Appendix I

Derivation of a Schematic for a DC Motor

The objective of this appendix is to derive a schematic for an armature-controlled dc motor, including parameter relationships.[1] This schematic can then be used to obtain the relationship between the input voltage and output angular displacement of a motor.

We begin by describing the behavior of electric currents in the presence of magnetic fields. From Figure I.1, an electric current, i, flowing in a wire of length, l, in the presence of a magnetic field of strength B feels a force, F, equal to

$$F = Bli \qquad (\text{I.1})$$

This principle can be used to build a motor, since the force produced can be used to turn a rotating member called a *rotor*. Basically, we attach the wire to the rotor, create a magnetic field with magnets, and allow the force to act to turn the rotor. In Figure I.2(a), we show the wire wrapped longitudinally around a cylindrical rotor, which is called an *armature*. The core is made of soft iron that easily conducts magnetic lines of force. The magnets are cut so that at the surface of the rotor the lines of magnetic flux are perpendicular. The force produced is tangent to the rotor and causes the armature to rotate. Each side of the loop feels a force as given by Eq. (I.1). Thus the total force, F, which acts tangential to the core, is

$$F = 2Bli \qquad (\text{I.2})$$

where i_a is the current through the wire on the armature. If the radius of the armature is r, then the developed torque, T_m, is

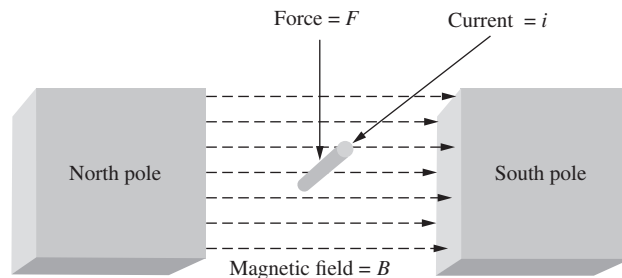$$T_m = rF = r2Bli_a = K_t i_a \qquad (\text{I.3})$$

Force = F Current = i

North pole   South pole

Magnetic field = B

FIGURE I.1   Current-carrying wire in a magnetic field

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[1] This appendix summarizes the derivations discussed in detail in the references listed in the Bibliography.

FIGURE I.2   a. Current-carrying wire on a rotor; **b.** current-carrying wire on a rotor with commutation and coils added to the permanent magnets to increase magnetic field strength
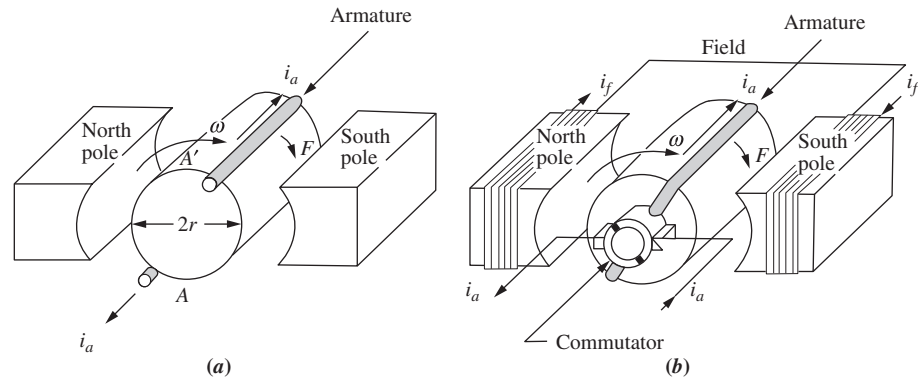
where $K_t = 2Blr$ is called the *torque constant*, and i_a is called the *armature current*. To develop more torque, we can increase the magnetic field strength, B, or equivalently K_t, by wrapping a coil around the permanent magnets as shown in Figure I.2(*b*).

There is a problem with the motor in Figure I.2(*a*). As the wire segment shown near point A' passes point A, the force will still be downward and the armature will begin to turn in the opposite direction. To correct for this reversal, we need to reverse the direction of the current flow, i_a when that wire segment reaches point A. Another reversal is required again at point A'. Figure I.2(*b*) shows how we can reverse the current in the wire every half turn to keep the armature turning in the same direction. Using *slip rings* to which the ends of the wire are connected, and *brushes*, which are stationary and rub against the slip rings, completing the circuit for i_a to flow, the current reverses every half turn. This arrangement of slip rings and brushes is called a *commutator*.

So far we have seen that a force exists on the wire wrapped around the armature. We now explore the possibility that a voltage is also induced across the terminals of this wire. Knowledge of this voltage will help us establish a circuit model for the motor. According to Faraday's law, if a loop of wire, such as a single loop wrapped around the armature, is placed in a changing magnetic field, a voltage

$$v_b(t) = -\frac{d\phi}{dt} \qquad (I.4)$$

will be induced across the loop, where $v_b(t)$ is the induced voltage and ϕ is the flux passing through the loop. Assuming that the flux density, B, is a constant anywhere along the surface of a half cylinder enclosed by the loop, and using $\phi = BA$, where A is the surface area of the enclosed half cylinder, the induced voltage for our single loop is

$$v_b(t) = -B\frac{dA}{dt} \qquad (I.5)$$

To determine the surface area of the half cylinder enclosed by the loop, we examine Figure I.3, which is an end view of the armature showing the ends of the wire and the magnetic lines of flux. The area, A_1, through which the flux passes from the center to the surface is the product of the circumference and length of the cylinder, or

$$A_1 = r(\pi - \theta)l \qquad (I.6)$$

The area, A_2, through which the flux passes from the surface to the center is

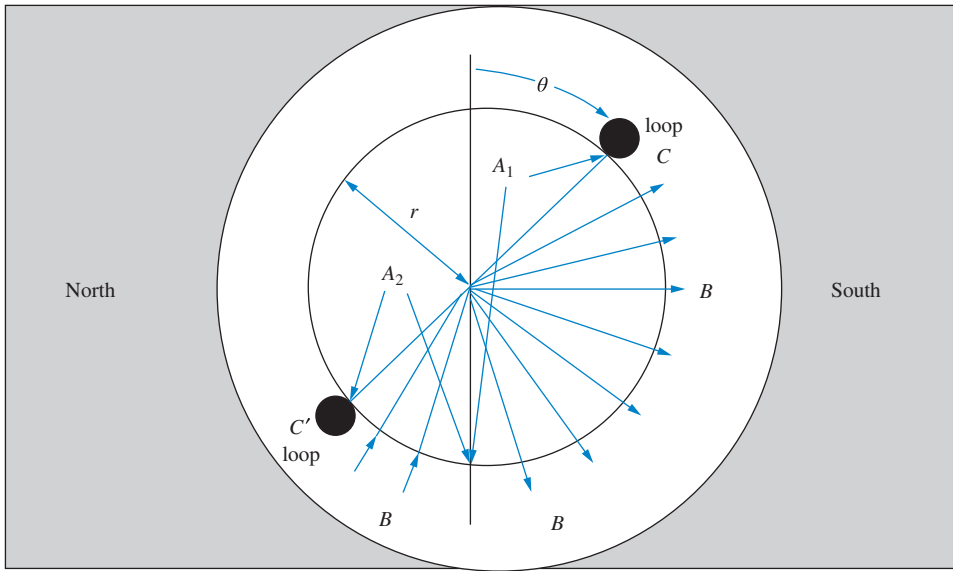$$A_2 = (r\theta)l \qquad (I.7)$$

FIGURE I.3   Magnetic flux density passing through a loop of wire on an armature

Thus,

$$v_b(t) = -\frac{dB(A_1 - A_2)}{dt} = -Brl\frac{d(\pi - 2\theta)}{dt} = 2Brl\frac{d\theta}{dt} = 2Brl\omega = K_b\omega \qquad (I.8)$$

For a motor, this induced voltage, $v_b(t)$, is called the *back electromagnetic force, or back emf*. The constant $K_b = 2Blr$ is called *the back emf constant*. Notice that $K_t = K_b$ in a consistent set of units.

Finally, the torque developed can be increased by winding a number of wire loops on the armature. This is called the *armature winding*. The back emf will also be increased because these loops can be thought of as a series connections where the voltages in each loop will add.

In summary, if a wire carrying current $i_a(t)$ passes through a magnetic field, a torque, $T_m(t) = K_t i_a(t)$, will be developed. Further, from Faraday's law the wire will have an induced voltage called the back emf. This voltage is given by $v_b(t) = K_b\omega(t)$.

From the relationships that we developed, we can create a circuit model for the motor as shown in Figure I.4. The armature winding is represented as having resistance, R_a, and, because it is wound around the armature, inductance, L_a. The back emf is shown as $v_b(t)$
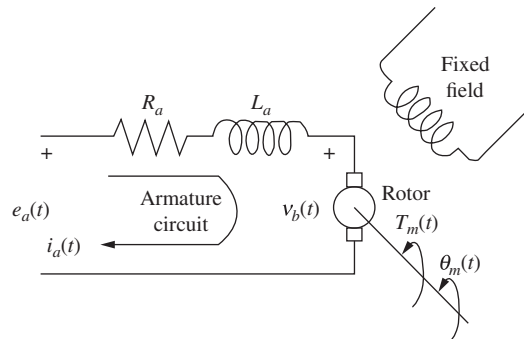

FIGURE I.4   DC motor circuit diagram

across the rotor. The fixed field is created by the electromagnets. Shown at the output of the rotor are the developed torque, $T_m(t)$, and the output angular displacement, $\theta_m(t)$.

Bibliography

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