LCP Notes

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1 Falling block

Consider a block of unit mass falling from an arbitrary height onto a static surface under the influence of a gravitational force with constant $-g$. We then have the following equations of motion:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (-g + \lambda_{k+1})\Delta t \end{aligned}$$

Where x_n represents the separation of the block from the surface at time step n, λ_n is the nonpenetration contact force, and Δt is our discrete time step. Note we are using implicit euler to prevent penetration of the block with the surface. Substituting the second equation into the first we have:

$$\begin{aligned} x_{k+1} &= x_k + (\dot{x}_k + (-g + \lambda_{k+1})\Delta t)\Delta t \\ x_{k+1} &= x_k + \dot{x}_k\Delta t + -g\Delta t^2 + \lambda_{k+1}\Delta t^2 \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}x_k - \frac{1}{\Delta t}\dot{x}_k + g \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2) \end{aligned}$$

Observe that we solve an LCP for each time step, and as such x_k and $\dot{x}_k$ are given to us from the previous time step. The problem therefore consists of solving for x_{k+1}, $\dot{x}_{k+1}$, and λ_{k+1}. Since λ_{k+1} determines $\dot{x}_{k+1}$ we can ignore $\dot{x}_{k+1}$ and instead consider the following LCP:

$$\begin{aligned} \lambda_{k+1} &\geq 0 \quad \text{only permit separating forces} \\ x_{k+1} &\geq 0 \quad \text{no penetration} \\ \lambda_{k+1}x_{k+1} &= 0 \quad \text{must have contact to have force} \\ \lambda_{k+1} &= Mx_{k+1} + q \end{aligned}$$

With $M = \frac{1}{\Delta t^2}$ and $q = -\frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2)$ from the equation above.

2 Sliding block

2.1 Forwards case

Consider a block of unit mass sliding in 1D on a surface under a positive external force u_n with static/dynamic coefficient of friction μ and friction force λ_n. We have the following EOM:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \end{aligned}$$

Observe that now the relevant constraint is that the velocity $\dot{x}_{k+1}$ must always be positive. Furthermore, to encode stick/slide transitions, we want:

$$0 \leq \mu g - \lambda_{k+1} \perp \dot{x}_{k+1} \geq 0$$

Such that when the block is moving, $\lambda_{k+1} = \mu g$ (i.e. the maximum friction force is being used to oppose the motion), and when the block is stationary $\lambda_{k+1} < \mu g$ (i.e. some smaller static friction force is being used). To encode this constraint as an LCP, we introduce the new variable $\lambda'_{k+1} = \mu g - \lambda_{k+1}$, where λ'_{k+1} intuitively represents the "unused" friction force. Furthermore, since we are interested in bounding velocity we will instead just use the second EOM:

$$\begin{aligned}
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \qquad (1)\\
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} + \lambda'_{k+1} - \mu g)\Delta t\\
\dot{x}_{k+1} &= \Delta t \cdot \lambda'_{k+1} + \dot{x}_k + \Delta t(u_{k+1} - \mu g)\\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}(\dot{x}_k + \Delta t(u_{k+1} - \mu g))\\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g
\end{aligned}$$

We can now write the LCP:

$$\begin{aligned}
\lambda'_{k+1} &\geq 0 \quad \text{friction force can't exceed } \mu \text{ times normal force}\\
\dot{x}_{k+1} &\geq 0 \quad \text{block is sliding to the right}\\
\lambda'_{k+1}\dot{x}_{k+1} &= 0 \quad \text{either block is stationary or using max friction force}\\
\lambda'_{k+1} &= M\dot{x}_{k+1} + q
\end{aligned}$$

With $M = \frac{1}{\Delta t}$ and $q = -\frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g$ from the equation above.
Remark 1. *It might seem that the above constraints are insufficient; what would prevent a negative friction force in sticking? Consider the case where $\lambda'_{k+1} > \mu g$; i.e., $\lambda_{k+1} < 0$. If $\lambda_{k+1} < 0$, we can immediately see that the right hand side of* (1) *must be strictly greater than zero because of our nonnegativity constraints on $\dot{x}_k$ and u_{k+1}. That means that $\dot{x}_{k+1} > 0$ and therefore $\lambda'_{k+1} = 0$. This proves that λ_{k+1} is always positive even though we never encoded that explicitely as a constraint. Intuitively, one can think of this as "our friction force can never propel the block, since if that happened the velocity would be immediately positive and friction must always oppose motion."*

2.2 Bidirectional case

For the bidirectional case we need to introduce more slack variables: $\dot{x}^+_{k+1}$, $\dot{x}^-_{k+1}$, λ^+_{k+1}, and λ^-_{k+1}, which are all positive numbers representing velocity / friction force in the + / - direction. We can then write:

$$\begin{aligned}
0 \leq \mu g - \lambda^-_{k+1} \perp \dot{x}^+_{k+1} \geq 0\\
0 \leq \mu g - \lambda^+_{k+1} \perp \dot{x}^-_{k+1} \geq 0
\end{aligned}$$

We again perform the substitution $\lambda'^+_{k+1} = \mu g - \lambda^+_{k+1}$ and $\lambda'^-_{k+1} = \mu g - \lambda^-_{k+1}$. We can now write the EOM for both $\dot{x}^+_{k+1}$ and $\dot{x}^-_{k+1}$. The first equation looks familiar to the forwards case, with the addition of a term involving $\dot{x}^-_k$. The importance of this term comes when the block changes direction, in this case from a negative velocity to a positive velocity. In order for the positive velocity component to begin increasing, we need to ensure that the negative velocity from the previous timestep has been driven completely to zero. This does not correspond to an actual physical force; it is more of a force threshold that the input u_{k+1} must overcome in order to begin increasing the block's positive velocity.

$$\begin{aligned}
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k - \lambda^-_{k+1})\Delta t\\
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + \lambda'^-_{k+1} - \mu g)\Delta t\\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k + \mu g\\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \mu g
\end{aligned}$$

Similarly, we can get:

$$\begin{aligned}
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - \lambda^+_{k+1})\Delta t \\
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k + \lambda'^+_{k+1} - \mu g)\Delta t \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k + \mu g \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \mu g
\end{aligned}$$

We can now put these in matrix form as:

$$\begin{bmatrix} \lambda'^-_{k+1} \\ \lambda'^+_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t} & 0 \\ 0 & \frac{1}{\Delta t} \end{bmatrix} \begin{bmatrix} \dot{x}^+_{k+1} \\ \dot{x}^-_{k+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\Delta t}\dot{x}^-_k - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \mu g \\ \frac{1}{\Delta t}\dot{x}^+_k - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \mu g \end{bmatrix}$$

This equation gives us the form of our M matrix and q vector. We also have the below constraints:

$$\begin{aligned}
\lambda'^+_{k+1} &\geq 0 \\
\lambda'^-_{k+1} &\geq 0 \\
\dot{x}^+_{k+1} &\geq 0 \\
\dot{x}^-_{k+1} &\geq 0 \\
\lambda'^-_{k+1}\dot{x}^+_{k+1} &= 0 \quad \text{block sliding to the right means all neg friction force used} \\
\lambda'^+_{k+1}\dot{x}^-_{k+1} &= 0 \quad \text{block sliding to the left means all pos friction force used}
\end{aligned}$$

Finally, we can just update the positions by:

$$x_{k+1} = x_k + (\dot{x}^+_{k+1} - \dot{x}^-_{k+1})\Delta t$$

2.3 Sanity checks

2.3.1 Velocity characteristics

Lemma 2.1. *The positive and negative velocities at any time step are complementary; i.e.,* $\dot{x}^+_k \cdot \dot{x}^-_k = 0$ *for all time steps* k.

Proof. Since $\dot{x}^+_k$ and $\dot{x}^-_k$ are both optimization variables they are not immediately constrained to be complimentary; however, this ends up being the case. Empirically, this constraint was never violated for about 30,000 time steps with random forces using the above formulation. It can also formally be proven via induction (assuming $\Delta t = 1$ for notational cleanliness):

Assume $\dot{x}^+_k \perp \dot{x}^-_k$; we want to show that $\dot{x}^+_{k+1} \perp \dot{x}^-_{k+1}$. Observe that the base case is trivial (assign all of $\dot{x}_0$ to the appropriate direction and set the other direction to zero). Now assume WLOG that from our induction hypothesis we have $\dot{x}^+_k = 0$. Also assume for sake of contradiction that both $\dot{x}^+_{k+1}$ and $\dot{x}^-_{k+1}$ are strictly positive. We can then write:

$$\begin{aligned}
\dot{x}^+_{k+1} &= \dot{x}^+_k + u_{k+1} - \dot{x}^-_k - \lambda^-_{k+1} \\
&= u_{k+1} - \dot{x}^-_k - (\mu g - \lambda'^-_{k+1}) \\
&= u_{k+1} - \dot{x}^-_k - \mu g \quad \text{by complimentarity} \\
&\leq u_{k+1} - \dot{x}^-_k \\
\dot{x}^-_{k+1} &= \dot{x}^-_k - u_{k+1} - \dot{x}^+_k - \lambda^+_{k+1} \\
&= -u_{k+1} + \dot{x}^-_k - (\mu g - \lambda'^+_{k+1}) \\
&= -u_{k+1} + \dot{x}^-_k - \mu g \quad \text{by complimentarity} \\
&\leq -(u_{k+1} - \dot{x}^-_k)
\end{aligned}$$

It is now easy to see that one of $\dot{x}^+_{k+1}$ or $\dot{x}^-_{k+1}$ must be nonpositive–which is a contradiction because we assumed them to be strictly positive. Therefore we have proven the induction step. ∎

Remark 2. *It seems that if* $u_{k+1} - \dot{x}^-_k \neq 0$ *the above two inequalities would force one of the velocities to be strictly negative, violating the nonnegativity constraints. However, note that this only occurs because of our contradictory assumption that both* $\dot{x}^+_{k+1}$ *and* $\dot{x}^-_{k+1}$ *are strictly positive. In reality, one would need to be zero, which would free up the complementary* λ' *to be arbitrarily positive.*

2.3.2 Summed velocity

Let's double check what happens when we add our velocities to get the overall velocity $\dot{x}_{k+1}$.

$$\begin{aligned} \dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k - \lambda^-_{k+1})\Delta t \\ -\Big(\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - \lambda^+_{k+1})\Delta t\Big) \\ \dot{x}^+_{k+1} - \dot{x}^-_{k+1} &= 2(\dot{x}^+_k - \dot{x}^-_k) + 2u_{k+1}\Delta t + (\lambda^+_{k+1} - \lambda^-_{k+1})\Delta t \end{aligned} \tag{2}$$

At first glance, the factors of 2 don't look quite right (why is velocity doubling every timestep?). However, this is actually not an issue.

The reason why is pretty unintuitive and has to do with the fact that our friction forces are being encoded as the slack variables $w = Mz + q$. The best way to see this is with a simple example. Assume that the block is moving with a constant rightwards velocity $\dot{x}^+_k = \dot{x}^+_{k+1} = 1$, and frictional / input forces are zero. Then by the above lemma, we have $\dot{x}^-_k = \dot{x}^-_{k+1} = 0$; the corresponding variable λ'^+_{k+1} is now free to vary. We can now satisfy equation (2) by letting $\lambda'^+_{k+1} = 1$ (this is what the LCP outputs as well). Observe that since $\mu g = 0$, this actual creates a fictitious force $\lambda^+_{k+1} = -1$ which prevents the block from increasing in velocity and keeps it moving in a straight line.

Essentially, we structured the problem in a way that the slack variable $w = Mz + q$ had rough physical meaning. Unless the block is at rest with no external force, there is always a physical positive friction force in one direction, and a fictitious negative friction force in the opposite direction. The nice thing about this formulation is that M is 2x2 and a multiple of the identity and therefore positive definite, so we're guaranteed a unique LCP solution. We also managed to use 2 complimentarity constraints instead of 3.

TODO: reformulate using the traditional slack variable method