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PROBLEMS

B.1-1 Given a complex number $w = x + jy$, the complex conjugate of w is defined in rectangular coordinates as $w^* = x - jy$. Use this fact to derive complex conjugation in polar form.

B.1-2 Express the following numbers in polar form:
(a) $w_a = 1 + j$
(b) $w_b = 1 + e^j$
(c) $w_c = -4 + j3$
(d) $w_d = (1 + j)(-4 + j3)$
(e) $w_e = e^{j\pi/4} + 2e^{-j\pi/4}$
(f) $w_f = \frac{1+j}{2j}$
(g) $w_g = (1 + j)/(-4 + j3)$
(h) $w_h = \frac{1-j}{\sin(j)}$

B.1-3 Express the following numbers in Cartesian (rectangular) form:
(a) $w_a = j + e^j$
(b) $w_b = 3e^{j\pi/4}$
(c) $w_c = 1/e^j$
(d) $w_d = (1 + j)(-4 + j3)$
(e) $w_e = e^{j\pi/4} + 2e^{-j\pi/4}$
(f) $w_f = e^j + 1$
(g) $w_g = 1/2^j$
(h) $w_h = j^{j^j}$ (j raised to the j raised to the j)

B.1-4 Showing all work and simplifying your answer, determine the real part of the following numbers:
(a) $w_a = \frac{1}{j}(j - 5e^{2-3j})$
(b) $w_b = (1 + j)\ln(1 + j)$

B.1-5 Showing all work and simplifying your answer, determine the imaginary part of the following numbers:
(a) $w_a = -je^{j\pi/4}$
(b) $w_b = 1 - 2je^{2-4j}$
(c) $w_c = \tan(j)$

B.1-6 For complex constant w, prove:
(a) $\mathrm{Re}(w) = (w + w^*)/2$
(b) $\mathrm{Im}(w) = (w - w^*)/2j$

B.1-7 Given $w = x - jy$, determine:
(a) $\mathrm{Re}(e^w)$
(b) $\mathrm{Im}(e^w)$

B.1-8 For arbitrary complex constants w_1 and w_2, prove or disprove the following:
(a) $\mathrm{Re}(jw_1) = -\mathrm{Im}(w_1)$
(b) $\mathrm{Im}(jw_1) = \mathrm{Re}(w_1)$
(c) $\mathrm{Re}(w_1) + \mathrm{Re}(w_2) = \mathrm{Re}(w_1 + w_2)$
(d) $\mathrm{Im}(w_1) + \mathrm{Im}(w_2) = \mathrm{Im}(w_1 + w_2)$
(e) $\mathrm{Re}(w_1)\mathrm{Re}(w_2) = \mathrm{Re}(w_1 w_2)$
(f) $\mathrm{Im}(w_1)/\mathrm{Im}(w_2) = \mathrm{Im}(w_1/w_2)$

B.1-9 Given $w_1 = 3 + j4$ and $w_2 = 2e^{j\pi/4}$.
(a) Express w_1 in standard polar form.
(b) Express w_2 in standard rectangular form.
(c) Determine $|w_1|^2$ and $|w_2|^2$.
(d) Express $w_1 + w_2$ in standard rectangular form.
(e) Express $w_1 - w_2$ in standard polar form.
(f) Express $w_1 w_2$ in standard rectangular form.
(g) Express w_1/w_2 in standard polar form.

B.1-10 Repeat Prob. B.1-9 using $w_1 = (3 + j4)^2$ and $w_2 = 2.5je^{-j40\pi}$.

B.1-11 Repeat Prob. B.1-9 using $w_1 = j + e^{\pi/4}$ and $w_2 = \cos(j)$.

B.1-12 Using the complex plane:
(a) Evaluate and locate the distinct solutions to $(w)^4 = -1$.
(b) Evaluate and locate the distinct solutions to $(w - (1 + j2))^5 = (32/\sqrt{2})(1 + j)$.
(c) Sketch the solution to $|w - 2j| = 3$.
(d) Graph $w(t) = (1 + t)e^{jt}$ for $(-10 \le t \le 10)$.

B.1-13 The distinct solutions to $(w - w_1)^n = w_2$ lie on a circle in the complex plane, as shown in Fig. PB.1-13. One solution is located on the real axis at $\sqrt{3} + 1 = 2.732$, and one solution is located on the imaginary axis at $\sqrt{3} - 1 = 0.732$. Determine w_1, w_2, and n.

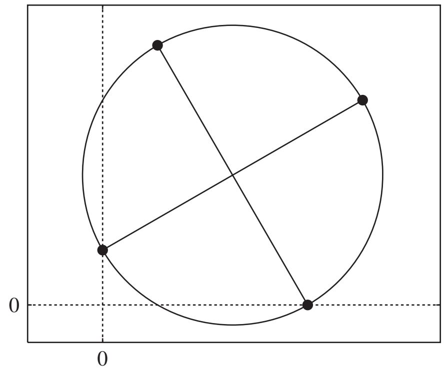

Figure PB.1-13

B.1-14 Find the distinct solutions to each of the following. Use MATLAB to graph each solution set in the complex plane.
(a) $w^3 = -\frac{8}{27}$
(b) $(w+1)^8 = 1$
(c) $w^2 + j = 0$
(d) $16(w-1)^4 + 81 = 0$
(e) $(w+2j)^3 = -8$
(f) $(j-w)^{1.5} = 2+j2$
(g) $(w-1)^{2.5} = j4\sqrt{2}$

B.1-15 If $j = \sqrt{-1}$, what is $\sqrt{j}$?

B.1-16 Find all the values of $\ln(-e)$, expressing your answer in Cartesian form.

B.1-17 Determine all values of $\log_{10}(-1)$, expressing your answer in Cartesian form. Notice that the logarithm has base 10, not e.

B.1-18 Express the following in standard rectangular coordinates:
(a) $w_a = \ln(1/(1+j))$
(b) $w_b = \cos(1+j)$
(c) $w_c = (1-j)^j$

B.1-19 By constraining w to be purely imaginary, show that the equation $\cos(w) = 2$ can be represented as a standard quadratic equation. Solve this equation for w.

B.1-20 Certain integrals, although expressed in relatively simple form, are quite difficult to solve. For example, $\int e^{-x^2} dx$ cannot be evaluated in terms of elementary functions; most calculators that perform integration cannot handle this indefinite integral. Fortunately, you are smarter than most calculators.
(a) Express e^{-x^2} using a Taylor series expansion.
(b) Using your series expansion for e^{-x^2}, determine $\int e^{-x^2} dx$.
(c) Using a suitably truncated series, evaluate the definite integral $\int_0^1 e^{-x^2} dx$.

B.1-21 Repeat Prob. B.1-20 for $\int e^{-x^3} dx$.

B.1-22 Repeat Prob. B.1-20 for $\int \cos x^2 \, dx$.

B.1-23 For each function, determine a suitable series expansion.
(a) $f_a(x) = (2-x^2)^{-1}$
(b) $f_b(x) = (0.5)^x$

B.1-24 Consider the function $f(x) = 1 + x + x^2 + x^3$.
(a) Express $f(x)$ using a Taylor series with expansion point of $a = 1$. Explicitly write out every term. [*Hint:* See Sec. B.8-4.]
(b) Describe a good reason why you might want to express a function that is already a simple polynomial using such a series.

B.1-25 Determine the Maclaurin series expansion of each of the following. [*Hint:* See Sec. B.8-4.]
(a) $f_a(x) = 2^x$
(b) $f_b(x) = \left(\frac{1}{3}\right)^x$

B.2-1 Determine the fundamental period T_0, frequency f_0, and radian frequency ω_0 for the following sinusoids:
(a) $\cos(5\pi t + 3)$
(b) $7\sin\left(\frac{2t-\pi}{3}\right)$

B.2-2 Determine an expression for a sinusoid that oscillates 15 times per second, that has a value of -1 at $t = 0$, and whose peak amplitude is 3. Use MATLAB to plot the signal over $0 \le t \le 1$.

B.2-3 Let $x_1(t) = 2\cos(3t+1)$ and $x_2(t) = -3\cos(3t-2)$.
(a) Determine a_1 and b_1 so that $x_1(t) = a_1\cos(3t) + b_1\sin(3t)$.
(b) Determine a_2 and b_2 so that $x_2(t) = a_2\cos(3t) + b_2\sin(3t)$.
(c) Determine C and θ so that $x_1(t) + x_2(t) = C\cos(3t+\theta)$.

B.2-4 In addition to the traditional sine and cosine functions, there are the *hyperbolic* sine and cosine functions, which are defined by $\sinh(w) = (e^w - e^{-w})/2$ and $\cosh(w) = (e^w + e^{-w})/2$. In general, the argument is a complex constant $w = x + jy$.

(a) Show that $\cosh(w) = \cosh(x)\cos(y) + j\sinh(x)\sin(y)$.
(b) Determine a similar expression for $\sinh(w)$ in rectangular form that only uses functions of real arguments, such as $\sin(x)$, $\cosh(y)$, and so on.

B.2-5 Use Euler's identity to solve or prove the following:
(a) Find real, positive constants c and ϕ for all real t such that $2.5\cos(3t) - 1.5\sin(3t + \pi/3) = c\cos(3t + \phi)$. Sketch the resulting sinusoid.
(b) Prove that $\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$.
(c) Given real constants a, b, and α, complex constant w, and the fact that

$$\int_a^b e^{wx}\,dx = \frac{1}{w}(e^{wb} - e^{wa})$$

evaluate the integral

$$\int_a^b e^{wx}\sin(\alpha x)\,dx$$

B.2-6 A particularly boring stretch of interstate highway has a posted speed limit of 70 mph. A highway engineer wants to install "rumble bars" (raised ridges on the side of the road) so that cars traveling the speed limit will produce quarter-second bursts of 1 kHz sound every second, a strategy that is particularly effective at startling sleepy drivers awake. Provide design specifications for the engineer.

B.3-1 By hand, accurately sketch the following signals over $(0 \le t \le 1)$:
(a) $x_a(t) = e^{-t}$
(b) $x_b(t) = \sin(2\pi 5t)$
(c) $x_c(t) = e^{-t}\sin(2\pi 5t)$

B.3-2 In 1950, the human population was approximately 2.5 billion people. Assuming a doubling time of 40 years, formulate an exponential model for human population in the form $p(t) = ae^{bt}$, where t is measured in years. Sketch $p(t)$ over the interval $1950 \le t \le 2100$. According to this model, in what year can we expect the population to reach the estimated 15 billion carrying capacity of the earth?

B.3-3 Determine an expression for an exponentially decaying sinusoid that oscillates three times per second and whose amplitude envelope decreases by 50% every 2 seconds. Use MATLAB to plot the signal over $-2 \le t \le 2$.

B.3-4 By hand, sketch the following against independent variable t:
(a) $x_a(t) = \mathrm{Re}\left(2e^{(-1+j2\pi)t}\right)$
(b) $x_b(t) = \mathrm{Im}\left(3 - e^{(1-j2\pi)t}\right)$
(c) $x_c(t) = 3 - \mathrm{Im}\left(e^{(1-j2\pi)t}\right)$

B.4-1 Consider the following system of equations:

$$\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Expressing all answers in rational form (ratio of integers), use Cramer's rule to determine x_1 and x_2. Perform all calculations by hand, including matrix determinants.

B.4-2 Consider the following system of equations:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

Expressing all answers in rational form (ratio of integers), use Cramer's rule to determine x_1, x_2, and x_3. Perform all calculations by hand, including matrix determinants.

B.4-3 Consider the following system of equations.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 2x_2 + 3x_3 &= 3 \\ x_1 - x_2 &= -3 \end{aligned}$$

Use Cramer's rule to determine x_1, x_2, and x_3. Matrix determinants can be computed by using MATLAB's `det` command.

B.5-1 Determine the constants a_0, a_1, and a_2 of the partial fraction expansion

$$\begin{aligned} F(s) &= \frac{s}{(s+1)^3} \\ &= \frac{a_0}{(s+1)^3} + \frac{a_1}{(s+1)^2} + \frac{a_2}{(s+1)} \end{aligned}$$

B.5-2 Compute by hand the partial fraction expansions of the following rational functions:
(a) $H_a(s) = \frac{s^2+5s+6}{s^3+s^2+s+1}$, which has denominator poles at $s = \pm j$ and $s = -1$
(b) $H_b(s) = \frac{1}{H_1(s)} = \frac{s^3+s^2+s+1}{s^2+5s+6}$

(c) $H_c(s) = \frac{1}{(s+1)^2(s^2+1)}$
(d) $H_d(s) = \frac{s^2+5s+6}{3s^2+2s+1}$

B.5-3 Compute by hand the partial fraction expansions of the following rational functions:
(a) $F_a(x) = \frac{(x-1)(x-2)}{(x-3)^2}$
(b) $F_b(x) = \frac{(x-1)^2}{(3x-1)(2x-1)}$
(c) $F_c(x) = \frac{(x-1)^2}{(3x-1)^2(2x-1)}$
(d) $F_d(x) = \frac{x^2-5x+6}{2x^2+8x+6}$
(e) $F_e(x) = \frac{2x^2-3x-11}{x^2-x-2}$
(f) $F_f(x) = \frac{3+2x^2}{-3+2x+x^2}$
(g) $F_g(x) = \frac{x^3+2x^2+3x+4}{x^2+1}$
(h) $F_h(x) = \frac{1+2x+3x^2}{x^2+5x+6}$
(i) $F_i(x) = \frac{3x^3-x^2+14x+4}{x^2+4}$
(j) $F_j(x) = \frac{2x^{-1}-1+2x}{x-5+6x^{-1}}$
(k) $F_k(x) = 3\frac{-5x^2-9x+23}{x^2+x-2}$

B.6-1 A system of equations in terms of unknowns x_1 and x_2 and arbitrary constants a, b, c, d, e, and f is given by

$$\begin{aligned} ax_1 + bx_2 &= c \\ dx_1 + ex_2 &= f \end{aligned}$$

(a) Represent this system of equations in matrix form.
(b) Identify specific constants a, b, c, d, e, and f such that $x_1 = 3$ and $x_2 = -2$. Are the constants you selected unique?
(c) Identify nonzero constants a, b, c, d, e, and f such that no solutions x_1 and x_2 exist.
(d) Identify nonzero constants a, b, c, d, e, and f such that an infinite number of solutions x_1 and x_2 exist.

B.6-2 Using a matrix approach, solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 - x_3 - x_4 &= 0 \\ x_1 - x_2 - x_3 - x_4 &= -2 \end{aligned}$$

B.6-3 Using a matrix approach, solve the following system of equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 - 2x_2 + 3x_3 &= 2 \\ x_1 - x_3 + 7x_4 &= 3 \\ -2x_2 + 3x_3 - 4x_4 &= 4 \end{aligned}$$

B.6-4 A signal $f(t) = a\cos(3t) + b\sin(3t)$ reaches a peak amplitude of 5 at $t = 1.8799$ and has a zero crossing at $t = 0.3091$. Use a matrix-based approach to determine the constants a and b.

B.6-5 Define

$$\mathbf{x} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -5 \\ 2 \end{bmatrix},$$

$$\text{and} \quad \mathbf{z} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

By hand, calculate the following:
(a) $\mathbf{f}_a = \mathbf{y}^T\mathbf{y}$
(b) $\mathbf{f}_b = \mathbf{y}\mathbf{y}^T$
(c) $\mathbf{f}_c = \mathbf{x}\mathbf{y}$
(d) $\mathbf{f}_d = \mathbf{x}^T\mathbf{y}$
(e) $\mathbf{f}_e = \mathbf{y}^T\mathbf{x}$
(f) $\mathbf{f}_f = \mathbf{x}\mathbf{z}$
(g) $\mathbf{f}_g = \mathbf{z}\mathbf{x}\mathbf{z}$
(h) $\mathbf{f}_h = \mathbf{x}^T - \mathbf{z}$

B.7-1 Use MATLAB to produce the plots requested in Prob. B.3-4.

B.7-2 Use MATLAB to plot the function $x(t) = t\sin(2\pi t)$ over $0 \le t \le 10$ using 501 equally spaced points. What is the maximum value of $x(t)$ over this range of t?

B.7-3 Use MATLAB to plot $x(t) = \cos(t)\sin(20t)$ over a suitable range of t.

B.7-4 Use MATLAB to plot $x(t) = \sum_{k=1}^{10} \cos(2\pi kt)$ over a suitable range of t. The MATLAB command `sum` may prove useful.

B.7-5 When a bell is struck with a mallet, it produces a ringing sound. Write an equation that approximates the sound produced by a small, light bell. Carefully identify your assumptions. How does your equation change if the bell is large and heavy? You can assess the quality of your models by using the MATLAB `sound` command to listen to your "bell."

B.7-6 You are working on a digital quadrature amplitude modulation (QAM) communication receiver. The QAM receiver requires a pair of quadrature signals: $\cos \Omega n$ and $\sin \Omega n$. These can be simultaneously generated by following a simple procedure: (1) choose a point w on the unit circle, (2) multiply w by itself and store the result, (3) multiply w by the last result and store, and (4) repeat step 3.

(a) Show that this method can generate the desired pair of quadrature sinusoids.

(b) Determine a suitable value of w so that good-quality, periodic, $2\pi \times 100{,}000$ rad/s signals can be generated. How much time is available for the processing unit to compute each sample?

(c) Simulate this procedure by using MATLAB and report your results.

(d) Identify as many assumptions and limitations to this technique as possible. For example, can your system operate correctly for an indefinite period of time?

B.7-7 Using MATLAB's `residue` command,

(a) Verify the results of Prob. B.5-2a.

(b) Verify the results of Prob. B.5-2b.

(c) Verify the results of Prob. B.5-2c.

(d) Verify the results of Prob. B.5-2d.

B.7-8 Using MATLAB's `residue` command,

(a) Verify the results of Prob. B.5-3a.

(b) Verify the results of Prob. B.5-3b.

(c) Verify the results of Prob. B.5-3c.

(d) Verify the results of Prob. B.5-3d.

(e) Verify the results of Prob. B.5-3e.

(f) Verify the results of Prob. B.5-3f.

(g) Verify the results of Prob. B.5-3g.

(h) Verify the results of Prob. B.5-3h.

(i) Verify the results of Prob. B.5-3i.

(j) Verify the results of Prob. B.5-3j.

(k) Verify the results of Prob. B.5-3k.

B.7-9 Determine the original length-3 vectors `a` and `b` need to produce the MATLAB output:

```
>>  [r,p,k] = residue(b,a)
    r =  0 + 2.0000i
         0 - 2.0000i
    p =  3
        -3
    k =  0 + 1.0000i
```

B.7-10 Let $N = [n_7, n_6, n_5, \ldots, n_2, n_1]$ represent the seven digits of your phone number. Construct a rational function according to

$$H_N(s) = \frac{n_7 s^2 + n_6 s + n_5 + n_4 s^{-1}}{n_3 s^2 + n_2 s + n_1}$$

Use MATLAB's `residue` command to compute the partial fraction expansion of $H_N(s)$.

B.7-11 When plotted in the complex plane for $-\pi \le \omega \le \pi$, the function $f(\omega) = \cos(\omega) + j0.1\sin(2\omega)$ results in a so-called Lissajous figure that resembles a two-bladed propeller.

(a) In MATLAB, create two row vectors `fr` and `fi` corresponding to the real and imaginary portions of $f(\omega)$, respectively, over a suitable number N samples of ω. Plot the real portion against the imaginary portion and verify the figure resembles a propeller.

(b) Let complex constant $w = x + jy$ be represented in vector form

$$\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Consider the 2×2 rotational matrix $\mathbf{R}$:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Show that $\mathbf{Rw}$ rotates vector $\mathbf{w}$ by θ radians.

(c) Create a rotational matrix `R` corresponding to 10° and multiply it by the $2 \times N$ matrix `f = [fr;fi];`. Plot the result to verify that the "propeller" has indeed rotated counterclockwise.

(d) Given the matrix `R` determined in part (c), what is the effect of performing `RRf`? How about `RRRf`? Generalize the result.

(e) Investigate the behavior of multiplying $f(\omega)$ by the function $e^{j\theta}$.