Some properties of the Gaussian distribution

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Denote the normal density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right).$$

Its derivative is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\phi(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \cdot (-x) \\ &= -x\phi(x) \end{aligned}$$

The CDF is

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\mathrm{d}t$$

and from the fundamental theorem of calculus, the derivative of the CDF is

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(x) = \Phi'(x) = \phi(x)$$

The derivative of the inverse CDF is found using the identity shown in Appendix A,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p}\Phi^{-1}(p) &= \frac{1}{\Phi'(\Phi^{-1}(p))} \\ &= \frac{1}{\phi(\Phi^{-1}(p))} \end{aligned}$$

A Derivative of an inverse function

For a one-to-one function f, the derivative of the inverse function is found as follows:

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{\mathrm{d}}{\mathrm{d}x} f(f^{-1}(x)) &= \frac{\mathrm{d}}{\mathrm{d}x} x \\ f'(f^{-1}(x)) \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) &= 1 \\ \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) &= \frac{1}{f'(f^{-1}(x))} \end{aligned}$$