Normal lemma

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Model

$$
\begin{aligned}
(y \mid \theta) &\sim \mathcal{N}(R\theta, \Omega) \\
\theta &\sim \mathcal{N}(m, \Sigma)
\end{aligned}
$$

Joint distribution Let $\mathcal{I}$ be the identity matrix, and $\mathcal{O}$ be a matrix of all zeros. Furthermore, define the random variable $x \sim \mathcal{N}(0, \Omega)$. The joint distribution of (y, θ) can be decomposed as follows:

$$
\begin{bmatrix} y \\ \theta \end{bmatrix} = \begin{bmatrix} R \\ \mathcal{I} \end{bmatrix} \theta + \begin{bmatrix} \mathcal{I} \\ \mathcal{O} \end{bmatrix} x.
$$

From this, the expectation is

$$
\begin{aligned}
\mathrm{E}\left(\begin{bmatrix} y \\ \theta \end{bmatrix} \right) &= \begin{bmatrix} R \\ \mathcal{I} \end{bmatrix} m + 0 \\
&= \begin{bmatrix} Rm \\ m \end{bmatrix},
\end{aligned}
$$

and the covariance is

$$
\begin{aligned}
\mathrm{Cov}\left(\begin{bmatrix} y \\ \theta \end{bmatrix} \right) &= \begin{bmatrix} R \\ \mathcal{I} \end{bmatrix} \Sigma \begin{bmatrix} R \\ \mathcal{I} \end{bmatrix}^T + \begin{bmatrix} \mathcal{I} \\ \mathcal{O} \end{bmatrix} \Omega \begin{bmatrix} \mathcal{I} \\ \mathcal{O} \end{bmatrix}^T \\
&= \begin{bmatrix} R^T \Sigma R + \Omega & R\Sigma \\ \Sigma R^T & \Sigma \end{bmatrix}.
\end{aligned}
$$

Therefore, the joint distribution of (y, θ) can be expressed as

$$
\begin{bmatrix} y \\ \theta \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} Rm \\ m \end{bmatrix}, \begin{bmatrix} R^T \Sigma R + \Omega & R\Sigma \\ \Sigma R^T & \Sigma \end{bmatrix} \right).
$$