Homework 7

Math 3607, Summer 2021

Spenser Smith

Table of Contents

Problem 1.

This problem asks for a script that calculates the interest rate necessary to reach an annuity value of 1,000,000 from various starting deposits. It asks for a table of these values.

```
format long
n = 300;
annuity = @(P,r) (12 * P / r) * ( ((1 + r / 12) ^ n) - 1);
fprintf('Monthly Deposit      Annual Interest Rate');
```

```
Monthly Deposit      Annual Interest Rate
```

```
for P = 500:50:1000
    new_ann = @(r) (annuity(P,r) - 1000000);
    zero_ann = fzero(new_ann, .1);
    fprintf('%d                  %f \n', P, zero_ann);
end
```

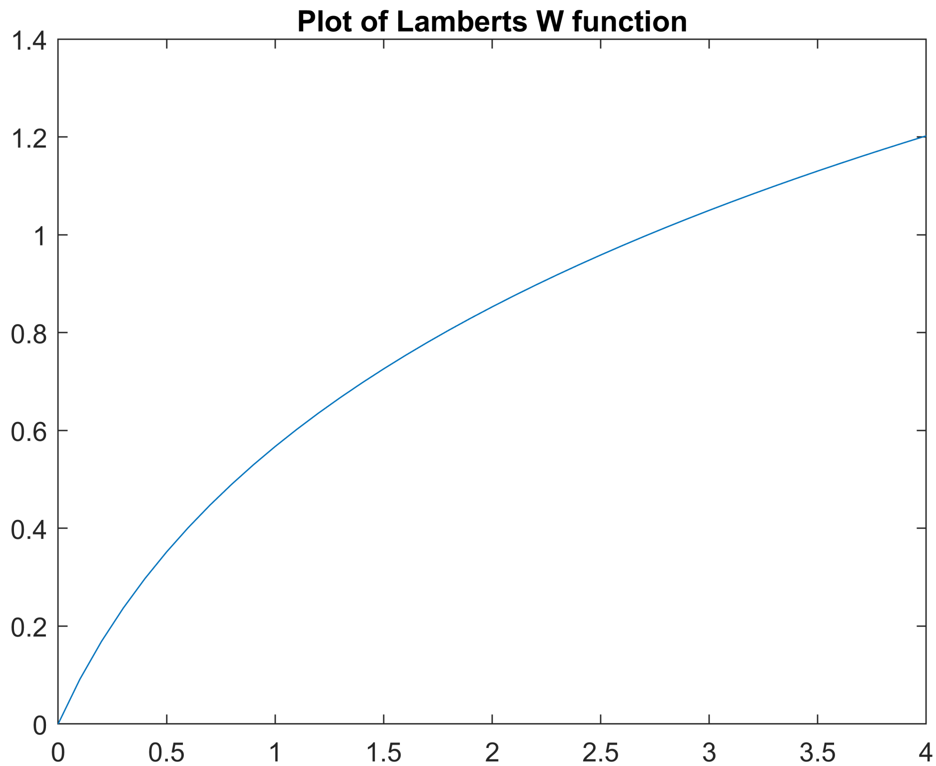
```
500                 0.123512
550                 0.118146
600                 0.113200
650                 0.108607
700                 0.104317
750                 0.100287
800                 0.096485
850                 0.092884
900                 0.089461
950                 0.086198
1000                 0.083078
```

Problem 2.

This problem asks us to create a function titled lambertW that takes an x-value and spits out the corresponding y-value using the function Lambert's W. It then asks us to graph said function from 0 to 4.

```
format short
xdata = [0:0.1:4]';
sz_x = size(xdata);
wdata = zeros(sz_x);
k=1;
for x = 0:.1:4
    w = lambertW(x);
```

```
    wdata(k) = w;
    k = k + 1;
end
plot(xdata, wdata)
title('Plot of Lamberts W function')
```



Problem 3.

This problem asks for a script that verifies that a given function has a fixed point and that FPI can converge to it. It also asks us to graph a semilogy plot of the error to verify its linear convergence and to determine an approximate value for σ.

3a) $g(x) = \frac{1}{2}(3 + \frac{9}{x})$ for $r = 3$;

$g(3) = \frac{1}{2}(3 + \frac{9}{3}) = \frac{1}{2}(3+3) = \frac{1}{2}(6) = 3$

Since $g(3) = 3$, 3 is a fixed point of g

$g'(3) = \frac{-9}{2(3)^2} = \frac{-9}{18} = -\frac{1}{2}$, $\sigma = \frac{1}{2}$

Since $g(r) = r$ and g is continuously diff. and $\sigma = |g'(r)| < 1$

then FPI converges to r with the rate σ.

3b) $g(x) = \pi + \frac{1}{4}\sin(x)$ for $r = \pi$;

$g(\pi) = \pi + \frac{1}{4}\sin(\pi) = \pi + 0 = \pi$

Since $g(\pi) = \pi$, π is a fixed point of g.

$g'(\pi) = \frac{1}{4}\cos(\pi) = -\frac{1}{4}$, $\sigma = \frac{1}{4}$

Since $g(r) = r$ and g is continuously diff. and $\sigma = |g'(r)| < 1$

then FPI converges to r with the rate σ.

3c) $g(x) = x + 1 - \tan(x/4)$ for $r = \pi$;

$g(\pi) = \pi + 1 - \tan(\pi/4) = \pi + 1 - 1 = \pi$

Since $g(\pi) = \pi$, π is a fixed point of g.

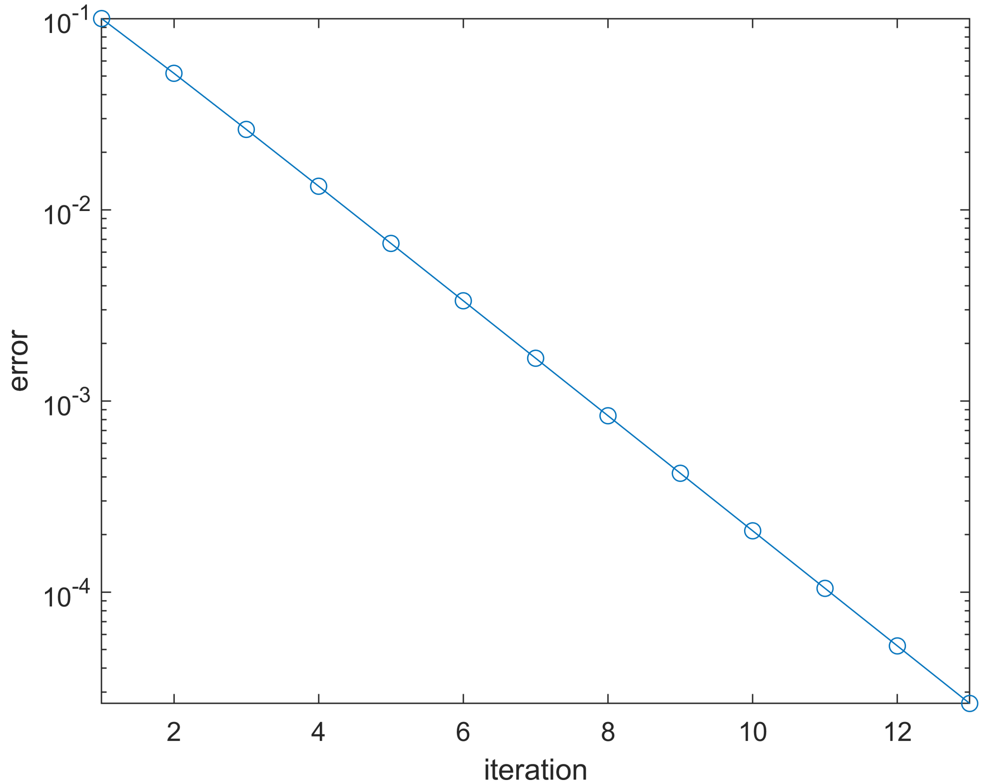
$g'(\pi) = 1 - \frac{1}{4}\sec^2(\frac{\pi}{4}) = 1 - \frac{1}{4}(2) = 1 - \frac{1}{2} = \frac{1}{2}$, $\sigma = \frac{1}{2}$

Since $g(r) = r$ and g is continuously diff. and $\sigma = |g'(r)| < 1$

then FPI converges to r with the rate σ.

```
clf
format long
r = 3;
x = 2.9;
f = @(x) 1/2 * (3 + (9 / x));
g = @(x) r - f(x);

for k = 1:12
    x(k + 1) = g(x(k)) + r;
end
err = abs(x-r);
semilogy(err, 'o-'), axis tight
xlabel('iteration'), ylabel('error')
```



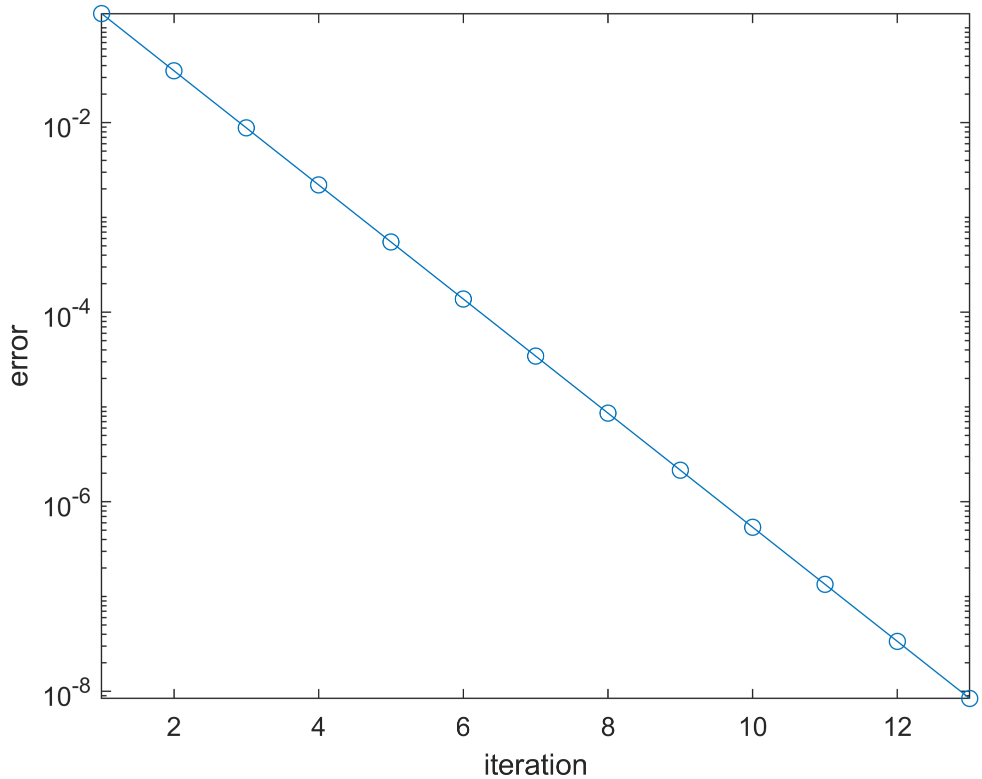

```
p = polyfit(1:13, log(err), 1);
sigma = exp(p(1))
```

```
sigma =
   0.501915541760583
```

```
% This values makes sense as it matches up very closely to that of 1/2 which
% we calculated above.

clf
format long
r = pi;
x = 3;
f = @(x) pi + (1/4 *sin(x));
g = @(x) r - f(x);

for k = 1:12
    x(k + 1) = g(x(k)) + r;
end
err = abs(x-r);
semilogy(err, 'o-'), axis tight
xlabel('iteration'), ylabel('error')
```



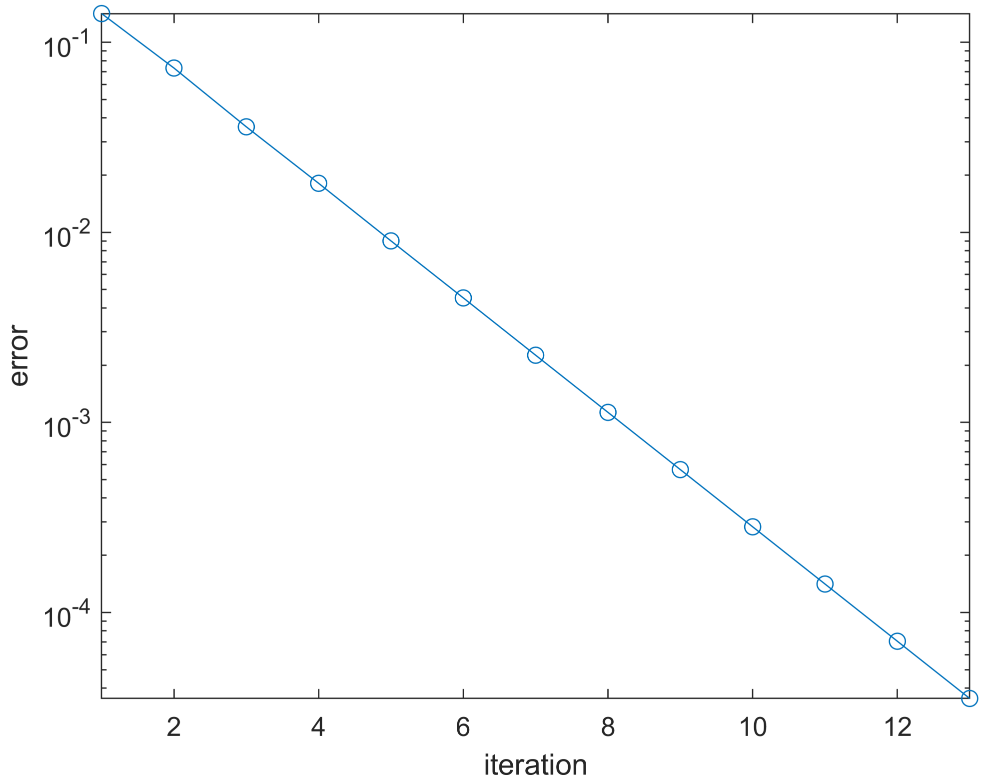

```
p = polyfit(1:13, log(err), 1);
sigma = exp(p(1))
```

```
sigma =
   0.249969021201731
```

```
% Again, this value makes sense as it matches up with that of 1/4 which we
% calculated above.

clf
format long
r = pi;
x = 3;
f = @(x) x + 1 - tan(x/4);
g = @(x) r - f(x);

for k = 1:12
    x(k + 1) = g(x(k)) + r;
end
err = abs(x-r);
semilogy(err, 'o-'), axis tight
xlabel('iteration'), ylabel('error')
```




```
p = polyfit(1:13, log(err), 1);
sigma = exp(p(1))
```

```
sigma =
   0.500193825121005
```

```
% This value also makes sense as it matches up with the 1/2 that we
% calculated above.
```

Problem 4.

This problem asks us to discuss what happens when Newton's Method is applied to f(x) and to show that the error of f(x) converges linearly in the case of a multiple root.

4a) $f(x) = \text{sign}(x)\sqrt{|x|}$

$$f'(x) = \cancel{\frac{d}{dx}(\text{sign}(x))} \cdot \sqrt{|x|} + (\text{sign}(x))\frac{d}{dx}(\sqrt{|x|})$$

but $\frac{d}{dx}(\text{sign}(x)) = 0$ since slope is constant

so $f'(x) = \text{sign}(x)\frac{d}{dx}(\sqrt{|x|})$

We know $|x| = \sqrt{x^2}$ so we now have

$$f'(x) = \text{sign}(x)\frac{d}{dx}\left(\sqrt{\sqrt{x^2}}\right)$$
$$= \text{sign}(x) \cdot \frac{1}{2\sqrt{\sqrt{x^2}}} \cdot \frac{x}{|x|}$$
$$= [\text{sign}(x)]^2 \cdot \frac{1}{2\sqrt{\sqrt{x^2}}}$$
$$= [\text{sign}(x)]^2 \cdot \frac{1}{2\sqrt{|x|}}$$

$$\boxed{\begin{gathered}(\sqrt{x^2})^{1/2} = \frac{1}{2}\sqrt{x^2}^{-1/2} \cdot \frac{d}{dx}\sqrt{x^2} \\ (x^2)^{1/2} = \frac{1}{\cancel{2}\sqrt{x^2}} \cdot \cancel{2}x \\ = \frac{x}{|x|}\end{gathered}}$$

So, let's start at $x_0 = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{\sqrt{2}}{\frac{1}{2\sqrt{2}}} = 2 - 4 = -2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{(-\sqrt{2})}{\frac{1}{2\sqrt{2}}} = -2 + 4 = 2$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2 - \frac{f(2)}{f'(2)} = -2$$

This process continues infinitely. Thus, NM does not work.

4b) Let $\epsilon_k = x_k - r$, then by substituting it into iterative formula we have

$$\epsilon_{k+1} = \epsilon_k - \frac{f(r+\epsilon_k)}{f'(r+\epsilon_k)}$$

Taylor-expand to get

$$\epsilon_{k+1} = \epsilon_k - \frac{\cancel{f(r)} + \cancel{\epsilon_k f'(r)} + \frac{1}{2}\epsilon_k^2 f''(r) + O(\epsilon_k)^3}{\cancel{f'(r)} + \epsilon_k f''(r) + O(\epsilon_k)^2}$$

But we know $f(r) = f'(r) = 0$

So we have

$$\epsilon_{k+1} = \epsilon_k - \frac{\frac{1}{2}\epsilon_k^2 f''(r) + \cancel{O(\epsilon_k)^3}}{\epsilon_k f''(r) + \cancel{O(\epsilon_k)^2}}$$

By ignoring $O(\epsilon_k)^3$ and $O(\epsilon_k)^2$ we have

$$\epsilon_{k+1} = \epsilon_k - \frac{1}{2}\frac{\epsilon_k^{\cancel{2}} \cancel{f''(r)}}{\cancel{\epsilon_k} \cancel{f''(r)}}$$

$$= \epsilon_k - \frac{1}{2}\epsilon_k$$

$$= \frac{1}{2}\epsilon_k$$

Thus, converges linearly.

```
function y = lambertW(x)
    problem = @(y) (x - (y* exp(1)^y));
    guess = 1;
    y = fzero(problem, guess);
end
```