Gaussian AR(1) model

Spencer Woody

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Model For a fixed ϕ and τ^2,

$$x_t \mid x_{t-1} \sim \mathcal{N}(\phi x_{t-1}, \tau^2)$$

Marginal mean Assume a stationary process and $\phi \neq 1$. The marginal expectation of x_t is

$$\begin{aligned} m := \mathrm{E}(x_t) &= \mathrm{E}(\mathrm{E}(x_t \mid x_{t-1})) \\ &= \mathrm{E}(\phi x_{t-1}) \\ &= \phi m, \end{aligned}$$

implying that $m = 0$.

Marginal variance

$$\begin{aligned} v := \mathrm{Var}(x_t) &= \mathrm{E}(\mathrm{Var}(x_t \mid x_{t-1})) + \mathrm{Var}(\mathrm{E}(x_t \mid x_{t-1})) \\ &= \mathrm{E}(\tau^2) + \mathrm{Var}\phi x_{t-1} \\ &= \tau^2 + \phi^2 v, \end{aligned}$$

implying that $v = \tau^2/(1-\phi^2)$, and this quantity is greater than zero only if $|\phi| < 1$.

Joint distribution

$$x_{1:n} \sim \mathcal{N}(0, v\Phi_n),$$

where Φ_n is the correlation matrix, and can be defined element-wise. The (i,j) element of this matrix is,

$$\{\Phi_n\}_{ij} = \phi^{|i-j|}.$$

The inverse of the correlation matrix, $\Omega_n := \Phi_n^{-1}$ is sparse (tri-diagonal), making for efficient likelihood evaluations. It can also be defined element-wise,

$$\{\Omega_n\}_{ij} = \frac{1}{1-\phi^2} \cdot \begin{cases} 1 & \text{if } i = j \\ -\phi & \text{if } i = j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$