$x = 0$

$\cos(-x) = \cos(x)$

$y = x$

1.5 Even and Odd Functions

A function $x_e(t)$ is said to be an *even function* of t if it is symmetrical about the vertical axis. A function $x_o(t)$ is said to be an *odd function* of t if it is antisymmetrical about the vertical axis. Mathematically expressed, these symmetry conditions require

$$\boxed{x_e(t) = x_e(-t)} \quad \text{and} \quad \boxed{x_o(t) = -x_o(-t)}$$

$x(t) = -2t$ $\quad$ $x(t) = 2t$

An even function has the same value at the instants t and $-t$ for all values of t. On the other hand, the value of an odd function at the instant t is the negative of its value at the instant $-t$. An example even signal and an example odd signal are shown in Figs. 1.23a and 1.23b, respectively.

1.5-1 Some Properties of Even and Odd Functions

Even and odd functions have the following properties:

even function × odd function = odd function

odd function × odd function = even function

even function × even function = even function

$\cos(t)$

$2\left[u(t) - \frac{1}{2}\right]$

$u(t)\, e^{-t}$

$u(t-1) \cdot e^{-t}$

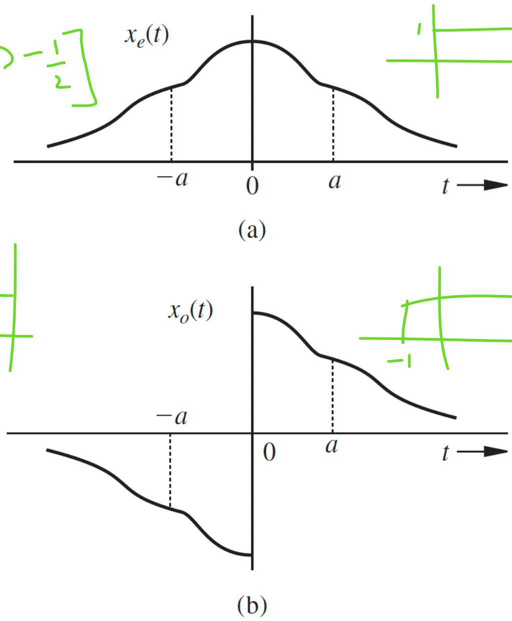


Figure 1.9 **(a)** Even and **(b)** odd functions.

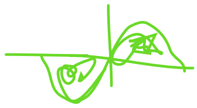
AREA

Because of the symmetries of even and odd functions about the vertical axis, we can infer

$$\int_{-a}^{a} x_e(t)\,dt = 2\int_{0}^{a} x_e(t)\,dt$$

$=\int_{-a}^{a} t \cdot$ [illegible] $(t)\ dt = 0$

$$\int_{-a}^{a} x_o(t)\,dt = 0$$

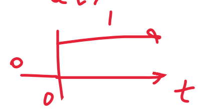


1.5-2 Even and Odd Components of a Signal

Every signal $x(t)$ can be expressed as a sum of even and odd components since

$$x(t) = \underbrace{\tfrac{1}{2}[x(t)+x(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[x(t)-x(-t)]}_{\text{odd}}$$

$u(t)$



EXAMPLE: Finding the Even and Odd Components of a Signal

$\int_{0}^{t_0} e^{-at}\, dt$

Find and sketch the even and odd components of $x(t) = e^{-at}u(t)$.

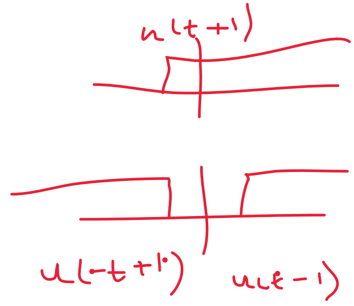
As described above, we can express $x(t)$ as a sum of the even component $x_e(t)$ and the odd component $x_o(t)$ as

$$x(t) = x_e(t) + x_o(t)$$

where,

$$x_e(t) = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$

$$x_o(t) = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$



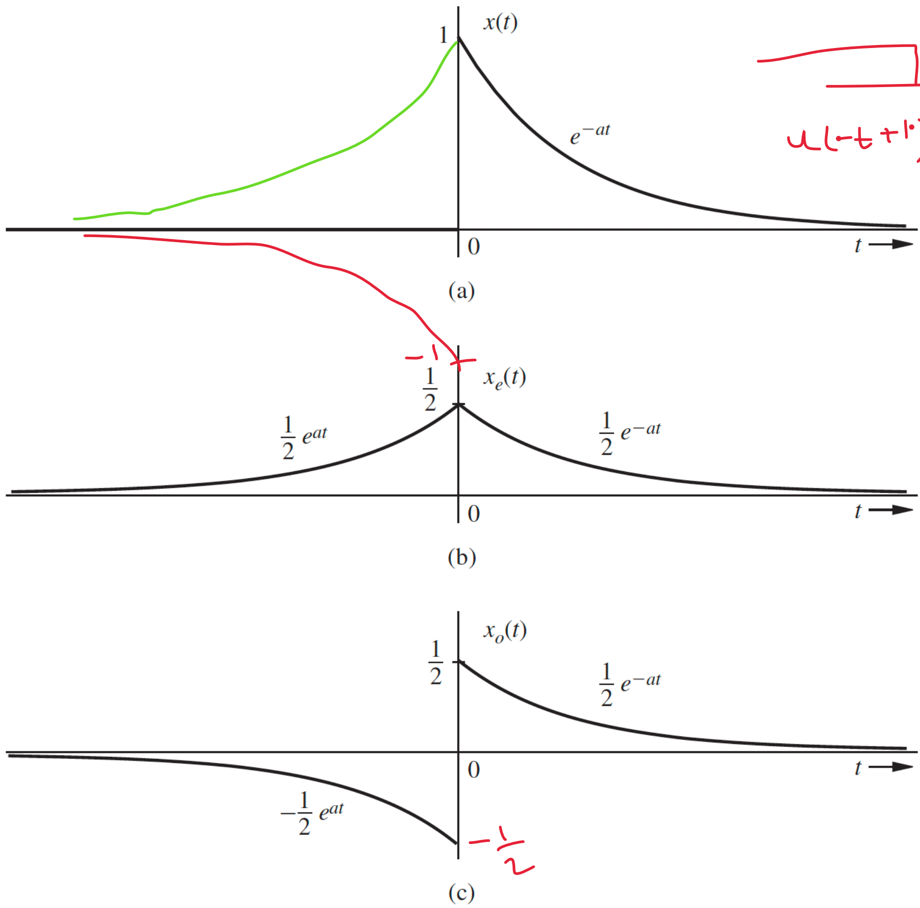


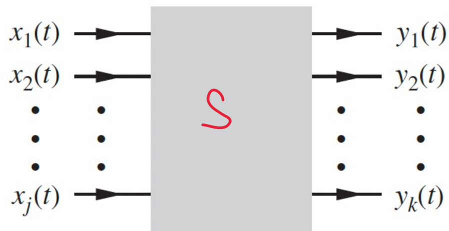
Figure 1.10 Finding even and odd components of a signal.

SEE EXAMPLE 1.9 (PAGE 94 OF THE TEXTBOOK)

1.6 Systems

Systems are used to process signals, allowing to modify or extract information from the signals. A system may consist of physical components or of an algorithm that computes the output signal from the input signal.

A system can be conveniently illustrated by a "black box" with one set of accessible terminals where the input variables $x_1(t), x_2(t), \ldots, x_j(t)$ are applied and another set of accessible terminals where the output variables $y_1(t),\ y_2(t), \ldots, y_k(t)$ are observed.



The study of systems consists of three major areas: mathematical modeling (the focus of this course), analysis, and design.

1.7 Classification of Systems

Systems may be classified broadly in the following categories:

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Other classifications, such as deterministic and probabilistic systems, are beyond the scope of this course.

1.7-1 Linear and Nonlinear Systems

The Concept of Linearity

For a linear system, if an input x_1 acting alone has an effect y_1, and if another input x_2, also acting alone, i.e.,

$$x_1 \longrightarrow y_1 \qquad \text{and} \qquad x_2 \longrightarrow y_2$$

then *additivity* requires that

$$\boxed{x_1 + x_2 \longrightarrow y_1 + y_2}$$

In addition, a linear system must satisfy the *homogeneity* or scaling property, which states that for arbitrary real or imaginary number k

$$kx \longrightarrow ky$$

Both homogeneity (scaling) and additivity properties can be combined into one property (*superposition*), which implies that if

$$x_1 \longrightarrow y_1 \qquad \text{and} \qquad x_2 \longrightarrow y_2$$

then for all inputs x_1 and x_2 and all constants k_1 and k_2,

$$k_1x_1 + k_2x_2 \longrightarrow k_1y_1 + k_2y_2$$

Response of a Linear System

A system's output for $t \geq 0$ is the result of two independent causes:

- the initial conditions of the system (or the system state) at $t = 0$, also known as the *zero-input response* (ZIR).
- the input $x(t)$ for $t \geq 0$, also called the *zero-state response* (ZSR). When all the appropriate initial conditions are zero, the system is said to be in *zero state*. The system output is zero when the input is zero only if the system is in zero state.

$x(t) = 0$
$y(0) = y_0$

$y(0) = 0$

In summary, the total response (TR) of a linear system is given as

$x(t) = u(t-1) - u(t-3) + 2u(t-4)$

$$\text{TR} = \text{ZIR} + \text{ZSR}$$

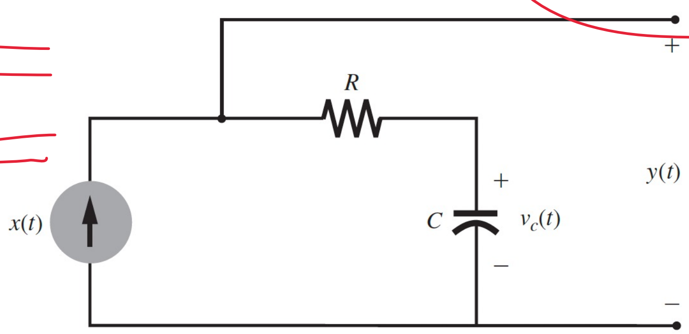
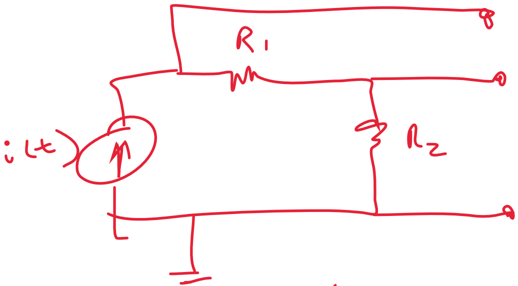


Figure 1.11 RC circuit.

$$y(t) = \underbrace{v_C(0)}_{\text{ZIR}} + \underbrace{Rx(t) + \frac{1}{C}\int_0^t x(\tau)\,d\tau}_{\text{ZSR}}$$

$V(t) = V_{ZIR} + V_{ZSR}$



$v_1 = i R_1$

$v_2 = i R_2$

$v_1(t) = i(t) R_1$

$v_2 \qquad R_2$

iR

$q \propto V$

$q = CV$

$i(t) = \frac{dq}{dt} = C\frac{dv}{dt} \longrightarrow$

$$\boxed{v_c(t) = \frac{1}{C}\int_0^t x(t)\,dt}$$

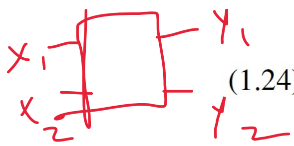
y(t) = i(t) * {R1 + R2} ← ZSR

ZIR = 0

EXAMPLE: Linearity of Constant-Coefficient Linear Differential Equations

Show that the system described by the equation

$$\boxed{\frac{dy(t)}{dt} + 3y(t) = x(t)} \tag{1.24}$$



is linear.

Let the system response to the inputs $x_1(t)$ and $x_2(t)$ be $y_1(t)$ and $y_2(t)$, respectively. Then

$$k_1 \quad \boxed{\frac{dy_1(t)}{dt} + 3y_1(t) = x_1(t)} \qquad \text{and} \qquad \boxed{\frac{dy_2(t)}{dt} + 3y_2(t) = x_2(t)} \quad k_2$$

Multiplying the first equation by k_1, the second by k_2, and adding them yield

$$\boxed{\frac{d}{dt}[k_1y_1(t) + k_2y_2(t)] + 3[k_1y_1(t) + k_2y_2(t)] = k_1x_1(t) + k_2x_2(t)}$$

But this equation is the system equation [Eq. (1.24)] with

$$x(t) = k_1x_1(t) + k_2x_2(t) \qquad \text{and} \qquad y(t) = k_1y_1(t) + k_2y_2(t)$$

Therefore, when the input is $k_1x_1(t) + k_2x_2(t)$, the system response is $k_1y_1(t) + k_2y_2(t)$. Consequently, the system is linear. Using this argument, we can readily generalize the result to

$$a_0\frac{d^Ny(t)}{dt^N} + a_1\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_Ny(t) = b_{N-M}\frac{d^Mx(t)}{dt^M} + \cdots + b_{N-1}\frac{dx(t)}{dt} + b_Nx(t)$$

DRILL: Linearity of a Differential Equation with Time-Varying Parameters

Show that the system described by the following equation is linear:

$$\boxed{\frac{dy(t)}{dt} + t^2 y(t) = (2t+3)x(t)}$$

$x_{1,2} \to \boxed{} \to y_{1,2}$

(x, y)

$x_1 \to y_1$
$x_2 \to y_2$

$$k_1 \left[\frac{dy_1}{dt} + t^2 y_1 = (2t+3)x_1 \right] +$$

$$k_2 \left[\frac{dy_2}{dt} + t^2 y_2 = (2t+3)x_2 \right]$$

$$\frac{d}{dt}\left[k_1 y_1 + k_2 y_2\right] + t^2\left[k_1 y_1 + k_2 y_2\right] = (2t+3)(k_1 x_1 + k_2 x_2)$$

$\Rightarrow$ Linear

DRILL 1.13 A Nonlinear Differential Equation

Show that the system described by the following equation is nonlinear:

$$y(t)\frac{dy(t)}{dt} + 3y(t) = x(t)$$

y * y'+3y = x

k_1 (y1 * y1' + 3y1 = x1) +

k_2 (y2 * y2' + 3y2 = x2)

k1 * y1 * y1' + k2 * y2 * y2' + 3 * [k1y1 + k2y2] = k1x1+k2x2

$$\underbrace{(k_1 y_1 + k_2 y_2)}_{\times}\ \underbrace{\frac{d}{dt}\left[k_1 y_1 + k_2 y_2\right]}\ + 3(\quad) = (\quad)$$

✓ ✓

1.7-2 Time-Invariant and Time-Varying Systems

y(t)

Systems whose parameters do not change with time are *time-invariant* (or *constant-parameter*) systems. For such a system, if the input is delayed by T seconds, the output is also delayed by T (assuming initial conditions are also delayed by T).

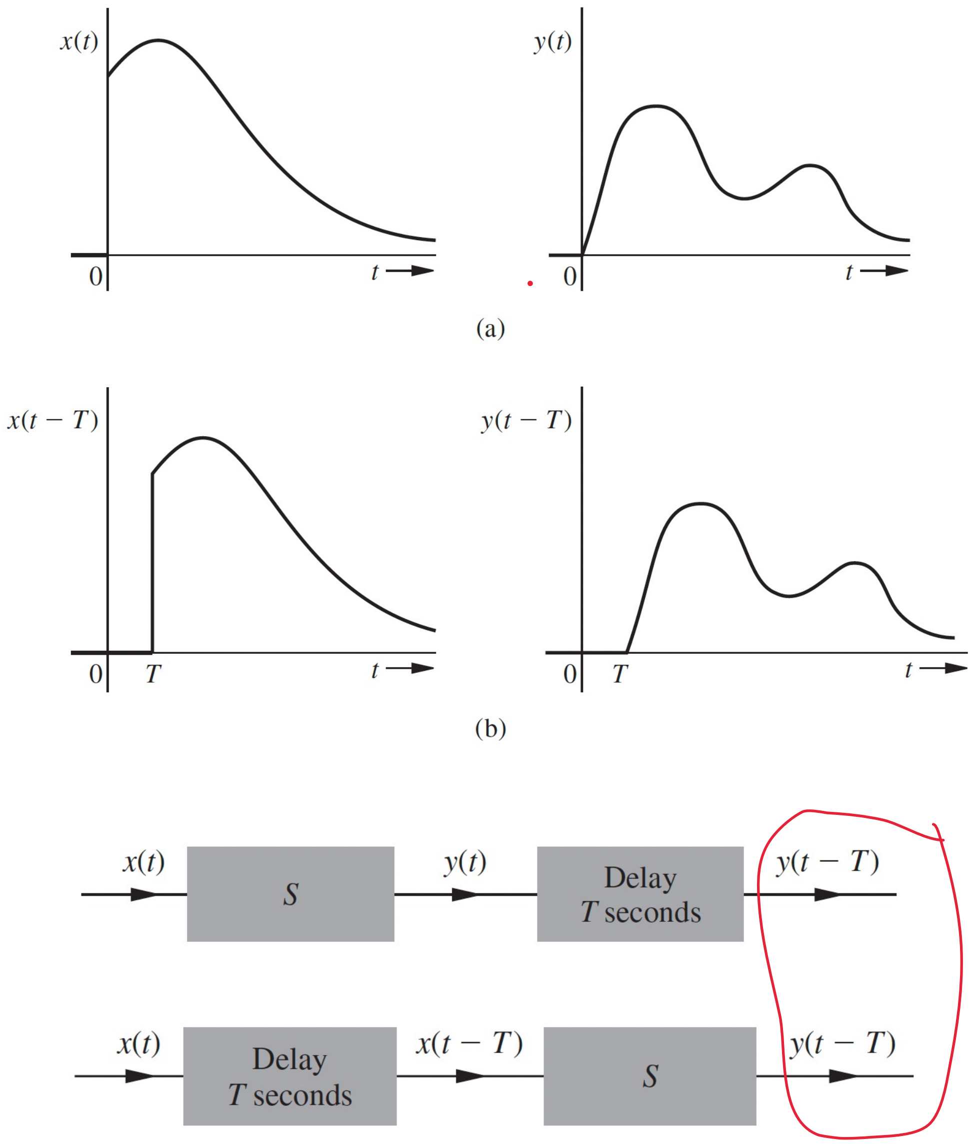
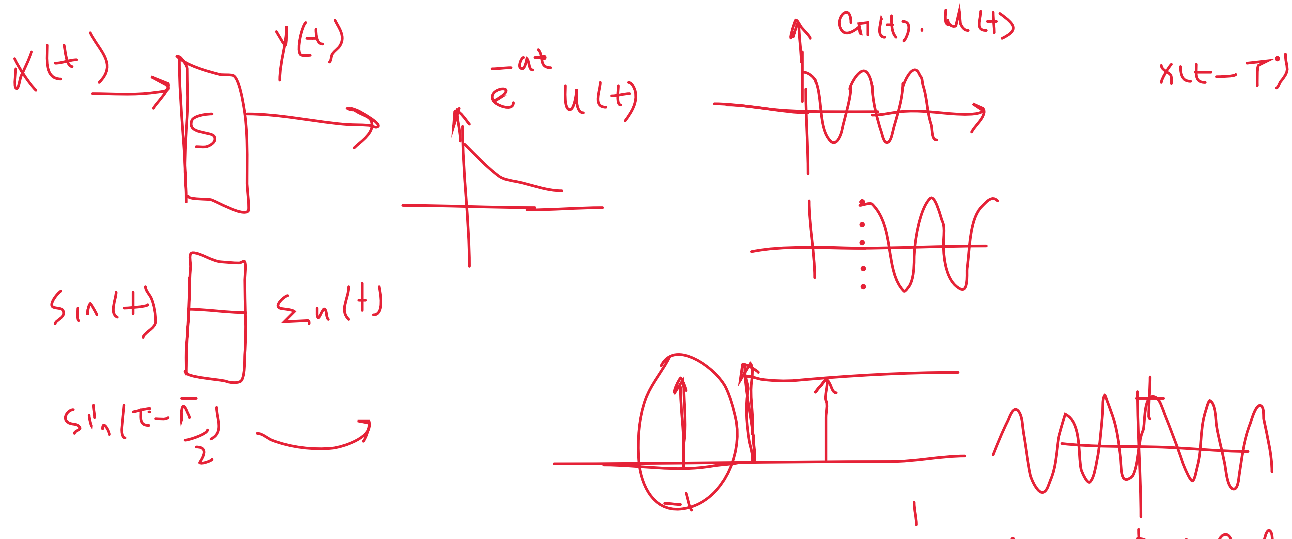
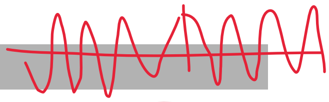


Figure 1.12 Illustration of time- invariance property.



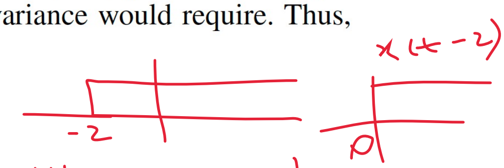


Assessing System Time Invariance

Determine the time invariance of the following systems: **(a)** $y(t) = x(t)u(t)$ and **(b)** $y(t) = \frac{d}{dt}x(t)$.

(a) In this case, the output equals the input for $t \geq 0$ and is otherwise zero. Clearly, the input is being modified by a time-dependent function, so the system is likely time variant. We can prove that the system is not time invariant through a counterexample. Letting $x_1(t) = \delta(t+1)$, we see that $y_1(t) = 0$. However, $x_2(t) = x_1(t-2) = \delta(t-1)$ produces an output of $y_2(t) = \delta(t-1)$, which does equal $y_1(t-2) = 0$ as time-invariance would require. Thus, $y(t) = x(t)u(t)$ is a time variant system.

not

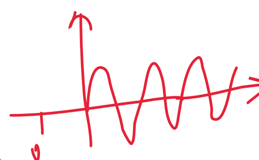


$x(t) = u(t+2)$

A Time-Variant System

Show that a system described by the following equation is a time-varying-parameter system:

$$y(t) = (\sin t)x(t-2)$$



$y_1 = \sin(t)\,\delta(t+1-2) = \sin(t)\,\delta(t-1)$

$= \sin(1)$

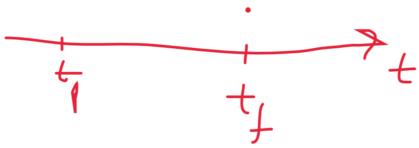
$y_2 = \sin(t)\cdot\delta(t-1-2) = \sin(t)\cdot\delta(t-3)$

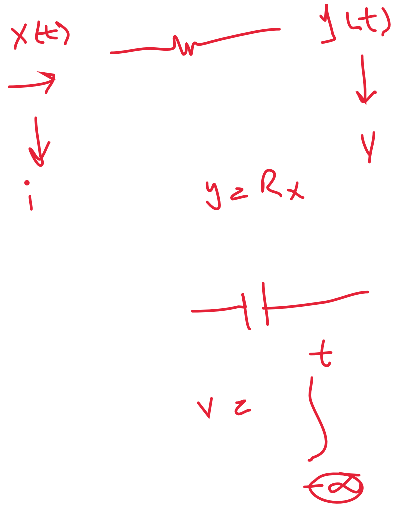
$= \sin(3)$

1.7-3 Instantaneous and Dynamic Systems

A system's output at any instant t generally depends on the entire past input. However, in a special class of systems, the output at any instant t depends only on its input at that instant. Such systems are said to be *instantaneous* or *memoryless* systems. For example, any output of a resistive network at some instant t depends only on the input at the instant t. Otherwise, the system is said to be *dynamic* (or a system with memory). Networks containing inductive and capacitive elements generally have infinite memory because the response of such networks at any instant t is determined by their inputs over the entire past $(-\infty, t)$.

RC Circuit





1.7-4 Causal and Noncausal Systems

A *causal* (also known as a *physical* or *nonanticipative*) system is one for which the output at any instant t_0 depends only on the value of the input $x(t)$ for $t \leq t_0$. In other words, the value of the output at the present instant depends only on the past and present values of the input $x(t)$, not on its future values.

Simply, in a causal system the output cannot start before the input is applied. If the response starts before the input, it means that the system knows the input in the future and acts on this knowledge before the input is applied.

Any practical system that operates in real time must necessarily be causal. We do not yet know how to build a system that can respond to future inputs. A noncausal system is a prophetic system that knows the future input and acts on it in the present. Thus, if we apply an input starting at t = 0 to a noncausal system, the output would begin even before t = 0.

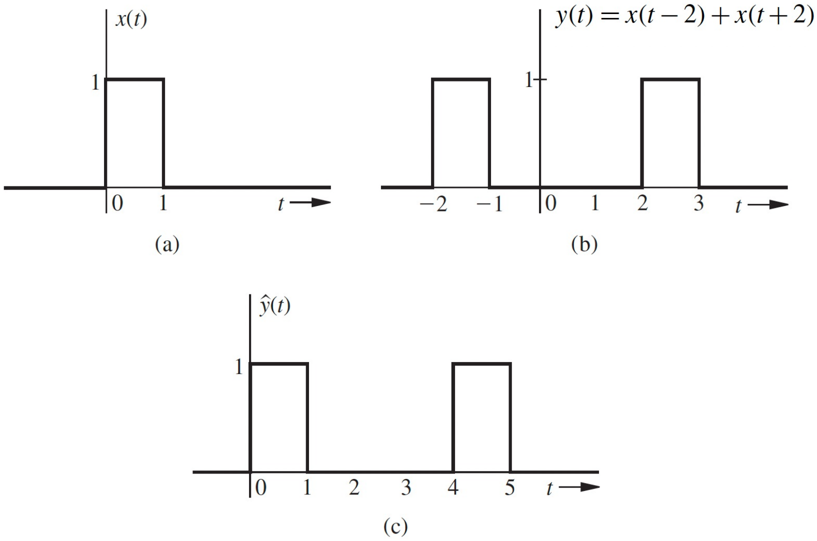


Figure 1.13 Input–output of a noncausal system and the causal output achieved by delay.

EXAMPLE 1.13 Assessing System Causality

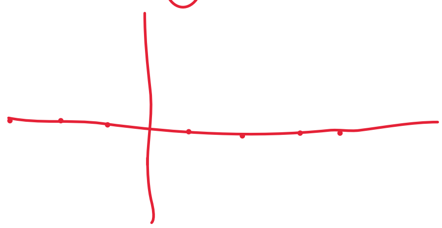
Determine whether the following systems are causal: **(a)** $y(t) = x(-t)$, **(b)** $y(t) = x(t + 1)$, and **(c)** $y(t + 1) = x(t)$.

NC

NC

C

(a)



DRILL 1.15 A Noncausal System

Show that a system described by the following equation is noncausal:

$$y(t) = \int_{t-5}^{t+5} x(\tau)\, d\tau$$

$x(t+5) - x(t-5)$

$-5 \qquad -5$

Show that this system can be realized physically if we accept a delay of 5 seconds in the output.

$$y(t) = \int_{t-10}^{t} x(\tau)\, d\tau$$

1.7-5 Continuous-Time and Discrete-Time Systems

- Signals defined or specified over a continuous range of time are *continuous-time signals,* denoted by symbols $x(t)$, $y(t)$, and so on.
- Systems whose inputs and outputs are continuous-time signals are *continuous-time systems*.
- Signals defined only at discrete instants of time t_0, t_1, t_2, . . . , t_n, . . . are *discrete-time signals,* denoted by the symbols $x(tn)$, $y(tn)$, and so on, n being integers.
- Discrete-time signals can arise from sampling continuous-time signals at uniformly spaced instances in time.
- Systems whose inputs and outputs are discrete-time signals are *discrete-time systems*.

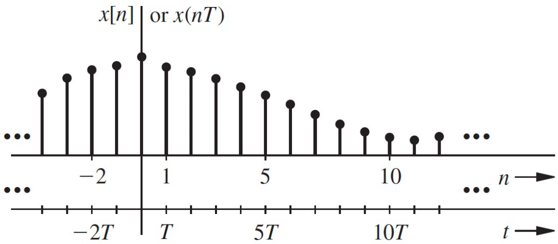


Figure 1.31 A discrete-time signal.

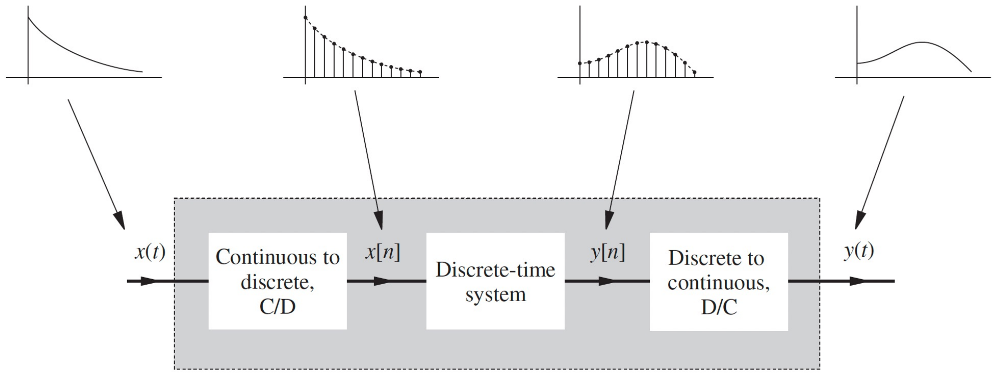


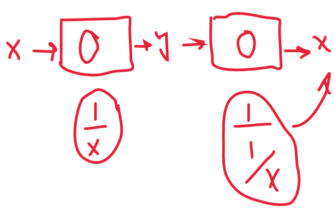
Figure 1.14 Processing continuous-time signals by discrete-time systems.

1.7-6 Analog and Digital Systems

- A system whose input and output signals are analog is an *analog system*
- A system whose input and output signals are digital is a *digital system*. A digital computer is an example of a digital (binary) system. Note that a digital computer is a digital as well as a discrete-time system.

1.7-7 Invertible and Noninvertible Systems

- Assume a system S that performs certain operation(s) on input signal(s). If we can obtain the input $x(t)$ back from the corresponding output $y(t)$ by some operation, the system S is said to be *invertible*.
- For example, when several different inputs result in the same output (as in a rectifier), it is impossible to obtain the input from the output, and the system is *noninvertible*.
- Consider a system S connected in tandem with its inverse S_i. The input $x(t)$ to this tandem system results in signal $y(t)$ at the output of S, and the signal $y(t)$, which now acts as an input to S_i, yields back the signal $x(t)$ at the output of S_i.



EXAMPLE 1.14 Assessing System Invertibility

Determine whether the following systems are invertible: **(a)** $y(t) = x(-t)$, **(b)** $y(t) = tx(t)$, and **(c)** $y(t) = \frac{d}{dt}x(t)$.

$y(t) =$

$1 + \sin(t) \to \cos(t) \to S + c$

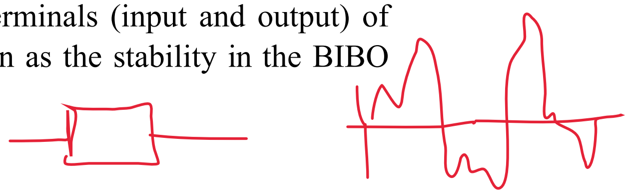
$x(t) \to \boxed{\quad} \to x(-t)$

$x(-t) \to \boxed{\quad} \to x(t)$

$x(t) = \frac{1}{t} y(t) = \frac{1}{t} \times \cancel{t}\, x(t)$

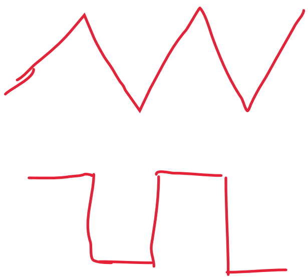
1.7-8 Stable and Unstable Systems

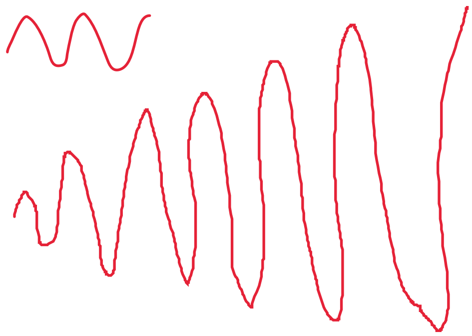
- Stability can be *internal* or *external*.
- If every *bounded input* applied at the input terminal results in a *bounded output*, the system is said to be stable *externally*. External stability can be ascertained by measurements at the external terminals (input and output) of the system. This type of stability is also known as the stability in the BIBO (bounded-input/bounded-output) sense.



EXAMPLE 1.15 Assessing System BIBO Stability

Determine whether the following systems are BIBO-stable: **(a)** $y(t) = x^2(t)$, **(b)** $y(t) = tx(t)$, and **(c)** $y(t) = \frac{d}{dt}x(t)$.





1.8 System Model: Input–Output Description

A system description in terms of the measurements at the input and output terminals is called the *input–output description*. The first step in analyzing any system is the construction of a system model, which is a mathematical expression or a rule that satisfactorily approximates the dynamical behavior of the system.

1.8-1 Electrical Systems

A system model studies the relationship(s) between different variables in the system. In electrical systems, for example, we must determine a satisfactory model for the voltage-current relationship of each element. In addition, we must determine the various constraints on voltages and currents when several electrical elements are interconnected. This can be accomplished using the well-known Kirchhoff laws for voltage and current (KVL and KCL).

EXAMPLE 1.16 Input–Output Equation of a Series *RLC* Circuit

For the series *RLC* circuit shown below, find the input–output equation relating the input voltage $x(t)$ to the output current (loop current) $y(t)$.

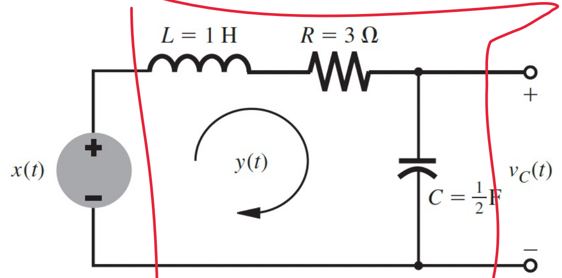


Figure 1.15 RLC circuit

$Q = CV$

$y = \frac{dQ}{dt} = C\frac{dV_c}{dt}$

$\boxed{V_c = \frac{1}{C}\int_{-\infty}^{t} y\,dt}$

$$v_L(t) + v_R(t) + v_C(t) = x(t)$$ ✓

$$L \cdot \frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$

Differentiating both sides of this equation, we obtain

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

It is convenient to use a compact notation D for the differential operator d/dt.

$$\frac{dy(t)}{dt} \equiv Dy(t), \qquad \frac{d^2y(t)}{dt^2} \equiv D^2y(t), \qquad \ldots, \qquad \frac{d^Ny(t)}{dt^N} \equiv D^N y(t)$$

With this notation, the last Eq. on the last page can be expressed as

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

The above equation is input-output equation of an RLC circuit with capacitor voltage as the output.

NOTE: The differential operator is the inverse of the integral operator, so we can use the operator $1/D$ to represent integration.

DRILL 1.17 Input–Output Equation of a Series *RLC* Circuit with Inductor Voltage as Output

If the inductor voltage $v_L(t)$ is taken as the output, show that the *RLC* circuit in Fig. 1.15 has an input–output equation of $(D^2 + 3D + 2)v_L(t) = D^2x(t)$.

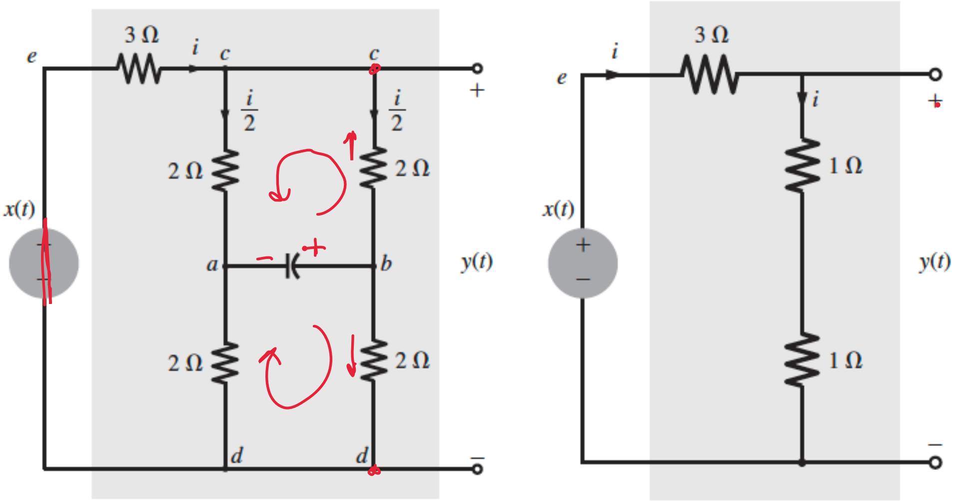
DRILL 1.18 Input–Output Equation of a Series *RLC* Circuit with Capacitor Voltage as Output

If the capacitor voltage $v_C(t)$ is taken as the output, show that the *RLC* circuit in Fig. 1.34 has an input–output equation of $(D^2 + 3D + 2)v_C(t) = 2x(t)$.

NOTE 1: While we have looked at electronic circuits as model systems, the systems could in fact be mechanical or even biological systems.

NOTE 2: The input–output relationship of a system is an *external description* of that system. The external description may not provide the complete picture of the system. In contrast, the *internal description* provides the complete information. An external description can always be found from an internal description, but the converse is not necessarily true.

NOTE 3: The internal description (also known as state-space description) is defined by what is known as state equations of the system.



ZSR

(a) (b)

Figure 1. 16 A system that cannot be described by external measurements.

Let the circuit in Fig. 1.16(a) be enclosed inside a "black box" with only the input and the output terminals accessible. To determine its external description, let us apply a known voltage $x(t)$ at the input terminals and measure the resulting output voltage $y(t)$.

<u>Scenario 1</u>: Non-zero initial charge on capacitor Q_0; and $x(t) = 0$
The currents in the two 2Ω resistors in the upper and the lower branches at the output terminals are equal and opposite because of the balanced nature of the circuit. Clearly, the "non-zero" capacitor charge results in zero voltage at the output.

<u>Scenario 2</u>: Zero initial charge on capacitor $Q_0 = 0$; and $x(t) \neq 0$
Refer to Fig. 1.16 (a)… the current $i(t)$ would divide equally between the two parallel branches as the circuit is balanced. Thus, the voltage across the capacitor will continue to remain zero. Therefore, for the purpose of computing the current $i(t)$, the capacitor may be removed or replaced by a short [see Fig. 1.16 (b)], which shows that the input $x(t)$ sees a load of 5Ω. Thus,

$$\boxed{i(t) = \frac{1}{5}x(t)}$$

Also, because $y(t) = 2\,i(t)$,

$$\boxed{y(t) = \frac{2}{5}x(t)}$$

ZSR

- For the external description, the capacitor does not exist
- No external measurement / observation can detect the presence of the capacitor

1.9 INTERNAL DESCRIPTION: THE STATE-SPACE DESCRIPTION

The *state-space* description of a linear system is an internal description of the system. The key variables here are called the *state variables,* which have the property that every possible signal in the system can be expressed as a linear combination of these state variables. To illustrate this point, consider the network shown below.

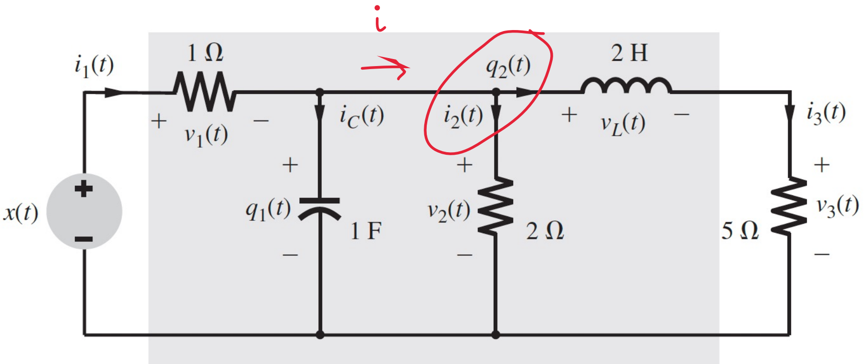


Figure 1. 17 Choosing suitable initial conditions in a network.

We identify two state variables: the capacitor voltage q_1 and the inductor current q_2. If the values of q_1, q_2, and the input $x(t)$ are known at some instant t, we can demonstrate that every possible signal (current or voltage) in the circuit can be determined at t. For example, if $q_1 = 10$, $q_2 = 1$, and the input $x = 20$ at some instant, the remaining voltages and currents at that instant will be

$$\begin{aligned}
i_1 &= (x - q_1)/1 = 20 - 10 = 10\,\mathrm{A} \\
v_1 &= x - q_1 = 20 - 10 = 10\,\mathrm{V} \\
v_2 &= q_1 = 10\,\mathrm{V} \\
i_2 &= q_1/2 = 5\,\mathrm{A} \\
i_C &= i_1 - i_2 - q_2 = 10 - 5 - 1 = 4\,\mathrm{A} \\
i_3 &= q_2 = 1\,\mathrm{A} \\
v_3 &= 5q_2 = 5\,\mathrm{V} \\
v_L &= q_1 - v_3 = 10 - 5 = 5\,\mathrm{V}
\end{aligned}$$

Thus, all signals in this circuit can be determined. Note that the *state-variable description is an internal description* of a system since it is capable of describing all possible signals in the system.

EXAMPLE 1.20 State-Space Description of a System

Consider again the network in Fig. 1.17 with q_1 and q_2 as the state variables and write the state equations. Since $\dot{q}_1$ is the current through the capacitor,

$C\dot{q}_1$

$$\begin{aligned}\dot{q}_1 &= i_C = i_1 - i_2 - q_2 \\ &= (x - q_1) - 0.5\,q_1 - q_2 \\ &= -1.5\,q_1 - q_2 + x\end{aligned}$$

$q_1 = i_2 R$

Also $2\dot{q}_2$, the voltage across the inductor, is given by

$$\begin{aligned}2\dot{q}_2 &= q_1 - v_3 \\ &= q_1 - 5q_2\end{aligned}$$

or

$$\dot{q}_2 = 0.5\,q_1 - 2.5\,q_2$$

Thus, the state equations are

$$\boxed{\begin{aligned}\dot{q}_1 &= -1.5\,q_1 - q_2 + x \\ \dot{q}_2 &= 0.5\,q_1 - 2.5\,q_2\end{aligned}}$$

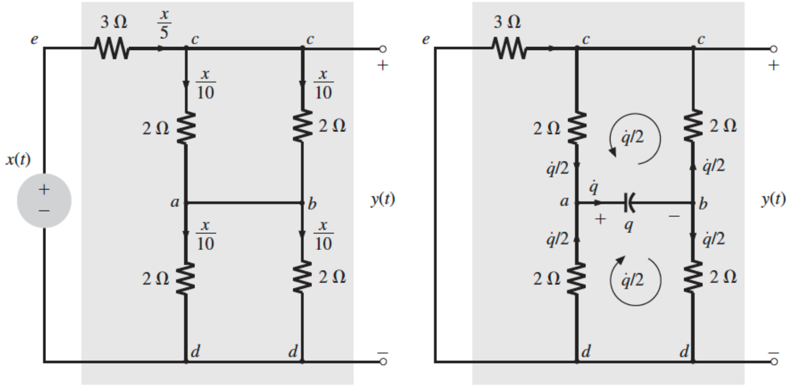
$\dot{q}_1 = f(x, -, \dot{-})$

$q_2($ $)$

This is a set of two simultaneous first-order differential equations, which comprise the *state equations*. Once these equations have been solved for q_1 and q_2, everything else in the circuit can be determined by the *output equations*. Thus, in this approach, we have two sets of equations, the state equations and the output equations.

State equations also help you figure out whether the system is observable and/or controllable.

EXAMPLE 1.21 Controllability and Observability



(a) (b)

This circuit has only one capacitor and no inductors. Hence, there is only one state variable, the capacitor voltage $q(t)$.

Since $C =$ 1 F, the capacitor current is $\dot{q}$.

Two sources in this circuit: the input $x(t)$ and the capacitor voltage $q(t)$. The response due to $x(t)$, assuming $q(t) = 0$, is the zero-state response, which can be found from Fig. 1.44 (a), where the capacitor has been shorted.

The response due to $q(t)$ assuming $x(t) =$ 0, is the zero-input response. This can be found from Fig. 1.44b, where input $x(t)$ has been shorted.

Figure 1.44a shows zero-state currents in every branch. It is clear that the input $x(t)$ sees an effective resistance of 5Ω, and, hence, the current through $x(t)$ is $x/5$ A, which divides in the two parallel branches, resulting in the current $x/10$ through each branch.

Examining the circuit in Fig. 1.44b for the zero-input response, we note that the capacitor voltage is q and the current is $\dot{q}$. We also observe that the capacitor sees two loops in parallel, each with resistance 4 Ω and current $\dot{q}/2$. Interestingly, the 3 Ω branch is effectively shorted because the circuit is balanced, and thus the voltage across the terminals cd is zero.

Branch	Current	Voltage
ca	$\frac{x}{10}+\frac{\dot{q}}{2}$	$2\left(\frac{x}{10}+\frac{\dot{q}}{2}\right)$
cb	$\frac{x}{10}-\frac{\dot{q}}{2}$	$2\left(\frac{x}{10}-\frac{\dot{q}}{2}\right)$
ad	$\frac{x}{10}-\frac{\dot{q}}{2}$	$2\left(\frac{x}{10}-\frac{\dot{q}}{2}\right)$
bd	$\frac{x}{10}+\frac{\dot{q}}{2}$	$2\left(\frac{x}{10}+\frac{\dot{q}}{2}\right)$
ec	$\frac{x}{5}$	$3\left(\frac{x}{5}\right)$
ed	$\frac{x}{5}$	x

1.11 MATLAB: WORKING WITH FUNCTIONS

Working with functions is fundamental to signals and systems applications. MATLAB provides several methods of defining and evaluating functions. An understanding and proficient use of these methods are therefore necessary and beneficial.

1.11-1 Anonymous Functions

Many simple functions are most conveniently represented by using MATLAB anonymous functions. An anonymous function provides a symbolic representation of a function defined in terms of MATLAB operators, functions, or other anonymous functions. For example, consider defining the exponentially damped sinusoid $f(t) = e^{-t}\cos(2\pi t)$.

```
>> f = @(t) exp(-t).*cos(2*pi*t);
```

In this context, the @symbol identifies the expression as an anonymous function, which is assigned a name of f. Parentheses following the @symbol are used to identify the function's independent variables (input arguments), which in this case is the single time variable t. Input arguments, such as t, are local to the anonymous function and are not related to any workspace variables with the same names.

Once defined, $f(t)$ can be evaluated simply by passing the input values of interest. For example,

```
>> t = 0; f(t)
   ans = 1
```

evaluates $f(t)$ at $t = 0$, confirming the expected result of unity. The same result is obtained by passing $t = 0$ directly.

```
>> f(0)
   ans = 1
```

Vector inputs allow the evaluation of multiple values simultaneously. Consider the task of plotting $f(t)$ over the interval $(-2 \leq t \leq 2)$. Gross function behavior is clear: $f(t)$ should oscillate four times with a decaying envelope. Since accurate hand sketches are cumbersome, MATLAB-generated plots are an attractive alternative. As the following example illustrates, care must be taken to ensure reliable results.

Suppose vector t is chosen to include only the integers contained in $(-2 \leq t \leq 2)$, namely, $[-2, -1, 0, 1, 2]$.

```
>> t = (-2:2);
```

This vector input is evaluated to form a vector output.

```
>> f(t)
   ans = 7.3891    2.7183    1.0000    0.3679    0.1353
```

The `plot` command graphs the result, which is shown in Fig. 1.46.

```
>> plot(t,f(t));
>> xlabel('t'); ylabel('f(t)'); grid;
```

Grid lines, added by using the `grid` command, aid feature identification. Unfortunately, the plot does not illustrate the expected oscillatory behavior. More points are required to adequately represent *f(t)*.

The question, then, is how many points is enough?† If too few points are chosen, information is lost. If too many points are chosen, memory and time are wasted. A balance is needed. For oscillatory functions, plotting 20 to 200 points per oscillation is normally adequate. For the present case, t is chosen to give 100 points per oscillation.

```
>> t = (-2:0.01:2);
```

Again, the function is evaluated and plotted.

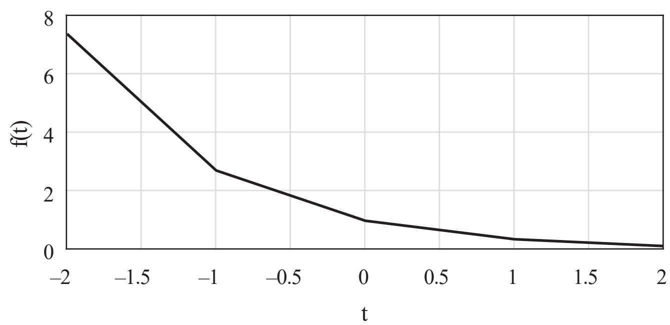


Figure 1.46 $f(t) = e^{-t}\cos(2\pi t)$ for `t = (-2:2)`.

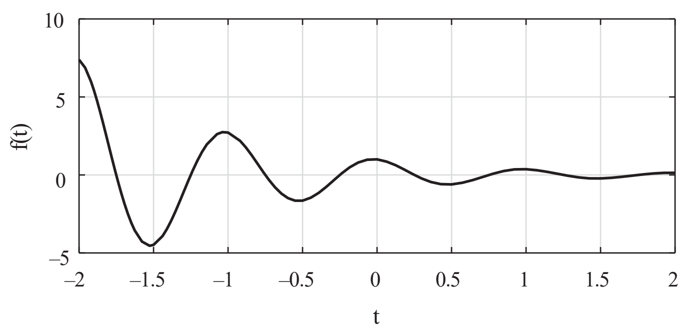


Figure 1.47 $f(t) = e^{-t}\cos(2\pi t)$ for `t = (-2:0.01:2)`.

† Sampling theory, presented later, formally addresses important aspects of this question.

```
>> plot(t,f(t));
>> xlabel('t'); ylabel('f(t)'); grid;
```

The result, shown in Fig. 1.47, is an accurate depiction of *f(t)*.

1.11-2 Relational Operators and the Unit Step Function

The unit step function *u(t)* arises naturally in many practical situations. For example, a unit step can model the act of turning on a system. With the help of relational operators, anonymous functions can represent the unit step function.

In MATLAB, a relational operator compares two items. If the comparison is true, a logical true (1) is returned. If the comparison is false, a logical false (0) is returned. Sometimes called indicator functions, relational operators indicates whether a condition is true. Six relational operators are available: <, >, <=, >=, ==, and ~=.

The unit step function is readily defined using the >= relational operator.

```
>> u = @(t) 1.0.*(t>=0);
```

Any function with a jump discontinuity, such as the unit step, is difficult to plot. Consider plotting *u(t)* by using `t = (-2:2)`.

```
>> t = (-2:2); plot(t,u(t));
>> xlabel('t'); ylabel('u(t)');
```

Two significant problems are apparent in the resulting plot, shown in Fig. 1.48. First, MATLAB automatically scales plot axes to tightly bound the data. In this case, this normally desirable feature obscures most of the plot. Second, MATLAB connects plot data with lines, making a true jump discontinuity difficult to achieve. The coarse resolution of vector `t` emphasizes the effect by showing an erroneous sloping line between $t = -1$ and $t = 0$.

The first problem is corrected by vertically enlarging the bounding box with the `axis` command. The second problem is reduced, but not eliminated, by adding points to vector `t`.

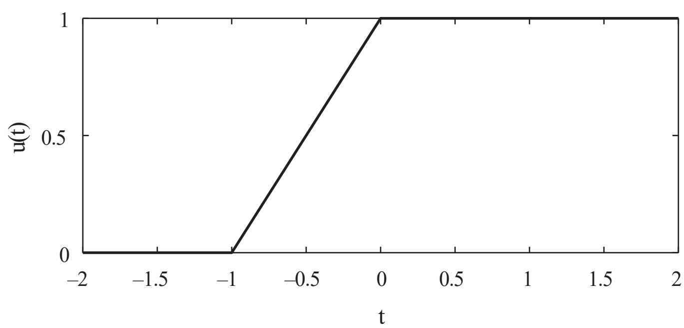


Figure 1.48 *u(t)* for `t = (-2:2)`.

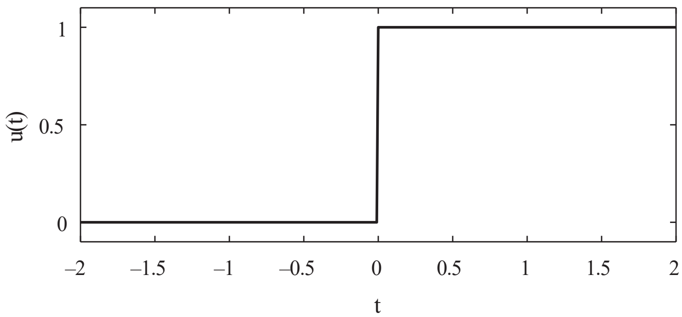


Figure 1.49 *u(t)* for `t = (-2:0.01:2)` with axis modification.

```
>> t = (-2:0.01:2);   plot(t,u(t));
>> xlabel('t'); ylabel('u(t)');
>> axis([-2 2 -0.1 1.1]);
```

The four-element vector argument of `axis` specifies x axis minimum, x axis maximum, y axis minimum, and y axis maximum, respectively. The improved results are shown in Fig. 1.49.

Relational operators can be combined using logical AND, logical OR, and logical negation: &, `|`, and ~, respectively. For example, `(t>0)&(t<1)` and `~((t<=0)|(t>=1))` both test if $0 < t < 1$. To demonstrate, consider defining and plotting the unit pulse $p(t) = u(t) - u(t - 1)$, as shown in Fig. 1.50:

```
>> p = @(t) 1.0.*((t>=0)&(t<1));
>> t = (-1:0.01:2); plot(t,p(t));
>> xlabel('t'); ylabel('p(t) = u(t)-u(t-1)');
>> axis([-1 2 -.1 1.1]);
```

Since anonymous functions can be constructed using other anonymous functions, we could have used our previously defined unit step anonymous function to define *p(t)* as `p = @(t) u(t)-u(t-1);`.

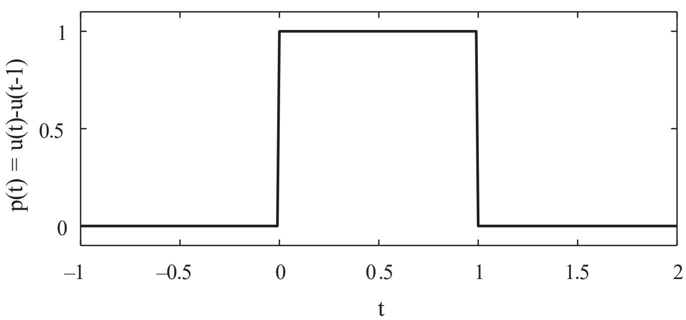


Figure 1.50 $p(t) = u(t) - u(t - 1)$ over $(-1 \leq t \leq 2)$.

For scalar operands, MATLAB also supports two short-circuit logical constructs. A short-circuit logical AND is performed by using &&, and a short-circuit logical OR is performed by using ||. Short-circuit logical operators are often more efficient than traditional logical operators because they test the second portion of the expression only when necessary. That is, when scalar expression A is found false in (A&&B), scalar expression B is not evaluated, since a false result is already guaranteed. Similarly, scalar expression B is not evaluated when scalar expression A is found true in (A||B), since a true result is already guaranteed.

1.11-3 Visualizing Operations on the Independent Variable

Two operations on a function's independent variable are commonly encountered: shifting and scaling. Anonymous functions are well suited to investigate both operations.

Consider $g(t) = f(t)u(t) = e^{-t}\cos(2\pi t)u(t)$, a causal version of $f(t)$. MATLAB easily multiplies anonymous functions. Thus, we create $g(t)$ by multiplying our anonymous functions for $f(t)$ and $u(t)$.[†]

```
>> g = @(t) f(t).*u(t);
```

A combined shifting and scaling operation is represented by $g(at + b)$, where a and b are arbitrary real constants. As an example, consider plotting $g(2t + 1)$ over $(-2 \le t \le 2)$. With $a = 2$, the function is compressed by a factor of 2, resulting in twice the oscillations per unit t. Adding the condition $b > 0$ shifts the waveform to the left. Given anonymous function g, an accurate plot is nearly trivial to obtain.

```
>> t = (-2:0.01:2);
>> plot(t,g(2*t+1)); xlabel('t'); ylabel('g(2t+1)'); grid;
```

Figure 1.51 confirms the expected waveform compression and left shift. As a final check, realize that function $g(\cdot)$ turns on when the input argument is zero. Therefore, $g(2t + 1)$ should turn on when $2t + 1 = 0$ or at $t = -0.5$, a fact again confirmed by Fig. 1.51.

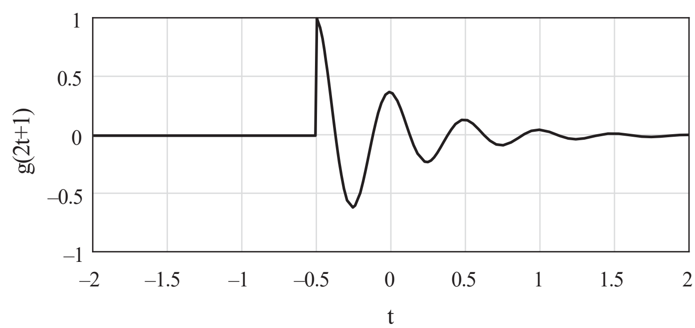


Figure 1.51 $g(2t + 1)$ over $(-2 \le t \le 2)$.

[†] Although we define g in terms of f and u, the function g will not change if we later change either f or u unless we subsequently redefine g as well.

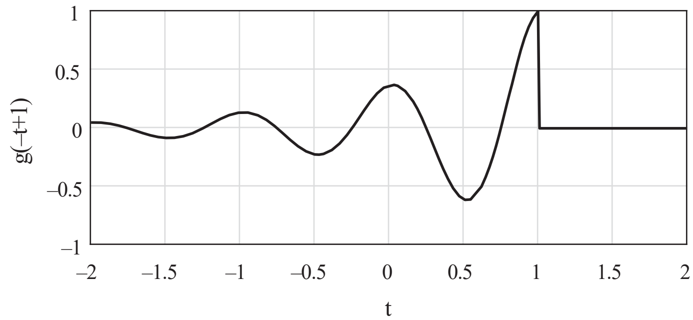


Figure 1.52 $g(-t+1)$ over $(-2 \leq t \leq 2)$.

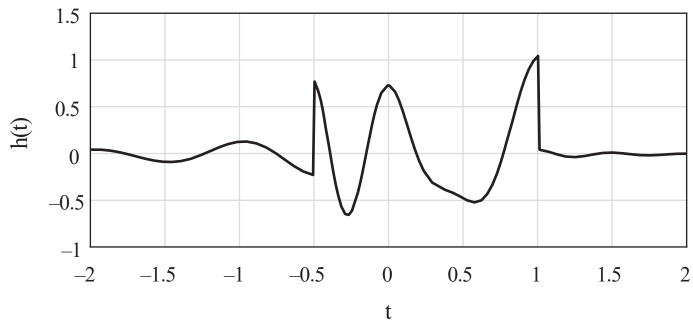


Figure 1.53 $h(t) = g(2t+1) + g(-t+1)$ over $(-2 \leq t \leq 2)$.

Next, consider plotting $g(-t+1)$ over $(-2 \leq t \leq 2)$. Since $a < 0$, the waveform will be reflected. Adding the condition $b > 0$ shifts the final waveform to the right.

```
>> plot(t,g(-t+1)); xlabel('t'); ylabel('g(-t+1)'); grid;
```

Figure 1.52 confirms both the reflection and the right shift.

Up to this point, Figs. 1.51 and 1.52 could be reasonably sketched by hand. Consider plotting the more complicated function $h(t) = g(2t+1) + g(-t+1)$ over $(-2 \leq t \leq 2)$ (Fig. 1.53); an accurate hand sketch would be quite difficult. With MATLAB, the work is much less burdensome.

```
>> plot(t,g(2*t+1)+g(-t+1)); xlabel('t'); ylabel('h(t)'); grid;
```

1.11-4 Numerical Integration and Estimating Signal Energy

Interesting signals often have nontrivial mathematical representations. Computing signal energy, which involves integrating the square of these expressions, can be a daunting task. Fortunately, many difficult integrals can be accurately estimated by means of numerical integration techniques.

Even if the integration appears simple, numerical integration provides a good way to verify analytical results.

To start, consider the simple signal $x(t) = e^{-t}(u(t) - u(t-1))$. The energy of $x(t)$ is expressed as $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^1 e^{-2t}\, dt$. Integrating yields $E_x = 0.5(1 - e^{-2}) \approx 0.4323$. The energy integral can also be evaluated numerically. Figure 1.27 helps illustrate the simple method of rectangular approximation: evaluate the integrand at points uniformly separated by *b..t*, multiply each by *b..t* to compute rectangle areas, and then sum over all rectangles. First, we create function $x(t)$.

```
>>  x = @(t) exp(-t).*((t>=0)&(t<1));
```

With *b..t* = 0.01, a suitable time vector is created.

```
>>  t = (0:0.01:1);
```

The final result is computed by using the sum command.

```
>>  E_x = sum(x(t).*x(t)*0.01)
    E_x = 0.4367
```

The result is not perfect, but at 1% relative error it is close. By reducing *b..t*, the approximation is improved. For example, *b..t* = 0.001 yields E_x = 0.4328, or 0.1% relative error.

Although simple to visualize, rectangular approximation is not the best numerical integration technique. The MATLAB function quad implements a better numerical integration technique called recursive adaptive Simpson quadrature.† To operate, quad requires a function describing the integrand, the lower limit of integration, and the upper limit of integration. Notice that no *b..t* needs to be specified.

To use quad to estimate E_x, the integrand must first be described.

```
>>  x_squared = @(t) x(t).*x(t);
```

Estimating E_x immediately follows.

```
>>  E_x = quad(x_squared,0,1)
    E_x = 0.4323
```

In this case, the relative error is −0.0026%.

The same techniques can be used to estimate the energy of more complex signals. Consider $g(t)$, defined previously. Energy is expressed as $E_g = \int_0^{\infty} e^{-2t} \cos^2(2\pi t)\, dt$. A closed-form solution exists, but it takes some effort. MATLAB provides an answer more quickly.

```
>>  g_squared = @(t) g(t).*g(t);
```

---

† A comprehensive treatment of numerical integration is outside the scope of this text. Details of this particular method are not important for the current discussion; it is sufficient to say that it is better than the rectangular approximation.

Although the upper limit of integration is infinity, the exponentially decaying envelope ensures $g(t)$ is effectively zero well before $t = 100$. Thus, an upper limit of $t = 100$ is used along with $\Delta t = 0.001$.

```
>>  t = (0:0.001:100);
>>  E_g = sum(g_squared(t)*0.001)
    E_g = 0.2567
```

A slightly better approximation is obtained with the quad function.

```
>>  E_g = quad(g_squared,0,100)
    E_g = 0.2562
```

DRILL 1.21 Computing Signal Energy with MATLAB

Use MATLAB to confirm that the energy of signal $h(t)$, defined previously as $h(t) = g(2t + 1) + g(-t + 1)$, is $E_h = 0.3768$.

1.12 SUMMARY

A *signal* is a set of data or information. A *system* processes input signals to modify them or extract additional information from them to produce output signals (response). A system may be made up of physical components (hardware realization), or it may be an algorithm that computes an output signal from an input signal (software realization).

A convenient measure of the size of a signal is its energy, if it is finite. If the signal energy is infinite, the appropriate measure is its power, if it exists. The signal power is the time average of its energy (averaged over the entire time interval from $-\infty$ to ∞). For periodic signals, the time averaging need be performed over only one period in view of the periodic repetition of the signal. Signal power is also equal to the mean squared value of the signal (averaged over the entire time interval from $t = -\infty$ to ∞).

Signals can be classified in several ways.

1. A *continuous-time signal* is specified for a continuum of values of the independent variable (such as time t). A *discrete-time signal* is specified only at a finite or a countable set of time instants.
2. An *analog signal* is a signal whose amplitude can take on any value over a continuum. On the other hand, a signal whose amplitudes can take on only a finite number of values is a *digital signal*. The terms *discrete-time* and *continuous-time* qualify the nature of a signal along the time axis (horizontal axis). The terms *analog* and *digital,* on the other hand, qualify the nature of the signal amplitude (vertical axis).
3. A *periodic signal* $x(t)$ is defined by the fact that $x(t) = x(t + T_0)$ for some T_0. The smallest positive value of T_0 for which this relationship is satisfied is called the *fundamental period*. A periodic signal remains unchanged when shifted by an integer multiple of its period. A periodic signal $x(t)$ can be generated by a periodic extension of any contiguous segment of $x(t)$ of duration T_0. Finally, a periodic signal, by definition, must exist over the entire time interval $-\infty < t < \infty$. A signal is *aperiodic* if it is not periodic.

4. An *everlasting signal* starts at $t = -\infty$ and continues forever to $t = \infty$. Hence, periodic signals are everlasting signals. A *causal signal* is a signal that is zero for $t < 0$.
5. A signal with finite energy is an *energy signal*. Similarly a signal with a finite and nonzero power (mean-square value) is a *power signal*. A signal can be either an energy signal or a power signal, but not both. However, there are signals that are neither energy nor power signals.
6. A signal whose physical description is known completely in a mathematical or graphical form is a *deterministic signal*. A *random signal* is known only in terms of its probabilistic description such as mean value or mean-square value, rather than by its mathematical or graphical form.

A signal $x(t)$ delayed by T seconds (right-shifted) can be expressed as $x(t - T)$; on the other hand, $x(t)$ advanced by T (left-shifted) is $x(t + T)$. A signal $x(t)$ time-compressed by a factor $a(a > 1)$ is expressed as $x(at)$; on the other hand, the same signal time-expanded by factor $a(a > 1)$ is $x(t/a)$. The signal $x(t)$ when time-reversed can be expressed as $x(-t)$.

The unit step function $u(t)$ is very useful in representing causal signals and signals with different mathematical descriptions over different intervals.

In the classical (Dirac) definition, the unit impulse function $\delta(t)$ is characterized by unit area and is concentrated at a single instant $t = 0$. The impulse function has a sampling (or sifting) property, which states that the area under the product of a function with a unit impulse is equal to the value of that function at the instant at which the impulse is located (assuming the function to be continuous at the impulse location). In the modern approach, the impulse function is viewed as a generalized function and is defined by the sampling property.

The exponential function e^{st}, where s is complex, encompasses a large class of signals that includes a constant, a monotonic exponential, a sinusoid, and an exponentially varying sinusoid.

A real signal that is symmetrical about the vertical axis ($t = 0$) is an *even* function of time, and a real signal that is antisymmetrical about the vertical axis is an *odd* function of time. The product of an even function and an odd function is an odd function. However, the product of an even function and an even function or an odd function and an odd function is an even function. The area under an odd function from $t = -a$ to a is always zero regardless of the value of a. On the other hand, the area under an even function from $t = -a$ to a is two times the area under the same function from $t = 0$ to a (or from $t = -a$ to 0). Every signal can be expressed as a sum of odd and even functions of time.

A system processes input signals to produce output signals (response). The input is the cause, and the output is its effect. In general, the output is affected by two causes: the internal conditions of the system (such as the initial conditions) and the external input.

Systems can be classified in several ways.

1. Linear systems are characterized by the linearity property, which implies superposition; if several causes (such as various inputs and initial conditions) are acting on a linear system, the total output (response) is the sum of the responses from each cause, assuming that all the remaining causes are absent. A system is nonlinear if superposition does not hold.
2. In time-invariant systems, system parameters do not change with time. The parameters of time-varying-parameter systems change with time.
3. For memoryless (or instantaneous) systems, the system response at any instant t depends only on the value of the input at t. For systems with memory (also known as dynamic

systems), the system response at any instant t depends not only on the present value of the input, but also on the past values of the input (values before t).

4. In contrast, if a system response at t also depends on the future values of the input (values of input beyond t), the system is noncausal. In causal systems, the response does not depend on the future values of the input. Because of the dependence of the response on the future values of input, the effect (response) of noncausal systems occurs before the cause. When the independent variable is time (temporal systems), the noncausal systems are prophetic systems, and therefore, unrealizable, although close approximation is possible with some time delay in the response. Noncausal systems with independent variables other than time (e.g., space) are realizable.
5. Systems whose inputs and outputs are continuous-time signals are continuous-time systems; systems whose inputs and outputs are discrete-time signals are discrete-time systems. If a continuous-time signal is sampled, the resulting signal is a discrete-time signal. We can process a continuous-time signal by processing the samples of the signal with a discrete-time system.
6. Systems whose inputs and outputs are analog signals are analog systems; those whose inputs and outputs are digital signals are digital systems.
7. If we can obtain the input $x(t)$ back from the output $y(t)$ of a system S by some operation, the system S is said to be invertible. Otherwise the system is noninvertible.
8. A system is stable if bounded input produces bounded output. This defines external stability because it can be ascertained from measurements at the external terminals of the system. External stability is also known as the stability in the BIBO (bounded-input/bounded-output) sense. Internal stability, discussed later in Ch. 2, is measured in terms of the internal behavior of the system.

The system model derived from a knowledge of the internal structure of the system is its internal description. In contrast, an external description is a representation of a system as seen from its input and output terminals; it can be obtained by applying a known input and measuring the resulting output. In the majority of practical systems, an external description of a system so obtained is equivalent to its internal description. At times, however, the external description fails to describe the system adequately. Such is the case with the so-called uncontrollable or unobservable systems.

A system may also be described in terms of certain set of key variables called state variables. In this description, an Nth-order system can be characterized by a set of N simultaneous first-order differential equations in N state variables. State equations of a system represent an internal description of that system.

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