Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A=\begin{bmatrix}1 & -4 & 2\\ 0 & 4 & -3\\ 0 & 0 & -2\end{bmatrix}$$

Answer choices:

A $\begin{bmatrix}0\\ \frac{1}{2}\\ 1\end{bmatrix}$, $\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$, and $\begin{bmatrix}\frac{4}{3}\\ -1\\ 0\end{bmatrix}$

B $\begin{bmatrix}0\\ \frac{1}{2}\\ 1\end{bmatrix}$, $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$, and $\begin{bmatrix}-\frac{4}{3}\\ 1\\ 0\end{bmatrix}$

C $\begin{bmatrix}0\\ 1\\ 3\end{bmatrix}$, $\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$, and $\begin{bmatrix}\frac{4}{3}\\ -1\\ 0\end{bmatrix}$

D $\begin{bmatrix}0\\ 1\\ 3\end{bmatrix}$, $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$, and $\begin{bmatrix}-\frac{4}{3}\\ 1\\ 0\end{bmatrix}$

Solution: B

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 2 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -4 & 2 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 1 & 0 - (-4) & 0 - 2 \\ 0 - 0 & \lambda - 4 & 0 - (-3) \\ 0 - 0 & 0 - 0 & \lambda - (-2) \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 1 & 4 & -2 \\ 0 & \lambda - 4 & 3 \\ 0 & 0 & \lambda + 2 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda - 1) \begin{vmatrix} \lambda - 4 & 3 \\ 0 & \lambda + 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & -2 \\ 0 & \lambda + 2 \end{vmatrix} + 0 \begin{vmatrix} 4 & -2 \\ \lambda - 4 & 3 \end{vmatrix}$$

$$(\lambda - 1)[(\lambda - 4)(\lambda + 2) - (3)(0)]$$

$$(\lambda - 1)(\lambda - 4)(\lambda + 2)$$

$\lambda = -2$ or $\lambda = 1$ or $\lambda = 4$

With $\lambda = -2$, $\lambda = 1$ and $\lambda = 4$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & 4 & -2 \\ 0 & \lambda - 4 & 3 \\ 0 & 0 & \lambda + 2 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 1 & 4 & -2 \\ 0 & -2 - 4 & 3 \\ 0 & 0 & -2 + 2 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -3 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 - 1 & 4 & -2 \\ 0 & 1 - 4 & 3 \\ 0 & 0 & 1 + 2 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 0 & 4 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 - 1 & 4 & -2 \\ 0 & 4 - 4 & 3 \\ 0 & 0 & 4 + 2 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 3 & 4 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -3 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 & -2 & | & 0 \\ 0 & -6 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & -6 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 - \frac{1}{2}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = \frac{1}{2}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 0 & 4 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & -2 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & \frac{3}{2} & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 3 & 4 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & -2 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{2}{3} & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{2}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{4}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{4}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

B $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

D $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ -1 \end{bmatrix}$

Solution: C

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - (-2) & 0 - 0 & 0 - 0 \\ 0 - 1 & \lambda - 3 & 0 - 0 \\ 0 - 0 & 0 - (-3) & \lambda - 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & 0 & 0 \\ -1 & \lambda - 3 & 0 \\ 0 & 3 & \lambda - 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 2) \begin{vmatrix} \lambda - 3 & 0 \\ 3 & \lambda - 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} + 0 \begin{vmatrix} -1 & \lambda - 3 \\ 0 & 3 \end{vmatrix}$$

$$(\lambda + 2)[(\lambda - 3)(\lambda - 1) - (0)(3)]$$

$$(\lambda + 2)(\lambda - 3)(\lambda - 1)$$

$\lambda = -2$ or $\lambda = 1$ or $\lambda = 3$

With $\lambda = -2$, $\lambda = 1$ and $\lambda = 3$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda+2 & 0 & 0 \\ -1 & \lambda-3 & 0 \\ 0 & 3 & \lambda-1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2+2 & 0 & 0 \\ -1 & -2-3 & 0 \\ 0 & 3 & -2-1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 3 & -3 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1+2 & 0 & 0 \\ -1 & 1-3 & 0 \\ 0 & 3 & 1-1 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 3 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 3 & 0 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3+2 & 0 & 0 \\ -1 & 3-3 & 0 \\ 0 & 3 & 3-1 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 5 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 3 & -3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ -1 & -5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + 5v_3 = 0$$

$$v_2 - v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -5v_3$$

$$v_2 = v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 5 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & | & 0 \\ -1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 + \frac{2}{3}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = -\frac{2}{3}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix}-5\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \text{ and } \begin{bmatrix}0\\-\frac{2}{3}\\1\end{bmatrix}$$

Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{2}{3} \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{2}{3} \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Solution: C

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 1 & 0 - 2 & 0 - (-1) \\ 0 - 0 & \lambda - (-2) & 0 - 0 \\ 0 - 0 & 0 - 0 & \lambda - 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda - 1)\begin{vmatrix} \lambda + 2 & 0 \\ 0 & \lambda - 1 \end{vmatrix} - 0 \begin{vmatrix} -2 & 1 \\ 0 & \lambda - 1 \end{vmatrix} + 0 \begin{vmatrix} -2 & 1 \\ \lambda + 2 & 0 \end{vmatrix}$$

$$(\lambda - 1)[(\lambda + 2)(\lambda - 1) - (0)(0)]$$

$$(\lambda - 1)(\lambda + 2)(\lambda - 1)$$

$$(\lambda + 2)(\lambda - 1)(\lambda - 1)$$

$\lambda = -2$ or $\lambda = 1$

With $\lambda = -2$ and $\lambda = 1$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 1 & -2 & 1 \\ 0 & -2 + 2 & 0 \\ 0 & 0 & -2 - 1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 - 1 & -2 & 1 \\ 0 & 1 + 2 & 0 \\ 0 & 0 & 1 - 1 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{2}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{2}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 1 & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & -2 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix}-\frac{2}{3}\\1\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\0\\0\end{bmatrix}$$