

```
module HuttonChap16 where
```

```
open import Haskell.Prelude
open import Haskell.Law.Equality using (sym; begin_; _≡⟨_⟩_; step-≡; _■; cong)
open import Haskell.Law.Eq.Def using (IsLawfulEq; eqReflexivity)
open import Haskell.Law.Num.Def using (+-assoc; +-comm)
open import Haskell.Law.Num.Int using (iLawfulNumInt)
```

## INDUCTION ON NUMBERS

Proving the first fact about replicate:

```
replicate : {a : Set} → Nat → a → List a
replicate zero _ = []
replicate (suc n) x = x :: replicate n x
```

```
len-repl : {A : Set} → (n : Nat) → (x : A) → lengthNat (replicate n x) ≡ n
len-repl zero x = refl
len-repl (suc n) x =
  begin
    lengthNat (replicate (suc n) x)
  ≡⟨ ⟩ -- Apply replicate
    lengthNat (x :: replicate n x)
  ≡⟨ ⟩ -- Apply lengthNat
    suc (lengthNat (replicate n x))
  ≡⟨ cong suc (len-repl n x) ⟩
    suc n
  ■
```

Some facts about append:

```
++-[] : {a : Set} → (xs : List a) → xs ++ [] ≡ xs
++-[] [] = begin ([] ++ []) ≡⟨ ⟩ [] ■
++-[] (x :: xs) =
  begin
    (x :: xs) ++ []
  ≡⟨ ⟩ -- Apply ++
    x :: (xs ++ [])
  ≡⟨ cong (x ::_) (++-[] xs) ⟩
    x :: xs
  ■
```

```

++-assoc : {a : Set} → (xs ys zs : List a)
  → (xs ++ ys) ++ zs ≡ xs ++ (ys ++ zs)
++-assoc [] ys zs =
  begin
    ([] ++ ys) ++ zs
  ≡⟨⟩ -- Apply ++
    ys ++ zs
  ≡⟨⟩ -- Unapply ++
    [] ++ (ys ++ zs)
  ■

++-assoc (x :: xs) ys zs =
  begin
    ((x :: xs) ++ ys) ++ zs
  ≡⟨⟩ -- Apply ++
    (x :: (xs ++ ys)) ++ zs
  ≡⟨⟩ -- Apply ++
    x :: ((xs ++ ys) ++ zs)
  ≡⟨ cong (x ::_) (++-assoc xs ys zs) ⟩
    x :: (xs ++ (ys ++ zs))
  ≡⟨⟩ -- Unapply ++
    (x :: xs) ++ (ys ++ zs)
  ■

```

Hutton's example of elimination of append from flattening a tree:

```

data Tree (a : Set) : Set where
  Leaf : a → Tree a
  Node : Tree a → Tree a → Tree a
{-# COMPILE AGDA2HS Tree #-}

flatten : {a : Set} → Tree a → List a
flatten (Leaf x) = x :: []
flatten (Node tl tr) = flatten tl ++ flatten tr
{-# COMPILE AGDA2HS flatten #-}

flatten' : {a : Set } → Tree a → List a → List a
flatten' (Leaf x) xs = x :: xs
flatten' (Node tl tr) xs = flatten' tl (flatten' tr xs)
{-# COMPILE AGDA2HS flatten' #-}

```

```

flatten'-flatten : {a : Set} → (t : Tree a) → (xs : List a)
  → flatten' t xs ≡ flatten t ++ xs
flatten'-flatten (Leaf x) xs = refl
flatten'-flatten (Node tl tr) xs =
  begin
    flatten' (Node tl tr) xs
  ≡⟨⟩ -- Apply flatten'
    flatten' tl (flatten' tr xs)
  ≡⟨ cong (flatten' tl) (flatten'-flatten tr xs) ⟩
    flatten' tl (flatten tr ++ xs)
  ≡⟨ flatten'-flatten tl (flatten tr ++ xs) ⟩
    flatten tl ++ (flatten tr ++ xs)
  ≡⟨ sym (++-assoc (flatten tl) (flatten tr) xs) ⟩
    (flatten tl ++ flatten tr) ++ xs
  ≡⟨⟩ -- Unapply flatten
    flatten (Node tl tr) ++ xs

```

■

```

flatten'-≡-flatten : {a : Set} → (t : Tree a)
  → flatten' t [] ≡ flatten t
flatten'-≡-flatten (Leaf x) = refl
flatten'-≡-flatten (Node tl tr) =
  begin
    flatten' (Node tl tr) []
  ≡⟨⟩ -- Apply flatten'
    flatten' tl (flatten' tr [])
  ≡⟨ cong (flatten' tl) (flatten'-flatten tr []) ⟩ -- Apply the above equality
    flatten' tl (flatten tr ++ [])
  ≡⟨ flatten'-flatten tl (flatten tr ++ []) ⟩ -- Apply it again
    flatten tl ++ (flatten tr ++ [])
  ≡⟨ cong (flatten tl ++_) (++-[] (flatten tr)) ⟩ -- Remove trailing []
    flatten tl ++ flatten tr
  ≡⟨⟩ -- Unapply flatten
    flatten (Node tl tr)

```

■

## COMPILER CORRECTNESS

```

data Expr : Set where
  Val : Int → Expr
  Add : Expr → Expr → Expr
{-# COMPILE AGDA2HS Expr #-}

eval : Expr → Int
eval (Val n) = n
eval (Add x y) = eval x + eval y
{-# COMPILE AGDA2HS eval #-}

Stack = List Int
{-# COMPILE AGDA2HS Stack #-}

data Op : Set where
  PUSH : Int → Op
  ADD : Op
{-# COMPILE AGDA2HS Op #-}

```

```
Code = List Op
{-# COMPILE AGDA2HS Code #-}
```

```
exec : Code → Stack → Stack
exec [] s = s
exec (PUSH n :: c) s = exec c $ n :: s
exec (ADD :: c) (m :: n :: s) = exec c $ n + m :: s
exec (ADD :: c) _ = []
{-# COMPILE AGDA2HS exec #-}
```

```
comp : Expr → Code → Code
comp (Val n) c = PUSH n :: c
comp (Add x y) c = comp x $ comp y $ ADD :: c
{-# COMPILE AGDA2HS comp #-}
```

```
comp-exec-eval : (e : Expr) → (c : Code) → (s : Stack)
→ exec (comp e c) s ≡ exec c (eval e :: s)
```

```
comp-exec-eval (Val n) c s =
  begin
    exec (comp (Val n) c) s
  ≡⟨⟩ -- Apply comp
    exec (PUSH n :: c) s
  ≡⟨⟩ -- Apply exec
    exec c (n :: s)
  ≡⟨⟩ -- Unapply eval
    exec c (eval (Val n) :: s)
```

■

```
comp-exec-eval (Add x y) c s =
  begin
    exec (comp (Add x y) c) s
  ≡⟨⟩ -- Apply comp
    exec (comp x $ comp y $ ADD :: c) s
  ≡⟨ comp-exec-eval x (comp y $ ADD :: c) s ⟩ -- Induction
    exec (comp y $ ADD :: c) (eval x :: s)
  ≡⟨ comp-exec-eval y (ADD :: c) (eval x :: s) ⟩ -- Induction Again
    exec (ADD :: c) (eval y :: eval x :: s)
  ≡⟨⟩ -- Apply exec
    exec c ((eval x) + (eval y) :: s)
  ≡⟨⟩ -- Unapply eval
    exec c (eval (Add x y) :: s)
```

■

```
compile : Expr → Code
compile e = comp e []
{-# COMPILE AGDA2HS compile #-}
```

```
compile-exec-eval : (e : Expr) → exec (compile e) [] ≡ eval e :: []
compile-exec-eval e =
```

```
  begin
    exec (compile e) []
  ≡⟨⟩ -- Apply compile
    exec (comp e []) []
  ≡⟨ comp-exec-eval e [] [] ⟩
    exec [] (eval e :: [])
  ≡⟨⟩ -- Apply exec
    eval e :: []
```

■

EXERCISE 1. Show that  $\text{add } n \ (\text{Suc } m) = \text{Suc } (\text{add } n \ m)$  by induction on  $n$

```

+-suc : (n m : Nat) → n + (suc m) ≡ suc (n + m)
+-suc zero m = refl
+-suc (suc n) m =
  begin
    (suc n) + (suc m)
  ≡⟨⟩ -- Apply +
    suc (n + suc m)
  ≡⟨⟩ -- cong suc (+-suc n m)
    suc (suc (n + m))
  ≡⟨⟩ -- Unapply +
    suc (suc n + m)
  ■

```

EXERCISE 2. Using this property, together with  $\text{add } n \ \text{zero} = n$ , show that addition is commutative,  $\text{add } n \ m = \text{add } m \ n$ , by induction on  $n$ .

```

+-zero : (n : Nat) → n + zero ≡ n
+-zero zero = refl
+-zero (suc n) =
  begin
    suc n + zero
  ≡⟨⟩ -- Apply +
    suc (n + zero)
  ≡⟨⟩ -- cong suc (+-zero n)
    suc n
  ■
+-commut : (n m : Nat) → n + m ≡ m + n
+-commut zero m =
  begin
    zero + m
  ≡⟨⟩ -- Apply +
    m
  ≡⟨⟩ -- sym (+-zero m)
    m + zero
  ■

```

```

+-commut (suc n) m =
  begin
    suc n + m
  ≡⟨⟩ -- Apply +
    suc (n + m)
  ≡⟨⟩ -- cong suc (+-commut n m)
    suc (m + n)
  ≡⟨⟩ -- sym (+-suc m n)
    m + suc n
  ■

```

EXERCISE 3. Complete the proof of the correctness of replicate by showing that it produces a list with identical elements,  $\text{all } (= x) \ (\text{replicate } n \ x)$ , by induction on  $n \geq 0$ . Hint: show that the property is always True.

```

all-repl : {α} iEq : Eq α → {β} IsLawfulEq α β → (n : Nat) → (x : α)
  → all (λ _ => x) (replicate n x) ≡ True
all-repl zero x = refl
all-repl (suc n) x =
  begin
    all (λ _ => x) (replicate (suc n) x)
  =<> -- Apply replicate
    all (λ _ => x) (x :: replicate n x)
  =<> -- Apply all
    (x == x) && (all (λ _ => x) (replicate n x))
  =<> cong ((x == x) &&_) (all-repl n x) -- Induction
    (x == x) && True
  =<> cong (λ _ => True) (eqReflexivity x) -- Reflexivity x == x
    True

```

EXERCISE 4. This is `++-[]` and `++-assoc` above.

EXERCISE 5. Using the above definition for `++`, together with the definitions for `take` and `drop` show that `take n xs ++ drop n xs = xs`, by simultaneous induction on the integer `n` and the list `xs`. Hint: there are three cases, one for each pattern of arguments in the definitions of `take` and `drop`.

```

take-drop-nat : {α : Set} → (n : Nat) → (xs : List α)
  → takeNat n xs ++ dropNat n xs ≡ xs
take-drop-nat n [] = refl
take-drop-nat zero (x :: xs) =
  begin
    takeNat zero (x :: xs) ++ dropNat zero (x :: xs)
  =<> -- Apply takeNat and dropNat
    [] ++ x :: xs
  =<>
    x :: xs

```

```

take-drop-nat (suc n) (x :: xs) =
  begin
    takeNat (suc n) (x :: xs) ++ dropNat (suc n) (x :: xs)
  =<> -- Apply takeNat and dropNat and ++
    x :: takeNat n xs ++ dropNat n xs
  =<> cong (x ::_) (take-drop-nat n xs)
    x :: xs

```

```

take-drop : {α : Set} → (n : Int) → {β} iNN : IsNonNegativeInt n β
  → (xs : List α) → take n xs ++ drop n xs ≡ xs
take-drop n xs =
  begin
    take n xs ++ drop n xs
  =<> -- Apply take and drop
    takeNat (intToNat n) xs ++ dropNat (intToNat n) xs
  =<> take-drop-nat (intToNat n) xs
    xs

```

EXERCISE 6. Given the `Tree` definition above, show that the number of leaves in such a tree is always one greater than the number of nodes, by induction on trees. Hint: start by defining functions that count the number of leaves and nodes in a tree.

```

nLeaves : {a : Set} → Tree a → Int
nLeaves (Leaf x) = 1
nLeaves (Node tl tr) = nLeaves tl + nLeaves tr
{-# COMPILE AGDA2HS nLeaves #-}

nNodes : {a : Set} → Tree a → Int
nNodes (Leaf x) = 0
nNodes (Node tl tr) = 1 + nNodes tl + nNodes tr
{-# COMPILE AGDA2HS nNodes #-}

leaves-nodes : {a : Set} → (t : Tree a)
  → nLeaves t ≡ 1 + nNodes t
leaves-nodes (Leaf x) = refl
leaves-nodes (Node tl tr) =
  begin
    nLeaves (Node tl tr)
  ≡⟨ ⟩
    nLeaves tl + nLeaves tr
  ≡⟨ cong (λ _ → nLeaves tr) (leaves-nodes tl) ⟩
    1 + nNodes tl + nLeaves tr
  ≡⟨ cong ((1 + nNodes tl) +_) (leaves-nodes tr) ⟩
    1 + nNodes tl + (1 + nNodes tr)
  ≡⟨ +-assoc 1 (nNodes tl) (1 + nNodes tr) ⟩
    1 + (nNodes tl + (1 + nNodes tr))
  ≡⟨ cong (1 +_) (sym (+-assoc (nNodes tl) 1 (nNodes tr))) ⟩
    1 + (nNodes tl + 1 + nNodes tr)
  ≡⟨ cong (1 +_) (cong (λ _ → nNodes tr) (+-comm (nNodes tl) 1)) ⟩
    1 + (1 + nNodes tl + nNodes tr)
  ≡⟨ ⟩
    1 + nNodes (Node tl tr)
  ■

```

**EXERCISE 7.** Verify the functor laws for the Maybe type. Hint: the proofs proceed by case analysis, and do not require the use of induction.

```

module LawfulFunctorMaybe where
open import Haskell.Law.Functor.Def
using (IsLawfulFunctor; identity; composition)

instance
  isLawful : IsLawfulFunctor Maybe
  identity  $\Downarrow$  isLawful  $\Downarrow$  Nothing =
    begin
      fmap id Nothing
     $\equiv$  ( $\Downarrow$ ) -- Apply fmap
      Nothing
     $\equiv$  ( $\Downarrow$ ) -- Unapply id
      id Nothing
    ■

  identity  $\Downarrow$  isLawful  $\Downarrow$  (Just x) =
    begin
      fmap id (Just x)
     $\equiv$  ( $\Downarrow$ ) -- Apply fmap
      Just (id x)
     $\equiv$  ( $\Downarrow$ ) -- Apply id
      Just x
     $\equiv$  ( $\Downarrow$ ) -- Unapply id
      id (Just x)
    ■

  composition  $\Downarrow$  isLawful  $\Downarrow$  Nothing f g =
    begin
      fmap (g  $\circ$  f) Nothing
     $\equiv$  ( $\Downarrow$ ) -- Apply fmap
      Nothing
     $\equiv$  ( $\Downarrow$ ) -- Unapply fmap
      fmap g Nothing
     $\equiv$  ( $\Downarrow$ ) -- Unapply fmap
      fmap g (fmap f Nothing)
     $\equiv$  ( $\Downarrow$ ) -- Unapply  $\circ$ 
      (fmap g  $\circ$  fmap f) Nothing
    ■

  composition  $\Downarrow$  isLawful  $\Downarrow$  (Just x) f g =
    begin
      fmap (g  $\circ$  f) (Just x)
     $\equiv$  ( $\Downarrow$ ) -- Apply fmap
      Just ((g  $\circ$  f) x)
     $\equiv$  ( $\Downarrow$ ) -- Apply  $\circ$ 
      Just (g (f x))
     $\equiv$  ( $\Downarrow$ ) -- Unapply fmap
      fmap g (Just (f x))
     $\equiv$  ( $\Downarrow$ ) -- Unapply fmap
      fmap g (fmap f (Just x))
     $\equiv$  ( $\Downarrow$ ) -- Unapply  $\circ$ 
      (fmap g  $\circ$  fmap f) (Just x)
    ■

```

EXERCISE 8. Given the instance declaration below, verify the functor laws for the `Tree` type, by induction on trees.



```

open import Haskell.Prim.Functor using (DefaultFunctor)

treeMap : (a → b) → (Tree a) → (Tree b)
treeMap f (Leaf x) = Leaf (f x)
treeMap f (Node tl tr) = Node (treeMap f tl) (treeMap f tr)
{-# COMPILE AGDA2HS treeMap #-}

dft : DefaultFunctor Tree
dft = record { fmap = treeMap }
instance
  iFunctorTree : Functor Tree
  iFunctorTree = record { DefaultFunctor dft }
  {-# COMPILE AGDA2HS iFunctorTree #-}

module LawfulFunctorTree where
  open import Haskell.Law.Functor.Def
  using (isLawfulFunctor; identity; composition)
  idTree : (t : Tree a) → (fmap id) t ≡ id t
  idTree (Leaf x) = refl
  idTree (Node tl tr) =
    begin
      fmap id (Node tl tr)
    ≡⟨⟩ -- Apply fmap
      Node (fmap id tl) (fmap id tr)
    ≡⟨ cong (λ x → Node x (fmap id tr)) (idTree tl) ⟩
      Node (id tl) (fmap id tr)
    ≡⟨ cong (Node (id tl)) (idTree tr) ⟩
      Node (id tl) (id tr)
    ≡⟨⟩ -- Apply and unapply id
      id (Node tl tr)
    ■

  compTree : (t : Tree a) → (f : a → b) → (g : b → c)
    → fmap (g ∘ f) t ≡ (fmap g ∘ fmap f) t
  compTree (Leaf x) f g = refl
  compTree (Node tl tr) f g =
    begin
      fmap (g ∘ f) (Node tl tr)
    ≡⟨⟩ -- Apply fmap
      Node (fmap (g ∘ f) tl) (fmap (g ∘ f) tr)
    ≡⟨ cong (λ x → Node x (fmap (g ∘ f) tr)) (compTree tl f g) ⟩
      Node ((fmap g ∘ fmap f) tl) (fmap (g ∘ f) tr)
    ≡⟨ cong (Node ((fmap g ∘ fmap f) tl)) (compTree tr f g) ⟩
      Node ((fmap g ∘ fmap f) tl) ((fmap g ∘ fmap f) tr)
    ≡⟨⟩ -- Unapply fmap
      fmap g (Node (fmap f tl) (fmap f tr))
    ≡⟨⟩ -- Unapply fmap
      fmap g (fmap f (Node tl tr))
    ≡⟨⟩ -- Unapply ∘
      (fmap g ∘ fmap f) (Node tl tr)
    ■

instance
  isLawful : IsLawfulFunctor Tree
  identity ∯ isLawful ∅ = idTree
  composition ∯ isLawful ∅ = compTree

```

## EXERCISE 9. Verify the applicative laws for the Maybe type.

```
idMaybe : (m : Maybe a) → (pure id <*> m) ≡ m
idMaybe Nothing =
```

```
  begin
    pure id <*> Nothing
  ≡⟨⟩ -- Apply pure and <*>
    Nothing
```

■

```
idMaybe (Just x) =
```

```
  begin
    pure id <*> Just x
  ≡⟨⟩ -- Apply pure
    Just id <*> Just x
  ≡⟨⟩ -- Apply <*>
    Just (id x)
  ≡⟨⟩ -- Apply id
    Just x
```

■

```
module LawfulApplicative where
```

```
  open import Haskell.Law.Applicative.Def
  using (IsLawfulApplicative; identity; composition;
    homomorphism; interchange; functor)
```

```
instance
```

```
  isLawful : IsLawfulApplicative Maybe
```

```
  identity ∷ isLawful ∅ = idMaybe
```

```
  composition ∷ isLawful ∅ Nothing y z =
```

```
    begin
```

```
      pure _ ∘_ <*> Nothing <*> y <*> z
```

```
    ≡⟨⟩ -- Apply pure and <*>
```

```
      Nothing <*> y <*> z
```

```
    ≡⟨⟩ -- Apply the rest of the <*>
```

```
      Nothing
```

```
    ≡⟨⟩ -- Unapply <*> on the right
```

```
      Nothing <*> (y <*> z)
```

■

```
  composition ∷ isLawful ∅ (Just x) Nothing z =
```

```
    begin
```

```
      pure _ ∘_ <*> Just x <*> Nothing <*> z
```

```
    ≡⟨⟩ -- Apply pure and <*>
```

```
      Just (x ∘_) <*> Nothing <*> z
```

```
    ≡⟨⟩ -- Apply <*>
```

```
      Nothing <*> z
```

```
    ≡⟨⟩ -- Apply <*>
```

```
      Nothing
```

```
    ≡⟨⟩ -- Unapply <*>
```

```
      Nothing <*> z
```

```
    ≡⟨⟩ -- Unapply <*>
```

```
      Just x <*> (Nothing <*> z)
```

■

```
  composition ∷ isLawful ∅ (Just x) (Just y) Nothing =
```

```
    refl -- Same kind of proof as above.
```

```

composition ≡ isLawful ≡ (Just x) (Just y) (Just z) =
  begin
    pure _◦_ <*> Just x <*> Just y <*> Just z
  ≡( ) -- Apply pure and <*>
    Just (x ◦_) <*> Just y <*> Just z
  ≡( ) -- Apply <*>
    Just (x ◦ y) <*> Just z
  ≡( ) -- Apply <*>
    Just ((x ◦ y) z)
  ≡( ) -- Apply ◦
    Just (x (y z))
  ≡( ) -- Unapply <*>
    Just x <*> Just (y z)
  ≡( ) -- Unapply <*>
    Just x <*> (Just y <*> Just z)

```

```

homomorphism ≡ isLawful ≡ f x =
  begin
    pure f <*> pure x
  ≡( ) -- Apply pure
    Just f <*> Just x
  ≡( ) -- Apply <*>
    Just (f x)
  ≡( ) -- Unapply pure
    pure (f x)

```

```

interchange ≡ isLawful ≡ Nothing y =
  begin
    Nothing <*> pure y
  ≡( ) -- Apply <*>
    Nothing
  ≡( ) -- Unapply <*>
    pure (_$ y) <*> Nothing

```

```

interchange ≡ isLawful ≡ (Just x) y =
  begin
    (Just x) <*> pure y
  ≡( ) -- Apply <*>
    Just (x y)
  ≡( ) -- Unapply $
    Just ((_ $ y) x)
  ≡( ) -- Unapply <*>
    pure (_$ y) <*> Just x

```

```

functor ≡ isLawful ≡ f Nothing = refl -- These are by definition.
functor ≡ isLawful ≡ f (Just x) = refl

```

EXERCISE 10. Verify the monad laws for the list type. Hint: the proofs can be completed using simple properties of list comprehensions.