Appendix K

Solution of State Equations for $t_0 \neq 0$

In Section 4.11 we used the state-transition matrix to perform a transformation taking $\mathbf{x}(t)$ from an initial time, $t_0 = 0$, to any time, $t \geq 0$, as defined in Eq. (4.109). What if we wanted to take $\mathbf{x}(t)$ from a different initial time, $t_0 \neq 0$, to any time $t \geq t_0$; would Eq. (4.109) and the state-transition matrix change? To find out, we need to convert Eq. (4.109) into a form that shows $t_0 \neq 0$ as the initial state rather than $t_0 = 0$ (*Kuo, 1991*).

Using Eq. (4.109), we find $\mathbf{x}(t)$ at t_0 to be

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{x}(0) + \int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{K.1}$$

Solving for $\mathbf{x}(0)$ by premultiplying both sides of Eq. (K.1) by $\mathbf{\Phi}^{-1}(t_0)$ and rearranging,

$$\mathbf{x}(0) = \mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \tag{K.2}$$

Substituting Eq. (K.2) into Eq. (4.109) yields

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t)(\mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\
&\quad + \int_0^{t_0} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\
&= \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\mathbf{x}(t_0) - \mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0)\int_0^{t_0} \mathbf{\Phi}(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\
&\quad + \int_0^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau
\end{aligned} \tag{K.3}$$

Since $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ and $\mathbf{\Phi}(-t) = e^{-\mathbf{A}t}, \mathbf{\Phi}(t)\mathbf{\Phi}(-t) = \mathbf{I}$. Hence,

$$\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t) \tag{K.4}$$

Therefore

$$\mathbf{\Phi}(t)\mathbf{\Phi}^{-1}(t_0) = e^{\mathbf{A}t}e^{-\mathbf{A}t_0} = e^{\mathbf{A}(t-t_0)} = \mathbf{\Phi}(t - t_0) \tag{K.5}$$

Substituting Eq. (K.5) into Eq. (K.3) yields

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0) - \int_0^{t_0} \mathbf{\Phi}(t - t_0)\mathbf{\Phi}(t_0 - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\
&\quad + \int_0^{t} \mathbf{\Phi}(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau
\end{aligned} \tag{K.6}$$

But

$$\mathbf{\Phi}(t-t_0)\mathbf{\Phi}(t_0-\tau) = e^{\mathbf{A}(t-t_0)}e^{\mathbf{A}(t_0-\tau)} = e^{\mathbf{A}(t_0-\tau)} = \mathbf{\Phi}(t-\tau) \tag{K.7}$$

Substituting Eq. (K.7) into Eq. (K.6),

$$\mathbf{x}(t) = \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) - \int_0^{t_0} \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)d\tau + \int_0^{t} \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)d\tau \tag{K.8}$$

Combining the two integrals finally yields

$$\mathbf{x}(t) = \mathbf{\Phi}(t-t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t-\tau)\mathbf{Bu}(\tau)d\tau \tag{K.9}$$

Equation (K.9) is more general than Eq. (4.109) in that it allows us to find $\mathbf{x}(t)$ after an initial time other than $t_0 = 0$. We can see that the state-transition matrix, $\mathbf{\Phi}(t-t_0)$, is of a more general form than previously described. In particular, the state-transition matrix is also a function of the initial time. We conclude this section by deriving some important properties of $\mathbf{\Phi}(t-t_0)$.

Using Eq. (K.4), the inverse of $\mathbf{\Phi}(t-t_0)$ is

$$\mathbf{\Phi}^{-1}(t-t_0) = \mathbf{\Phi}(t_0-t) \tag{K.10}$$

Also, from Eq. (K.7),

$$\mathbf{\Phi}(t_2-t_0) = \mathbf{\Phi}(t_2-t_1)\mathbf{\Phi}(t_1-t_0) \tag{K.11}$$

which states that the transformation from t_0 to t_2 is the product of the transformation from t_0 to t_1 and the transformation from t_1 to t_2.

Bibliography

Kuo, B. *Automatic Control Systems*, 6th ed. Prentice-Hall, Englewood Cliffs, NJ, 1991.