LCP Notes

Samuel Pfrommer

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1 Falling block

Consider a block of unit mass falling from an arbitrary height onto a static surface under the influence of a gravitational force with constant $-g$. We then have the following equations of motion:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (-g + \lambda_{k+1})\Delta t \end{aligned}$$

Where x_n represents the separation of the block from the surface at time step n, λ_n is the nonpenetration contact force, and Δt is our discrete time step. Note we are using implicit euler to prevent penetration of the block with the surface. Substituting the second equation into the first we have:

$$\begin{aligned} x_{k+1} &= x_k + (\dot{x}_k + (-g + \lambda_{k+1})\Delta t)\Delta t \\ x_{k+1} &= x_k + \dot{x}_k\Delta t + -g\Delta t^2 + \lambda_{k+1}\Delta t^2 \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}x_k - \frac{1}{\Delta t}\dot{x}_k + g \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2) \end{aligned}$$

Observe that we solve an LCP for each time step, and as such x_k and $\dot{x}_k$ are given to us from the previous time step. The problem therefore consists of solving for x_{k+1}, $\dot{x}_{k+1}$, and λ_{k+1}. Since λ_{k+1} determines $\dot{x}_{k+1}$ we can ignore $\dot{x}_{k+1}$ and instead consider the following LCP:

$$\begin{aligned} \lambda_{k+1} &\geq 0 \quad \text{only permit separating forces} \\ x_{k+1} &\geq 0 \quad \text{no penetration} \\ \lambda_{k+1}x_{k+1} &= 0 \quad \text{must have contact to have force} \\ \lambda_{k+1} &= Mx_{k+1} + q \end{aligned}$$

With $M = \frac{1}{\Delta t^2}$ and $q = -\frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2)$ from the equation above.

2 Sliding block

2.1 Forwards case

Consider a block of unit mass sliding in 1D on a surface under a positive external force u_n with static/dynamic coefficient of friction μ and friction force λ_n. We have the following EOM:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \end{aligned}$$

Observe that now the relevant constraint is that the velocity $\dot{x}_{k+1}$ must always be positive. Furthermore, to encode stick/slide transitions, we want:

$$0 \leq \mu g - \lambda_{k+1} \perp \dot{x}_{k+1} \geq 0$$

Such that when the block is moving, $\lambda_{k+1} = \mu g$ (i.e. the maximum friction force is being used to oppose the motion), and when the block is stationary $\lambda_{k+1} < \mu g$ (i.e. some smaller static friction force is being used). To encode this constraint as an LCP, we introduce the new variable $\lambda'_{k+1} = \mu g - \lambda_{k+1}$, where λ'_{k+1} intuitively represents the "unused" friction force. Furthermore, since we are interested in bounding velocity we will instead just use the second EOM:

$$\begin{aligned}
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \\
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} + \lambda'_{k+1} - \mu g)\Delta t \\
\dot{x}_{k+1} &= \Delta t \cdot \lambda'_{k+1} + \dot{x}_k + \Delta t(u_{k+1} - \mu g) \\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}(\dot{x}_k + \Delta t(u_{k+1} - \mu g)) \\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g
\end{aligned}$$

We can now write the LCP:

$$\begin{aligned}
\lambda'_{k+1} &\geq 0 && \text{friction force can't exceed } \mu \text{ times normal force} \\
\dot{x}_{k+1} &\geq 0 && \text{block is sliding to the right} \\
\lambda'_{k+1}\dot{x}_{k+1} &= 0 && \text{either block is stationary or using max friction force} \\
\lambda'_{k+1} &= M\dot{x}_{k+1} + q
\end{aligned}$$

With $M = \frac{1}{\Delta t}$ and $q = -\frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g$ from the equation above.

2.2 Bidirectional case

For the bidirectional case we need to introduce more slack variables: $\dot{x}^+_{k+1}$, $\dot{x}^-_{k+1}$, λ^+_{k+1}, and λ^-_{k+1}, which are all positive numbers representing velocity / friction force in the + / - direction. We can then write:

$$\begin{aligned}
0 \leq \mu g - \lambda^-_{k+1} \perp \dot{x}^+_{k+1} \geq 0 \\
0 \leq \mu g - \lambda^+_{k+1} \perp \dot{x}^-_{k+1} \geq 0
\end{aligned}$$

We again perform the substitution $\lambda'^+_{k+1} = \mu g - \lambda^+_{k+1}$ and $\lambda'^-_{k+1} = \mu g - \lambda^-_{k+1}$. We can now solve the EOM for both $\dot{x}^+_{k+1}$ and $\dot{x}^-_{k+1}$. Observe that instead of just u_{k+1}, we also need to subtract the force required to bring the complementary velocity to zero in one time step.

$$\begin{aligned}
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k - \lambda^-_{k+1})\Delta t \\
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + \lambda'^-_{k+1} - \mu g)\Delta t \\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k + \mu g \\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \mu g
\end{aligned}$$

Similarly, we can get:

$$\begin{aligned}
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - \lambda^+_{k+1})\Delta t \\
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k + \lambda'^+_{k+1} - \mu g)\Delta t \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k + \mu g \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \mu g
\end{aligned}$$

We can now put these in matrix form as:

$$\begin{bmatrix} \lambda'^{-}_{k+1} \\ \lambda'^{+}_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t} & 0 \\ 0 & \frac{1}{\Delta t} \end{bmatrix} \begin{bmatrix} \dot{x}^{+}_{k+1} \\ \dot{x}^{-}_{k+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\Delta t}\dot{x}^{-}_{k} - \frac{1}{\Delta t}\dot{x}^{+}_{k} - u_{k+1} + \mu g \\ \frac{1}{\Delta t}\dot{x}^{+}_{k} - \frac{1}{\Delta t}\dot{x}^{-}_{k} + u_{k+1} + \mu g \end{bmatrix}$$

This equation gives us the form of our M matrix and q vector. Note that the order of the equations is important; if we had stacked the λ'^{+}_{k+1} on top of λ'^{-}_{k+1}, our resulting M matrix would not be positive semidefinite (Lemke algorithm gives secondary rays). Furthermore, we would not get the desired complementarity constraints below:

$$\begin{aligned} \lambda'^{+}_{k+1} &\geq 0 \\ \lambda'^{-}_{k+1} &\geq 0 \\ \dot{x}^{+}_{k+1} &\geq 0 \\ \dot{x}^{-}_{k+1} &\geq 0 \\ \lambda'^{-}_{k+1}\dot{x}^{+}_{k+1} &= 0 \quad \text{block sliding to the right means all neg friction force used} \\ \lambda'^{+}_{k+1}\dot{x}^{-}_{k+1} &= 0 \quad \text{block sliding to the left means all pos friction force used} \end{aligned}$$

Finally, we can just update the positions by:

$$x_{k+1} = x_k + (\dot{x}^{+}_{k+1} - \dot{x}^{-}_{k+1})\Delta t$$