Linear Algebra Workbook Solutions

Matrix-vector products

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MATH

MULTIPLYING MATRICES BY VECTORS

1. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\vec{x} = (4, -1)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 2 columns and the vector has 2 rows.

$$A\vec{x} = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0(4) + 2(-1) \\ -1(4) + 1(-1) \\ 0(4) - 2(-1) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0 - 2 \\ -4 - 1 \\ 0 + 2 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}$$

2. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x} = (-2,3)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 2 columns and the matrix has 2 rows.

$$\vec{x}A = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -2(3)+3(0) & -2(-1)+3(0) & -2(0)+3(4) \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -6+0 & 2+0 & 0+12 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -6 & 2 & 12 \end{bmatrix}$$

3. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\vec{x} = (2,0,1)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 3 columns and the vector has 3 rows.

$$A\vec{x} = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 4(2) - 2(0) + 1(1) \\ 0(2) + 0(0) - 1(1) \\ -3(2) + 1(0) + 2(1) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 8 - 0 + 1 \\ 0 + 0 - 1 \\ -6 + 0 + 2 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 9 \\ -1 \\ -4 \end{bmatrix}$$

■ 4. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 1 & -1 & 0 & -2 \\ -3 & 0 & -2 & 1 \end{bmatrix}$$

$$\vec{x} = (2, -6)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 2 columns and the matrix has 2 rows.

$$\vec{x}A = \begin{bmatrix} 2 & -6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & -2 \\ -3 & 0 & -2 & 1 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 2(1) - 6(-3) & 2(-1) - 6(0) & 2(0) - 6(-2) & 2(-2) - 6(1) \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 2 + 18 & -2 - 0 & 0 + 12 & -4 - 6 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 20 & -2 & 12 & -10 \end{bmatrix}$$

■ 5. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & 6 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{x} = (3,3)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 2 columns and the vector has 2 rows.

$$A\vec{x} = \begin{bmatrix} 4 & 6 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 4(3) + 6(3) \\ -2(3) - 3(3) \\ 1(3) + 0(3) \\ 0(3) + 1(3) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 12 + 18 \\ -6 - 9 \\ 3 + 0 \\ 0 + 3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 30 \\ -15 \\ 3 \\ 3 \end{bmatrix}$$

■ 6. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 6 & -4 & -4 \\ 1 & -4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = (-3,1,1)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 3 columns and the matrix has 3 rows.

$$\vec{x}A = \begin{bmatrix} -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -4 & -4 \\ 1 & -4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -3(6) + 1(1) + 1(0) & -3(-4) + 1(-4) + 1(0) & -3(-4) + 1(-4) + 1(1) \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -18 + 1 + 0 & 12 - 4 + 0 & 12 - 4 + 1 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -17 & 8 & 9 \end{bmatrix}$$

THE NULL SPACE AND AX=O

1. Is $\vec{x} = (1,2)$ in the null space of A?

$$A = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$$

Solution:

If $\vec{x} = (1,2)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x} = \vec{O}$$

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 4(1) - 2(2) \\ 2(1) - 1(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - 4 \\ 2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x}=(1,2)$ is in the null space of A.

■ 2. Is $\vec{x}=(5,-8,-9)$ in the null space of A?

$$A=\begin{bmatrix}6 & 1 & 1\\ 0 & -2 & 3\\ -1 & 0 & 4\end{bmatrix}$$

Solution:

If $\vec{x}=(5,-8,-9)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}6 & 1 & 1\\ 0 & -2 & 3\\ -1 & 0 & 4\end{bmatrix}\begin{bmatrix}5\\ -8\\ -9\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix}6(5)+1(-8)+1(-9)\\ 0(5)-2(-8)+3(-9)\\ -1(5)+0(-8)+4(-9)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix} 30 - 8 - 9 \\ 0 + 16 - 27 \\ -5 + 0 - 36 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ -11 \\ -41 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we don't get the zero vector on the left side, we know that $\overrightarrow{x} = (5, -8, -9)$ is not in the null space of A.

■ 3. Is $\overrightarrow{x} = (1,1,1)$ in the null space of A?

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & -5 \\ 1 & -6 & 5 \end{bmatrix}$$

Solution:

If $\overrightarrow{x} = (1,1,1)$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & -5 \\ 1 & -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 2(1) - 3(1) + 1(1) \\ 1(1) + 4(1) - 5(1) \\ 1(1) - 6(1) + 5(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 3 + 1 \\ 1 + 4 - 5 \\ 1 - 6 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x} = (1,1,1)$ is in the null space of A.

■ 4. Is $\vec{x} = (4, -2)$ in the null space of A?

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \\ -2 & -4 \end{bmatrix}$$

Solution:

If $\vec{x} = (4, -2)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x} = \vec{O}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 1(4) + 2(-2) \\ -1(4) - 2(-2) \\ 2(4) + 4(-2) \\ -2(4) - 4(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - 4 \\ -4 + 4 \\ 8 - 8 \\ -8 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x} = (4, -2)$ is in the null space of A.

■ 5. Is $\vec{x} = (1,1,2,1)$ in the null space of A?

$$A = \begin{bmatrix} 1 & -7 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Solution:

If $\overrightarrow{x} = (1,1,2,1)$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} 1 & -7 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 1(1) - 7(1) + 3(2) + 0(1) \\ 0(1) + 1(1) - 1(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - 7 + 6 + 0 \\ 0 + 1 - 2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\overrightarrow{x} = (1,1,2,1)$ is in the null space of A.

■ 6. Is $\overrightarrow{x} = (-1,-3,1)$ in the null space of A?

$$A = \begin{bmatrix} -4 & 3 & 5 \\ 3 & 1 & 6 \\ 0 & -2 & -6 \end{bmatrix}$$

Solution:

If $\overrightarrow{x} = (-1, -3, 1)$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} -4 & 3 & 5 \\ 3 & 1 & 6 \\ 0 & -2 & -6 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} -4(-1) + 3(-3) + 5(1) \\ 3(-1) + 1(-3) + 6(1) \\ 0(-1) - 2(-3) - 6(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - 9 + 5 \\ -3 - 3 + 6 \\ 0 + 6 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x}=(-1,-3,1)$ is in the null space of A.

NULL SPACE OF A MATRIX

1. Find the null space of A.

$$A = \begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 = 0$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the null space of A is only the zero vector.

2. Find the null space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & 1 \\ 5 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & 9 & -9 \\ 5 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & 9 & -9 \\ 0 & -19 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -1 \\ 0 & -19 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -19 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

which we can solve for the pivot variables.

$$x_1 = x_3$$

$$x_2 = x_3$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

3. Find the null space of A.

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the equation,

$$x_1 - \frac{1}{3}x_2 = 0$$

which we can solve for the pivot variable.

$$x_1 = \frac{1}{3}x_2$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right)$$

4. Find the null space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 + \frac{7}{2}x_4 = 0$$

$$x_3 + \frac{1}{2}x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = 0$$

$$x_2 = -\frac{7}{2}x_4$$

$$x_3 = -\frac{1}{2}x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -\frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

■ 5. Find the null space of A.

$$A = \begin{bmatrix} 4 & -2 & 1 & 1 \\ -1 & 0 & 3 & -3 \\ 0 & 0 & -4 & 6 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 4 & -2 & 1 & 1 \\ -1 & 0 & 3 & -3 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 3 & -3 \\ 4 & -2 & 1 & 1 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 4 & -2 & 1 & 1 \\ 0 & 0 & -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & -2 & 13 & -11 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{17}{4} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{17}{4} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 - \frac{3}{2}x_4 = 0$$

$$x_2 - \frac{17}{4}x_4 = 0$$

$$x_3 - \frac{3}{2}x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = \frac{3}{2}x_4$$

$$x_2 = \frac{17}{4}x_4$$

$$x_3 = \frac{3}{2}x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{2} \\ \frac{17}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} \frac{3}{2} \\ \frac{17}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right)$$

■ 6. Find the null space of A.

$$A = \begin{bmatrix} -2 & 0 & 7 \\ 3 & -1 & 4 \\ 0 & 3 & -2 \\ 1 & 4 & -5 \\ 2 & 2 & 1 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix}-2&0&7\\3&-1&4\\0&3&-2\\1&4&-5\\2&2&1\end{bmatrix}\to\begin{bmatrix}1&4&-5\\-2&0&7\\3&-1&4\\0&3&-2\\2&2&1\end{bmatrix}\to\begin{bmatrix}1&4&-5\\0&8&-3\\3&-1&4\\0&3&-2\\2&2&1\end{bmatrix}\to\begin{bmatrix}1&4&-5\\0&8&-3\\0&-13&19\\0&3&-2\\2&2&1\end{bmatrix}$$

$$\begin{bmatrix}1&4&-5\\0&8&-3\\0&-13&19\\0&3&-2\\0&-6&11\end{bmatrix}\to\begin{bmatrix}1&4&-5\\0&1&-\frac{3}{8}\\0&-13&19\\0&3&-2\\0&-6&11\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{7}{2}\\0&1&-\frac{3}{8}\\0&-13&19\\0&3&-2\\0&-6&11\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{7}{2}\\0&1&-\frac{3}{8}\\0&0&\frac{113}{8}\\0&3&-2\\0&-6&11\end{bmatrix}$$

$$\begin{bmatrix}1&0&-\frac{7}{2}\\0&1&-\frac{3}{8}\\0&0&\frac{113}{8}\\0&0&-\frac{7}{8}\\0&-6&11\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{7}{2}\\0&1&-\frac{3}{8}\\0&0&\frac{113}{8}\\0&0&-\frac{7}{8}\\0&0&\frac{35}{4}\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{7}{2}\\0&1&-\frac{3}{8}\\0&0&1\\0&0&-\frac{7}{8}\\0&0&\frac{35}{4}\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&-\frac{3}{8}\\0&0&1\\0&0&-\frac{7}{8}\\0&0&\frac{35}{4}\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&-\frac{7}{8}\\0&0&\frac{35}{4}\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&\frac{35}{4}\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the null space of A is only the zero vector.

THE COLUMN SPACE AND AX=B

1. Find the column space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

Solution:

The column space of a matrix is all the possible linear combinations of its columns, which we can also say is the span of its columns.

$$C(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}\right)$$

2. Find the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

The column space of a matrix is all the possible linear combinations of its columns, which we can also say is the span of its columns.

$$C(A) = \text{Span}\left(\begin{bmatrix}-1\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\1\\4\end{bmatrix}, \begin{bmatrix}3\\4\\2\end{bmatrix}\right)$$

■ 3. Find a basis for the column space of A.

$$A = \begin{bmatrix}4 & -3\\0 & 4\end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix}4 & -3\\0 & 4\end{bmatrix} \to \begin{bmatrix}1 & -\frac{3}{4}\\0 & 4\end{bmatrix} \to \begin{bmatrix}1 & -\frac{3}{4}\\0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Because the first (and only) two columns of rref(A) are pivot columns, that means the first two columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix}\right)$$

4. Find a basis for the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Because the first three columns of rref(A) are pivot columns, that means the first three columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} \right)$$

5. Find a basis for the column space of A.

$$A = \begin{bmatrix} 5 & -2 & 6 \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 5 & -2 & 6 \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 1 & -18 \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & -22 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & -22 \\ 0 & 0 & 86 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the first three (and only) columns of rref(A) are pivot columns, that means the first three columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}5\\-3\\0\\8\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}6\\0\\-4\\2\end{bmatrix}\right)$$

■ 6. Find a basis for the column space of A.

$$A = \begin{bmatrix}2 & -4 & 3 & -6\\1 & -2 & 0 & 0\\4 & -8 & 5 & -10\end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & -4 & 3 & -6\\1 & -2 & 0 & 0\\4 & -8 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\2 & -4 & 3 & -6\\4 & -8 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 3 & -6\\4 & -8 & 5 & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 3 & -6\\0 & 0 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & -2\\0 & 0 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & -2\\0 & 0 & 0 & 0\end{bmatrix}$$

Because the first and third columns of rref(A) are pivot columns, that means the first and third columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}\right)$$

SOLVING AX=B

1. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2 & -4 & 3 & -6\\ 1 & -2 & 0 & 0\\ 4 & -8 & 5 & -10\end{bmatrix} \text{ with } \vec{b}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Solution:

Solve $A\vec{x}=\vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & -4 & 3 & -6 & | & 0\\ 1 & -2 & 0 & 0 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 2 & -4 & 3 & -6 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 3 & -6 & | & 0\\ 4 & -8 & 5 & -10 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 3 & -6 & | & 0\\ 0 & 0 & 5 & -10 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 1 & -2 & | & 0\\ 0 & 0 & 5 & -10 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0 & | & 0\\ 0 & 0 & 1 & -2 & | & 0\\ 0 & 0 & 0 & 0 & | & 0\end{bmatrix}$$

The first and third columns are pivot columns and the second and fourth columns are free columns. Which means x_1 and x_3 are pivot variables, and x_2 and x_4 are free variables. Pull out a system of equations,

$$x_1-2x_2=0$$

$$x_3-2x_4=0$$

then solve it for the pivot variables in terms of the free variables.

$$x_1 = 2x_2$$

$$x_3 = 2x_4$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2, b_3)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & -4 & 3 & -6 & | & b_1 \\ 1 & -2 & 0 & 0 & | & b_2 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 2 & -4 & 3 & -6 & | & b_1 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 3 & -6 & | & b_1 - 2b_2 \\ 4 & -8 & 5 & -10 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 3 & -6 & | & b_1 - 2b_2 \\ 0 & 0 & 5 & -10 & | & b_3 - 4b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}b_1 - \frac{2}{3}b_2 \\ 0 & 0 & 5 & -10 & | & b_3 - 4b_2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & b_2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}b_1 - \frac{2}{3}b_2 \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3}b_1 - \frac{2}{3}b_2 + b_3 \end{bmatrix}$$

Rewrite rref(A) with $\overrightarrow{b} = (1,2,-3)$.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}(1) - \frac{2}{3}(2) \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3}(1) - \frac{2}{3}(2) + 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3} - \frac{4}{3} \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3} - \frac{4}{3} + \frac{9}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -\frac{3}{3} \\ 0 & 0 & 0 & 0 & | & \frac{0}{3} \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_3 are pivot variables, and x_2 and x_4 are free variables. Pull a system of equations from the matrix,

$$x_1 - 2x_2 = 2$$

$$x_3 - 2x_4 = -1$$

then set the free variables equal to 0 to simplify the system to

$$x_1 - 2(0) = 2$$

$$x_3 - 2(0) = -1$$

and then

$$x_1 = 2$$

$$x_3 = -1$$

So the particular solution then is $x_1 = 2$, $x_2 = 0$, $x_3 = -1$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

■ 2. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution:

Solve $A\vec{x}=\vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}3 & 6 & | & 0\\ 6 & 12 & | & 0\\ 1 & 1 & | & 0\\ 2 & 2 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 6 & 12 & | & 0\\ 1 & 1 & | & 0\\ 2 & 2 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 0 & 0 & | & 0\\ 1 & 1 & | & 0\\ 2 & 2 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 0 & 0 & | & 0\\ 0 & -1 & | & 0\\ 2 & 2 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & | & 0\\ 0 & 0 & | & 0\\ 0 & -1 & | & 0\\ 0 & -2 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 0 & -1 & | & 0\\ 0 & -2 & | & 0\\ 0 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 0 & 1 & | & 0\\ 0 & -2 & | & 0\\ 0 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & 0\\ 0 & 1 & | & 0\\ 0 & 0 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & 0\\ 0 & 1 & | & 0\\ 0 & 0 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

The first and second columns are pivot columns and there are no free columns. Which means x_1 and x_2 are pivot variables, and there are no free variables. Pull out a system of equations.

$$x_1 = 0$$

$$x_2 = 0$$

Express the system as a linear combination.

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 3 & 6 & | & b_1 \\ 6 & 12 & | & b_2 \\ 1 & 1 & | & b_3 \\ 2 & 2 & | & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 6 & 12 & | & b_2 \\ 1 & 1 & | & b_3 \\ 2 & 2 & | & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 0 & 0 & | & b_2 - 2b_1 \\ 1 & 1 & | & b_3 \\ 2 & 2 & | & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & -1 & | & b_3 - \frac{1}{3}b_1 \\ 2 & 2 & | & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & -1 & | & b_3 - \frac{1}{3}b_1 \\ 0 & -2 & | & b_4 - \frac{2}{3}b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 0 & -1 & | & b_3 - \frac{1}{3}b_1 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & -2 & | & b_4 - \frac{2}{3}b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & | & \frac{1}{3}b_1 - b_3 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & -2 & | & b_4 - \frac{2}{3}b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{1}{3}b_1 + 2b_3 \\ 0 & 1 & | & \frac{1}{3}b_1 - b_3 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & -2 & | & b_4 - \frac{2}{3}b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -\frac{1}{3}b_1 + 2b_3 \\ 0 & 1 & | & \frac{1}{3}b_1 - b_3 \\ 0 & 0 & | & b_2 - 2b_1 \\ 0 & 0 & | & b_4 - 2b_3 \end{bmatrix}$$

Rewrite rref(A) with $\vec{b} = (1,2,1,2)$.

$$\begin{bmatrix} 1 & 0 & | & -\frac{1}{3}(1) + 2(1) \\ 0 & 1 & | & \frac{1}{3}(1) - 1 \\ 0 & 0 & | & 2 - 2(1) \\ 0 & 0 & | & 2 - 2(1) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{1}{3} + 2 \\ 0 & 1 & | & \frac{1}{3} - 1 \\ 0 & 0 & | & 2 - 2 \\ 0 & 0 & | & 2 - 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{5}{3} \\ 0 & 1 & | & -\frac{2}{3} \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_2 are pivot variables, and there are no free variables. Pull a system of equations from the matrix.

$$x_1 = \frac{5}{3}$$

$$x_2 = -\frac{2}{3}$$

So the particular solution then is $x_1 = 5/3$ and $x_2 = -2/3$.

$$\vec{x}_p = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But adding the zero vector won't ever affect the solution, so the general solution is just

$$\vec{x} = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

■ 3. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & -5 & 3 \\ -1 & 4 & 0 \\ 3 & -16 & 12 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Solution:

Solve $A\vec{x} = \vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -5 & 3 & | & 0 \\ -1 & 4 & 0 & | & 0 \\ 3 & -16 & 12 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & 0 \\ 0 & -1 & 3 & | & 0 \\ 3 & -16 & 12 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & 0 \\ 0 & -1 & 3 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -12 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first and second columns are pivot columns, and the third column is a free column. Which means x_1 and x_2 are pivot variables, and x_3 is free variable. Pull out a system of equations.

$$x_1 - 12x_3 = 0$$

$$x_2 - 3x_3 = 0$$

Then solve it for the pivot variables in terms of the free variable.

$$x_1 = 12x_3$$

$$x_2 = 3x_3$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2, b_3)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ -1 & 4 & 0 & | & b_2 \\ 3 & -16 & 12 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & -1 & 3 & | & b_1+b_2 \\ 3 & -16 & 12 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & -1 & 3 & | & b_1+b_2 \\ 0 & -1 & 3 & | & -3b_1+b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & -1 & 3 & | & -3b_1+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & 0 & 0 & | & -4b_1-b_2+b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -12 & | & -4b_1-5b_2 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & 0 & 0 & | & -4b_1-b_2+b_3 \end{bmatrix}$$

Rewrite rref(A) with $\vec{b}=(1,-1,3)$.

$$\begin{bmatrix} 1 & 0 & -12 & | & -4(1)-5(-1) \\ 0 & 1 & -3 & | & -1-(-1) \\ 0 & 0 & 0 & | & -4(1)-(-1)+3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -12 & | & -4+5 \\ 0 & 1 & -3 & | & -1+1 \\ 0 & 0 & 0 & | & -4+1+3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -12 & | & 1 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_2 are pivot variables, and x_3 is a free variable. Pull a system of equations from the matrix,

$$x_1 - 12x_3 = 1$$

$$x_2 - 3x_3 = 0$$

then set the free variables equal to 0 to simplify the system to

$$x_1 - 12(0) = 1$$

$$x_2 - 3(0) = 0$$

and then

$$x_1 = 1$$

$$x_2 = 0$$

So the particular solution then is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

■ 4. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} -2 & 10 & -6 & 2 \\ 1 & -5 & 3 & -1 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solution:

Solve $A\vec{x} = \vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 10 & -6 & 2 & | & 0 \\ 1 & -5 & 3 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & -1 & | & 0 \\ -2 & 10 & -6 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first column is pivot column, and the second, third, and fourth columns are free columns. Which means x_1 is pivot variable, and x_2, x_3, and x_4 are free variables. Pull out an equation,

$$x_1 - 5x_2 + 3x_3 - x_4 = 0$$

then solve it for the pivot variable in terms of the free variables.

$$x_1 = 5x_2 - 3x_3 + x_4$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 10 & -6 & 2 & | & b_1 \\ 1 & -5 & 3 & -1 & | & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & -1 & | & b_2 \\ -2 & 10 & -6 & 2 & | & b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & b_2 \\ 0 & 0 & 0 & 0 & | & 2b_2 + b_1 \end{bmatrix}$$

Rewrite rref(A) with $\vec{b} = (-2,1)$.

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 2(1) - 2 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 is a pivot variable, and x_2, x_3, and x_4 are free variables. Pull an equation from the matrix,

$$x_1 - 5x_2 + 3x_3 - x_4 = 1$$

then set the free variables equal to 0 to simplify the system to

$$x_1 - 5(0) + 3(0) - 0 = 1$$

and then

$$x_1 = 1$$

So the particular solution then is $x_1 = 1$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

■ 5. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 2 & 0 & 0 & 12 \\ -1 & 2 & -1 & 4 \\ 5 & -6 & 3 & 0 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Solution:

Solve $A\vec{x} = \vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 0 & 0 & 12 & | & 0 \\ -1 & 2 & -1 & 4 & | & 0 \\ 5 & -6 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & -1 & 4 & | & 0 \\ 2 & 0 & 0 & 12 & | & 0 \\ 5 & -6 & 3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & 0 \\ 2 & 0 & 0 & 12 & | & 0 \\ 5 & -6 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & 0 \\ 0 & 4 & -2 & 20 & | & 0 \\ 5 & -6 & 3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & 0 \\ 0 & 4 & -2 & 20 & | & 0 \\ 0 & 4 & -2 & 20 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 5 & | & 0 \\ 0 & 4 & -2 & 20 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 5 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 5 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first and second columns are pivot columns and the third and fourth columns are free columns. Which means x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. Pull out a system of equations,

$$x_1 + 6x_4 = 0$$

$$x_2 - \frac{1}{2}x_3 + 5x_4 = 0$$

then solve it for the pivot variables in terms of the free variables.

$$x_1 = -6x_4$$

$$x_2 = \frac{1}{2}x_3 - 5x_4$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2, b_3)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 0 & 0 & 12 & | & b_1 \\ -1 & 2 & -1 & 4 & | & b_2 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & -1 & 4 & | & b_2 \\ 2 & 0 & 0 & 12 & | & b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 2 & 0 & 0 & 12 & | & b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 4 & -2 & 20 & | & 2b_2 + b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 4 & -2 & 20 & | & 2b_2 + b_1 \\ 0 & 4 & -2 & 20 & | & 5b_2 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2 + b_1}{4} \\ 0 & 4 & -2 & 20 & | & 5b_2 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2+b_1}{4} \\ 0 & 0 & 0 & 0 & | & -b_1+3b_2+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{b_1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2+b_1}{4} \\ 0 & 0 & 0 & 0 & | & -b_1+3b_2+b_3 \end{bmatrix}$$

Rewrite rref(A) with $\overrightarrow{b}=(1,1,-2)$.

$$\begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2(1)+1}{4} \\ 0 & 0 & 0 & 0 & | & -1+3(1)-2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{3}{4} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1 and x_2 are pivot variables, and x_3 and x_4 are free variables. Pull a system of equations from the matrix,

$$x_1+6x_4=\frac{1}{2}$$

$$x_2-\frac{1}{2}x_3+5x_4=\frac{3}{4}$$

then set the free variables equal to 0 to simplify the system to

$$x_1+6(0)=\frac{1}{2}$$

$$x_2-\frac{1}{2}(0)+5(0)=\frac{3}{4}$$

and then

$$x_1=\frac{1}{2}$$

$$x_2 = \frac{3}{4}$$

So the particular solution then is $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = 0$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

■ 6. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & 0 & 3 & -5 \\ 4 & -2 & 2 & 0 \\ -1 & 2 & -1 & 1 \\ 3 & 2 & -1 & 5 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

Solution:

Solve $A\vec{x}=\vec{O}$ by augmenting A with $\vec{O}$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 4 & -2 & 2 & 0 & | & 0 \\ -1 & 2 & -1 & 1 & | & 0 \\ 3 & 2 & -1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & -2 & -10 & 20 & | & 0 \\ -1 & 2 & -1 & 1 & | & 0 \\ 3 & 2 & -1 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & -2 & -10 & 20 & | & 0 \\ 0 & 2 & 2 & -4 & | & 0 \\ 3 & 2 & -1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & -2 & -10 & 20 & | & 0 \\ 0 & 2 & 2 & -4 & | & 0 \\ 0 & 2 & -10 & 20 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 2 & 2 & -4 & | & 0 \\ 0 & 2 & -10 & 20 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 0 & -8 & 16 & | & 0 \\ 0 & 2 & -10 & 20 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 0 & -8 & 16 & | & 0 \\ 0 & 0 & -20 & 40 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & -20 & 40 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 5 & -10 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The first, second, and third columns are pivot columns, and the fourth column is a free column. Which means x_1, x_2, and x_3 are pivot variables, and x_4 is a free variable. Pull out a system of equations,

$$x_1 + x_4 = 0$$

$$x_2 = 0$$

$$x_3 - 2x_4 = 0$$

then solve it for the pivot variables in terms of the free variables.

$$x_1 = -x_4$$

$$x_2 = 0$$

$$x_3 = 2x_4$$

Express the system as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Then the complementary solution is

$$\vec{x}_n = c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Find the particular solution that satisfies $A\vec{x}_p = \vec{b}$ by augmenting A with $\vec{b} = (b_1, b_2, b_3, b_4)$ and then putting the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 4 & -2 & 2 & 0 & | & b_2 \\ -1 & 2 & -1 & 1 & | & b_3 \\ 3 & 2 & -1 & 5 & | & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & -2 & -10 & 20 & | & -4b_1 + b_2 \\ -1 & 2 & -1 & 1 & | & b_3 \\ 3 & 2 & -1 & 5 & | & b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & -2 & -10 & 20 & | & -4b_1 + b_2 \\ 0 & 2 & 2 & -4 & | & b_1 + b_3 \\ 3 & 2 & -1 & 5 & | & b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & -2 & -10 & 20 & | & -4b_1 + b_2 \\ 0 & 2 & 2 & -4 & | & b_1 + b_3 \\ 0 & 2 & -10 & 20 & | & -3b_1 + b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1 + b_2}{-2} \\ 0 & 2 & 2 & -4 & | & b_1 + b_3 \\ 0 & 2 & -10 & 20 & | & -3b_1 + b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1 + b_2}{-2} \\ 0 & 0 & -8 & 16 & | & -3b_1 + b_2 + b_3 \\ 0 & 2 & -10 & 20 & | & -3b_1 + b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1 + b_2}{-2} \\ 0 & 0 & -8 & 16 & | & -3b_1 + b_2 + b_3 \\ 0 & 0 & -20 & 40 & | & -7b_1 + b_2 + b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1 + b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1 + b_2 + b_3}{-8} \\ 0 & 0 & -20 & 40 & | & -7b_1 + b_2 + b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-b_1+3b_2+3b_3}{8} \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-b_1+3b_2+3b_3}{8} \\ 0 & 1 & 0 & 0 & | & \frac{b_1+b_2+5b_3}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix}$$

Rewrite rref(A) with $\vec{b} = (2,1,1,3)$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-2+3(1)+3(1)}{8} \\ 0 & 1 & 0 & 0 & | & \frac{2+1+5(1)}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-3(2)+1+1}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{2-3(1)-5(1)+2(3)}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-2+3+3}{8} \\ 0 & 1 & 0 & 0 & | & \frac{2+1+5}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-6+1+1}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{2-3-5+6}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{4}{8} \\ 0 & 1 & 0 & 0 & | & \frac{8}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-4}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{0}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{1}{2} \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & -2 & | & \frac{1}{2} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Again, just like with the null space, x_1, x_2, and x_3 are pivot variables, and x_4 is a free variable. Pull a system of equations from the matrix,

$$x_1 + x_4 = \frac{1}{2}$$

$$x_2 = 1$$

$$x_3 - 2x_4 = \frac{1}{2}$$

then set the free variables equal to 0 to simplify the system to

$$x_1 + 0 = \frac{1}{2}$$

$$x_2 = 1$$

$$x_3 - 2(0) = \frac{1}{2}$$

and then

$$x_1 = \frac{1}{2}$$

$$x_2 = 1$$

$$x_3 = \frac{1}{2}$$

So the particular solution then is $x_1 = 1/2$, $x_2 = 1$, $x_3 = 1/2$, and $x_4 = 0$, or

$$\overrightarrow{x}_p = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\overrightarrow{x} = \overrightarrow{x}_p + \overrightarrow{x}_n$$

$$\overrightarrow{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

DIMENSIONALITY, NULLITY, AND RANK

■ 1. Find the nullity of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ 3 & -7 & 0 & 1 \end{bmatrix}$$

Solution:

To find the nullity of the matrix A, we need to first find the null space, so we'll set up the augmented matrix for $A\vec{x} = \vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 3 & -7 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 0 & 2 & -6 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 0 & 1 & -3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7 & 5 & | & 0 \\ 0 & 1 & -3 & 2 & | & 0 \end{bmatrix}$$

Pull out a system of equations,

$$x_1 - 7x_3 + 5x_4 = 0$$

$$x_2 - 3x_3 + 2x_4 = 0$$

then solve it for the pivot variables.

$$x_1 = 7x_3 - 5x_4$$

$$x_2 = 3x_3 - 2x_4$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 7 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

We found 2 spanning vectors that form a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(A)) = \text{nullity}(A) = 2$$

We can also get the nullity of A from the number of free variables in rref(A). Because there were 2 free variables, x_3 and x_4, nullity$(A) = 2$.

■ 2. Find the rank of X.

$$X = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 0 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix}-2 & 3 & 1\\ -1 & 0 & -4\\ 2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}-1 & 0 & -4\\ -2 & 3 & 1\\ 2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\ -2 & 3 & 1\\ 2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\ 0 & 3 & 9\\ 2 & -4 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 4\\ 0 & 3 & 9\\ 0 & -4 & -5\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 3\\ 0 & -4 & -5\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 3\\ 0 & 0 & 7\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that all three columns are pivot columns. So because there are three pivot variables, the rank is

$$\text{Dim}(C(X)) = \text{rank}(X) = 3$$

■ 3. Find the nullity and the rank of A.

$$A = \begin{bmatrix}-1 & -3 & 2 & 4 & -2\\ -3 & -5 & -2 & 1 & 4\\ 0 & 4 & -8 & -11 & 10\\ 1 & 3 & -2 & -4 & 5\end{bmatrix}$$

Solution:

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -1 & -3 & 2 & 4 & -2 \\ -3 & -5 & -2 & 1 & 4 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ -3 & -5 & -2 & 1 & 4 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & 0 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & 0 \\ 0 & 1 & -2 & -\frac{11}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rref(A) has three pivot columns and two free columns,

rank$(A) = 3$

nullity$(A) = 2$

■ 4. Find the nullity of M.

$$M = \begin{bmatrix} -4 & 2 & -2 & 1 \\ -1 & 0 & -3 & 2 \\ 3 & -2 & 5 & 0 \end{bmatrix}$$

Solution:

To find the nullity of the matrix M, we need to first find the null space, so we'll set up the augmented matrix for $M\vec{x} = \vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix} -4 & 2 & -2 & 1 & | & 0 \\ -1 & 0 & -3 & 2 & | & 0 \\ 3 & -2 & 5 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & -3 & 2 & | & 0 \\ -4 & 2 & -2 & 1 & | & 0 \\ 3 & -2 & 5 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ -4 & 2 & -2 & 1 & | & 0 \\ 3 & -2 & 5 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ 0 & 2 & 10 & -7 & | & 0 \\ 3 & -2 & 5 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ 0 & 2 & 10 & -7 & | & 0 \\ 0 & -2 & -4 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ 0 & 1 & 5 & -\frac{7}{2} & | & 0 \\ 0 & -2 & -4 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ 0 & 1 & 5 & -\frac{7}{2} & | & 0 \\ 0 & 0 & 6 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -2 & | & 0 \\ 0 & 1 & 5 & -\frac{7}{2} & | & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & | & 0 \\ 0 & 1 & 5 & -\frac{7}{2} & | & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & | & 0 \\ 0 & 1 & 0 & -\frac{8}{3} & | & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & | & 0 \end{bmatrix}$$

Pull out a system of equations,

$$x_1 - \frac{3}{2}x_4 = 0$$

$$x_2 - \frac{8}{3}x_4 = 0$$

$$x_3 - \frac{1}{6}x_4 = 0$$

then solve it for the pivot variables.

$$x_1 = \frac{3}{2}x_4$$

$$x_2 = \frac{8}{3}x_4$$

$$x_3 = \frac{1}{6}x_4$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{2} \\ \frac{8}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

Then the null space of M is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left(\begin{bmatrix} \frac{3}{2} \\ \frac{8}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}\right)$$

We found one spanning vector that forms a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(M)) = \text{nullity}(M) = 1$$

We can also get the nullity of M from the number of free variables in rref(M). Because there was one free variable, x_4, nullity$(M) = 1$.

■ 5. Find the rank of M.

$$M = \begin{bmatrix} -2 & 0 & -5 & 6 & 2 \\ 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix}$$

Solution:

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -2 & 0 & -5 & 6 & 2 \\ 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ -2 & 0 & -5 & 6 & 2 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -3 & -1 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that two columns are pivot columns. So because there are two pivot columns, and therefore two pivot variables, the rank is

$$\text{Dim}(C(M)) = \text{rank}(M) = 2$$

■ 6. Find the nullity and the rank of M.

$$M = \begin{bmatrix} -1 & 2 & 0 & 3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix}$$

Solution:

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 0 & 4 & 0 & 5 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 0 & 4 & 0 & 5 \\ 0 & -2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 4 & 0 & 5 \\ 0 & -2 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since rref(A) has four pivot columns and zero free columns,

rank$(M) = 4$

nullity$(M = 0$

W W W . K R I S T A K I N G M A T H . C O M