Homework 6

Math 3607, Summer 2021

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Problem 1.

THis problems asks for a proof that if a given matrix X has an EVD then p(X) can be found using the evaluations of p at the eigenvalues and two matrix multiplications. It also asks for a function that will evaluate the polynomial for a given coefficient vector and either a scalar, vector, or matrix.

Part A

a) If $X \in \mathbb{R}^{k \times k}$ has an EVD then X can be expressed by $X = VDV^{-1}$ where V is square matrix whose columns are the eigenvectors of X and D is the diagonal matrix whose diagonal elements are the eigenvalues of X.

$$\text{So, } X = \underbrace{\begin{bmatrix} & n\times n & \\ & & \\ & & \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} & n\times n & \\ & & \\ & & \end{bmatrix}}_{V^{-1}}$$

We know
$$p(x) = c_1 + \sum_{i=1}^{n-1} c_{i+1} x^i$$
$$= c_1 + \sum_{i=1}^{n-1} c_{i+1} [V D V^{-1}]^i$$
$$= V c_1 I V^{-1} + \sum_{i=1}^{n-1} c_{i+1} V D^i V^{-1}$$
$$= V\left[c_1 I + \sum_{i=1}^{n-1} c_{i+1} D^i\right] V^{-1}$$

Since D is diagonal matrix containing eigenvalues of X then

$$p(X) = V\left[c_1 I + \sum_{i=1}^{n-1} c_{i+1} D^i\right] V^{-1} = V\left[c_1 I + \sum_{i=1}^{n-1} c_{i+1} \lambda_k^i\right] V^{-1}$$

where $c_1 I + \sum_{i=1}^{n-1} c_{i+1} \lambda_k^i$ is a diagonal matrix and k is the number of eigenvalues.

```
%Part B
```

```
c = [2 6 3];
x = [5 3 4 2];
mypolyval(c, x)
```

```
ans =
     107             47             74             26
```

```
polyval(flip(c),x) %test
```

```
ans =
     107             47             74             26
```

```
c = [2 6 3];
x = [5 3; 5 4];
mypolyval(c, x)
```

```
ans =
     152             99
     165            119
```

```
polyvalm(flip(c),x) %test
```

```
ans =
     152             99
     165            119
```

Problem 2.

This problem asks us to calculate the singular values of a given matrix A by hand.

2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}_{4\times 2}$$

$\underset{2\times 4}{A^T}\underset{4\times 2}{A}$ yields 2×2 matrix

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$(4-4\lambda+\lambda^2)-1 = 0$$

$$\lambda^2-4\lambda+3 = 0$$

$$(\lambda-3)(\lambda-1) = 0$$

$\lambda = 3$
$\lambda = 1$ are the eigenvalues

Thus, by theorem 5 we have the singular values are $\sqrt{3}$, 1

Problem 3.

This problem asks us to prove that A and its transpose have the same singular values and the same 2-norm.

③ a) show A and A^T have same sing. values

Consider SVD of $A = U\Sigma V^T$. Then we know that singular values of A are the diagonal values of Σ.

Now, consider $A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$

Since the transpose of a diagonal matrix is just the very same diagonal matrix, $\Sigma^T = \Sigma$ then the singular values of A^T are the diagonal values of Σ^T which equals diagonal values of Σ.

Thus A and A^T have the same diagonal values.

b) show $\|A\|_2 = \|A^T\|_2$

We know that $\|A\|_2 = \sigma_1$, that is the largest singular value of A. Since A^T and A have the same singular values (which we just proved above), it follows that they share the largest singular value as well. Thus,

$\|A\|_2 = \sigma_1 = \|A^T\|_2$.

Problem 4.

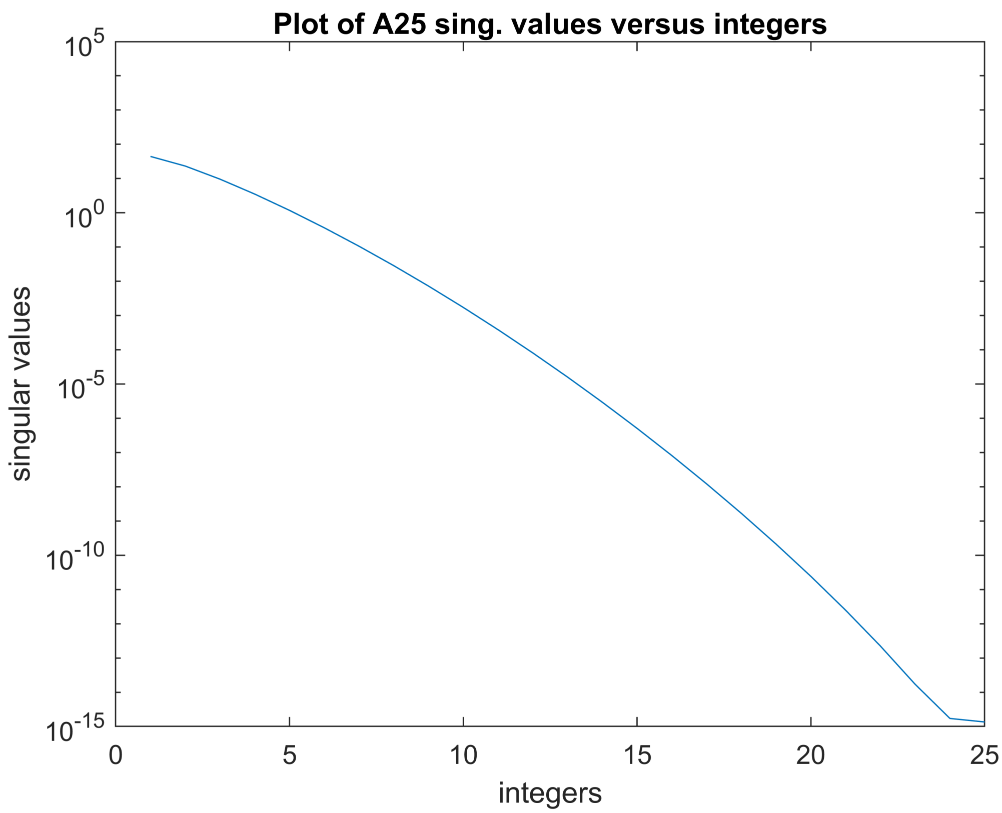
This problem asks us to generate a Vandermonde matrix where x is a vector containing 1000 linearly spaced points between 0 and 1. Then, print out the singular values of A1, A2, and A3. Next, make a semi-log plot of the singular values of A25 and compute its rank.

```
x=linspace(0,1,1000)';
for j=1:3
    A=x.^(0:j-1);
    s=svd(A);
    fprintf('The singular values of A%d are: \n',j)
    disp(s)
end
```

```
The singular values of A1 are:
    3605/114
The singular values of A2 are:
```

```
    1887/53
    2167/267
The singular values of A3 are:
    1839/49
    2989/270
    2137/1301
```

```
A=x.^(0:24);
s=svd(A);
semilogy([1:25], s)
title('Plot of A25 sing. values versus integers')
xlabel('integers');
ylabel('singular values');
```




```
rank25=rank(A);
fprintf('The rank of matrix A25 is: %d\n',rank25)
```

```
The rank of matrix A25 is: 20
```

```
% The rank provides the number of singular values
% of a matrix that are larger than the tolerance.
% This means that there are only 20 singular values
% that are larger than the tolerance which is given
% by tol=max(size(A)) * eps(norm(A)).
% Thus, the singular values at integers 21 through
% 25 are smaller than the tolerance and are not
% contributing to the rank of the matrix.
```

```
function y = mypolyval(c, x)

    if size(x,1) == size(x,2) %check if square matrix
        t=eig(x); %grabs eigenvalues of matrix
        w=zeros(1,length(t)); %creates vector for storage
        for j=1:length(t) %same number of iterations as eigenvalues
                y=c(1); %initialize y to be the first coefficient
                for i=1:length(c)-1 %one less than c b/c we already used first coefficient
                    y=c(i+1)*(t(j)^(i))+y; %formula from part A
                    w(j)=y; %stores the eigenvalue evaluations into the zero vector
                end
        end
        D = diag(w);
        [V, ~] = eig(x);
        y = V *D / V;


    elseif (size(x,1)>1) && (size(x,1) ~= size(x,2)) %if it's a matrix and not square
        fprintf('Error: Not a square matrix!\n')


    else   %from 12b Module 2 problems
        n=length(c);
        y = c(n);
        for j = n-1:-1:1
            y = y.*x + c(j);
        end
    end
end
```