SDS 383D: Exercises 2 – Bayes and the Gaussian linear model

February 5, 2017

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Problem 1

The conjugate Gaussian linear model

We have a model where

$$(y_i|\theta, \sigma^2),\ i = 1, \dots n,$$
$$(\theta|\sigma) \sim \mathcal{N}(\mu, \tau^2\sigma^2),$$
$$\sigma^2 \sim \text{IG}(\frac{d}{2}, \frac{\eta}{2}).$$

Define $\omega = \frac{1}{\sigma^2}$ and $\kappa = \frac{1}{\tau^2}$, so now

$$(\theta|\omega) \sim \mathcal{N}(\mu, (\omega\kappa)^{-1}),$$

and

$$\omega \sim \text{Gamma}(\frac{d}{2}, \frac{\eta}{2}).$$

(A) Taking advantage that (**??**) below is the integral of the kernel of a gamma distribution, the marginal prior distribution of θ is

$$\begin{aligned}
p(\theta) &= \int_0^\infty p(\theta, \omega) d\omega \\
&\propto \int_0^\infty p(\theta|\omega) p(\omega) d\omega \\
&\propto \int_0^\infty (\omega\kappa)^{1/2} \exp\left[-\frac{1}{2}\omega\kappa(\theta-\mu)^2\right] \omega^{d/2-1} \exp\left[-\frac{\eta}{2}\omega\right] d\omega \\
&= \int_0^\infty \omega^{(d+1)/2-1} \exp\left[-\left(\frac{1}{2}\kappa(\theta-\mu)^2 + \frac{\eta}{2}\right)\omega\right] d\omega \\
&\propto \left(\frac{1}{2}\kappa(\theta-\mu)^2 + \frac{\eta}{2}\right)^{-\left(\frac{d+1}{2}\right)} \\
&\propto \left(1 + \frac{1}{d} \cdot \frac{(\theta-\mu)^2}{\frac{\eta}{d\kappa}}\right)^{-\left(\frac{d+1}{2}\right)},
\end{aligned}$$

(**??**) which is the kernel of the Student's t-distribution with degrees of freedom d, center parameter μ, and scale paramater $\sqrt{\frac{\eta}{d\kappa}}$.

Problem 2

Bayesian inference in simple conjugate families

(A)

R code for `myfuns02.R`

```
##########################################################
######### Created by Spencer Woody on 04 Feb 2017 #########
##########################################################
```

R code for `exercises02.R`

```
##########################################################
######### Created by Spencer Woody on 04 Feb 2017 #########
##########################################################
```