Appendix M

Root Locus Rules: Derivations

M.1 Derivation of the Behavior of the Root Locus at Infinity (Kuo, 1987)

Let the open-loop transfer function be represented as follows:

$$KG(s)H(s) = \frac{K(s^m + a_1 s^{m-1} + \ldots + a_m)}{(s^{m+n} + b_1 s^{m+n-1} + \ldots + b_{m+n})} \tag{M.1}$$

or

$$KG(s)H(s) = \frac{K}{\left(\dfrac{s^{m+n} + b_1 s^{m+n-1} + \ldots + b_{m+n}}{s^m + a_1 s^{m-1} + \ldots + a_m}\right)} \tag{M.2}$$

Performing the indicated division in the denominator, we obtain

$$KG(s)H(s) = \frac{K}{s^n + (b_1 - a_1)s^{n-1} + \ldots} \tag{M.3}$$

In order for a pole of the closed-loop transfer function to exist,

$$KG(s)H(s) = -1 \tag{M.4}$$

Assuming large values of s that would exist as the locus moves toward infinity, Eq. (M.3) becomes

$$s^n + (b_1 - a_1)s^{n-1} = -K \tag{M.5}$$

Factoring out s^n, Eq. (M.5) becomes

$$s^n\left(1 + \frac{b_1 - a_1}{s}\right) = -K \tag{M.6}$$

Taking the nth root of both sides, we have

$$s\left(1 + \frac{b_1 - a_1}{s}\right)^{1/n} = -K^{1/n} \tag{M.7}$$

If the term

$$\left(1+\frac{b_1-a_1}{s}\right)^{1/n} \tag{M.8}$$

is expanded into an infinite series where only the first two terms are significant,[1] we obtain

$$s\left(1+\frac{b_1-a_1}{ns}\right)=(-K)^{1/n} \tag{M.9}$$

Distributing the factor S on the left-hand side yields

$$s+\frac{b_1-a_1}{n}=(-K)^{1/n} \tag{M.10}$$

Now, letting $s=\sigma+j\omega$ and $(-K)^{1/n}=|K^{1/n}|e^{j(2k+1)\pi/n}$, where

$$(-1)^{1/n}=e^{j(2k+1)\pi/n}=\cos\left(\frac{(2k+1)\pi}{n}\right)+j\sin\left(\frac{(2k+1)\pi}{n}\right) \tag{M.11}$$

Eq. (M.10) becomes

$$\sigma+j\omega+\frac{b_1-a_1}{n}=\left|K^{1/n}\right|\left[\cos\frac{(2k+1)\pi}{n}+j\sin\frac{(2k+1)\pi}{n}\right] \tag{M.12}$$

where $k=0,\pm 1,\pm 2,\pm 3,\ldots$ Setting the real and imaginary parts of both sides equal to each other, we obtain

$$\sigma+\frac{b_1-a_1}{n}=\left|K^{1/n}\right|\cos\frac{(2k+1)\pi}{n} \tag{M.13a}$$

$$\omega=|k^{1/n}|\sin-\frac{(2k+1)\pi}{n} \tag{M.13b}$$

Dividing the two equations to eliminate $|K^{1/n}|$, we obtain

$$\frac{\sigma+\dfrac{b_1-a_1}{n}}{\omega}=\frac{\cos\dfrac{(2k+1)\pi}{n}}{\sin\dfrac{(2k+1)\pi}{n}} \tag{M.14}$$

Finally, solving for ω, we find

$$\omega=\left[\tan\frac{(2k+1)\pi}{n}\right]\left[\sigma+\frac{b_1-a_1}{n}\right] \tag{M.15}$$

The form of this equation is that of a straight line,

$$\omega=M(\sigma-\sigma_0) \tag{M.16}$$

where the slope of the line, M, is

$$M=\tan\frac{(2k+1)\pi}{n} \tag{M.17}$$

[1] This is a good approximation since s is approaching infinity for the region applicable to the derivation.

Thus, the angle of the line in radians with respect to the positive extension of the real axis is

$$\theta = \frac{(2k+1)\pi}{n} \tag{M.18}$$

and the σ intercept is

$$\sigma_0 = -\left[\frac{b_1 - a_1}{n}\right] \tag{M.19}$$

From the theory of equations,[2]

$$b_1 = -\sum \text{finite poles} \tag{M.20a}$$

$$a_1 = -\sum \text{finite zeros} \tag{M.20b}$$

Also, from Eq. (M.1),

$$\begin{aligned} n &= \text{number of finite poles} - \text{number of finite zeros} \\ &= \#\text{finite poles} - \#\text{finite zeros} \end{aligned} \tag{M.21}$$

By examining Eq. (M.16), we conclude that the root locus approaches a straight line as the locus approaches infinity. Further, this straight line intersects the σ axis at

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.22}$$

which is obtained by substituting Eqs. (M.20)

Let us summarize the results: *The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept and the angle with respect to the real axis as follows:*

$$\sigma_0 = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.23}$$

$$\theta = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{M.24}$$

where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$ Notice that the running index, k, in Eq. (M.24) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity.

M.2 Derivation of Transition Method for Breakaway and Break-in Points

The *transition* method for finding real-breakaway and break-in points without differentiating can be derived by showing that the natural log of $1/[G(\sigma)H(\sigma)]$ has a zero derivative at the same value of σ as $1/[G(\sigma)H(\sigma)]$ (*Franklin, 1991*).

We now show that if we work with the natural log we can eliminate the step of differentiation.

[2] Given an nth-order polynomial of the form $s^n + a_{n-1}s^{n-1} + \cdots$, the coefficient, a_{n-1}, is the negative sum of the roots of the polynomial.

First find the derivative of the natural log of $1/[G(\sigma)H(\sigma)]$ and set it equal to zero. Thus,

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = G(\sigma)H(\sigma)\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.25}$$

Since $G(\sigma)H(\sigma)$ is not zero at the breakaway or break-in points, letting

$$\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.26}$$

will thus yield the same value of σ as letting

$$\frac{d}{d\sigma}\left[\frac{1}{G(\sigma)H(\sigma)}\right] = 0 \tag{M.27}$$

Hence,

$$\begin{aligned}
\frac{d}{d\sigma}\ln\left[\frac{1}{G(\sigma)H(\sigma)}\right] &= \frac{d}{d\sigma}\ln\left[\frac{(\sigma+p_1)(\sigma+p_2)\ldots(\sigma+p_n)}{(\sigma+z_1)(\sigma+z_2)\ldots(\sigma+z_m)}\right] \\
&= \frac{d}{d\sigma}\left[\ln(\sigma+p_1)+\ln(\sigma+p_2)\ldots\ln(\sigma+p_n)\right. \\
&\quad \left. -\ln(\sigma+z_1)-\ln(\sigma+z_2)\ldots-\ln(\sigma+z_m)\right] \\
&= \frac{1}{\sigma+p_1}+\frac{1}{\sigma+p_2}\ldots+\frac{1}{\sigma+p_n}-\frac{1}{\sigma+z_1}-\frac{1}{\sigma+z_2}\ldots \\
&\quad -\frac{1}{\sigma+z_m} = 0
\end{aligned} \tag{M.28}$$

or

$$\sum_{i=1}^{n}\frac{1}{\sigma+p_i} = \sum_{i=1}^{m}\frac{1}{\sigma+z_i} \tag{M.29}$$

where z_i and p_i are the negatives of the zero and pole values of $G(s)H(s)$, respectively. Equation (M.29) can be solved for σ, the real axis values that minimize or maximize K, yielding the breakaway and break-in points without differentiating.

Bibliography

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Kuo, B. *Automatic Control Systems*, 5th ed. Prentice-Hall, Englewood Cliffs, NJ, 1987.