CHAPTER B

BACKGROUND

The topics discussed in this chapter are not entirely new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful later, as reference material in your professional career.

B.1 COMPLEX NUMBERS

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly *imaginary numbers,* sometimes seem mysterious and unreal. This feeling of unreality derives from their unfamiliarity and novelty rather than their supposed nonexistence! Mathematicians blundered in calling these numbers “imaginary,” for the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts have been made to ascribe some physical meaning to imaginary numbers. However, this effort is needless. In mathematics we assign symbols and operations any meaning we wish as long as internal consistency is maintained. The history of mathematics is full of entities that were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) needed to express the number of children, cattle, and quivers of arrows. These people had no need for fractions. Whoever heard of two and one-half children or three and one-fourth cows!

However, with the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field and the weight of a quantity of butter. The number system, therefore, was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but *Pythagoras* discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic, who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and imposed a death penalty for divulging this secret [1]. These numbers, however, were included in the number system by the time of Descartes, and they are now known as *irrational numbers*.

Until recently, *negative numbers* were not a part of the number system. The concept of negative numbers must have appeared absurd to early man. However, the medieval Hindus had a clear understanding of the significance of positive and negative numbers [2, 3]. They were also the first to recognize the existence of absolute negative quantities [4]. The works of *Bhaskar* (1114–1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots [5]. Much later, in Europe, the men who developed the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) are credited with introducing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger [6].

Thus, the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician *Leonhard Euler* introduced the notation i (for *imaginary*) around 1777 to represent $\sqrt{-1}$. Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current. Thus,

$$j^2=-1 \qquad \text{and} \qquad \sqrt{-1}=\pm j$$

This notation allows us to determine the square root of any negative number. For example,

$$\sqrt{-4}=\sqrt{4}\times\sqrt{-1}=\pm 2j$$

When imaginary numbers are included in the number system, the resulting numbers are called *complex numbers*.

ORIGINS OF COMPLEX NUMBERS

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2+1=0$ because this equation has no real solution. But in 1545, *Gerolamo Cardano* of Milan published *Ars Magna* (The Great Art), the most important algebraic work of the Renaissance. In this book, he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3+ax+b=0$$

is given by

$$x=\sqrt[3]{-\frac{b}{2}+\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}+\sqrt[3]{-\frac{b}{2}-\sqrt{\frac{b^2}{4}+\frac{a^3}{27}}}$$

For example, to find a solution of $x^3+6x-20=0$, we substitute $a=6, b=-20$ in the foregoing equation to obtain

$$x=\sqrt[3]{10+\sqrt{108}}+\sqrt[3]{10-\sqrt{108}}=\sqrt[3]{20.392}-\sqrt[3]{0.392}=2$$

We can readily verify that 2 is indeed a solution of $x^3+6x-20=0$. But when Cardano tried to solve the equation $x^3-15x-4=0$ by this formula, his solution was

$$x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days, negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today, we know that

$$(2\pm j)^3=2\pm j11=2\pm\sqrt{-121}$$

Therefore, Cardano's formula gives

$$x=(2+j)+(2-j)=4$$

We can readily verify that $x=4$ is indeed a solution of $x^3-15x-4=0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, *Raphael Bombelli* (1526–1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, although we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this proposal seemed incredibly strange [7]. Yet they could not dismiss the idea of imaginary numbers so easily because this concept yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra [7].

† This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3+py^2+qy+r=0$$

can always be reduced to a depressed cubic form by substituting $y=x-(p/3)$. Therefore, any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by *Scipione del Ferro* (1465–1526) and then by *Niccolo Fontana* (1499–1557). The latter is better known in the history of mathematics as *Tartaglia* ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y=x-(p/3)$, a general cubic is reduced to a depressed cubic.

In 1799 the German mathematician *Karl Friedrich Gauss,* at the ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for their general and systematic use. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y, as illustrated in Fig. B.1. If we want to travel from City a to City b (both in

Gerolamo Cardano

Karl Friedrich Gauss

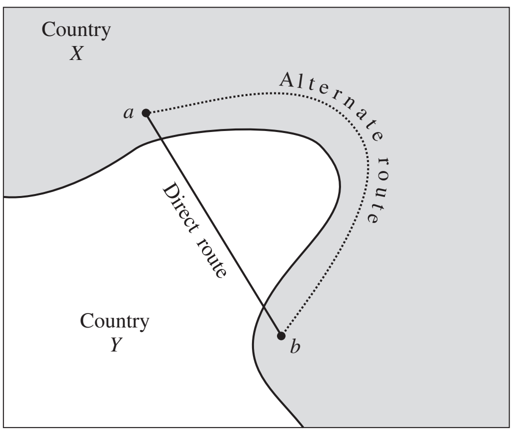

Figure B.1 Use of complex numbers can reduce the work.

Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). Most real-world problems start with real numbers, and the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve any real-world problem by an alternate method, using real numbers exclusively, but such procedures would increase the work needlessly.

B.1-2 Algebra of Complex Numbers

A complex number (a,b) or $a+jb$ can be represented graphically by a point whose Cartesian coordinates are (a,b) in a complex plane (Fig. B.2). Let us denote this complex number by z so that

$$z = a + jb \tag{B.1}$$

This representation is the Cartesian (or rectangular) form of complex number z. The numbers a and b (the abscissa and the ordinate) of z are the *real part* and the *imaginary part*, respectively, of z. They are also expressed as

$$\operatorname{Re} z = a \qquad \text{and} \qquad \operatorname{Im} z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

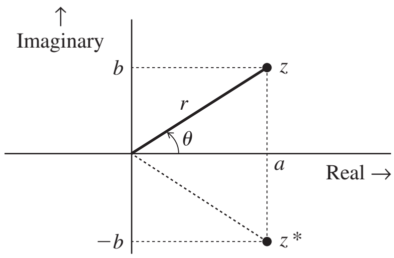

Figure B.2 Representation of a number in the complex plane.

Complex numbers may also be expressed in terms of polar coordinates. If (r,θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

Consequently,

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta) \tag{B.2}$$

Euler's formula states that

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{B.3}$$

To prove Euler's formula, we use a Maclaurin series to expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$:

$$\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} \cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots
\end{aligned}$$

Clearly, it follows that $e^{j\theta} = \cos\theta + j\sin\theta$. Using Eq. (B.3) in Eq. (B.2) yields

$$z = re^{j\theta} \tag{B.4}$$

This representation is the polar form of complex number z.

Summarizing, a complex number can be expressed in rectangular form $a + jb$ or polar form $re^{j\theta}$ with

$$\begin{aligned} a &= r\cos\theta \\ b &= r\sin\theta \end{aligned} \qquad \text{and} \qquad \begin{aligned} r &= \sqrt{a^2+b^2} \\ \theta &= \tan^{-1}\left(\tfrac{b}{a}\right) \end{aligned} \tag{B.5}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the *magnitude* (or *absolute value*) of z and is denoted by $|z|$. Similarly, θ is called the angle of z and is denoted by $\angle z$. Therefore, we can also write polar form of Eq. (B.4) as

$$z = |z|e^{j\angle z} \qquad \text{where } |z| = r \text{ and } \angle z = \theta$$

Using polar form, we see that the reciprocal of a complex number is given by

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z}$$

Conjugate of a Complex Number

We define z^*, the *conjugate* of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} = |z|e^{-j\angle z} \tag{B.6}$$

The graphical representations of a number z and its conjugate z^* are depicted in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only replace j with $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z$$

Thus, we see that the real part of complex number z can be computed as

$$\mathrm{Re}\,z = \frac{z + z^*}{2} \tag{B.7}$$

Similarly, the imaginary part of complex number z can be computed as

$$\operatorname{Im} z = \frac{z - z^*}{2j} \tag{B.8}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = |z|e^{j\angle z}|z|e^{-j\angle z} = |z|^2 \tag{B.9}$$

Understanding Some Useful Identities

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis, as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (more generally, π plus any integer multiple of 2π), as seen from Fig. B.3b. Therefore,

$$-1 = e^{j(\pi+2\pi n)} \qquad n \text{ integer}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 0 (more generally, 0 plus any integer multiple of 2π). Therefore,

$$1 = e^{j2\pi n} \qquad n \text{ integer} \tag{B.10}$$

The number j is at a unit distance from the origin and its angle is $\frac{\pi}{2}$ (more generally, $\frac{\pi}{2}$ plus any integer multiple of 2π), as seen from Fig. B.3b. Therefore,

$$j = e^{j(\frac{\pi}{2}+2\pi n)} \qquad n \text{ integer}$$

Similarly,

$$-j = e^{j(-\frac{\pi}{2}+2\pi n)} \qquad n \text{ integer}$$

Notice that the angle of any complex number is only known within an integer multiple of 2π.

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t}$$

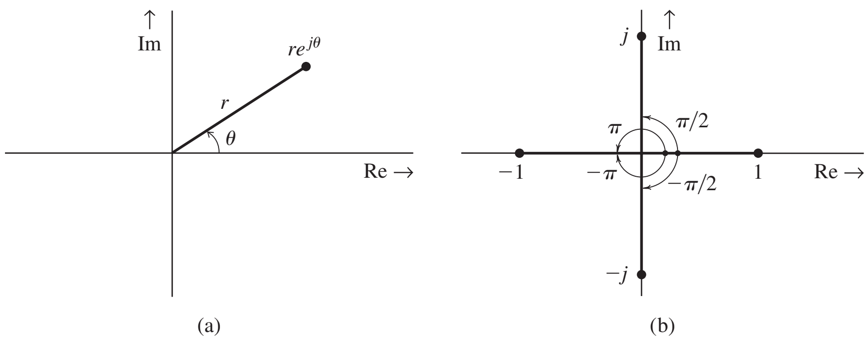

Figure B.3 Understanding some useful identities in terms of $re^{j\theta}$.

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t} e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases}$$

In future discussions, you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning About Computing Angles with Calculators

From the Cartesian form $a + jb$, we can readily compute the polar form $re^{j\theta}$ [see Eq. (B.5)]. Calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator computes an angle of a complex number by using an inverse tangent function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(-3/-2)$. This result is not the same as $\tan^{-1}(3/2)$. The former is $-123.7°$, whereas the latter is $56.3°$. A calculator cannot make this distinction and can give a correct answer only for angles in the first and fourth quadrants.† A calculator will read $\tan^{-1}(-3/-2)$ as $\tan^{-1}(3/2)$, which is clearly wrong. When you are computing inverse trigonometric functions, if the angle appears in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason, it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

EXAMPLE B.1 Cartesian to Polar Form

Express the following numbers in polar form: **(a)** $2+j3$, **(b)** $-2+j1$, **(c)** $-2-j3$, and **(d)** $1-j3$.

(a)

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3°$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore (see Fig. B.4a), we can write

$$2 + j3 = \sqrt{13}\, e^{j56.3°}$$

(b)

$$|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4°$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer given by the calculator, $\tan^{-1}(1/-2) = -26.6°$, is off by 180°. The correct answer is

---

† Calculators with two-argument inverse tangent functions will correctly compute angles.

$(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. It is a common practice to choose an angle whose numerical value is less than 180°. Such a value is called the *principal value* of the angle, which in this case is 153.4°. Therefore,

$$-2 + j1 = \sqrt{5}e^{j153.4^\circ}$$

(c)

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{-3}{-2}\right) = -123.7^\circ$$

In this case the angle appears in the third quadrant (see Fig. B.4c), and therefore the answer obtained by the calculator ($\tan^{-1}(-3/-2) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. We choose the principal value -123.7° so that (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)

$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}\left(\tfrac{-3}{1}\right) = -71.6^\circ$$

In this case the angle appears in the fourth quadrant (see Fig. B.4d), and therefore the answer given by the calculator, $\tan^{-1}(-3/1) = -71.6^\circ$, is correct (see Fig. B.4d):

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ}$$

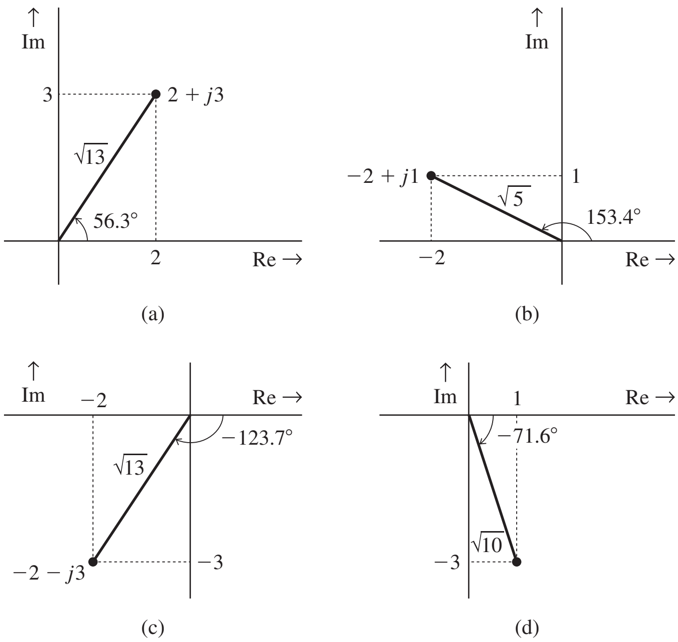

Figure B.4 From Cartesian to polar form.

We can easily verify these results using the MATLAB `abs` and `angle` commands. To obtain units of degrees, we must multiply the radian result of the `angle` command by $\frac{180}{\pi}$. Furthermore, the `angle` command correctly computes angles for all four quadrants of the complex plane. To provide an example, let us use MATLAB to verify that $-2+j1 = \sqrt{5}e^{j153.4^\circ} = 2.2361e^{j153.4^\circ}$.

```
>>  abs(-2+1j)
    ans = 2.2361
>>  angle(-2+1j)*180/pi
    ans = 153.4349
```

One can also use the `cart2pol` command to convert Cartesian to polar coordinates. Readers, particularly those who are unfamiliar with MATLAB, will benefit by reading the overview in Sec. B.7.

EXAMPLE B.2 Polar to Cartesian Form

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{j\pi/3}$, **(b)** $4e^{-j3\pi/4}$, **(c)** $2e^{j\pi/2}$, **(d)** $3e^{-j3\pi}$, **(e)** $2e^{j4\pi}$, and **(f)** $2e^{-j4\pi}$.

(a) $2e^{j\pi/3} = 2(\cos\pi/3 + j\sin\pi/3) = 1 + j\sqrt{3}$ (see Fig. B.5a)
(b) $4e^{-j3\pi/4} = 4(\cos 3\pi/4 - j\sin 3\pi/4) = -2\sqrt{2} - j2\sqrt{2}$ (see Fig. B.5b)
(c) $2e^{j\pi/2} = 2(\cos\pi/2 + j\sin\pi/2) = 2(0 + j1) = j2$ (see Fig. B.5c)
(d) $3e^{-j3\pi} = 3(\cos 3\pi - j\sin 3\pi) = 3(-1 + j0) = -3$ (see Fig. B.5d)
(e) $2e^{j4\pi} = 2(\cos 4\pi + j\sin 4\pi) = 2(1 + j0) = 2$ (see Fig. B.5e)
(f) $2e^{-j4\pi} = 2(\cos 4\pi - j\sin 4\pi) = 2(1 - j0) = 2$ (see Fig. B.5f)

We can readily verify these results using MATLAB. First, we use the `exp` function to represent a number in polar form. Next, we use the `real` and `imag` commands to determine the real and imaginary components of that number. To provide an example, let us use MATLAB to verify the result of part (a): $2e^{j\pi/3} = 1 + j\sqrt{3} = 1 + j1.7321$.

```
>>  real(2*exp(1j*pi/3))
    ans = 1.0000
>>  imag(2*exp(1j*pi/3))
    ans = 1.7321
```

Since MATLAB defaults to Cartesian form, we could have verified the entire result in one step.

```
>>  2*exp(1j*pi/3)
    ans = 1.0000 + 1.7321i
```

One can also use the `pol2cart` command to convert polar to Cartesian coordinates.

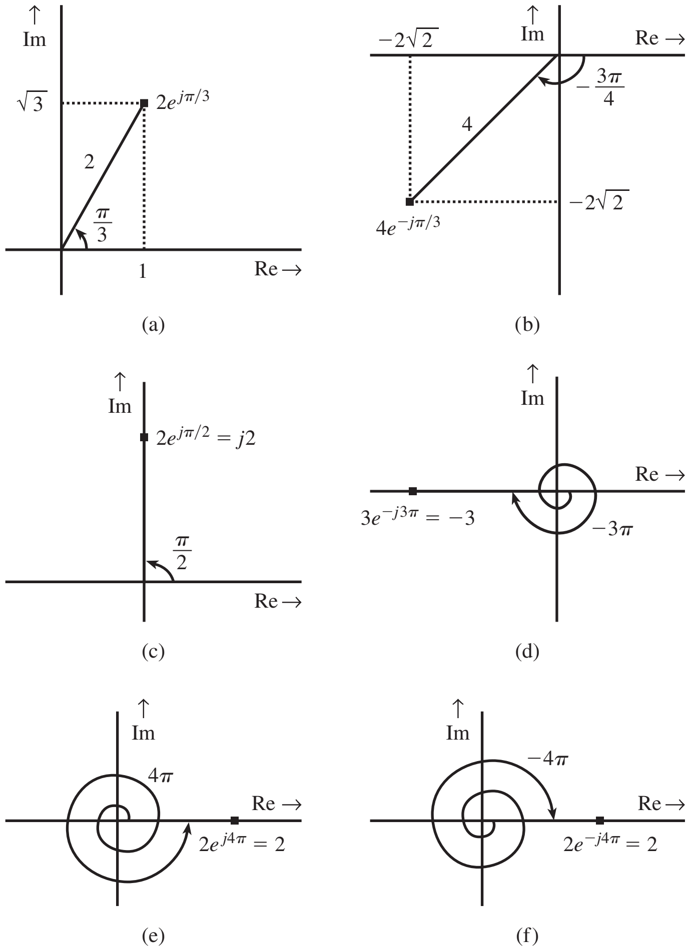

Figure B.5 From polar to Cartesian form.

Arithmetical Operations, Powers, and Roots of Complex Numbers

To conveniently perform addition and subtraction, complex numbers should be expressed in Cartesian form. Thus, if

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

and

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

then

$$z_1 + z_2 = (3 + j4) + (2 + j3) = 5 + j7$$

If z_1 and z_2 are given in polar form, we would need to convert them into Cartesian form for the purpose of adding (or subtracting). Multiplication and division, however, can be carried out in either Cartesian or polar form, although the latter proves to be much more convenient. This is because if z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = (r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

Moreover,

$$z^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

and

$$z^{1/n} = (re^{j\theta})^{1/n} = r^{1/n} e^{j\theta/n} \tag{B.11}$$

This shows that the operations of multiplication, division, powers, and roots can be carried out with remarkable ease when the numbers are in polar form.

Strictly speaking, there are n values for $z^{1/n}$ (the nth root of z). To find all the n roots, we reexamine Eq. (B.11):

$$z^{1/n} = [re^{j\theta}]^{1/n} = \left[re^{j(\theta+2\pi k)}\right]^{1/n} = r^{1/n} e^{j(\theta+2\pi k)/n} \qquad k = 0, 1, 2, \ldots, n-1 \tag{B.12}$$

The value of $z^{1/n}$ given in Eq. (B.11) is the *principal value* of $z^{1/n}$, obtained by taking the nth root of the principal value of z, which corresponds to the case $k = 0$ in Eq. (B.12).

EXAMPLE B.3 Multiplication and Division of Complex Numbers

Using both polar and Cartesian forms, determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ} \qquad \text{and} \qquad z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = (5e^{j53.1^\circ})\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3+j4}{2+j3}$$

To eliminate the complex number in the denominator, we multiply both the numerator and the denominator of the right-hand side by $2-j3$, the denominator's conjugate. This yields

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ-56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ}$$

It is clear from this example that multiplication and division are easier to accomplish in polar form than in Cartesian form.

These results are also easily verified using MATLAB. To provide one example, let us use Cartesian forms in MATLAB to verify that $z_1z_2 = -6+j17$.

```
>>  z1 = 3+4j; z2 = 2+3j;
>>  z1*z2
    ans = -6.0000 + 17.0000i
```

As a second example, let us use polar forms in MATLAB to verify that $z_1/z_2 = 1.3868e^{-j3.2^\circ}$. Since MATLAB generally expects angles be represented in the natural units of radians, we must use appropriate conversion factors in moving between degrees and radians (and vice versa).

```
>>  z1 = 5*exp(1j*53.1*pi/180); z2 = sqrt(13)*exp(1j*56.3*pi/180);
>>  abs(z1/z2)
    ans = 1.3868
>>  angle(z1/z2)*180/pi
    ans = -3.2000
```

EXAMPLE B.4 Working with Complex Numbers

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find the following: **(a)** $2z_1 - z_2$, **(b)** $1/z_1$, **(c)** z_1/z_2^2, and **(d)** $\sqrt[3]{z_2}$.

(a) Since subtraction cannot be performed directly in polar form, we convert z_1 and z_2 to Cartesian form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = \sqrt{2}+j\sqrt{2}$$
$$z_2 = 8e^{j\pi/3} = 8\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 4+j4\sqrt{3}$$

Therefore,

$$2z_1 - z_2 = 2\left(\sqrt{2} + j\sqrt{2}\right) - \left(4 + j4\sqrt{3}\right) = \left(2\sqrt{2} - 4\right) + j\left(2\sqrt{2} - 4\sqrt{3}\right) = -1.17 - j4.1$$

(b)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

(c)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\pi/4 - 2\pi/3)} = \frac{1}{32}e^{-j(5\pi/12)}$$

(d) There are three cube roots of $8e^{j(\pi/3)} = 8e^{j(\pi/3 + 2\pi k)}$, $k = 0, 1, 2$.

$$\sqrt[3]{z_2} = z_2^{1/3} = \left[8e^{j(\pi/3 + 2\pi k)}\right]^{1/3} = 8^{1/3}\left(e^{j[(6\pi k + \pi)/3]}\right)^{1/3} = \begin{cases} 2e^{j\pi/9} & k = 0 \\ 2e^{j7\pi/9} & k = 1 \\ 2e^{j13\pi/9} & k = 2 \end{cases}$$

The value corresponding to $k = 0$ is termed the *principal value*.

EXAMPLE B.5 Standard Forms of Complex Numbers

Consider $X(\omega)$, a complex function of a real variable ω:

$$X(\omega) = \frac{2 + j\omega}{3 + j4\omega}$$

(a) Express $X(\omega)$ in Cartesian form, and find its real and imaginary parts.

(b) Express $X(\omega)$ in polar form, and find its magnitude $|X(\omega)|$ and angle $\angle X(\omega)$.

(a) To obtain the real and imaginary parts of $X(\omega)$, we must eliminate imaginary terms in the denominator of $X(\omega)$. This is readily done by multiplying both the numerator and the denominator of $X(\omega)$ by $3 - j4\omega$, the conjugate of the denominator $3 + j4\omega$ so that

$$X(\omega) = \frac{(2 + j\omega)(3 - j4\omega)}{(3 + j4\omega)(3 - j4\omega)} = \frac{(6 + 4\omega^2) - j5\omega}{9 + 16\omega^2} = \frac{6 + 4\omega^2}{9 + 16\omega^2} - j\frac{5\omega}{9 + 16\omega^2}$$

This is the Cartesian form of $X(\omega)$. Clearly, the real and imaginary parts $X_r(\omega)$ and $X_i(\omega)$ are given by

$$X_r(\omega) = \frac{6 + 4\omega^2}{9 + 16\omega^2} \qquad \text{and} \qquad X_i(\omega) = \frac{-5\omega}{9 + 16\omega^2}$$

(b)

$$X(\omega) = \frac{2+j\omega}{3+j4\omega} = \frac{\sqrt{4+\omega^2}\, e^{j\tan^{-1}(\omega/2)}}{\sqrt{9+16\omega^2}\, e^{j\tan^{-1}(4\omega/3)}} = \sqrt{\frac{4+\omega^2}{9+16\omega^2}}\, e^{j[\tan^{-1}(\omega/2)-\tan^{-1}(4\omega/3)]}$$

This is the polar representation of $X(\omega)$. Observe that

$$|X(\omega)| = \sqrt{\frac{4+\omega^2}{9+16\omega^2}} \qquad \text{and} \qquad \angle X(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right)$$

LOGARITHMS OF COMPLEX NUMBERS

To take the natural logarithm of a complex number z, we first express z in general polar form as

$$z = re^{j\theta} = re^{j(\theta \pm 2\pi k)} \qquad k = 0, 1, 2, 3, \ldots$$

Taking the natural logarithm, we see that

$$\ln z = \ln\left(re^{j(\theta \pm 2\pi k)}\right) = \ln r \pm j(\theta + 2\pi k) \qquad k = 0, 1, 2, 3, \ldots$$

The value of $\ln z$ for $k = 0$ is called the *principal value* of $\ln z$ and is denoted by $\operatorname{Ln} z$. In this way, we see that

$$\begin{aligned}
\ln 1 &= \ln(1e^{\pm j2\pi k}) = \pm j2\pi k \qquad k = 0, 1, 2, 3, \ldots \\
\ln(-1) &= \ln[1e^{\pm j\pi(2k+1)}] = \pm j(2k+1)\pi \qquad k = 0, 1, 2, 3, \ldots \\
\ln j &= \ln\left(e^{j\pi(1\pm 4k)/2}\right) = j\frac{\pi(1\pm 4k)}{2} \qquad k = 0, 1, 2, 3, \ldots \\
j^j &= e^{j\ln j} = e^{-\pi(1\pm 4k)/2} \qquad k = 0, 1, 2, 3, \ldots
\end{aligned}$$

In all of these cases, setting $k = 0$ yields the principal value of the expression.

We can further our logarithm skills by noting that the familiar properties of logarithms hold for complex arguments. Therefore, we have

$$\begin{aligned}
\log(z_1 z_2) &= \log z_1 + \log z_2 \\
\log(z_1/z_2) &= \log z_1 - \log z_2 \\
a^{(z_1+z_2)} &= a^{z_1} \times a^{z_2} \\
z^c &= e^{c\ln z} \\
a^z &= e^{z\ln a}
\end{aligned}$$

B.2 SINUSOIDS

Consider the sinusoid

$$x(t) = C\cos(2\pi f_0 t + \theta) \tag{B.13}$$

We know that

$$\cos\varphi = \cos(\varphi + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Therefore, $\cos\varphi$ repeats itself for every change of 2π in the angle φ. For the sinusoid in Eq. (B.13), the angle $2\pi f_0 t + \theta$ changes by 2π when t changes by $1/f_0$. Clearly, this sinusoid repeats every $1/f_0$ seconds. As a result, there are f_0 repetitions per second. This is the *frequency* of the sinusoid, and the repetition interval T_0 given by

$$T_0 = \frac{1}{f_0} \tag{B.14}$$

is the *period*. For the sinusoid in Eq. (B.13), C is the *amplitude*, f_0 is the *frequency* (in hertz), and θ is the phase. Let us consider two special cases of this sinusoid when $\theta = 0$ and $\theta = -\pi/2$ as follows:

$$x(t) = C\cos 2\pi f_0 t \qquad (\theta = 0)$$

and

$$x(t) = C\cos(2\pi f_0 t - \pi/2) = C\sin 2\pi f_0 t \qquad (\theta = -\pi/2)$$

The angle or phase can be expressed in units of degrees or radians. Although the radian is the proper unit, in this book we shall often use the degree unit because students generally have a better feel for the relative magnitudes of angles expressed in degrees rather than in radians. For example, we relate better to the angle 24° than to 0.419 radian. Remember, however, when in doubt, use the radian unit and, above all, be consistent. In other words, in a given problem or an expression, do not mix the two units.

It is convenient to use the variable ω_0 (*radian frequency*) to express $2\pi f_0$:

$$\omega_0 = 2\pi f_0 \tag{B.15}$$

With this notation, the sinusoid in Eq. (B.13) can be expressed as

$$x(t) = C\cos(\omega_0 t + \theta)$$

in which the period T_0 and frequency ω_0 are given by [see Eqs. (B.14) and (B.15)]

$$T_0 = \frac{1}{\omega_0/2\pi} = \frac{2\pi}{\omega_0} \qquad \text{and} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Although we shall often refer to ω_0 as the frequency of the signal $\cos(\omega_0 t + \theta)$, it should be clearly understood that ω_0 is the *radian frequency*; the *hertzian frequency* of this sinusoid is $f_0 = \omega_0/2\pi$).

The signals $C\cos\omega_0 t$ and $C\sin\omega_0 t$ are illustrated in Figs. B.6a and B.6b, respectively. A general sinusoid $C\cos(\omega_0 t + \theta)$ can be readily sketched by shifting the signal $C\cos\omega_0 t$ in Fig. B.6a by the appropriate amount. Consider, for example,

$$x(t) = C\cos(\omega_0 t - 60°)$$

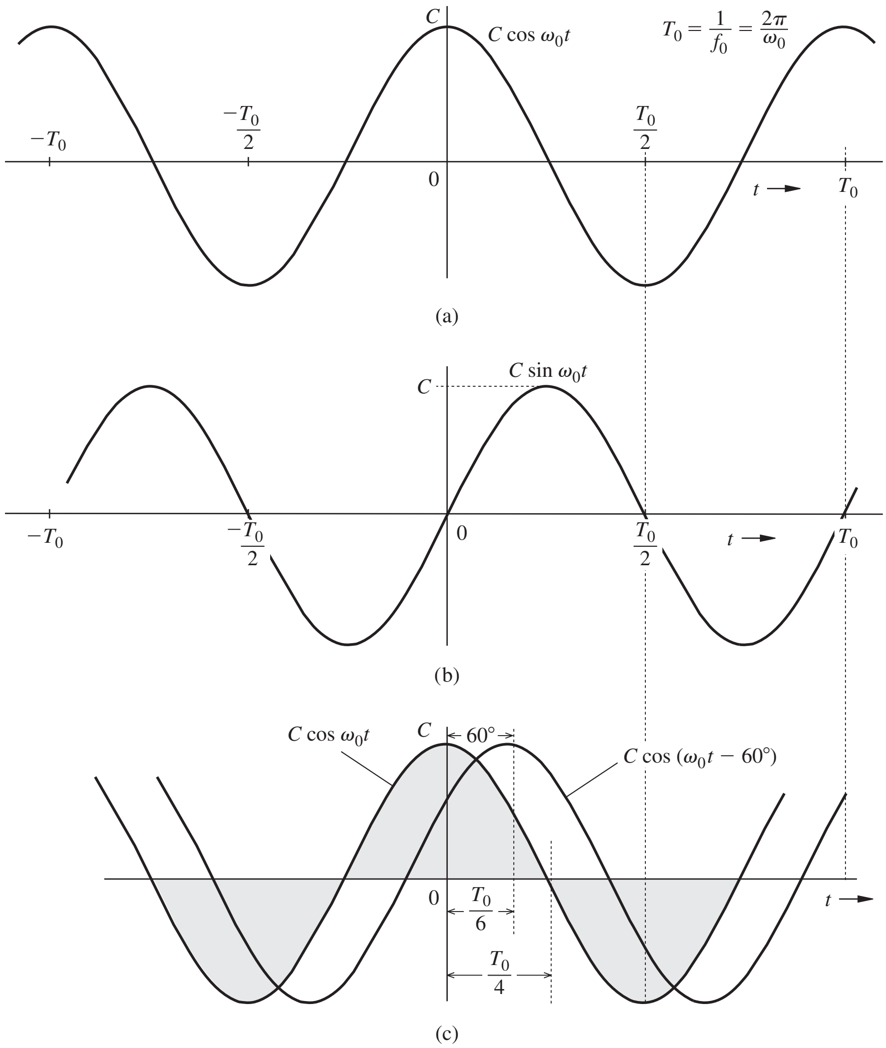

Figure B.6 Sketching a sinusoid.

This signal can be obtained by shifting (delaying) the signal $C \cos \omega_0 t$ (Fig. B.6a) to the right by a phase (angle) of 60°. We know that a sinusoid undergoes a 360° change of phase (or angle) in one cycle. A quarter-cycle segment corresponds to a 90° change of angle. We therefore shift (delay) the signal in Fig. B.6a by two-thirds of a quarter-cycle segment to obtain $C \cos(\omega_0 t - 60°)$, as shown in Fig. B.6c.

Observe that if we delay $C \cos \omega_0 t$ in Fig. B.6a by a quarter-cycle (angle of 90° or $\pi/2$ radians), we obtain the signal $C \sin \omega_0 t$, depicted in Fig. B.6b. This verifies the well-known trigonometric identity

$$C \cos(\omega_0 t - \pi/2) = C \sin \omega_0 t$$

Alternatively, if we advance $C \sin \omega_0 t$ by a quarter-cycle, we obtain $C \cos \omega_0 t$. Therefore,

$$C \sin(\omega_0 t + \pi/2) = C \cos \omega_0 t$$

These observations mean that $\sin \omega_0 t$ lags $\cos \omega_0 t$ by $90°(\pi/2$ radians) and that $\cos \omega_0 t$ leads $\sin \omega_0 t$ by $90°$.

B.2-1 Addition of Sinusoids

Two sinusoids having the same frequency but different phases add to form a single sinusoid of the same frequency. This fact is readily seen from the well-known trigonometric identity

$$C \cos \theta \cos \omega_0 t - C \sin \theta \sin \omega_0 t = C \cos(\omega_0 t + \theta)$$

Setting $a = C \cos \theta$ and $b = -C \sin \theta$, we see that

$$a \cos \omega_0 t + b \sin \omega_0 t = C \cos(\omega_0 t + \theta) \tag{B.16}$$

From trigonometry, we know that

$$C = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{-b}{a}\right) \tag{B.17}$$

Equation (B.17) shows that C and θ are the magnitude and angle, respectively, of a complex number $a - jb$. In other words, $a - jb = Ce^{j\theta}$. Hence, to find C and θ, we convert $a - jb$ to polar form and the magnitude and the angle of the resulting polar number are C and θ, respectively.

The process of adding two sinusoids with the same frequency can be clarified by using *phasors* to represent sinusoids. We represent the sinusoid $C \cos(\omega_0 t + \theta)$ by a phasor of length C at an angle θ with the horizontal axis. Clearly, the sinusoid $a \cos \omega_0 t$ is represented by a horizontal phasor of length a $(\theta = 0)$, while $b \sin \omega_0 t = b \cos(\omega_0 t - \pi/2)$ is represented by a vertical phasor of length b at an angle $-\pi/2$ with the horizontal (Fig. B.7). Adding these two phasors results in a phasor of length C at an angle θ, as depicted in Fig. B.7. From this figure, we verify the values of C and θ found in Eq. (B.17). Proper care should be exercised in computing θ, as explained on page 8 ("A Warning About Computing Angles with Calculators").

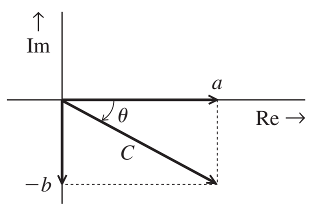

Figure B.7 Phasor addition of sinusoids.

EXAMPLE B.6 Addition of Sinusoids

In the following cases, express $x(t)$ as a single sinusoid:

(a) $x(t) = \cos\omega_0 t - \sqrt{3}\sin\omega_0 t$

(b) $x(t) = -3\cos\omega_0 t + 4\sin\omega_0 t$

(a) In this case, $a = 1$ and $b = -\sqrt{3}$. Using Eq. (B.17) yields

$$C = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$$

Therefore,

$$x(t) = 2\cos(\omega_0 t + 60^\circ)$$

We can verify this result by drawing phasors corresponding to the two sinusoids. The sinusoid $\cos\omega_0 t$ is represented by a phasor of unit length at a zero angle with the horizontal. The phasor $\sin\omega_0 t$ is represented by a unit phasor at an angle of -90° with the horizontal. Therefore, $-\sqrt{3}\sin\omega_0 t$ is represented by a phasor of length $\sqrt{3}$ at 90° with the horizontal, as depicted in Fig. B.8a. The two phasors added yield a phasor of length 2 at 60° with the horizontal (also shown in Fig. B.8a).

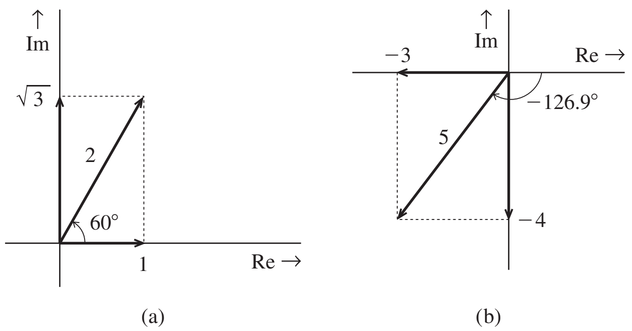

Figure B.8 Phasor addition of sinusoids.

Alternately, we note that $a - jb = 1 + j\sqrt{3} = 2e^{j\pi/3}$. Hence, $C = 2$ and $\theta = \pi/3$.

Observe that a phase shift of $\pm\pi$ amounts to multiplication by -1. Therefore, $x(t)$ can also be expressed alternatively as

$$x(t) = -2\cos(\omega_0 t + 60^\circ \pm 180^\circ) = -2\cos(\omega_0 t - 120^\circ) = -2\cos(\omega_0 t + 240^\circ)$$

In practice, the principal value, that is, -120°, is preferred.

(b) In this case, $a = -3$ and $b = 4$. Using Eq. (B.17) yields

$$C = \sqrt{(-3)^2 + 4^2} = 5 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{-4}{-3}\right) = -126.9^\circ$$

Observe that

$$\tan^{-1}\left(\tfrac{-4}{-3}\right) \neq \tan^{-1}\left(\tfrac{4}{3}\right) = 53.1^\circ$$

Therefore,

$$x(t) = 5\cos(\omega_0 t - 126.9^\circ)$$

This result is readily verified in the phasor diagram in Fig. B.8b. Alternately, $a - jb = -3 - j4 = 5e^{-j126.9^\circ}$, a fact readily confirmed using MATLAB.

```
>>  C = abs(-3+4j)
    C = 5
>>  theta = angle(-3+4j)*180/pi
    theta = 126.8699
```

Hence, $C = 5$ and $\theta = -126.8699^\circ$.

We can also perform the reverse operation, expressing $C\cos(\omega_0 t + \theta)$ in terms of $\cos\,\omega_0 t$ and $\sin\,\omega_0 t$ by again using the trigonometric identity

$$C\cos(\omega_0 t + \theta) = C\cos\theta\cos\,\omega_0 t - C\sin\theta\sin\,\omega_0 t$$

For example,

$$10\cos(\omega_0 t - 60^\circ) = 5\cos\,\omega_0 t + 5\sqrt{3}\sin\,\omega_0 t$$

B.2-2 Sinusoids in Terms of Exponentials

From Eq. (B.3), we know that $e^{j\varphi} = \cos\varphi + j\sin\varphi$ and $e^{-j\varphi} = \cos\varphi - j\sin\varphi$. Adding these two expressions and dividing by 2 provide an expression for cosine in terms of complex exponentials, while subtracting and scaling by $2j$ provide an expression for sine. That is,

$$\cos\varphi = \frac{1}{2}(e^{j\varphi} + e^{-j\varphi}) \qquad \text{and} \qquad \sin\varphi = \frac{1}{2j}(e^{j\varphi} - e^{-j\varphi}) \tag{B.18}$$

B.3 Sketching Signals

In this section, we discuss the sketching of a few useful signals, starting with exponentials.

B.3-1 Monotonic Exponentials

The signal e^{-at} decays monotonically, and the signal e^{at} grows monotonically with t (assuming $a > 0$), as depicted in Fig. B.9. For the sake of simplicity, we shall consider an exponential e^{-at} starting at $t = 0$, as shown in Fig. B.10a.

The signal e^{-at} has a unit value at $t = 0$. At $t = 1/a$, the value drops to $1/e$ (about 37% of its initial value), as illustrated in Fig. B.10a. This time interval over which the exponential reduces by

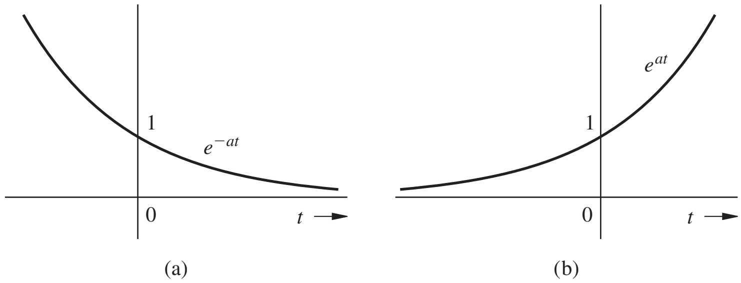

Figure B.9 Monotonic exponentials.

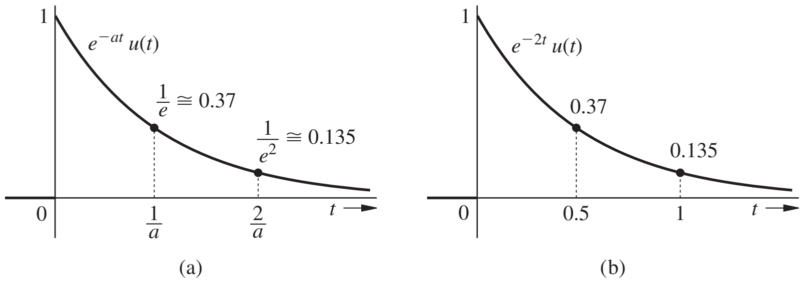

Figure B.10 Sketching **(a)** e^{-at} and **(b)** e^{-2t}.

a factor e (i.e., drops to about 37% of its value) is known as the *time constant* of the exponential. Therefore, the time constant of e^{-at} is $1/a$. Observe that the exponential is reduced to 37% of its initial value over any time interval of duration $1/a$. This can be shown by considering any set of instants t_1 and t_2 separated by one time constant so that

$$t_2 - t_1 = \frac{1}{a}$$

Now the ratio of e^{-at_2} to e^{-at_1} is given by

$$\frac{e^{-at_2}}{e^{-at_1}} = e^{-a(t_2-t_1)} = \frac{1}{e} \approx 0.37$$

We can use this fact to sketch an exponential quickly. For example, consider

$$x(t) = e^{-2t}$$

The time constant in this case is 0.5. The value of $x(t)$ at $t = 0$ is 1. At $t = 0.5$ (one time constant), it is $1/e$ (about 0.37). The value of $x(t)$ continues to drop further by the factor $1/e$ (37%) over the next half-second interval (one time constant). Thus, $x(t)$ at $t = 1$ is $(1/e)^2$. Continuing in this manner, we see that $x(t) = (1/e)^3$ at $t = 1.5$, and so on. A knowledge of the values of $x(t)$ at $t = 0$, 0.5, 1, and 1.5 allows us to sketch the desired signal, as shown in Fig. B.10b.[†]

For a monotonically growing exponential e^{at}, the waveform increases by a factor e over each interval of $1/a$ seconds.

B.3-2 The Exponentially Varying Sinusoid

We now discuss sketching an exponentially varying sinusoid

$$x(t) = Ae^{-at}\cos(\omega_0 t + \theta)$$

Let us consider a specific example:

$$x(t) = 4e^{-2t}\cos(6t - 60°)$$

We shall sketch $4e^{-2t}$ and $\cos(6t - 60°)$ separately and then multiply them:

(a) Sketching $4e^{-2t}$. This monotonically decaying exponential has a time constant of 0.5 second and an initial value of 4 at $t = 0$. Therefore, its values at $t = 0.5$, 1, 1.5, and 2 are $4/e$, $4/e^2$, $4/e^3$, and $4/e^4$, or about 1.47, 0.54, 0.2, and 0.07, respectively. Using these values as a guide, we sketch $4e^{-2t}$, as illustrated in Fig. B.11a.

(b) Sketching $\cos(6t - 60°)$. The procedure for sketching $\cos(6t - 60°)$ is discussed in Sec. B.2 (Fig. B.6c). Here, the period of the sinusoid is $T_0 = 2\pi/6 \approx 1$, and there is a phase delay of 60°, or two-thirds of a quarter-cycle, which is equivalent to a delay of about $(60/360)(1) \approx 1/6$ seconds (see Fig. B.11b).

(c) Sketching $4e^{-2t}\cos(6t - 60°)$. We now multiply the waveforms in steps (a) and (b). This multiplication amounts to forcing the sinusoid $4\cos(6t - 60°)$ to decrease exponentially with a time constant of 0.5. The initial amplitude (at $t = 0$) is 4, decreasing to $4/e$ $(= 1.47)$ at $t = 0.5$, to $1.47/e\,(= 0.54)$ at $t = 1$, and so on. This is depicted in Fig. B.11c. Note that when $\cos(6t - 60°)$ has a value of unity (peak amplitude),

$$4e^{-2t}\cos(6t - 60°) = 4e^{-2t}$$

Therefore, $4e^{-2t}\cos(6t - 60°)$ touches $4e^{-2t}$ at the instants at which the sinusoid $\cos(6t - 60°)$ is at its positive peaks. Clearly, $4e^{-2t}$ is an envelope for positive amplitudes of $4e^{-2t}\cos(6t - 60°)$. Similar argument shows that $4e^{-2t}\cos(6t - 60°)$ touches $-4e^{-2t}$ at its negative peaks. Therefore, $-4e^{-2t}$ is an envelope for negative amplitudes of $4e^{-2t}\cos(6t - 60°)$. Thus, to sketch $4e^{-2t}\cos(6t - 60°)$, we first draw the envelopes $4e^{-2t}$ and $-4e^{-2t}$ (the mirror image of $4e^{-2t}$ about the horizontal axis), and then sketch the sinusoid $\cos(6t - 60°)$, with these envelopes acting as constraints on the sinusoid's amplitude (see Fig. B.11c).

In general, $Ke^{-at}\cos(\omega_0 t + \theta)$ can be sketched in this manner, with Ke^{-at} and $-Ke^{-at}$ constraining the amplitude of $\cos(\omega_0 t + \theta)$.

[†] If we wish to refine the sketch further, we could consider intervals of half the time constant over which the signal decays by a factor $1/\sqrt{e}$. Thus, at $t = 0.25$, $x(t) = 1/\sqrt{e}$, and at $t = 0.75$, $x(t) = 1/e\sqrt{e}$, and so on.

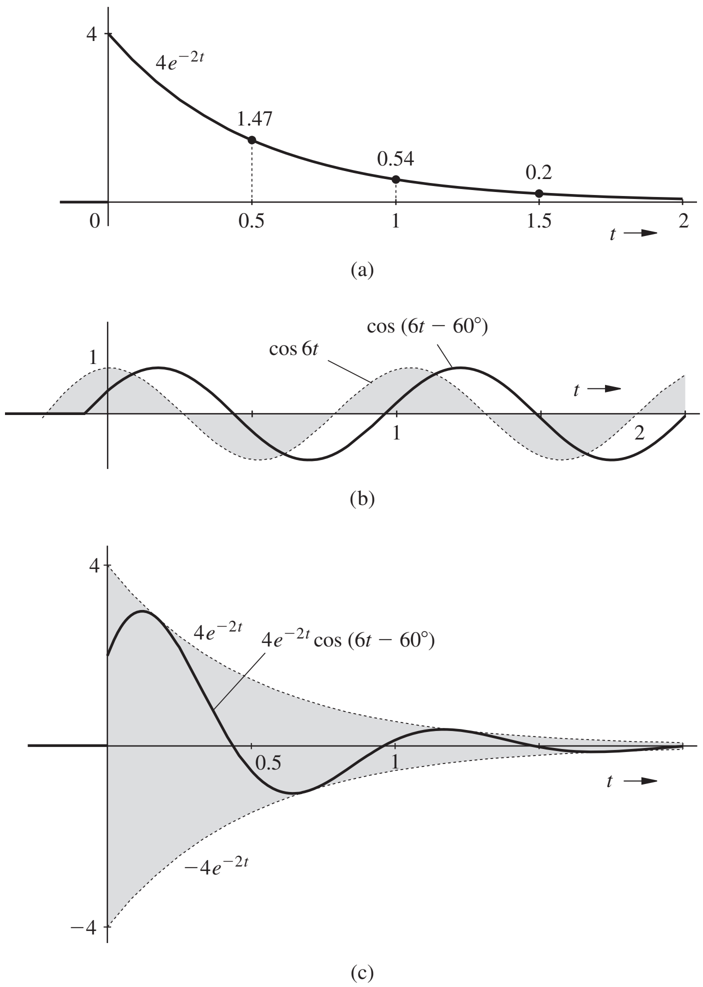

Figure B.11 Sketching an exponentially varying sinusoid.

B.4 Cramer's Rule

Cramer's rule offers a very convenient way to solve simultaneous linear equations. Consider a set of n linear simultaneous equations in n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n
\end{gathered} \tag{B.19}$$

These equations can be expressed in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{B.20}$$

We denote the matrix on the left-hand side formed by the elements a_{ij} as $\mathbf{A}$. The determinant of $\mathbf{A}$ is denoted by $|\mathbf{A}|$. If the determinant $|\mathbf{A}|$ is not zero, Eq. (B.19) has a unique solution given by Cramer's formula

$$x_k = \frac{|\mathbf{D}_k|}{|\mathbf{A}|} \qquad k = 1, 2, \ldots, n \tag{B.21}$$

where $|\mathbf{D}_k|$ is obtained by replacing the kth column of $|\mathbf{A}|$ by the column on the right-hand side of Eq. (B.20) (with elements $y_1, y_2, \ldots, y_n$).

We shall demonstrate the use of this rule with an example.

EXAMPLE B.7 Using Cramer's Rule to Solve a System of Equations

Use Cramer's rule to solve the following simultaneous linear equations in three unknowns:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 3 \\ x_1 + 3x_2 - x_3 &= 7 \\ x_1 + x_2 + x_3 &= 1 \end{aligned}$$

In matrix form, these equations can be expressed as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

Here,

$$|\mathbf{A}| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$$

Since $|\mathbf{A}| = 4 \neq 0$, a unique solution exists for x_1, x_2, and x_3. This solution is provided by Cramer's rule [Eq. (B.21)] as follows:

$$x_1 = \frac{1}{|\mathbf{A}|} \begin{vmatrix} 3 & 1 & 1 \\ 7 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

$$x_2 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \frac{4}{4} = 1$$

$$x_3 = \frac{1}{|\mathbf{A}|}\begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{-8}{4} = -2$$

MATLAB is well suited to compute Cramer's formula, so these results are easy to verify. To provide an example, let us verify that $x_1 = 2$ using MATLAB's `det` command to compute the needed matrix determinants.

```
>>  x1 = det([3 1 1;7 3 -1;1 1 1])/det([2 1 1;1 3 -1;1 1 1])
    x1 = 2.0000
```

B.5 Partial Fraction Expansion

In the analysis of linear time-invariant systems, we encounter functions that are ratios of two polynomials in a certain variable, say, x. Such functions are known as *rational functions*. A rational function $F(x)$ can be expressed as

$$F(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} = \frac{P(x)}{Q(x)} \tag{B.22}$$

The function $F(x)$ is *improper* if $m \geq n$ and *proper* if $m < n$.[†] An improper function can always be separated into the sum of a polynomial in x and a proper function. Consider, for example, the function

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3}$$

Because this is an improper function, we divide the numerator by the denominator until the remainder has a lower degree than the denominator.

$$\begin{array}{r|l} & \quad 2x+1 \\ \cline{2-2} x^2+4x+3 & 2x^3+9x^2+11x+2 \\ & \underline{2x^3+8x^2+6x} \\ & \qquad\quad x^2+5x+2 \\ & \qquad\quad \underline{x^2+4x+3} \\ & \qquad\qquad\quad x-1 \end{array}$$

[†] Some sources classify $F(x)$ as strictly proper if $m < n$, proper if $m \leq n$, and improper if $m > n$.

Therefore, $F(x)$ can be expressed as

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} = \underbrace{2x + 1}_{\text{polynomial in } x} + \underbrace{\frac{x - 1}{x^2 + 4x + 3}}_{\text{proper function}}$$

A proper function can be further expanded into partial fractions. The remaining discussion in this section is concerned with various ways of doing this.

B.5-1 Method of Clearing Fractions

A rational function can be written as a sum of appropriate partial fractions with unknown coefficients, which are determined by clearing fractions and equating the coefficients of similar powers on the two sides. This procedure is demonstrated by the following example.

EXAMPLE B.8 Method of Clearing Fractions

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2}$$

This function can be expressed as a sum of partial fractions with denominators $(x + 1)$, $(x+2)$, $(x+3)$, and $(x+3)^2$, as follows:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3} + \frac{k_4}{(x+3)^2}$$

To determine the unknowns k_1, k_2, k_3, and k_4, we clear fractions by multiplying both sides by $(x+1)(x+2)(x+3)^2$ to obtain

$$\begin{aligned} x^3 + 3x^2 + 4x + 6 &= k_1(x^3 + 8x^2 + 21x + 18) + k_2(x^3 + 7x^2 + 15x + 9) \\ &\quad + k_3(x^3 + 6x^2 + 11x + 6) + k_4(x^2 + 3x + 2) \\ &= x^3(k_1 + k_2 + k_3) + x^2(8k_1 + 7k_2 + 6k_3 + k_4) \\ &\quad + x(21k_1 + 15k_2 + 11k_3 + 3k_4) + (18k_1 + 9k_2 + 6k_3 + 2k_4) \end{aligned}$$

Equating coefficients of similar powers on both sides yields

$$\begin{aligned} k_1 + k_2 + k_3 &= 1 \\ 8k_1 + 7k_2 + 6k_3 + k_4 &= 3 \\ 21k_1 + 15k_2 + 11k_3 + 3k_4 &= 4 \\ 18k_1 + 9k_2 + 6k_3 + 2k_4 &= 6 \end{aligned}$$

Solution of these four simultaneous equations yields

$$k_1 = 1, \qquad k_2 = -2, \qquad k_3 = 2, \qquad k_4 = -3$$

Therefore,

$$F(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x+3} - \frac{3}{(x+3)^2}$$

Although this method is straightforward and applicable to all situations, it is not necessarily the most efficient. We now discuss other methods that can reduce numerical work considerably.

B.5-2 The Heaviside "Cover-Up" Method

DISTINCT FACTORS OF $Q(x)$

We shall first consider the partial fraction expansion of $F(x) = P(x)/Q(x)$, in which all the factors of $Q(x)$ are distinct (not repeated). Consider the proper function

$$\begin{aligned} F(x) &= \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \qquad m < n \\ &= \frac{P(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} \end{aligned}$$

As seen in Ex. B.8, $F(x)$ can be expressed as the sum of partial fractions

$$F(x) = \frac{k_1}{x-\lambda_1} + \frac{k_2}{x-\lambda_2} + \cdots + \frac{k_n}{x-\lambda_n} \tag{B.23}$$

To determine the coefficient k_1, we multiply both sides of Eq. (B.23) by $x-\lambda_1$ and then let $x = \lambda_1$. This yields

$$(x-\lambda_1)F(x)|_{x=\lambda_1} = k_1 + \frac{k_2(x-\lambda_1)}{(x-\lambda_2)} + \frac{k_3(x-\lambda_1)}{(x-\lambda_3)} + \cdots + \left.\frac{k_n(x-\lambda_1)}{(x-\lambda_n)}\right|_{x=\lambda_1}$$

On the right-hand side, all the terms except k_1 vanish. Therefore,

$$k_1 = (x-\lambda_1)F(x)|_{x=\lambda_1}$$

Similarly, we can show that

$$k_r = (x-\lambda_r)F(x)|_{x=\lambda_r} \qquad r = 1, 2, \ldots, n \tag{B.24}$$

This procedure also goes under the name *method of residues*.

EXAMPLE B.9 Heaviside "Cover-Up" Method

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{k_1}{x+1} + \frac{k_2}{x-2} + \frac{k_3}{x+3}$$

To determine k_1, we let $x = -1$ in $(x+1)F(x)$. Note that $(x+1)F(x)$ is obtained from $F(x)$ by omitting the term $(x+1)$ from its denominator. Therefore, to compute k_1 corresponding to the factor $(x+1)$, we cover up the term $(x+1)$ in the denominator of $F(x)$ and then substitute $x = -1$ in the remaining expression. [Mentally conceal the term $(x+1)$ in $F(x)$ with a finger and then let $x = -1$ in the remaining expression.] The steps in covering up the function

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}$$

are as follows.

Step 1. Cover up (conceal) the factor $(x+1)$ from $F(x)$:

$$\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}$$

Step 2. Substitute $x = -1$ in the remaining expression to obtain k_1:

$$k_1 = \frac{2 - 9 - 11}{(-1-2)(-1+3)} = \frac{-18}{-6} = 3$$

Similarly, to compute k_2, we cover up the factor $(x-2)$ in $F(x)$ and let $x = 2$ in the remaining function, as follows:

$$k_2 = \left.\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}\right|_{x=2} = \frac{8 + 18 - 11}{(2+1)(2+3)} = \frac{15}{15} = 1$$

and

$$k_3 = \left.\frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)}\right|_{x=-3} = \frac{18 - 27 - 11}{(-3+1)(-3-2)} = \frac{-20}{10} = -2$$

Therefore,

$$F(x) = \frac{2x^2 + 9x - 11}{(x+1)(x-2)(x+3)} = \frac{3}{x+1} + \frac{1}{x-2} - \frac{2}{x+3}$$

Complex Factors of $Q(x)$

The procedure just given works regardless of whether the factors of $Q(x)$ are real or complex. Consider, for example,

$$\begin{aligned} F(x) &= \frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{4x^2+2x+18}{(x+1)(x+2-j3)(x+2+j3)} \\ &= \frac{k_1}{x+1} + \frac{k_2}{x+2-j3} + \frac{k_3}{x+2+j3} \end{aligned} \tag{B.25}$$

where

$$k_1 = \left[\frac{4x^2+2x+18}{\boxed{(x+1)}\,(x^2+4x+13)}\right]_{x=-1} = 2$$

Similarly,

$$k_2 = \left[\frac{4x^2+2x+18}{(x+1)\,\boxed{(x+2-j3)}\,(x+2+j3)}\right]_{x=-2+j3} = 1+j2 = \sqrt{5}e^{j63.43^\circ}$$

$$k_3 = \left[\frac{4x^2+2x+18}{(x+1)(x+2-j3)\,\boxed{(x+2+j3)}}\right]_{x=-2-j3} = 1-j2 = \sqrt{5}e^{-j63.43^\circ}$$

Therefore,

$$F(x) = \frac{2}{x+1} + \frac{\sqrt{5}e^{j63.43^\circ}}{x+2-j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{x+2+j3}$$

The coefficients k_2 and k_3 corresponding to the complex-conjugate factors are also conjugates of each other. This is generally true when the coefficients of a rational function are real. In such a case, we need to compute only one of the coefficients.

Quadratic Factors

Often we are required to combine the two terms arising from complex-conjugate factors into one quadratic factor. For example, $F(x)$ in Eq. (B.25) can be expressed as

$$F(x) = \frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{k_1}{x+1} + \frac{c_1x+c_2}{x^2+4x+13}$$

The coefficient k_1 is found by the Heaviside method to be 2. Therefore,

$$\frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{2}{x+1} + \frac{c_1x+c_2}{x^2+4x+13} \tag{B.26}$$

The values of c_1 and c_2 are determined by clearing fractions and equating the coefficients of similar powers of x on both sides of the resulting equation. Clearing fractions on both sides of Eq. (B.26) yields

$$\begin{aligned} 4x^2+2x+18 &= 2(x^2+4x+13) + (c_1x+c_2)(x+1) \\ &= (2+c_1)x^2 + (8+c_1+c_2)x + (26+c_2) \end{aligned}$$

Equating terms of similar powers yields $c_1 = 2$, $c_2 = -8$, and

$$\frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{2}{x+1} + \frac{2x-8}{x^2+4x+13}$$

SHORTCUTS

The values of c_1 and c_2 in Eq. (B.26) can also be determined by using shortcuts. After computing $k_1 = 2$ by the Heaviside method as before, we let $x = 0$ on both sides of Eq. (B.26) to eliminate c_1. This gives us

$$\frac{18}{13} = 2 + \frac{c_2}{13} \qquad \Rightarrow \qquad c_2 = -8$$

To determine c_1, we multiply both sides of Eq. (B.26) by x and then let $x \to \infty$. Remember that when $x \to \infty$, only the terms of the highest power are significant. Therefore,

$$4 = 2 + c_1 \qquad \Rightarrow \qquad c_1 = 2$$

In the procedure discussed here, we let $x = 0$ to determine c_2 and then multiply both sides by x and let $x \to \infty$ to determine c_1. However, nothing is sacred about these values ($x = 0$ or $x = \infty$). We use them because they reduce the number of computations involved. We could just as well use other convenient values for x, such as $x = 1$. Consider the case

$$F(x) = \frac{2x^2+4x+5}{x(x^2+2x+5)} = \frac{k}{x} + \frac{c_1x+c_2}{x^2+2x+5}$$

We find $k = 1$ by the Heaviside method in the usual manner. As a result,

$$\frac{2x^2+4x+5}{x(x^2+2x+5)} = \frac{1}{x} + \frac{c_1x+c_2}{x^2+2x+5} \tag{B.27}$$

If we try letting $x = 0$ to determine c_1 and c_2, we obtain ∞ on both sides. So let us choose $x = 1$. This yields

$$\frac{11}{8} = 1 + \frac{c_1+c_2}{8} \qquad \text{or} \qquad c_1 + c_2 = 3$$

We can now choose some other value for x, such as $x = 2$, to obtain one more relationship to use in determining c_1 and c_2. In this case, however, a simple method is to multiply both sides of Eq. (B.27) by x and then let $x \to \infty$. This yields

$$2 = 1 + c_1 \qquad \Rightarrow \qquad c_1 = 1$$

Since $c_1 + c_2 = 3$, we see that $c_2 = 2$ and therefore,

$$F(x) = \frac{1}{x} + \frac{x+2}{x^2+2x+5}$$

B.5-3 Repeated Factors of $Q(x)$

If a function $F(x)$ has a repeated factor in its denominator, it has the form

$$F(x) = \frac{P(x)}{(x-\lambda)^r(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_j)}$$

Its partial fraction expansion is given by

$$\begin{aligned} F(x) &= \frac{a_0}{(x-\lambda)^r} + \frac{a_1}{(x-\lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{(x-\lambda)} \\ &\quad + \frac{k_1}{x-\alpha_1} + \frac{k_2}{x-\alpha_2} + \cdots + \frac{k_j}{x-\alpha_j} \end{aligned} \tag{B.28}$$

The coefficients $k_1, k_2, \ldots, k_j$ corresponding to the unrepeated factors in this equation are determined by the Heaviside method, as before [Eq. (B.24)]. To find the coefficients $a_0, a_1, a_2, \ldots, a_{r-1}$, we multiply both sides of Eq. (B.28) by $(x-\lambda)^r$. This gives us

$$\begin{aligned} (x-\lambda)^r F(x) &= a_0 + a_1(x-\lambda) + a_2(x-\lambda)^2 + \cdots + a_{r-1}(x-\lambda)^{r-1} \\ &\quad + k_1\frac{(x-\lambda)^r}{x-\alpha_1} + k_2\frac{(x-\lambda)^r}{x-\alpha_2} + \cdots + k_n\frac{(x-\lambda)^r}{x-\alpha_n} \end{aligned} \tag{B.29}$$

If we let $x = \lambda$ on both sides of Eq. (B.29), we obtain

$$(x-\lambda)^r F(x)|_{x=\lambda} = a_0$$

Therefore, a_0 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$ and letting $x = \lambda$ in the remaining expression (the Heaviside "cover-up" method). If we take the derivative (with respect to x) of both sides of Eq. (B.29), the right-hand side is a_1+ terms containing a factor $(x-\lambda)$ in their numerators. Letting $x = \lambda$ on both sides of this equation, we obtain

$$\frac{d}{dx}\left[(x-\lambda)^r F(x)\right]\Big|_{x=\lambda} = a_1$$

Thus, a_1 is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the derivative of the remaining expression, and then letting $x = \lambda$. Continuing in this manner, we find

$$a_j = \frac{1}{j!}\frac{d^j}{dx^j}\left[(x-\lambda)^r F(x)\right]\Big|_{x=\lambda} \tag{B.30}$$

Observe that $(x-\lambda)^r F(x)$ is obtained from $F(x)$ by omitting the factor $(x-\lambda)^r$ from its denominator. Therefore, the coefficient a_j is obtained by concealing the factor $(x-\lambda)^r$ in $F(x)$, taking the jth derivative of the remaining expression, and then letting $x = \lambda$ (while dividing by $j!$).

EXAMPLE B.10 Partial Fraction Expansion with Repeated Factors

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}$$

The partial fractions are

$$F(x) = \frac{a_0}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{k}{x+2}$$

The coefficient k is obtained by concealing the factor $(x+2)$ in $F(x)$ and then substituting $x = -2$ in the remaining expression:

$$k = \left.\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}\right|_{x=-2} = 1$$

To find a_0, we conceal the factor $(x+1)^3$ in $F(x)$ and let $x = -1$ in the remaining expression:

$$a_0 = \left.\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}\right|_{x=-1} = 2$$

To find a_1, we conceal the factor $(x+1)^3$ in $F(x)$, take the derivative of the remaining expression, and then let $x = -1$:

$$a_1 = \frac{d}{dx}\left.\left[\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}\right]\right|_{x=-1} = 1$$

Similarly,

$$a_2 = \frac{1}{2!}\frac{d^2}{dx^2}\left.\left[\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)}\right]\right|_{x=-1} = 3$$

Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

B.5-4 A Combination of Heaviside "Cover-Up" and Clearing Fractions

For multiple roots, especially of higher order, the Heaviside expansion method, which requires repeated differentiation, can become cumbersome. For a function that contains several repeated and unrepeated roots, a hybrid of the two procedures proves to be the best. The simpler coefficients are determined by the Heaviside method, and the remaining coefficients are found by clearing fractions or shortcuts, thus incorporating the best of the two methods. We demonstrate this procedure by solving Ex. B.10 once again by this method.

In Ex. B.10, coefficients k and a_0 are relatively simple to determine by the Heaviside expansion method. These values were found to be $k_1 = 1$ and $a_0 = 2$. Therefore,

$$\frac{4x^3+16x^2+23x+13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

We now multiply both sides of this equation by $(x+1)^3(x+2)$ to clear the fractions. This yields

$$\begin{aligned} 4x^3+16x^2+23x+13 &= 2(x+2) + a_1(x+1)(x+2) + a_2(x+1)^2(x+2) + (x+1)^3 \\ &= (1+a_2)x^3 + (a_1+4a_2+3)x^2 + (5+3a_1+5a_2)x + (4+2a_1+2a_2+1) \end{aligned}$$

Equating coefficients of the third and second powers of x on both sides, we obtain

$$\left.\begin{aligned} 1+a_2 &= 4 \\ a_1+4a_2+3 &= 16 \end{aligned}\right\} \implies \begin{aligned} a_1 &= 1 \\ a_2 &= 3 \end{aligned}$$

We may stop here if we wish because the two desired coefficients, a_1 and a_2, are now determined. However, equating the coefficients of the two remaining powers of x yields a convenient check on the answer. Equating the coefficients of the x^1 and x^0 terms, we obtain

$$\begin{aligned} 23 &= 5+3a_1+5a_2 \\ 13 &= 4+2a_1+2a_2+1 \end{aligned}$$

These equations are satisfied by the values $a_1 = 1$ and $a_2 = 3$, found earlier, providing an additional check for our answers. Therefore,

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

which agrees with the earlier result.

A Combination of Heaviside "Cover-Up" and Shortcuts

In Ex. B.10, after determining the coefficients $a_0 = 2$ and $k = 1$ by the Heaviside method as before, we have

$$\frac{4x^3+16x^2+23x+13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

There are only two unknown coefficients, a_1 and a_2. If we multiply both sides of this equation by x and then let $x \to \infty$, we can eliminate a_1. This yields

$$4 = a_2 + 1 \implies a_2 = 3$$

Therefore,

$$\frac{4x^3+16x^2+23x+13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}$$

There is now only one unknown a_1, which can be readily found by setting x equal to any convenient value, say, $x = 0$. This yields

$$\frac{13}{2} = 2 + a_1 + 3 + \frac{1}{2} \quad \Longrightarrow \quad a_1 = 1$$

which agrees with our earlier answer.

There are other possible shortcuts. For example, we can compute a_0 (coefficient of the highest power of the repeated root), subtract this term from both sides, and then repeat the procedure.

B.5-5 Improper $F(x)$ with $m = n$

A general method of handling an improper function is indicated in the beginning of this section. However, for the special case of when the numerator and denominator polynomials of $F(x)$ have the same degree ($m = n$), the procedure is the same as that for a proper function. We can show that for

$$\begin{aligned} F(x) &= \frac{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \\ &= b_n + \frac{k_1}{x - \lambda_1} + \frac{k_2}{x - \lambda_2} + \cdots + \frac{k_n}{x - \lambda_n} \end{aligned}$$

the coefficients $k_1, k_2, \ldots, k_n$ are computed as if $F(x)$ were proper. Thus,

$$k_r = (x - \lambda_r)F(x)|_{x=\lambda_r}$$

For quadratic or repeated factors, the appropriate procedures discussed in Secs. B.5-2 or B.5-3 should be used as if $F(x)$ were proper. In other words, when $m = n$, the only difference between the proper and improper case is the appearance of an extra constant b_n in the latter. Otherwise, the procedure remains the same. The proof is left as an exercise for the reader.

EXAMPLE B.11 Partial Fraction Expansion of Improper Rational Function

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{3x^2 + 9x - 20}{x^2 + x - 6} = \frac{3x^2 + 9x - 20}{(x-2)(x+3)}$$

Here, $m = n = 2$ with $b_n = b_2 = 3$. Therefore,

$$F(x) = \frac{3x^2 + 9x - 20}{(x-2)(x+3)} = 3 + \frac{k_1}{x-2} + \frac{k_2}{x+3}$$

in which

$$k_1 = \left.\frac{3x^2+9x-20}{(x-2)(x+3)}\right|_{x=2} = \frac{12+18-20}{(2+3)} = \frac{10}{5} = 2$$

and

$$k_2 = \left.\frac{3x^2+9x-20}{(x-2)(x+3)}\right|_{x=-3} = \frac{27-27-20}{(-3-2)} = \frac{-20}{-5} = 4$$

Therefore,

$$F(x) = \frac{3x^2+9x-20}{(x-2)(x+3)} = 3 + \frac{2}{x-2} + \frac{4}{x+3}$$

B.5-6 Modified Partial Fractions

In finding the inverse z-transform (Ch. 5), we require partial fractions of the form $kx/(x-\lambda_i)^r$ rather than $k/(x-\lambda_i)^r$. This can be achieved by expanding $F(x)/x$ into partial fractions. Consider, for example,

$$F(x) = \frac{5x^2+20x+18}{(x+2)(x+3)^2}$$

Dividing both sides by x yields

$$\frac{F(x)}{x} = \frac{5x^2+20x+18}{x(x+2)(x+3)^2}$$

Expansion of the right-hand side into partial fractions as usual yields

$$\frac{F(x)}{x} = \frac{5x^2+20x+18}{x(x+2)(x+3)^2} = \frac{a_1}{x} + \frac{a_2}{x+2} + \frac{a_3}{(x+3)} + \frac{a_4}{(x+3)^2}$$

Using the procedure discussed earlier, we find $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = 1$. Therefore,

$$\frac{F(x)}{x} = \frac{1}{x} + \frac{1}{x+2} - \frac{2}{x+3} + \frac{1}{(x+3)^2}$$

Now multiplying both sides by x yields

$$F(x) = 1 + \frac{x}{x+2} - \frac{2x}{x+3} + \frac{x}{(x+3)^2}$$

This expresses $F(x)$ as the sum of partial fractions having the form $kx/(x-\lambda_i)^r$.

B.6 VECTORS AND MATRICES

An entity specified by n numbers in a certain order (ordered n-tuple) is an n-dimensional *vector*. Thus, an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ represents an n-dimensional vector $\mathbf{x}$. A vector may be represented as a row (*row vector*):

$$\mathbf{x} = [\ x_1 \quad x_2 \quad \cdots \quad x_n\ ]$$

or as a column (*column vector*):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Simultaneous linear equations can be viewed as the transformation of one vector into another. Consider, for example, the m simultaneous linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned} \tag{B.31}$$

If we define two column vectors $\mathbf{x}$ and $\mathbf{y}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

then Eq. (B.31) may be viewed as the relationship or the function that transforms vector $\mathbf{x}$ into vector $\mathbf{y}$. Such a transformation is called a *linear transformation* of vectors. To perform a linear transformation, we need to define the array of coefficients a_{ij} appearing in Eq. (B.31). This array is called a *matrix* and is denoted by $\mathbf{A}$ for convenience:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A matrix with m rows and n columns is called a matrix of order (m, n) or an $(m \times n)$ matrix. For the special case of $m = n$, the matrix is called a *square matrix* of order n.

It should be stressed at this point that a matrix is not a number such as a determinant, but an array of numbers arranged in a particular order. It is convenient to abbreviate the representation of matrix $\mathbf{A}$ with the form $(a_{ij})_{m\times n}$, implying a matrix of order $m \times n$ with a_{ij} as its ijth element. In practice, when the order $m \times n$ is understood or need not be specified, the notation can be abbreviated to (a_{ij}). Note that the first index i of a_{ij} indicates the row and the second index j indicates the column of the element a_{ij} in matrix $\mathbf{A}$.

Equation (B.31) may now be expressed in a matrix form as

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{or} \qquad \mathbf{y} = \mathbf{A}\mathbf{x} \tag{B.32}$$

At this point, we have not defined the multiplication of a matrix by a vector. The quantity $\mathbf{Ax}$ is not meaningful until such an operation has been defined.

B.6-1 Some Definitions and Properties

A square matrix whose elements are zero everywhere except on the main diagonal is a *diagonal matrix*. An example of a diagonal matrix is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

A diagonal matrix with unity for all its diagonal elements is called an *identity matrix* or a *unit matrix*, denoted by $\mathbf{I}$. This is a square matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The order of the unit matrix is sometimes indicated by a subscript. Thus, $\mathbf{I}_n$ represents the $n \times n$ unit matrix (or identity matrix). However, we shall omit the subscript since order is easily understood by context.

A matrix having all its elements zero is a *zero matrix*.

A square matrix $\mathbf{A}$ is a *symmetric matrix* if $a_{ij} = a_{ji}$ (symmetry about the main diagonal).

Two matrices of the same order are said to be *equal* if they are equal element by element. Thus, if

$$\mathbf{A} = (a_{ij})_{m\times n} \qquad \text{and} \qquad \mathbf{B} = (b_{ij})_{m\times n}$$

then $\mathbf{A} = \mathbf{B}$ only if $a_{ij} = b_{ij}$ for all i and j.

If the rows and columns of an $m \times n$ matrix $\mathbf{A}$ are interchanged so that the elements in the ith row now become the elements of the ith column (for $i = 1, 2, \ldots, m$), the resulting matrix is called the *transpose* of $\mathbf{A}$ and is denoted by $\mathbf{A}^T$. It is evident that $\mathbf{A}^T$ is an $n \times m$ matrix. For example, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}, \qquad \text{then} \qquad \mathbf{A}^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Using the abbreviated notation, if $\mathbf{A} = (a_{ij})_{m\times n}$, then $\mathbf{A}^T = (a_{ji})_{n\times m}$. Intuitively, further notice that $(\mathbf{A}^T)^T = \mathbf{A}$.

B.6-2 Matrix Algebra

We shall now define matrix operations, such as addition, subtraction, multiplication, and division of matrices. The definitions should be formulated so that they are useful in the manipulation of matrices.

Addition of Matrices

For two matrices $\mathbf{A}$ and $\mathbf{B}$, both of the same order $(m \times n)$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

we define the sum $\mathbf{A}+\mathbf{B}$ as

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) & \cdots & (a_{1n}+b_{1n}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) & \cdots & (a_{2n}+b_{2n}) \\ \vdots & \vdots & \cdots & \vdots \\ (a_{m1}+b_{m1}) & (a_{m2}+b_{m2}) & \cdots & (a_{mn}+b_{mn}) \end{bmatrix}$$

or

$$\mathbf{A}+\mathbf{B} = (a_{ij}+b_{ij})_{m\times n}$$

Note that two matrices can be added only if they are of the same order.

Multiplication of a Matrix by a Scalar

We multiply a matrix $\mathbf{A}$ by a scalar c as follows:

$$c\mathbf{A} = c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix} = \mathbf{A}c$$

Thus, we also observe that the scalar c and the matrix $\mathbf{A}$ commute: $c\mathbf{A} = \mathbf{A}c$.

Matrix Multiplication

We define the product

$$\mathbf{AB} = \mathbf{C}$$

in which c_{ij}, the element of $\mathbf{C}$ in the ith row and jth column, is found by adding the products of the elements of $\mathbf{A}$ in the ith row multiplied by the corresponding elements of $\mathbf{B}$ in the jth column.

Thus,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \tag{B.33}$$

This result is expressed as follows:

$$\underbrace{\begin{bmatrix} & & & \\ & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \\ & & & \end{bmatrix}}_{\mathbf{A}(m\times n)} \underbrace{\begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ \cdots & b_{ij} & \cdots \\ & \vdots & \\ & b_{nj} & \end{bmatrix}}_{\mathbf{B}(n\times p)} = \underbrace{\begin{bmatrix} & & \\ & & \\ \cdots & c_{ij} & \cdots \\ & & \\ & & \end{bmatrix}}_{\mathbf{C}(m\times p)}$$

Note carefully that if this procedure is to work, the number of columns of **A** must be equal to the number of rows of **B**. In other words, **AB**, the product of matrices **A** and **B**, is defined only if the number of columns of **A** is equal to the number of rows of **B**. If this condition is not satisfied, the product **AB** is not defined and is meaningless. When the number of columns of **A** is equal to the number of rows of **B**, matrix **A** is said to be *conformable* to matrix **B** for the product **AB**. Observe that if **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, **A** and **B** are conformable for the product, and **C** is an $m \times p$ matrix.

We demonstrate the use of the rule in Eq. (B.33) with the following examples.

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 5 & 7 \\ 3 & 4 & 2 & 3 \\ 5 & 10 & 4 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 8$$

In both cases, the two matrices are conformable. However, if we interchange the order of the first matrices as follows:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$$

the matrices are no longer conformable for the product. It is evident that, in general,

$$\mathbf{AB} \neq \mathbf{BA}$$

Indeed, **AB** may exist and **BA** may not exist, or vice versa, as in our examples. We shall see later that for some special matrices, $\mathbf{AB} = \mathbf{BA}$. When this is true, matrices **A** and **B** are said to *commute*. We re-emphasize that in general, matrices do not commute.

In the matrix product **AB**, matrix **A** is said to be *postmultiplied* by **B** or matrix **B** is said to be *premultiplied* by **A**. We may also verify the following relationships:

$$(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$
$$\mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$$

We can verify that any matrix **A** premultiplied or postmultiplied by the identity matrix **I** remains unchanged:

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Of course, we must make sure that the order of **I** is such that the matrices are conformable for the corresponding product.

We give here, without proof, another important property of matrices:

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

where $|\mathbf{A}|$ and $|\mathbf{B}|$ represent determinants of matrices **A** and **B**.

Multiplication of a Matrix by a Vector

Consider Eq. (B.32), which represents Eq. (B.31). The right-hand side of Eq. (B.32) is a product of the $m \times n$ matrix **A** and a vector **x**. If, for the time being, we treat the vector **x** as if it were an $n \times 1$ matrix, then the product **Ax**, according to the matrix multiplication rule, yields the right-hand side of Eq. (B.31). Thus, we may multiply a matrix by a vector by treating the vector as if it were an $n \times 1$ matrix. Note that the constraint of conformability still applies. Thus, in this case, **xA** is not defined and is meaningless.

Matrix Inversion

To define the inverse of a matrix, let us consider the set of equations represented by Eq. (B.32) when $m = n$:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{B.34}$$

We can solve this set of equations for $x_1, x_2, \ldots, x_n$ in terms of $y_1, y_2, \ldots, y_n$ by using Cramer's rule [see Eq. (B.21)]. This yields

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{|\mathbf{D}_{11}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{21}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{n1}|}{|\mathbf{A}|} \\ \frac{|\mathbf{D}_{12}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{22}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{n2}|}{|\mathbf{A}|} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{|\mathbf{D}_{1n}|}{|\mathbf{A}|} & \frac{|\mathbf{D}_{2n}|}{|\mathbf{A}|} & \cdots & \frac{|\mathbf{D}_{nn}|}{|\mathbf{A}|} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{B.35}$$

in which $|\mathbf{A}|$ is the determinant of the matrix **A** and $|\mathbf{D}_{ij}|$ is the *cofactor* of element a_{ij} in the matrix **A**. The cofactor of element a_{ij} is given by $(-1)^{i+j}$ times the determinant of the

$(n-1) \times (n-1)$ matrix that is obtained when the ith row and the jth column in matrix $\mathbf{A}$ are deleted.

We can express Eq. (B.34) in compact matrix form as

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{B.36}$$

We now define $\mathbf{A}^{-1}$, the inverse of a square matrix $\mathbf{A}$, with the property

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \qquad \text{(unit matrix)}$$

Then, premultiplying both sides of Eq. (B.36) by $\mathbf{A}^{-1}$, we obtain

$$\mathbf{A}^{-1}\mathbf{y} = \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{I}\mathbf{x} = \mathbf{x}$$

or

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{B.37}$$

A comparison of Eq. (B.37) with Eq. (B.35) shows that

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} |\mathbf{D}_{11}| & |\mathbf{D}_{21}| & \cdots & |\mathbf{D}_{n1}| \\ |\mathbf{D}_{12}| & |\mathbf{D}_{22}| & \cdots & |\mathbf{D}_{n2}| \\ \vdots & \vdots & \cdots & \vdots \\ |\mathbf{D}_{1n}| & |\mathbf{D}_{2n}| & \cdots & |\mathbf{D}_{nn}| \end{bmatrix}$$

One of the conditions necessary for a unique solution of Eq. (B.34) is that the number of equations must equal the number of unknowns. This implies that the matrix $\mathbf{A}$ must be a square matrix. In addition, we observe from the solution as given in Eq. (B.35) that if the solution is to exist, $|\mathbf{A}| \neq 0$.† Therefore, the inverse exists only for a square matrix and only under the condition that the determinant of the matrix be nonzero. A matrix whose determinant is nonzero is a *nonsingular* matrix. Thus, an inverse exists only for a nonsingular, square matrix. Since $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, we further note that the matrices $\mathbf{A}$ and $\mathbf{A}^{-1}$ commute.‡

The operation of matrix division can be accomplished through matrix inversion.

EXAMPLE B.12 Computing the Inverse of a Matrix

Let us find $\mathbf{A}^{-1}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

† These two conditions imply that the number of equations is equal to the number of unknowns and that all the equations are independent.

‡ To prove $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, notice first that we define $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Thus, $\mathbf{IA} = \mathbf{AI} = \mathbf{A}(\mathbf{A}^{-1}\mathbf{A}) = (\mathbf{A}\mathbf{A}^{-1})\mathbf{A}$. Subtracting $(\mathbf{A}\mathbf{A}^{-1})\mathbf{A}$, we see that $\mathbf{IA} - (\mathbf{A}\mathbf{A}^{-1})\mathbf{A} = 0$ or $(\mathbf{I} - \mathbf{A}\mathbf{A}^{-1})\mathbf{A} = 0$. This requires $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

Here,

$$\begin{array}{lll} |\mathbf{D}_{11}| = -4, & |\mathbf{D}_{12}| = 8, & |\mathbf{D}_{13}| = -4 \\ |\mathbf{D}_{21}| = 1, & |\mathbf{D}_{22}| = -1, & |\mathbf{D}_{23}| = -1 \\ |\mathbf{D}_{31}| = 1, & |\mathbf{D}_{32}| = -5, & |\mathbf{D}_{33}| = 3 \end{array}$$

and $|\mathbf{A}| = -4$. Therefore,

$$\mathbf{A}^{-1} = -\frac{1}{4}\begin{bmatrix} -4 & 1 & 1 \\ 8 & -1 & -5 \\ -4 & -1 & 3 \end{bmatrix}$$

B.7 MATLAB: Elementary Operations

B.7-1 MATLAB Overview

Although MATLAB® (a registered trademark of The MathWorks, Inc.) is easy to use, it can be intimidating to new users. Over the years, MATLAB has evolved into a sophisticated computational package with thousands of functions and thousands of pages of documentation. This section provides a brief introduction to the software environment.

When MATLAB is first launched, its command window appears. When MATLAB is ready to accept an instruction or input, a command prompt (>>) is displayed in the command window. Nearly all MATLAB activity is initiated at the command prompt.

Entering instructions at the command prompt generally results in the creation of an object or objects. Many classes of objects are possible, including functions and strings, but usually objects are just data. Objects are placed in what is called the MATLAB workspace. If not visible, the workspace can be viewed in a separate window by typing `workspace` at the command prompt. The workspace provides important information about each object, including the object's name, size, and class.

Another way to view the workspace is the `whos` command. When `whos` is typed at the command prompt, a summary of the workspace is printed in the command window. The `who` command is a short version of `whos` that reports only the names of workspace objects.

Several functions exist to remove unnecessary data and help free system resources. To remove specific variables from the workspace, the `clear` command is typed, followed by the names of the variables to be removed. Just typing `clear` removes all objects from the workspace. Additionally, the `clc` command clears the command window, and the `clf` command clears the current figure window.

Often, important data and objects created in one session need to be saved for future use. The `save` command, followed by the desired filename, saves the entire workspace to a file, which has the .mat extension. It is also possible to selectively save objects by typing `save` followed by the filename and then the names of the objects to be saved. The `load` command followed by the filename is used to load the data and objects contained in a MATLAB data file (.mat file).

Although MATLAB does not automatically save workspace data from one session to the next, lines entered at the command prompt are recorded in the command history. Previous command lines can be viewed, copied, and executed directly from the command history window. From the command window, pressing the up or down arrow key scrolls through previous commands and redisplays them at the command prompt. Typing the first few characters and then pressing the arrow keys scrolls through the previous commands that start with the same characters. The arrow keys allow command sequences to be repeated without retyping.

Perhaps the most important and useful command for new users is `help`. To learn more about a function, simply type `help` followed by the function name. Helpful text is then displayed in the command window. The obvious shortcoming of `help` is that the function name must first be known. This is especially limiting for MATLAB beginners. Fortunately, help screens often conclude by referencing related or similar functions. These references are an excellent way to learn new MATLAB commands. Typing `help help`, for example, displays detailed information on the `help` command itself and also provides reference to relevant functions, such as the `lookfor` command. The `lookfor` command helps locate MATLAB functions based on a keyword search. Simply type `lookfor` followed by a single keyword, and MATLAB searches for functions that contain that keyword.

MATLAB also has comprehensive HTML-based help. The HTML help is accessed by using MATLAB's integrated help browser, which also functions as a standard web browser. The HTML help facility includes a function and topic index as well as full text-searching capabilities. Since HTML documents can contain graphics and special characters, HTML help can provide more information than the command-line help. After a little practice, it is easy to find information in MATLAB.

When MATLAB graphics are created, the `print` command can save figures in a common file format such as postscript, encapsulated postscript, JPEG, or TIFF. The format of displayed data, such as the number of digits displayed, is selected by using the `format` command. MATLAB help provides the necessary details for both these functions. When a MATLAB session is complete, the `exit` command terminates MATLAB.

B.7-2 Calculator Operations

MATLAB can function as a simple calculator, working as easily with complex numbers as with real numbers. Scalar addition, subtraction, multiplication, division, and exponentiation are accomplished using the traditional operator symbols `+`, `-`, `*`, `/`, and `^`. Since MATLAB predefines `i` $=$ `j` $= \sqrt{-1}$, a complex constant is readily created using Cartesian coordinates. For example,

```
>>  z = -3-4j
    z = -3.0000 - 4.0000i
```

assigns the complex constant $-3 - j4$ to the variable z.

The real and imaginary components of z are extracted by using the `real` and `imag` operators. In MATLAB, the input to a function is placed parenthetically following the function name.

```
>>  z_real = real(z); z_imag = imag(z);
```

When a command is terminated with a semicolon, the statement is evaluated but the results are not displayed to the screen. This feature is useful when one is computing intermediate results, and it allows multiple instructions on a single line. Although not displayed, the results `z_real = -3` and `z_imag = -4` are calculated and available for additional operations such as computing $|z|$.

There are many ways to compute the modulus, or magnitude, of a complex quantity. Trigonometry confirms that $z = -3 - j4$, which corresponds to a 3-4-5 triangle, has modulus

$|z| = |-3 - j4| = \sqrt{(-3)^2 + (-4)^2} = 5$. The MATLAB sqrt command provides one way to compute the required square root.

```
>>  z_mag = sqrt(z_real^2 + z_imag^2)
    z_mag = 5
```

In MATLAB, most commands, including sqrt, accept inputs in a variety of forms, including constants, variables, functions, expressions, and combinations thereof.

The same result is also obtained by computing $|z| = \sqrt{zz^*}$. In this case, complex conjugation is performed by using the conj command.

```
>>  z_mag = sqrt(z*conj(z))
    z_mag = 5
```

More simply, MATLAB computes absolute values directly by using the abs command.

```
>>  z_mag = abs(z)
    z_mag = 5
```

In addition to magnitude, polar notation requires phase information. The angle command provides the angle of a complex number.

```
>>  z_rad = angle(z)
    z_rad = -2.2143
```

MATLAB expects and returns angles in a radian measure. Angles expressed in degrees require an appropriate conversion factor.

```
>>  z_deg = angle(z)*180/pi
    z_deg = -126.8699
```

Notice, MATLAB predefines the variable pi $= \pi$.

It is also possible to obtain the angle of z using a two-argument arc-tangent function, atan2.

```
>>  z_rad = atan2(z_imag,z_real)
    z_rad = -2.2143
```

Unlike a single-argument arctangent function, the two-argument arctangent function ensures that the angle reflects the proper quadrant. MATLAB supports a full complement of trigonometric functions: standard trigonometric functions cos, sin, tan; reciprocal trigonometric functions sec, csc, cot; inverse trigonometric functions acos, asin, atan, asec, acsc, acot; and hyperbolic variations cosh, sinh, tanh, sech, csch, coth, acosh, asinh, atanh, asech, acsch, and acoth. Of course, MATLAB comfortably supports complex arguments for any trigonometric function. As with the angle command, MATLAB trigonometric functions utilize units of radians.

The concept of trigonometric functions with complex-valued arguments is rather intriguing. The results can contradict what is often taught in introductory mathematics courses. For example, a common claim is that $|\cos(x)| \leq 1$. While this is true for real x, it is not necessarily true for complex x. This is readily verified by example using MATLAB and the cos function.

```
>>  cos(1j)
    ans = 1.5431
```

Problem B.1-19 investigates these ideas further.

Similarly, the claim that it is impossible to take the logarithm of a negative number is false. For example, the principal value of $\ln(-1)$ is $j\pi$, a fact easily verified by means of Euler's equation. In MATLAB, base-10 and base-e logarithms are computed by using the `log10` and `log` commands, respectively.

```
>>  log(-1)
    ans = 0 + 3.1416i
```

B.7-3 Vector Operations

The power of MATLAB becomes apparent when vector arguments replace scalar arguments. Rather than computing one value at a time, a single expression computes many values. Typically, vectors are classified as row vectors or column vectors. For now, we consider the creation of row vectors with evenly spaced, real elements. To create such a vector, the notation `a:b:c` is used, where `a` is the initial value, `b` designates the step size, and `c` is the termination value. For example, `0:2:11` creates the length-6 vector of even-valued integers ranging from 0 to 10.

```
>>  k = 0:2:11
    k =  0    2    4    6    8   10
```

In this case, the termination value does not appear as an element of the vector. Negative and noninteger step sizes are also permissible.

```
>>  k = 11:-10/3:0
    k = 11.0000    7.6667    4.3333    1.0000
```

If a step size is not specified, a value of 1 is assumed.

```
>>  k = 0:11
    k =  0    1    2    3    4    5    6    7    8    9   10   11
```

Vector notation provides the basis for solving a wide variety of problems.

For example, consider finding the three cube roots of minus one, $w^3 = -1 = e^{j(\pi+2\pi k)}$ for integer k. Taking the cube root of each side yields $w = e^{j(\pi/3+2\pi k/3)}$. To find the three unique solutions, use any three consecutive integer values of k and MATLAB's `exp` function.

```
>>  k = 0:2; w = exp(1j*(pi/3 + 2*pi*k/3))
    w = 0.5000 + 0.8660i  -1.0000 + 0.0000i   0.5000 - 0.8660i
```

The solutions, particularly $w = -1$, are easy to verify.

Finding the 100 unique roots of $w^{100} = -1$ is just as simple.

```
>>  k = 0:99; w = exp(1j*(pi/100 + 2*pi*k/100));
```

A semicolon concludes the final instruction to suppress the inconvenient display of all 100 solutions. To view a particular solution, the user must use an index to specify desired elements.

MATLAB indices are integers that increase from a starting value of 1. For example, the fifth element of w is extracted using an index of 5.[†]

```
>>  w(5)
    ans = 0.9603 + 0.2790i
```

Notice that this solution corresponds to $k = 4$. The independent variable of a function, in this case k, rarely serves as the index. Since k is also a vector, it can likewise be indexed. In this way, we can verify that the fifth value of k is indeed 4.

```
>>  k(5)
    ans = 4
```

It is also possible to use a vector index to access multiple values. For example, index vector `98:100` identifies the last three solutions corresponding to $k = [97, 98, 99]$.

```
>>  w(98:100)
    ans = 0.9877 - 0.1564i   0.9956 - 0.0941i   0.9995 - 0.0314i
```

Vector representations provide the foundation to rapidly create and explore various signals. Consider the simple 10 Hz sinusoid described by $f(t) = \sin(2\pi 10t + \pi/6)$. Two cycles of this sinusoid are included in the interval $0 \leq t < 0.2$. A vector t is used to uniformly represent 500 points over this interval.

```
>>  t = 0:0.2/500:0.2-0.2/500;
```

Next, the function $f(t)$ is evaluated at these points.

```
>>  f = sin(2*pi*10*t+pi/6)
```

The value of $f(t)$ at $t = 0$ is the first element of the vector and is thus obtained by using an index of 1.

```
>>  f(1)
    ans = 0.5000
```

Unfortunately, MATLAB's indexing syntax conflicts with standard equation notation.[‡] That is, the MATLAB indexing command `f(1)` is not the same as the standard notation $f(1) = f(t)|_{t=1}$. Care must be taken to avoid confusion; remember that the index parameter rarely reflects the independent variable of a function.

B.7-4 Simple Plotting

MATLAB's `plot` command provides a convenient way to visualize data, such as graphing $f(t)$ against the independent variable t.

```
>>  plot(t,f);
```

[†] Some other programming languages, such as C, begin indexing at 0. Careful attention is warranted.
[‡] MATLAB anonymous functions, considered in Sec. 1.11, are an important and useful exception.

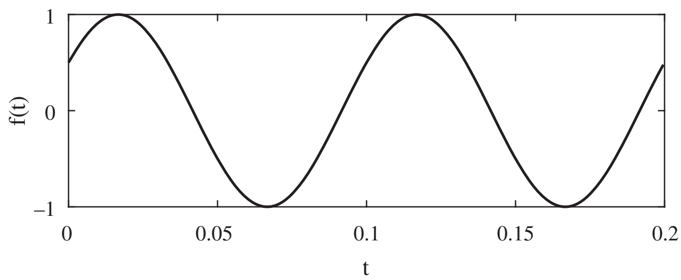

Figure B.12 $f(t) = \sin(2\pi 10t + \pi/6)$.

Axis labels are added using the `xlabel` and `ylabel` commands, where the desired string must be enclosed by single quotation marks. The result is shown in Fig. B.12.

```
>>  xlabel('t'); ylabel('f(t)')
```

The `title` command is used to add a title above the current axis.

By default, MATLAB connects data points with solid lines. Plotting discrete points, such as the 100 unique roots of $w^{100} = -1$, is accommodated by supplying the `plot` command with an additional string argument. For example, the string `'o'` tells MATLAB to mark each data point with a circle rather than connecting points with lines. A full description of the supported plot options is available from MATLAB's help facilities.

```
>>  plot(real(w),imag(w),'o');
>>  xlabel('Re(w)'); ylabel('Im(w)'); axis equal
```

The `axis equal` command ensures that the scale used for the horizontal axis is equal to the scale used for the vertical axis. Without `axis equal`, the plot would appear elliptical rather than circular. Figure B.13 illustrates that the 100 unique roots of $w^{100} = -1$ lie equally spaced on the unit circle, a fact not easily discerned from the raw numerical data.

MATLAB also includes many specialized plotting functions. For example, MATLAB commands `semilogx`, `semilogy`, and `loglog` operate like the `plot` command but use base-10 logarithmic scales for the horizontal axis, vertical axis, and the horizontal and vertical axes,

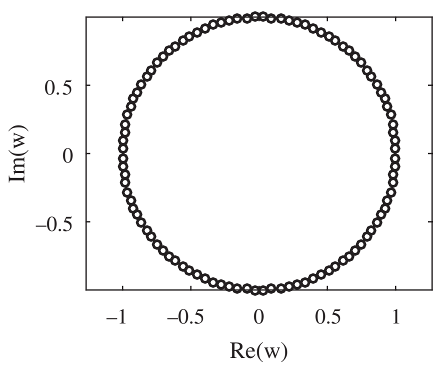

Figure B.13 Unique roots of $w^{100} = -1$.

respectively. Monochrome and color images can be displayed by using the `image` command, and contour plots are easily created with the `contour` command. Furthermore, a variety of three-dimensional plotting routines are available, such as `plot3`, `contour3`, `mesh`, and `surf`. Information about these instructions, including examples and related functions, is available from MATLAB help.

B.7-5 Element-by-Element Operations

Suppose a new function $h(t)$ is desired that forces an exponential envelope on the sinusoid $f(t)$, $h(t) = f(t)g(t)$, where $g(t) = e^{-10t}$. First, row vector $g(t)$ is created.

```
>>  g = exp(-10*t);
```

Given MATLAB's vector representation of $g(t)$ and $f(t)$, computing $h(t)$ requires some form of vector multiplication. There are three standard ways to multiply vectors: inner product, outer product, and element-by-element product. As a matrix-oriented language, MATLAB defines the standard multiplication operator `*` according to the rules of matrix algebra: the multiplicand must be conformable to the multiplier. A $1 \times N$ row vector times an $N \times 1$ column vector results in the scalar-valued inner product. An $N \times 1$ column vector times a $1 \times M$ row vector results in the outer product, which is an $N \times M$ matrix. Matrix algebra prohibits multiplication of two row vectors or multiplication of two column vectors. Thus, the `*` operator is not used to perform element-by-element multiplication.†

Element-by-element operations require vectors to have the same dimensions. An error occurs if element-by-element operations are attempted between row and column vectors. In such cases, one vector must first be transposed to ensure both vector operands have the same dimensions. In MATLAB, most element-by-element operations are preceded by a period. For example, element-by-element multiplication, division, and exponentiation are accomplished using `.*`, `./`, and `.^`, respectively. Vector addition and subtraction are intrinsically element-by-element operations and require no period. Intuitively, we know $h(t)$ should be the same size as both $g(t)$ and $f(t)$. Thus, $h(t)$ is computed using element-by-element multiplication.

```
>>  h = f.*g;
```

The `plot` command accommodates multiple curves and also allows modification of line properties. This facilitates side-by-side comparison of different functions, such as $h(t)$ and $f(t)$. Line characteristics are specified by using options that follow each vector pair and are enclosed in single quotes.

```
>>  plot(t,f,'-k',t,h,':k');
>>  xlabel('t'); ylabel('Amplitude');
>>  legend('f(t)','h(t)');
```

Here, `'-k'` instructs MATLAB to plot $f(t)$ using a solid black line, while `':k'` instructs MATLAB to use a dotted black line to plot $h(t)$. A legend and axis labels complete the plot, as shown in

† While grossly inefficient, element-by-element multiplication can be accomplished by extracting the main diagonal from the outer product of two N-length vectors.

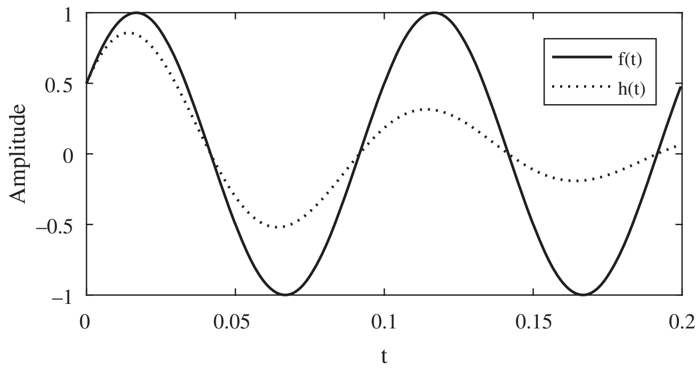

Figure B.14 Graphical comparison of $f(t)$ and $h(t)$.

Fig. B.14. It is also possible, although more cumbersome, to use pull down menus to modify line properties and to add labels and legends directly in the figure window.

B.7-6 Matrix Operations

Many applications require more than row vectors with evenly spaced elements; row vectors, column vectors, and matrices with arbitrary elements are typically needed.

MATLAB provides several functions to generate common, useful matrices. Given integers `m`, `n`, and vector `x`, the function `eye(m)` creates the $m \times m$ identity matrix; the function `ones(m,n)` creates the $m \times n$ matrix of all ones; the function `zeros(m,n)` creates the $m \times n$ matrix of all zeros; and the function `diag(x)` uses vector `x` to create a diagonal matrix. The creation of general matrices and vectors, however, requires each individual element to be specified.

Vectors and matrices can be input spreadsheet style by using MATLAB's array editor. This graphical approach is rather cumbersome and is not often used. A more direct method is preferable.

Consider a simple row vector $\mathbf{r}$,

$$\mathbf{r} = [\begin{array}{ccc} 1 & 0 & 0 \end{array}]$$

The MATLAB notation `a:b:c` cannot create this row vector. Rather, square brackets are used to create $\mathbf{r}$.

```
>>  r = [1 0 0]
    r = 1    0    0
```

Square brackets enclose elements of the vector, and spaces or commas are used to separate row elements.

Next, consider the 3×2 matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 0 & 6 \end{bmatrix}$$

Matrix **A** can be viewed as a three-high stack of two-element row vectors. With a semicolon to separate rows, square brackets are used to create the matrix.

```
>>  A = [2 3;4 5;0 6]
    A = 2    3
        4    5
        0    6
```

Each row vector needs to have the same length to create a sensible matrix.

In addition to enclosing string arguments, a single quote performs the complex conjugate transpose operation. In this way, row vectors become column vectors and vice versa. For example, a column vector **c** is easily created by transposing row vector **r**.

```
>>  c = r'
    c = 1
        0
        0
```

Since vector **r** is real, the complex-conjugate transpose is just the transpose. Had **r** been complex, the simple transpose could have been accomplished by either `r.'` or `(conj(r))'`.

More formally, square brackets are referred to as a concatenation operator. A concatenation combines or connects smaller pieces into a larger whole. Concatenations can involve simple numbers, such as the six-element concatenation used to create the 3×2 matrix **A**. It is also possible to concatenate larger objects, such as vectors and matrices. For example, vector **c** and matrix **A** can be concatenated to form a 3×3 matrix **B**.

```
>>  B = [c A]
    B = 1    2    3
        0    4    5
        0    0    6
```

Errors will occur if the component dimensions do not sensibly match; a 2×2 matrix would not be concatenated with a 3×3 matrix, for example.

Elements of a matrix are indexed much like vectors, except two indices are typically used to specify row and column.[†] Element (1, 2) of matrix **B**, for example, is 2.

```
>>  B(1,2)
    ans = 2
```

Indices can likewise be vectors. For example, vector indices allow us to extract the elements common to the first two rows and last two columns of matrix **B**.

```
>>  B(1:2,2:3)
    ans = 2    3
          4    5
```

[†] Matrix elements can also be accessed by means of a single index, which enumerates along columns. Formally, the element from row m and column n of an $M \times N$ matrix may be obtained with a single index $(n-1)M+m$. For example, element (1, 2) of matrix **B** is accessed by using the index $(2-1)3+1=4$. That is, `B(4)` yields 2.

One indexing technique is particularly useful and deserves special attention. A colon can be used to specify all elements along a specified dimension. For example, `B(2,:)` selects all column elements along the second row of **B**.

```
>>  B(2,:)
    ans = 0    4    5
```

Now that we understand basic vector and matrix creation, we turn our attention to using these tools on real problems. Consider solving a set of three linear simultaneous equations in three unknowns.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 1 \\ -\sqrt{3}x_1 + x_2 - \sqrt{5}x_3 &= \pi \\ 3x_1 - \sqrt{7}x_2 + x_3 &= e \end{aligned}$$

This system of equations is represented in matrix form according to $\mathbf{Ax} = \mathbf{y}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -\sqrt{3} & 1 & -\sqrt{5} \\ 3 & -\sqrt{7} & 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \qquad \mathbf{y} = \begin{bmatrix} 1 \\ \pi \\ e \end{bmatrix}$$

Although Cramer's rule can be used to solve $\mathbf{Ax} = \mathbf{y}$, it is more convenient to solve by multiplying both sides by the matrix inverse of $\mathbf{A}$. That is, $\mathbf{x} = \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{y}$. Solving for $\mathbf{x}$ by hand or by calculator would be tedious at best, so MATLAB is used. We first create $\mathbf{A}$ and $\mathbf{y}$.

```
>>  A = [1 -2 3;-sqrt(3) 1 -sqrt(5);3 -sqrt(7) 1]; y = [1;pi;exp(1)];
```

The vector solution is found by using MATLAB's `inv` function.

```
>>  x = inv(A)*y
    x = -1.9999
        -3.8998
        -1.5999
```

It is also possible to use MATLAB's left divide operator `x = A\y` to find the same solution. The left divide is generally more computationally efficient than matrix inverses. As with matrix multiplication, left division requires that the two arguments be conformable.

Of course, Cramer's rule can be used to compute individual solutions, such as x_1, by using vector indexing, concatenation, and MATLAB's `det` command to compute determinants.

```
>>  x1 = det([y,A(:,2:3)])/det(A)
    x1 = -1.9999
```

Another nice application of matrices is the simultaneous creation of a family of curves. Consider $h_\alpha(t) = e^{-\alpha t}\sin(2\pi 10t + \pi/6)$ over $0 \le t \le 0.2$. Figure B.14 shows $h_\alpha(t)$ for $\alpha = 0$ and $\alpha = 10$. Let's investigate the family of curves $h_\alpha(t)$ for $\alpha = [0, 1, \ldots, 10]$.

An inefficient way to solve this problem is create $h_\alpha(t)$ for each α of interest. This requires 11 individual cases. Instead, a matrix approach allows all 11 curves to be computed simultaneously. First, a vector is created that contains the desired values of α.

```
>>  alpha = (0:10);
```

By using a sampling interval of one millisecond, $\Delta t = 0.001$, a time vector is also created.

```
>>  t = (0:0.001:0.2)';
```

The result is a length-201 column vector. By replicating the time vector for each of the 11 curves required, a time matrix `T` is created. This replication can be accomplished by using an outer product between `t` and a 1×11 vector of ones.†

```
>>  T = t*ones(1,11);
```

The result is a 201×11 matrix that has identical columns. Right multiplying `T` by a diagonal matrix created from α, columns of `T` can be individually scaled and the final result is computed.

```
>>  H = exp(-T*diag(alpha)).*sin(2*pi*10*T+pi/6);
```

Here, `H` is a 201×11 matrix, where each column corresponds to a different value of α. That is, $\mathbf{H} = [\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{10}]$, where $\mathbf{h}_\alpha$ are column vectors. As shown in Fig. B.15, the 11 desired curves are simultaneously displayed by using MATLAB's plot command, which allows matrix arguments.

```
>>  plot(t,H); xlabel('t'); ylabel('h(t)');
```

This example illustrates an important technique called vectorization, which increases execution efficiency for interpretive languages such as MATLAB. Algorithm vectorization uses matrix and

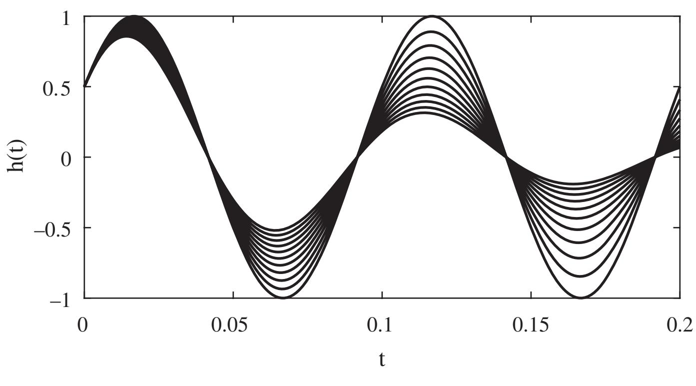

Figure B.15 $h_\alpha(t)$ for $\alpha = [0, 1, \ldots, 10]$.

† The `repmat` command provides a more flexible method to replicate or tile objects. Equivalently, `T = repmat(t,1,11)`.

vector operations to avoid manual repetition and loop structures. It takes practice and effort to become proficient at vectorization, but the worthwhile result is efficient, compact code.†

B.7-7 Partial Fraction Expansions

There are a wide variety of techniques and shortcuts to compute the partial fraction expansion of rational function $F(x) = B(x)/A(x)$, but few are more simple than the MATLAB `residue` command. The basic form of this command is

```
>>  [R,P,K] = residue(B,A)
```

The two input vectors `B` and `A` specify the polynomial coefficients of the numerator and denominator, respectively. These vectors are ordered in descending powers of the independent variable. Three vectors are output. The vector `R` contains the coefficients of each partial fraction, and vector `P` contains the corresponding roots of each partial fraction. For a root repeated r times, the r partial fractions are ordered in ascending powers. When the rational function is not proper, the vector `K` contains the direct terms, which are ordered in descending powers of the independent variable.

To demonstrate the power of the `residue` command, consider finding the partial fraction expansion of

$$F(x) = \frac{x^5+\pi}{\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)^3} = \frac{x^5+\pi}{x^4-\sqrt{8}x^3+\sqrt{32}x-4}$$

By hand, the partial fraction expansion of $F(x)$ is difficult to compute. MATLAB, however, makes short work of the expansion.

```
>>  [R,P,K] = residue([1 0 0 0 0 pi],[1 -sqrt(8) 0 sqrt(32) -4]); R.', P.', K
    R = 7.8888    5.9713    3.1107    0.1112
    P = 1.4142    1.4142    1.4142   -1.4142
    K = 1.0000    2.8284
```

Written in standard form, the partial fraction expansion of $F(x)$ is

$$F(x) = x + 2.8284 + \frac{7.8888}{x-\sqrt{2}} + \frac{5.9713}{(x-\sqrt{2})^2} + \frac{3.1107}{(x-\sqrt{2})^3} + \frac{0.1112}{x+\sqrt{2}}$$

The signal–processing toolbox function `residuez` is similar to the `residue` command and offers more convenient expansion of certain rational functions, such as those commonly encountered in the study of discrete-time systems. Additional information about the `residue` and `residuez` commands is available from MATLAB's help facilities.

† The benefits of vectorization are less pronounced in recent versions of MATLAB.

B.8 Appendix: Useful Mathematical Formulas

We conclude this chapter with a selection of useful mathematical facts.

B.8-1 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 \approx 0.30103$

$\log_{10} 3 \approx 0.47712$

B.8-2 Complex Numbers

$e^{\pm j\pi/2} = \pm j$

$e^{\pm jn\pi} = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

$e^{\pm j\theta} = \cos\theta \pm j \sin\theta$

$a + jb = re^{j\theta} \qquad r = \sqrt{a^2 + b^2},\ \theta = \tan^{-1}\left(\frac{b}{a}\right)$

$(re^{j\theta})^k = r^k e^{jk\theta}$

$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$

B.8-3 Sums

$$\sum_{k=m}^{n} r^k = \frac{r^{n+1} - r^m}{r - 1} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k\, r^k = \frac{r + [n(r-1) - 1]\, r^{n+1}}{(r-1)^2} \qquad r \neq 1$$

$$\sum_{k=0}^{n} k^2\, r^k = \frac{r[(1+r)(1-r^n) - 2n(1-r)r^n - n^2(1-r)^2 r^n]}{(1-r)^3} \qquad r \neq 1$$

B.8-4 Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0)$$

B.8-5 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$(1+x)^n \approx 1 + nx \qquad |x| \ll 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.8-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j \sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp \sin x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm \cos x$$

$$2 \sin x \cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$

$\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$

$a\cos x + b\sin x = C\cos(x+\theta) \qquad C = \sqrt{a^2+b^2},\ \theta = \tan^{-1}\left(\frac{-b}{a}\right)$

B.8-7 Common Derivative Formulas

$\dfrac{d}{dx}f(u) = \dfrac{d}{du}f(u)\dfrac{du}{dx}$

$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dx^n}{dx} = nx^{n-1}$

$\dfrac{d}{dx}\ln(ax) = \dfrac{1}{x}$

$\dfrac{d}{dx}\log(ax) = \dfrac{\log e}{x}$

$\dfrac{d}{dx}e^{bx} = be^{bx}$

$\dfrac{d}{dx}a^{bx} = b(\ln a)a^{bx}$

$\dfrac{d}{dx}\sin ax = a\cos ax$

$\dfrac{d}{dx}\cos ax = -a\sin ax$

$\dfrac{d}{dx}\tan ax = \dfrac{a}{\cos^2 ax}$

$\dfrac{d}{dx}(\sin^{-1} ax) = \dfrac{a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\cos^{-1} ax) = \dfrac{-a}{\sqrt{1-a^2x^2}}$

$\dfrac{d}{dx}(\tan^{-1} ax) = \dfrac{a}{1+a^2x^2}$

B.8-8 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)$$

B.8-9 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form $0/0$ or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.8-10 Solution of Quadratic and Cubic Equations

Any *quadratic* equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is provided by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A general *cubic* equation

$$y^3 + py^2 + qy + r = 0$$

may be reduced to the *depressed cubic* form

$$x^3 + ax + b = 0$$

by substituting

$$y = x - \frac{p}{3}$$

This yields

$$a = \tfrac{1}{3}(3q - p^2) \qquad b = \tfrac{1}{27}(2p^3 - 9pq + 27r)$$

Now let

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The solution of the depressed cubic is

$$x = A + B, \qquad x = -\frac{A+B}{2} + \frac{A-B}{2}\sqrt{-3}, \qquad x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{-3}$$

and

$$y = x - \frac{p}{3}$$

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