BACKGROUND

B.1 COMPLEX NUMBERS

B.1-1 History

In 1799, the German mathematician *Karl Friedrich Gauss* proved the fundamental theorem of algebra… that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less.

B.1-2 Algebra of Complex Numbers

A complex number (a, b) or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are (a, b) in a complex plane (Fig. B.1). Let us denote this complex number by z so that

$$z = a + jb$$

This representation is the **Cartesian (or rectangular) form** of complex number z. The numbers a and b (the abscissa and the ordinate) of z are the *real part* and the *imaginary part*, respectively, of z. They are also expressed as

$$\text{Re}\,\{z\} = a \text{ and } \quad \text{Im}\,\{z\} = b$$

Compared to ordinary numbers that only have real part (which can be realized on a line), complex numbers have both real and imaginary parts and can be graphically realized in a plane (real part on the horizontal axis and imaginary part along the vertical axis).

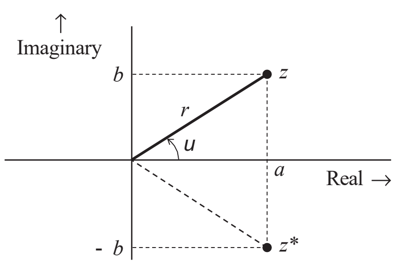

Figure B.1 Representation of a number in the complex plane.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.1), then

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta$$

Consequently,

$$z = a + jb = r\cos\theta + jr\sin\theta = r(\cos\theta + j\sin\theta)$$

Euler's formula (see page 6 of your textbook for detail) states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

which implies

$$z = re^{j\theta}$$

The above representation is the polar form of complex number z. A complex number can be expressed in rectangular or polar form as summarized below:

$$\begin{array}{ccc} a = r\cos\theta & & r = \sqrt{a^2 + b^2} \\ b = r\sin\theta & \text{and} & \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{array}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the *magnitude* (or *absolute value*) of z and is denoted by $|z|$. Similarly, θ is called the angle of z and is denoted by $\angle z$

$$z = |z|e^{j\angle z} \qquad \text{where } |z| = r \text{ and } \angle z = \theta$$

Using polar form, we see that the reciprocal of a complex number is given by

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z}$$

Conjugate of a Complex Number

We define z^*, the *conjugate* of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} = |z|e^{-j\angle z}$$

The graphical representations of a number z and its conjugate z^* are depicted in Fig. B.1. Note that conjugate z^* is a mirror image of z about the horizontal axis. *To find the conjugate of any number, we need only replace j with $-j$ in that number* (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z$$

Thus, we see that the real part of complex number z can be computed as

$$\mathrm{Re}\,z = \frac{z + z^*}{2}$$

Similarly, the imaginary part of complex number z can be computed as

$$\mathrm{Im}\,z = \frac{z - z^*}{2j}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = |z|e^{j\angle z}|z|e^{-j\angle z} = |z|^2$$

Useful identities

$$-1 = e^{j(\pi + 2\pi n)}$$

$$1 = e^{j2\pi n}$$

$$j = e^{j(\frac{\pi}{2} + 2\pi n)}$$

$$-j = e^{j(-\frac{\pi}{2} + 2\pi n)}$$

where n is an integer.

SEE EXAMPLES B.1 & B.2 (PAGE 8 TO 11 OF THE TEXTBOOK)

ARITHMETICAL OPERATIONS ON COMPLEX NUMBERS

- Addition and subtraction can be easily performed in CARTESIAN coordinates
- Multiplication, division, powers, and roots can be easily carried in POLAR coordinates

See examples of arithmetical operations on pages 12-15 of the textbook.

NATURAL LOGARITHMS OF COMPLEX NUMBERS

- Convert the complex number into polar form, if needed
- Next, take the natural logarithm

Suppose, we have a complex number $z = a + jb$, then first convert to polar form

$$z = re^{j\theta} = re^{j(\theta \pm 2\pi k)} \qquad k = 0, 1, 2, 3, \ldots$$

Taking the natural logarithm, we see that

$$\ln z = \ln\left(re^{j(\theta \pm 2\pi k)}\right) = \ln r \pm j(\theta + 2\pi k) \qquad k = 0, 1, 2, 3, \ldots$$

The value of $\ln z$ for $k = 0$ is called the *principal value* of $\ln z$ and is denoted by $\mathrm{Ln}\, z$. Using the above expression, we can show

$$\ln 1 = \ln(1e^{\pm j2\pi k}) = \pm j2\pi k \qquad k = 0, 1, 2, 3, \ldots$$

$$\ln(-1) = \ln[1e^{\pm j\pi(2k+1)}] = \pm j(2k+1)\pi \qquad k = 0, 1, 2, 3, \ldots$$

$$\ln j = \ln\left(e^{j\pi(1 \pm 4k)/2}\right) = j\frac{\pi(1 \pm 4k)}{2} \qquad k = 0, 1, 2, 3, \ldots$$

$$j^j = e^{j \ln j} = e^{-\pi(1 \pm 4k)/2} \qquad k = 0, 1, 2, 3, \ldots$$

ADDITIONAL PROPERTIES OF LOGARITHMS

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log(z_1/z_2) = \log z_1 - \log z_2$$

$$a^{(z_1+z_2)} = a^{z_1} \times a^{z_2}$$

$$z^c = e^{c \ln z}$$

$$a^z = e^{z \ln a}$$

B.2 SINUSOIDS

Consider the sinusoid

$$x(t) = C\cos(2\pi f_0 t + \theta)$$

We know that

$$\cos\phi = \cos(\phi + 2n\pi) \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

This shows that $\cos\phi$ repeats itself for every change of 2π in the angle ϕ. For the sinusoid $x(t)$, the angle $2\pi f_0 t + \theta$ changes by 2π when t changes by $1/f_0$. As a result, there are f_0 repetitions per second. This is the *frequency* of the sinusoid, and the repetition interval T_0 given by

$$T_0 = \frac{1}{f_0}$$

is the *period*. Moreover, C is the *amplitude* of the sinusoid, f_0 is the *frequency* (in hertz), and θ is the phase.

SPECIAL CASE: $\theta = 0$

$$x(t) = C\cos 2\pi f_0 t$$

SPECIAL CASE: $\theta = -\pi/2$

$$x(t) = C\cos\left(2\pi f_0 t - \frac{\pi}{2}\right) = C\sin 2\pi f_0 t$$

The angle or phase can be expressed in units of degrees or radians. It is convenient to use the variable ω_0 (*radian frequency*) to express $2\pi f_0$:

$$\omega_0 = 2\pi f_0$$

With this notation, the sinusoid $x(t)$ can be expressed as

$$x(t) = C\cos(\omega_0 t + \theta)$$

in which the period T_0 and frequency ω_0 are given by

$$T_0 = \frac{1}{\omega_0/2\pi} = \frac{2\pi}{\omega_0} \quad \text{and} \quad \omega_0 = \frac{2\pi}{T_0}$$

B.2-1 Sketching and Shifting Sinusoids

Two sinusoids having the same frequency but different phases add to form a single sinusoid of the same frequency. This fact is readily seen from the well-known trigonometric identity

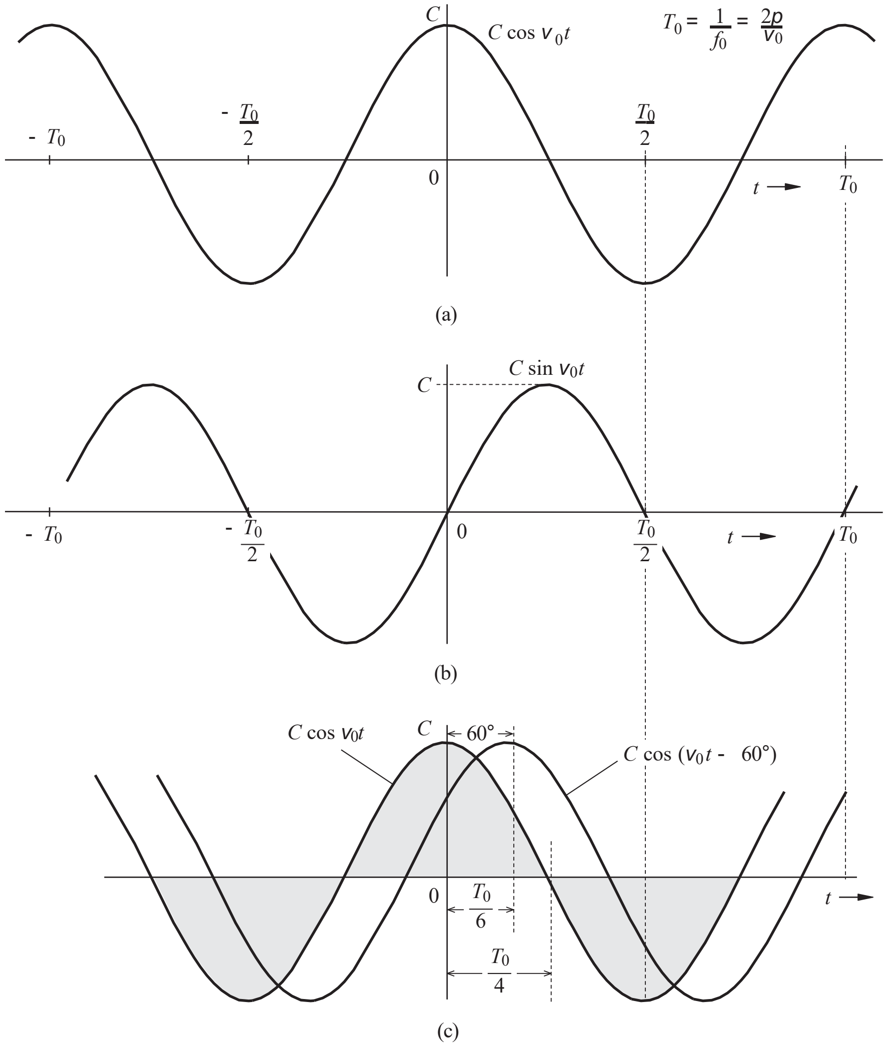

Figure B.2 Sketching a sinusoid.

The signal $C\cos(\omega_0 t - 60^\circ)$ can be obtained by shifting (delaying) the signal $C\cos\omega_0 t$ to the right by a phase (angle) of 60°.

B.2-2 Addition of Sinusoids

Two sinusoids having the same frequency but different phases add to form a single sinusoid of the same frequency. This fact is readily seen from the well-known trigonometric identity

$$C\cos\theta\cos\omega_0 t - C\sin\theta\sin\omega_0 t = C\cos(\omega_0 t + \theta)$$

Setting $a = C\cos\theta$ and $b = -C\sin\theta$, we see that

$$a\cos\omega_0 t + b\sin\omega_0 t = C\cos(\omega_0 t + \theta)$$

where,

$$C = \sqrt{a^2+b^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

This shows that C and θ are the magnitude and angle, respectively, of a complex number $a - jb$.

The process of adding two sinusoids with the same frequency can be clarified by using *phasors* to represent sinusoids. We represent the sinusoid $C\cos(\omega_0 t + \theta)$ by a phasor of length C at an angle θ with the horizontal axis. Clearly, the sinusoid $a\cos\omega_0 t$ is represented by a horizontal phasor of length $a(\theta = 0)$, while $b\sin\omega_0 t = b\cos(\omega_0 t - \pi/2)$ is represented by a vertical phasor of length b at an angle $-\pi/2$ with the horizontal (Fig. B.3). Adding these two phasors results in a phasor of length C at an angle θ.

We can also perform the reverse operation, expressing $C\cos(\omega_0 t + \theta)$ in terms of $\cos\omega_0 t$ and $\sin\omega_0 t$ by again using the trigonometric identity

$$C\cos(\omega_0 t + \theta) = C\cos\theta\cos\omega_0 t - C\sin\theta\sin\omega_0 t$$

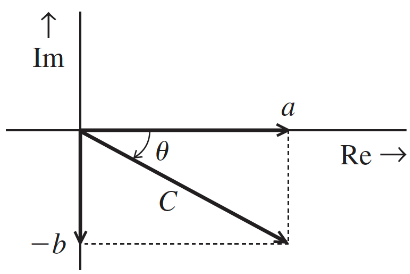

Figure B.3 Phasor addition of sinusoids.

SEE EXAMPLE B.6 (PAGE 19 TO 20 OF THE TEXTBOOK)

B.2-3 Sinusoids in Terms of Exponentials

From Euler's equation, we know that

$$e^{j\phi} = \cos\phi + j\sin\phi$$

and

$$e^{-j\phi} = \cos\phi - j\sin\phi.$$

Adding these two expressions and dividing by 2 provides an expression for cosine in terms of complex exponentials, while subtracting and scaling by $2j$ provides an expression for sine, as below:

$$\cos\varphi = \frac{1}{2}(e^{j\varphi} + e^{-j\varphi}) \qquad \text{and} \qquad \sin\varphi = \frac{1}{2j}(e^{j\varphi} - e^{-j\varphi})$$

B.3 Sketching Signals

This section discusses sketching a few useful signals. These include exponentials, exponentials and exponentially varying sinusoids. Please see page 20 to 23 of your textbook.

You should also know how to use MATLAB to sketch different signals.

B.4 Cramer's Rule

Cramer's rule offers a very convenient way to solve simultaneous linear equations. Consider a set of n linear simultaneous equations in n unknowns x_1, x_2, . . ., x_n:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n
\end{aligned}$$

These equations can be expressed in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

We denote the matrix on the left-hand side formed by the elements a_{ij} as **A**. The determinant of **A** is denoted by $|\mathbf{A}|$. If the determinant $|\mathbf{A}|$ is not zero, the system of equations has a unique solution given by Cramer's formula

$$x_k = \frac{|\mathbf{D}_k|}{|\mathbf{A}|} \qquad k = 1, 2, \ldots, n$$

where $|\mathbf{D}_k|$ is obtained by replacing the kth column of $|\mathbf{A}|$ by the column on the right-hand side.

SEE EXAMPLE B.7 (PAGE 24 TO 25 OF THE TEXTBOOK)

B.5 Partial Fraction Expansion

Functions that are ratios of two polynomials in a certain variable, say, x. Such functions are known as *rational functions*. A rational function $F(x)$ can be expressed as

$$F(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} = \frac{P(x)}{Q(x)}$$

The function $F(x)$ is *improper* if $m \geq n$ and *proper* if $m < n$.† An improper function can always be separated into the sum of a polynomial in x and a proper function. Consider, for example, the function

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3}$$

Because this is an improper function, we divide the numerator by the denominator until the remainder has a lower degree than the denominator.

$$\begin{array}{r|l} & \quad 2x + 1 \\ \hline x^2 + 4x + 3 & 2x^3 + 9x^2 + 11x + 2 \\ & \underline{2x^3 + 8x^2 + 6x} \\ & \qquad\quad x^2 + 5x + 2 \\ & \qquad\quad \underline{x^2 + 4x + 3} \\ & \qquad\qquad\quad x - 1 \end{array}$$

B.5-1 Method of Clearing Fractions

A rational function can be written as a sum of appropriate partial fractions with unknown coefficients, which are determined by clearing fractions and equating the coefficients of similar powers on the two sides. This procedure is demonstrated by the following example.

EXAMPLE:

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2}$$

This function can be expressed as a sum of partial fractions with denominators $(x+1)$, $(x+2)$, $(x+3)$, and $(x+3)^2$, as follows:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x+1)(x+2)(x+3)^2} = \frac{k_1}{x+1} + \frac{k_2}{x+2} + \frac{k_3}{x+3} + \frac{k_4}{(x+3)^2}$$

To determine the unknowns k_1, k_2, k_3, and k_4, we clear fractions by multiplying both sides by $(x+1)(x+2)(x+3)^2$ to obtain

$$\begin{aligned} x^3 + 3x^2 + 4x + 6 &= k_1(x^3 + 8x^2 + 21x + 18) + k_2(x^3 + 7x^2 + 15x + 9) \\ &\quad + k_3(x^3 + 6x^2 + 11x + 6) + k_4(x^2 + 3x + 2) \\ &= x^3(k_1 + k_2 + k_3) + x^2(8k_1 + 7k_2 + 6k_3 + k_4) \\ &\quad + x(21k_1 + 15k_2 + 11k_3 + 3k_4) + (18k_1 + 9k_2 + 6k_3 + 2k_4) \end{aligned}$$

Equating coefficients of similar powers on both sides yields

$$\begin{aligned} k_1 + k_2 + k_3 &= 1 \\ 8k_1 + 7k_2 + 6k_3 + k_4 &= 3 \\ 21k_1 + 15k_2 + 11k_3 + 3k_4 &= 4 \\ 18k_1 + 9k_2 + 6k_3 + 2k_4 &= 6 \end{aligned}$$

Solving simultaneously, we get

$$k_1 = 1, \qquad k_2 = -2, \qquad k_3 = 2, \qquad k_4 = -3$$

Therefore,

$$F(x) = \frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x+3} - \frac{3}{(x+3)^2}$$

B.5-2 The Heaviside "Cover-Up" Method

<u>DISTINCT FACTORS OF *Q(x)*</u>

We shall first consider the partial fraction expansion of $F(x) = P(x)/Q(x)$, in which all the factors of $Q(x)$ are distinct (not repeated). Consider the proper function

$$\begin{aligned} F(x) &= \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0} \qquad m < n \\ &= \frac{P(x)}{(x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)} \end{aligned}$$

As seen in previous example, $F(x)$ can be expressed as the sum of partial fractions

$$F(x) = \frac{k_1}{x-\lambda_1} + \frac{k_2}{x-\lambda_2} + \cdots + \frac{k_n}{x-\lambda_n}$$

To determine the coefficient k_1, we multiply both sides of Eq. (B.23) by $(x - \lambda_1)$ and then let $x = \lambda_1$. This yields

$$(x-\lambda_1)F(x)|_{x=\lambda_1} = k_1 + \frac{k_2(x-\lambda_1)}{(x-\lambda_2)} + \frac{k_3(x-\lambda_1)}{(x-\lambda_3)} + \cdots + \left.\frac{k_n(x-\lambda_1)}{(x-\lambda_n)}\right|_{x=\lambda_1}$$

On the right-hand side, all the terms except k_1 vanish. Therefore,

$$k_1 = (x-\lambda_1)F(x)|_{x=\lambda_1}$$

Similarly, we can show that

$$k_r = (x-\lambda_r)F(x)|_{x=\lambda_r} \qquad r = 1, 2, \ldots, n$$

This procedure is also known as the *method of residues*.

SEE EXAMPLE B.9 (PAGE 28 OF THE TEXTBOOK)

Complex Factors of *Q(x)*

The Heaviside "Cover-up" method method works just fine… regardless of whether the factors of *Q(x)* are real or complex. Consider the following function with complex factors in the denominator:

$$F(x) = \frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{4x^2+2x+18}{(x+1)(x+2-j3)(x+2+j3)}$$
$$= \frac{k_1}{x+1} + \frac{k_2}{x+2-j3} + \frac{k_3}{x+2+j3}$$

where

$$k_1 = \left[\frac{4x^2+2x+18}{(x+1)\,(x^2+4x+13)}\right]_{x=-1} = 2$$

Similarly,

$$k_2 = \left[\frac{4x^2+2x+18}{(x+1)\,(x+2-j3)\,(x+2+j3)}\right]_{x=-2+j3} = 1+j2 = \sqrt{5}e^{j63.43^\circ}$$

$$k_3 = \left[\frac{4x^2+2x+18}{(x+1)(x+2-j3)\,(x+2+j3)}\right]_{x=-2-j3} = 1-j2 = \sqrt{5}e^{-j63.43^\circ}$$

Therefore,

$$F(x) = \frac{2}{x+1} + \frac{\sqrt{5}e^{j63.43^\circ}}{x+2-j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{x+2+j3}$$

Note that the factor k_2 and k_3 are not only complex, but also complex conjugate of each other.

Quadratic Factors

Often we are required to combine the two terms arising from complex-conjugate factors into one quadratic factor. For example, function $F(x)$ in the above example can also be expressed as:

$$F(x) = \frac{4x^2+2x+18}{(x+1)(x^2+4x+13)} = \frac{k_1}{x+1} + \frac{c_1x+c_2}{x^2+4x+13}$$

Here, the factor k_1 can be determined by using the Heaviside "Cover-up" method and factors c_1 and c_2 can determined using the "Clearing" method.

SEE EXAMPLE (ON PAGE 29 AND 30 OF THE TEXTBOOK)

SHORTCUTS

The values of c_1 and c_2 in the example with quadratic factor in the denominator can also be determined by using shortcuts.

SEE EXAMPLE (ON PAGE 30 OF THE TEXTBOOK)

B.5-3 Repeated Factors of *Q(x)*

If a function $F(x)$ has a repeated factor in its denominator, it has the form

$$F(x) = \frac{P(x)}{(x-\lambda)^r(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_j)} \tag{5.3.1)}$$

Its partial fraction expansion is given by

$$\begin{aligned} F(x) = {} & \frac{a_0}{(x-\lambda)^r} + \frac{a_1}{(x-\lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{(x-\lambda)} \\ & + \frac{k_1}{x-\alpha_1} + \frac{k_2}{x-\alpha_2} + \cdots + \frac{k_j}{x-\alpha_j} \end{aligned} \tag{5.3.2)}$$

The coefficients $k_1, k_2, \ldots, k_j$ corresponding to the unrepeated factors are determined using the Heaviside method

To find the coefficients $a_0, a_1, a_2, \ldots, a_{r-1}$, we multiply both sides by the factor $(x - \lambda)^r$, which yields

$$\begin{aligned} (x-\lambda)^r F(x) = {} & a_0 + a_1(x-\lambda) + a_2(x-\lambda)^2 + \cdots + a_{r-1}(x-\lambda)^{r-1} \\ & + k_1\frac{(x-\lambda)^r}{x-\alpha_1} + k_2\frac{(x-\lambda)^r}{x-\alpha_2} + \cdots + k_n\frac{(x-\lambda)^r}{x-\alpha_n} \end{aligned} \tag{5.3.3)}$$

If we let $x = \lambda$ on both sides, we obtain

$$(x-\lambda)^r F(x)|_{x=\lambda} = a_0$$

Therefore, a_0 is obtained by concealing the factor $(x - \lambda)^r$ in $F(x)$ and letting $x = \lambda$ in the remaining expression (the Heaviside "cover-up" method).

If we take the derivative (with respect to x) of both sides of Eq. (5.3.3), on the right-hand side the term a_0 vanishes and what remains is a_1+ terms containing a factor $(x - \lambda)$ in their numerators. Letting $x = \lambda$ on both sides of this equation, we obtain

$$\frac{d}{dx}\left[(x-\lambda)^r F(x)\right]\Big|_{x=\lambda} = a_1$$

Thus, a_1 is obtained by concealing the factor $(x - \lambda)^r$ in $F(x)$, taking the derivative of the remaining expression, and then letting $x = \lambda$.

Continuing in this manner, we find

$$a_j = \frac{1}{j!} \frac{d^j}{dx^j} \left[(x-\lambda)^r F(x) \right] \Big|_{x=\lambda}$$

Thus, a_r is obtained by concealing the factor $(x - \lambda)^r$ in $F(x)$, taking the j^{th} derivative of the remaining expression, and then letting $x = \lambda$ while dividing by $j!$

SEE EXAMPLE B.10 (ON PAGE 32 OF THE TEXTBOOK)

B.5-4 Combination of Heaviside "Cover-Up" and Clearing Fractions with Shortcuts

Sometime, taking derivatives can be cumbersome. In this case, one can use:

- Combination of Heaviside "cover-up" and "clearing" fractions
- Combination of Heaviside "cover-up" and shortcuts

SEE EXAMPLES (ON PAGE 32 TO 34 OF THE TEXTBOOK)

B.5-5 Improper *F(x)* with *m* = *n*

For the special case when the numerator and denominator polynomials of $F(x)$ have the same degree ($m = n$), the procedure is the same as that for a proper function. The function

$$F(x) = \frac{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}$$

can be rewritten as

$$= b_n + \frac{k_1}{x - \lambda_1} + \frac{k_2}{x - \lambda_2} + \cdots + \frac{k_n}{x - \lambda_n}$$

the coefficients $k_1, k_2, \ldots, k_n$ are computed as if $F(x)$ were proper. Thus

$$k_r = (x - \lambda_r) F(x) |_{x=\lambda_r}$$

SEE EXAMPLE B.11 (ON PAGE 34 TO 35 OF THE TEXTBOOK)

B.5-6 Modified Partial Fractions

In some cases, partial fractions of the form $kx/(x - \lambda_i)^r$ are required rather than of the form $k/(x - \lambda_i)^r$. This can be achieved by expanding $F(x)/x$ into partial fractions. Consider, for example,

$$F(x) = \frac{5x^2 + 20x + 18}{(x+2)(x+3)^2}$$

Dividing both sides by x yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2}$$

Expansion of the right-hand side into partial fractions yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x+2)(x+3)^2} = \frac{a_1}{x} + \frac{a_2}{x+2} + \frac{a_3}{(x+3)} + \frac{a_4}{(x+3)^2}$$

Using the procedure discussed earlier, we find $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = 1$.

Therefore,

$$\frac{F(x)}{x} = \frac{1}{x} + \frac{1}{x+2} - \frac{2}{x+3} + \frac{1}{(x+3)^2}$$

which yields

$$F(x) = 1 + \frac{x}{x+2} - \frac{2x}{x+3} + \frac{x}{(x+3)^2}$$

B.6 Vectors and Matrices

An entity specified by n numbers in a certain order (ordered n-tuple) is an n-dimensional *vector*. Thus, an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ represents an n-dimensional vector **x**. A vector may be represented as a row (*row vector*):

$$\mathbf{x} = [\ x_1\ \ x_2\ \ \cdots\ \ x_n\ ]$$

or as a column (*column vector*):

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Simultaneous linear equations can be viewed as transformation of one vector to another. Consider the following m simultaneous equations:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

We can write the above simultaneous linear equations in matrix form as below:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad \mathbf{y} = \mathbf{A}\mathbf{x}$$

where **x** and **y** are column vectors and **A** is a matrix of order (m, n) or $(m \times n)$ indicating m rows and n columns.

B.6-1 Definitions and Examples

Covers definitions and examples of most common matrices, including:

- Square matrix
- Identity matrix
- Zero matrix

SEE EXAMPLES (ON PAGE 37 OF THE TEXTBOOK)

B.6-2 Matrix Algebra

Covers operations, such as:

- Addition of matrix
- Subtraction of matrix
- Multiplication and division of a matrix with a scalar
- Matrix multiplication
- Determinant of a matrix
- Matrix inversion
- Division of matrices

SEE EXAMPLES AND DETAILS (ON PAGE 38 TO 42 OF THE TEXTBOOK)

B.7 MATLAB: ELEMENTARY OPERATIONS

B.7-1 MATLAB Overview

- MATLAB (a registered trademark of The MathWorks, Inc.) is easy to use scientific computing software
- When MATLAB is first launched, its command window appears. When MATLAB is ready to accept an instruction or input, a command prompt (>>) is displayed in the command window. Nearly all MATLAB activity is initiated at the command prompt
- Several (thousands of) functions are available to compute different operations from matrix algebra to different kind of transforms, filter design, and graphs for sake of visualization
- Use this link to launch and complete the MATLAB Onramp:
  https://www.mathworks.com/products/matlab/getting-started.html