SIGNALS AND SYSTEMS

This chapter discusses the basic aspects of signals and systems. The material is presented to build a solid foundation for understanding the quantitative analysis in the following chapters. For simplicity, the chapter focuses on continuous-time signals and systems. Chapter 3 presents the same ideas for discrete-time signals and systems.

SIGNALS

A *signal* is a set of data or information. Examples include a telephone or a television signal, monthly sales of a corporation, or daily closing prices of a stock market (e.g., the Dow Jones averages). In all these examples, the signals are functions of the independent variable *time*. While this is not always the case, this course deals almost exclusively with signals that are functions of time.

SYSTEMS

Signals may be processed further by *systems,* which may modify them or extract additional information from them. Thus, a system is an entity that *processes* a set of signals (*inputs*) to yield another set of signals (*outputs*). A system may be made up of physical components as in an electrical system (hardware realization) or it may be an algorithm that computes an output from an input signal (software realization).

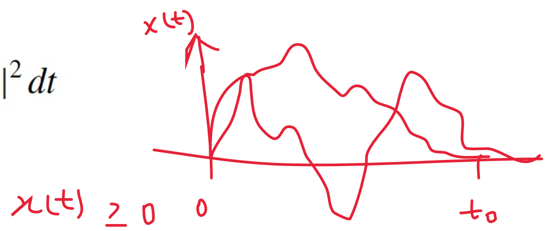
1.1 SIZE OF A SIGNAL

The size of any entity is a number that indicates the strength of that entity. Generally speaking, the signal amplitude varies with time. How can a signal that exists over a certain time interval with varying amplitude be measured by one number that will indicate the signal size or signal strength?

1.1-1 Signal Energy

Using area under a signal $x(t)$ as a possible measure of its size may be problematic since even for a large signal $x(t)$, its positive and negative areas could cancel each other, indicating a signal of small size. This difficulty can be corrected by defining the signal size as the area under $|x(t)|^2$. We call this measure the *signal energy*

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$



For a real-valued signal $x(t)$, the above definition simplifies to:

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt$$

J/cm^2

1.1-2 Signal Power

Note that signal energy must be finite for it to be a meaningful measure of signal size. A necessary condition for the energy to be finite is that the signal amplitude $\rightarrow 0$ as $|t| \rightarrow \infty$. When this condition is not satisfied, the signal energy is infinite. Therefore, a more meaningful measure of the signal size would be the time average of the energy, if it exists. Such a measure is called the *power* of the signal. For a signal $x(t)$, we define its power P_x as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt$$

$\frac{\alpha}{\alpha} \rightarrow F$

For a real-valued signal $x(t)$, the above definition simplifies to

$\lim_{t\to a} \frac{t^2+2t+3}{5t^2-5t}$

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$$

$\frac{1+0+0}{5-0} = \frac{1}{5}$

Generally, the mean of an entity averaged over a large time interval approaching infinity exists if the entity either is periodic or has a statistical regularity. The units of energy and power depend on the nature of the signal $x(t)$. If $x(t)$ is a voltage signal, its energy E_x has units of volts squared-seconds (V^2 s), and its power P_x has units of volts squared. If $x(t)$ is a current signal, these units will be amperes squared-seconds (A^2 s) and amperes squared, respectively.

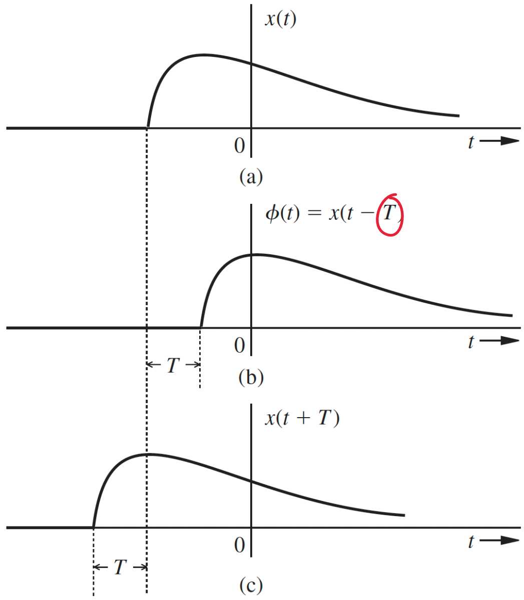
SEE EXAMPLES 1.1 AND 1.2 (PAGE 69 OF THE TEXTBOOK)

1.2 Useful Signal Operations

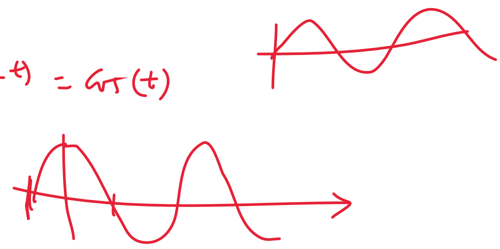
1.2-1 Time Shifting

Consider a signal $x(t)$ and the same signal delayed by T seconds $\varphi(t)$. Whatever happens in $x(t)$ (Fig. 1.1a) at some instant t also happens in $\varphi(t)$ (Fig. 1.1b) T seconds later at the instant $t + T$. Therefore

$$\varphi(t+T) = x(t) \quad \text{and} \quad \varphi(t) = x(t-T)$$



$x(t) = \cos(t)$



$\cos(t - \pi/2)$

$x(t - \pi/2) = \varphi(t)$

$\boxed{+\sin(t)}$

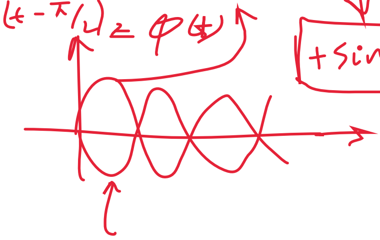
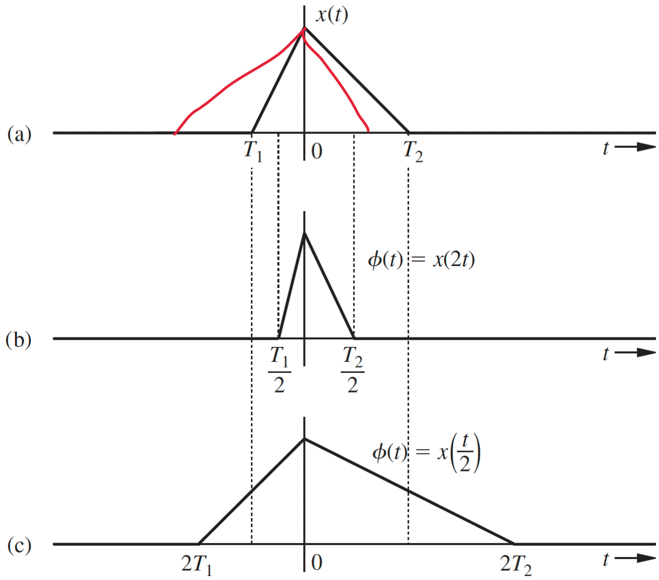


Figure 1.1 Time-shifting a signal.

SEE EXAMPLES 1.3 (PAGE 72-73 OF THE TEXTBOOK)

1.2-2 Time Scaling

The compression or expansion of a signal in time is known as *time scaling*. Consider the signal $x(t)$ of Fig. 1.2a. The signal *φ(t)* in Fig. 1.2b is $x(t)$ compressed in time by a factor of 2. Therefore, whatever happens in $x(t)$ at some instant t also happens to $\varphi(t)$ at the instant $t/2$ so that



$x(t)$

$\alpha\, x(t)$

$x(\alpha t)$

$\sin(t)$

$\sin(2t)$

$-t_0$ t_0

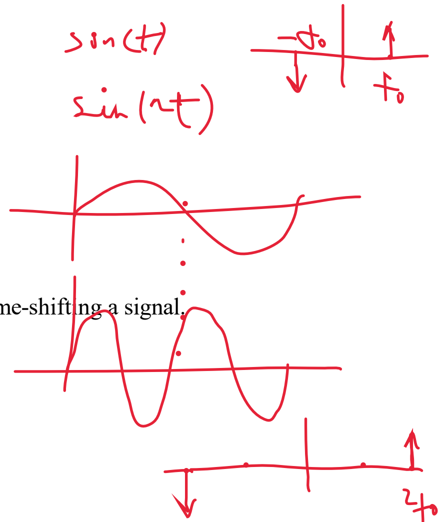


Figure 1.2 Time-shifting a signal.

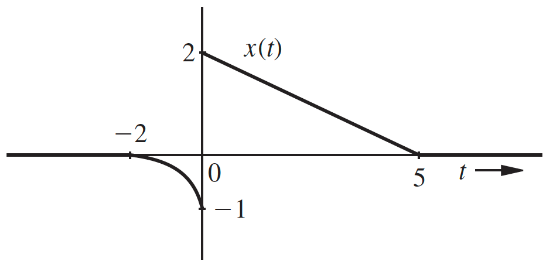
SEE EXAMPLES 1.4 (PAGE 74-75 OF THE TEXTBOOK)

$2t_0$

1.2-3 Time Reversal

Consider the signal $x(t)$ in Fig. 1.3a. To time-reverse $x(t)$, we flip across the vertical axis. This time reversal [a horizontal flip] gives us the signal $\varphi(t)$ (Fig. 1.3b), such that whatever happens in Fig. 1.3a at some instant t also happens in Fig. 1.3b at the instant $-t$, and vice versa. Therefore,

$$\varphi(t) = x(-t)$$



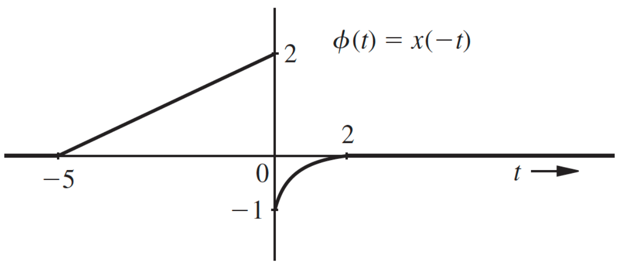


Figure 1.3 Time reversal of a signal.

SEE EXAMPLES 1.5 (PAGE 77 OF THE TEXTBOOK)

1.2-4 Combined Operations

Complex operations including more than one of the above operations can be performed. The most general operation involving all the three operations is $x(at - b)$, which is realized in two possible sequences of operation:

1. Time-shift $x(t)$ by b to obtain $x(t - b)$. Next, time-scale the shifted signal $x(t - b)$ by a [i.e., replace t with at] to obtain $x(at - b)$.
2. Time-scale $x(t)$ by a to obtain $x(at)$. Now time-shift $x(at)$ by b/a [i.e., replace t with $t - (b/a)$] to obtain $x[a(t - b/a)] = x(at - b)$. If a is negative, time scaling involves time reversal as well.

1.3 Classification of Signals

There are several classes of signals… some classes are more suitable for certain applications than others. Here we shall consider only the following classes of signals:

1. Analog and digital signals
2. Continuous-time and discrete-time signals
3. Periodic and aperiodic signals
4. Energy and power signals
5. Deterministic and probabilistic signals

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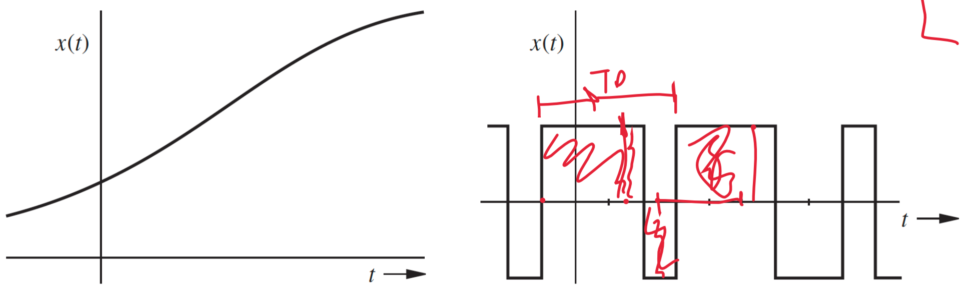
1.3-1 Analog and Digital Signals

1.3-2 Periodic and aperiodic Signals

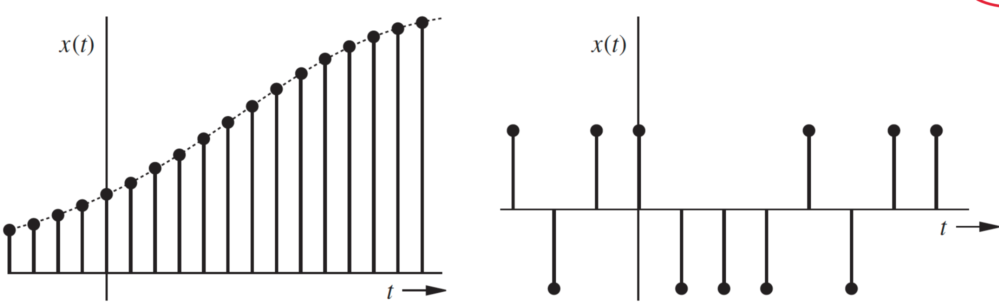


(a) (b)

For periodic signals, $$\int_a^{a+T_0} x(t)\,dt = \int_b^{b+T_0} x(t)\,dt$$

$x(t) = x(t+T_0)$ $\quad x(t) = \begin{cases} 2t & t \geq 0 \\ -1 & t < 0 \end{cases}$

1.3-3 Continuous-Time and Discrete-Time Signals

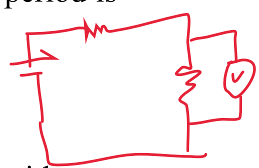


(c) (d)

1.3-4 Energy and Power Signals

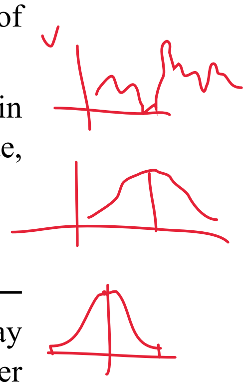
1. A signal with finite energy is an *energy signal*
2. A signal with finite and nonzero power is a *power signal*

Comments: All practical signals have finite energies and are therefore energy signals. A power signal must necessarily have infinite duration; otherwise, its power, which is its energy averaged over an infinitely large interval, will not approach a (nonzero) limit. Clearly, it is impossible to generate a true power signal in practice because such a signal has infinite duration and infinite energy. Also, periodic signals for which the area under $|x(t)|^2$ over one period is finite are power signals; however, not all power signals are periodic.



1.3-5 Deterministic and Random Signals

1. A signal whose physical description is known completely, in either a mathematical form or a graphical form, is a *deterministic signal* (scope of this course)
2. A signal whose values cannot be predicted precisely but are known only in terms of probabilistic description, such as mean value or mean-squared value, is a *random signal*.



$\frac{1}{N}\sum(x-\bar{x})^2$

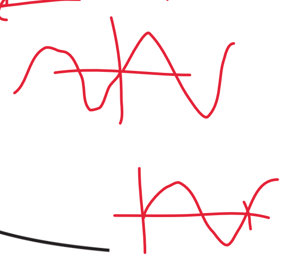
1.4 Some Useful Signal Models

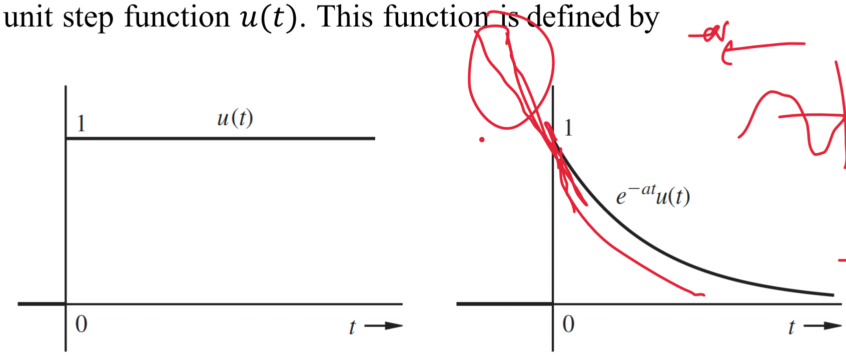
In signals and systems, the step, the impulse, and the exponential functions play very important roles as not only do they serve as a basis for representing other signals, but their use can simplify many aspects of the signals and systems.

1.4-1 The Unit Step Function $u(t)$

Causal signals (the signals begin at $t = 0$) can be conveniently described in terms of unit step function $u(t)$. This function is defined by

$-\infty \leftarrow \quad \rightarrow \infty$

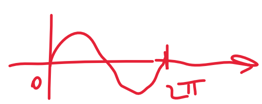




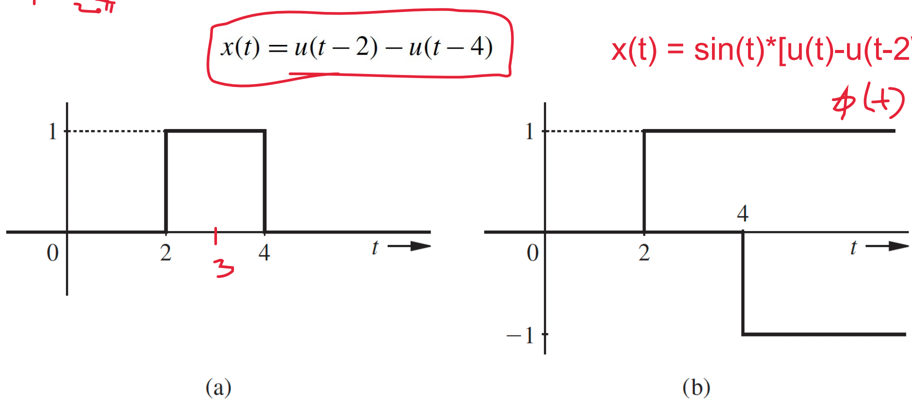
(a) (b)

Figure 1.4. **(a)** Unit step function $u(t)$. **(b)** Exponential $e^{-at}\,u(t)$.

$\sin(t).u(t)$



$u(t)$ $u(t-2\pi)$



x(t) = sin(t)*[u(t)-u(t-2\pi)]

$\phi(t) = \begin{cases} 1 & 0 \le t \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

$\mathrm{rect}(t/T)$

$-T \quad 0 \quad T$

Figure 1.5 Representation of a rectangular pulse by step functions.

SEE EXAMPLES 1.6 AND 1.7 (PAGE 84-85 OF THE TEXTBOOK)

SEE DRILL 1.8 (PAGE 86 OF THE TEXTBOOK)

1.4-2 The Unit Impulse Function $\delta(t)$

The unit impulse function $\delta(t)$, one of the most important functions in the study of signals and systems, was first defined by Dirac as

$$\delta(t) = 0 \quad t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$\frac{0}{0} \quad \frac{\alpha}{\alpha} \quad 0.\alpha$

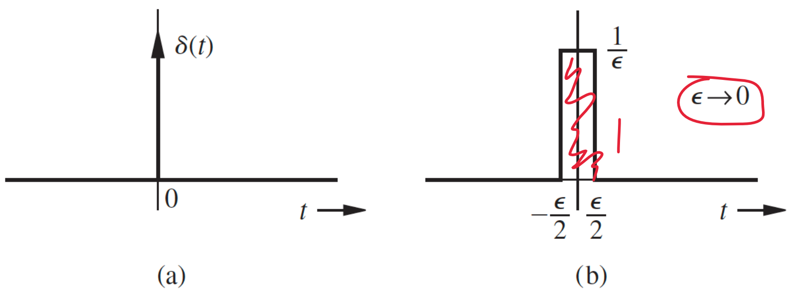


Figure 1.6 A unit impulse and its approximation.

$$\alpha\int_0^\infty e^{-\alpha t}\,dt = \cancel{\alpha}\,\frac{e^{-\alpha t}}{-\cancel{\alpha}}\Big|_0^\infty = -\left[e^{-\infty}-e^{0}\right] = 1$$

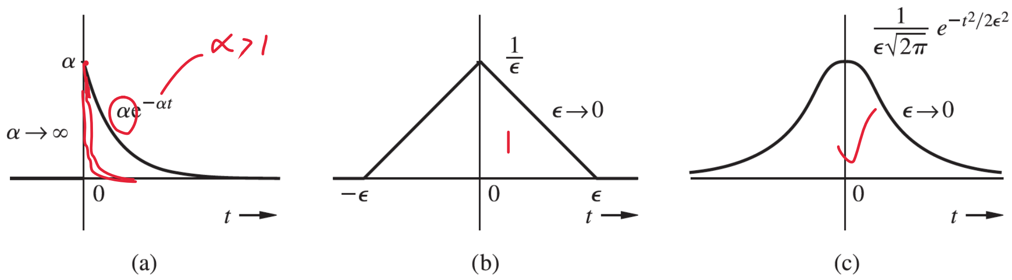


Figure 1.7 Other possible approximations to a unit impulse.

Multiplication of a Function by an Impulse

When we multiply the unit impulse $\delta(t)$ by a function $\varphi(t)$ that is known to be continuous at $t = 0$, we obtain the value of $\varphi(t)$ at $t = 0$

$$\varphi(t)\delta(t) = \varphi(0)\delta(t)$$

Likewise, provided $\varphi(t)$ is continuous at $t = T$, $\varphi(t)$ multiplied by an impulse $\delta(t - T)$ (impulse located at $t = T$) results in an impulse located at $t = T$ and having strength $\varphi(T)$.

$$\varphi(t)\delta(t - T) = \varphi(T)\delta(t - T)$$

Sampling Property of the Unit Impulse Function

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)\boxed{\int_{-\infty}^{\infty}\delta(t)\,dt} = \boxed{\phi(T)}$$

Indicating that the area under the product of a function with an impulse $\delta(t - T)$ is equal to the value of that function at the instant at which the unit impulse is located. This important property is known as the *sampling* or *sifting property* of the unit impulse.

Unit Impulse as a Generalized Function

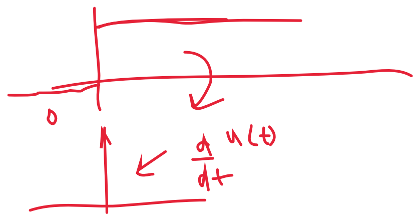
A *generalized function* is defined by its effect on other functions instead of by its value at every instant of time. For example, *the sampling property mentioned previously defines the impulse function in the generalized function approach.*

Next, let's consider the unit step function $u(t)$. Since it is discontinuous at $t = 0$, its derivative du/dt does not exist at $t = 0$ in the ordinary sense. However, its derivative *does* exist in the generalized sense, and it is, in fact, $\delta(t)$. As a proof, we evaluate the integral of $(du/dt)\varphi(t)$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{du(t)}{dt}\phi(t)\,dt &= u(t)\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\dot{\phi}(t)\,dt \\ &= \phi(\infty) - 0 - \int_{0}^{\infty} \dot{\phi}(t)\,dt \\ &= \phi(\infty) - \phi(t)|_0^{\infty} = \phi(0)\end{aligned}$$

This result shows that that du/dt satisfies the sampling property of $\delta(t)$. In other words,

$$\frac{du(t)}{dt} = \delta(t)$$



which implies

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

These results can also be obtained graphically from Fig. 1.6b. Note that the area from $-\infty$ to t under the limiting form of $\delta(t)$ is zero if $t < -\epsilon/2$ and unity if $t \geq \epsilon/2$ with $\epsilon \to 0$. Consequently,

$$\begin{aligned}\int_{-\infty}^{t} \delta(\tau)\,d\tau &= \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \\ &= u(t)\end{aligned}$$

SEE DRILLS 1.9 & 1.10 (PAGE 89 OF THE TEXTBOOK)

1.4-3 The Exponential Function e^{st}

Another important function in the area of signals and systems is the exponential signal e^{st}, where s is complex in general, given by

$$s = \sigma + j\omega$$

Therefore,

$$e^{st} = e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

Since $s^* = \sigma - j\omega$ (the conjugate of s), then

$$e^{s^*t} = e^{(\sigma-j\omega)t} = e^{\sigma t}e^{-j\omega t} = e^{\sigma t}(\cos\omega t - j\sin\omega t)$$

and

$$e^{\sigma t}\cos\omega t = \tfrac{1}{2}(e^{st} + e^{s^*t})$$

The following functions are either special cases of or can be expressed in terms of e^{st}:

1. A constant $k = ke^{0t}$ $(s = 0)$
2. A monotonic exponential $e^{\sigma t}$ $(\omega = 0, s = \sigma)$
3. A sinusoid $\cos\omega t$ $(\sigma = 0, s = \pm j\omega)$
4. An exponentially varying sinusoid $e^{\sigma t}\cos\omega t$ $(s = \sigma \pm j\omega)$

These functions are illustrated in Fig. 1.8.

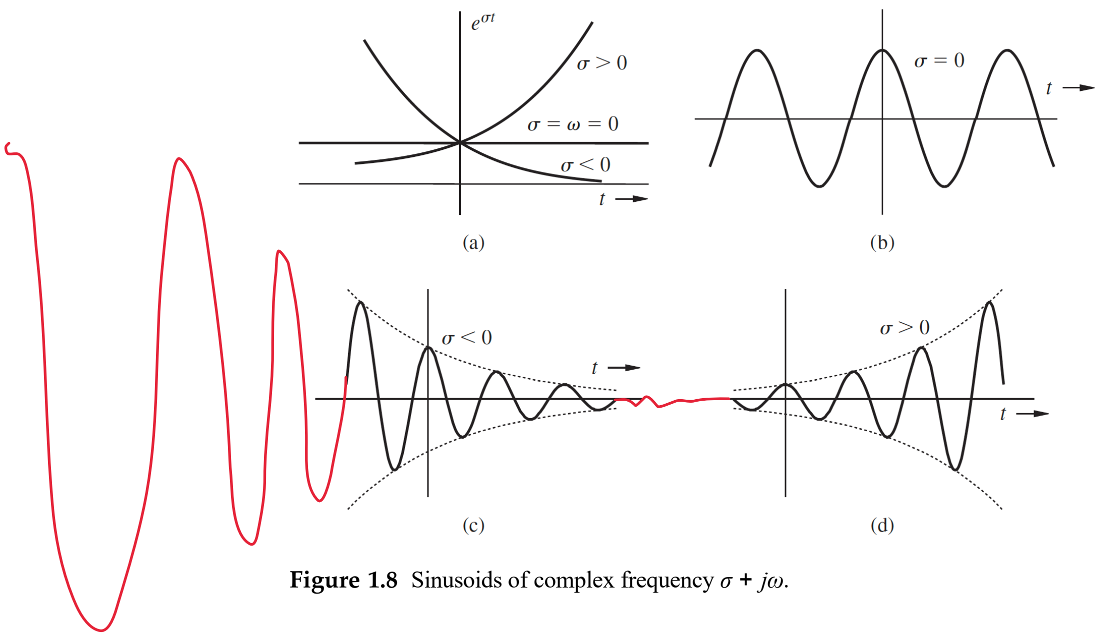


Figure 1.8 Sinusoids of complex frequency $\sigma + j\omega$.

1.5 Even and Odd Functions

A function $x_e(t)$ is said to be an *even function* of t if it is symmetrical about the vertical axis. A function $x_o(t)$ is said to be an *odd function* of t if it is antisymmetrical about the vertical axis. Mathematically expressed, these symmetry conditions require

$$x_e(t) = x_e(-t) \quad \text{and} \quad x_o(t) = -x_o(-t)$$

An even function has the same value at the instants t and $-t$ for all values of t. On the other hand, the value of an odd function at the instant t is the negative of its value at the instant $-t$. An example even signal and an example odd signal are shown in Figs. 1.23a and 1.23b, respectively.

1.5-1 Some Properties of Even and Odd Functions

Even and odd functions have the following properties:

$$\begin{aligned} \text{even function} \times \text{odd function} &= \text{odd function} \\ \text{odd function} \times \text{odd function} &= \text{even function} \\ \text{even function} \times \text{even function} &= \text{even function} \end{aligned}$$

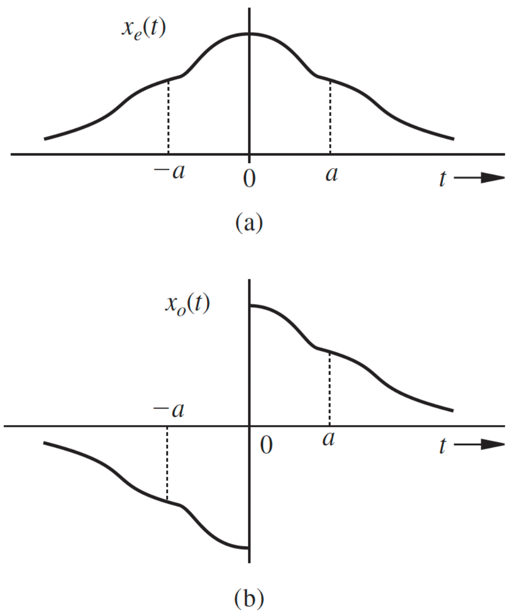


Figure 1.9 **(a)** Even and **(b)** odd functions.

AREA

Because of the symmetries of even and odd functions about the vertical axis, we can infer

$$\int_{-a}^{a} x_e(t)\,dt = 2\int_{0}^{a} x_e(t)\,dt$$

$$\int_{-a}^{a} x_o(t)\,dt = 0$$

1.5-2 Even and Odd Components of a Signal

Every signal $x(t)$ can be expressed as a sum of even and odd components since

$$x(t) = \underbrace{\tfrac{1}{2}[x(t)+x(-t)]}_{\text{even}} + \underbrace{\tfrac{1}{2}[x(t)-x(-t)]}_{\text{odd}}$$

EXAMPLE: Finding the Even and Odd Components of a Signal

Find and sketch the even and odd components of $x(t) = e^{-at}u(t)$.

As described above, we can express $x(t)$ as a sum of the even component $x_e(t)$ and the odd component $x_o(t)$ as

$$x(t) = x_e(t) + x_o(t)$$

where,

$$x_e(t) = \tfrac{1}{2}[e^{-at}u(t) + e^{at}u(-t)]$$

$$x_o(t) = \tfrac{1}{2}[e^{-at}u(t) - e^{at}u(-t)]$$

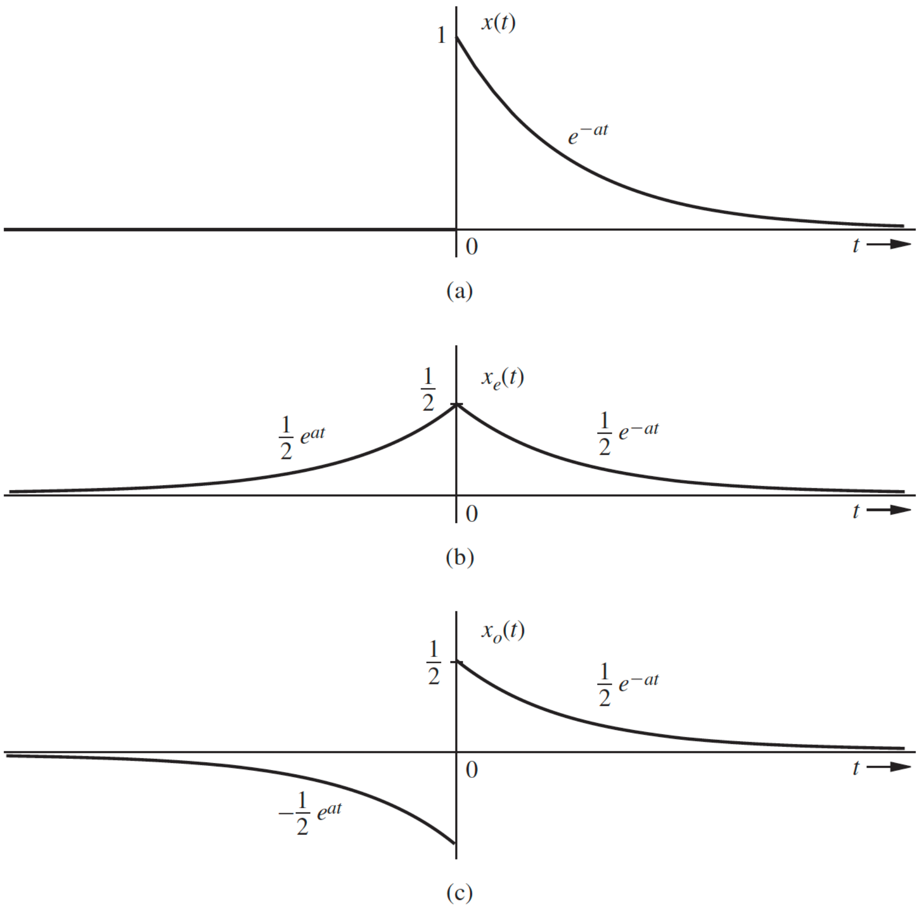


Figure 1.10 Finding even and odd components of a signal.

SEE EXAMPLE 1.9 (PAGE 94 OF THE TEXTBOOK)