Solutions to Skill-Assessment Exercises

CHAPTER 2

2.1

The Laplace transform of t is $\frac{1}{s^2}$ using Table 2.1, Item 3. Using Table 2.2, Item 4, $F(s) = \frac{1}{(s+5)^2}$.

2.2

Expanding $F(s)$ by partial fractions yields:

$$F(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+3)^2} + \frac{D}{(s+3)}$$

where,

$$A = \left.\frac{10}{(s+2)(s+3)^2}\right|_{S\to 0} = \frac{5}{9} \quad B = \left.\frac{10}{s(s+3)^2}\right|_{S\to -2} = -5$$

$$C = \left.\frac{10}{s(s+2)}\right|_{S\to -3} = \frac{10}{3}, \text{ and } D = \left.(s+3)^2 \frac{dF(s)}{ds}\right|_{s\to -3} = \frac{40}{9}$$

Taking the inverse Laplace transform yields,

$$f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$$

2.3

Taking the Laplace transform of the differential equation assuming zero initial conditions yields:

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$

Collecting terms,

$$\left(s^3 + 3s^2 + 7s + 5\right)C(s) = \left(s^2 + 4s + 3\right)R(s)$$

Thus,

$$\frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

2.4

$$G(s) = \frac{C(s)}{R(s)} = \frac{2s+1}{s^2+6s+2}$$

Cross multiplying yields,

$$\frac{d^2c}{dt^2} + 6\frac{dc}{dt} + 2c = 2\frac{dr}{dt} + r$$

2.5

$$C(s) = R(s)G(s) = \frac{1}{s^2} * \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)} = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

where

$$A = \left.\frac{1}{(s+4)(s+8)}\right|_{S\to 0} = \frac{1}{32} \quad B = \left.\frac{1}{s(s+8)}\right|_{S\to -4} = -\frac{1}{16}, \text{ and } C = \left.\frac{1}{s(s+4)}\right|_{S\to -8} = \frac{1}{32}$$

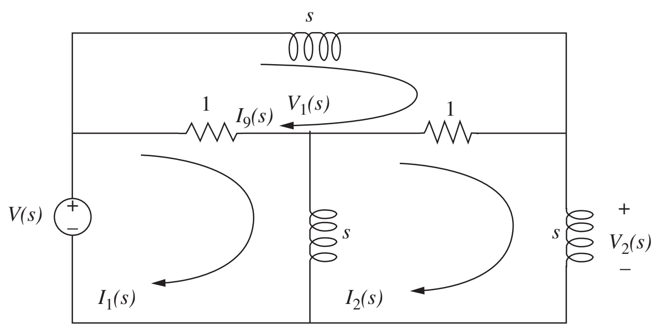
Thus,

$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

2.6

Mesh Analysis

Transforming the network yields,



Now, writing the mesh equations,

$$\begin{aligned}
&(s+1)I_1(s) - sI_2(s) - I_3(s) = V(s) \\
&-sI_1(s) + (2s+1)I_2(s) - I_3(s) = 0 \\
&-I_1(s) - I_2(s) + (s+2)I_3(s) = 0
\end{aligned}$$

Solving the mesh equations for $I_2(s)$,

$$I_2(s) = \frac{\begin{vmatrix} (s+1) & V(s) & -1 \\ -s & 0 & -1 \\ -1 & 0 & (s+2) \end{vmatrix}}{\begin{vmatrix} (s+1) & -s & -1 \\ -s & (2s+1) & -1 \\ -1 & -1 & (s+2) \end{vmatrix}} = \frac{(s^2+2s+1)V(s)}{s(s^2+5s+2)}$$

But, $V_L(s) = sI_2(s)$
Hence,

$$V_L(s) = \frac{\left(s^2+2s+1\right)V(s)}{\left(s^2+5s+2\right)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2+2s+1}{s^2+5s+2}$$

Nodal Analysis

Writing the nodal equations,

$$\left(\frac{1}{s}+2\right)V_1(s) - V_L(s) = V(s)$$
$$-V_1(s) + \left(\frac{2}{s}+1\right)V_L(s) = \frac{1}{s}V(s)$$

Solving for $V_L(s)$,

$$V_L(s) = \frac{\begin{vmatrix} \left(\frac{1}{s}+2\right) & V(s) \\ -1 & \frac{1}{s}V(s) \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{s}+2\right) & -1 \\ -1 & \left(\frac{2}{s}+1\right) \end{vmatrix}} = \frac{\left(s^2+2s+1\right)V(s)}{\left(s^2+5s+2\right)}$$

or

$$\frac{V_L(s)}{V(s)} = \frac{s^2+2s+1}{s^2+5s+2}$$

2.7

Inverting

$$G(s) = -\frac{Z_2(s)}{Z_1(s)} = \frac{-100000}{\left(10^5/s\right)} = -s$$

Noninverting

$$G(s) = \frac{[Z_1(s)+Z(s)]}{Z_1(s)} = \frac{\left(\frac{10^5}{s}+10^5\right)}{\left(\frac{10^5}{s}\right)} = s+1$$

2.8

Writing the equations of motion,

$$\left(s^2+3s+1\right)X_1(s) - (3s+1)X_2(s) = F(s)$$
$$-(3s+1)X_1(s) + \left(s^2+4s+1\right)X_2(s) = 0$$

Solving for $X_2(s)$,

$$X_2(s) = \frac{\begin{vmatrix} (s^2+3s+1) & F(s) \\ -(3s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+3s+1) & -(3s+1) \\ -(3s+1) & (s^2+4s+1) \end{vmatrix}} = \frac{(3s+1)F(s)}{s(s^3+7s^2+5s+1)}$$

Hence,

$$\frac{X_2(s)}{F(s)} = \frac{(3s+1)}{s(s^3+7s^2+5s+1)}$$

2.9

Writing the equations of motion,

$$(s^2+s+1)\theta_1(s) - (s+1)\theta_2(s) = T(s)$$
$$-(s+1)\theta_1(s) + (2s+2)\theta_2(s) = 0$$

where $\theta_1(s)$ is the angular displacement of the inertia.
Solving for $\theta_2(s)$,

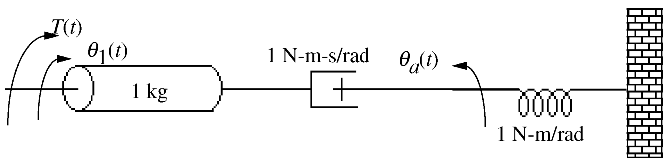
$$\theta_2(s) = \frac{\begin{vmatrix} (s^2+s+1) & T(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s+1) & -(s+1) \\ -(s+1) & (2s+2) \end{vmatrix}} = \frac{(s+1)T(s)}{2s^3+3s^2+2s+1}$$

From which, after simplification,

$$G(s) = \frac{\theta_2(s)}{T(s)} = \frac{1}{2s^2+s+1}$$

2.10

Transforming the network to one without gears by reflecting the 4 N-m/rad spring to the left and multiplying by $(25/50)^2$, we obtain,



Writing the equations of motion,

$$(s^2+s)\theta_1(s) - s\theta_a(s) = T(s)$$
$$-s\theta_1(s) + (s+1)\theta_a(s) = 0$$

where $\theta_1(s)$ is the angular displacement of the 1-kg inertia.
Solving for $\theta_a(s)$,

$$\theta_a(s) = \frac{\begin{vmatrix} (s^2+s) & T(s) \\ -s & 0 \end{vmatrix}}{\begin{vmatrix} (s^2+s) & -s \\ -s & (s+1) \end{vmatrix}} = \frac{sT(s)}{s^3+s^2+s}$$

From which,

$$\frac{\theta_a(s)}{T(s)} = \frac{1}{s^2 + s + 1}$$

But, $\theta_2(s) = \frac{1}{2}\theta_a(s)$.

Thus,

$$\frac{\theta_2(s)}{T(s)} = \frac{1/2}{s^2 + s + 1}$$

2.11

First find the mechanical constants.

$$J_m = J_a + J_L\left(\frac{1}{5} * \frac{1}{4}\right)^2 = 1 + 400\left(\frac{1}{400}\right) = 2$$

$$D_m = D_a + D_L\left(\frac{1}{5} * \frac{1}{4}\right)^2 = 5 + 800\left(\frac{1}{400}\right) = 7$$

Now find the electrical constants. From the torque-speed equation, set $\omega_{\mathrm{m}} = 0$ to find stall torque and set $T_m = 0$ to find no-load speed. Hence,

$$T_{stall} = 200$$
$$\omega_{no-load} = 25$$

which,

$$\frac{K_t}{R_a} = \frac{T_{stall}}{E_a} = \frac{200}{100} = 2$$
$$K_b = \frac{E_a}{\omega_{no-load}} = \frac{100}{25} = 4$$

Substituting all values into the motor transfer function,

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{K_T}{R_a J_m}}{s\left(s + \dfrac{1}{J_m}\right)\left(D_m + \dfrac{K_T K_b}{R_a}\right)} = \frac{1}{s\left(s + \dfrac{15}{2}\right)}$$

where $\theta_m(s)$ is the angular displacement of the armature.

Now $\theta_L(s) = \frac{1}{20}\theta_m(s)$. Thus,

$$\frac{\theta_L(s)}{E_a(s)} = \frac{1/20}{s\left(s + \dfrac{15}{2}\right)}$$

2.12

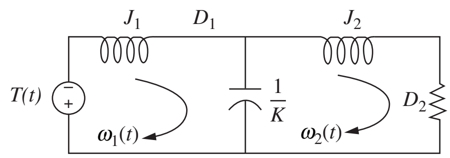
Letting

$$\theta_1(s) = \omega_1(s)/s$$
$$\theta_2(s) = \omega_2(s)/s$$

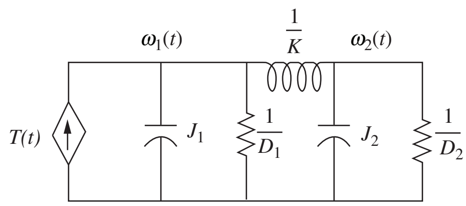
in Eqs. 2.127, we obtain

$$\left(J_1 s + D_1 + \frac{K}{s}\right)\omega_1(s) - \frac{K}{s}\omega_2(s) = T(s)$$
$$-\frac{K}{s}\omega_1(s) + \left(J_2 s + D_2 + \frac{K}{s}\right)\omega_2(s)$$

From these equations we can draw both series and parallel analogs by considering these to be mesh or nodal equations, respectively.



Series analog



Parallel analog

2.13

Writing the nodal equation,

$$C\frac{dv}{dt} + i_r - 2 = i(t)$$

But,

$$C = 1$$
$$v = v_o + \delta v$$
$$i_r = e^{v_r} = e^{v} = e^{v_o + \delta v}$$

Substituting these relationships into the differential equation,

$$\frac{d(v_o + \delta v)}{dt} + e^{v_o + \delta v} - 2 = i(t) \tag{1}$$

We now linearize e^v.

The general form is

$$f(v) - f(v_o) \approx \left.\frac{df}{dv}\right|_{v_o} \delta v$$

Substituting the function, $f(v) = e^v$, with $v = v_o + \delta v$ yields,

$$e^{v_o + \delta v} - e^{v_o} \approx \left.\frac{de^v}{dv}\right|_{v_o} \delta v$$

Solving for $e^{v_o+\delta v}$,

$$e^{v_o+\delta v} = e^{v_o} + \left.\frac{de^v}{dv}\right|_{v_o} \delta v = e^{v_o} + e^{v_o}\delta v$$

Substituting into Eq. (1)

$$\frac{d\delta v}{dt} + e^{v_o} + e^{v_o}\delta v - 2 = i(t) \tag{2}$$

Setting $i(t) = 0$ and letting the circuit reach steady state, the capacitor acts like an open circuit. Thus, $v_o = v_r$ with $i_r = 2$. But, $i_r = e^{v_r}$ or $v_r = \ln i_r$.

Hence, $v_o = \ln 2 = 0.693$. Substituting this value of v_o into Eq. (2) yields

$$\frac{d\delta v}{dt} + 2\delta v = i(t)$$

Taking the Laplace transform,

$$(s+2)\delta v(s) = I(s)$$

Solving for the transfer function, we obtain

$$\frac{\delta v(s)}{I(s)} = \frac{1}{s+2}$$

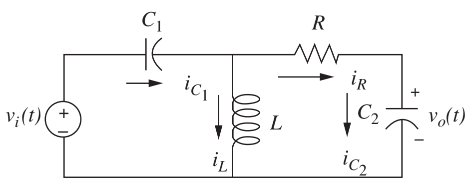
or

$\dfrac{V(s)}{I(s)} = \dfrac{1}{s+2}$ about equilibrium.

CHAPTER 3

3.1

Identifying appropriate variables on the circuit yields



Writing the derivative relations

$$C_1\frac{dv_{C_1}}{dt} = i_{C_1}$$
$$L\frac{di_L}{dt} = v_L \tag{1}$$
$$C_2\frac{dv_{C_2}}{dt} = i_{C_2}$$

Using Kirchhoff's current and voltage laws,

$$
\begin{aligned}
i_{C_1} &= i_L + i_R = i_L + \frac{1}{R}(v_L - v_{C_2}) \\
v_L &= -v_{C_1} + v_i \\
i_{C_2} &= i_R = \frac{1}{R}(v_L - v_{C_2})
\end{aligned}
$$

Substituting these relationships into Eqs. (1) and simplifying yields the state equations as

$$
\begin{aligned}
\frac{dv_{C_1}}{dt} &= -\frac{1}{RC_1}v_{C_1} + \frac{1}{C_1}i_L - \frac{1}{RC_1}v_{C_2} + \frac{1}{RC_1}v_i \\
\frac{di_L}{dt} &= -\frac{1}{L}v_{C_1} + \frac{1}{L}v_i \\
\frac{dv_{C_2}}{dt} &= -\frac{1}{RC_2}v_{C_1} - \frac{1}{RC_2}v_{C_2} + \frac{1}{RC_2}v_i
\end{aligned}
$$

where the output equation is

$$v_o = v_{C_2}$$

Putting the equations in vector-matrix form,

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{C_1} & -\frac{1}{RC_1} \\ -\frac{1}{L} & 0 & 0 \\ -\frac{1}{RC_2} & 0 & -\frac{1}{RC_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{L} \\ \frac{1}{RC_2} \end{bmatrix} v_i(t) \\
y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\mathbf{x}
\end{aligned}
$$

3.2

Writing the equations of motion

$$
\begin{aligned}
(s^2+s+1)X_1(s) \quad -sX_2(s) \quad &= F(s) \\
-sX_1(s) + (s^2+s+1)X_2(s) \quad -X_3(s) \quad &= 0 \\
-X_2(s) + (s^2+s+1)X_3(s) &= 0
\end{aligned}
$$

Taking the inverse Laplace transform and simplifying,

$$
\begin{aligned}
\ddot{x}_1 &= -\dot{x}_1 - x_1 + \dot{x}_2 + f \\
\ddot{x}_2 &= \dot{x}_1 - \dot{x}_2 - x_2 + x_3 \\
\ddot{x}_3 &= -\dot{x}_3 - x_3 + x_2
\end{aligned}
$$

Defining state variables, z_i,

$$z_1 = x_1; z_2 = \dot{x}_1; z_3 = x_2; z_4 = \dot{x}_2; z_5 = x_3; z_6 = \dot{x}_3$$

Writing the state equations using the definition of the state variables and the inverse transform of the differential equation,

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= \ddot{x}_1 = -\dot{x}_1 - x_1 + \dot{x}_2 + f = -z_2 - z_1 + z_4 + f \\
\dot{z}_3 &= \dot{x}_2 = z_4 \\
\dot{z}_4 &= \ddot{x}_2 = \dot{x}_1 - \dot{x}_2 - x_2 + x_3 = z_2 - z_4 - z_3 + z_5 \\
\dot{z}_5 &= \dot{x}_3 = z_6 \\
\dot{z}_6 &= \ddot{x}_3 = -\dot{x}_3 - x_3 + x_2 = -z_6 - z_5 + z_3
\end{aligned}$$

The output is z_5. Hence, $y = z_5$. In vector-matrix form,

$$\dot{\mathbf{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t); y = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]\mathbf{z}$$

3.3

First derive the state equations for the transfer function without zeros.

$$\frac{X(s)}{R(s)} = \frac{1}{s^2 + 7s + 9}$$

Cross multiplying yields

$$(s^2 + 7s + 9)X(s) = R(s)$$

Taking the inverse Laplace transform assuming zero initial conditions, we get

$$\ddot{x} + 7\dot{x} + 9x = r$$

Defining the state variables as,

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

Hence,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x} = -7\dot{x} - 9x + r = -9x_1 - 7x_2 + r \end{aligned}$$

Using the zeros of the transfer function, we find the output equation to be,

$$c = 2\dot{x} + x = x_1 + 2x_2$$

Putting all equation in vector-matrix form yields,

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -9 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r \\ c &= [1 \quad 2]\mathbf{x} \end{aligned}$$

3.4

The state equation is converted to a transfer function using

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} -4 & -1.5 \\ 4 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{C} = [1.5 \quad 0.625].$$

Evaluating $(s\mathbf{I} - \mathbf{A})$ yields

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s+4 & 1.5 \\ -4 & s \end{bmatrix}$$

Taking the inverse we obtain

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 4s + 6}\begin{bmatrix} s & -1.5 \\ 4 & s+4 \end{bmatrix}$$

Substituting all expressions into Eq. (1) yields

$$G(s) = \frac{3s + 5}{s^2 + 4s + 6}$$

3.5

Writing the differential equation we obtain

$$\frac{d^2x}{dt^2} + 2x^2 = 10 + \delta f(t) \tag{1}$$

Letting $x = x_o + \delta x$ and substituting into Eq. (1) yields

$$\frac{d^2(x_o + \delta x)}{dt^2} + 2(x_o + \delta x)^2 = 10 + \delta f(t) \tag{2}$$

Now, linearize x^2.

$$(x_o + \delta x)^2 - x_o^2 = \left.\frac{d(x^2)}{dx}\right|_{x_o} \delta x = 2x_o \delta x$$

from which

$$(x_o + \delta x)^2 = x_o^2 + 2x_o\delta x \tag{3}$$

Substituting Eq. (3) into Eq. (1) and performing the indicated differentiation gives us the linearized intermediate differential equation,

$$\frac{d^2\delta x}{dt^2} + 4x_o\delta x = -2x_o^2 + 10 + \delta f(t) \tag{4}$$

The force of the spring at equilibrium is 10 N. Thus, since $F = 2x^2, 10 = 2x_o^2$ from which

$$x_o = \sqrt{5}$$

Substituting this value of x_o into Eq. (4) gives us the final linearized differential equation.

$$\frac{d^2\delta x}{dt^2} + 4\sqrt{5}\,\delta x = \delta f(t)$$

Selecting the state variables,

$$\begin{aligned} x_1 &= \delta x \\ x_2 &= \dot{\delta}x \end{aligned}$$

Writing the state and output equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{\delta}x = -4\sqrt{5}x_1 + \delta f(t) \\ y &= x_1 \end{aligned}$$

Converting to vector-matrix form yields the final result as

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -4\sqrt{5} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta f(t) \\ y &= [1 \quad 0]\mathbf{x} \end{aligned}$$

CHAPTER 4

4.1

For a step input

$$C(s) = \frac{10(s+4)(s+6)}{s(s+1)(s+7)(s+8)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+7} + \frac{D}{s+8} + \frac{E}{s+10}$$

Taking the inverse Laplace transform,

$$c(t) = A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$$

4.2

Since $a = 50$, $T_c = \dfrac{1}{a} = \dfrac{1}{50} = 0.02\text{s}$; $T_s = \dfrac{4}{a} = \dfrac{4}{50} = 0.08$ s; and $T_r = \dfrac{2.2}{a} = \dfrac{2.2}{50} = 0.044$ s.

4.3

a. Since poles are at $-6 \pm j19.08$, $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$.
b. Since poles are at -78.54 and -11.46, $c(t) = A + Be^{-78.54t} + Ce^{-11.4t}$.
c. Since poles are double on the real axis at -15 $c(t) = A + Be^{-15t} + Cte^{-15t}$.
d. Since poles are at $\pm j25$, $c(t) = A + B\cos(25t + \phi)$.

4.4

a. $\omega_n = \sqrt{400} = 20$ and $2\zeta\omega_n = 12$; $\therefore \zeta = 0.3$ and system is underdamped.
b. $\omega_n = \sqrt{900} = 30$ and $2\zeta\omega_n = 90$; $\therefore \zeta = 1.5$ and system is overdamped.
c. $\omega_n = \sqrt{225} = 15$ and $2\zeta\omega_n = 30$; $\therefore \zeta = 1$ and system is critically damped.
d. $\omega_n = \sqrt{625} = 25$ and $2\zeta\omega_n = 0$; $\therefore \zeta = 0$ and system is undamped.

4.5

$$\omega_n = \sqrt{361} = 19 \text{ and } 2\zeta\omega_n = 16; \quad \therefore \zeta = 0.421.$$

Now, $T_s = \dfrac{4}{\zeta\omega_n} = 0.5\,s$ and $T_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.182$ s.

From Figure 4.16, $\omega_n T_r = 1.4998$. Therefore, $T_r = 0.079$ s.

Finally, $\%\text{os} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 = 23.3\%$

4.6

a. The second-order approximation is valid, since the dominant poles have a real part of -2 and the higher-order pole is at -15, i.e. more than five-times further.

b. The second-order approximation is not valid, since the dominant poles have a real part of -1 and the higher-order pole is at -4, i.e. not more than five-times further.

4.7

a. Expanding $\dfrac{1}{s}\,G(s)$ by partial fractions yields $G(s) = \dfrac{1}{s} + \dfrac{0.8942}{s+20} - \dfrac{1.5918}{s+10} - \dfrac{0.3023}{s+6.5}$. But -0.3023 is not an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is not valid.

b. Expanding $\dfrac{1}{s}\,G(s)$ by partial fractions yields $G(s) = \dfrac{1}{s} + \dfrac{0.9782}{s+20} - \dfrac{1.9078}{s+10} - \dfrac{0.0704}{s+6.5}$. Because 0.0704 is an order of magnitude less than residues of second-order terms (term 2 and 3). Therefore, a second-order approximation is valid.

4.8

See Figure 4.31 in the textbook for the Simulink block diagram and the output responses.

4.9

a. Since $s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}$, $(s\mathbf{I} - \mathbf{A})^{-1} = \dfrac{1}{s^2+5s+6}\begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}$. Also, $\mathbf{BU}(s) = \begin{bmatrix} 0 \\ 1/(s+1) \end{bmatrix}$.

The state vector is $\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{BU}(s)] = \dfrac{1}{(s+1)(s+2)(s+3)} \times \begin{bmatrix} 2(s^2+7s+7) \\ s^2-4s-6 \end{bmatrix}$. The output is $Y(s) = [1 \quad 3]\mathbf{X}(s) = \dfrac{5s^2+2s-4}{(s+1)(s+2)(s+3)} = -\dfrac{0.5}{s+1} - \dfrac{12}{s+2} + \dfrac{17.5}{s+3}$. Taking the inverse Laplace transform yields $y(t) = -0.5e^{-t} - 12e^{-2t} + 17.5e^{-3t}$.

b. The eigenvalues are given by the roots of $|s\mathbf{I} - \mathbf{A}| = s^2 + 5s + 6$, or -2 and -3.

4.10

a. Since $(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & -2 \\ 2 & s+5 \end{bmatrix}$, $(s\mathbf{I} - \mathbf{A})^{-1} = \dfrac{1}{s^2+5s+4}\begin{bmatrix} s+5 & 2 \\ -2 & s \end{bmatrix}$. Taking the Laplace transform of each term, the state transition matrix is given by

$$\Phi(t) = \begin{bmatrix} \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} & \frac{2}{3}e^{-t} - \frac{2}{3}e^{-4t} \\ -\frac{2}{3}e^{-t} + \frac{2}{3}e^{-4t} & -\frac{1}{3}e^{-t} + \frac{4}{3}e^{-4t} \end{bmatrix}.$$

b. Since $\Phi(t-\tau)=\begin{bmatrix}\frac{4}{3}e^{-(t-\tau)}-\frac{1}{3}e^{-4(t-\tau)} & \frac{2}{3}e^{-(t-\tau)}-\frac{2}{3}e^{-4(t-\tau)}\\ -\frac{2}{3}e^{-(t-\tau)}+\frac{2}{3}e^{-4(t-\tau)} & -\frac{1}{3}e^{-(t-\tau)}+\frac{4}{3}e^{-4(t-\tau)}\end{bmatrix}$ and

$$\mathbf{Bu}(\tau)=\begin{bmatrix}0\\ e^{-2\tau}\end{bmatrix},\ \Phi(t-\tau)\mathbf{Bu}(\tau)=\begin{bmatrix}\frac{2}{3}e^{-\tau}e^{-t}-\frac{2}{3}e^{2\tau}e^{-4t}\\ -\frac{1}{3}e^{-\tau}e^{-t}+\frac{4}{3}e^{2\tau}e^{-4t}\end{bmatrix}.$$

Thus, $\mathbf{x}(t)=\Phi(t)\mathbf{x}(0)+\int_0^t\Phi(t-\tau)$

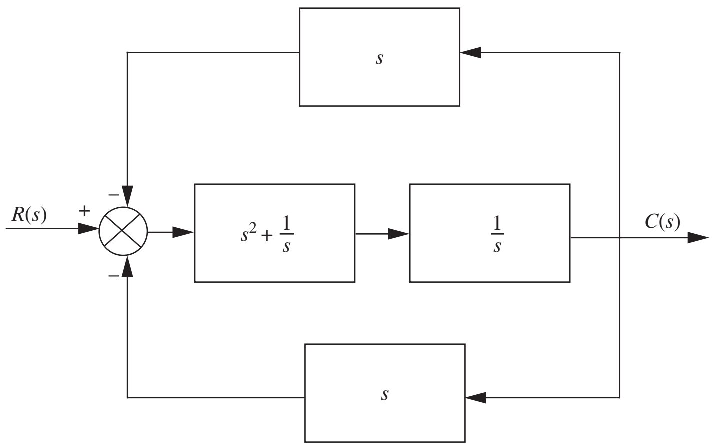
$$\mathbf{BU}(\tau)d\tau=\begin{bmatrix}\frac{10}{3}e^{-t}-e^{-2t}-\frac{4}{3}e^{-4t}\\ -\frac{5}{3}e^{-t}+e^{-2t}+\frac{8}{3}e^{-4t}\end{bmatrix}$$

c. $y(t)=[2\quad 1]\mathbf{x}=5e^{-t}-e^{-2t}$

CHAPTER 5

5.1

Combine the parallel blocks in the forward path. Then, push $\frac{1}{s}$ to the left past the pickoff point.



Combine the parallel feedback paths and get $2s$. Then, apply the feedback formula, simplify, and get, $T(s)=\dfrac{s^3+1}{2s^4+s^2+2s}$.

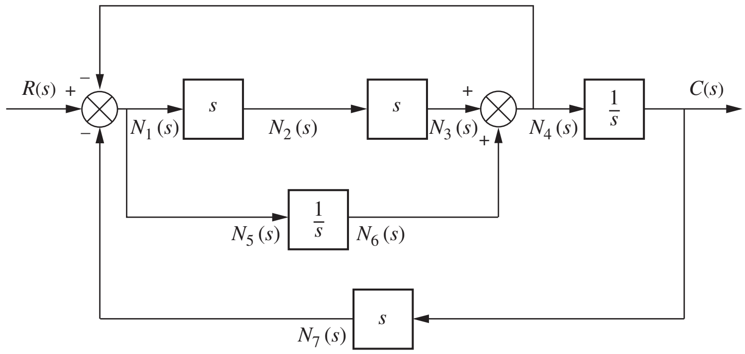
5.2

Find the closed-loop transfer function, $T(s)=\dfrac{G(s)}{1+G(s)H(s)}=\dfrac{16}{s^2+as+16}$, where and $G(s)=\dfrac{16}{s(s+a)}$ and $H(s)=1$. Thus, $\omega_n=4$ and $2\zeta\omega_n=a$, from which $\zeta=\dfrac{a}{8}$.

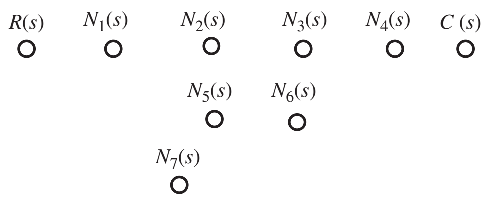
But, for 5% overshoot, $\zeta=\dfrac{-\ln\left(\frac{\%}{100}\right)}{\sqrt{\pi^2+\ln^2\left(\frac{\%}{100}\right)}}=0.69$. Since, $\zeta=\dfrac{a}{8}$, $a=5.52$.

5.3

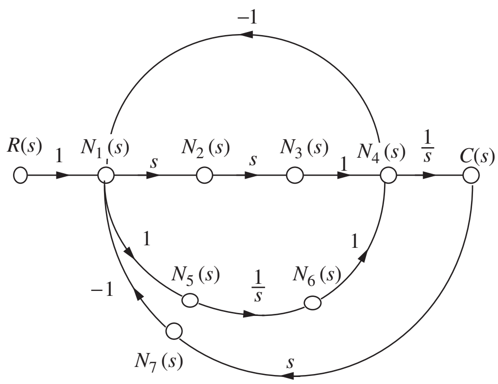
Label nodes.



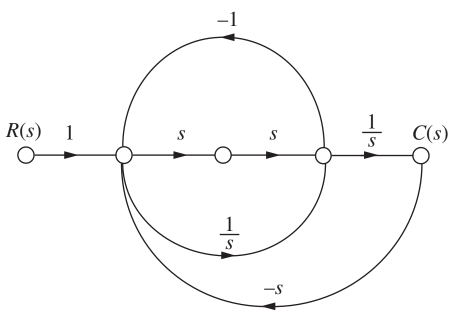
Draw nodes.



Connect nodes and label subsystems.



Eliminate unnecessary nodes.



5.4

Forward-path gains are $G_1G_2G_3$ and G_1G_3.
Loop gains are $-G_1G_2H_1$, $-G_2H_2$, and $-G_3H_3$.
Nontouching loops are $[-G_1G_2H_1][-G_3H_3] = G_1G_2G_3H_1H_3$ and $[-G_2H_2][-G_3H_3] = G_2G_3H_2H_3$.
Also, $\Delta = 1 + G_1G_2H_1 + G_2H_2 + G_3H_3 + G_1G_2G_3H_1H_3 + G_2G_3H_2H_3$.
Finally, $\Delta_1 = 1$ and $\Delta_2 = 1$.

Substituting these values into $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{\sum_k T_k\Delta_k}{\Delta}$ yields

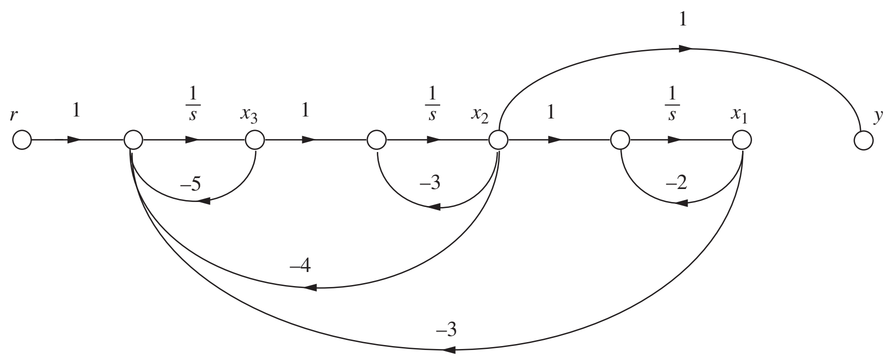
$$T(s) = \frac{G_1(s)G_3(s)[1 + G_2(s)]}{[1 + G_2(s)H_2(s) + G_1(s)G_2(s)H_1(s)][1 + G_3(s)H_3(s)]}$$

5.5

The state equations are,

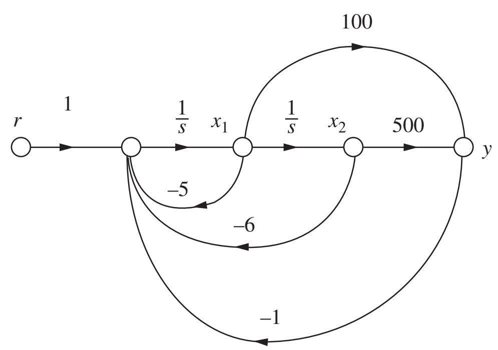
$$\begin{aligned}
\dot{x}_1 &= -2x_1 + x_2\\
\dot{x}_2 &= -3x_2 + x_3\\
\dot{x}_3 &= -3x_1 - 4x_2 - 5x_3 + r\\
y &= x_2
\end{aligned}$$

Drawing the signal-flow diagram from the state equations yields



5.6

From $G(s) = \dfrac{100(s+5)}{s^2 + 5s + 6}$ we draw the signal-flow graph in controller canonical form and add the feedback.



Writing the state equations from the signal-flow diagram, we obtain

$$\mathbf{x} = \begin{bmatrix} -105 & -506 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$$
$$y = [100 \quad 500]\mathbf{x}$$

5.7

From the transformation equations,

$$\mathbf{P}^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}$$

Taking the inverse,

$$\mathbf{P} = \begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix}$$

Now,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}$$
$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 3 & -2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$$
$$\mathbf{C}\mathbf{P} = [1 \quad 4]\begin{bmatrix} 0.4 & -0.2 \\ 0.1 & -0.3 \end{bmatrix} = [0.8 \quad -1.4]$$

Therefore,

$$\dot{\mathbf{z}} = \begin{bmatrix} 6.5 & -8.5 \\ 9.5 & -11.5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -3 \\ -11 \end{bmatrix} u$$
$$y = [0.8 \quad -1.4]\mathbf{z}$$

5.8

First find the eigenvalues.

$$|\lambda\mathbf{I} - \mathbf{A}| = \left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\right| = \begin{vmatrix} \lambda - 1 & -3 \\ 4 & \lambda + 6 \end{vmatrix} = \lambda^2 + 5\lambda + 6$$

From which the eigenvalues are -2 and -3.

Now use $\mathbf{A}x_i = \lambda x_i$ for each eigenvalue, λ.

Thus,

$$\begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For $\lambda = -2$,

$$3x_1 + 3x_2 = 0$$
$$-4x_1 - 4x_2 = 0$$

Thus $x_1 = -x_2$

For $\lambda = -3$

$$\begin{aligned} 4x_1 + 3x_2 &= 0 \\ -4x_1 - 3x_2 &= 0 \end{aligned}$$

Thus $x_1 = -x_2$ and $x_1 = -0.75x_2$; from which we let

$$\mathbf{P} = \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix}$$

Taking the inverse yields

$$\mathbf{P}^{-1} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix}$$

Hence,

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\mathbf{P}^{-1}\mathbf{B} = \begin{bmatrix} 5.6577 & 4.2433 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 18.39 \\ 20 \end{bmatrix}$$

$$\mathbf{CP} = [1 \quad 4] \begin{bmatrix} 0.707 & -0.6 \\ -0.707 & 0.8 \end{bmatrix} = [-2.121 \quad 2.6]$$

Finally,

$$\dot{\mathbf{z}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 18.39 \\ 20 \end{bmatrix} u$$

$$\mathbf{y} = [-2.121 \quad 2.6]\mathbf{z}$$

CHAPTER 6

6.1

Make a Routh table.

s^7	3	6	7	2
s^6	9	4	8	6
s^5	4.666666667	4.333333333	0	0
s^4	−4.35714286	8	6	0
s^3	12.90163934	6.426229508	0	0
s^2	10.17026684	6	0	0
s^1	−1.18515742	0	0	0
s^0	6	0	0	0

Since there are four sign changes and no complete row of zeros, there are four right half-plane poles and three left half-plane poles.

6.2

Make a Routh table. We encounter a row of zeros on the s^3 row. The even polynomial is contained in the previous row as $-6s^4 + 0s^2 + 6$. Taking the derivative yields

$-24s^3 + 0s$. Replacing the row of zeros with the coefficients of the derivative yields the s^3 row. We also encounter a zero in the first column at the s^2 row. We replace the zero with ε and continue the table. The final result is shown now as

s^6	1	−6	−1	6	
s^5	1	0	−1	0	
s^4	−6	0	6	0	
s^3	−24	0	0	0	ROZ
s^2	ε	6	0	0	
s^1	$144/\varepsilon$	0	0	0	
s^0	6	0	0	0	

There is one sign change below the even polynomial. Thus the even polynomial (4^{th} order) has one right half-plane pole, one left half-plane pole, and 2 imaginary axis poles. From the top of the table down to the even polynomial yields one sign change. Thus, the rest of the polynomial has one right half-plane root, and one left half-plane root. The total for the system is two right half-plane poles, two left half-plane poles, and 2 imaginary poles.

6.3

Since $G(s) = \dfrac{K(s+20)}{s(s+2)(s+3)}$, $T(s) = \dfrac{G(s)}{1+G(s)} = \dfrac{K(s+20)}{s^3+5s^2+(6+K)s+20K}$

Form the Routh table.

s^3	1	$(6+K)$
s^2	5	$20K$
s^1	$\dfrac{30-15K}{5}$	
s^0	$20K$	

From the s^1 row, $K < 2$. From the s^0 row, $K > 0$. Thus, for stability, $0 < K < 2$.

6.4

First find

$$|s\mathbf{I}-\mathbf{A}| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix}\right| = \begin{vmatrix} (s-2) & -1 & -1 \\ -1 & (s-7) & -1 \\ 3 & -4 & (s+5) \end{vmatrix}$$

$$= s^3 - 4s^2 - 33s + 51$$

Now form the Routh table.

s^3	1	−33
s^2	−4	51
s^1	−20.25	
s^0	51	

There are two sign changes. Thus, there are two rhp poles and one lhp pole.

CHAPTER 7

7.1

a. First check stability.

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{10s^2+500s+6000}{s^3+70s^2+1375s+6000}=\frac{10(s+30)(s+20)}{(s+26.03)(s+37.89)(s+6.085)}$$

Poles are in the lhp. Therefore, the system is stable. Stability also could be checked via Routh-Hurwitz using the denominator of $T(s)$. Thus,

$$15u(t): \qquad e_{step}(\infty)=\frac{15}{1+\lim\limits_{s\to 0}G(s)}=\frac{15}{1+\infty}=0$$

$$15tu(t): \qquad e_{ramp}(\infty)=\frac{15}{\lim\limits_{s\to 0}sG(s)}=\frac{15}{\frac{10^*20^*30}{25^*35}}=2.1875$$

$$15t^2u(t): \quad e_{parabola}(\infty)=\frac{15}{\lim\limits_{s\to 0}s^2G(s)}=\frac{30}{0}=\infty,\ \text{since } L\left[15t^2\right]=\frac{30}{s^3}$$

b. First check stability.

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{10s^2+500s+6000}{s^5+110s^4+3875s^3+4.37e04s^2+500s+6000}$$
$$=\frac{10(s+30)(s+20)}{(s+50.01)(s+35)(s+25)(s^2-7.189e-04s+0.1372)}$$

From the second-order term in the denominator, we see that the system is unstable. Instability could also be determined using the Routh-Hurwitz criteria on the denominator of $T(s)$. Since the system is unstable, calculations about steady-state error cannot be made.

7.2

a. The system is stable, since

$$T(s)=\frac{G(s)}{1+G(s)}=\frac{1000(s+8)}{(s+9)(s+7)+1000(s+8)}=\frac{1000(s+8)}{s^2+1016s+8063}$$

and is of Type 0. Therefore,

$$K_p=\lim_{s\to 0}G(s)=\frac{1000^*8}{7^*9}=127;\ K_v=\lim_{s\to 0}sG(s)=0;$$

and $K_a=\lim\limits_{s\to 0}s^2G(s)=0$

b.

$$e_{step}(\infty)=\frac{1}{1+\lim\limits_{s\to 0}G(s)}=\frac{1}{1+127}=7.8e-03$$

$$e_{ramp}(\infty)=\frac{1}{\lim\limits_{s\to 0}sG(s)}=\frac{1}{0}=\infty$$

$$e_{parabola}(\infty)=\frac{1}{\lim\limits_{s\to 0}s^2G(s)}=\frac{1}{0}=\infty$$

7.3

System is stable for positive K. System is Type 0. Therefore, for a step input $e_{step}(\infty) = \frac{1}{1+K_p} = 0.1$. Solving for K_p yields $K_p = 9 = \lim_{s\to 0} G(s) = \frac{12K}{14^*18}$; from which we obtain $K = 189$.

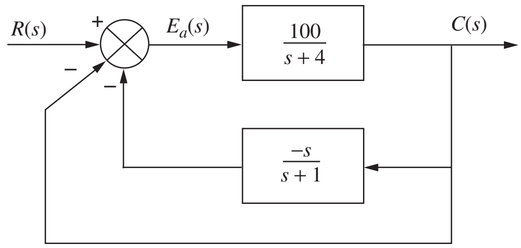
7.4

System is stable. Since $G_1(s) = 1000$, and $G_2(s) = \frac{(s+2)}{(s+4)}$,

$$e_D(\infty) = -\frac{1}{\lim_{s\to 0}\frac{1}{G_2(s)} + \lim_{s\to 0} G_1(s)} = \frac{1}{2+1000} = -9.98e-04$$

7.5

System is stable. Create a unity-feedback system, where $H_e(s) = \frac{1}{s+1} - 1 = \frac{-s}{s+1}$ The system is as follows:



Thus,

$$G_e(s) = \frac{G(s)}{1+G(S)H_e(s)} = \frac{\frac{100}{(s+4)}}{1-\frac{100s}{(s+1)(s+4)}} = \frac{100(s+1)}{S^2-95s+4}$$

Hence, the system is Type 0. Evaluating K_p yields

$$K_p = \frac{100}{4} = 25$$

The steady-state error is given by

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+25} = 3.846e-02$$

7.6

Since $G(s) = \frac{K(s+7)}{s^2+2s+10}$, $e(\infty) = \frac{1}{1+K_p} = \frac{1}{1+\frac{7K}{10}} = \frac{10}{10+7K}$.

Calculating the sensitivity, we get

$$S_{e:K} = \frac{K}{e}\frac{\partial e}{\partial K} = \frac{K}{\left(\frac{10}{10+7K}\right)}\frac{(-10)7}{(10+7K)^2} = -\frac{7K}{10+7K}$$

7.7

Given

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}; \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ \mathbf{C} = [1 \quad 1]; \ \mathrm{R(s)} = \frac{1}{\mathrm{s}}.$$

Using the final value theorem,

$$e_{step}(\infty) = \lim_{s\to 0} sR(s)\left[1 - \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right] = \lim_{s\to 0}\left[1 - [1 \quad 1]\begin{bmatrix} s & -1 \\ 3 & s+6 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$= \lim_{s\to 0}\left[1 - [1 \quad 1]\frac{\begin{bmatrix} s+6 & 1 \\ -3s & s \end{bmatrix}}{s^2 + 6s + 3}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \lim_{s\to 0}\frac{s^2 + 5s + 2}{s^2 + 6s + 3} = \frac{2}{3}$$

Using input substitution,

$$_{step}(\infty) = 1 + \mathbf{CA}^{-1}\mathbf{B} = 1 - [1 \quad 1]\begin{bmatrix} 0 & 1 \\ -3 & -6 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 1 + [1 \quad 1]\frac{\begin{bmatrix} -6 & -1 \\ 3 & 0 \end{bmatrix}}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 + [1 \quad 1]\begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} = \frac{2}{3}$$

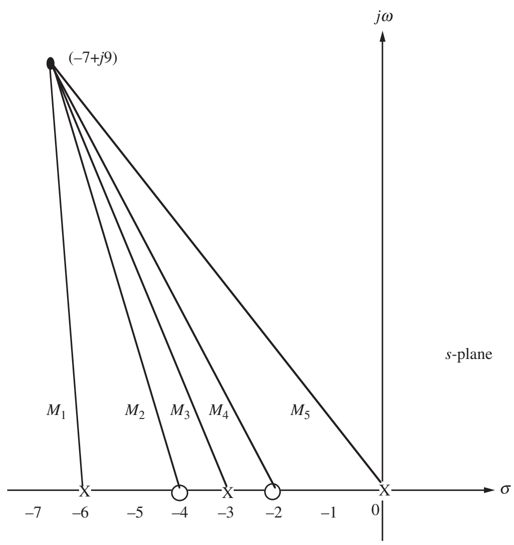
CHAPTER 8

8.1

a. $$F(-7 + j9) = \frac{(-7 + j9 + 2)(-7 + j9 + 4)0.0339}{(-7 + j9)(-7 + j9 + 3)(-7 + j9 + 6)} = \frac{(-5 + j9)(-3 + j9)}{(-7 + j9)(-4 + j9)(-1 + j9)}$$

$$= \frac{(-66 - j72)}{(944 - j378)} = -0.0339 - j0.0899 = 0.096 < -110.7^\circ$$

b. The arrangement of vectors is shown as follows:



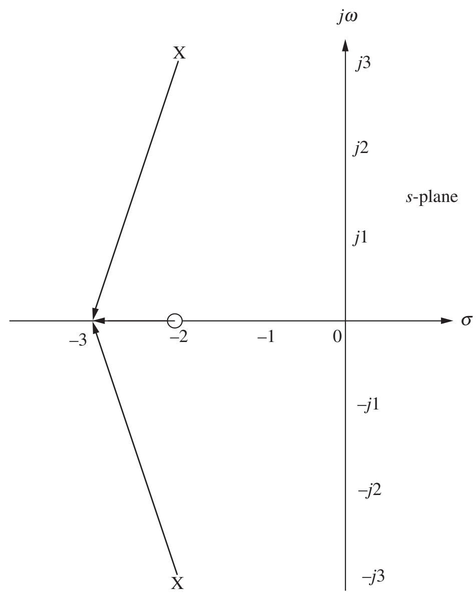
From the diagram,

$$F(-7+j9) = \frac{M_2 M_4}{M_1 M_3 M_5} = \frac{(-3+j9)(-5+j9)}{(-1+j9)(-4+j9)(-7+j9)} = \frac{(-66-j72)}{(944-j378)}$$

$$= -0.0339 - j0.0899 = 0.096 <; -110.7^\circ$$

8.2

a. First draw the vectors.



From the diagram,

$$\sum \text{angles} = 180^\circ - \tan^{-1}\left(\frac{-3}{-1}\right) - \tan^{-1}\left(\frac{-3}{1}\right) = 180^\circ - 108.43^\circ + 108.43^\circ = 180^\circ.$$

b. Since the angle is 180°, the point is on the root locus.

c. $K = \dfrac{\Pi\,\text{pole lengths}}{\Pi\,\text{zero lengths}} = \dfrac{\left(\sqrt{1^2+3^2}\right)\left(\sqrt{1^2+3^2}\right)}{1} = 10$

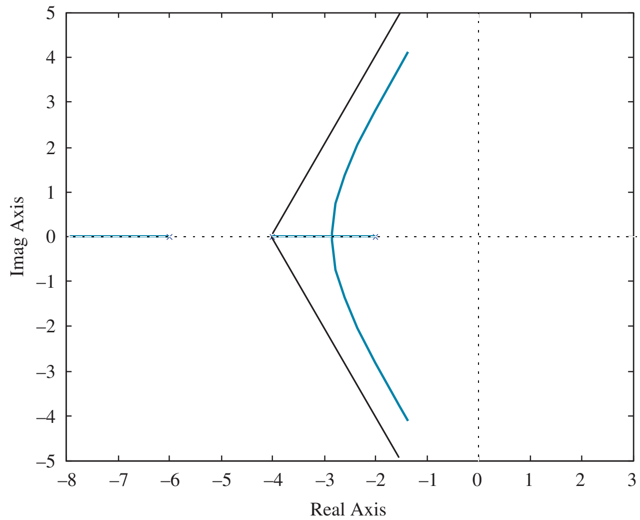
8.3

First, find the asymptotes.

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{\#\,\text{poles} - \#\,\text{zeros}} = \frac{(-2-4-6)-(0)}{3-0} = -4$$

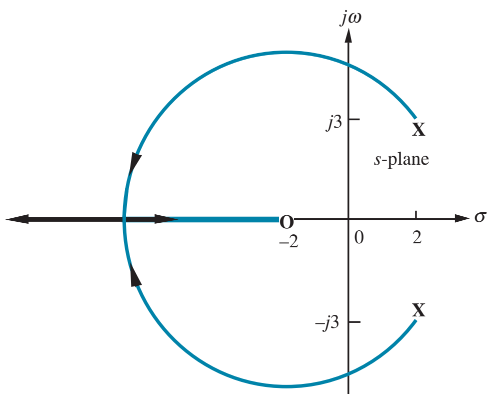
$$\theta_a = \frac{(2k+1)\pi}{3} = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$$

Next draw root locus following the rules for sketching.



8.4

a.



b. Using the Routh-Hurwitz criteria, we first find the closed-loop transfer function.

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K(s+2)}{s^2+(K-4)s+(2K+13)}$$

Using the denominator of T(s), make a Routh table.

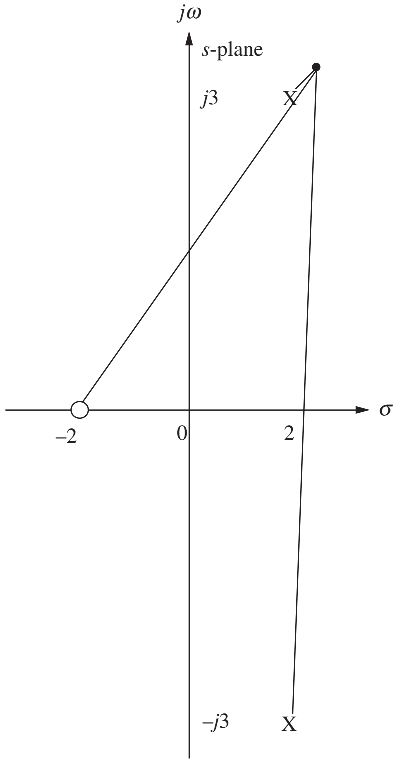
s^2	1	$2K+13$
s^1	$K-40$	0
s^0	$2K+13$	0

We get a row of zeros for $K=4$. From the s^2 row with $K=4$, $s^2+21=0$. From which we evaluate the imaginary axis crossing at $\sqrt{21}$.

c. From part (b), $K=4$.

d. Searching for the minimum gain to the left of -2 on the real axis yields -7 at a gain of 18. Thus the break-in point is at -7.

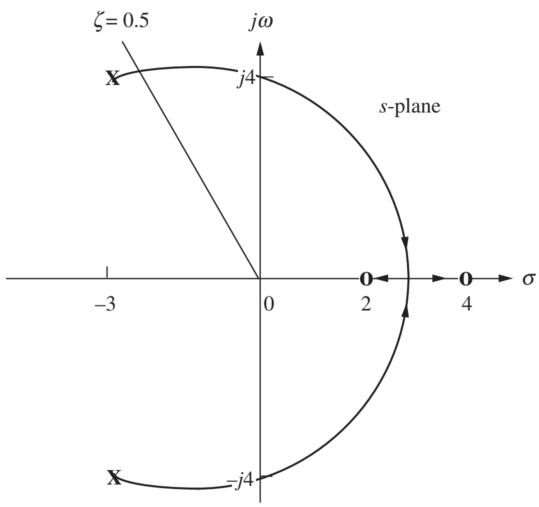
e. First, draw vectors to a point ε close to the complex pole.



At the point ε close to the complex pole, the angles must add up to zero. Hence, angle from zero – angle from pole in 4th quadrant – angle from pole in 1st quadrant = 180°, or $\tan^{-1}\left(\frac{3}{4}\right) - 90^\circ - \theta = 180^\circ$. Solving for the angle of departure, $\theta = -233.1$.

8.5

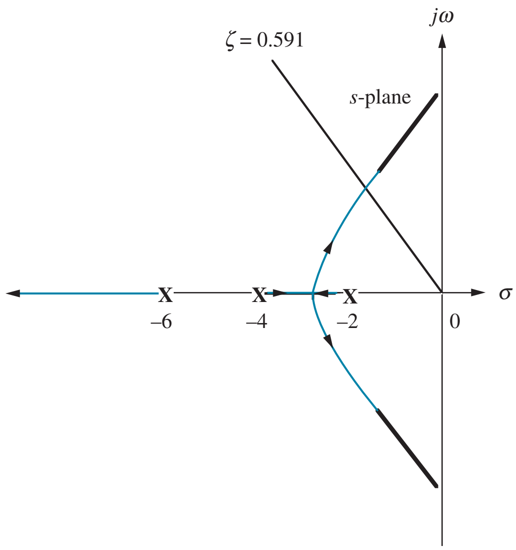
a.



b. Search along the imaginary axis and find the 180° point at $s = \pm j4.06$.
c. For the result in part (b), $K = 1$.
d. Searching between 2 and 4 on the real axis for the minimum gain yields the break-in at $s = 2.89$.
e. Searching along $\zeta = 0.5$ for the 180° point we find $s = -2.42 + j4.18$.
f. For the result in part (*e*), $K = 0.108$.
g. Using the result from part (*c*) and the root locus, $K < 1$.

8.6

a.



b. Searching along the $\zeta = 0.591$ (10% overshoot) line for the 180° point yields $-2.028 + j2.768$ with $K = 45.55$.

c. $T_s = \dfrac{4}{|\text{Re}|} = \dfrac{4}{2.028} = 1.97\,s$; $T_p = \dfrac{\pi}{|\text{Im}|} = \dfrac{\pi}{2.768} = 1.13\,s$; $\omega_n T_r = 1.8346$ from the rise-time chart and graph in Chapter 4. Since ω_n is the radial distance to the pole, $\omega_n = \sqrt{2.028^2 + 2.768^2} = 3.431$. Thus, $T_r = 0.53\,s$; since the system is Type 0, $K_p = \dfrac{K}{2^*4^*6} = \dfrac{45.55}{48} = 0.949$. Thus,

$$e_{step}(\infty) = \frac{1}{1 + K_p} = 0.51.$$

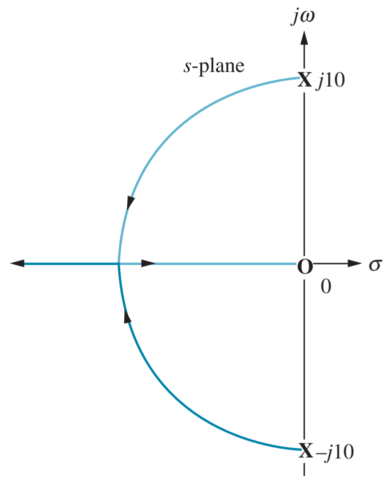
d. Searching the real axis to the left of –6 for the point whose gain is 45.55, we find -7.94. Comparing this value to the real part of the dominant pole, -2.028, we find that it is not five times further. The second-order approximation is not valid.

8.7

Find the closed-loop transfer function and put it the form that yields p_i as the root locus variable. Thus,

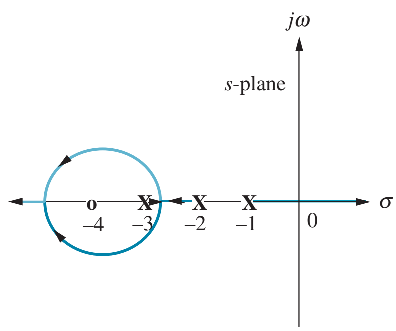
$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{100}{s^2 + p_i s + 100} = \frac{100}{(s^2 + 100) + p_i s} = \frac{\dfrac{100}{s^2 + 100}}{1 + \dfrac{p_i s}{s^2 + 100}}$$

Hence, $KG(s)H(s) = \dfrac{p_i s}{s^2 + 100}$. The following shows the root locus.



8.8

Following the rules for plotting the root locus of positive-feedback systems, we obtain the following root locus:



8.9

The closed-loop transfer function is $T(s) = \dfrac{K(s+1)}{s^2 + (K+2)s + K}$. Differentiating the denominator with respect to K yields

$$2s\frac{\partial s}{\partial K} + (K+2)\frac{\partial s}{\partial K} + (s+1) = (2s+K+2)\frac{\partial s}{\partial K} + (s+1) = 0$$

Solving for $\dfrac{\partial s}{\partial K}$, we get $\dfrac{\partial s}{\partial K} = \dfrac{-(s+1)}{(2s+K+2)}$. Thus, $S_{s:K} = \dfrac{K}{s}\dfrac{\partial s}{\partial K} = \dfrac{-K(s+1)}{s(2s+K+2)}$.

Substituting $K = 20$ yields $S_{s:K} = \dfrac{-10(s+1)}{s(s+11)}$.

Now find the closed-loop poles when $K = 20$. From the denominator of $T(s)$, $s_{1,2} = -21.05, \ -0.95$, when $K = 20$.

For the pole at -21.05,

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = -12.05\left(\frac{-10(-21.05+1)}{-21.05(-21.05+11)}\right)0.05 = -0.9975.$$

For the pole at −0.95,

$$\Delta s = s(S_{s:K})\frac{\Delta K}{K} = 0.95\left(\frac{-10(-0.95+1)}{-0.95(-0.95+11)}\right)0.05 = -0.0025.$$

CHAPTER 9

9.1

a. Searching along the 15% overshoot line, we find the point on the root locus at $-3.5 + j5.8$ at a gain of $K = 45.84$. Thus, for the uncompensated system, $K_v = \lim_{s\to 0} sG(s) = K/7 = 45.84/7 = 6.55$.

Hence, $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1527$.

b. Compensator zero should be 20x further to the left than the compensator pole. Arbitrarily select $G_c(s) = \dfrac{(s+0.2)}{(s+0.01)}$.

c. Insert compensator and search along the 15% overshoot line and find the root locus at $-3.4 + j5.63$ with a gain, $K = 44.64$. Thus, for the compensated system, $K_v = \dfrac{44.64(0.2)}{(7)(0.01)} = 127.5$ and $e_{ramp\_compensated}(\infty) = \dfrac{1}{K_v} = 0.0078$.

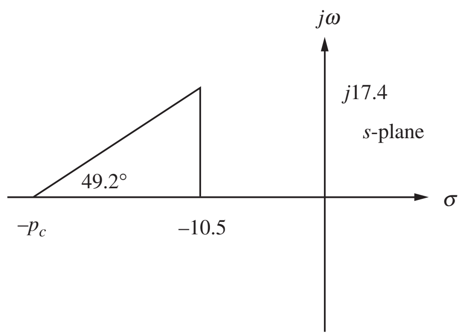
d. $\dfrac{e_{ramp\_uncompensated}}{e_{ramp\_compensated}} = \dfrac{0.1527}{0.0078} = 19.58$

9.2

a. Searching along the 15% overshoot line, we find the point on the root locus at $-3.5 + j5.8$ at a gain of $K = 45.84$. Thus, for the uncompensated system,

$$T_s = \frac{4}{|Re|} = \frac{4}{3.5} = 1.143\, s.$$

b. The real part of the design point must be three times larger than the uncompensated pole's real part. Thus the design point is $3(-3.5)+\ j3(5.8) = -10.5 + j17.4$. The angular contribution of the plant's poles and compensator zero at the design point is 130.8°. Thus, the compensator pole must contribute $180° - 130.8° = 49.2°$. Using the following diagram,



we find $\dfrac{17.4}{P_c - 10.5} = \tan 49.2°$, from which, $p_c = 25.52$. Adding this pole, we find the gain at the design point to be $K = 476.3$. A higher-order closed-loop pole is found to be at −11.54. This pole may not be close enough to the closed-loop zero at −10. Thus, we should simulate the system to be sure the design requirements have been met.

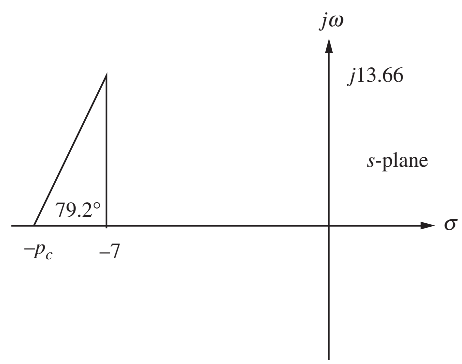
9.3

a. Searching along the 20% overshoot line, we find the point on the root locus at $-3.5 + 6.83$ at a gain of $K = 58.9$. Thus, for the uncompensated system,

$$T_s = \frac{4}{|\text{Re}|} = \frac{4}{3.5} = 1.143\,s.$$

b. For the uncompensated system, $K_v = \lim_{s\to 0} sG(s) = K/7 = 58.9/7 = 8.41$. Hence, $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1189$.

c. In order to decrease the settling time by a factor of 2, the design point is twice the uncompensated value, or $-7 + j\,13.66$. Adding the angles from the plant's poles and the compensator's zero at -3 to the design point, we obtain $-100.8°$. Thus, the compensator pole must contribute $180° - 100.8° = 79.2°$. Using the following diagram,



we find $\dfrac{13.66}{P_c - 7} = \tan 79.2°$, from which, $p_c = 9.61$. Adding this pole, we find the gain at the design point to be $K = 204.9$.
Evaluating K_v for the lead-compensated system:

$$K_v = \lim_{s\to 0} sG(s)G_{lead} = K(3)/[(7)(9.61)] = (204.9)(3)/[(7)(9.61)] = 9.138.$$

K_v for the uncompensated system was 8.41. For a $10x$ improvement in steady-state error, K_v must be $(8.41)(10) = 84.1$. Since lead compensation gave us $K_v = 9.138$, we need an improvement of $84.1/9.138 = 9.2$. Thus, the lag compensator zero should be $9.2x$ further to the left than the compensator pole. Arbitrarily select $G_c(s) = \dfrac{(s + 0.092)}{(s + 0.01)}$.

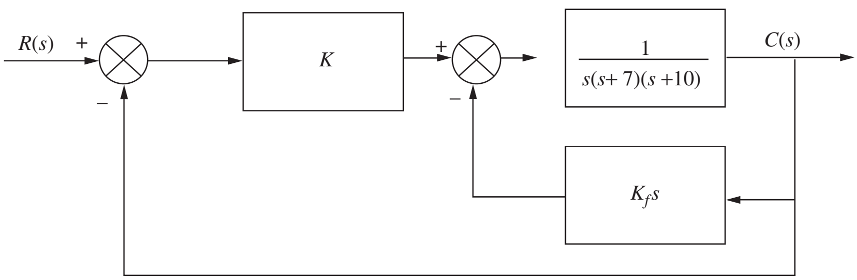
Using all plant and compensator poles, we find the gain at the design point to be $K = 205.4$. Summarizing the forward path with plant, compensator, and gain yields

$$G_e(s) = \frac{205.4(s+3)(s+0.092)}{s(s+7)(9.61)(s+0.01)}.$$

Higher-order poles are found at -0.928 and -2.6. It would be advisable to simulate the system to see if there is indeed pole-zero cancellation.

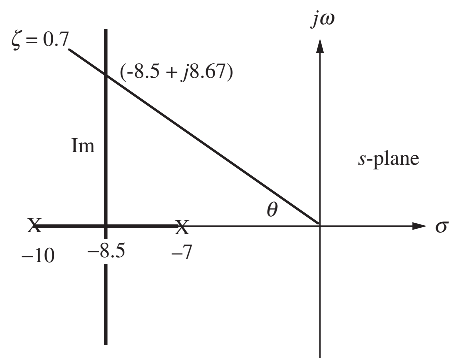
9.4

The configuration for the system is shown in the figure below.



Minor-Loop Design:

For the minor loop, $G(s)H(s) = \dfrac{K_f}{(s+7)(s+10)}$. Using the following diagram, we find that the minor-loop root locus intersects the 0.7 damping ratio line at $-8.5 + j8.67$. The imaginary part was found as follows: $\theta = \cos^{-1}\zeta = 45.57°$. Hence, $\dfrac{\text{Im}}{8.5} = \tan 45.57°$, from which $\text{Im} = 8.67$.



The gain, K_f, is found from the vector lengths as

$$K_f = \sqrt{1.5^2 + 8.67^2}\sqrt{1.5^2 + 8.67^2} = 77.42$$

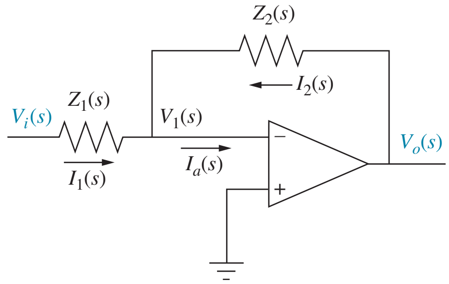
Major-Loop Design:
Using the closed-loop poles of the minor loop, we have an equivalent forward-path transfer function of

$$G_e(s) = \frac{K}{s(s + 8.5 + j8.67)(s + 8.5 - j8.67)} = \frac{K}{s(s^2 + 17s + 147.4)}.$$

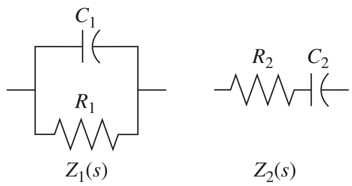
Using the three poles of $G_e(s)$ as open-loop poles to plot a root locus, we search along $\zeta = 0.5$ and find that the root locus intersects this damping ratio line at $-4.34 + j7.51$ at a gain, $K = 626.3$.

9.5

a. An active PID controller must be used. We use the circuit shown in the following figure:



where the impedances are shown below as follows:



Matching the given transfer function with the transfer function of the PID controller yields

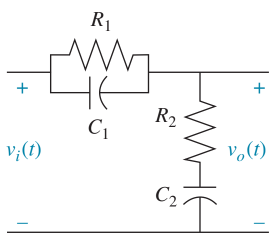
$$G_c(s) = \frac{(s+0.1)(s+5)}{s} = \frac{s^2+5.1s+0.5}{s} = s+5.1+\frac{0.5}{8}$$
$$= -\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right) + R_2C_1s + \frac{\frac{1}{R_1C_2}}{s}\right]$$

Equating coefficients

$$\frac{1}{R_1C_2} = 0.5 \qquad (1)$$
$$R_2C_1 = 1 \qquad (2)$$
$$\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right) = 5.1 \qquad (3)$$

In Eq. (2) we arbitrarily let $C_1 = 10^{-5}$. Thus, $R_2 = 10^5$. Using these values along with Eqs. (1) and (3) we find $C_2 = 100\mu F$ and $R_1 = 20\,\text{k}\Omega$.

b. The lag-lead compensator can be implemented with the following passive network, since the ratio of the lead pole-to-zero is the inverse of the ratio of the lag pole-to-zero:



Matching the given transfer function with the transfer function of the passive lag-lead compensator yields

$$G_c(s) = \frac{(s+0.1)(s+2)}{(s+0.01)(s+20)} = \frac{(s+0.1)(s+2)}{s^2+20.01s+0.2}$$

$$= \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$$

Equating coefficients

$$\frac{1}{R_1C_1} = 0.1 \qquad (1)$$

$$\frac{1}{R_2C_2} = 0.1 \qquad (2)$$

$$\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right) = 20.01 \qquad (3)$$

Substituting Eqs. (1) and (2) in Eq. (3) yields

$$\frac{1}{R_2C_1} = 17.91 \qquad (4)$$

Arbitrarily letting $C_1 = 100\,\mu F$ in Eq. (1) yields $R_1 = 100\,k\Omega$.
Substituting $C_1 = 100\,\mu F$ into Eq. (4) yields $R_2 = 558\,k\Omega$.
Substituting $R_2 = 558\,k\Omega$ into Eq. (2) yields $C_2 = 900\,\mu F$.

CHAPTER 10

10.1

a.

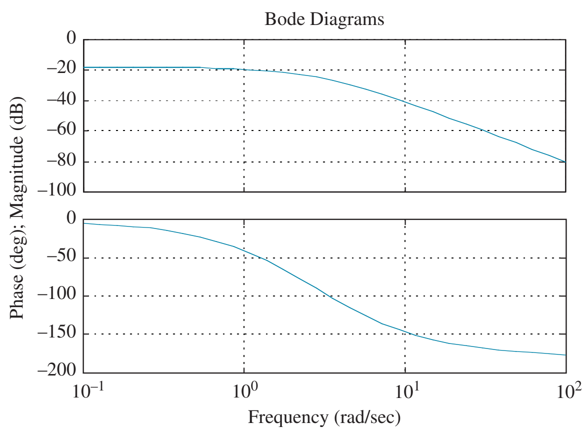
$$G(s) = \frac{1}{(s+2)(s+4)};\ G(j\omega) = \frac{1}{(8+\omega^2)+j6\omega}$$

$$M(\omega) = \sqrt{(8-\omega^2)^2+(6\omega)^2}$$

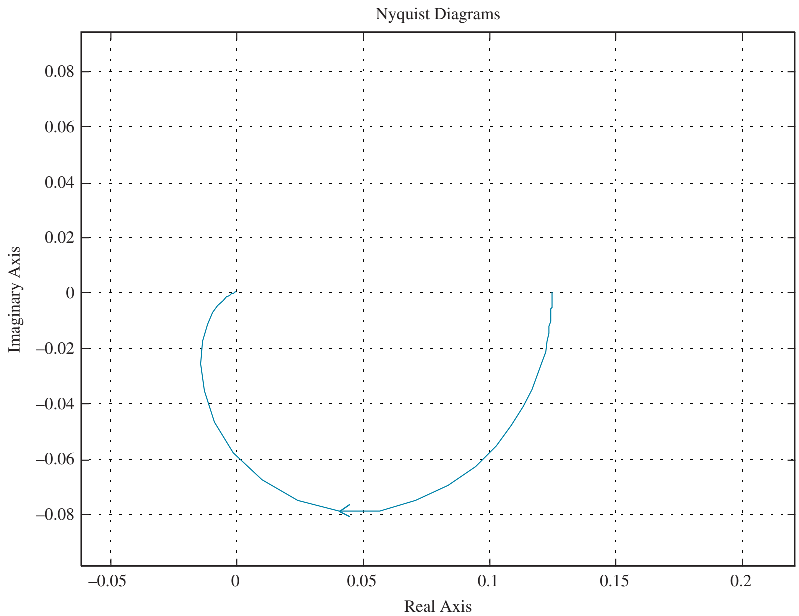
For $\omega < \sqrt{8}$, $\phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{8-\omega^2}\right)$.

For $\omega < \sqrt{8}$, $\phi(\omega) = -\left(\pi + \tan^{-1}\left[\frac{6\omega}{8-\omega^2}\right]\right)$.

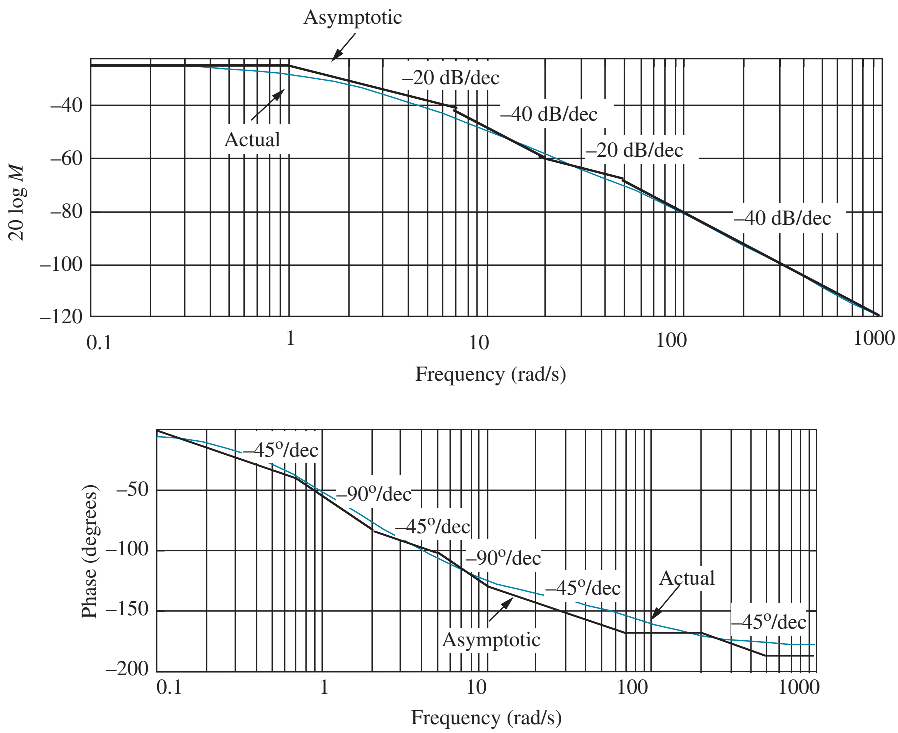
b.



c.

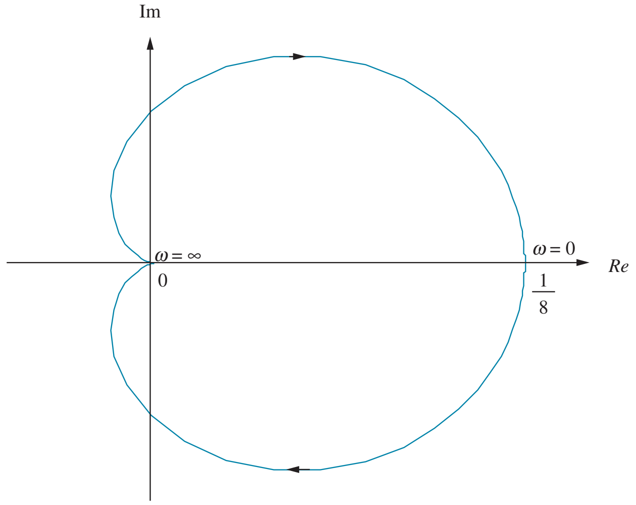

Nyquist Diagrams
Imaginary Axis
Real Axis
0.08
0.06
0.04
0.02
0
–0.02
–0.04
–0.06
–0.08
–0.05
0
0.05
0.1
0.15
0.2

10.2


Asymptotic
–20 dB/dec
–40 dB/dec
–20 dB/dec
–40 dB/dec
Actual
20 log M
–40
–60
–80
–100
–120
0.1
1
10
100
1000
Frequency (rad/s)
–45°/dec
–90°/dec
–45°/dec
–90°/dec
–45°/dec
Actual
–45°/dec
Asymptotic
Phase (degrees)
–50
–100
–150
–200
0.1
1
10
100
1000
Frequency (rad/s)

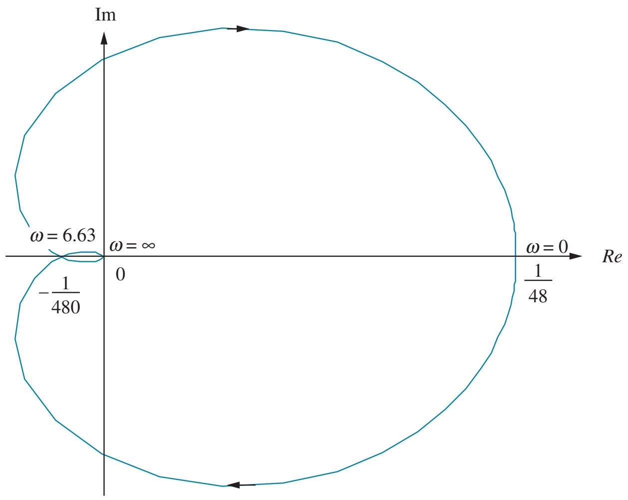
10.3

The frequency response is 1/8 at an angle of zero degrees at $\omega = 0$. Each pole rotates $90°$ in going from $\omega = 0$ to $\omega = \infty$. Thus, the resultant rotates $-180°$ while its magnitude goes to zero. The result is shown below.



10.4

a. The frequency response is 1/48 at an angle of zero degrees at $\omega = 0$. Each pole rotates $90°$ in going from $\omega = 0$ to $\omega = \infty$. Thus, the resultant rotates $-270°$ while its magnitude goes to zero. The result is shown below.



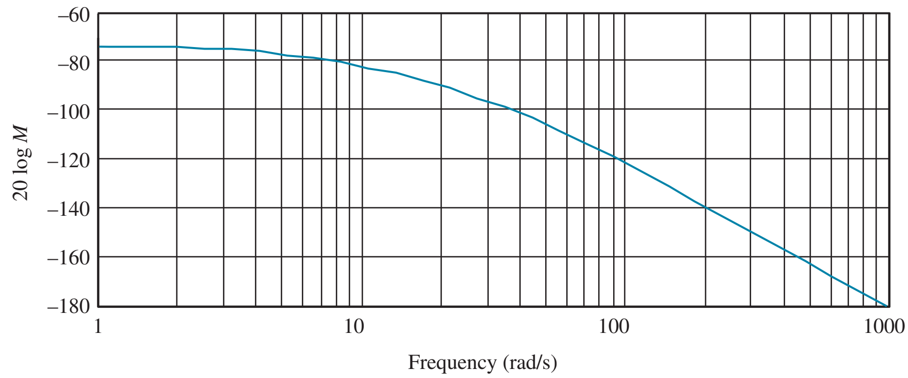
b. Substituting $j\omega$ into $G(s) = \dfrac{1}{(s+2)(s+4)(s+6)} = \dfrac{1}{s^3 + 12s^2 + 44s + 48}$ and simplifying, we obtain $G(j\omega) = \dfrac{(48 - 12\omega^2) - j(44\omega - \omega^3)}{\omega^6 + 56\omega^4 + 784\omega^2 + 2304}$. The Nyquist diagram crosses the real axis when the imaginary part of $G(j\omega)$ is zero. Thus, the Nyquist diagram crosses the real axis at $\omega^2 = 44$, or $\omega = \sqrt{44} = 6.63$ rad/s. At this frequency $G(j\omega) = -\dfrac{1}{480}$. Thus, the system is stable for $K < 480$.

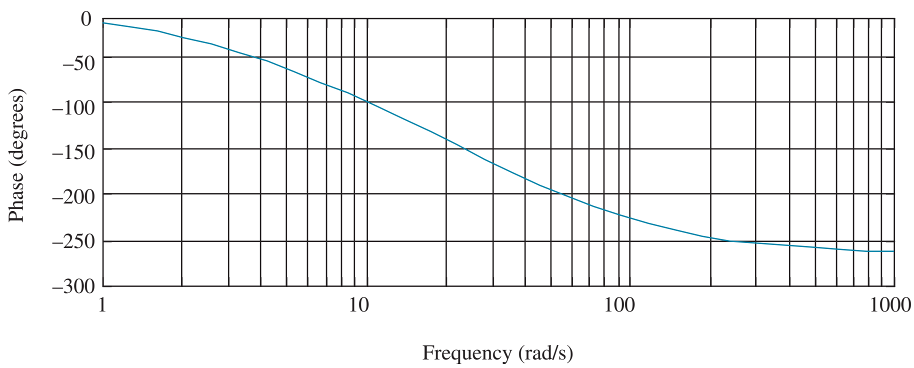
10.5

If $K = 100$, the Nyquist diagram will intersect the real axis at $-100/480$. Thus, $G_M = 20\log\frac{480}{100} = 13.62$ dB. From Skill-Assessment Exercise Solution 10.4, the 180° frequency is 6.63 rad/s.

10.6

a.





b. The phase angle is 180° at a frequency of 36.74 rad/s. At this frequency the gain is -99.67 dB. Therefore, $20\log K = 99.67$, or $K = 96,270$. We conclude that the system is stable for $K < 96,270$.

c. For $K = 10,000$, the magnitude plot is moved up $20\log 10,000 = 80$ dB. Therefore, the gain margin is $99.67 - 80 = 19.67$ dB. The 180° frequency is 36.7 rad/s. The gain curve crosses 0 dB at $\omega = 7.74$ rad/s, where the phase is 87.1°. We calculate the phase margin to be $180° - 87.1° = 92.9°$.

10.7

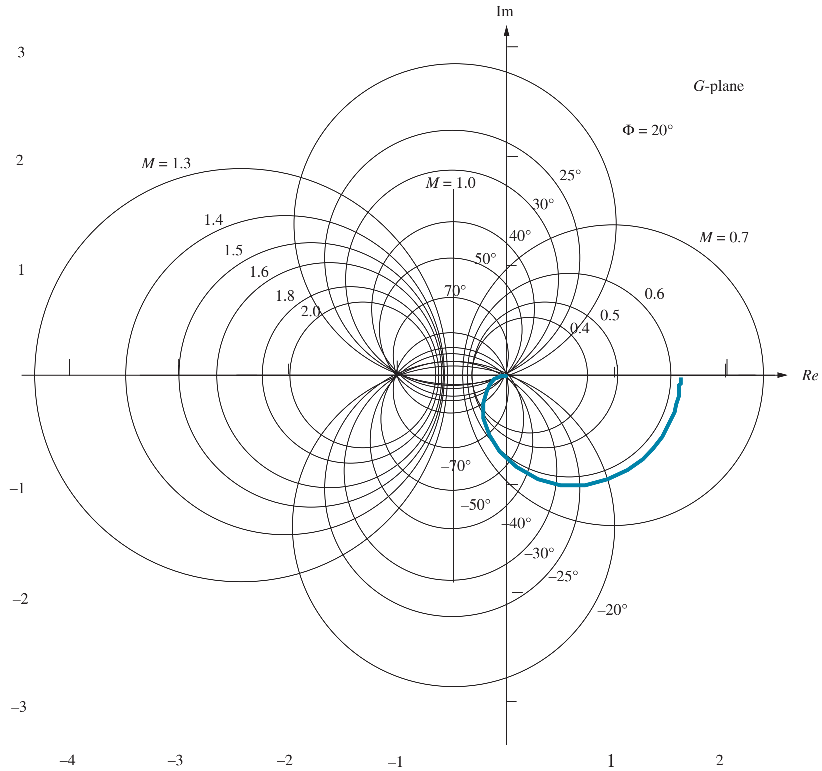
Using $\zeta = \frac{-\ln(\%/100)}{\sqrt{\pi^2 + \ln^2(\%100)}}$, we find $\zeta = 0.456$, which corresponds to 20% overshoot. Using $T_s = 2$, $\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 5.79$ rad/s.

10.8

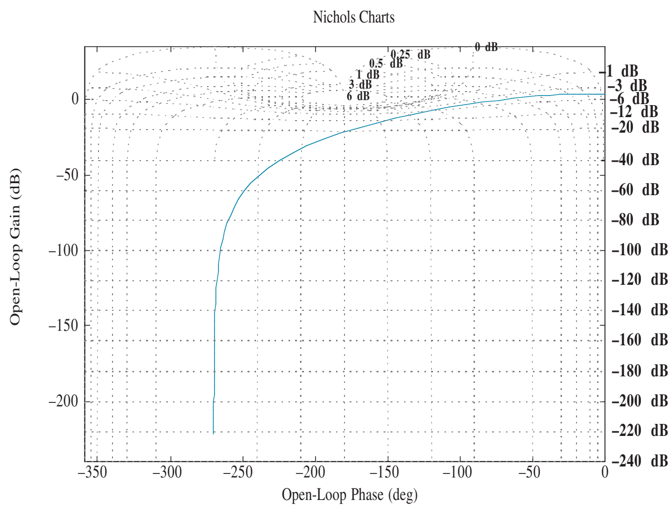
For both parts find that $G(j\omega) = \dfrac{160}{27} * \dfrac{(6750000 - 101250\omega^2) + j1350(\omega^2 - 1350)\omega}{\omega^6 + 2925\omega^4 + 1072500\omega^2 + 25000000}$.

For a range of values for ω, superimpose $G(j\omega)$ on the **a.** M and N circles, and on the **b.** Nichols chart.

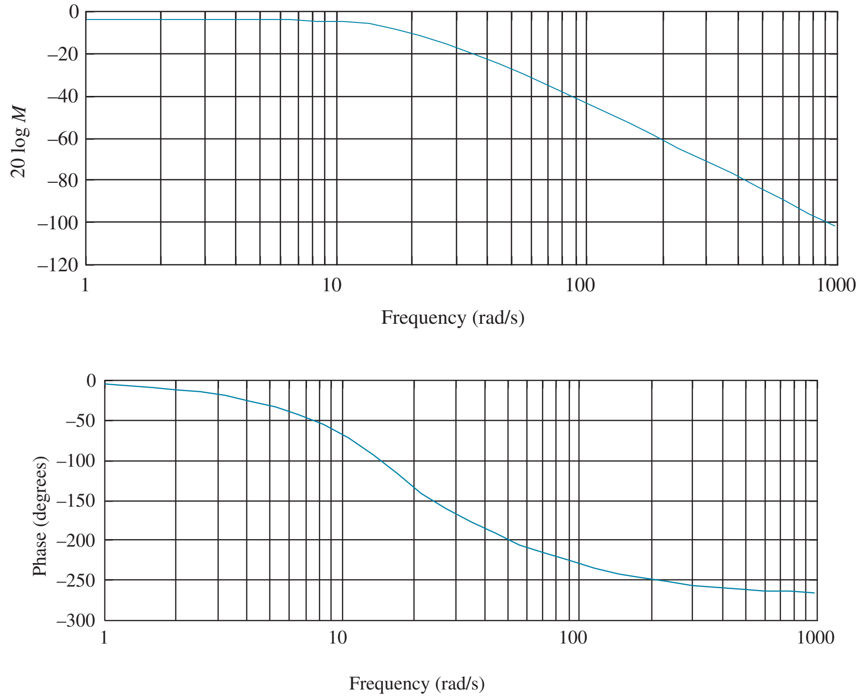
a.



b.

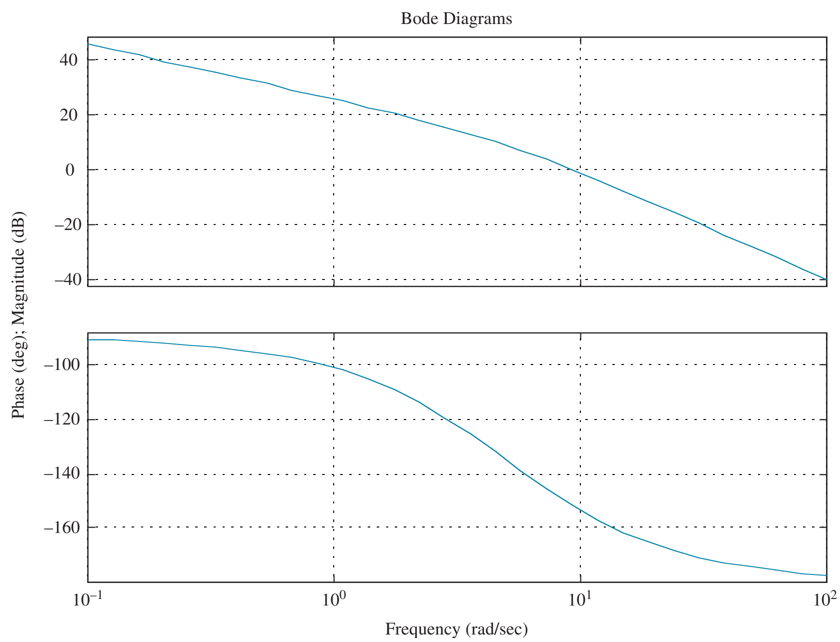


Plotting the closed-loop frequency response from **a.** or **b.** yields the following plot:



10.9

The open-loop frequency response is shown in the following figure:



The open-loop frequency response is -7 at $\omega = 14.5$ rad/s. Thus, the estimated bandwidth is $\omega_{WB} = 14.5$ rad/s. The open-loop frequency response plot goes through zero dB at a frequency of 9.4 rad/s, where the phase is 151.98°. Hence, the phase margin is $180° - 151.98° = 28.02°$. This phase margin corresponds to

$$\zeta = 0.25.\ \text{Therefore,}\ \ \%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100 = 44.4\%$$

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} = 1.64\ \text{s and}$$

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} = 0.33\ \text{s}$$

10.10

The initial slope is 40 dB/dec. Therefore, the system is Type 2. The initial slope intersects 0 dB at $\omega = 9.5$ rad/s. Thus, $K_a = 9.5^2 = 90.25$ and $K_p = K_v = \infty$.

10.11

a. Without delay, $G(j\omega) = \dfrac{10}{j\omega(j\omega+1)} = \dfrac{10}{\omega(-\omega+j)}$, from which the zero dB frequency is found as follows: $M = \dfrac{10}{\omega\sqrt{\omega^2+1}} = 1$. Solving for ω, $\omega\sqrt{\omega^2+1} = 10$, or after squaring both sides and rearranging, $\omega^4 + \omega^2 - 100 = 0$. Solving for the roots, $\omega^2 = -10.51,\ 9.51$. Taking the square root of the positive root, we find the 0 dB frequency to be 3.08 rad/s. At this frequency, the phase angle, $\phi = -\angle(-\omega + j) = -\angle(-3.08 + j) = -162°$. Therefore the phase margin is $180° - 162° = 18°$.

b. With a delay of 0.1 s,

$$\begin{aligned}\phi &= -\angle(-\omega+j) - \omega T = -\angle(-3.08+j) - (3.08)(0.1)(180/\text{pi}) \\ &= -162 - 17.65 = -179.65°\end{aligned}$$

Therefore the phase margin is $180° - 179.65° = 0.35°$. Thus, the system is table.

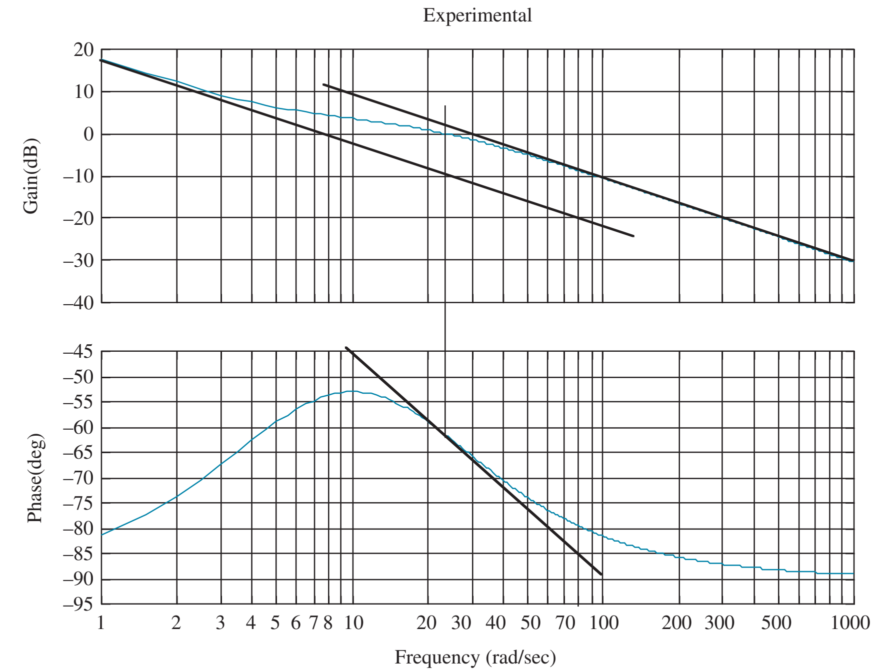
c. With a delay of 3 s,

$$\begin{aligned}\phi &= -\angle(-\omega+j) - \omega T = -\angle(-3.08+j) - (3.08)(3)(180/\text{pi}) = -162° - 529.41° \\ &= -691.41° = 28.59\ \text{deg.}\end{aligned}$$

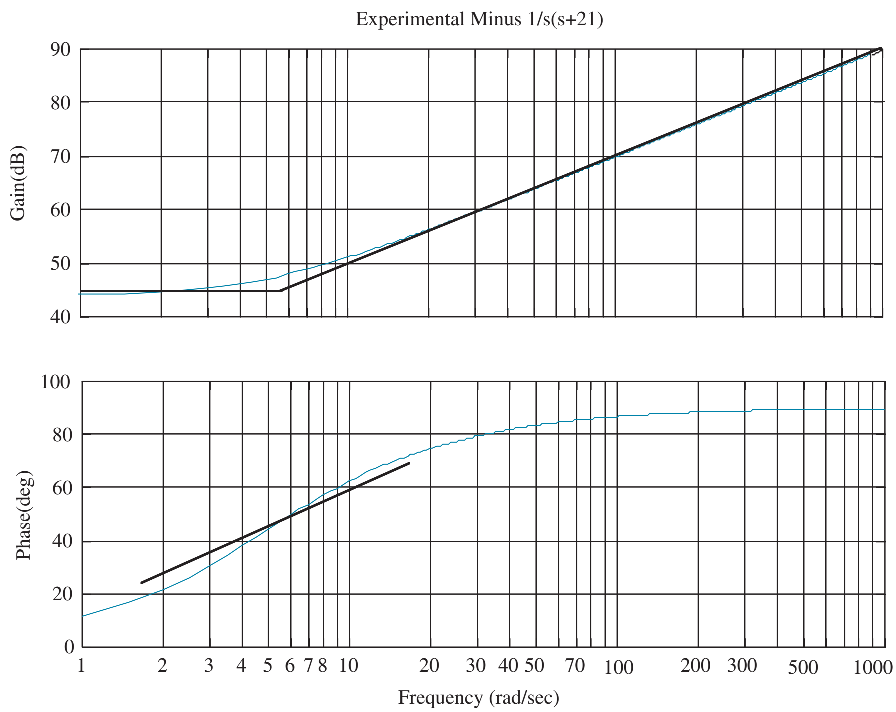
Therefore the phase margin is $28.59 - 180 = -151.41$ deg. Thus, the system is unstable.

10.12

Drawing judicially selected slopes on the magnitude and phase plot as shown below yields a first estimate.



We see an initial slope on the magnitude plot of –20 dB/dec. We also see a final –20 dB/dec slope with a break frequency around 21 rad/s. Thus, an initial estimate is $G_1(s) = \frac{1}{s(s+21)}$. Subtracting $G_1(s)$ from the original frequency response yields the frequency response shown below.

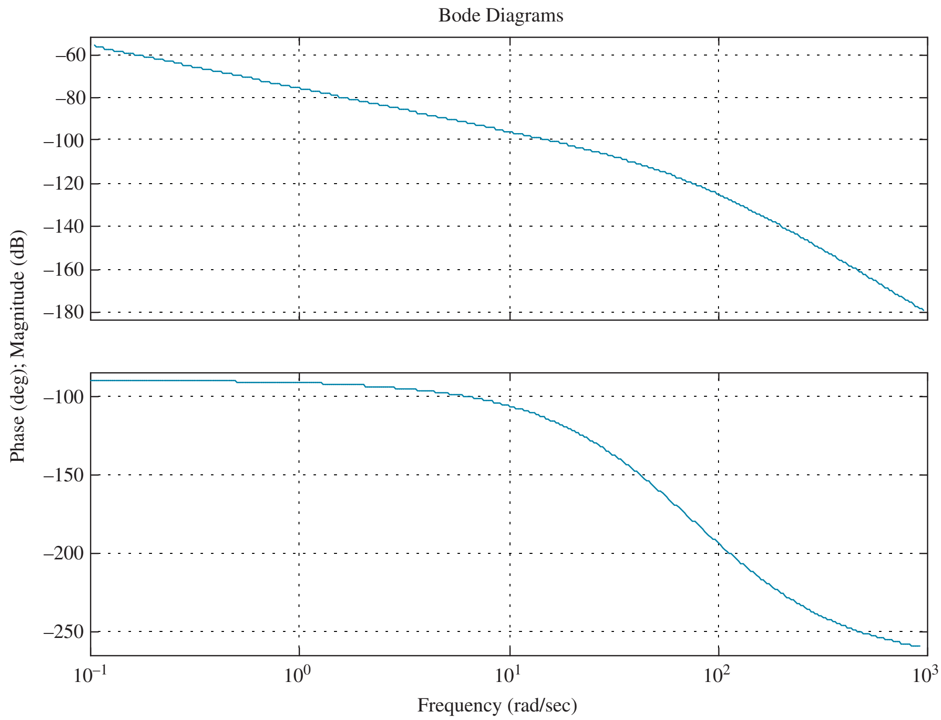


Drawing judicially selected slopes on the magnitude and phase plot as shown yields a final estimate. We see first-order zero behavior on the magnitude and phase plots with a break frequency of about 5.7 rad/s and a dc gain of about 44 dB $= 20\log(5.7K)$, or $K = 27.8$. Thus, we estimate $G_2(s) = 27.8(s+7)$. Thus, $G(s) = G_1(s)G_2(s) = \dfrac{27.8(s+5.7)}{s(s+21)}$. It is interesting to note that the original problem was developed from $G(s) = \dfrac{30(s+5)}{s(s+20)}$.

CHAPTER 11

11.1

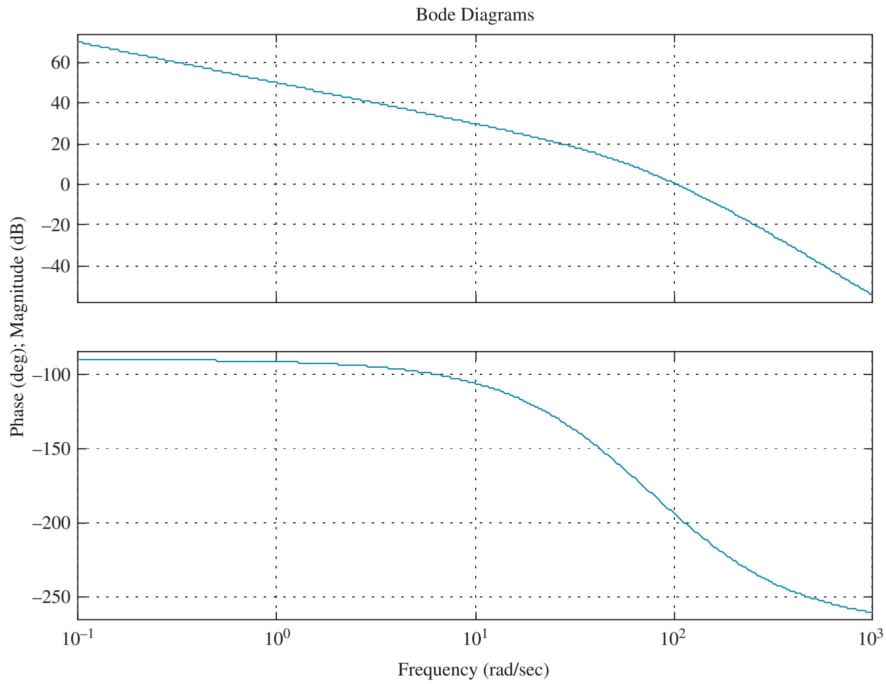
The Bode plot for $K = 1$ is shown below.



A 20% overshoot requires $\zeta = \dfrac{-\log\left(\dfrac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\dfrac{\%}{100}\right)}} = 0.456$. This damping ratio implies a phase margin of 48.10, which is obtained when the phase angle $= -1800 + 48.10 = 131.9°$. This phase angle occurs at $\omega = 27.6$ rad/s. The magnitude at this frequency is 5.15×10^{-6}. Since the magnitude must be unity $K = \dfrac{1}{5.15 \times 10^{-6}} = 194,200$.

11.2

To meet the steady-state error requirement, $K = 1,942,000$. The Bode plot for this gain is shown below.

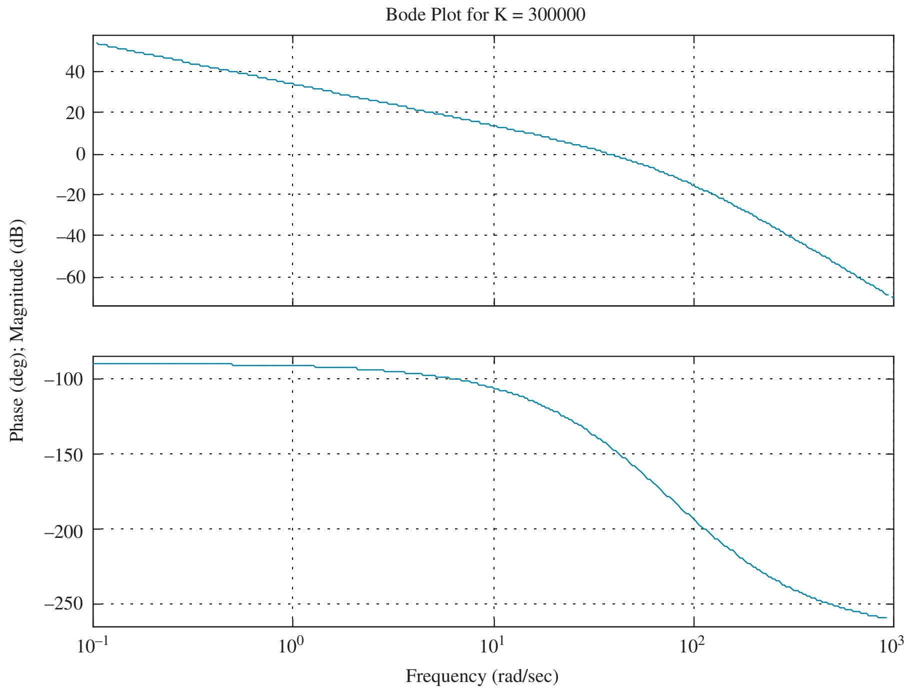


A 20% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$. This damping ratio implies a phase margin of 48.1°. Adding 10° to compensate for the phase angle contribution of the lag, we use 58.1°. Thus, we look for a phase angle of $-180° + 58.1° = -129.9°$. The frequency at which this phase occurs is 20.4 rad/s. At this frequency the magnitude plot must go through zero dB. Presently, the magnitude plot is 23.2 dB. Therefore draw the high frequency asymptote of the lag compensator at -23.2 dB. Insert a break at $0.1(20.4) = 2.04$ rad/s. At this frequency, draw -23.2 dB/dec slope until it intersects 0 dB. The frequency of intersection will be the low frequency break or 0.141 rad/s. Hence the compensator is $G_c(s) = K_c\dfrac{(s+2.04)}{(s+0.141)}$, where the gain is chosen to yield 0 dB at low frequencies, or $K_c = 0.141/2.04 = 0.0691$. In summary,

$$G_c(s) = 0.0691\,\frac{(s+2.04)}{(s+0.141)} \text{ and } G(s) = \frac{1{,}942{,}000}{s(s+50)(s+120)}$$

11.3

A 20% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$. The required bandwidth is then calculated as $\omega_{BW} = \dfrac{4}{T_s\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 57.9$ rad/s. In order to meet the steady-state error requirement of $K_v = 50 = \dfrac{K}{(50)(120)}$, we calculate $K = 300,000$. The uncompensated Bode plot for this gain is shown below.



The uncompensated system's phase margin measurement is taken where the magnitude plot crosses 0 dB. We find that when the magnitude plot crosses 0 dB, the phase angle is $-144.8°$. Therefore, the uncompensated system's phase margin is $-180° + 144.8° = 35.2°$. The required phase margin based on the required damping ratio is $\Phi_M = \tan^{-1}\dfrac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 48.1°$. Adding a $10°$ correction factor, the required phase margin is $58.1°$. Hence, the compensator must contribute $\phi_{\max} = 58.1° - 35.2° = 22.9°$. Using $\phi_{\max} = \sin^{-1}\dfrac{1-\beta}{1+\beta}$, $\beta = \dfrac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}} = 0.44$. The compensator's peak magnitude is calculated as $M_{\max} = \dfrac{1}{\sqrt{\beta}} = 1.51$. Now find the frequency at which the uncompensated system has a magnitude $1/M_{\max}$, or -3.58 dB. From the Bode plot, this magnitude occurs at $\omega_{\max} = 50\,\text{rad/s}$. The compensator's zero is at $z_c = \dfrac{1}{T}$. $\omega_{\max} = \dfrac{1}{T\sqrt{\beta}}$ Therefore, $z_c = 33.2$.

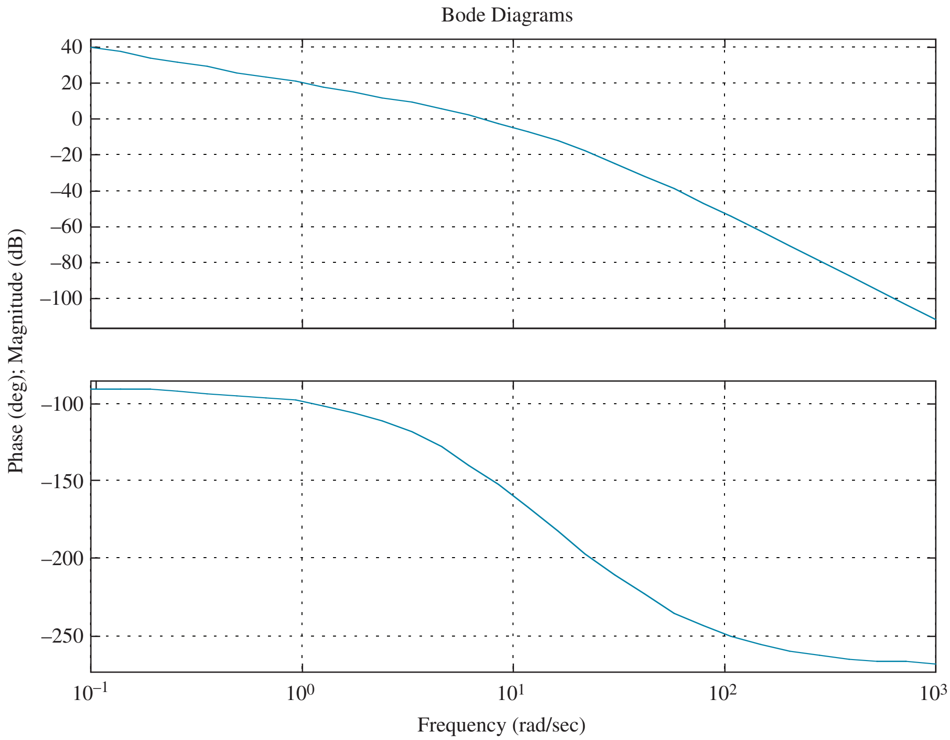
The compensator's pole is at $P_c = \dfrac{1}{\beta T} = \dfrac{z_c}{\beta} = 75.4$. The compensator gain is chosen to yield unity gain at dc.

Hence, $K_c = 75.4/33.2 = 2.27$. Summarizing, $G_c(s) = 2.27\dfrac{(s+33.2)}{(s+75.4)}$, and $G(s)\dfrac{300{,}000}{s(s+50)(s+120)}$.

11.4

A 10% overshoot requires $\zeta = \dfrac{-\log\left(\dfrac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\dfrac{\%}{100}\right)}} = 0.591$. The required bandwidth is then calculated as $\omega_{BW} = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} = 7.53$ rad/s.

In order to meet the steady-state error requirement of $K_v = 10 = \dfrac{K}{(8)(30)}$, we calculate $K = 2400$. The uncompensated Bode plot for this gain is shown below.



Let us select a new phase-margin frequency at $0.8\omega_{BW} = 6.02$ rad/s. The required phase margin based on the required damping ratio is $\Phi_M = \tan^{-1}\dfrac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} = 58.6^\circ$. Adding a 5° correction factor, the required phase margin is 63.6°. At 6.02 rad/s, the new phase-margin frequency, the phase angle is–which represents a phase margin of $180^\circ - 138.3^\circ = 41.7^\circ$. Thus, the lead compensator must contribute $\phi_{\max} = 63.6^\circ - 41.7^\circ = 21.9^\circ$.

Using $\phi_{\max} = \sin^{-1}\dfrac{1-\beta}{1+\beta}$, $\beta = \dfrac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}} = 0.456$.

We now design the lag compensator by first choosing its higher break frequency one decade below the new phase-margin frequency, that is, $z_{lag} = 0.602$ rad/s. The lag compensator's pole is $p_{lag} = \beta z_{lag} = 0.275$. Finally, the lag compensator's gain is $K_{lag} = \beta = 0.456$.

Now we design the lead compensator. The lead zero is the product of the new phase margin frequency and $\sqrt{\beta}$, or $z_{lead} = 0.8\omega_{BW}\sqrt{\beta} = 4.07$. Also, $p_{lead} = \dfrac{z_{lead}}{\beta}$ $= 8.93$. Finally, $K_{lead} = \dfrac{1}{\beta} = 2.19$. Summarizing,

$$G_{lag} = (s) = 0.456\frac{(s+0.602)}{(s+0.275)};\ G_{lead}(s) = 2.19\frac{(s+4.07)}{(s+8.93)};\quad \text{and } k = 2400.$$

CHAPTER 12

12.1

We first find the desired characteristic equation. A 5% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2+\log^2\left(\frac{\%}{100}\right)}} = 0.69$. Also, $\omega_n = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}} = 14.47\ \text{rad/s}$. Thus, the characteristic equation is $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 19.97s + 209.4$. Adding a pole at -10 to cancel the zero at -10 yields the desired characteristic equation, $(s^2+19.97s+209.4)(s+10) = s^3 + 29.97s^2 + 409.1s + 2094$. The compensated system matrix in phase-variable form is

$\mathbf{A}-\mathbf{BK} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(k_1) & -(36+k_2) & -(15+k_3) \end{bmatrix}$. The characteristic equation for this system is $|s\mathbf{I}-(\mathbf{A}-\mathbf{BK})| = s^3 + (15+k_3)s^2 + (36+k_2)s + (k_1)$. Equating coefficients of this equation with the coefficients of the desired characteristic equation yields the gains as

$$\mathbf{K} = [k_1\quad k_2\quad k_3] = [2094\quad 373.1\quad 14.97].$$

12.2

The controllability matrix is $\mathbf{C_M} = [\mathbf{B}\quad \mathbf{AB}\quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & -9 \\ 1 & -1 & 16 \end{bmatrix}$. Since $|\mathbf{C_M}| = 80$, $\mathbf{C_M}$ is full rank, that is, rank 3. We conclude that the system is controllable.

12.3

First check controllability. The controllability matrix is $C_{\mathbf{M}z} = [\mathbf{B}\quad \mathbf{AB}\quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix}$. Since $|\mathbf{C}_{\mathbf{M}z}| = -1$, $\mathbf{C_{Mz}}$ is full rank, that is, rank 3. We conclude that the system is controllable. We now find the desired characteristic equation. A 20% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2+\log^2\left(\frac{\%}{100}\right)}} = 0.456$. Also, $\omega_n = \dfrac{4}{\zeta T_s} = 4.386\ \text{rad/s}$.

Thus, the characteristic equation is $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 4s + 19.24$. Adding a pole at -6 to cancel the zero at -6 yields the resulting desired characteristic equation,

$$(s^2 + 4s + 19.24)(s + 6) = s^3 + 10s^2 + 43.24s + 115.45.$$

Since $G(s) = \dfrac{(s+6)}{(s+7)(s+8)(s+9)} = \dfrac{s+6}{s^3 + 24s^2 + 191s + 504}$, we can write the phase-variable representation as $\mathbf{A_p} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -504 & -191 & -24 \end{bmatrix}$; $\mathbf{B_p} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{C_p} = [6 \quad 1 \quad 0]$. The compensated system matrix in phase-variable form is $\mathbf{A_p} - \mathbf{B_p K_p} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(504 + k_1) & -(191 + k_2) & -(24 + k_3) \end{bmatrix}$. The characteristic equation for this system is $|s\mathbf{I} - (\mathbf{A_p} - \mathbf{B_p K_p})| = s^3 + (24 + k_3)s^2 + (191 + k_2)s + (504 + k_1)$. Equating coefficients of this equation with the coefficients of the desired characteristic equation yields the gains as $\mathbf{K_p} = [k_1 \quad k_2 \quad k_3] = [-388.55 \quad -147.76 \quad -14]$. We now develop the transformation matrix to transform back to the z-system.

$$\mathbf{C_{Mz}} = \begin{bmatrix} \mathbf{B}_z & \mathbf{A}_z\mathbf{B}_z & \mathbf{A}_z^2\mathbf{B}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix} \text{ and}$$

$$\mathbf{C_{Mp}} = \begin{bmatrix} \mathbf{B_p} & \mathbf{A_p B_p} & \mathbf{A_p^2 B_p} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -24 \\ 1 & -24 & 385 \end{bmatrix}.$$

Therefore,

$$\mathbf{P} = \mathbf{C_{Mz}} \mathbf{C_{Mx}^{-1}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -17 \\ 1 & -9 & 81 \end{bmatrix} \begin{bmatrix} 191 & 24 & 1 \\ 24 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix}$$

Hence, $\mathbf{K}_z = \mathbf{K_p P^{-1}} = [-388.55 \quad -147.76 \quad -14] \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}$

$= [-40.23 \quad 62.24 \quad -14]$.

12.4

For the given system $\mathbf{e_{\dot{x}}} = (\mathbf{A} - \mathbf{LC})\mathbf{e_x} = \begin{bmatrix} -(24 + l_1) & 1 & 0 \\ -(191 + l_2) & 0 & 1 \\ -(504 + l_3) & 0 & 0 \end{bmatrix} \mathbf{e_x}$. The characteristic polynomial is given by $|[s\mathbf{I} - (\mathbf{A} - \mathbf{LC})]| = s^3 + (24 + l_1)s^2 + (191 + l_2)s + (504 + l_3)$. Now we find the desired characteristic equation. The dominant poles from Skill-Assessment Exercise 12.3 come from $(s^2 + 4s + 19.24)$. Factoring yields $(-2 + j3.9)$ and $(-2 - j3.9)$. Increasing these poles by a factor of 10 and adding a third pole 10 times the real part of the dominant second-order poles yields the desired characteristic polynomial, $(s+20+j39)(s+20-j39)(s+200) = s^3+240s^2+9921s+384200$. Equating coefficients of the desired characteristic equation to the system's characteristic equation yields $\mathbf{L} = \begin{bmatrix} 216 \\ 9730 \\ 383696 \end{bmatrix}$.

12.5

The observability matrix is $\mathbf{O_M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ -64 & -80 & -78 \\ 674 & 848 & 814 \end{bmatrix}$, where $\mathbf{A}^2 = \begin{bmatrix} 25 & 28 & 32 \\ -7 & -4 & -11 \\ 77 & 95 & 94 \end{bmatrix}$. The matrix is of full rank, that is, rank 3, since $|\mathbf{O_M}| = -1576$. Therefore the system is observable.

12.6

The system is represented in cascade form by the following state and output equations:

$$\dot{\mathbf{z}} = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -8 & 1 \\ 0 & 0 & -9 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0]\mathbf{z}$$

The observability matrix is $\mathbf{O_{Mz}} = \begin{bmatrix} \mathbf{C_z} \\ \mathbf{C_z A_z} \\ \mathbf{C_z A_z^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}$,

where $\mathbf{A_z^2} = \begin{bmatrix} 49 & -15 & 1 \\ 0 & 64 & -17 \\ 0 & 0 & 81 \end{bmatrix}$. Since $G(s) = \dfrac{1}{(s+7)(s+8)(s+9)}$

$= \dfrac{1}{s^3+24s^2+191s+504}$, we can write the observable canonical form as

$$\dot{\mathbf{x}} = \begin{bmatrix} -24 & 1 & 0 \\ -191 & 0 & 1 \\ -504 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0 \quad 0]\mathbf{x}$$

The observability matrix for this form is $\mathbf{O_{Mx}} = \begin{bmatrix} \mathbf{C_x} \\ \mathbf{C_x A_x} \\ \mathbf{C_x A_x^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix}$, where

$$\mathbf{A}_{\mathbf{x}}^2 = \begin{bmatrix} 385 & -24 & 1 \\ 4080 & -191 & 0 \\ 12096 & 504 & 0 \end{bmatrix}.$$

We next find the desired characteristic equation. A 10% overshoot requires $\zeta = \dfrac{-\log\left(\dfrac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\dfrac{\%}{100}\right)}} = 0.591$. Also, $\omega_n = \dfrac{4}{\zeta T_s} = 67.66$ rad/s. Thus, the characteristic equation is $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 80s + 4578.42$. Adding a pole at -400, or 10 times the real part of the dominant second-order poles, yields the resulting desired characteristic equation, $(s^2 + 80s + 4578.42)(s + 400) = s^3 + 480s^2 + 36580s + 1.831x10^6$. For the system represented in observable canonical form $\mathbf{e}_{\dot{\mathbf{x}}} = (\mathbf{A}_{\mathbf{x}} - \mathbf{L}_{\mathbf{x}}\mathbf{C}_{\mathbf{x}})\,\mathbf{e}_{\mathbf{x}} = \begin{bmatrix} -(24+l_1) & 1 & 0 \\ -(191+l_2) & 0 & 1 \\ -(504+l_3) & 0 & 0 \end{bmatrix}\mathbf{e}_{\mathbf{x}}$. The characteristic polynomial is given by $|[s\mathbf{I} - (\mathbf{A}_{\mathbf{x}} - \mathbf{L}_{\mathbf{x}}\mathbf{C}_{\mathbf{x}})]| = s^3 + (24+l_1)s^2 + (191+l_2)s + (504+l_3)$. Equating coefficients of the desired characteristic equation to the system's characteristic equation yields $\mathbf{L}_{\mathbf{x}} = \begin{bmatrix} 456 \\ 36,389 \\ 1,830,496 \end{bmatrix}$.

Now, develop the transformation matrix between the observer canonical and cascade forms.

$$\mathbf{P} = \mathbf{O}_{\mathbf{Mz}}^{-1}\mathbf{O}_{\mathbf{Mx}} = \begin{bmatrix} 1 & 0 & 0 \\ -7 & 1 & 0 \\ 49 & -15 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 56 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -24 & 1 & 0 \\ 385 & -24 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -17 & 1 & 0 \\ 81 & -9 & 1 \end{bmatrix}$$

Finally,

$$\mathbf{L}_{\mathbf{z}} = \mathbf{P}\mathbf{L}_{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ -17 & 1 & 0 \\ 81 & -9 & 1 \end{bmatrix} \begin{bmatrix} 456 \\ 36,389 \\ 1,830,496 \end{bmatrix} = \begin{bmatrix} 456 \\ 28,637 \\ 1,539,931 \end{bmatrix} \approx \begin{bmatrix} 456 \\ 28,640 \\ 1,540,000 \end{bmatrix}.$$

12.7

We first find the desired characteristic equation. A 10% overshoot requires

$$\zeta = \frac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.591$$

Also, $\omega_n = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}} = 1.948\,\text{rad/s}$. Thus, the characteristic equation is $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2.3s + 3.79$. Adding a pole at -4, which corresponds to the original system's zero location, yields the resulting desired characteristic equation, $(s^2 + 2.3s + 3.79)(s+4) = s^3 + 6.3s^2 + 13s + 15.16$.

Now, $\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B}K_e \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r$; and $y = [\mathbf{C} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_N \end{bmatrix}$,

where

$$\mathbf{A}-\mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ -7 & -9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -(7+k_1) & -(9+k_2) \end{bmatrix}$$

$$\mathbf{C} = [4 \quad 1]$$

$$\mathbf{B}k_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix} k_e = \begin{bmatrix} 0 \\ k_e \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -(7+k_1) & -(9+k_2) & k_e \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r;\ y = [4 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix}.$$

Finding the characteristic equation of this system yields

$$\left| s\mathbf{I} - \begin{bmatrix} (\mathbf{A}-\mathbf{B}\mathbf{K}) & \mathbf{B}K_e \\ -\mathbf{C} & 0 \end{bmatrix} \right| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ -(7+k_1) & -(9+k_2) & k_e \\ -4 & -1 & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & -1 & 0 \\ (7+k_1) & s+(9+k_2) & -k_e \\ 4 & 1 & s \end{bmatrix} \right| = s^3 + (9+k_2)s^2 + (7+k_1+k_e)s + 4k_e$$

Equating this polynomial to the desired characteristic equation,

$$s^3 + 6.3s^2 + 13s + 15.16 = s^3 + (9+k_2)s^2 + (7+k_1+\ k_e)s + 4k_e$$

Solving for the k's,

$$\mathbf{K} = [2.21 \quad -2.7] \text{ and } k_e = 3.79.$$

CHAPTER 13

13.1

$f(t) = \sin(\omega kT);\ f^*(t) = \sum_{k=0}^{\infty} \sin(\omega kT)\delta(t - kT);$

$$F^*(s) = \sum_{k=0}^{\infty} \sin(\omega kT)e^{-kTs} = \sum_{k=0}^{\infty} \frac{\left(e^{j\omega kT} - e^{-j\omega kT}\right)e^{-kTs}}{2j}$$

$$= \frac{1}{2j}\sum_{k=0}^{\infty} \left(e^{T(s-j\omega)}\right)^{-k} - \left(e^{T(s+j\omega)}\right)^{-k}$$

But, $\sum_{k=0}^{\infty} x^{-k} = \frac{1}{1 - x^{-1}}$

Thus,

$$F^*(s) = \frac{1}{2j}\left[\frac{1}{1 - e^{-T(s-j\omega)}} - \frac{1}{1 - e^{-T(s+j\omega)}}\right] = \frac{1}{2j}\left[\frac{e^{-Ts}e^{j\omega T} - e^{-Ts}e^{j\omega T}}{1 - \left(e^{-Ts}e^{j\omega T} - e^{-Ts}e^{j\omega T}\right) + e^{-2Ts}}\right]$$

$$= e^{-Ts}\left[\frac{\sin(\omega T)}{1 - e^{-Ts}2\cos(\omega T) + e^{-2Ts}}\right] = \frac{z^{-1}\sin(\omega T)}{1 - 2z^{-1}\cos(\omega T) + z^{-2}}$$

13.2

$$F(z) = \frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

$$\frac{F(z)}{z} = \frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

$$= 46.875\frac{z}{z-0.5} - 114.75\frac{1}{z-0.7} + 68.875\frac{z}{z-0.9}$$

$$F(z) = 46.875\frac{z}{z-0.5} - 114.75\frac{z}{z-0.7} + 68.875\frac{z}{z-0.9},$$

$$f(kT) = 46.875(0.5)^k - 114.75(0.7)^k + 68.875(0.9)^k$$

13.3

Since $G(s) = \left(1 - e^{-Ts}\right)\frac{8}{s(s+4)}$,

$$G(z) = \left(1 - z^{-1}\right)z\left\{\frac{8}{s(s+4)}\right\} = \frac{z-1}{z}z\left\{\frac{A}{s} + \frac{B}{s+4}\right\} = \frac{z-1}{z}z\left\{\frac{2}{s} + \frac{2}{s+4}\right\}.$$

Let $G_2(s) = \frac{2}{s} + \frac{2}{s+4}$. Therefore, $g_2(t) = 2 - 2e^{-4t}$, or $g_2(kT) = 2 - 2e^{-4kT}$.

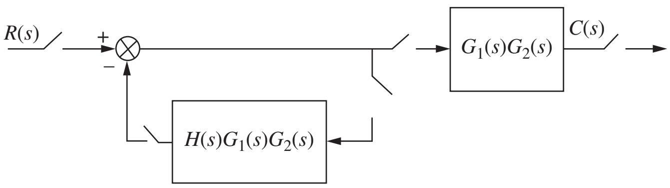
Hence, $G_2(z) = \frac{2z}{z-1} - \frac{2z}{z - \mathrm{e}^{-4T}} = \frac{2z\left(1 - \mathrm{e}^{-4T}\right)}{(z-1)(z - e^{-4T})}$.

Therefore, $G(z) = \dfrac{z-1}{z}G_2(z) = \dfrac{2\left(1-e^{-4T}\right)}{\left(z-e^{-4T}\right)}$.

For $T = \dfrac{1}{4}s$, $G(z) = \dfrac{1.264}{z-0.3679}$.

13.4

Add phantom samplers to the input, feedback after $H(s)$, and to the output. Push $G_1(s)G_2(s)$, along with its input sampler, to the right past the pickoff point and obtain the block diagram shown below.



Hence, $T(z) = \dfrac{G_1G_2(z)}{1+HG_1G_2(z)}$.

13.5

Let $G(s) = \dfrac{20}{s+5}$. Let $G_2(s) = \dfrac{G(s)}{s} = \dfrac{20}{s(s+5)} = \dfrac{4}{s} - \dfrac{4}{s+5}$. Taking the inverse Laplace transform and letting $t = kT$, $g_2(kT) = 4 - 4e^{-5kT}$. Taking the z-transform yields $G_2(z) = \dfrac{4z}{z-1} - \dfrac{4z}{z-e^{-5T}} = \dfrac{4z\left(1-e^{-5T}\right)}{(z-1)\left(z-e^{-5T}\right)}$.

Now, $G(z) = \dfrac{z-1}{z} - G_2(z) = \dfrac{4\left(1-e^{-5T}\right)}{\left(z-e^{-5T}\right)}$.

Finally, $T(z) = \dfrac{G(z)}{1+G(z)} = \dfrac{4\left(1-e^{-5T}\right)}{z-5e^{-5T}+4}$.

The pole of the closed-loop system is at $5e^{-5T} - 4$. Substituting values of T, we find that the pole is greater than 1 if $T > 0.1022$ s. Hence, the system is stable for $0 < T < 0.1022$ s.

13.6

Substituting $z = \dfrac{s+1}{s-1}$ into $D(z) = z^3 - z^2 - 0.5z + 0.3$, we obtain $D(s) = s^3 - 8s^2 - 27s - 6$. The Routh table for this polynomial is shown below.

s^3	1	−27
s^2	−8	−6
s^1	−27.75	0
s^0	−6	0

Since there is one sign change, we conclude that the system has one pole outside the unit circle and two poles inside the unit circle. The table did not produce a row of zeros and thus, there are no $j\omega$ poles. The system is unstable because of the pole outside the unit circle.

13.7

Defining $G(s)$ as $G_1(s)$ in cascade with a zero-order-hold,

$$G(s) = 20\left(1 - e^{-Ts}\right)\left[\frac{(s+3)}{s(s+4)(s+5)}\right] = 20\left(1 - e^{-Ts}\right)\left[\frac{3/20}{s} + \frac{1/4}{(s+4)} - \frac{2/5}{(s+5)}\right].$$

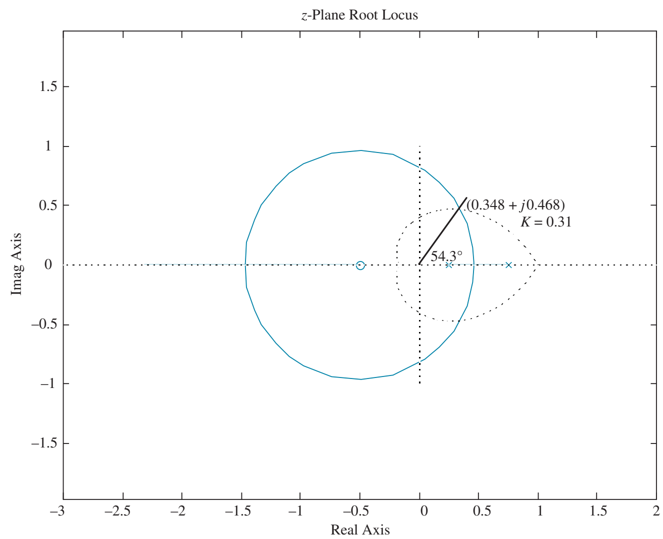
Taking the z-transform yields

$$G(z) = 20\left(1 - z^{-1}\right)\left[\frac{(3/20)z}{z-1} + \frac{(1/4)z}{z - e^{-4T}} - \frac{(2/5)z}{z - e^{-5T}}\right] = 3 + \frac{5(z-1)}{z - e^{-4T}} - \frac{8(z-1)}{z - e^{-5T}}.$$

Hence for $T = 0.1$ second, $K_p = \lim_{z\to 1} G(z) = 3$, and $K_v = \frac{1}{T}\lim_{z\to 1}(z-1)G(z) = 0$, and $K_a = \frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z) = 0$. Checking for stability, we find that the system is stable for $T = 0.1$ second, since $T(z) = \frac{G(z)}{1+G(z)} = \frac{1.5z - 1.109}{z^2 + 0.222z - 0.703}$ has poles inside the unit circle at -0.957 and $+0.735$. Again, checking for stability, we find that the system is unstable for $T = 0.5$ second, since $T(z) = \frac{G(z)}{1+G(z)} = \frac{3.02z - 0.6383}{z^2 + 2.802z - 0.6272}$ has poles inside and outside the unit circle at $+0.208$ and -3.01, respectively.

13.8

Draw the root locus superimposed over the $\zeta = 0.5$ curve shown below. Searching along a 54.3° line, which intersects the root locus and the $\zeta = 0.5$ curve, we find the point $0.587\angle 54.3^\circ = (0.348 + j0.468)$ and $K = 0.31$.

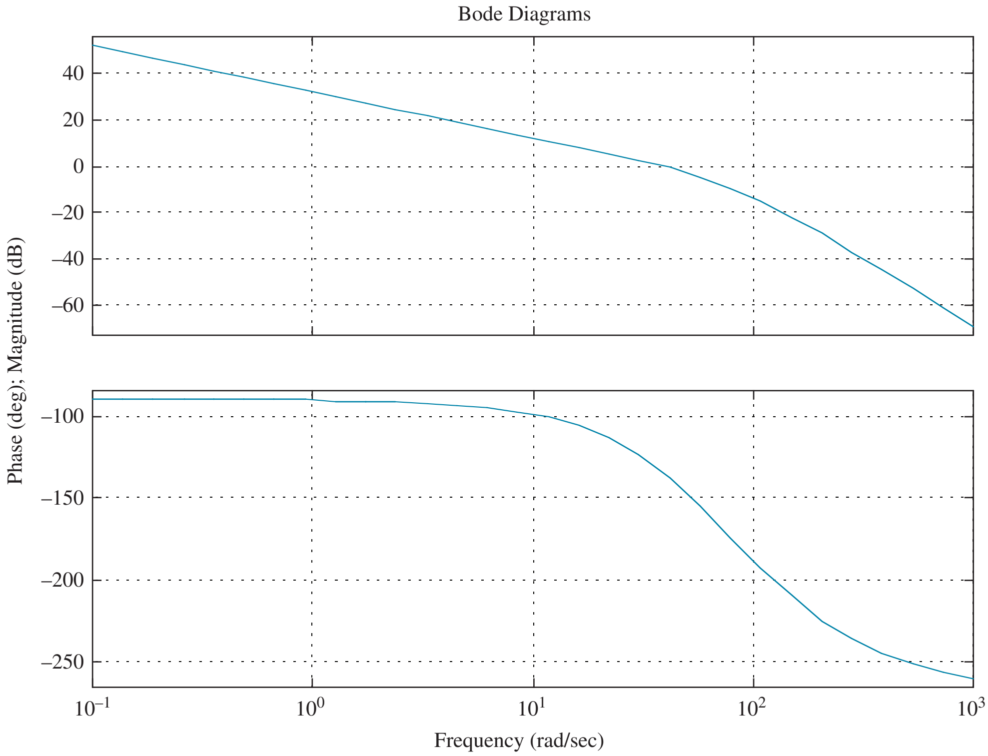


13.9

Let

$$G_e(s) = G(s)G_c(s) = \frac{100K}{s(s+36)(s+100)}\frac{2.38(s+25.3)}{(s+60.2)} = \frac{342720(s+25.3)}{s(s+36)(s+100)(s+60.2)}.$$

The following shows the frequency response of $G_e(j\omega)$.



We find that the zero dB frequency, ω_{Φ_M}, for $G_e(j\omega)$ is 39 rad/s. Using Astrom's guideline the value of T should be in the range, $0.15/\omega_{\Phi_M} = 0.0038$ second to $0.5/\omega_{\Phi_M} = 0.0128$ second. Let us use $T = 0.001$ second. Now find the Tustin transformation for the compensator. Substituting $s = \dfrac{2(z-1)}{T(z-1)}$ into $G_c(s) = \dfrac{2.38(s+25.3)}{(s+60.2)}$ with $T = 0.001$ second yields

$$G_c(z) = 2.34\frac{(z-0.975)}{(z-0.9416)}.$$

13.10

$G_c(z) = \dfrac{X(z)}{E(z)} = \dfrac{1899z^2 - 3761z + 1861}{z^2 - 1.908z + 0.9075}$. Cross-multiply and obtain $\left(z^2 - 1.908z + 0.9075X(z) = \left(1899z^2 - 3761z + 1861\right)E(z)\right.$. Solve for the highest power of z operating on the output, $X(z)$, and obtain $z^2X(z) = \left(1899z^2 - 3761z + 1861\right) E(z) - (-1.908z + 0.9075)X(z)$. Solving for $X(z)$ on the left-hand side yields

$X(z) = \left(1899 - 3761z^{-1} + 1861z^{-2}\right)\ E(z) - \ \left(-1.908z^{-1} + 0.9075z^{-2}\right)X(z)$. Finally, we implement this last equation with the following flow chart: