Chapter 1

Exercise 1.1.i.i: Show that a morphism can have at most one inverse isomorphism.

Given $f : x \to y$ and $g, g' : y \to x$ with $fg = 1_y, gf = 1_x, fg' = 1_y$ and $g'f = 1_x$, then $g = 1_x g = g'fg = g'1_y = g'$

Exercise 1.1.i.ii: Consider a morphism $f : x \to y$. Show that if there exists a pair of morphisms $g, h : y \rightrightarrows x$ so that $gf = 1_x$ and $fh = 1_y$, then $g = h$ and f is an isomorphism.

Then $g = g1_y = gfh = 1_x h = h$ so that $fg = fh = 1_y$ and we already know that $gf = 1_x$ hence f is an isomorphism.

Exercise 1.1.1.iii: For any category $\mathbf{C}$ and any object $c \in \mathbf{C}$, show that:

i. There is a category $c/\mathbf{C}$ whose objects are morphisms $f : c \to x$ with domain c and in which a morphism from $f : c \to x$ to $g : c \to y$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[dr, "g"] & \\ x \arrow[rr, "h"'] & & y \end{tikzcd}$$

commutes, i.e., so that $g = hf$

Suppose $f : c \to x, g : c \to y, h : c \to z$ are objects of $c/\mathbf{C}$ and $\alpha : x \to y, \beta : y \to z$ are morphisms $f \to g$ and $g \to h$ in $c/\mathbf{C}$. In that case we have $\alpha f = g$ and $\beta g = h$. Then define composition $\beta\alpha$ in $c/\mathbf{C}$ as composition in $\mathbf{C}$. This is a morphism $f \to h$ in $c/\mathbf{C}$ because

$$(\beta\alpha)f = \beta(\alpha f) = \beta g = h$$

Associativity follows from associativity in $\mathbf{C}$.

Define the identity 1_f for $f : c \to x$ as the identity 1_x in $\mathbf{C}$. Then given $\alpha : f \to g$ ($\alpha : x \to y$ and $\alpha f = g$), we have $\alpha 1_f = \alpha 1_x = \alpha$ and $1_g \alpha = 1_y \alpha = \alpha$.

ii. There is a category $\mathbf{C}/c$ whose objects are morphisms $f : x \to c$ with codomain c and in which a morphism from $f : x \to c$ to $g : y \to c$ is a map $h : x \to y$ between the codomains so that the triangle

$$\begin{tikzcd} x \arrow[rr, "h"] \arrow[dr, "f"'] & & y \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commutes, i.e., so that $f = gh$.

Exercise 1.2.i: Show that $\mathbf{C}/c \cong (c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$. Defining $\mathbf{C}/c$ to be $(c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$, deduce Exercise 1.1.iii(ii) from Exercise 1.1.iii(i).

Say f is an object of $(c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$ which is, by definition, simply an object of $c/\mathbf{C}^{\mathrm{op}}$ which is a morphism $f^{\mathrm{op}} : c \to x$ in $\mathbf{C}^{\mathrm{op}}$ which is simply a morphism $f : x \to c$ in $\mathbf{C}$. This is the definition of objects in $\mathbf{C}/c$.

Now say f and g are objects of $(c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$ which means they are morphisms $f^{\mathrm{op}} : c \to x$ and $g^{\mathrm{op}} : c \to y$ and say $\alpha^{\mathrm{op}} : f \to g$ is a morphism in $(c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$. This means that $\alpha : g^{\mathrm{op}} \to f^{\mathrm{op}}$ is a morphism in $c/\mathbf{C}^{\mathrm{op}}$. This means α is a morphism $\alpha^{\mathrm{op}} : y \to x$ in $\mathbf{C}^{\mathrm{op}}$ such that $\alpha^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}}$. Then

$$\alpha^{\mathrm{op}} g^{\mathrm{op}} = f^{\mathrm{op}} \Leftrightarrow (g\alpha)^{\mathrm{op}} = f^{\mathrm{op}} \Leftrightarrow g\alpha = f$$

which means α is a morphism $f \to g$ in $\mathbf{C}/c$.

We deduce that $\mathbf{C}/c \cong (c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$ is a category as follows: $c/\mathbf{C}^{\mathrm{op}}$ is a category because $\mathbf{C}$ is and $(c/\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$ is a category because $c/\mathbf{C}^{\mathrm{op}}$ is.

Exercise 1.2.ii:

i. Show that a morphism $f : x \to y$ is a split epimorphism in a category $\mathbf{C}$ if and only if for all $c \in \mathbf{C}$, post-composition $f_* : \mathbf{C}(c, x) \to \mathbf{C}(c, y)$ defines a surjective function.

   Proof: If f is a split epi then we have $f' : y \to x$ such that $ff' = 1_y$. Given $g : c \to y$ let $g' = f'g$ in which case post-composition gives $f_*(g') = fg' = ff'g = 1_y g = g$ so that f_* is a surjection.

   In the other direction, if f_* is a surjection then $1_y : y \to y$ is in its image which is to say there exists $f' : y \to x$ such that $f_*(f') = ff' = 1_y$. Thus f is a split epi.

ii. Argue by duality that f is a split monomorphism if and only if for all $c \in \mathbf{C}$, pre-composition $f^* : \mathbf{C}(y, c) \to \mathbf{C}(x, c)$ defines a surjective function.

   By definition, $f : x \to y$ is a split mono if and only if $f^{\mathrm{op}} : y \to x$ is a split epi in $\mathbf{C}$. This is the case if and only if post-composition $f_*^{\mathrm{op}} : \mathbf{C}^{\mathrm{op}}(c, y) \to \mathbf{C}^{\mathrm{op}}(c, x)$ is a surjection by the previous exercise. This is saying $f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}$ is a surjection on morphisms $g'^{\mathrm{op}} : c \to x$ which is the same as pre-composition gf being a surjection to $g' : x \to c$.