Appendix J

Derivation of the Time Domain Solution of State Equations

J.1 Derivation

Rather than using the Laplace transformation, we can solve the equations directly in the time domain using a method closely allied to the classical solution of differential equations. We will find that the final solution consists of two parts that are different from the forced and natural responses

First, assume a homogeneous state equation of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{J.1}$$

Since we want to solve for $\mathbf{x}$, we assume a series solution, just as we did in elementary scalar differential equations. Thus,

$$\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_\mathbf{k} t^k + \mathbf{b}_{\mathbf{k}+1} t^{k+1} + \cdots \tag{J.2}$$

Substituting Eq. (J.2) into (J.1) we get

$$\begin{aligned} &\mathbf{b}_1 + 2\mathbf{b}_2 t + \cdots + k\mathbf{b}_\mathbf{k} t^{k-1} + (k+1)\mathbf{b}_{\mathbf{k}+1} t^k + \cdots \\ &\qquad = \mathbf{A}(\mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_\mathbf{k} t^k + \mathbf{b}_{\mathbf{k}+1} t^{k+1} + \cdots) \end{aligned} \tag{J.3}$$

Equating like coefficients yields

$$\mathbf{b}_1 = \mathbf{A}\mathbf{b}_0 \tag{J.4a}$$

$$\mathbf{b}_2 = \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{b}_0 \tag{J.4b}$$

$$\vdots$$

$$\mathbf{b}_\mathbf{k} = \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0 \tag{J.4c}$$

$$\mathbf{b}_{\mathbf{k}+1} = \frac{1}{(k+1)!}\mathbf{A}^{k+1}\mathbf{b}_0 \tag{J.4d}$$

$$\vdots$$

Substituting these values into Eq. (J.2) yields

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{b}_0 + \mathbf{A}\mathbf{b}_0 t + \frac{1}{2}\mathbf{A}^2\mathbf{b}_0 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0 t^k + \frac{1}{(k+1)!}\mathbf{A}^{k+1}\mathbf{b}_0 t^{k+1} + \cdots \\ &= \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \frac{1}{(k+1)!}\mathbf{A}^{k+1} t^{k+1} + \cdots\right)\mathbf{b}_0 \end{aligned} \tag{J.5}$$

But, from Eq. (J.2),

$$\mathbf{x}(0) = \mathbf{b}_0 \tag{J.6}$$[1]

Therefore,

$$\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \frac{1}{(k+1)!}\mathbf{A}^{k+1}t^{k+1} + \cdots\right)\mathbf{x}(0) \tag{J.7}$$

Let

$$e^{\mathbf{A}t} = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \frac{1}{(k+1)!}\mathbf{A}^{k+1}t^{k+1} + \cdots\right) \tag{J.8}$$

where $e^{\mathbf{A}t}$ is simply a notation for the matrix formed by the right-hand side of Eq. (J.8). We use this definition because the right-hand side of Eq. (J.8) resembles a power series expansion of e^{at}, or

$$e^{at} = \left(1 + at + \frac{1}{2}a^2t^2 + \cdots + \frac{1}{k!}a^k t^k + \frac{1}{(k+1)!}a^{k+1}t^{k+1} + \cdots\right) \tag{J.9}$$

Using Eq. (J.7), we have

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) \tag{J.10}$$

We give a special name to $e^{\mathbf{A}t}$: it is called the *state-transition matrix*[2], since it performs a transformation on $\mathbf{x}(0)$, taking $\mathbf{x}$ from the initial state, $\mathbf{x}(0)$, to the state $\mathbf{x}(t)$ at any time, t. The symbol, $\mathbf{\Phi}(t)$, is used to denote $e^{\mathbf{A}t}$. Thus,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} \tag{J.11}$$

and

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{J.12}$$

There are some properties of $\mathbf{\Phi}(t)$ that we will use later when we solve for $\mathbf{x}(t)$ in the text. From Eq. (J.12),

$$\mathbf{x}(0) = \mathbf{\Phi}(0)\mathbf{x}(0) \tag{J.13}$$

Hence, the first property of $\mathbf{\Phi}(t)$ is

$$\mathbf{\Phi}(0) = \mathbf{I} \tag{J.14}$$

where $\mathbf{I}$ is the identity matrix. Also, differentiating Eq. (J.12) and setting this equal to Eq. (J.1) yields

$$\dot{\mathbf{x}}(t) = \dot{\mathbf{\Phi}}(t)\mathbf{x}(0) = \mathbf{A}\mathbf{x}(t) \tag{J.15}$$

which, at $t = 0$, yields

$$\dot{\mathbf{\Phi}}(0)\mathbf{x}(0) = \mathbf{A}\mathbf{x}(0) \tag{J.16}$$

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[1] In this development we consider the initial time, t_0, to be 0. More generally, $t_0 \neq 0$. After completing this development, the interested reader should consult Appendix K on www.wiley.com/college/nise for the more general solution in terms of initial time $t_0 \neq 0$.

[2] The state-transition matrix here is for the initial time $t_0 = 0$. The derivation in Appendix K on at www.wiley.com/go/Nise/ControlSystemsEngineering8e for $t_0 \neq 0$ yields $\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0)$.

Thus, the second property of $\mathbf{\Phi}(t)$ follows from Eq. (J.16):

$$\dot{\mathbf{\Phi}}(0) = \mathbf{A} \tag{J.17}$$

In summary, the solution to the homogeneous, or unforced, system is

$$x(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{J.18}$$

where

$$\mathbf{\Phi}(0) = \mathbf{I} \tag{J.19}$$

and

$$\dot{\mathbf{\Phi}}(0) = \mathbf{A} \tag{J.20}$$

Let us now solve the forced, or nonhomogeneous, problem. Given the forced state equation

$$\dot{\mathbf{x}}(t)\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{J.21}$$

rearrange and multiply both sides by $e^{-\mathbf{A}t}$:

$$e^{-\mathbf{A}t}[\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{J.22}$$

Realizing that the left-hand side is equal to the derivative of the product $e^{-\mathbf{A}t}\mathbf{x}(t)$, we obtain

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \tag{J.23}$$

Integrating both sides yields

$$\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right]\Big/_0^t = e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{J.24}$$

since $e^{-\mathbf{A}t}$ evaluated at $t = 0$ is the identity matrix (from Eq. (J.8)). Solving for $\mathbf{x}(t)$ in Eq. (J.24) we obtain

$$\begin{aligned} \mathbf{x}(t) &= e^{-\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{-\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\ &= \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \end{aligned} \tag{J.25}$$

where $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ by definition.

Bibliography

Timothy, L. K., and Bona, B. E. *State Space Analysis: An Introduction*. McGraw-Hill, New York, 1968.