LCP Notes

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May 21, 2019

1 Falling block

Consider a block of unit mass falling from an arbitrary height onto a static surface under the influence of a gravitational force with constant $-g$. We then have the following equations of motion:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (-g + \lambda_{k+1})\Delta t \end{aligned}$$

Where x_n represents the separation of the block from the surface at time step n, λ_n is the nonpenetration contact force, and Δt is our discrete time step. Note we are using implicit euler to prevent penetration of the block with the surface. Substituting the second equation into the first we have:

$$\begin{aligned} x_{k+1} &= x_k + (\dot{x}_k + (-g + \lambda_{k+1})\Delta t)\Delta t \\ x_{k+1} &= x_k + \dot{x}_k\Delta t + -g\Delta t^2 + \lambda_{k+1}\Delta t^2 \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}x_k - \frac{1}{\Delta t}\dot{x}_k + g \\ \lambda_{k+1} &= \frac{1}{\Delta t^2}x_{k+1} - \frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2) \end{aligned}$$

Observe that we solve an LCP for each time step, and as such x_k and $\dot{x}_k$ are given to us from the previous time step. The problem therefore consists of solving for x_{k+1}, $\dot{x}_{k+1}$, and λ_{k+1}. Since λ_{k+1} determines $\dot{x}_{k+1}$ we can ignore $\dot{x}_{k+1}$ and instead consider the following LCP:

$$\begin{aligned} \lambda_{k+1} &\geq 0 \quad \text{only permit separating forces} \\ x_{k+1} &\geq 0 \quad \text{no penetration} \\ \lambda_{k+1}x_{k+1} &= 0 \quad \text{must have contact to have force} \\ \lambda_{k+1} &= Mx_{k+1} + q \end{aligned}$$

With $M = \frac{1}{\Delta t^2}$ and $q = -\frac{1}{\Delta t^2}(x_k + \Delta t\dot{x}_k - g\Delta t^2)$ from the equation above.

2 Sliding block

2.1 Forwards case

Consider a block of unit mass sliding in 1D on a surface under a positive external force u_n with static/dynamic coefficient of friction μ and friction force λ_n. We have the following EOM:

$$\begin{aligned} x_{k+1} &= x_k + \dot{x}_{k+1}\Delta t \\ \dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \end{aligned}$$

Observe that now the relevant constraint is that the velocity $\dot{x}_{k+1}$ must always be positive. Furthermore, to encode stick/slide transitions, we want:

$$0 \leq \mu g - \lambda_{k+1} \perp \dot{x}_{k+1} \geq 0$$

Such that when the block is moving, $\lambda_{k+1} = \mu g$ (i.e. the maximum friction force is being used to oppose the motion), and when the block is stationary $\lambda_{k+1} < \mu g$ (i.e. some smaller static friction force is being used). To encode this constraint as an LCP, we introduce the new variable $\lambda'_{k+1} = \mu g - \lambda_{k+1}$, where λ'_{k+1} intuitively represents the "unused" friction force. Furthermore, since we are interested in bounding velocity we will instead just use the second EOM:

$$\begin{aligned}
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} - \lambda_{k+1})\Delta t \\
\dot{x}_{k+1} &= \dot{x}_k + (u_{k+1} + \lambda'_{k+1} - \mu g)\Delta t \\
\dot{x}_{k+1} &= \Delta t \cdot \lambda'_{k+1} + \dot{x}_k + \Delta t(u_{k+1} - \mu g) \\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}(\dot{x}_k + \Delta t(u_{k+1} - \mu g)) \\
\lambda'_{k+1} &= \frac{1}{\Delta t}\dot{x}_{k+1} - \frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g
\end{aligned}$$

We can now write the LCP:

$$\begin{aligned}
\lambda'_{k+1} &\geq 0 \quad \text{friction force can't exceed } \mu \text{ times normal force} \\
\dot{x}_{k+1} &\geq 0 \quad \text{block is sliding to the right} \\
\lambda'_{k+1}\dot{x}_{k+1} &= 0 \quad \text{either block is stationary or using max friction force} \\
\lambda'_{k+1} &= M\dot{x}_{k+1} + q
\end{aligned}$$

With $M = \frac{1}{\Delta t}$ and $q = -\frac{1}{\Delta t}\dot{x}_k - u_{k+1} + \mu g$ from the equation above.

2.2 Bidirectional case

For the bidirectional case we need to introduce more slack variables: $\dot{x}^+_{k+1}$, $\dot{x}^-_{k+1}$, λ^+_{k+1}, and λ^-_{k+1}, which are all positive numbers representing velocity / friction force in the + / - direction. We can then write:

$$\begin{aligned}
0 \leq \mu g - \lambda^-_{k+1} \perp \dot{x}^+_{k+1} \geq 0 \\
0 \leq \mu g - \lambda^+_{k+1} \perp \dot{x}^-_{k+1} \geq 0
\end{aligned}$$

We again perform the substitution $\lambda'^+_{k+1} = \mu g - \lambda^+_{k+1}$ and $\lambda'^-_{k+1} = \mu g - \lambda^-_{k+1}$. We can now solve the EOM for both $\dot{x}^+_{k+1}$ and $\dot{x}^-_{k+1}$. Observe that instead of just u_{k+1}, we also need to subtract the force required to bring the complementary velocity to zero in one time step.

$$\begin{aligned}
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k - \lambda^-_{k+1})\Delta t \\
\dot{x}^+_{k+1} &= \dot{x}^+_k + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + \lambda'^-_{k+1} - \mu g)\Delta t \\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k + \mu g \\
\lambda'^-_{k+1} &= \frac{1}{\Delta t}\dot{x}^+_{k+1} + \frac{1}{\Delta t}\dot{x}^-_k - \frac{1}{\Delta t}\dot{x}^+_k - u_{k+1} + \mu g
\end{aligned}$$

Similarly, we can get:

$$\begin{aligned}
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k - \lambda^+_{k+1})\Delta t \\
\dot{x}^-_{k+1} &= \dot{x}^-_k + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^+_k + \lambda'^+_{k+1} - \mu g)\Delta t \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k + \mu g \\
\lambda'^+_{k+1} &= \frac{1}{\Delta t}\dot{x}^-_{k+1} + \frac{1}{\Delta t}\dot{x}^+_k - \frac{1}{\Delta t}\dot{x}^-_k + u_{k+1} + \mu g
\end{aligned}$$

We can now put these in matrix form as:

$$\begin{bmatrix} \lambda'^{-}_{k+1} \\ \lambda'^{+}_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t} & 0 \\ 0 & \frac{1}{\Delta t} \end{bmatrix} \begin{bmatrix} \dot{x}^{+}_{k+1} \\ \dot{x}^{-}_{k+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\Delta t}\dot{x}^{-}_{k} - \frac{1}{\Delta t}\dot{x}^{+}_{k} - u_{k+1} + \mu g \\ \frac{1}{\Delta t}\dot{x}^{+}_{k} - \frac{1}{\Delta t}\dot{x}^{-}_{k} + u_{k+1} + \mu g \end{bmatrix}$$

This equation gives us the form of our M matrix and q vector. Note that the order of the equations is important; if we had stacked the λ'^{+}_{k+1} on top of λ'^{-}_{k+1}, our resulting M matrix would not be positive semidefinite (Lemke algorithm gives secondary rays). Furthermore, we would not get the desired complementarity constraints below:

$$\begin{aligned} \lambda'^{+}_{k+1} &\geq 0 \\ \lambda'^{-}_{k+1} &\geq 0 \\ \dot{x}^{+}_{k+1} &\geq 0 \\ \dot{x}^{-}_{k+1} &\geq 0 \\ \lambda'^{-}_{k+1}\dot{x}^{+}_{k+1} &= 0 \quad \text{block sliding to the right means all neg friction force used} \\ \lambda'^{+}_{k+1}\dot{x}^{-}_{k+1} &= 0 \quad \text{block sliding to the left means all pos friction force used} \end{aligned}$$

Finally, we can just update the positions by:

$$x_{k+1} = x_k + (\dot{x}^{+}_{k+1} - \dot{x}^{-}_{k+1})\Delta t$$

2.3 Sanity checks

2.3.1 Velocity characteristics

Lemma 2.1. *The positive and negative velocities at any time step are complementary; i.e.,* $\dot{x}^{+}_{k} \cdot \dot{x}^{-}_{k} = 0$ *for all time steps* k.

Proof. Since $\dot{x}^{+}_{k}$ and $\dot{x}^{-}_{k}$ are both optimization variables they are not immediately constrained to be complimentary; however, this ends up being the case. Empirically, this constraint was never violated for about 30,000 time steps with random forces using the above formulation. It can also formally be proven via induction (assuming $\Delta t = 1$ for notational cleanliness):

Assume $\dot{x}^{+}_{k} \perp \dot{x}^{-}_{k}$; we want to show that $\dot{x}^{+}_{k+1} \perp \dot{x}^{-}_{k+1}$. Observe that the base case is trivial (assign all of $\dot{x}_0$ to the appropriate direction and set the other direction to zero). Now assume WLOG of generality that from our induction hypothesis we have $\dot{x}^{+}_{k} = 0$. We can then write:

$$\begin{aligned} \dot{x}^{+}_{k+1} &= \dot{x}^{+}_{k} + u_{k+1} - \dot{x}^{-}_{k} - \lambda^{-}_{k+1} = u_{k+1} - \dot{x}^{-}_{k} - \lambda^{-}_{k+1} \leq u_{k+1} - \dot{x}^{-}_{k} \\ \dot{x}^{-}_{k+1} &= \dot{x}^{-}_{k} - u_{k+1} - \dot{x}^{+}_{k} - \lambda^{+}_{k+1} = -u_{k+1} + \dot{x}^{-}_{k} - \lambda^{+}_{k+1} \leq -(u_{k+1} - \dot{x}^{-}_{k}) \end{aligned}$$

It is now easy to see that one of $\dot{x}^{+}_{k+1}$ or $\dot{x}^{-}_{k+1}$ must be nonpositive; the only way this can satisfy the nonnegativity constraint is for the corresponding velocity in the next time step to be zero, satisfying $\dot{x}^{+}_{k+1} \perp \dot{x}^{-}_{k+1}$. ■

2.3.2 Summed velocity

Let's double check what happens when we add our velocities to get the overall velocity $\dot{x}_{k+1}$.

$$\begin{aligned} \dot{x}^{+}_{k+1} &= \dot{x}^{+}_{k} + (u_{k+1} - \frac{1}{\Delta t}\dot{x}^{-}_{k} - \lambda^{-}_{k+1})\Delta t \\ -\Big(\dot{x}^{-}_{k+1} &= \dot{x}^{-}_{k} + (-u_{k+1} - \frac{1}{\Delta t}\dot{x}^{+}_{k} - \lambda^{+}_{k+1})\Delta t\Big) \\ \dot{x}^{+}_{k+1} - \dot{x}^{-}_{k+1} &= 2(\dot{x}^{+}_{k} - \dot{x}^{-}_{k}) + 2u_{k+1}\Delta t + (\lambda^{+}_{k+1} - \lambda^{-}_{k+1})\Delta t \qquad (1) \end{aligned}$$

At first glance, the factors of 2 don't look quite right (why is velocity doubling every timestep?). However, this is actually not an issue.

The reason why is pretty unintuitive and has to do with the fact that our friction forces are being encoded as the slack variables $w = Mz + q$. The best way to see this is with a simple example. Assume that the block is moving with a constant rightwards velocity $\dot{x}^{+}_{k} = \dot{x}^{+}_{k+1} = 1$, and frictional / input forces are zero. Then by the above lemma, we have $\dot{x}^{-}_{k} = \dot{x}^{-}_{k+1} = 0$; the corresponding variables λ'^{+}_{k} and λ'^{+}_{k+1} are now free to vary. We can now satisfy equation (1) by letting $\lambda'^{+}_{k+1} = 1$ (this is what the LCP outputs as well). Observe that since $\mu g = 0$, this actual creates a fictitious force $\lambda_{k+1}^{+} = -1$ which prevents the block from increasing in velocity and keeps it moving in a straight line.

Essentially, we structured the problem in a way that the slack variable $w = Mz + q$ had rough physical meaning; it's not trivial to extract the corresponding friction forces but is also not too tricky (see process_sliding_solution in dynamics.py). The nice thing about this formulation is that M is 2x2 and a multiple of the identity and therefore positive definite, so we're guaranteed a unique LCP solution. We also managed to use 2 complimentarity constraints instead of 3.

2.4 Traditional formulation

The traditional formulation of the above bidirectional sliding block problem involves a third variables, γ, which is effectively an upper bound on the absolute value of the sliding velocity. We still decompose λ into λ_+ and λ_-, but these are now z variables instead of w variables in the LCP. Velocity isn't a decision variable in the formulation. Observe that we can write (assuming $\Delta t = 1$):

$$v_{k+1} = v_k + \lambda_{k+1}^{+} - \lambda_{k+1}^{-} + u_{k+1}$$

And now have the following complimentarity constraints:

$$\begin{aligned} &0 \leq \lambda_{k+1}^{+} \perp \gamma_{k+1} + v_{k+1} \geq 0 \quad \text{use positive friction force if block is moving to the left} \\ &0 \leq \lambda_{k+1}^{-} \perp \gamma_{k+1} - v_{k+1} \geq 0 \quad \text{use negative friction force if block is moving to the right} \\ &0 \leq \gamma_{k+1} \perp mg\mu - \lambda_{k+1}^{+} - \lambda_{k+1}^{-} \geq 0 \quad \text{use max friction force if block is moving} \end{aligned}$$

We can now substitute in for v_{k+1} to only include our decision variables and constants:

$$\begin{aligned} 0 \leq \lambda_{k+1}^{+} \perp \gamma_{k+1} + v_k + \lambda_{k+1}^{+} - \lambda_{k+1}^{-} + u_{k+1} \geq 0 \\ 0 \leq \lambda_{k+1}^{-} \perp \gamma_{k+1} - v_k - \lambda_{k+1}^{+} + \lambda_{k+1}^{-} - u_{k+1} \geq 0 \\ 0 \leq \gamma_{k+1} \perp mg\mu - \lambda_{k+1}^{+} - \lambda_{k+1}^{-} \geq 0 \end{aligned}$$

Now this can be written in matrix form as:

$$0 \leq \begin{bmatrix} \lambda_{k+1}^{+} \\ \lambda_{k+1}^{-} \\ \gamma_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{k+1}^{+} \\ \lambda_{k+1}^{-} \\ \gamma_{k+1} \end{bmatrix} + \begin{bmatrix} v_k + u_{k+1} \\ -v_k - u_{k+1} \\ mg\mu \end{bmatrix} \geq 0$$

In this case I believe M is a P-matrix and Lemke's algorithm is guaranteed to have a unique solution.

3 Learning

We want to ultimately learn the dynamics for a full Kuka arm interacting with an object, but we start with just the sliding block example. The most obvious idea is to learn the unknown friction force $\lambda = f(x, u)$ by fitting a neural network or some other function approximator to the raw data. However, this approach won't be able to properly capture sticking behavior. We need some way of embedding the complementary nature of the problem into our learning algorithm.

3.1 Known frictional forces

Assume that we know the friction forces vector $\boldsymbol{\lambda} = \begin{bmatrix} \lambda^- & \lambda^+ \end{bmatrix}$, which contains the complementary positive/negative friction components. We want to predict the velocity vector $\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x}^+ & \dot{x}^- \end{bmatrix}$ (order flipped to preserve complementarity); we will denote the scalar velocity as $\dot{x}$ without boldface. Let's claim that $\dot{\boldsymbol{x}}$ is affine in $\boldsymbol{\lambda}$, as is the case from teh underlying dynamics. Then we can learn two functions $\boldsymbol{f}$ (2x1 vector) and G (2x2 matrix) such that:

$$\dot{\boldsymbol{x}}_{k+1} = \boldsymbol{f}(\dot{x}_k, u_{k+1}) + G(\dot{x}_k, u_{k+1})\boldsymbol{\lambda}_{k+1}$$

Let's consider the true system we're trying to learn:

$$\begin{aligned} \dot{x}_{k+1}^{+} = \dot{x}_k^{+} + \Delta t \cdot u_{k+1} - \dot{x}_k^{-} - \Delta t \cdot \lambda_{k+1}^{-} \\ \dot{x}_{k+1}^{-} = \dot{x}_k^{-} - \Delta t \cdot u_{k+1} - \dot{x}_k^{+} - \Delta t \cdot \lambda_{k+1}^{+} \end{aligned}$$

This can be written in matrix form as:

$$\dot{\boldsymbol{x}}_{k+1} = \begin{bmatrix} \dot{x}_{k+1}^{+} \\ \dot{x}_{k+1}^{-} \end{bmatrix} = \begin{bmatrix} \Delta t \cdot u_{k+1} \\ -\Delta t \cdot u_{k+1} \end{bmatrix} + \begin{bmatrix} -\Delta t & 0 \\ 0 & -\Delta t \end{bmatrix} \begin{bmatrix} \lambda_{k+1}^{-} \\ \lambda_{k+1}^{+} \end{bmatrix}$$