Servo algorithm simulation for $^{88}Sr^{+}$ion clock

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1 Basic outline

The aim of this document is to describe the servo simulator for a $^{88}Sr^{+}$ion clock. The clock functions by matching a laser to a narrow transition of an $^{88}Sr^{+}$ion as accurately as possible.

We simulate the probing of a two-state system with a Rabi π-pulse. The system is assumed to be in the ground state initially, so the probability distribution is

$$\rho_{ee}(f, \omega_0, \tau, S) = S \left(\frac{\pi}{2}\right)^2 \mathrm{sinc}^2 \left(\frac{\sqrt{\pi^2 + (f - \omega_0)^2 \tau^2}}{2}\right), \tag{1}$$

where f is the laser wavelength, ω_0 the linecenter of the distribution, τ the length of the π-pulse and S the state prep factor. If the ion is state-prepared i.e. we ensure that the ion is in the ground state before the probe pulse, $S = 1$; otherwise $S = 1/2$.

Key assumptions:

- The system is always in the ground state before the pulse. To simulate the lack of state preparation, the probability distribution is simply divided by 2.
- All laser pulses are π-pulses.
- The laser line is an ideal Dirac distribution. Thus, we know exactly where we are probing in the frequency space.

To simulate a clock cycle with laser at frequency f, we take the ideal excitation probability $\rho_{ee}(f)$ and take n draws from the binomial distribution $B(n, \rho_{ee}(f))$. This way, we can simulate the random shot noise from a cycle of multiple pulses.

In addition to shot noise, the model also includes laser and magnetic field drift. To complement these drifts, a servo algorithm is included to correct the drift.

1.1 Servo algorithm

Naïvely, one would think that we could simply probe the ion at the transition line center and attempt to maximize the amount of cycles where the ion is in the clock state. However, a problem immediately presents itself: the probability of excitation of a two-state system is symmetric as a function of detuning as seen in Fig. 1. If local oscillator does not maximize the transition probability, should it be tuned toward blue or red? Additionally, due to small perturbations in environmental conditions, the laser line and Zeeman peaks are constantly drifting and need periodic correction to remain at the reference as precisely as possible. To solve this problem, the ion is instead probed from both sides of the linecenter at the assumed points where the spectrum reaches half maximum. By sampling both of these multiple points many times, we can approximate the transition probabilities on the red and blue side, p_R and p_B. These can then be used to compose an algorithm for counteracting the detuning by tuning the laser frequency using a servo [1]:

$$f_{i+1} = f_i + E \tag{2}$$

$$E = G\,\frac{p_B - p_R}{p_B + p_R} \tag{3}$$

where f_i is the frequency of the laser at time step i, p_B and p_R the blue- and red-detuned transition probabilities and G the gain in Hz.

To choose the optimal value for the gain G such that it would always correct the detuning in a single step, we must first revisit the definition of Allan deviation, given by the square root of Eq. **??**. As we cannot directly measure the transition probability of at detuning δ, we instead perform many cycles and use the statistics to approximate the probability. These measurements follow the binomial distribution, the variance of which is $\sigma_{p_X} = p_X(1 - p_X)$ where p_X represents either p_B or p_R. Now we can redefine the Allan deviation as

$$\sigma_y(\tau) = \frac{G}{\nu_0}\sqrt{\left(\frac{1 - p_X}{p_X}\right)\frac{T_c}{\tau}} \tag{4}$$

where ν_0 is the transition frequency, τ the averaging time and T_c the cycle time of a single interrogation of the transition. To arrive at the optimal G value, we must minimize the residual servo tracking error. We define k_p as

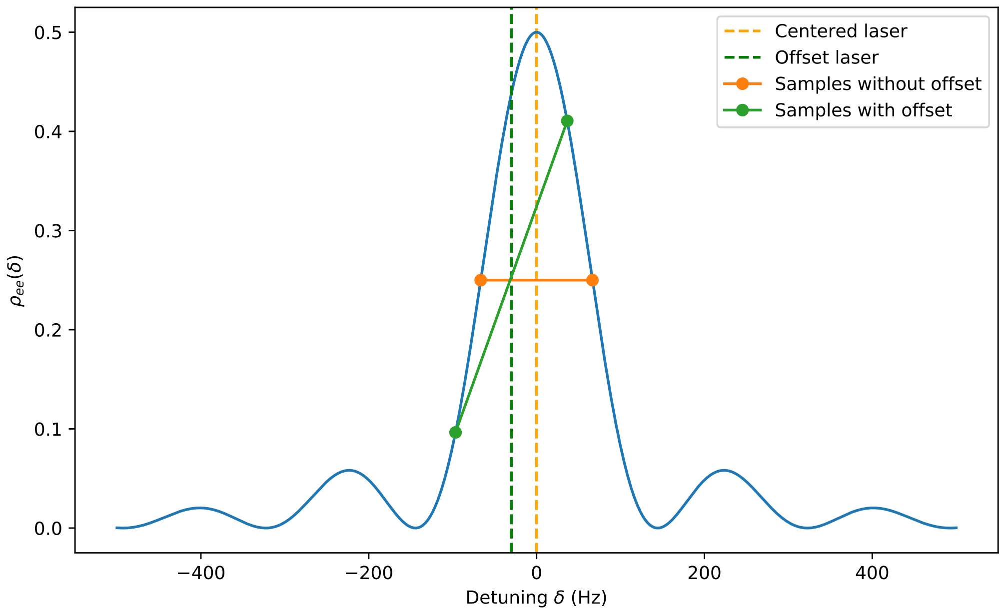

Figure 1: Excitation spectrum of an ideal two state system, along with the blue- and red-detuned sampling without and with offset.

$k_p \frac{\partial}{\partial \delta}(p_B - p_R)|_{\delta=0}$, which we can numerically evaluate by sampling p_B and p_R from the lineshape function. Then $G = -2p_X/k_p$ and (4) becomes

$$\sigma_y(\tau) = \frac{-2\sigma_{p_X}}{k_p \nu_0}\sqrt{\frac{T_c}{\tau}}. \tag{5}$$

If state preparation is not possible, the average excitation probability at the center of the transition line is only 50%. Due to the nature of binomial sampling, this increases the effect of shot noise which in turn can cause erroneous servo corrections and technical shifts. However, we can exploit the Zeeman effect of $^{88}Sr^{+}$ion to bypass this issue to achieve better statistics. In bichromatic (BC) sampling a laser pulse at frequency f_0 is split in two pulses which target different Zeeman peaks of a single Zeeman pair. The splitting is done using an acousto-optic modulator. Instead of one servo correcting the laser linecenter, we now have two. One servo tracks the linear Zeeman splitting, $\Delta\nu_B$ and the other tracks the frequency of the center of the peak pair $\Delta\nu_C$. For a graphical representation, see Fig. 2

To control these servos, we interrogate the beaks on the blue and red sides, similarly as in the single peak case. Now that we have four such points, two for each peaks, we will use + and − to label which peak we are sampling. + corresponds to the peak with $m_J > 0$ and − to the $m_J < 0$ one. By splitting the pulse to two different frequencies, we are effectively probing the distribution at two points simultaneously. Thus, we can sum the transition probabilities of the two probing locations to get new distributions with which we can control the servos. The error signals associated with the magnetic field E_B and center E_C servos are

$$p_{\text{outer}} = p_{R-} + p_{B+} \tag{6}$$

$$p_{\text{inner}} = p_{B-} + p_{R+} \tag{7}$$

$$E_B = G_B \frac{p_{\text{outer}} - p_{\text{inner}}}{p_{\text{outer}} + p_{\text{inner}}} \tag{8}$$

$$p_{\text{BB}} = p_{B-} + p_{B+} \tag{9}$$

$$p_{\text{RR}} = p_{R-} + p_{R+} \tag{10}$$

$$E_C = G_C \frac{p_{\text{BB}} - p_{\text{RR}}}{p_{\text{BB}} + p_{\text{RR}}} \tag{11}$$

The benefit of bichromatic sampling lies in the fact that $p_{\text{outer}}, p_{\text{inner}}, p_{\text{BB}}, p_{\text{RR}} \in [0, 1]$. Thus the error signals are less impacted by the random shot noise. The servo gains G_B and G_C are calculated in a similar manner as the gain for a single peak servo; only now p_X represents one of the derived probabilities and k_p has to be redefined: $k_p = \frac{\partial}{\partial \delta}(p_X - p_Y)|_{\delta=0}$, where p_X and p_Y correspond to the probabilities in the numerators of the magnetic field and center error signals.

The functions used in the module are self-documented.

2 Example simulation workflow

1. The user provides general settings. These may include ambient magnetic field, magnetic field drift, laser line drift etc.

2. Using `sample_initial_values`, the finding and characterizing the transition lineshape is performed. The ideal lineshape is sampled at different points using binomial sampling. Afterwards, a lineshape function is fit to the sampled points using `scipy.optimize`'s `curve_fit`.

3. The main simulation loop. For each main cycle step, the lineshape is probed numerous times to approximate transition probabilities at red and blue detuned points. While sampling, the possibly drifting laser line and magnetic field change values Then, using the gain setting

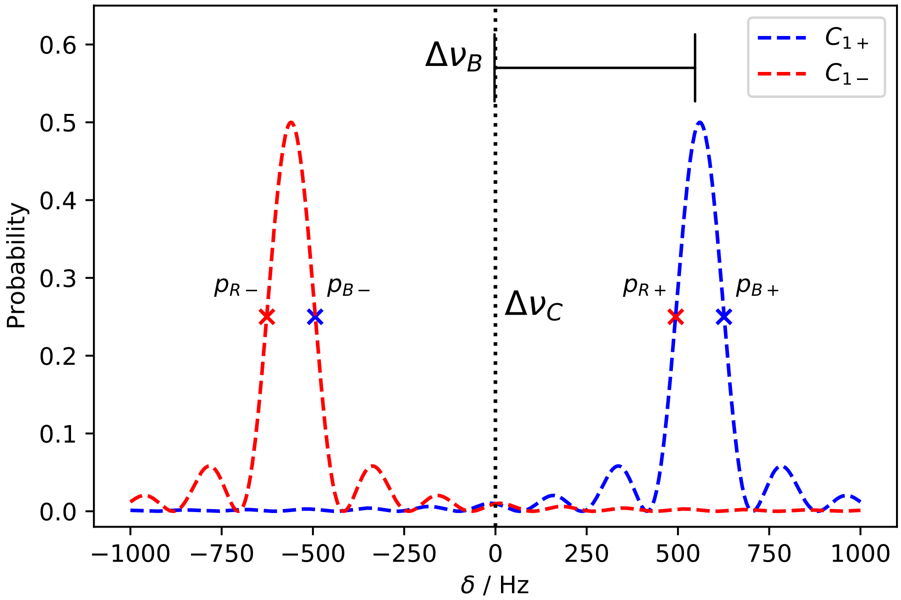

Figure 2: An example of bichromatic sampling. The two peaks of a Zeeman pair are shown with dashed lines, the red being the $m_J < 0$ and the blue the $m_J > 0$ peak. The crosses show the locations of the ideal sampling points for red- and blue-detuned samples for the servo algorithm. The line with endcaps shows the separation tracked by the magnetic field servo and the dotted line shows the frequency tracked by the center servo.

calculated in the initialization stage, the servos are corrected and the intermediary results saved.

References

[1] Pierre Dubé, Alan A. Madej, Andrew Shiner, and Bin Jian. $^{88}Sr^{+}$ single-ion optical clock with a stability approaching the quantum projection noise limit. *Phys. Rev. A*, 92(4):042119, Oct 2015.