Notation

Y_{it} = ordinal response for subject i on day t with levels $k \in \{1, 2, \dots \mathcal{K}\}$

$$Y_{it,k} = I(Y_{it} = k)$$

$$Z_{it,k} = \sum_{l=1}^{k} Y_{it,k} = I(Y_{it} \le k)$$

$i \in \{1, \dots N\}$

$t \in \{1, \dots T_i\}$

Note that conditioning on X_i is implicit

$\mu^m_{it,k} = pr(Y_{it} = k)$ marginal mean

$\mu^c_{it,k} = pr(Y_{it} = k \mid Y_{it-1})$ conditional mean

$\pi^m_{it,k} = pr(Y_{it} \le k)$ & $\mu^m_{it,k} = \pi^m_{it,k} - \pi^m_{it,k-1}$

$h_{it,kl} = pr(Y_{it} = k \mid Y_{it-1} = l)$

$$\log\left(\frac{\pi^m_{it,k}}{1-\pi^m_{it,k}}\right) = \eta_{it,k} = \alpha_k + X_{it}\beta$$

$$\log\left(\frac{\mu^c_{it,k}}{\mu^c_{it,\mathcal{K}}}\right) = \Delta_{it,k}(X_i) + \sum_{l=1}^{\mathcal{K}-1} \gamma^{kl}\, Y_{it-1,l}$$

$\Delta_{it,k}$ calculation

$$\mu^{m}_{it,k} = \sum_{g=1}^{\mathcal{K}} h_{it,kg}\, \mu^{m}_{it-1,g}$$

$$= \sum_{g=1}^{\mathcal{K}} \frac{\exp(\Delta_{it,k} + \phi^{kg})}{1 - \sum_{l=1}^{\mathcal{K}-1} \exp(\Delta_{it,k} + \phi^{lg})} \mu^{m}_{it-1,g}$$

Need to solve for $\Delta_{it,k}$ $\forall$ i,t,k

$$f(\Delta_{it}) = \{f_k(\Delta_{it})\}_{k=1}^{\mathcal{K}} = \left\{ \sum_{g=1}^{\mathcal{K}} h_{it,kg}\, \mu^{m}_{it-1,g} - \mu^{m}_{it,k} \right\}_{k=1}^{\mathcal{K}}$$

For all $k = \{1, .. \mathcal{K}\}$, solve $f_k(\Delta_{it}) = 0$

Using Newton Raphson, at iteration m

$$\Delta^{(m)}_{it,k} = \Delta^{(m-1)}_{it,k} - \left(\frac{\partial f(\Delta^{(m-1)}_{it})}{\partial \Delta^{(m-1)}_{it,k}}\right)^{-1} f_k(\Delta^{(m-1)}_{it})$$

where $$\frac{\partial f_k}{\partial \Delta_{it,j}} = \begin{cases} \sum_{g=1}^{\mathcal{K}} h_{it,kg}(1-h_{it,kg})\, \mu^{m}_{it-1,g} & k=j \\ \sum_{g=1}^{\mathcal{K}} -h_{it,kg}\, h_{it,jg}\, \mu^{m}_{it-1,g} & k \neq j \end{cases}$$

Notes: ① $$pr(Y_{it}=k \mid Y_{it-1} = \mathcal{K}) = \frac{1}{1 + \sum_{l=1}^{\mathcal{K}-1} \exp(\Delta_{it,k} + \phi^{kl})}$$

② $$pr(Y_{it}=\mathcal{K} \mid Y_{it-1}=g) = 1 - \sum_{k=1}^{\mathcal{K}-1} pr(Y_{it}=k \mid Y_{it-1}=g) = 1 - \sum_{k=1}^{\mathcal{K}-1} h_{it,kg}$$

③ $\phi^{k\mathcal{K}} = 0$ since state $\mathcal{K}$ is the reference state in the transition

$$
\begin{array}{c|c|cccc}
 & & & & g & \\
 & & 1 & \cdots & \mathcal{K}-1 & \mathcal{K} \\
\hline
 & 1 & \begin{array}{c} pr(Y_{it}=1 \mid Y_{it-1}=1) \\ h_{it11}(\Delta_{it1}, \theta^{11}) \end{array} & \cdots & \begin{array}{c} pr(Y_{it}=1 \mid Y_{it-1}=\mathcal{K}-1) \\ h_{it,1\mathcal{K}-1}(\Delta_{it1}, \theta^{1,\mathcal{K}-1}) \end{array} & \begin{array}{c} pr(Y_{it}=1 \mid Y_{it-1}=\mathcal{K}) \\ h_{it1\mathcal{K}}(\Delta_{it1}, \theta^{1\mathcal{K}}=0) \end{array} \\
 & & \vdots & & \vdots & \vdots \\
k & \mathcal{K}-1 & \begin{array}{c} pr(Y_{it}=\mathcal{K}-1 \mid Y_{it-1}=1) \\ h_{it\,\mathcal{K}-1,1}(\Delta_{it\mathcal{K}-1}, \theta^{\mathcal{K}-1,1}) \end{array} & \cdots & \begin{array}{c} pr(Y_{it}=\mathcal{K}-1 \mid Y_{it-1}=\mathcal{K}-1) \\ h_{it\,\mathcal{K}-1,\mathcal{K}-1}(\Delta_{it\mathcal{K}-1}, \theta^{\mathcal{K}-1,\mathcal{K}-1}) \end{array} & \begin{array}{c} pr(Y_{it}=\mathcal{K}-1 \mid Y_{it-1}=\mathcal{K}) \\ h_{it\mathcal{K}-1\,\mathcal{K}}(\Delta_{it\mathcal{K}-1}, \\ \theta^{\mathcal{K}-1,\mathcal{K}}=0) \end{array} \\
 & \mathcal{K} & 1-\sum_{l=1}^{\mathcal{K}-1} h_{it,l1} & \cdots & 1-\sum_{l=1}^{\mathcal{K}-1} h_{itl,\mathcal{K}-1} & 1-\sum_{l=1}^{\mathcal{K}-1} h_{it,l\mathcal{K}} \\
\hline
\end{array}
$$

$$\theta^{kg} = \log\left(\frac{\text{RR of } Y_{it}=k \text{ (vs } \mathcal{K}) \text{ for } Y_{it-1}=g)}{\text{RR of } Y_{it}=k \text{ (vs } \mathcal{K}) \text{ for } Y_{it-1}=\mathcal{K})}\right)$$

$$\log\left(\frac{pr(Y_{it}=k \mid Y_{it-1}=g)}{pr(Y_{it}=\mathcal{K} \mid Y_{it-1}=g)}\right) = \Delta_{itk} + \theta^{kg} \quad b/c \quad \log\left(\frac{pr(Y_{it}=k \mid Y_{it-1})}{pr(Y_{it}=\mathcal{K} \mid Y_{it-1})}\right) = \Delta_{itk} + \sum_{l=1}^{\mathcal{K}-1} \theta^{kl} Y_{it-1,l}$$

$$\log\left(\frac{pr(Y_{it}=k \mid Y_{it-1}=\mathcal{K})}{pr(Y_{it}=\mathcal{K} \mid Y_{it-1}=\mathcal{K})}\right) = \Delta_{itk}$$

$$L(\theta) = \prod_{i=1}^{N} L_i(\theta)$$

$$\ell(\theta) = \sum_{i=1}^{N} \ell_i(\theta) = \sum_{i=1}^{N} \log L_i(\theta)$$

$$\ell_i(\theta) = \ell_i^{(1)}(\theta) + \ell_i^{(2)}(\theta)$$

$$= \sum_{k=1}^{\mathcal{K}} y_{i1,k} \log \mu_{i1,k}^{m} + \sum_{t=2}^{T_i} \sum_{k=1}^{\mathcal{K}} y_{it,k} \log \mu_{it,k}^{c}$$

$$\ell_i^{(1)}(\theta) = \sum_{k=1}^{\mathcal{K}-1} \left[Z_{it,k} \phi_{it,k} - Z_{it,k+1}\, g(\phi_{it,k}) \right]$$

<u>From M1980 + LD2007.</u>
Not sure why they did this, but it is the same log-likelihood as above.

where $\phi_{it,k} = \log\left(\dfrac{\pi_{it,k}^{m}}{\pi_{it,k+1}^{m} - \pi_{it,k}^{m}} \right)$

$$g(\phi_{it,k}) = \log(1 + \exp(\phi_{it,k})) = \log\left(\frac{\pi_{it,k+1}^{m}}{\pi_{it,k+1}^{m} - \pi_{it,k}^{m}} \right)$$

$$\ell_i^{(2)}(\theta) = \sum_{j=2}^{T_i} \left[\sum_{k=1}^{\mathcal{K}-1} y_{it,k} \left(\Delta_{it,k} + \sum_{\ell=1}^{\mathcal{K}-1} \gamma^{k\ell} y_{it-1,\ell} \right) - \log\left(1 + \sum_{k=1}^{\mathcal{K}-1} \exp\left(\Delta_{it,k} + \sum_{\ell=1}^{\mathcal{K}-1} \gamma^{k\ell} y_{it-1,\ell} \right) \right) \right]$$

$$\frac{\partial l_i^{(1)}(\theta)}{\partial \theta_1} = \sum_{k=1}^{K-1}\left(Z_{1,k}\frac{\partial \phi_{1,k}}{\partial \theta_1} - Z_{1,k+1}\frac{\partial g(\phi_{1k})}{\partial \theta_1}\right)$$

$$\frac{\partial \phi_{1k}}{\partial \theta_1} = \frac{1}{\pi_{1k}^m}\frac{\partial \pi_{1k}^m}{\partial \theta_1} - \frac{\frac{\partial \pi_{1k+1}^m}{\partial \theta_1} - \frac{\partial \pi_{1k}^m}{\partial \theta_1}}{\pi_{1k+1}^m - \pi_{1k}^m} = \frac{\pi_{1k}^m(1-\pi_{1k}^m)\frac{\partial \eta_{1k}}{\partial \theta}}{\pi_{1k}^m} - \frac{\pi_{1k+1}^m(1-\pi_{1k+1}^m)\frac{\partial \eta_{1k+1}}{\partial \theta_1} - \pi_{k}^m(1-\pi_{k}^m)\frac{\partial \eta_{1k}}{\partial \theta_1}}{\pi_{1k+1}^m - \pi_{1k}^m}$$

$$= \frac{\pi_{1k+1}^m(1-\pi_{1k}^m)\frac{\partial \eta_{1k}}{\partial \theta_1} - \pi_{1k+1}^m(1-\pi_{1k+1}^m)\frac{\partial \eta_{1k+1}}{\partial \theta_1}}{\pi_{k+1} - \pi_k} = \frac{\pi_{1k+1}^m}{\pi_{1k+1}^m - \pi_k^m}\left((1-\pi_{1k}^m)\frac{\partial \eta_{1k}}{\partial \theta_1} - (1-\pi_{1k+1}^m)\frac{\partial \eta_{1k+1}}{\partial \theta_1}\right)$$

$$= \exp(g(\phi_{1k}))\left[(1-\pi_{1k}^m)\frac{\partial \eta_{1k}}{\partial \theta_1} - (1-\pi_{1k+1}^m)\frac{\partial \eta_{1k+1}}{\partial \theta_1}\right]$$

$$\frac{\partial g(\phi_{1k})}{\partial \theta_1} = \frac{1}{\pi_{k+1}^m}\frac{\partial \pi_{k+1}^m}{\partial \theta_1} - \frac{\frac{\partial \pi_{k+1}^m}{\partial \theta_1} - \frac{\partial \pi_k^m}{\partial \theta_1}}{\pi_{k+1}^m - \pi_k^m} = \frac{\pi_{k+1}(1-\pi_{k+1})\frac{\partial \eta_{k+1}}{\partial \theta_1}}{\pi_{k+1}} - \frac{\pi_{k+1}(1-\pi_{k+1})\frac{\partial \eta_{k+1}}{\partial \theta_1} - \pi_k^m(1-\pi_k)\frac{\partial \eta_k}{\partial \theta_1}}{\pi_{k+1} - \pi_k}$$

$$= \frac{(\pi_{k+1} - \pi_k - \pi_{k+1})(1-\pi_{k+1})\frac{\partial \eta_{k+1}}{\partial \theta_1} + \pi_k(1-\pi_k)\frac{\partial \eta_k}{\partial \theta_1}}{\pi_{k+1} - \pi_k} = \frac{\pi_k}{\pi_{k+1} - \pi_k}\left((1-\pi_k)\frac{\partial \eta_k}{\partial \theta} - (1-\pi_{k+1})\frac{\partial \eta_{k+1}}{\partial \theta_1}\right)$$

$$= \exp(\phi_{1k})\left((1-\pi_k)\frac{\partial \eta_k}{\partial \theta_1} - (1-\pi_{k+1})\frac{\partial \eta_{k+1}}{\partial \theta_1}\right) = \frac{\exp(\phi_k)}{\exp(g(\phi_k))}\frac{\partial \phi_k}{\partial \theta_1} = \frac{\pi_k}{\pi_{k+1}}\frac{\partial \phi_k}{\partial \theta_1}$$

$$\frac{\partial l_i^{(1)}(\theta)}{\partial \theta_1} = \sum_{k=1}^{K-1}\left(Z_{1k} - Z_{1k+1}\underbrace{\frac{\pi_{1k}^m}{\pi_{1k+1}^m}}_{\downarrow\ \frac{\exp(\phi_{1k})}{1+\exp(\phi_{1k})}}\right)\frac{\partial \phi_{1k}}{\partial \theta_1} \qquad \left(\frac{\partial \phi_{1k}}{\partial \theta_1} = \frac{\partial \phi_{1k}}{\partial \pi_{1k}}\frac{\partial \pi_{1k}}{\partial \eta_{1k}}\frac{\partial \eta_{1k}}{\partial \theta_1} + \frac{\partial \phi_{1k}}{\partial \pi_{1k+1}}\frac{\partial \pi_{1k+1}}{\partial \eta_{1k+1}}\frac{\partial \eta_{1k+1}}{\partial \theta_1}\right)$$

Checks out W/ LD2007

$$\ell_i^{(2)}(\theta) = \sum_{t=2}^{T_i}\left[\sum_{k=1}^{K-1} y_{t,k}\left(\Delta_{t,k} + \sum_{\ell=1}^{K-1} \phi^{k\ell}\, y_{t-1,\ell}\right) - \log\left(1+\sum_{k=1}^{K-1}\exp\left(\Delta_{t,k}+\sum_{\ell=1}^{K-1}\phi^{k\ell}\, y_{t-1,\ell}\right)\right)\right]$$

$$\frac{d\ell_i^{(2)}(\theta)}{d\theta_1} = \sum_{t=2}^{T_i}\left[\sum_{k=1}^{K-1} y_{t,k}\left(\frac{d\Delta_{t,k}}{d\theta_1} + \sum_{\ell=1}^{K-1}\frac{d\phi^{k\ell}}{d\theta_1}\, y_{t-1,\ell}\right) - \frac{\sum_{k=1}^{K-1}\left\{\exp\left(\Delta_{t,k}+\sum_{\ell=1}^{K-1}\phi^{k\ell}\, y_{t-1,\ell}\right)\left(\frac{d\Delta_{t,k}}{d\theta_1}+\sum_{\ell=1}^{K-1}\frac{d\phi^{k\ell}}{d\theta_1}\, y_{t-1,\ell}\right)\right\}}{1+\sum_{k=1}^{K-1}\exp\left(\Delta_{t,k}+\sum_{\ell=1}^{K-1}\phi^{k\ell}\, y_{t-1,\ell}\right)}\right]$$

$$= \sum_{t=2}^{T_i}\left[\sum_{k=1}^{K-1}\left(y_{t,k}-\mu_{t,k}^{c}\right)\left(\frac{d\Delta_{t,k}}{d\theta_1}+\sum_{\ell=1}^{K-1}\frac{d\phi^{k\ell}}{d\theta_1}\, y_{t-1,\ell}\right)\right]$$

$$\frac{d\ell_i^{(2)}(\theta)}{d\alpha_j} = \sum_{t=2}^{T_i}\sum_{k=1}^{K-1}\left(y_{t,k}-\mu_{t,k}^{c}\right)\frac{d\Delta_{t,k}}{d\alpha_j}$$

$$\frac{d\ell_i^{(2)}(\theta)}{d\beta_j} = \sum_{t=2}^{T_i}\sum_{k=1}^{K-1}\left(y_{t,k}-\mu_{t,k}^{c}\right)\frac{d\Delta_{t,k}}{d\beta_j}$$

b/c $\frac{d\phi^{k\ell}}{d\phi^{db}} = \begin{cases} 1 & \text{if } k\ell = db \\ 0 & \text{else}\end{cases}$

$$\frac{d\ell_i^{(2)}(\theta)}{d\phi^{db}} = \sum_{t=2}^{T_i}\left[\sum_{k=1}^{K-1}\left(y_{t,k}-\mu_{t,k}^{c}\right)\left[\frac{d\Delta_{t,k}}{d\phi^{db}}\right] + \left(y_{t,d}-\mu_{t,d}^{c}\right) y_{t-1,b}\right]$$

N.

Calculate $\frac{d\Delta_{tk}}{d\theta}$:

$$\mu^m_{tk} = \sum_{g=1}^{\mathcal{K}} h_{tkg}\, \mu^m_{t-1,g}$$

$$\frac{d\mu^m_{tk}}{d\theta} = \sum_{g=1}^{\mathcal{K}} \left[h_{tkg} \frac{d\mu^m_{t-1,g}}{d\theta} + \frac{dh_{tkg}}{d\theta} \mu^m_{t-1,g} \right]$$

$$\sum_{g=1}^{\mathcal{K}} \frac{dh_{tkg}}{d\theta} \mu^m_{t-1,g} = \frac{d\mu^m_{tk}}{d\theta} - \sum_{g=1}^{\mathcal{K}} h_{tkg} \frac{d\mu^m_{t-1,g}}{d\theta}$$

Note: $h_{tkg} = h(\Delta_{t1g}, \Delta_{t2g}, \ldots \Delta_{t\mathcal{K}-1,g}, \phi^{1g}, \phi^{2g}, \ldots \phi^{\mathcal{K}-1,g})$

$$\frac{\partial h_{tkg}}{\partial\theta} = \sum_{j=1}^{\mathcal{K}-1} \frac{\partial h_{tkg}}{\partial\Delta_{tj}} \frac{\partial\Delta_{tj}}{\partial\theta} + \sum_{j=1}^{\mathcal{K}-1} \frac{\partial h_{tkg}}{\partial\phi^{jg}} \frac{\partial\phi^{jg}}{\partial\theta}$$

$$\sum_{g=1}^{\mathcal{K}} \Bigg[\underbrace{\sum_{j=1}^{\mathcal{K}-1} \frac{\partial h_{tkg}}{\partial\Delta_{tj}} \frac{\partial\Delta_{tj}}{\partial\theta} \mu^m_{t-1,g}}_{ⓐ} + \underbrace{\sum_{j=1}^{\mathcal{K}-1} \frac{\partial h_{tkg}}{\partial\phi^{jg}} \frac{\partial\phi^{jg}}{\partial\theta} \mu^m_{t-1,g}}_{ⓑ} \Bigg] = \underbrace{\frac{\partial\mu^m_{tk}}{\partial\theta}}_{ⓒ} - \underbrace{\sum_{g=1}^{\mathcal{K}} h_{tkg} \frac{\partial\mu^m_{t-1,g}}{\partial\theta}}_{ⓓ}$$

For $\theta \in \{\alpha_1, \ldots \alpha_{\mathcal{K}-1}, \beta_1, \ldots \beta_p\}$, ⓑ $= 0$

$$\frac{\partial\Delta_{t1}}{\partial\theta} \sum_{g=1}^{\mathcal{K}} \frac{\partial h_{tkg}}{\partial\Delta_{t1}} \mu^m_{t-1,g} + \ldots + \frac{\partial\Delta_{t\mathcal{K}-1}}{\partial\theta} \sum_{g=1}^{\mathcal{K}} \frac{\partial h_{tkg}}{\partial\Delta_{t\mathcal{K}-1}} \mu^m_{t-1,g} = \frac{\partial\mu^m_{tk}}{\partial\theta} - \sum_{g=1}^{\mathcal{K}} h_{tkg} \frac{\partial\mu^m_{t-1,g}}{\partial\theta}$$

↳ for each element of θ, there are $\mathcal{K}-1$ unknown values of $\left(\frac{\partial\Delta_{t1}}{\partial\theta_1}, \ldots \frac{\partial\Delta_{t\mathcal{K}-1}}{\partial\theta_1}\right)$

For $\theta \in \{\phi^{11}, \phi^{12}, \ldots \phi^{1\mathcal{K}-1}, \phi^{21}, \phi^{22}, \ldots \ldots \phi^{\mathcal{K}-1,\mathcal{K}-2}, \phi^{\mathcal{K}-1,\mathcal{K}-1}\}$, ⓒ + ⓓ are 0

$$\frac{\partial\Delta_{t1}}{\partial\theta} \sum_{g=1}^{\mathcal{K}} \frac{\partial h_{tkg}}{\partial\Delta_{t1}} \mu_{t-1,g} + \ldots + \frac{\partial\Delta_{t\mathcal{K}-1}}{\partial\theta} \sum_{g=1}^{\mathcal{K}} \frac{\partial h_{tkg}}{\partial\Delta_{t\mathcal{K}-1}} \mu^m_{t-1,g} = -\sum_{g=1}^{\mathcal{K}} \sum_{j=1}^{\mathcal{K}-1} \frac{\partial h_{tkg}}{\partial\phi^{jg}} \frac{\partial\phi^{jg}}{\partial\theta} \mu^m_{t-1,g}$$

$$= -\sum_{g=1}^{\mathcal{K}} \frac{\partial h_{tkg}}{\partial\phi^{jg}} \frac{\partial\phi^{jg}}{\partial\theta} \mu^m_{t-1,g}$$

& there are $\mathcal{K}-1$ equations, So we solve the system of equations. With npar parameters, we have npar$\cdot(\mathcal{K}-1)$ values of $\frac{\partial\Delta_t}{\partial\vec{\theta}}$. Yikes! for all i & all t.

$$h_{tkg} = \frac{\exp(\Delta_{tk} + b^{kg})}{1 + \sum_{\ell=1}^{K-1} \exp(\Delta_{t\ell} + b^{\ell g})} = f(\Delta_{t1}, \Delta_{t2}, \ldots \Delta_{tK-1}, b^{1g}, b^{2g}, \ldots b^{K-1,g})$$

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$$\frac{\partial h_{tkg}}{\partial \theta} = \sum_{j=1}^{K-1} \frac{\partial h_{tkg}}{\partial \Delta_{tj}} \cdot \frac{\partial \Delta_{tj}}{\partial \theta} + \sum_{j=1}^{K-1} \frac{\partial h_{tkg}}{\partial b^{jg}} \frac{\partial b^{jg}}{\partial \theta}$$

$$\frac{\partial h_{tkg}}{\partial \Delta_{tj}} = \begin{cases} h_{tkg}(1 - h_{tkg}) & \text{if } j = k \\ -h_{tkg} h_{tjg} & \text{if } j \neq k \end{cases}$$

$$\frac{\partial h_{tkg}}{\partial b^{jg}} = \begin{cases} h_{tkg}(1 - h_{tkg}) & \text{if } j = k \\ -h_{tkg} h_{tjg} & \text{if } j \neq k \end{cases}$$

$$\frac{\partial b^{jg}}{\partial \theta} = \begin{cases} 1 & \text{if } \theta = b^{jg} \\ 0 & \text{else} \end{cases}$$

For all parameters, we solve $\underset{K-1 \times K-1}{A} \, \underset{K-1 \times 1}{X} = \underset{K-1 \times 1}{B}$ for X where

$h \sim (K-1) \times K$ matrix

$\mu \sim K$ - vector

We want a $(K-1) \times (K-1)$ matrix w/ elements:

ℓ^{th} diagonal: $\sum_{g=1}^{K} h_{\ell g}(1-h_{\ell g})\mu_g = \text{sum}(\underbrace{h[\ell,] \cdot (1-h[\ell,]) \cdot \mu}_{\text{elementwise multiplication}})$

ℓ, m element: $\sum_{g=1}^{K} -h_{\ell g} h_{mg} \mu_g = \text{sum}(\underbrace{h[\ell,] \cdot h[m,] \cdot \mu}_{\text{elementwise multiplication}})$

lagged value of μ_t

$$\begin{bmatrix} h_{1\cdot} & h_{1\cdot} & h_{1\cdot} \\ h_{2\cdot} & \boxed{h_{2\cdot}} & h_{2\cdot} \\ h_{3\cdot} & h_{3\cdot} & h_{3\cdot} \end{bmatrix} \cdot \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} h_{1\cdot} & h_{2\cdot} & h_{3\cdot} \\ h_{1\cdot} & \boxed{h_{2\cdot}} & h_{3\cdot} \\ h_{1\cdot} & h_{2\cdot} & h_{3\cdot} \end{pmatrix} \right] \cdot \begin{bmatrix} \mu_{t-1} & \mu_{t-1} & \mu_{t-1} \\ \mu_{t-1} & \boxed{\mu_{t-1}} & \mu_{t-1} \\ \mu_{t-1} & \mu_{t-1} & \mu_{t-1} \end{bmatrix}$$

Let h be a 3×4 matrix. To create the 3×3 matrix we want, we can do the above. Each element in every matrix is a 4-vector.

To get the 2,2 element, we do: $\text{sum}(h_{2\cdot}[1-h_{2\cdot}]\mu)$

" " " 2,3 " " " $\text{sum}(h_{3\cdot}[0-h_{2\cdot}]\mu)$

To calculate A on page $\boxed{7a}$

ϑ_{11}	ϑ_{12}	ϑ_{13}	ϑ_{21}	ϑ_{22}	ϑ_{23}	ϑ_{31}	ϑ_{32}	ϑ_{33}
$h_{11}\mu_1$	$h_{12}\mu_2$	$h_{13}\mu_3$	$h_{21}\mu_1$	$h_{22}\mu_2$	$h_{23}\mu_3$	$h_{31}\mu_1$	$h_{32}\mu_2$	$h_{32}\mu_3$
$h_{11}\mu_1$	$h_{12}\mu_2$	$h_{13}\mu_3$	$h_{21}\mu_1$	$h_{22}\mu_2$	$h_{23}\mu_3$	$h_{31}\mu_1$	$h_{32}\mu_2$	$h_{33}\mu_3$
$h_{11}\mu_1$	$h_{12}\mu_2$	$h_{13}\mu_3$	$h_{21}\mu_1$	$h_{22}\mu_2$	$h_{23}\mu_3$	$h_{31}\mu_1$	$h_{32}\mu_2$	$h_{33}\mu_3$
$(1-h_{11})$	$(1-h_{12})$	$(1-h_{13})$	$-h_{11}$	$-h_{12}$	$-h_{13}$	$-h_{11}$	$-h_{12}$	$-h_{13}$
$-h_{21}$	$-h_{22}$	$-h_{23}$	$(1-h_{21})$	$(1-h_{22})$	$(1-h_{23})$	$-h_{21}$	$-h_{22}$	$-h_{23}$
$-h_{31}$	$-h_{32}$	$-h_{33}$	$-h_{31}$	$-h_{32}$	$-h_{33}$	$(1-h_{31})$	$(1-h_{32})$	$(1-h_{33})$

$0 \cdot kl +$ $\quad 0 + 1:kl$

$1 \cdot kl +$ $\quad kl + 1:$

$2 \cdot kl +$

$kl \cdot (kl-1)\ 1:3$

rep(1, kl), rep(0, kl^2)

111 000 000 000

1 1 1

11 00 00 00, 11 00 00 00 11

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7c

	α_1	α_2	α_3	β_1	β_2
$\frac{\partial \mu_1}{\partial \theta} \rightarrow$	$\frac{\partial \pi_1}{\partial \alpha_1} - 0$	$\frac{\partial \pi_1}{\partial \alpha_2} - 0$	$\frac{\partial \pi_1}{\partial \alpha_3} - 0$	$\frac{\partial \pi_1}{\partial \beta_1} - 0$	$\frac{\partial \pi_1}{\partial \beta_2} - 0$
$\frac{\partial \mu_2}{\partial \theta} \rightarrow$	$\frac{\partial \pi_2}{\partial \alpha_1} - \frac{\partial \pi_1}{\partial \alpha_1}$	$\frac{\partial \pi_2}{\partial \alpha_2} - \frac{\partial \pi_1}{\partial \alpha_2}$	$\frac{\partial \pi_2}{\partial \alpha_3} - \frac{\partial \pi_1}{\partial \alpha_3}$	$\frac{\partial \pi_2}{\partial \beta_1} - \frac{\partial \pi_1}{\partial \beta_1}$	$\frac{\partial \pi_2}{\partial \beta_2} - \frac{\partial \pi_1}{\partial \beta_2}$
$\frac{\partial \mu_3}{\partial \theta} \rightarrow$	$\frac{\partial \pi_3}{\partial \alpha_1} - \frac{\partial \pi_2}{\partial \alpha_1}$	$\frac{\partial \pi_3}{\partial \alpha_2} - \frac{\partial \pi_2}{\partial \alpha_2}$	$\frac{\partial \pi_3}{\partial \alpha_3} - \frac{\partial \pi_2}{\partial \alpha_7}$	$\frac{\partial \pi_3}{\partial \beta_1} - \frac{\partial \pi_2}{\partial \beta_1}$	$\frac{\partial \pi_3}{\partial \beta_2} - \frac{\partial \pi_2}{\partial \beta_2}$
$\frac{\partial \mu_4}{\partial \theta} \rightarrow$	$0 - \frac{\partial \pi_3}{\partial \alpha_1}$	$0 - \frac{\partial \pi_3}{\partial \alpha_2}$	$0 - \frac{\partial \pi_3}{\partial \alpha_3}$	$0 - \frac{\partial \pi_3}{\partial \beta_1}$	$0 - \frac{\partial \pi_3}{\partial \beta_2}$

$\sum h_{t+1g}(1-h_{t+1g})\mu_{t-1,g}$

$$\frac{\partial \mu_t}{\partial \theta} + \frac{\partial \mu_{t-1}}{\partial \theta}$$

$$\begin{array}{c} \frac{\partial \pi_1}{\partial \alpha_1} \\ \frac{\partial \pi_1}{\partial \alpha_2} \\ \frac{\partial \pi_2}{\partial \alpha_3} \\ 0 \end{array} \begin{pmatrix} 1 & 0 & 0 & X_1 & X_2 \\ 0 & 1 & 0 & X_1 & X_2 \\ 0 & 0 & 1 & X_1 & X_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{c} 0 \\ \frac{\partial \pi_1}{\partial \alpha_1} \\ \frac{\partial \pi_2}{\partial \alpha_2} \\ \frac{\partial \pi_3}{\partial \alpha_3} \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & X_1 & X_2 \\ 0 & 1 & 0 & X_1 & X_2 \\ 0 & 0 & 1 & X_1 & X_2 \end{pmatrix}$$

$$= \begin{matrix} \pi_1(1-\pi_1) & 0 & 0 & 0 & 0 \\ 0 & \pi_2(1-\pi_2)-\pi_1(1-\pi_1) & & 0 & 0 \\ 0 & 0 & \pi_3(1-\pi_3)-\pi_2(1-\pi_2) & & \end{matrix}$$