z_i b_i)\right]\right) \cdot \lambda^{-1} \\
 &\sim \text{Gamma}\left(\frac{N}{2}, \frac{1}{2} \sum_{i=1}^I \|y_i - X_i\beta - Z_i b_i\|_2^2\right)
 \end{aligned}$$

- For  $\beta$ , define  $w_i := y_i - Z_i b_i$ .

$$\begin{aligned}
 \pi(\beta|y, \lambda, b) &\propto p(y|\lambda, \beta) \cdot \pi(\beta) \\
 &\propto \left(\prod_{i=1}^I \exp\left[-\frac{1}{2}\lambda(y_i - X_i\beta - Z_i b_i)^T(y_i - X_i\beta - Z_i b_i)\right]\right) \cdot 1 \\
 &= \prod_{i=1}^I \exp\left[-\frac{1}{2}\lambda(X_i\beta - w_i)^T(X_i\beta - w_i)\right] \\
 &\propto \prod_{i=1}^I \exp\left[-\frac{1}{2}\lambda\left(\beta^T X_i^T X_i \beta - 2\beta^T X_i^T w_i\right)\right] \\
 &= \exp\left(-\frac{1}{2}\lambda\left[\beta^T \left(\sum_{i=1}^I X_i^T X_i\right) \beta - 2\beta^T \sum_{i=1}^I X_i^T w_i\right]\right) \\
 &= \exp\left(-\frac{1}{2}\lambda\left[\beta^T \left(\sum_{i=1}^I X_i^T X_i\right) \beta - 2\beta^T \sum_{i=1}^I X_i^T (y_i - Z_i b_i)\right]\right) \\
 &\sim \mathcal{N}\left(\left[\sum_{i=1}^I X_i^T X_i\right]^{-1} \sum_{i=1}^I X_i^T (y_i - Z_i b_i), \left[\lambda \sum_{i=1}^I X_i^T X_i\right]^{-1}\right).
 \end{aligned}$$



- For  $D$ ,

$$\begin{aligned}
 \pi(D|b) &\propto p(b|D) \cdot \pi(D) \\
 &\propto \left( \prod_{i=1}^I [\det(D)]^{-1/2} \exp \left[ -\frac{1}{2} b_i^T D^{-1} b_i \right] \right) \cdot [\det(D)]^{-\frac{\nu+q+1}{2}} \exp \left[ -\frac{1}{2} \text{tr}(\Psi D^{-1}) \right] \\
 &\sim \text{IW} \left( I + \nu, \Psi + \sum_{i=1}^I b_i b_i^T \right)
 \end{aligned}$$

The most computationally intensive part of this Gibbs sampler scheme is sampling each  $b_i$ , and I chose to do this by exploiting a block-diagonal matrix of each  $Z_i$  and drawing each  $b_i$  simultaneously as a long vector called  $b$ . For this application specifically, the  $X_i$  and  $Z_i$  are identical, with a column of 1's for the intercept, a column of log-prices, a column of indicator variables for display, and a column of interaction terms for log-price and display. We run 6000 iterations of the Gibbs sampler with the first 1000 draws discarded as burn-in. The `mix` folder within the `img` folder shows traceplots of  $\lambda$ , each component in  $\beta$ , and four randomly selected columns of posterior draws of  $b$ , which all show a good degree of mixing. Histograms for  $\lambda$  and each component of  $\beta$  are shown below. Figure 8 shows a grid of plots, each of which has 95% credible intervals of all the subject-level effects on a given covariate terms, arranged in increasing order by posterior median. Note that on the  $x$ -axis is different for each plot in order to have each one ordered by posterior median.

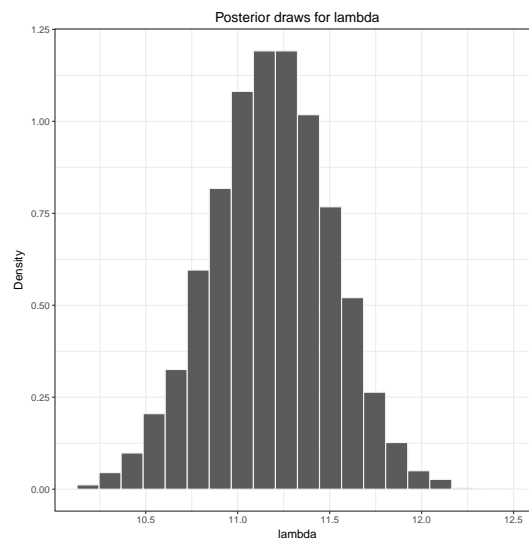


Figure 6: Histogram of posterior draws of  $\lambda$

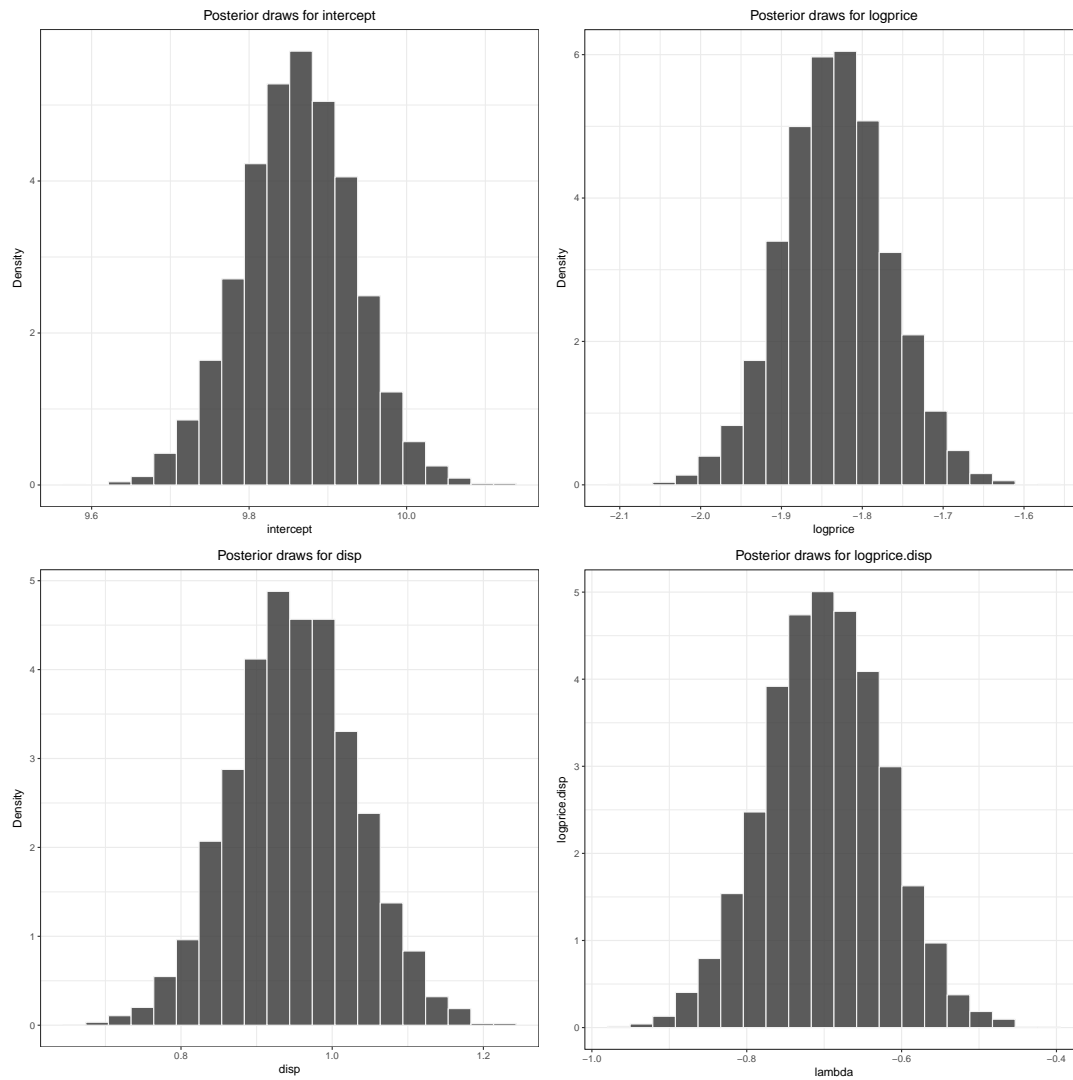


Figure 7: Histogram of posterior draws of each term in  $\beta$

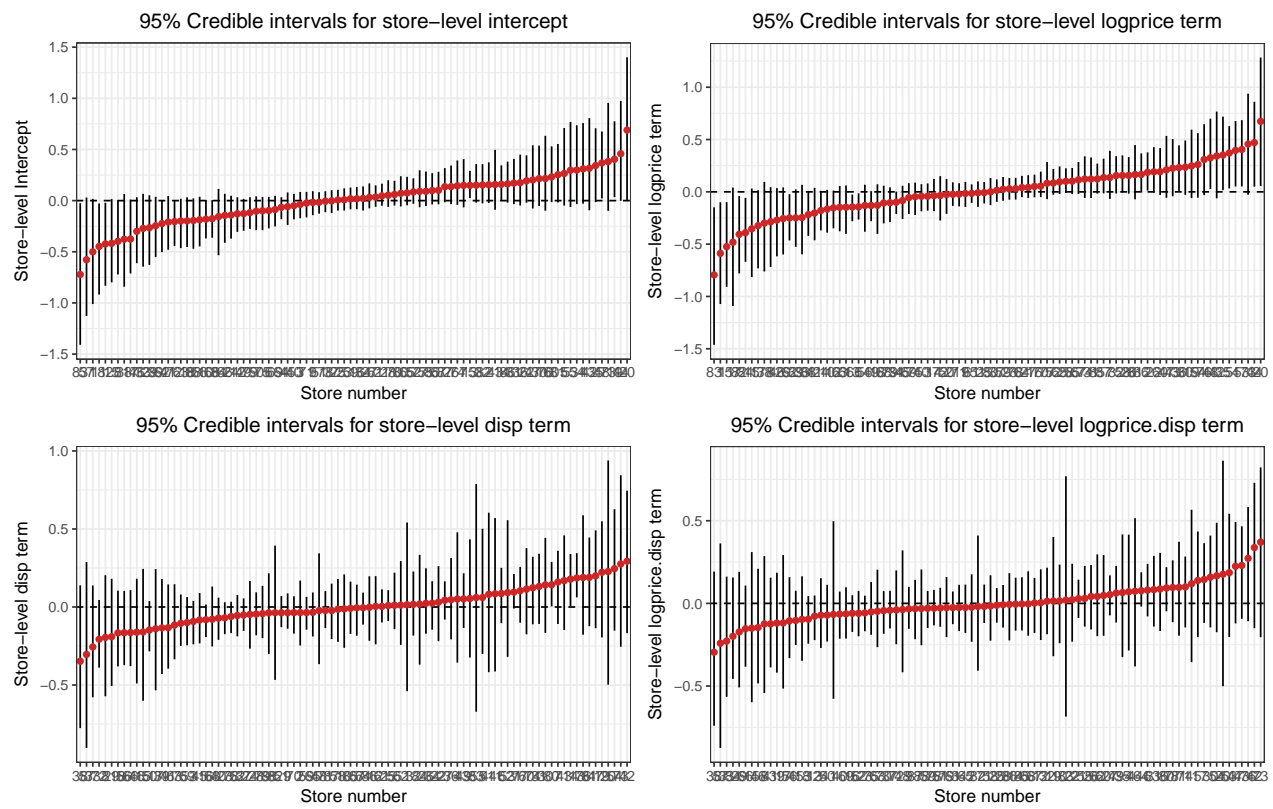


Figure 8: Ordered 95% credible intervals of store-level each store

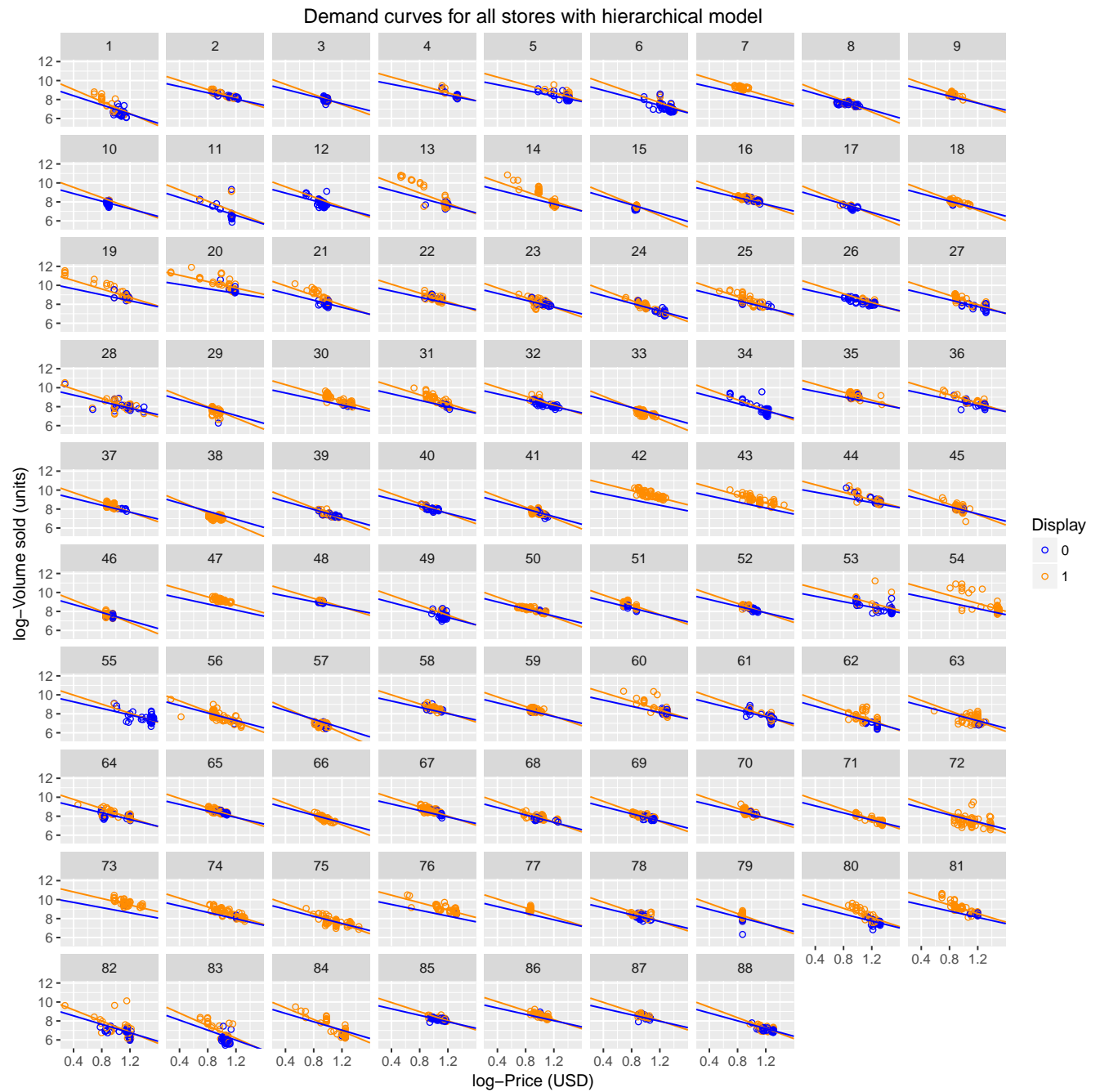


Figure 9: Each store's demand curves with fitted line from Bayesian hierarchical model

### Problem 3

#### A hierarchical probit model via data augmentation

For this model we model  $y_{ij}$ , the  $j$ th binary 0-1 response,  $j \in \{1, 2, \dots, n_i\}$ , within group  $i \in \{1, 2, \dots, I\}$  through the utilization of data augmentation whereby we introduce a latent variable  $z_{ij}$ ,

$$(z_{ij}|\beta, \gamma_i) \sim N(x_{ij}^T\beta + w_{ij}^T\gamma_i, 1)$$

$$y_{ij} = \mathbf{1}(z_{ij} > 0) = \begin{cases} 1 & \text{if } z_{ij} > 0 \\ 0 & \text{if } z_{ij} \leq 0 \end{cases},$$

where  $x_{ij}$  is a vector of covariate features and  $w_{ij}$  is a subset of these features whose effects vary at the subject level, captured through  $\gamma_i$ . We can see that this implies a probit link function so that

$$p_{ij} = P(y_{ij} = 1) = \Phi(x_{ij}^T\beta + w_{ij}^T\gamma_i),$$

where  $\Phi(\cdot)$  is the CDF of the standard normal distribution. Let  $z_i$  be the  $n_i$ -length vector of responses from subject  $i$ , and similarly,  $X_i$  is a  $n_i \times p$  design matrix of subject  $i$  and  $W_i$  is a  $n_i \times q$  design matrix with  $q \leq p$  whose columns are a subset of the columns of  $X_i$ . We then see that

$$(z_i|\beta, \gamma_i) \sim \mathcal{N}_{n_i}(X_i\beta + W_i\gamma_i, \mathcal{I}_{n_i}),$$

and furthermore we model the subject-level responses as coming from a multivariate normal distribution

$$\gamma_i \stackrel{\text{iid}}{\sim} \mathcal{N}_q(0, D),$$

where  $D$  is some  $q \times q$  covariance matrix. We set the priors for our parameters,

$$\pi(\beta) \propto 1$$

$$\pi(D) \sim \text{IW}(v, \Psi),$$

and now we can show the full conditionals for the Gibbs sampler. At each iteration we also need to generate values for the latent variables  $z_i$ .

$$\pi(\Sigma|\gamma_i) \sim \text{IW}\left(v + I, \Psi + \sum_{i=1}^I \gamma_i \gamma_i^T\right)$$

$$\pi(\gamma_i|z_i, \beta, D) \propto \pi(\gamma_i|D) p(z_i)$$

$$\sim \mathcal{N}\left(\left[W_i^T W_i + D^{-1}\right]^{-1} W_i^T (z_i - X_i \beta), \left[W_i^T W_i + D^{-1}\right]^{-1}\right)$$

$$\pi(\beta|z, \gamma) \propto \pi(\beta) \prod_{i=1}^I p(z_i|\beta, \gamma_i)$$

$$\propto \prod_{i=1}^I \exp\left[-\frac{1}{2}(z_i - X_i \beta - W_i \gamma_i)^T (z_i - X_i \beta - W_i \gamma_i)\right]$$

$$\sim \mathcal{N}\left(\left[\sum_{i=1}^I X_i^T X_i\right]^{-1} \sum_{i=1}^I X_i^T (z_i - W_i \gamma_i), \left[\sum_{i=1}^I X_i^T X_i\right]^{-1}\right)$$

Finally, the latent variables are generated as follows:

(1)

$$\tilde{z}_i \sim \mathcal{N}_{n_i}(X_i\beta + W_ib_i, I_{n_i})$$

(2) For each  $z_{ij}$ ,

$$z_{ij}|y_{ij} = \begin{cases} \min\{0, \tilde{z}_{ij}\} & \text{if } y_{ij} = 1 \\ \max\{0, \tilde{z}_{ij}\} & \text{if } y_{ij} = 0 \end{cases}$$

## Problem 4

### Gene expression over time

For this problem, we have measurements of the gene-expression profiles of 14 genes in the *Drosophila* genome tracked over time during embryogenesis. Figure 10 shows the data, faceted by each gene. There are two levels of hierarchy in the data, as demonstrated. Each gene belongs to a cluster, or “group” as it is called in this specific context, and each gene has three biological replicates. Figure 11 demonstrates this two-level hierarchical structure; the left column shows the expression profiles for all the genes in each group, and the right column shows the replicates of each gene for a given group. To accomodate the hierarchical and nonlinear time series nature of the data, we introduce a Bayesian hierarchical non-parametric model.

Let  $i$  be the subscript for clusters of genes,  $n$  in the subscript genes,  $r$  is the subscript for replicates. If gene  $n$  belongs to cluster  $i$  we denote this as  $n \in c_i$ , and  $N_i = \# \{n \in c_i\}$ . Each gene  $n$  has  $N_n$  replicates, and each replicate  $r$  of gene  $n$  has  $N_{nr}$  measurements across time. Note that because every array of genes is measured all at once, so each gene has the same  $D = \sum_{r=1}^{N_n} N_{nr}$  total measurements across time for all replicates.

We can say, for every replicate  $r$  of gene  $n$ , the data we observe take the form of  $\mathbf{y}_{nr}$ , a  $N_{nr} \times 1$  vector observed at times  $\mathbf{t}_{nr}$ . Define the following Gaussian processes,

$$\begin{aligned} h_i(t) &\sim \text{GP}(\mathbf{0}, k_h(t, t')) \\ g_n(t) &\sim \text{GP}(h_i(t), k_g(t, t')) \text{ for } n \in c_i \\ f_{nr}(t) &\sim \text{GP}(g_n(t), k_f(t, t')) \end{aligned}$$

for some covariance functions  $k_h(t, t')$ ,  $k_g(t, t')$ , and  $k_f(t, t')$ . Suppose we have  $\mathbf{h}_i$ ,  $\mathbf{g}_n$ , and  $\mathbf{f}_{nr}$  which is draws from  $h_i(t)$ ,  $g_n(t)$ , and  $f_{nr}(t)$ , respectively, at times  $\mathbf{t}_{nr}$ . Define  $\mathbf{K}_f(\mathbf{t}_{nr}, \mathbf{t}_{nr'})$  to be the  $N_{nr} \times N_{nr'}$  matrix such that its  $(i, j)$  element is  $k_f(\mathbf{t}_{nr}[i], \mathbf{t}_{nr'}[j])$ , and define the matrices  $\mathbf{K}_g(\mathbf{t}_{nr}, \mathbf{t}_{nr'})$  and  $\mathbf{K}_h(\mathbf{t}_{nr}, \mathbf{t}_{nr'})$  likewise. Then we model the data  $\mathbf{y}_{nr}$  as

$$\mathbf{y}_{nr} = \mathbf{f}_{nr} + \mathbf{e}, \quad \mathbf{e} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathcal{I}).$$

We can see the following conditional distributions,

$$\begin{aligned} (\mathbf{y}_{nr} | \mathbf{f}_{nr}, \sigma^2) &\sim \mathcal{N}(\mathbf{f}_{nr}, \sigma^2 \mathcal{I}) \\ (\mathbf{f}_{nr} | \mathbf{g}_n, \theta_f) &\sim \mathcal{N}(\mathbf{g}_n, \mathbf{K}_f(\mathbf{t}_{nr}, \mathbf{t}_{nr})) \\ (\mathbf{g}_n | \mathbf{h}_i, \theta_g) &\sim \mathcal{N}(\mathbf{h}_i, \mathbf{K}_g(\mathbf{t}_{nr}, \mathbf{t}_{nr})) \\ (\mathbf{h}_i | \mathbf{t}_{nr}, \theta_h) &\sim \mathcal{N}(\mathbf{0}, \mathbf{K}_h(\mathbf{t}_{nr}, \mathbf{t}_{nr})), \end{aligned}$$

where  $\theta_f$ ,  $\theta_g$ , and  $\theta_h$  are vectors containing the respective parameters for the covariance functions  $k_f$ ,  $k_g$ , and  $k_h$ . Let  $\theta = (\theta_f, \theta_g, \theta_h, \sigma^2)^T$ . It is straightforward to find the marginal likelihood of the data  $\mathbf{y}_{nr}$ ,

$$(\mathbf{y}_{nr} | \mathbf{t}_{nr}, \theta) \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_h(\mathbf{t}_{nr}, \mathbf{t}_{nr}) + \mathbf{K}_g(\mathbf{t}_{nr}, \mathbf{t}_{nr}) + \mathbf{K}_f(\mathbf{t}_{nr}, \mathbf{t}_{nr}) + \sigma^2 \mathcal{I}).$$

Now we consider the full data vector for all genes in cluster  $i$ ,  $\mathbf{Y}_i = \{\mathbf{y}_n\}_{n \in c_i}$  where each  $\mathbf{y}_n$  is a concatenation of the replicates in gene  $n$ ,  $\mathbf{y}_n = \{\mathbf{y}_{nr}\}_{r=1}^{N_n}$ . We can also write  $\mathbf{Y}_i = \left\{ \{\mathbf{y}_{nr}\}_{r=1}^{N_n} \right\}_{n \in c_i}$ . Note that  $\mathbf{t}_n = \{\mathbf{t}_{nr}\}_{r=1}^{N_n} =: \mathbf{t}$  is the same for each  $n$  as explained above, and has length  $D$ . Finally, define  $\mathbf{T}_i = \{\mathbf{t}_n\}_{n \in c_i}$ . This will have a marginal full likelihood,

$$(\mathbf{Y}_i | \mathbf{T}_i, \theta) \sim \mathcal{N}(\mathbf{0}, \Sigma_i),$$

where  $\Sigma_i$  has a matrix which is  $N_i \times N_i$  arrangement of block matrices, each of which is of dimension  $D \times D$ ,

$$\Sigma_i[n, n'] = \text{cov}(\mathbf{y}_n, \mathbf{y}_{n'}) = \begin{cases} \mathbf{K}_h(\mathbf{t}, \mathbf{t}) + \Sigma_n & \text{if } n = n' \\ \mathbf{K}_h(\mathbf{t}, \mathbf{t}) & \text{otherwise} \end{cases},$$

where each  $\Sigma_n$  is a covariance matrix representing the with-in gene variance for gene  $n$ , i.e. the marginal covariance matrix of  $\mathbf{y}_n$ ,

$$\Sigma_n[r, r'] = \text{cov}(\mathbf{y}_{nr}, \mathbf{y}_{nr'}) = \begin{cases} \mathbf{K}_g(\mathbf{t}_{nr}, \mathbf{t}_{nr}) + \mathbf{K}_f(\mathbf{t}_{nr}, \mathbf{t}_{nr}) + \sigma^2 \mathcal{I} & \text{if } r = r' \\ \mathbf{K}_g(\mathbf{t}_{nr}, \mathbf{t}_{nr'}) & \text{otherwise} \end{cases}$$

and also notice that each block  $\Sigma_n[r, r']$  is of dimension  $N_{nr} \times N_{nr'}$ .

Now suppose we want to find the conditional distribution of “new” draws from the Gaussian processes given the data we observe. Specifically we want to find the distribution of  $\mathbf{h}_i^*$  drawn at  $\mathbf{t}_i^*$ ,  $\mathbf{g}_n^*$  at  $\mathbf{t}_n^*$ , and  $\mathbf{f}_{nr}^*$  at  $\mathbf{t}_{nr}^*$ , conditional on the data. First, it is easy to find the respective marginal distributions of each of these,

$$\begin{aligned} (\mathbf{h}_i^* | \mathbf{t}_i^*) &\sim \mathcal{N}(\mathbf{0}, \mathbf{K}_h(\mathbf{t}_i^*, \mathbf{t}_i^*)) \\ (\mathbf{g}_n^* | \mathbf{t}_n^*) &\sim \mathcal{N}(\mathbf{0}, \mathbf{K}_h(\mathbf{t}_n^*, \mathbf{t}_n^*) + \mathbf{K}_g(\mathbf{t}_n^*, \mathbf{t}_n^*)) \\ (\mathbf{f}_{nr}^* | \mathbf{t}_{nr}^*) &\sim \mathcal{N}(\mathbf{0}, \mathbf{K}_h(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*) + \mathbf{K}_g(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*) + \mathbf{K}_f(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*)). \end{aligned}$$

Conditioned on  $\mathbf{y}_i$ , the distribution of each becomes

$$\begin{aligned} \begin{bmatrix} \mathbf{Y}_i \\ \mathbf{h}_i^* \end{bmatrix} &\sim \mathcal{N}\left(0, \begin{bmatrix} \Sigma_i & \mathbf{K}_{i*}^T \\ \mathbf{K}_{i*} & \mathbf{K}_{i**} \end{bmatrix}\right) \\ \begin{bmatrix} \mathbf{Y}_i \\ \mathbf{g}_n^* \end{bmatrix} &\sim \mathcal{N}\left(0, \begin{bmatrix} \Sigma_i & \mathbf{K}_{n*}^T \\ \mathbf{K}_{n*} & \mathbf{K}_{n**} \end{bmatrix}\right) \\ \begin{bmatrix} \mathbf{Y}_i \\ \mathbf{f}_{nr}^* \end{bmatrix} &\sim \mathcal{N}\left(0, \begin{bmatrix} \Sigma_i & \mathbf{K}_{nr*}^T \\ \mathbf{K}_{nr*} & \mathbf{K}_{nr**} \end{bmatrix}\right), \end{aligned}$$

where

$$\begin{aligned} \mathbf{K}_{i**} &= \mathbf{K}_h(\mathbf{t}_i^*, \mathbf{t}_i^*) \\ \mathbf{K}_{n**} &= \mathbf{K}_h(\mathbf{t}_n^*, \mathbf{t}_n^*) + \mathbf{K}_g(\mathbf{t}_n^*, \mathbf{t}_n^*) \\ \mathbf{K}_{nr**} &= \mathbf{K}_h(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*) + \mathbf{K}_g(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*) + \mathbf{K}_f(\mathbf{t}_{nr}^*, \mathbf{t}_{nr}^*) \end{aligned}$$

and the elements of the off-diagonal matrices are given as

$$\begin{aligned} \mathbf{K}_{i*}[t, t'] &= \text{cov}(\mathbf{h}_i^*[t], \mathbf{Y}_i[t']) = k_h(t, t') \\ \mathbf{K}_{n*}[t, t'] &= \text{cov}(\mathbf{g}_n^*[t], \mathbf{Y}_i[t'] \in \mathbf{y}_n) = \begin{cases} k_h(t, t') + k_g(t, t') & \text{if } n = n' \\ k_h(t, t') & \text{otherwise} \end{cases} \\ \mathbf{K}_{nr*}[t, t'] &= \text{cov}(\mathbf{f}_{nr}^*[t], \mathbf{Y}_i[t'] \in \mathbf{y}_{nr'}) = \begin{cases} k_h(t, t') + k_g(t, t') + k_f(t, t') & \text{if } n = n' \text{ and } r = r' \\ k_h(t, t') + k_g(t, t') & \text{if } n = n' \text{ and } r \neq r' \\ k_h(t, t') & \text{otherwise.} \end{cases} \end{aligned}$$

With all this in hand, the conditional distributions may be written explicitly, e.g.

$$(\mathbf{h}_i^* | \mathbf{Y}_i) \sim \mathcal{N}\left(\mathbf{K}_{i*} \Sigma_i^{-1} \mathbf{Y}_i, \mathbf{K}_{i**} - \mathbf{K}_{i*} \Sigma_i^{-1} \mathbf{K}_{i*}^T\right).$$



Now the concern is choice of hyperparameters,  $\theta$ . We have two options here.

- (1) **Fully Bayesian:** Place a hyperprior on  $\theta$  (perhaps a joint independent half-Cauchy priors for each component) and then find the posterior using Bayes' Rule.
- (2) **Marginal Likelihood Maximization:** Maximize  $\log p(\mathbf{Y}_i|\theta, \mathbf{T}_i)$  w.r.t.  $\theta$ .

For the sake of simplicity in implementation, I chose the latter approach. The downside is that the likelihood is not convex, so we are not guaranteed to find a global minimum. One approach to this problem is using gradient based methods, using the partial derivatives,

$$\begin{aligned}\frac{\partial}{\partial \theta_j} \log p(\mathbf{Y}_i|\theta, \mathbf{T}_i) &= \frac{\partial}{\partial \theta_j} \left[ -\frac{1}{2} \mathbf{Y}_i^T \Sigma_i^{-1} \mathbf{Y}_i - \frac{1}{2} \log |\Sigma_i| - \frac{N_i D}{2} \log 2\pi \right] \\ &= -\frac{1}{2} \mathbf{Y}_i \Sigma_i^{-1} \frac{\partial \Sigma_i}{\partial \theta_j} \Sigma_i^{-1} \mathbf{Y}_i - \frac{1}{2} \text{tr} \left( \Sigma_i^{-1} \frac{\partial \Sigma_i}{\partial \theta_j} \right) \\ &= -\frac{1}{2} \text{tr} \left( \left[ \boldsymbol{\eta} \boldsymbol{\eta}^T - \Sigma_i^{-1} \right] \frac{\partial \Sigma_i}{\partial \theta_j} \right), \text{ where } \boldsymbol{\eta} = \Sigma_i^{-1} \mathbf{Y}_i\end{aligned}$$

for each component  $\theta_j$  in  $\theta$  (notice that  $\frac{\partial \Sigma_i}{\partial \theta_j}$  is a matrix derivative). In my R script, however, I used the `optim` command to do this, which by default uses the Nelder-Mead method of optimization. For this application, we use the squared-exponential covariance function with zero “nugget” parameter, i.e.,

$$\begin{aligned}k_h(t, t') &= \alpha_h \cdot \exp \left[ -\left( \frac{t - t'}{\gamma_h} \right)^2 \right] \\ k_g(t, t') &= \alpha_g \cdot \exp \left[ -\left( \frac{t - t'}{\gamma_g} \right)^2 \right] \\ k_f(t, t') &= \alpha_f \cdot \exp \left[ -\left( \frac{t - t'}{\gamma_f} \right)^2 \right],\end{aligned}$$

where each  $\alpha_\bullet$  is called the *amplitude* parameter, and each  $\gamma_\bullet$  is called the *relative length* parameter.

Results of the HGP regression are shown below, at the group, gene, and replicate level for all three groups. The estimated time series functions are shown, along with a 95% confidence band. Not that we have strange results for Group 1. The estimated optimal  $\gamma_h$  parameter obtained by `optim` for Group 1 is very high, around  $e^{13}$ . This is likely due to a very flat likelihood function, due to the fact that the data all look very similar to each other. The high value of  $\gamma_h$  gives a matrix  $\mathbf{K}_h(\mathbf{t}, \mathbf{t})$  which has very large numbers in most of its elements, which may have been a source of numerical instability in R.

Parameter	MLE (group 1)	MLE (group 2)	MLE (group 3)
$\gamma_f$	$1.96 \times 10^{+2}$	$1.53 \times 10^{-2}$	$2.26 \times 10^{-1}$
$\alpha_f$	$1.44 \times 10^{-4}$	$2.24 \times 10^{-2}$	$1.01 \times 10^{-2}$
$\gamma_g$	$3.48 \times 10^{+0}$	$2.10 \times 10^{+0}$	$1.38 \times 10^{+0}$
$\alpha_g$	$3.68 \times 10^{-2}$	$1.26 \times 10^{+0}$	$1.42 \times 10^{+0}$
$\gamma_h$	$7.85 \times 10^{+1}$	$2.96 \times 10^{+1}$	$1.13 \times 10^{+1}$
$\alpha_h$	$2.20 \times 10^{+4}$	$1.48 \times 10^{+2}$	$1.41 \times 10^{+2}$
$\sigma^2$	$1.99 \times 10^{-3}$	$1.91 \times 10^{-1}$	$3.72 \times 10^{-2}$

Table 2: Results of Nelder-Mead method for finding MLEs of hyperparameters

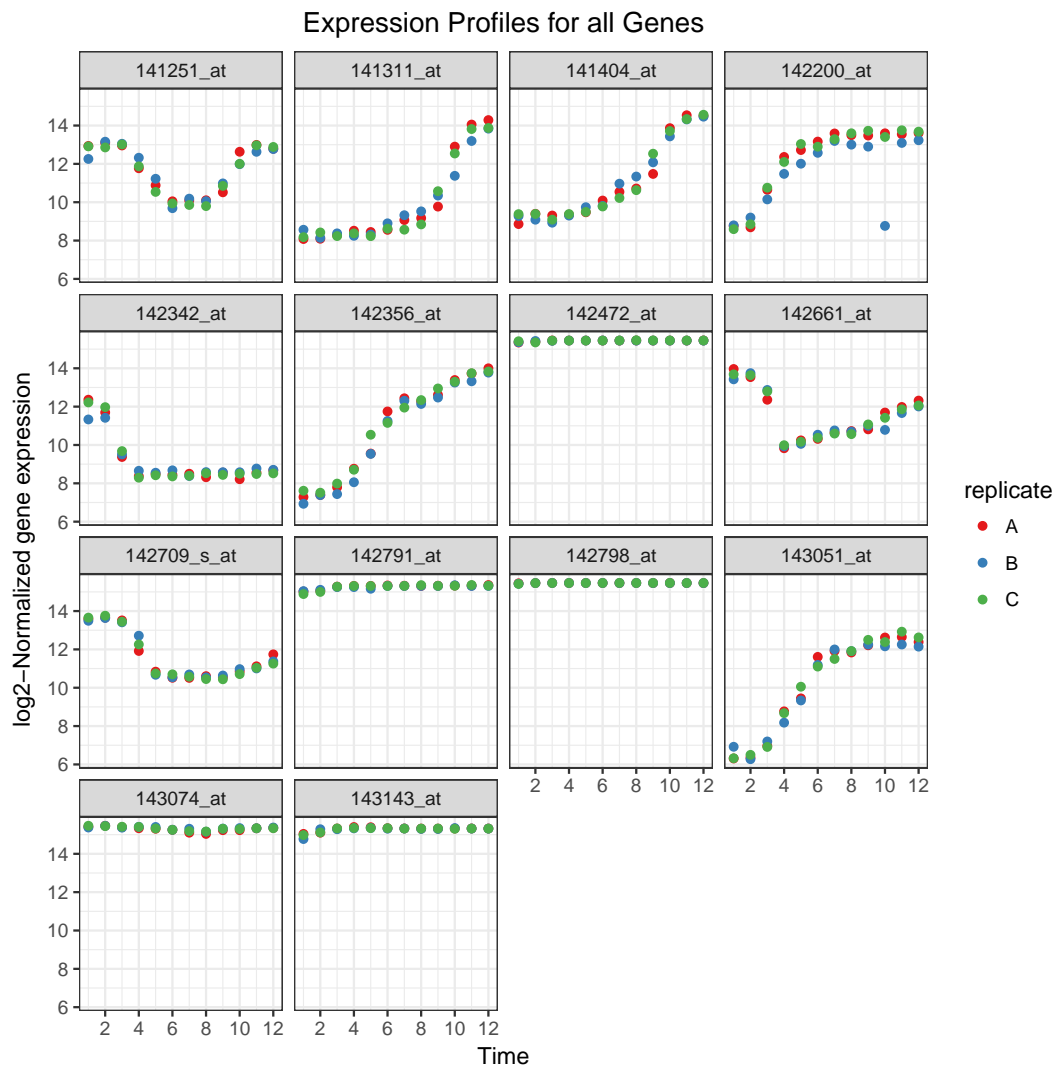


Figure 10: Expression profiles for each gene across all replicates

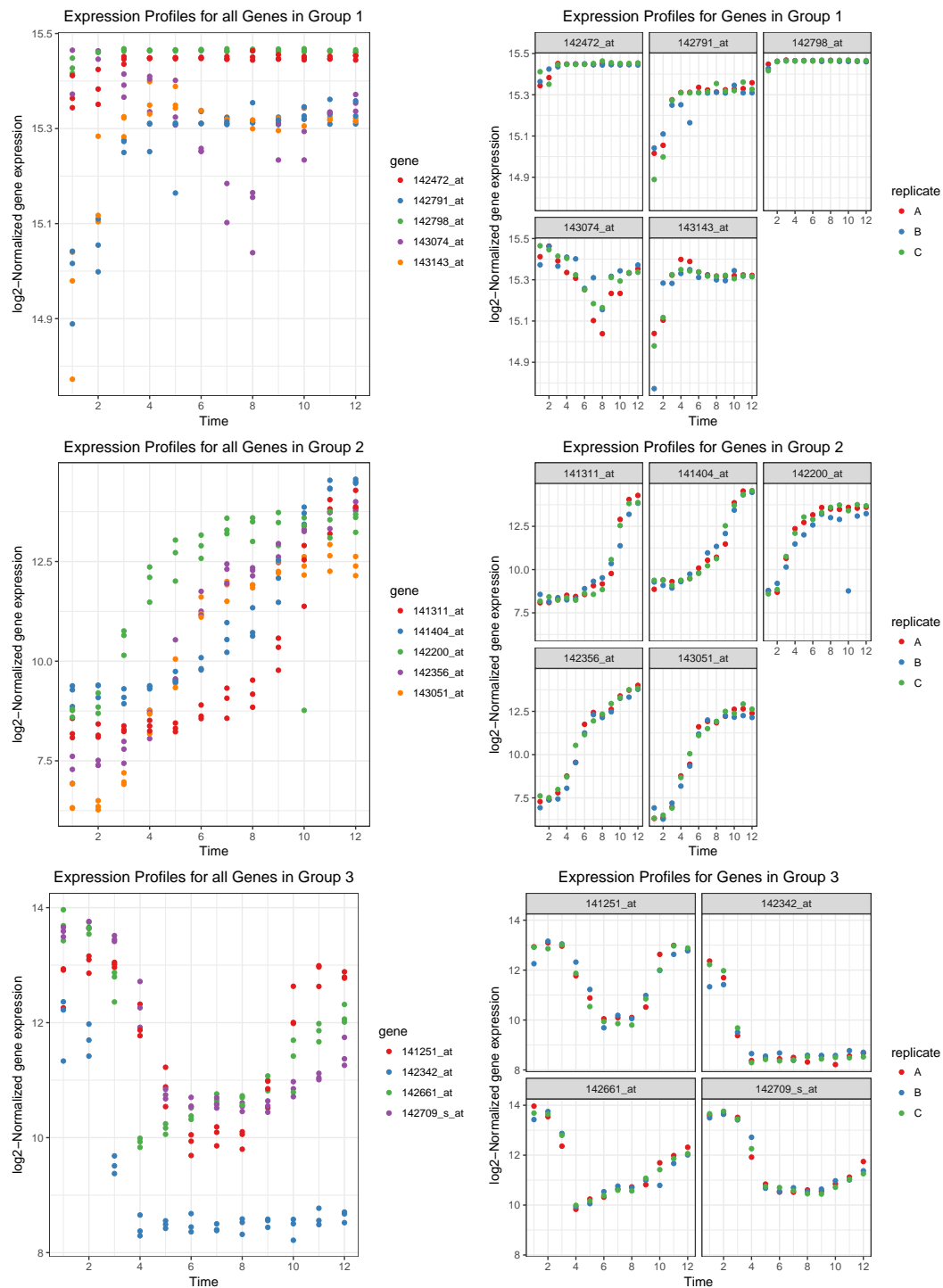


Figure 11: Expression profiles of all genes, accounting for clusters (or "groups") and replicates

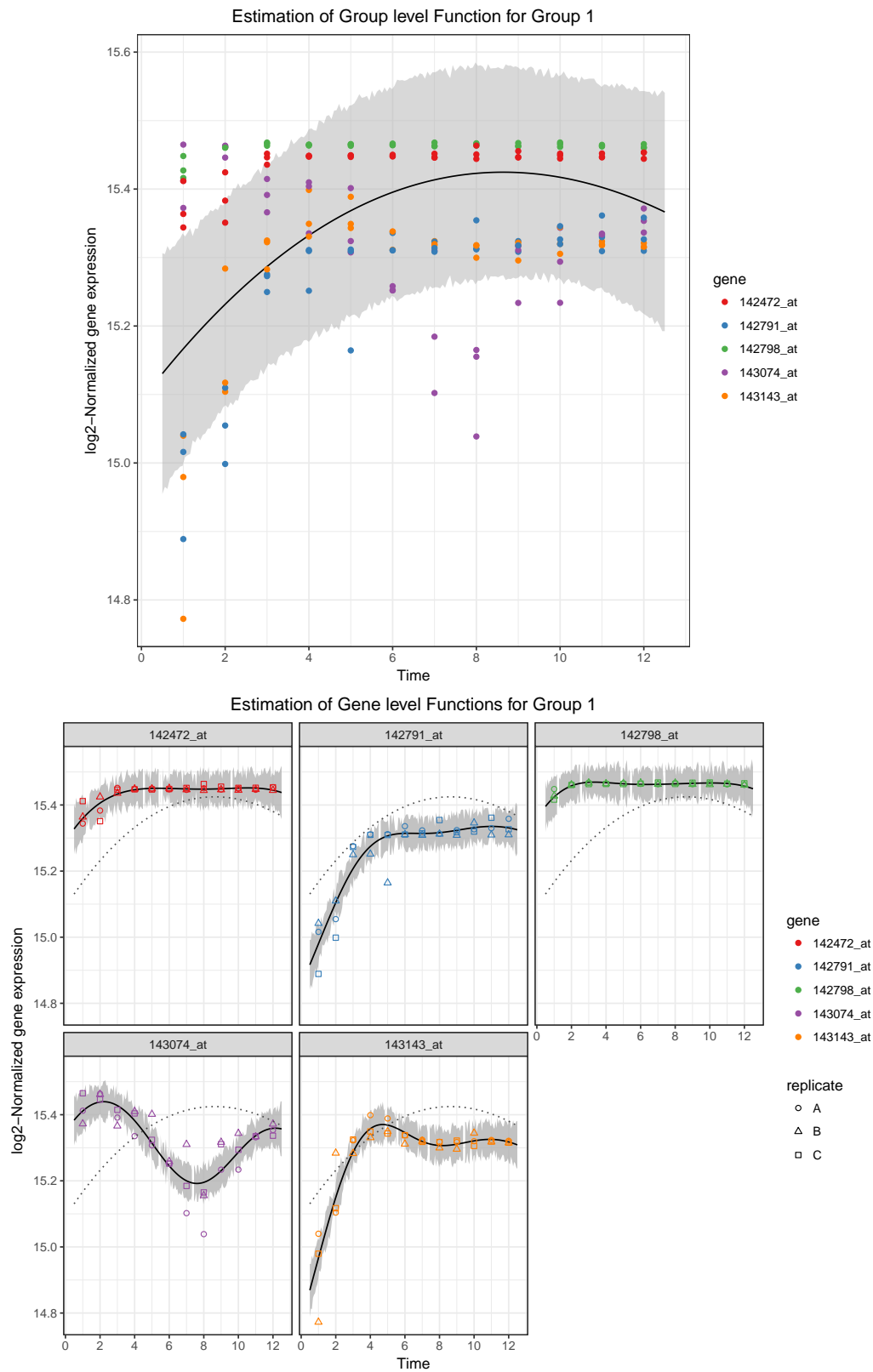


Figure 12: Estimation of group- and gene-level gene expression time series functions for Group 1

## Estimation of Gene–Replicate level Functions for Group 1

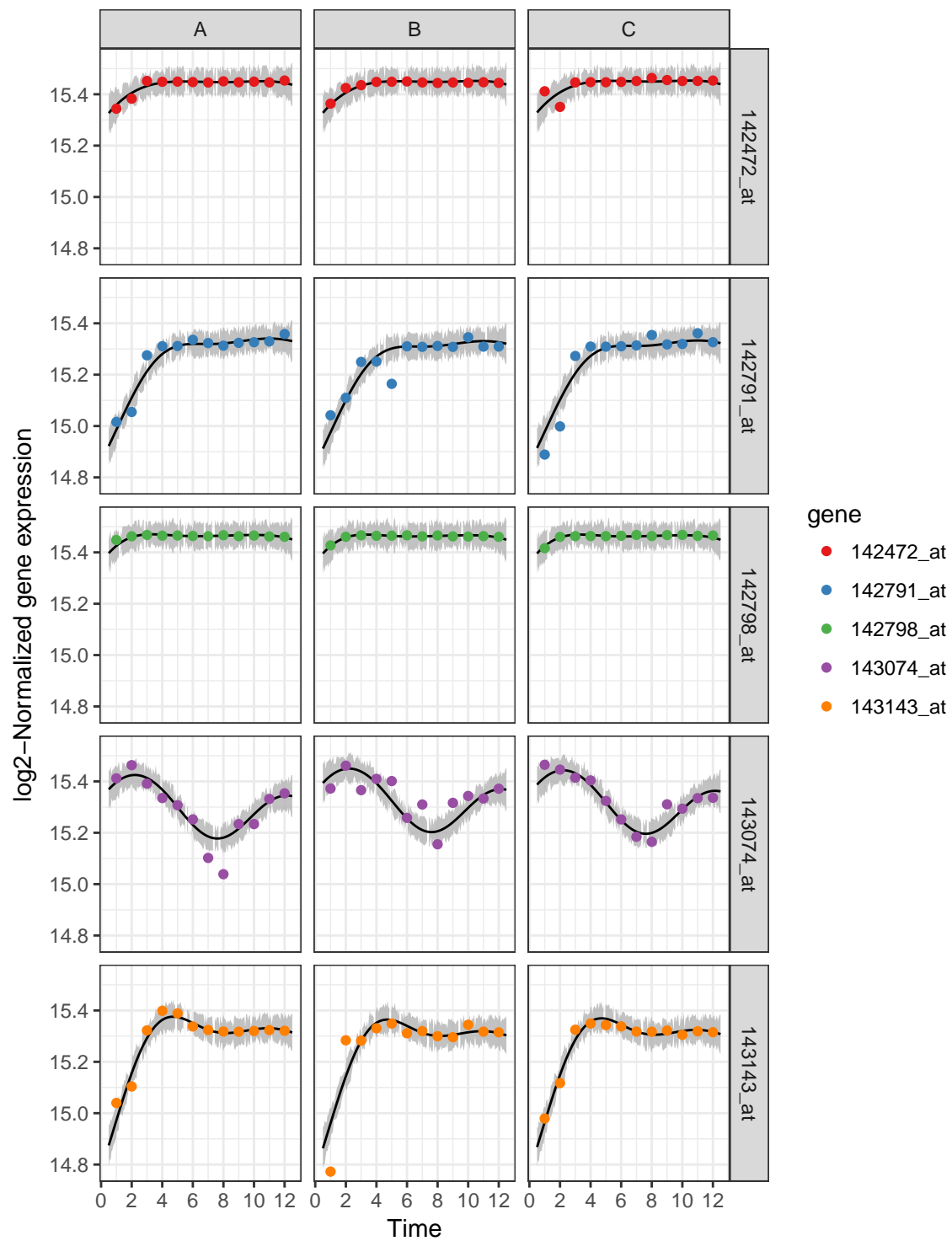
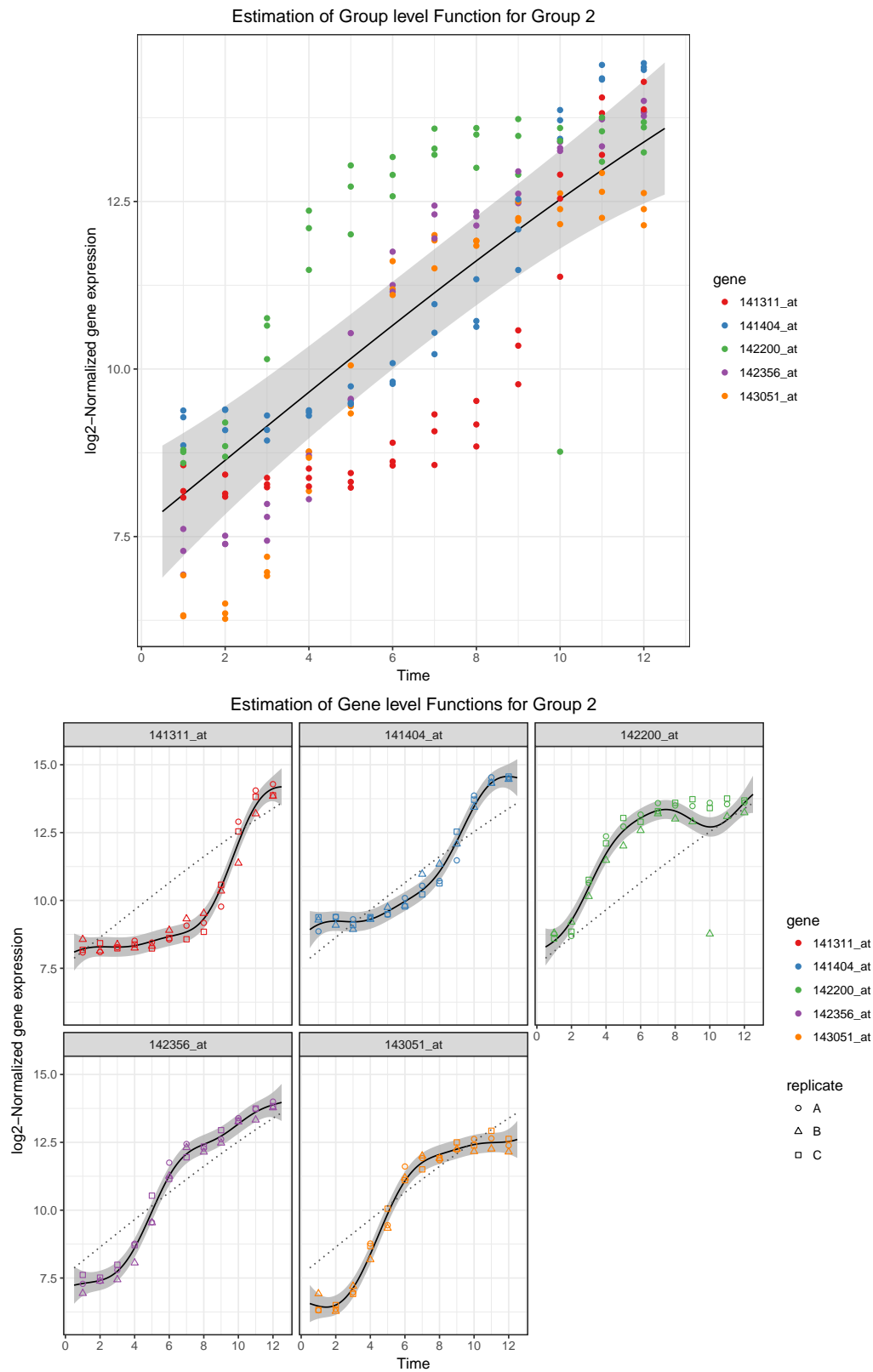


Figure 13: Estimation of gene, replicate-level gene expression time series functions for genes in Group 1



### Estimation of Gene–Replicate level Functions for Group 2

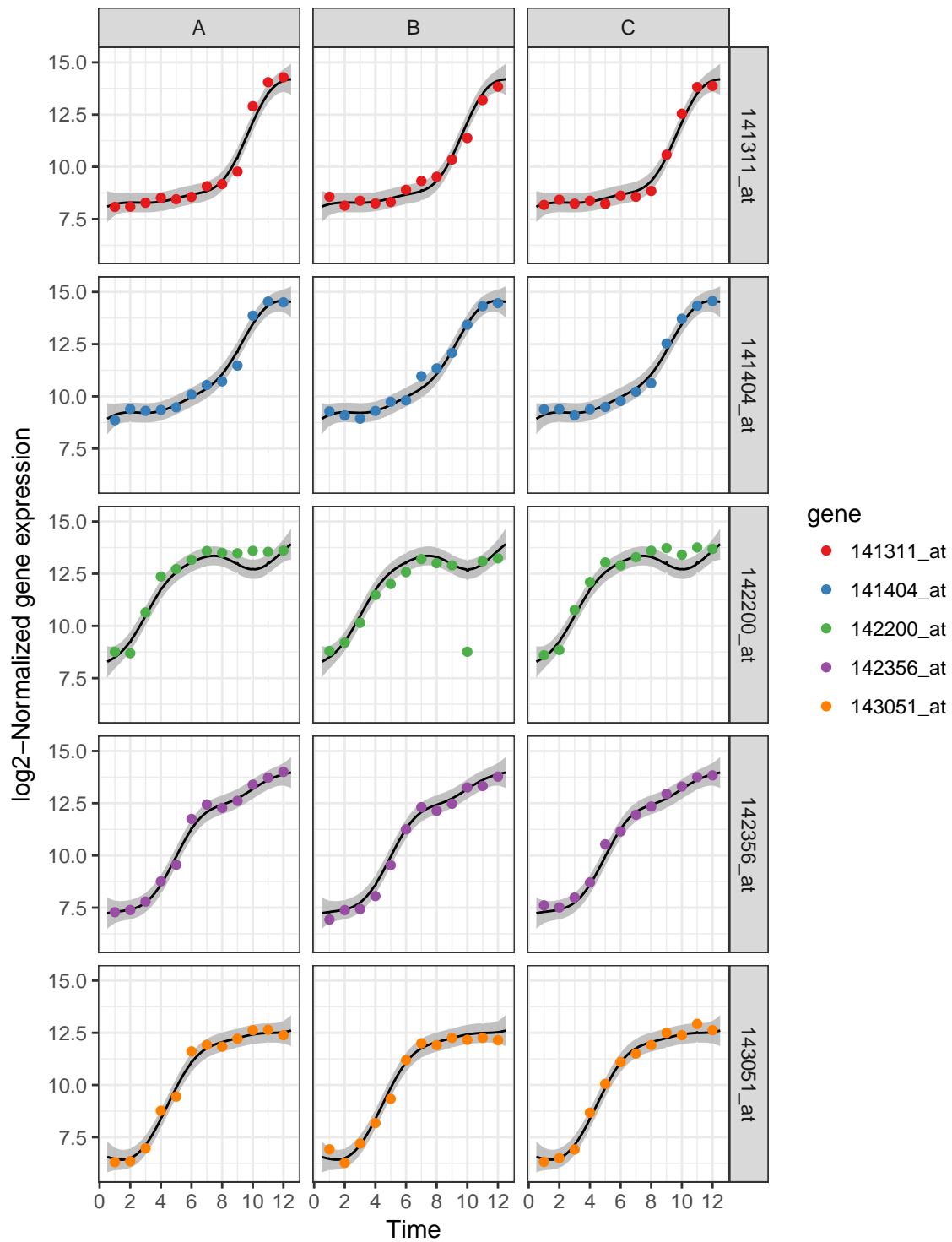


Figure 15: Estimation of gene, replicate-level gene expression time series functions for genes in Group 2

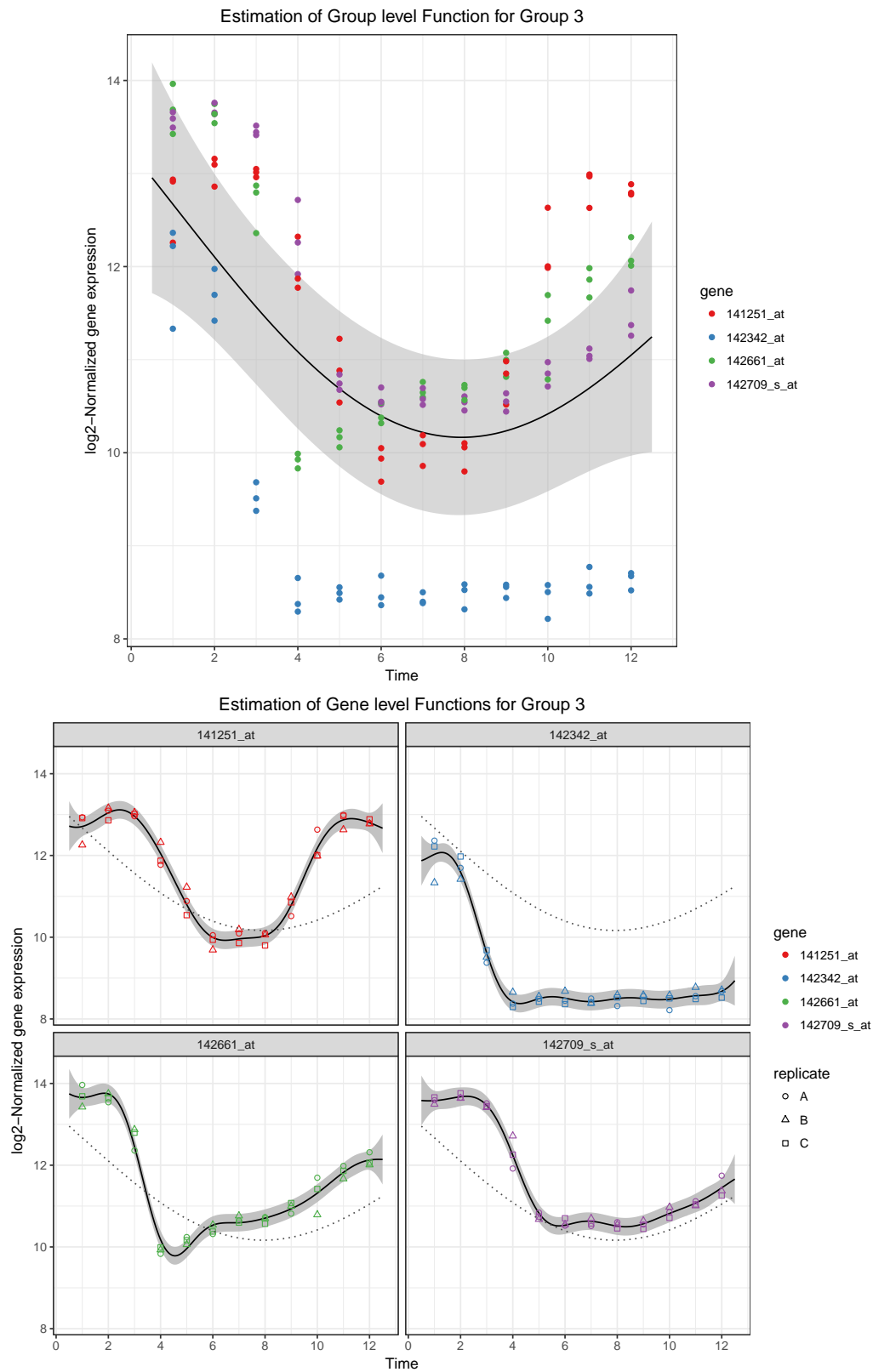


Figure 16: Estimation of group- and gene-level gene expression time series functions for Group 3



## Estimation of Gene–Replicate level Functions for Group 3

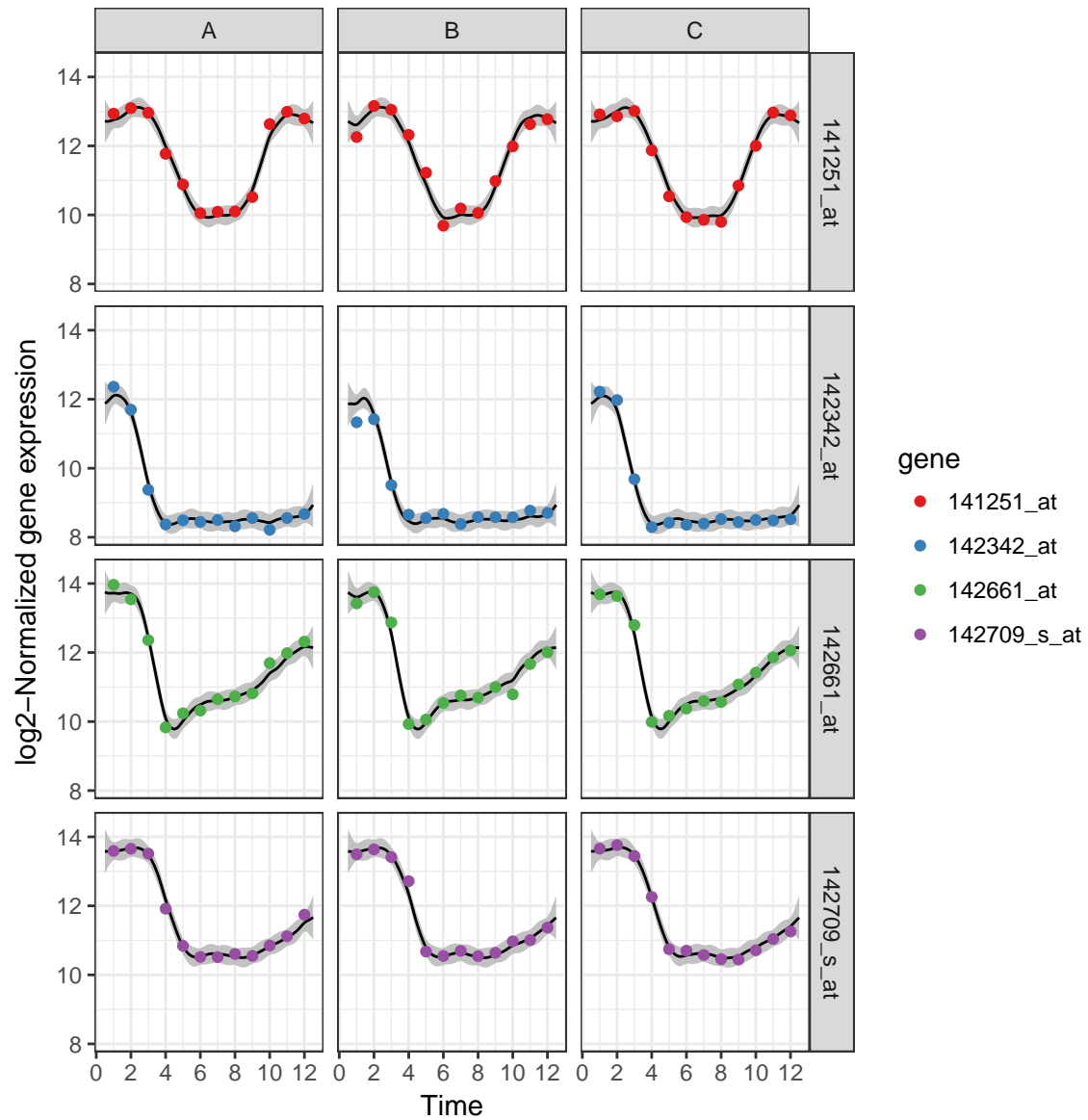


Figure 17: Estimation of gene, replicate-level gene expression time series functions for genes in Group 3