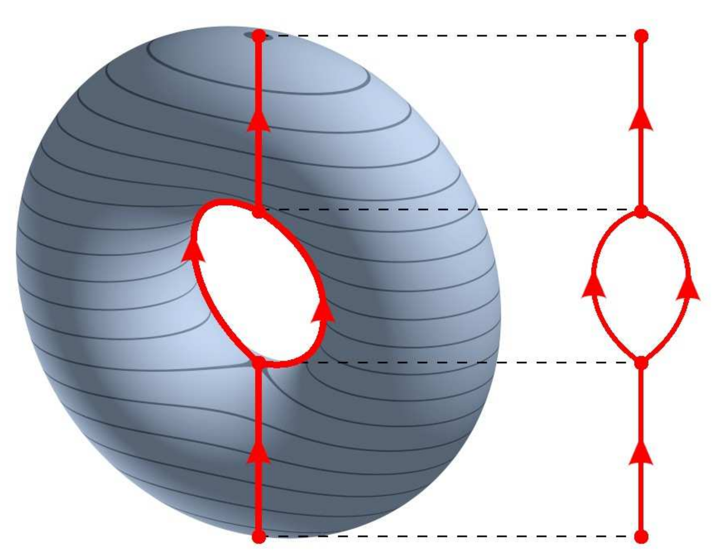
Reeb Spaces of Definable Maps

Saugata Basu, Nathanael Cox, and Sarah Percival
Department of Mathematics, Purdue University

The Reeb Space

Given a topological space X and a continuous function $f: X \to \mathbb{R}$, define an equivalence relation $\sim$ on X by setting $x_1 \sim x_2$ if $f(x_1) = f(x_2)$ and x_1 and x_2 are in the same connected component of $f^{-1}(f(x_1)) = f^{-1}(f(x_2)) \in X$. The space $X/\sim$ is called the *Reeb graph* of f, denoted $\mathrm{Reeb}(f)$. The concept of the Reeb graph was introduced by Georges Reeb as a tool in Morse theory. The notion of the Reeb graph can be generalized to the notion of *Reeb space* by letting $f: X \to Y$, where Y is any topological space. Our motivation is to understand how topologically complicated the Reeb space of a map can become in terms of the complexity of the map itself. In order to obtain meaningful results we restrict ourselves to the category of maps *definable in an o-minimal expansion of* $\mathbb{R}$ and in particular to *semi-algebraic* maps.



The Reeb graph of the height function of the upright torus (Ilya Voyager)

O-minimal Structures

An *o-minimal structure* over a real closed field R is a sequence $\mathscr{S}(\mathrm{R}) = (\mathscr{S}_n)_{n\in\mathbb{N}}$ of collections of subsets of R^n (called the *definable sets* in the structure) where each $\mathscr{S}_n$ satisfies the following axioms (following the exposition by Coste):

1. All algebraic subsets of R^n are in $\mathscr{S}_n$.
2. The class $\mathscr{S}_n$ is closed under complementation and finite unions and intersections.
3. If $A \in \mathscr{S}_m$ and $B \in \mathscr{S}_n$ then $A \times B \in \mathscr{S}_{m+n}$.
4. If $\pi : \mathrm{R}^{n+1} \to \mathrm{R}^n$ is the projection map on the first n co-ordinates and $A \in \mathscr{S}_{n+1}$, then $\pi(A) \in \mathscr{S}_n$.
5. The elements of $\mathscr{S}_1$ are finite unions of points and intervals. (Note that these are precisely the subsets of $\mathbb{R}$ which are definable by a first-order formula in the language of the reals with one free variable.)

Quotients by Definable Equivalence Relations

Let $E \subset X \times X$ be a definable equivalence relation on a definable set X. A *definable quotient* of X by E is a pair (p, Y) consisting of a definable set Y and a definable surjective map $p : X \to Y$ such that:

1. $(x_1, x_2) \in E \Leftrightarrow p(x_1) = p(x_2)$, for all $x_1, x_2 \in X$.
2. p is definably identifying: for all definable $K \subset Y$, if $p^{-1}(K)$ is closed in X, then K is closed in Y.

The definable quotient (p, Y) is *definably proper* if p is *definably proper*; that is, for every definable $K \subset Y$, with K closed and bounded in R^n, $p^{-1}(K) \subset X$ is closed and bounded in R^m.

A definable equivalence relation $E \subset X \times X$ is *definably proper* if the two maps $\mathrm{pr}_1, \mathrm{pr}_2 : E \to R$ are proper.

$\mathrm{Reeb}(f)$ is Homeomorphic to a Definable Set.

The space $\mathrm{Reeb}(f) \triangleq X/\sim$ exists as a definably proper quotient. In other words, there exists a proper definable map $\psi : X \to Z$, and a homeomorphism $\theta : \mathrm{Reeb}(f) \to Z$ such that the following diagram commutes, where ϕ is the quotient map:

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ \mathrm{Reeb}(f) = X/\sim \arrow[rr, "\theta"] & & Z \end{tikzcd}$$

Bounds on Betti Numbers

For any topological space X, and $i \geq 0$, let $b_i(X)$ denote the i-th Betti number (that is, the rank of the i-th singular homology group of X), and let $b(X) = \sum_i b_i(X)$. Cole-McLaughlin et al. proved that for a manifold M and a Morse function $f: M \to \mathbb{R}$, $b_1(\mathrm{Reeb}(f)) \leq b_1(M)$, and hence, for such a function,

$$b(\mathrm{Reeb}(f)) \leq b(M).$$

More generally, Edelsbrunner and Harer noted that the inequality $b(\mathrm{Reeb}(f)) \leq b(X)$ holds for arbitrary maps $f: X \to \mathbb{R}$,

The Betti Numbers of the Reeb Space of $f: X \to Y$ Can Exceed Those of X

Consider the closed n-dimensional disk $\mathbf{D}^n$ with $n > 1$, and let $\sim$ be the equivalence relation identifying all points on the boundary of $\mathbf{D}^n$. Then $\mathbf{D}^n/\sim \cong \mathbf{S}^n$, where $\mathbf{S}^n$ is the n-dimensional sphere. Let f_n denote the quotient map $f_n : \mathbf{D}^n \to \mathbf{S}^n$. The fibers of f_n consist of either one point or the boundary $\mathbf{S}^{n-1}$ of $\mathbf{D}^n$, hence $\mathrm{Reeb}(f_n) \cong \mathbf{S}^n$ for all $n > 1$. Note that $b_0(\mathbf{D}^n) = 1$ and $b_i(\mathbf{D}^n) = 0$ for all $i > 0$. Moreover, $b_0(\mathbf{S}^n) = 1$, $b_n(\mathbf{S}^n) = 1$, and $b_i(\mathbf{S}^n) = 0$ for $i \neq 0, n$. Thus, we have for $n > 1$,

$$b(\mathbf{D}^n) = 1, \text{ and } b(\mathrm{Reeb}(f_n)) = 2.$$

More generally, for any $k \geq 0$, let

$$f_{n,k} = \underbrace{f \times \cdots \times f}_{k \text{ times}} : \underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}} \longrightarrow \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}.$$

Using the same argument as before, for $n > 1$ and all $k > 0$, $\mathrm{Reeb}(f_{n,k}) \cong \underbrace{\mathbf{S}^n \times \cdots \times \mathbf{S}^n}_{k \text{ times}}$. Thus,

$$b_0(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1,\ b_i((\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 0, i > 0,$$

and hence

$$b(\underbrace{\mathbf{D}^n \times \cdots \times \mathbf{D}^n}_{k \text{ times}}) = 1.$$

Moreover, for $n > 1$,

$$b_i(\mathrm{Reeb}(f_{n,k})) = 0, \text{ if } n \nmid i \text{ or if } i > nk,$$
$$b_i(\mathrm{Reeb}(f_{n,k})) = \binom{k}{i/n}, \text{ otherwise},$$

and hence for $n > 1$,

$$b(\mathrm{Reeb}(f_{n,k})) = 2^k.$$

Definitions and Notation

For any finite family of polynomials $\mathscr{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call an element $\sigma \in \{0, 1, -1\}^{\mathscr{P}}$ a *sign condition* on $\mathscr{P}$. For any semi-algebraic set $Z \subset \mathbb{R}^k$ and sign condition $\sigma \in \{0, 1, -1\}^{\mathscr{P}}$, the *realization* of σ on Z is the semi-algebraic set defined by

$$\{\mathbf{x} \in Z \mid \mathbf{sign}(P(\mathbf{x})) = \sigma(P), P \in \mathscr{P}\},$$

denoted $\mathscr{R}(\sigma, Z)$. More generally, a $\mathscr{P}$*-formula* is any Boolean formula Φ with atoms $P\{=, >, <\}0, P \in \mathscr{P}$. We call the realization of Φ, namely the semi-algebraic set

$$\mathscr{R}(\Phi, \mathbb{R}^k) = \{\mathbf{x} \in \mathbb{R}^k \mid \Phi(\mathbf{x})\}$$

a $\mathscr{P}$*-semi-algebraic set*. Finally, we call a Boolean formula without negations and with atoms $P\{\geq, \leq\}0$, $P \in \mathscr{P}$ a $\mathscr{P}$*-closed formula*, and we call the realization, $\mathscr{R}(\Phi, \mathbb{R}^k)$, a $\mathscr{P}$*-closed semi-algebraic set*.

Bounding the Betti Numbers of $\mathrm{Reeb}(f)$

Let $S \subset \mathbb{R}^n$ be a bounded $\mathscr{P}$-closed semi-algebraic set, and let $f = (f_1, \ldots, f_m) : S \to \mathbb{R}^m$ be a polynomial map. Suppose that $s = \mathrm{card}(\mathscr{P})$, and the maximum of the degrees of the polynomials in $\mathscr{P}$ and $f_1, \ldots, f_m$ is bounded by d. Then

$$b(\mathrm{Reeb}(f)) \leq (sd)^{(n+m)^{O(1)}}.$$

Conclusions and Future Work

We have shown that the Reeb space of a proper definable map in an o-minimal structure is homeomorphic to a definable set. Furthermore, we have produced a singly exponential upper bound on the Betti numbers of the Reeb spaces of proper semi-algebraic maps.

Because Reeb spaces of proper semi-algebraic maps can be realized as semi-algebraic quotients, it is possible to pursue the question of creating an algorithm to describe this quotient semi-algebraically. Moreover, a singly exponential upper bound on the Betti numbers of the Reeb space of a semi-algebraic map gives rise to the possibility of finding an algorithm with a singly exponential complexity bound to compute the Betti numbers and a semi-algebraic description of the Reeb space.

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