Exam One

March 2020

I use the notation $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}'\boldsymbol{y}$ for the inner-product of euclidean vectors.

Question 1

- The Lagrangian $\mathcal{L}$ of the minimum variance optimisation problem can be written as:

$$\mathcal{L}(\boldsymbol{w}, \lambda, \gamma) = \frac{1}{2} \langle \boldsymbol{w}, \boldsymbol{\Sigma w} \rangle + \lambda(m - \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle) + \gamma(1 - \langle \boldsymbol{w}, \boldsymbol{1} \rangle)$$

  It follows that its partial derivatives at an arbitrary point $(\boldsymbol{w_0}, \lambda_0, \gamma_0)$ are:

$$\nabla_w \mathcal{L}(\boldsymbol{w_0}, \lambda_0, \gamma_0) = \boldsymbol{\Sigma w_0} - \lambda_0 \boldsymbol{\mu} - \gamma_0 \boldsymbol{1}$$

$$\partial_\lambda \mathcal{L}(\boldsymbol{w_0}, \lambda_0, \gamma_0) = m - \langle \boldsymbol{w_0}, \boldsymbol{\mu} \rangle$$

$$\partial_\gamma \mathcal{L}(\boldsymbol{w_0}, \lambda_0, \gamma_0) = 1 - \langle \boldsymbol{w_0}, \boldsymbol{1} \rangle$$

- Recall the formula provided in the Portfolio Optimisation lecture for the (unique) allocation solving the minimum variance problem:

$$\boldsymbol{w}^* = \boldsymbol{\Sigma^{-1}}(\lambda \boldsymbol{\mu} + \gamma \boldsymbol{1}) \tag{1}$$

  Fixing $m = 4.5\%$ and using (1) we obtain the below portfolio allocations and risks:

Stress Factor	Asset A	Asset B	Asset C	Asset D	Risk
1	78.51%	5.39%	13.36%	2.75%	5.84%
1.25	81.82%	-0.94%	17.90%	1.22%	6.07%
1.50	87.62%	-14.61%	32.57%	-5.57%	6.11%

  Table 1: Minimum Variance Portfolio Allocations and Risks

  The 'Stress Factor' column above refers to the factor by which all entries of the correlation matrix, used for the calculation of items in that row, have been multiplied excluding those on the diagonal. That is to say, I have assumed that the correlation of any asset with itself is always 1. Denoting the stressed correlation matrix by $\boldsymbol{R}'$, the identity matrix of the relevant size by $\boldsymbol{I}$ and the stress factor by s, we could write:

$$\boldsymbol{R}' = s\boldsymbol{R} + (1-s)\boldsymbol{I}$$

- To generate random portfolios I sampled independent standard normal random variables and then re-scaled by their sum. That is to say, for $i \in \{1, 2, 3, 4\}$

$$w_i \sim N(0,1),\ w_i' = \frac{w_i}{\sum_{j=1}^{4} w_j}$$

Since the sum of independent, normal random variables also has a normal distribution we know that each weight, as the ratio of two zero mean normal random variables, will be distributed like a Cauchy random variable. Forming approximately 6000 such portfolios gives the below figure. The global minimum variance portfolio can be seen as the leftmost orange

dot. Technically, the efficient frontier only includes the branch of the red curve above this portfolio. However, I have included both branches in the figure to illustrate the bounding nature of the curve. I have restricted the image to only show portfolios whose return is between −50% and +50%. Portfolios with risk greater than 100% are also omitted from the figure.

Question 2

- The search for the tangency portfolio $\boldsymbol{w}^*$, the portfolio formed from risky assets of maximal Sharpe ratio, can be formulated as the below optimisation problem:

$$\boldsymbol{w}^* = \text{argmax}\, \frac{\mu_\Pi - r}{\sigma_\Pi} \text{ subject to } \langle \boldsymbol{w}, \mathbf{1} \rangle = 1$$

where the (ex-ante) portfolio return μ_Π and risk σ_Π are defined by:

$$\mu_\Pi = \langle \boldsymbol{w}, \boldsymbol{\mu} \rangle\,,\ \sigma_\Pi = \sqrt{\langle \boldsymbol{w}, \boldsymbol{\Sigma w} \rangle}$$

The Lagrangian is given by:

$$\mathcal{L}(\boldsymbol{w}, \gamma) = \frac{\langle \boldsymbol{w}, \boldsymbol{\mu} \rangle - r}{\sqrt{\langle \boldsymbol{w}, \boldsymbol{\Sigma w} \rangle}} + \gamma(1 - \langle \boldsymbol{w}, \mathbf{1} \rangle)$$

The partial derivatives can be computed as:

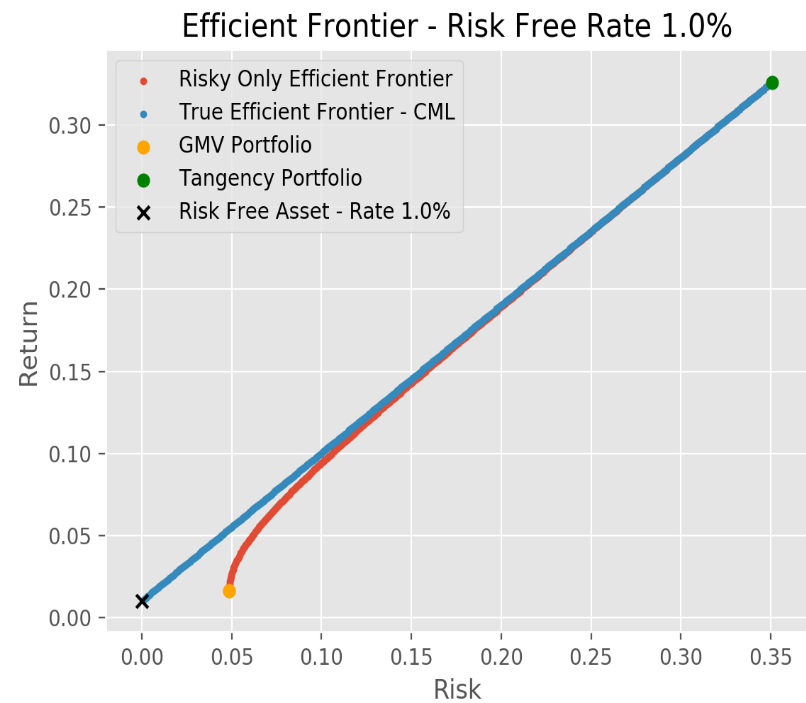
$$\begin{aligned} \nabla_w \mathcal{L}(\boldsymbol{w_0}, \gamma_0) &= \sigma_\Pi^{-2}[\sigma_\Pi \boldsymbol{\mu} - (\mu_\Pi - r)\frac{\boldsymbol{\Sigma w_0}}{\sigma_\Pi}] - \gamma_0 \mathbf{1} \\ &= \sigma_\Pi^{-3}[\sigma_\Pi^2 \boldsymbol{\mu} - (\mu_\Pi - r)\boldsymbol{\Sigma w_0}] - \gamma_0 \mathbf{1} \\ \partial_\gamma \mathcal{L}(\boldsymbol{w_0}, \gamma_0) &= 1 - \langle \boldsymbol{w_0}, \mathbf{1} \rangle \end{aligned}$$

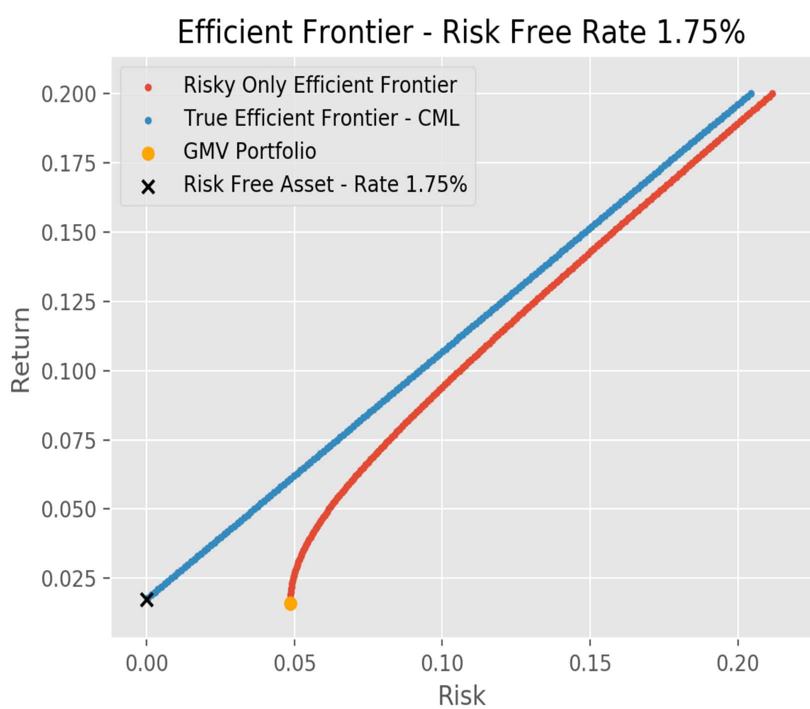
- The below results show the allocations and risk of the tangency portfolio formed for a given level of the risk-free rate:

Risk-Free Rate	Asset A	Asset B	Asset C	Asset D	Risk
0.50%	1.68%	-22.94%	81.43%	39.82%	19.65%
1.00%	-74.59%	-51.06%	149.02%	76.63%	35.07%
1.50%	-864.49%	-342.26%	848.97%	457.78%	197.24%
1.75%	810.35%	275.19%	-635.14%	-350.39%	147.35%

Table 2: Tangency Portfolio Allocations and Risks

- Below we plot the true efficient frontier in the presence of a risk free asset. The major qualitative difference between the cases where the risk-free rate is 1.00% or 1.75% is that the tangency portfolio no longer lies on the Capital Market Line in the latter. When the tangency portfolio does lie on the CML we see an intersection between the CML and the risky-only efficient frontier (as the tangency portfolio lies on this frontier). However, when the tangency portfolio does not lie on the CML we find that the Sharpe ratio of any portfolio on the CML is strictly greater than that of any portfolio on the risky-only efficient-frontier. The portfolios on the CML in this case are actually short the tangency portfolio. The switch between the two cases has occurred because the risk-free rate his risen above the key value $\mu_{GMV} = \left\langle \boldsymbol{\mu}, \boldsymbol{\Sigma^{-1}1} \right\rangle / \left\langle \mathbf{1}, \boldsymbol{\Sigma^{-1}1} \right\rangle \sim 1.61\%$.


Efficient Frontier - Risk Free Rate 1.0%
Risky Only Efficient Frontier
True Efficient Frontier - CML
GMV Portfolio
Tangency Portfolio
Risk Free Asset - Rate 1.0%
Return
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
Risk


Efficient Frontier - Risk Free Rate 1.75%
Risky Only Efficient Frontier
True Efficient Frontier - CML
GMV Portfolio
Risk Free Asset - Rate 1.75%
Return
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
0.00
0.05
0.10
0.15
0.20
Risk

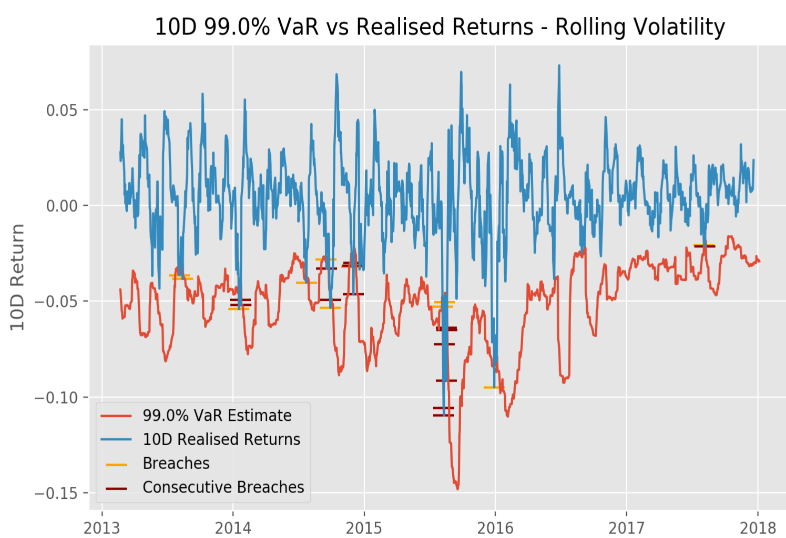
Question 3

The original data set contains 1250 prices, from which 1249 returns can be constructed. The use of a rolling 21 day window to compute standard deviations removes 20 returns from the start of the series from breach consideration. Finally, making use of a 10 day window for realised return computation removes 10 further observations at the end of the series from consideration. This means we have 1219 days on which a VaR can be computed and tested against a realised return.

- Using the rolling volatility method gives the following VaR backtest results:

Breaches	% Breaches	Consec. Breaches	Cond. Prob. Consec. Breach
25	2.05%	14	56%

Table 3: Rolling Volatility 99% VaR Backtest

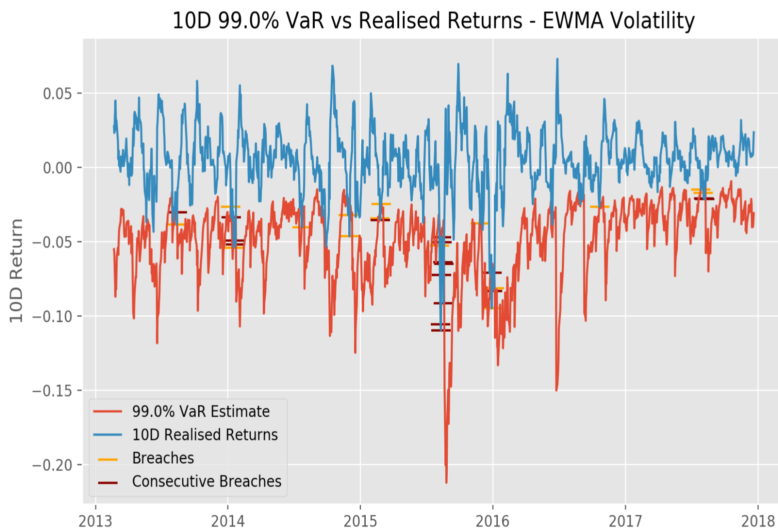


Stand alone breaches are marked with orange lines, while consecutive breaches (defined as those occurring on a day that follows a breach) are marked with a dark red line. The proportion of observed breaches is slightly higher than would be expected with a perfect model: 2.05% versus 1%. It is clear from the figure that breaches tend to be clustered together in time and this is also borne out by the conditional probability of consecutive breach.

- Using the EWMA methodology for estimating volatility, again with a 21 day window and using a 'persistence factor' of $\lambda = 0.72$, gives the following VaR backtest. I have used the sample volatility over the whole period as a starting point for the recursive EWMA calculation. However, explicitly applying the relevant weights to the squared returns of the first 21 days leads to a series that is almost identical after a few terms.

Breaches	% Breaches	Consec. Breaches	Cond. Prob. Consec. Breach
32	2.63%	17	53.13%

Table 4: EWMA Volatility 99% VaR Backtest



Using an EWMA estimation method for volatility seems to slightly decrease the performance of the VaR model over this history. We notice, in particular, a large number of (consecutive) breaches in August 2015 around the Chinese market disturbance. Several days of returns of small magnitude cause the EWMA volatility to drop. Large returns that lead to the 10 day breaches did not occur for almost a week after this low level of volatility had been reached, meaning several days were caught as breaches. This pattern can also be observed elsewhere within the series. Again, we see that breaches are much more likely on days following breaches.

Question 4

- For the technology portfolio, I find a total VaR (standard VaR plus additional liquidity VaR) of -7.90%. The breakdown below shows the notional total VaR, those components attributable to the standard measure and the liquidity specific measure, as well as the relative proportions of the total VaR attributable to each type:

Total LVaR	Asset VaR	Liq. VaR	% Asset VaR	% Liq. VaR
-1,263,808.72	-956,646.98	-307,161.74	75.70%	24.30%

Table 5: 99% LVaR - Technology Portfolio

- For the Gilt portfolio I have assumed a daily mean return of 0%, which seems reasonable to me for highly rated government debt. Under this assumption, the results are as follows:

Spread	Total LVaR	Asset VaR	Liq. VaR	% Asset VaR	% Liq. VaR
15	-2,821,617.45	-2,791,617.45	-30,000.0	98.94%	1.06%
125	-3,041,617.45	-2,791,617.45	-250,000.0	91.78%	8.22%

Table 6: 99% LVaR - Gilt Portfolio

I find total VaRs of -7.01% and -7.60% when the bid/ask spread volatility ('Spread' column of the table) is 15/125bp, respectively. As expected with a constant bid/ask spread, the majority of the overall VaR can be attributed to market moves, as opposed to liquidity concerns. As an aside, I would generally expect to see much lower volatility from a gilt portfolio than a technology portfolio.

Question 5

- Firm-wide backtesting operates with both Actual and Hypothetical P&L over the prior 12 month period, as explained in section 32.4.

- The metrics chosen for comparing RTPL and HPL are Spearman's rank correlation coefficient and the Kolmogorov-Smirnov test metric, namely the supremum norm of the difference in sample CDFs.

- If a trading desk experiences more than 12 exceptions at the 99th percentile in the most recent 12 month period then it will be forced to use the standardised approach, as described in section 32.19.