Topic: Upper and lower triangular matrices

Question: Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer choices:

A $|A| = 0$

B $|A| = 4$

C $|A| = 8$

D $|A| = 12$

Solution: D

Because A is an upper-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. Looking at A,

$$A = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

the determinant is given by

$$|A| = (-2)(-1)(3)(2)$$

$$|A| = 12$$

Topic: Upper and lower triangular matrices

Question: Calculate the determinant $|M|$ in two ways.

$$M=\begin{bmatrix}1&0&0&0\\2&-1&0&0\\-2&0&2&0\\1&2&3&-1\end{bmatrix}$$

Answer choices:

A $\quad|M|=0$

B $\quad|M|=1$

C $\quad|M|=2$

D $\quad|M|=4$

Solution: C

Given any triangular matrix, we can calculate the determinant using the traditional method, where we break the entire determinant down into a sum of 2×2 determinants, or we can simply multiply the values along the main diagonal.

With the matrix M,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix}$$

first calculate the determinant by multiplying the values along the main diagonal.

$$|M| = (1)(-1)(2)(-1)$$

$$|M| = 2$$

Second, calculate the determinant by breaking down the 4×4 determinant eventually into 2×2 determinants. We'll work across the first row, since it includes lots of 0 entries.

$$|M| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{vmatrix}$$

$$|M| = 1\begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 0 \\ -2 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

The last three determinants cancel out, because of the 0 scalars.

$$|M| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & -1 \end{vmatrix}$$

Now break the remaining 3×3 determinant into 2×2 determinants.

$$|M| = -1 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}$$

$$|M| = - \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$|M| = -[(2)(-1) - (0)(3)]$$

$$|M| = -(-2 - 0)$$

$$|M| = 2$$

Topic: Upper and lower triangular matrices

Question: Put Z into upper- or lower-triangular form in order to find the determinant.

$$Z = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix}$$

Answer choices:

A $\quad |Z| = 12$

B $\quad |Z| = -12$

C $\quad |Z| = 0$

D $\quad |Z| = -2$

Solution: B

There are four 0 entries above the main diagonal, and only one 0 below the main diagonal, so it'll be easier to turn this into a lower-triangular matrix, in which all the entries above the main diagonal are 0.

To get the matrix in lower-triangular form, we'll work in the opposite order that we normally use to find upper-triangular form. First, find a pivot of 1 in the $z_{4,4}$ position.

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_2 - R_4 \to R_2$, Z is

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ -2 & 0 & 2 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

Find a pivot of 1 in the $z_{3,3}$ position.

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, Z is

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, Z is

$$\begin{bmatrix} -6 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

The matrix is now in lower triangular form. Because we never switched a pair of rows, we never had to multiply the determinant by -1. But we did multiply a row in the matrix by -1, and another row by $1/2$, so if we call the reduced row-echelon matrix the matrix B, then

$$|B| = \frac{1}{2}(-1)|Z|$$

$$|B| = -\frac{1}{2}|Z|$$

$$(-6)(-1)(1)(1) = -\frac{1}{2}|Z|$$

$$6 = -\frac{1}{2}|Z|$$

$$|Z| = -12$$