

```

module HuttonChap17 where
open import Haskell.Prelude
open import Haskell.Law.Equality using (sym; begin_;  $\equiv$ {_}_; step- $\equiv$ ;  $\dashv$ ; cong)

```

1. SYNTAX AND SEMANTICS

```

module SyntaxAndSemantics where
  data Expr : Set where
    Val : Int → Expr
    Add : Expr → Expr → Expr
  {-# COMPILE AGDA2HS Expr #-}

  eval : Expr → Int
  eval (Val n) = n
  eval (Add e1 ex) = eval e1 + eval ex
  {-# COMPILE AGDA2HS eval #-}

```

2. ADDING A STACK

```

module AddingAStack where
  open SyntaxAndSemantics
  Stack = List Int
  {-# COMPILE AGDA2HS Stack #-}

  push : Int → Stack → Stack
  push n s = n :: s
  {-# COMPILE AGDA2HS push #-}

  add : Stack → Stack
  add [] = []
  add (x :: []) = []
  add (x :: y :: s) = y + x :: s
  {-# COMPILE AGDA2HS add #-}

  module DefineEval' where
    postulate
      eval' : Expr → Stack → Stack
      eval'  $\equiv$  eval : (e : Expr) → (s : Stack) → eval' e s  $\equiv$  eval e :: s

    eval'-val : (n : Int) → (s : Stack) → eval' (Val n) s  $\equiv$  push n s
    eval'-val n s =
      begin
        eval' (Val n) s
       $\equiv$  { eval'  $\equiv$  eval (Val n) s } -- Specification
        eval (Val n) :: s
       $\equiv$  { } -- Apply eval
        n :: s
       $\equiv$  { } -- Unapply push
        push n s
      ■

    eval'-add : (x y : Expr) → (s : Stack)
      → eval' (Add x y) s  $\equiv$  add (eval' y (eval' x s))

```

```

eval'-add x y s =
  begin
    eval' (Add x y) s
  ≡{ eval'≡eval (Add x y) s } -- Specification
    eval (Add x y) :: s
  ≡{ } -- Apply eval
    eval x + eval y :: s
  ≡{ } -- Unapply add
    add (eval y :: eval x :: s)
  ≡{ cong (λ s → add (eval y :: s)) (sym (eval'≡eval x s)) } -- Induction
    add (eval y :: eval' x s)
  ≡{ cong add (sym (eval'≡eval y (eval' x s))) } -- Induction
    add (eval' y (eval' x s))
  ■

```

```

eval' : Expr → Stack → Stack
eval' (Val n) s = push n s
eval' (Add el er) s = add (eval' er (eval' el s))
{-# COMPILE AGDA2HS eval' #-}

```

```

eval'≡eval : (e : Expr) → (s : Stack) → eval' e s ≡ eval e :: s
eval'≡eval (Val n) s = refl
eval'≡eval (Add x y) s =
  begin
    eval' (Add x y) s
  ≡{ } -- Apply eval'
    add (eval' y (eval' x s))
  ≡{ cong (λ s → add (eval' y s)) (eval'≡eval x s) } -- Induction
    add (eval' y (eval x :: s))
  ≡{ cong add (eval'≡eval y (eval x :: s)) } -- Induction
    add (eval y :: eval x :: s)
  ≡{ } -- Unapply add and eval
    eval (Add x y) :: s
  ■

```

```

eval≡eval' : (e : Expr) → (s : Stack) → eval e :: s ≡ eval' e s
eval≡eval' e s = sym $ eval'≡eval e s

```

Since $\text{eval}' e s$ is the same as $\text{eval } e :: s$, this is evidence that $\text{eval}' e s$ is a non-empty list:

```

open import Haskell.Prim using (NonEmpty; itsNonEmpty)
open import Haskell.Law.Equality using (subst)

instance
  eval'-nonempty : {e : Expr} → NonEmpty (eval' e [])
  eval'-nonempty {e} = subst NonEmpty (eval≡eval' e []) itsNonEmpty

```

Since we know that $\text{eval}' e []$ is always non-empty, it is safe to apply head to it and get an equivalent definition of eval:

```

eval1 : {e : Expr} → Int
eval1 {e} = head (eval' e [])
{-# COMPILE AGDA2HS eval1 #-}

```

3. ADDING A CONTINUATION

```

module AddingAContinuation where
  open SyntaxAndSemantics
  open AddingAStack
  Cont = Stack → Stack
  {-# COMPILE AGDA2HS Cont #-}

module DefineEval'' where
  postulate
    eval'' : Expr → Cont → Cont
    eval''≡eval' : (e : Expr) → (c : Cont) → (s : Stack)
      → eval'' e c s ≡ c (eval' e s)

  eval''-val : (n : Int) → (c : Cont) → (s : Stack)
    → eval'' (Val n) c s ≡ c (push n s)
  eval''-val n c s =
    begin
      eval'' (Val n) c s
    ≡⟨ eval''≡eval' (Val n) c s ⟩ -- Postulate
      c (eval' (Val n) s)
    ≡⟨ ⟩ -- Apply eval'
      c (push n s)
    ≡⟨ ⟩ -- Unapply ∘
      (c ∘ push n) s
    ■

  eval''-add : (x y : Expr) → (c : Cont) → (s : Stack)
    → eval'' (Add x y) c s ≡ eval'' x (eval'' y (c ∘ add)) s
  eval''-add x y c s =
    begin
      eval'' (Add x y) c s
    ≡⟨ eval''≡eval' (Add x y) c s ⟩
      c (eval' (Add x y) s)
    ≡⟨ ⟩ -- Apply eval'
      c (add (eval' y (eval' x s)))
    ≡⟨ ⟩ -- Unapply ∘
      (c ∘ add) (eval' y (eval' x s))
    ≡⟨ sym (eval''≡eval' y (c ∘ add) (eval' x s)) ⟩ -- Induction y
      eval'' y (c ∘ add) (eval' x s)
    ≡⟨ sym (eval''≡eval' x (eval'' y (c ∘ add)) s) ⟩ -- Induction x
      eval'' x (eval'' y (c ∘ add)) s
    ■

  eval'' : Expr → Cont → Cont
  eval'' (Val n) c = c ∘ push n
  eval'' (Add x y) c = eval'' x (eval'' y (c ∘ add))
  {-# COMPILE AGDA2HS eval'' #-}

  eval''≡eval' : (e : Expr) → (c : Cont) → (s : Stack)
    → eval'' e c s ≡ c (eval' e s)
  eval''≡eval' (Val x) c s = refl

```

```

eval''≡eval' (Add x y) c s =
  begin
    eval'' (Add x y) c s
  ≡⟨⟩ -- Apply eval''
    eval'' x (eval'' y (c ∘ add)) s
  ≡⟨ eval''≡eval' x (eval'' y (c ∘ add)) s ⟩ -- Induction
    eval'' y (c ∘ add) (eval' x s)
  ≡⟨ eval''≡eval' y (c ∘ add) (eval' x s) ⟩ -- Induction
    (c ∘ add) (eval' y (eval' x s))
  ≡⟨⟩ -- Apply add
    c (eval' (Add x y) s)
  ■

```

Thus `eval'` is simply redefined as follows:

```

eval'1 : Expr → Cont
eval'1 e = eval'' e id
{-# COMPILE AGDA2HS eval'1 #-}

```

4. DEFUNCTIONALISING

```
module Defunctionalising where
  open SyntaxAndSemantics
  open AddingAStack
  open AddingAContinuation
  haltC : Cont
  haltC = id
  {-# COMPILE AGDA2HS haltC #-}

  pushC : Int → Cont → Cont
  pushC n c = c ∘ push n
  {-# COMPILE AGDA2HS pushC #-}

  addC : Cont → Cont
  addC c = c ∘ add
  {-# COMPILE AGDA2HS addC #-}

  data Code : Set where
    HALT : Code
    PUSH : Int → Code → Code
    ADD : Code → Code
  {-# COMPILE AGDA2HS Code deriving Show #-}

  exec : Code → Cont
  exec HALT = haltC
  exec (PUSH n c) = pushC n (exec c)
  exec (ADD c) = addC (exec c)
  {-# COMPILE AGDA2HS exec #-}

  comp' : Expr → Code → Code
  comp' (Val n) c = PUSH n c
  comp' (Add x y) c = comp' x (comp' y (ADD c))
  {-# COMPILE AGDA2HS comp' #-}

  comp : Expr → Code
  comp e = comp' e HALT
  {-# COMPILE AGDA2HS comp #-}

  exec-comp'≡eval'' : (e : Expr) → (c : Code)
    → exec (comp' e c) ≡ eval'' e (exec c)
  exec-comp'≡eval'' (Val n) c = refl
  exec-comp'≡eval'' (Add x y) c =
    begin
      exec (comp' (Add x y) c)
    ≡⟨⟩ -- Apply comp'
      exec (comp' x (comp' y (ADD c)))
    ≡⟨ exec-comp'≡eval'' x (comp' y (ADD c)) ⟩ -- Induction
      eval'' x (exec (comp' y (ADD c)))
    ≡⟨ cong (eval'' x) (exec-comp'≡eval'' y (ADD c)) ⟩ -- Induction
      eval'' x (eval'' y (exec (ADD c)))
    ≡⟨⟩ -- Apply exec
      eval'' x (eval'' y (addC (exec c)))
    ≡⟨⟩ -- Unapply eval''
      eval'' (Add x y) (exec c)
```

■

```

exec-comp≡eval' : (e : Expr) → (s : Stack) → exec (comp e) s ≡ eval' e s
exec-comp≡eval' e s =
  begin
    exec (comp e) s
  ≡⟨⟩ -- Apply comp
    exec (comp' e HALT) s
  ≡⟨ cong (_$ s) (exec-comp'≡eval'' e HALT) ⟩
    eval'' e (exec HALT) s
  ≡⟨⟩ -- Apply exec
    eval'' e id s
  ≡⟨ eval''≡eval' e id s ⟩
    id (eval' e s)
  ≡⟨⟩ -- Apply id
    eval' e s
  ■

```

Alternatively, explicitly with lists:

```

data Op : Set where
  PUSHOP : Int → Op
  ADDOP   : Op
{-# COMPILE AGDA2HS Op #-}

Prog = List Op
{-# COMPILE AGDA2HS Prog #-}

execute : Prog → Cont
execute [] = haltC
execute (PUSHOP n :: os) = pushC n (execute os)
execute (ADDOP :: os) = addC (execute os)
{-# COMPILE AGDA2HS execute #-}

compile' : Expr → Prog → Prog
compile' (Val n) p = PUSHOP n :: p
compile' (Add x y) p = compile' x (compile' y (ADDOP :: p))
{-# COMPILE AGDA2HS compile' #-}

compile : Expr → Prog
compile e = compile' e []
{-# COMPILE AGDA2HS compile #-}

```

```

execute-compile'≡eval'' : (e : Expr) → (p : Prog)
  → execute (compile' e p) ≡ eval'' e (execute p)
execute-compile'≡eval'' (Val n) p = refl
execute-compile'≡eval'' (Add x y) p =
  begin
    execute (compile' (Add x y) p)
  ≡⟨⟩ -- Apply compile'
    execute (compile' x (compile' y (ADDOP :: p)))
  ≡⟨ execute-compile'≡eval'' x (compile' y (ADDOP :: p)) ⟩ -- Induction
    eval'' x (execute (compile' y (ADDOP :: p)))
  ≡⟨ cong (eval'' x) (execute-compile'≡eval'' y (ADDOP :: p)) ⟩ -- Induction
    eval'' x (eval'' y (execute $ ADDOP :: p))
  ≡⟨⟩ -- Apply execute
    eval'' x (eval'' y (addC (execute p)))
  ≡⟨⟩ -- apply addC
    eval'' x (eval'' y ((execute p) ◦ add))
  ≡⟨⟩ -- Unapply eval''
    eval'' (Add x y) (execute p)
  ■

```

```

execute-compile≡eval' : (e : Expr) → (s : Stack)
  → execute (compile e) s ≡ eval' e s
execute-compile≡eval' e s =
  begin
    execute (compile e) s
  ≡⟨⟩ -- Apply compile
    execute (compile' e []) s
  ≡⟨ cong (_$ s) (execute-compile'≡eval'' e []) ⟩
    eval'' e (execute []) s
  ≡⟨⟩ -- Apply execute
    eval'' e haltC s
  ≡⟨ eval''≡eval' e haltC s ⟩
    haltC (eval' e s)
  ≡⟨⟩ -- Apply haltC ≡ id
    eval' e s
  ■

```

```

execute-compile≡eval : (e : Expr) → (s : Stack)
  → execute (compile e) s ≡ eval e :: s
execute-compile≡eval e s =
  begin
    execute (compile e) s
  ≡⟨ execute-compile≡eval' e s ⟩
    eval' e s
  ≡⟨ eval'≡eval e s ⟩
    eval e :: s
  ■

```

5. COMBINING THE STEPS

```
module CombiningTheSteps where
  open SyntaxAndSemantics using (Expr; eval)
  open AddingAStack using (Stack)
  module DefineComp where
    postulate
      Code : Set
      exec : Code → Stack → Stack
      comp : Expr → Code
      comp' : Expr → Code → Code
      exec-comp≡eval : (e : Expr) → (s : Stack)
        → exec (comp e) s ≡ eval e :: s
      exec-comp'≡eval : (e : Expr) → (c : Code) → (s : Stack)
        → exec (comp' e c) s ≡ exec c (eval e :: s)
```