CH. nth-order linear Differential Equation

§. General Form

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)\,y = Q(x)$$

General Form of nth-order linear Differential Eq.

If $Q(x) \neq 0$, nth-order non-homogeneous linear Differential Eq.

If $Q(x) = 0$, nth-order homogeneous linear Differential Eq.

note)

If $Q(x) = 0$ and $P_i(x) = a_i$ $(i = 1, 2, \cdots, n)$, one can get general solution by applying CH. 2.

§ Series Solution for nth-order homogeneous linear Differential Eq.

Series

$$f(x) = \sum_{n=0}^{\infty} C_n (x - x_0)^n = C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + \cdots$$

x_0: analytic point of $f(x)$

Theorem 1

Consider nth-order homogeneous linear Differential Eq

$$y^{(n)} + P_1(x)\,y^{(n-1)} + \cdots + P_{n-1}(x)\,y' + P_n(x)\,y = 0 \qquad -(1)$$

If $P_1(x), P_2(x), \cdots, P_n(x)$ have same analytic point $x = x_0$, Then the general solution of Eq. (1) is of the form

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n$$

x_0: ordinary point of Eq. (1)

(Ex) $\dfrac{d^2y}{dx^2} - xy = 0 \quad -①$

$P_1(x) = 0 = \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 = a_1 = \cdots = 0)$

$P_2(x) = -x = \sum_{n=0}^{\infty} b_n x^n \qquad (b_0 = b_2 = \cdots = 0,\ b_1 = -1)$

$x=0$: analytic point.

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \\ y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = 2c_2 + 6c_3 x + 12 c_4 x^2 + \cdots \end{aligned}\right\} -②$$

② → ①

$$0 = y'' - xy$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{x^0,\ x,\ x^2,\ x^3,\ \cdots} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{x,\ x^2,\ x^3,\ \cdots}$$

$$= 2c_2 + \underbrace{\sum_{n=3}^{\infty} n(n-1) c_n x^{n-2}}_{\sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1}} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= 2c_2 + \sum_{n=0}^{\infty} \left[(n+3)(n+2) c_{n+3} - c_n \right] x^{n+1}$$

$$\left.\begin{aligned} &\Rightarrow c_2 = 0 \\ &c_{n+3} = \frac{1}{(n+3)(n+2)} c_n \quad : \text{recurrence formula} \end{aligned}\right\} -③$$

$$C_3 = \frac{C_0}{3\times 2}$$

$$C_4 = \frac{C_1}{4\times 3}$$

$$C_5 = \frac{C_2}{5\times 4} = 0$$

$$C_6 = \frac{1}{6\times 5} C_3 = \frac{1}{6\times 5}\frac{1}{3\times 2} C_0 = \frac{4}{6!} C_0$$

$$C_7 = \frac{1}{7\times 6} C_4 = \frac{1}{7\times 6}\frac{1}{4\times 3} C_1 = \frac{5\times 2}{7!} C_1$$

$$C_8 = \frac{1}{8\times 7} C_5 = 0$$

$$C_9 = \frac{4\times 7}{9!} C_0$$

$$C_{10} = \frac{2\times 5\times 7}{10!} C_1$$

$$\vdots$$

$$y = \sum_m C_m x^m$$

$$= C_0 \left[1 + \frac{1}{3!} x^3 + \frac{1\times 4}{6!} x^6 + \frac{1\times 4\times 7}{9!} x^9 + \cdots \right]$$

$$+ C_1 \left[x + \frac{2}{4!} x^4 + \frac{2\times 5}{7!} x^7 + \frac{2\times 5\times 7}{10!} x^{10} + \cdots \right] \divideontimes$$

Theorem 2

Consider n^{th}-order homogeneous linear Differential Eq

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots + P_{n-1}(x)\, y' + P_n(x)\, y = 0 \qquad - ①$$

If $x = x_0$ is analytic point for all $(x-x_0)P_1(x)$, $(x-x_0)^2 P_2(x)$, $\cdots$, $(x-x_0)^n P_n(x)$,

$x = x_0$ is called " regular singular point " of Eq. ①.

If $x = x_0$ is regular singular point, then the solution is

$$y = (x-x_0)^{\lambda} \sum_{m=0}^{\infty} c_m (x-x_0)^m$$

λ: constant

$c_0 \neq 0$

Ex)

$$4x y'' + 2y' + y = 0 \qquad -①$$

$$P_1(x) = \frac{2}{4x}, \quad P_2(x) = \frac{1}{4x}$$

$$xP_1(x) = \frac{1}{2} = \sum_{n=0}^{\infty} a_n x^n \qquad (a_0 = \frac{1}{2},\ a_1 = a_2 = \cdots = 0)$$

$$x^2 P_2(x) = \frac{x}{4} = \sum_{n=0}^{\infty} b_n x^n \qquad (b_1 = \frac{1}{4},\ b_0 = b_2 = \cdots = 0)$$

$x = 0$: regular singular point of Eq. ①

$$\left.\begin{aligned} y &= x^\lambda \sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} C_n x^{n+\lambda} \\ y' &= \sum_{n=0}^{\infty} (n+\lambda) C_n x^{n+\lambda-1} \\ y'' &= \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) C_n x^{n+\lambda-2} \end{aligned}\right\} \quad -②$$

② → ①

$$0 = 4x y'' + 2y' + y$$

$$= 4x \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) C_n x^{n+\lambda-2} + 2\sum_{n=0}^{\infty} (n+\lambda) C_n x^{n+\lambda-1} + \sum_{n=0}^{\infty} C_n x^{n+\lambda}$$

$$= \underbrace{4\sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) C_n x^{n+\lambda-1}}_{x^{\lambda-1},\ x^\lambda,\ \cdots} + \underbrace{2\sum_{n=0}^{\infty} (n+\lambda) C_n x^{n+\lambda-1}}_{x^{\lambda-1},\ x^\lambda,\ \cdots} + \underbrace{\sum_{n=0}^{\infty} C_n x^{n+\lambda}}_{x^\lambda,\ x^{\lambda+1},\ \cdots}$$

$$= 4\lambda(\lambda-1) C_0 x^{\lambda-1} + 4\sum_{n=1}^{\infty} (n+\lambda)(n+\lambda-1) C_n x^{n+\lambda-1}$$

$$+ 2\lambda C_0 x^{\lambda-1} + 2\sum_{n=1}^{\infty} (n+\lambda) C_n x^{n+\lambda-1} + \underbrace{\sum_{n=0}^{\infty} C_n x^{n+\lambda}}_{\sum_{n=1}^{\infty} C_{n-1} x^{n+\lambda-1}}$$

$$= 2C_0 \lambda (2\lambda-1) x^{\lambda-1}$$

$$+ \sum_{n=1}^{\infty} \left[2(n+\lambda)(2n+2\lambda-1) C_n + C_{n-1} \right] x^{n+\lambda-1}$$

$$\lambda(2\lambda-1) = 0$$: indicial Equation

— ②

$$C_n = -\frac{1}{2(n+\lambda)(2n+2\lambda-1)} C_{n-1}$$

: recurrence formula

$\lambda = 0$ or $\frac{1}{2}$

[1] $\lambda = 0$

$$C_n = -\frac{1}{2n(2n-1)} C_{n-1}$$

$$C_1 = -\frac{1}{2} C_0$$

$$C_2 = -\frac{C_1}{4\times 3} = \frac{1}{4!} C_0$$

$$C_3 = -\frac{C_2}{6\times 5} = -\frac{1}{6!} C_0$$

$\vdots$

$$y_1 = x^\lambda \sum_{n=0}^{\infty} C_n x^n$$

$$= A\left[1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \cdots \right]$$

[2] $n = \frac{1}{2}$

$$C_m = -\frac{1}{2m(2m+1)} C_{m-1}$$

$$C_1 = -\frac{1}{3\times 2} C_0$$

$$C_2 = -\frac{C_1}{5\times 4} = \frac{C_0}{5!}$$

$$C_3 = -\frac{C_2}{7\times 6} = -\frac{C_0}{7!}$$

$$\vdots$$

$$y_2 = x^n \sum_{m=0}^{\infty} C_m x^m$$

$$= B\sqrt{x}\left[1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \cdots\right]$$

* solution

$$\boxed{\begin{aligned} y = & A\left[1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \cdots\right] \\ & + B\sqrt{x}\left[1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \cdots\right] \end{aligned}}$$

note)

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$y = A\cos\sqrt{x} + B\sin\sqrt{x}$$ *

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§ Legendre Polynomial

$$\left.\begin{aligned}&(1-x^2)y'' - 2xy' + \lambda y = 0 \\ &-1 \le x \le 1 \\ &\lambda : \text{real}\end{aligned}\right\} \quad -①$$

Legendre Differential Eq.

$$P_1(x) = -\frac{2x}{1-x^2} = -2x(1+x^2+x^4+\cdots) = \sum_{n=0}^{\infty} a_n x^n$$

$$P_2(x) = \frac{\lambda}{1-x^2} = \lambda(1+x^2+x^4+\cdots) = \sum_{n=0}^{\infty} b_n x^n$$

$$\left(\because \frac{1}{1-x} = 1+x+x^2+\cdots \quad \text{when } -1 \le x \le 1\right)$$

$\Rightarrow x=0$ is a ordinary point of Eq. ①.

$$\left.\begin{aligned}y &= \sum_{n=0}^{\infty} C_n x^n \\ y' &= \sum_{n=1}^{\infty} n C_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}\end{aligned}\right\} \quad -②$$

② → ①

$$0 = (1-x^2)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - 2x\sum_{n=1}^{\infty} nC_n x^{n-1} + \lambda \sum_{n=0}^{\infty} C_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}}_{x^0,\, x,\, x^2,\, \cdots} - \underbrace{\sum_{n=2}^{\infty} n(n-1)C_n x^n}_{x^2,\, x^3,\, \cdots} - 2\underbrace{\sum_{n=1}^{\infty} nC_n x^n}_{x,\, x^2,\, x^3,\, \cdots} + \lambda \underbrace{\sum_{n=0}^{\infty} C_n x^n}_{x^0,\, x,\, x^2,\, \cdots}$$

$$= 2C_2 + 6C_3 x + \underline{\sum_{n=4}^{\infty} n(n-1)C_n x^{n-2}} \quad \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n$$

$$- \sum_{n=2}^{\infty} n(n-1)C_n x^n$$

$$-2C_1 x \quad - 2\sum_{n=2}^{\infty} nC_n x^n$$

$$+\lambda C_0 + \lambda C_1 x \quad + \lambda \sum_{n=2}^{\infty} C_n x^n$$

$$= (2C_2 + \lambda C_0) + [6C_3 + (\lambda - 2) C_1]\, x$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1) C_{n+2} - \{ n(n+1) - \lambda \} C_n \right] x^n$$

$$\Rightarrow \quad C_2 = -\frac{\lambda}{2} C_0 = -\frac{\lambda}{2!} C_0$$

$$C_3 = \frac{2-\lambda}{6} C_1 = \frac{2-\lambda}{3!} C_1 \qquad - ③$$

$$C_{n+2} = \frac{n(n+1) - \lambda}{(n+2)(n+1)} C_n$$

recurrence formula

$$C_4 = \frac{6-\lambda}{4 \times 3} C_2 = -\frac{\lambda(6-\lambda)}{4!} C_0$$

$$C_5 = \frac{12-\lambda}{5 \times 4} C_3 = \frac{(2-\lambda)(12-\lambda)}{5!} C_1$$

$$C_6 = \frac{20-\lambda}{6 \times 5} C_4 = -\frac{\lambda(6-\lambda)(20-\lambda)}{6!} C_0$$

$$C_7 = \frac{30-\lambda}{7 \times 6} C_5 = \frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!} C_1$$

solution

$$\boxed{\begin{aligned} y = {} & C_0 \left[1 - \frac{\lambda}{2!} x^2 - \frac{\lambda(6-\lambda)}{4!} x^4 - \frac{\lambda(6-\lambda)(20-\lambda)}{6!} x^6 + \cdots \right] \\ & + C_1 \left[x + \frac{2-\lambda}{3!} x^3 + \frac{(2-\lambda)(12-\lambda)}{5!} x^5 + \frac{(2-\lambda)(12-\lambda)(30-\lambda)}{7!} x^7 + \cdots \right] \end{aligned}}$$

Special solutions

[1] $\lambda = 0$ and $C_1 = 0$

$$y_0(x) = A_0$$

[2] $\lambda = 2 = 1\times 2$ and $C_0 = 0$

$$y_1(x) = A_1 x$$

[3] $\lambda = 6 = 2\times 3$ and $C_1 = 0$

$$y_2(x) = A_2 (1 - 3x^2)$$

[4] $\lambda = 12 = 3\times 4$ and $C_0 = 0$

$$y_3(x) = A_3 (x - \frac{5}{3} x^3)$$

$\vdots$

definition of Legendre Polynomial

$$\boxed{P_n(x) = y_n(x) \quad \text{with} \quad P_n(1) = 1}$$

① $P_0(x) = A_0$ $\quad\quad A_0 = 1$

$$P_0(x) = 1$$

② $P_1(x) = A_1 x$, $P_1(1) = A_1 = 1$

$$P_1(x) = x$$

③ $P_2(x) = A_2 (1 - 3x^2)$, $P_2(1) = -2A_2 = 1 \Rightarrow A_2 = -\frac{1}{2}$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

④ $P_3(x) = A_3 (x - \frac{5}{3} x^3)$, $P_3(1) = -\frac{2}{3} A_3 = 1 \Rightarrow A_3 = -\frac{3}{2}$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Legendre Polynomial

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

....

Property of Legendre Polynomial

[1] Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

(Ex) $P_0(x) = (x^2-1)^0 = 1$

$$P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2-1) = x$$

$$P_2(x) = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2} (x^2-1)^2$$

$$= \frac{1}{8} \frac{d}{dx} \left[2(x^2-1) \cdot 2x \right]$$

$$= \frac{1}{2} \frac{d}{dx} (x^3 - x)$$

$$= \frac{1}{2}(3x^2-1)$$

$\vdots$

[2]

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{when } m \neq n \\ \dfrac{2}{2n+1} & \text{when } m = n \end{cases} = \frac{2}{2n+1}\,\delta_{mn}$$

⇒ orthogonal polynomial

[3] Expansion

Any function $f(x)$ defined at $-1 \leq x \leq 1$ can be expanded as a linear combination of the Legendre Polynomials

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx$$

pf)

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

$$\Rightarrow P_m(x)\, f(x) = \sum_{n=0}^{\infty} C_n P_m(x)\, P_n(x) \quad — ①$$

Integrate Eq. ① from $x=-1$ to $x=1$

$$\int_{-1}^{1} P_m(x)\, f(x)\, dx = \sum_{n=0}^{\infty} C_n \underbrace{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx}_{\frac{2}{2n+1}\delta_{mn}} = \sum_{n=0}^{\infty} C_n \frac{2}{2n+1}\,\delta_{mn} = \frac{2}{2m+1} C_m$$

$$\Rightarrow C_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, P_m(x)\, dx$$

$$\Rightarrow C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx \quad ※$$

(Ex) $f(x) = x^5 + 2x^2$

$$x^5 + 2x^2 = \sum_{n=0}^{\infty} C_n P_n(x)$$

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

$$C_0 = \frac{2}{3}, \quad C_1 = \frac{3}{7}, \quad C_2 = \frac{4}{3} \quad C_3 = \frac{4}{9}, \quad C_4 = 0, \quad C_5 = \frac{8}{63}$$

$$C_6 = C_7 = \cdots = 0$$

$$\Rightarrow \quad x^5 + 2x^2 = \frac{2}{3} P_0(x) + \frac{3}{7} P_1(x) + \frac{4}{3} P_2(x) + \frac{4}{9} P_3(x) + \frac{8}{63} P_5(x)$$ ※

(숙제) 다음을 Legendre Polynomial 의 linear combination 으로

나타내시오

① $3x^3 + x$

② $7x^4 - 3x + 1$

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§ Bessel function

[1] Gamma Function

$$\Gamma(p) = \int_0^{\infty} t^{p-1} e^{-t}\, dt$$

① $\Gamma(p) = (p-1)\,\Gamma(p-1)$

(pf) $\Gamma(p) = \int_0^{\infty} \underbrace{t^{p-1}}_{u}\, \underbrace{e^{-t}}_{v'}\, dt$

$$= -t^{p-1} e^{-t} \Big|_{t=0}^{t=\infty} - \int_0^{\infty} (p-1)\, t^{p-2} \left(-e^{-t}\right) dt$$

$$= (p-1) \int_0^{\infty} t^{p-2} e^{-t}\, dt$$

$$= (p-1)\,\Gamma(p-1) \quad \#$$

② $\Gamma(1) = 1$

③ If $p = N$ (integer), $\Gamma(N) = (N-1)!$

④ $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin p\pi}$

(Ex) $\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}) = \dfrac{\pi}{\sin\frac{\pi}{2}} = \pi$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(Ex) $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$

(Ex) $\Gamma(-\frac{3}{2}) = \dfrac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \dfrac{1}{-\frac{3}{2}}\;\dfrac{1}{-\frac{1}{2}}\,\Gamma(\frac{1}{2}) = \frac{4}{3}\sqrt{\pi}$ #.

[2] Bessel Differential Equation

$$\underline{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$ Bessel Differential Equation

$$P_1(x) = \frac{1}{x} \qquad x P_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$P_2(x) = 1 - \frac{\nu^2}{x^2} \qquad x^2 P_2(x) = -\nu^2 + x^2 = \sum_{n=0}^{\infty} b_n x^n$$

$x = 0$: regular singular point of Bessel Differential Eq.

$$\left.\begin{aligned} y &= x^\lambda \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+\lambda} \\ y' &= \sum_{n=0}^{\infty} (n+\lambda) c_n x^{n+\lambda-1} \end{aligned}\right\} \quad -①$$

$$y'' = \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) c_n x^{n+\lambda-2}$$

① → Bessel Differential Eq.

$$0 = x^2 \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) c_n x^{n+\lambda-2} + x \sum_{n=0}^{\infty} (n+\lambda) c_n x^{n+\lambda-1}$$

$$+ x^2 \sum_{n=0}^{\infty} c_n x^{n+\lambda} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+\lambda}$$

$$= \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) c_n x^{n+\lambda} + \sum_{n=0}^{\infty} (n+\lambda) c_n x^{n+\lambda}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+\lambda+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+\lambda}$$

$$= \lambda(\lambda-1) c_0 x^\lambda + (\lambda+1)\lambda c_1 x^{\lambda+1} + \sum_{n=2}^{\infty} (n+\lambda)(n+\lambda-1) c_n x^{n+\lambda}$$

$$+ \lambda c_0 x^\lambda + (\lambda+1) c_1 x^{\lambda+1} \qquad + \sum_{n=2}^{\infty} (n+\lambda) c_n x^{n+\lambda}$$

$$+ \underline{\sum_{n=0}^{\infty} c_n x^{n+\lambda+2}} \quad \sum_{n=2}^{\infty} c_{n-2} x^{n+\lambda}$$

$$- \nu^2 \left[c_0 x^\lambda + c_1 x^{\lambda+1} \right] \qquad - \nu^2 \sum_{n=2}^{\infty} c_n x^{n+\lambda}$$

$$= c_0 x^{\lambda} (\lambda^2 - \nu^2) + c_1 x^{\lambda+1} [(\lambda+1)^2 - \nu^2]$$

$$+ \sum_{n=2}^{\infty} \left[(n+\lambda+\nu)(n+\lambda-\nu) c_n + c_{n-2} \right] x^{n+\lambda}$$

$$\Rightarrow \quad \left.\begin{aligned} &\lambda^2 - \nu^2 = 0 \quad : \text{indicial Equation} \\ &c_1 = 0 \\ &c_n = - \frac{c_{n-2}}{(n+\lambda+\nu)(n+\lambda-\nu)} \quad : \text{recurrence formula} \end{aligned}\right\} \quad - ②$$

(i) $\lambda = \nu$

$$c_n = - \frac{c_{n-2}}{n(2\nu+n)}$$

$$c_2 = - \frac{c_0}{2^2(\nu+1)}, \quad c_3 = 0$$

$$c_4 = \frac{1}{2^5(\nu+1)(\nu+2)} c_0, \quad c_5 = 0$$

$$c_6 = - \frac{1}{2^6(3!)(\nu+1)(\nu+2)(\nu+3)} c_0, \quad c_7 = 0$$

$$\vdots$$

$$c_{2n} = (-1)^n \frac{1}{2^{2n} n! (1+\nu)(2+\nu)\cdots(n+\nu)} c_0, \quad c_{2n+1} = 0$$

$$\Rightarrow y_1(x) = c_0 x^{\nu} \left[1 - \frac{1}{2^2 \cdot 1! (1+\nu)} x^2 + \frac{x^4}{2^4 \cdot 2! (1+\nu)(2+\nu)} - \frac{x^6}{2^6 \cdot 3! (1+\nu)(2+\nu)(3+\nu)} + \cdots \right]$$

$$= c_0 x^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot n! (1+\nu)(2+\nu)\cdots(n+\nu)} x^{2n}$$

definition: First-Kind Bessel function

$$\boxed{J_\nu(x) \equiv y_1(x) \quad \text{with} \quad c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}}$$

$$J_\nu(x) = \frac{1}{2^\nu \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot n! \, (1+\nu)(2+\nu)\cdots(n+\nu)} x^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \, \underbrace{\Gamma(\nu+1)\left[(\nu+1)(\nu+2)\cdots(\nu+n)\right]}_{\Gamma(n+\nu+1)}} x^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \cdot n! \, \Gamma(n+\nu+1)} x^{2n+\nu}$$

$$\boxed{J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} \, n! \, \Gamma(n+\nu+1)} x^{2n+\nu}}$$

Bessel function of the first kind of order ν

Ex)

① $$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}\, n!\, \Gamma(n+1)}\, x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

② $$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\frac{1}{2}}\, n!\, \Gamma(n+\frac{3}{2})}\, x^{2n+\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, (n+\frac{1}{2})(n+\frac{1}{2}-1)\cdots \frac{3}{2}\cdot\frac{1}{2}\,\Gamma(\frac{1}{2})} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, \frac{2n+1}{2}\, \frac{2n-1}{2} \cdots \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\underbrace{\left(\frac{2n}{2}\,\frac{2n-2}{2}\cdots\frac{2}{2}\right)\left(\frac{2n+1}{2}\,\frac{2n-1}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\right)}_{\frac{(2n+1)!}{2^{2n+1}}}\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n+1}}{(2n+1)!} \frac{1}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1} \sqrt{\frac{2}{x}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}$$

$$= \sqrt{\frac{2}{\pi x}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}}\, \sin x$$

* Property of Bessel function

p417

(i) If ν is not integer, the general solution of Bessel Differential Equation is

$$y(x) = A J_\nu(x) + B J_{-\nu}(x)$$

note) If $\nu = n$ is integer, ~~$J_{-n}(x)$ is not defined !!~~ $J_{-n}(x) = (-1)^n J_n(x)$.

(ii) Recursion formula of Bessel function

p417

식 (8.42)

(3), (4), (5)

① $J_\nu'(x) + \frac{\nu}{x} J_\nu(x) = J_{\nu-1}(x)$

② $J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x)$

③ $2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x)$

④ $\frac{2\nu}{x} J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x)$

(pf) 숙제

p418 (예제 8.31)

(Ex) $J_{\frac{3}{2}}(x) = -J_{\frac{1}{2}}'(x) + \frac{1}{2x} J_{\frac{1}{2}}(x)$

$$= -\frac{d}{dx}\left(\sqrt{\frac{2}{\pi x}}\sin x\right) + \frac{1}{2x}\left(\sqrt{\frac{2}{\pi x}}\sin x\right)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

(iii) $e^{x\left(\frac{t}{2}-\frac{1}{2t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

$e^{x\left(\frac{t}{2}-\frac{1}{2t}\right)}$: generating function of Bessel function

(iv) $J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos[n\theta - x\sin\theta]\, d\theta \qquad (n = 0, 1, 2, \cdots)$

p419

(v) orthogonality of Bessel function

Let $J_p(a) = J_p(b) = 0$

Then
$$\boxed{\int_0^1 x J_p(ax) J_p(bx)\, dx = \begin{cases} 0 & \text{if } a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a) & \text{if } a = b \end{cases}}$$

(vi) Expansion (Fourier - Bessel Series)

Let $J_\nu(j_n) = 0 \qquad (n = 1, 2, \cdots)$

If $f(x)$ is defined at $[0, 1]$,

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} a_n J_\nu(j_n x) \\ a_n &= \frac{\int_0^1 dx\, x f(x) J_\nu(j_n x)}{\int_0^1 x J_\nu^2(j_n x)} \end{aligned}}$$

Fourier - Bessel Expansion

$$* \quad a_n = \frac{\int_0^1 dx\, x f(x) J_\nu(j_n x)}{\frac{1}{2} J_{\nu+1}^2(j_n)}$$

pf) $f(x) = \sum_{n=1}^{\infty} a_n J_\nu(j_n x)$

$$\int_0^1 dx\, x J_\nu(j_m x) f(x)$$

$$= \sum_{n=1}^{\infty} a_n \int_0^1 x J_\nu(j_m x) J_\nu(j_n x)\, dx$$

$$= \sum_{n=1}^{\infty} a_n \left[\int_0^1 x J_\nu^2(j_n x)\, dx\right] \delta_{mn} = a_m \int_0^1 dx\, x J_\nu^2(j_m x)$$

$$\Rightarrow \quad a_m = \frac{\int_0^1 dx\, x\, J_\nu(j_m x)\, f(x)}{\int_0^1 dx\, x\, J_\nu^2(j_m x)}$$

$$\Rightarrow \quad a_m = \frac{\int_0^1 dx\, x\, J_\nu(j_m x)\, f(x)}{\int_0^1 dx\, x\, J_\nu^2(j_m x)} \qquad \#$$

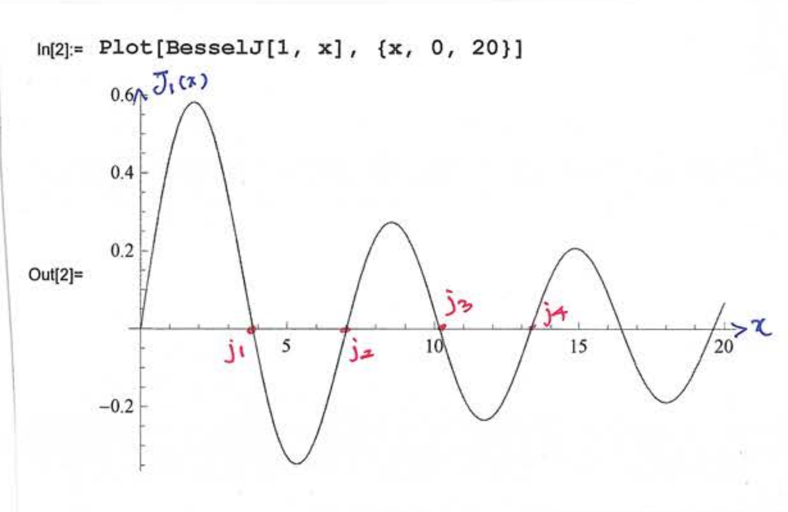
p400

(예제 8.32)

$$f(x) = x(1-x) \qquad 0 \le x \le 1$$

$$x(1-x) = \sum_{m=1}^{\infty} a_m J_1(j_m x)$$

$$a_m = \frac{2}{J_2^2(j_m)} \int_0^1 dx\, x^2 (1-x) J_1(j_m x)$$



$j_1 = 3.83170597$, $j_2 = 7.01558667$, $j_3 = 10.17346814$, $j_4 = 13.32369194$

$$a_1 = \frac{2}{[J_2(j_1)]^2} \int_0^1 dx\ x^2(1-x) J_1(j_1 x) = 0.45221702$$

$$a_2 = -0.03151859$$

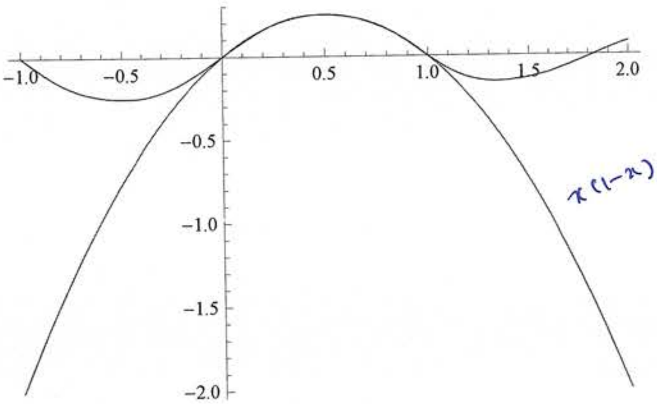
$$a_3 = 0.03201789$$

$$a_4 = -0.00768864$$

$$\Rightarrow x(1-x) \approx a_1 J_1(j_1 x) + a_2 J_1(j_2 x) + a_3 J_3(j_3 x) + a_4 J_4(j_4 x)$$

```
In[58]:= j1 = 3.83170597;
         j2 = 7.01558667;
         j3 = 10.17346814;
         j4 = 13.32369194;
         a1 = 0.45221702;
         a2 = -0.03151859;
         a3 = 0.03201789;
         a4 = -0.00768864;
         f[x_] := x (1 - x);
         g[x_] :=
           a1 BesselJ[1, j1 x] + a2 BesselJ[1, j2 x] + a3 BesselJ[1, j3 x] + a4 BesselJ[1, j4 x];
         Plot[{f[x], g[x]}, {x, -1, 2}, PlotStyle → {Red, Blue}]
```

Out[68]=



[3] Bessel function of second kind

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}\left[J_\nu(x)\cos\nu\pi - J_{-\nu}(x)\right]$$

Bessel function of the second kind of order ν

or

Neumann function of order ν

The general solution of Bessel differential Eq.

$$y(x) = A J_\nu(x) + B Y_\nu(x)$$

p.406

(예제 8.29)

$$9x^2 y'' - 27xy' + (9x^2+35)y = 0 \quad -①$$

put

$$\left.\begin{aligned} y &= x^2 u \\ y' &= x^2 u' + 2xu \\ y'' &= x^2 u'' + 4xu' + 2u \end{aligned}\right\} -②$$

② → ①

$$u'' + \frac{1}{x}u' + \left(1 - \frac{1}{9x^2}\right)u = 0 \quad -③$$

Eq. ③ is Bessel Differential Eq. with $\nu = \frac{1}{3}$.

$$u = A J_{\frac{1}{3}}(x) + B J_{-\frac{1}{3}}(x) \quad -④$$

④ → ②

$$y = x^2\left[A J_{\frac{1}{3}}(x) + B J_{-\frac{1}{3}}(x)\right] \quad \#$$

- General Bessel Differential Eq.

$$y'' + \frac{1-2a}{x} y' + \left[b^2 c^2 x^{2c-2} + \frac{a^2 - \nu^2 c^2}{x^2} \right] y = 0$$

solution

$$y = x^a \left[A J_\nu (b x^c) + B Y_\nu (b x^c) \right]$$

(Ex) $$y'' - \frac{2\sqrt{3} - 1}{x} y' + \left(784 x^6 - \frac{61}{x^2} \right) y = 0$$

$a = \sqrt{3}$, $c = 4$, $b = 7$, $\nu = 2$

$$\Rightarrow y = x^{\sqrt{3}} \left[A J_2 (7x^4) + B Y_2 (7x^4) \right]$$

※

[4] Modified Bessel function

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)\,y = 0$$

Modified Bessel Differential Eq.

solution

$$y = c_1\, I_\nu(x) + c_2\, K_\nu(x)$$

$I_\nu(x) = (-i)^\nu J_\nu(ix)$: Modified Bessel function of first kind

$K_\nu(x) = \frac{1}{2}\pi \dfrac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$: Modified Bessel function of second kind

or

Mc Donald function

[5] Asymptotic limit

(i) $z \to 0$

$$J_\nu(z) \sim \frac{1}{\Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu$$

$$Y_0(z) \sim \frac{2}{\pi} \ln z$$

$$Y_\nu(z) \sim -\frac{1}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}$$

$$I_\nu(z) \sim \frac{1}{\Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu$$

$$K_0(z) \sim -\ln z$$

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}$$

(ii) $z \to \infty$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$Y_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

[6] Application : Skin Effect

Skin Effect

▪ Skin Effect(표면효과)란?

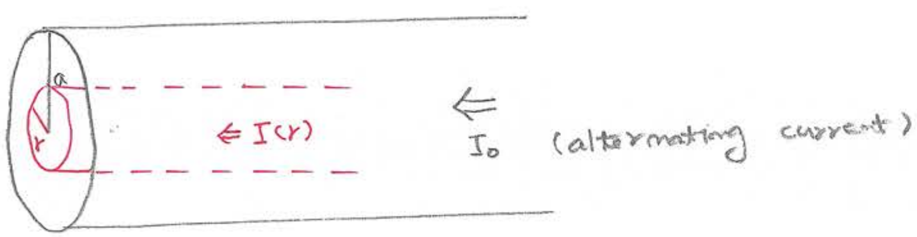
고주파 신호를 다룰 때 한가지 알아두어야 할 것이 바로 금속에서의 표면효과이다. 표면효과란 **주파수가 올라갈수록 교류전류는 금속의 내부가 아니라 표면에 집중되어 흐른다**는 것을 의미한다. 저주파에서는 금속의 내부를 전하가 이동하면서 신호가 오가지만, 고주파에서는 도체 표면을 따라 전하가 오간다.

이 성질이 특별히 설계에 제한이 되는 요소는 아니지만, 구조설계시 나름대로 고려해야할 부분이 된다. 예를 들어 이 성질을 잘 이용하면 굳이 도전율이 좋은 금이나 은을 전체에 사용할 필요 없이 외부에 coating함으로써 고주파의 전도율을 높일 수 있다. 이것은 3D simulation시에 더욱 유용하게 활용되는데, 고주파에서는 금속 내부에 field가 존재하지 않기 때문에 금속내부는 아예 계산하지 않음으로써 많은 시간을 절약할 수 있다. 반면 이문제 때문에 저주파에서의 해석 정확도를 보장하기 어려운 문제점도 있긴 하다.

▪ Skin depth

표면효과를 수치적으로 나타낸 수식으로서, 주파수와 금속성분의 관계에 따라 전류가 어느정도 깊이까지 침투하느냐를 나타낸 수식이다. 이것을 통해 고주파 전도에 적절한 금속의 두께를 알아낼 수 있다.

$$\delta_s = \frac{1}{\alpha} = \sqrt{\frac{2}{2\pi f \mu \sigma}}$$



We will compute

$$\gamma(r) \equiv \frac{I(r)}{I_0}$$

Let us consider Ampere law

$$\oint \vec{B} \cdot d\vec{l} = \mu I_{inside} \quad — ①$$

(μ: 투자율 (permeability))

We assume $B = B(r,t)$ and $J = J(r,t)$.

Then Eq. ① gives

$$B \cdot 2\pi r = \mu I_{inside} = \mu \int_0^r J(\xi,t)\, 2\pi \xi \, d\xi$$

$$\Rightarrow \quad rB = \mu \int_0^r J(\xi,t)\, \xi \, d\xi$$

$$\Rightarrow \quad \frac{\partial}{\partial r}(rB) = \mu J(r,t)\, r$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(rB) = \mu J(r,t) \quad — ②$$

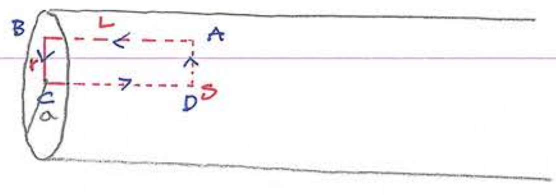
Next, we consider Faraday law

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} \quad — ③$$

$\mathcal{E}$: 유도기전력

Φ_B: magnetic flux

Consider the following open surface



Then the magnetic flux for surface S is

$$\Phi_B = \int_0^r B(\xi,t)\, L\, d\xi = L \int_0^r B(\xi,t)\, d\xi \qquad - ④$$

Therefore

$$\mathcal{E} = -\frac{\partial \Phi_B}{\partial t} = -L \frac{\partial}{\partial t} \int_0^r B(\xi,t)\, d\xi \qquad - ⑤$$

Since

$$\mathcal{E} = \oint \vec{E}\cdot d\vec{l} \quad \text{and} \quad \vec{E} = \rho \vec{J} \quad (\rho: \text{resitivity}),$$

$$\mathcal{E} = \rho \oint \vec{J}\cdot d\vec{l}$$

$$= \rho \Big[\underbrace{\int_A^B \vec{J}\cdot d\vec{l}}_{\substack{J(r,t)\int_L^0 dl \\ = -L\, J(r,t)}} + \underbrace{\overbrace{\int_B^C \vec{J}\cdot d\vec{l}}^{=0}} + \underbrace{\int_C^D \vec{J}\cdot d\vec{l}}_{\substack{J(0,t)\int_0^L dl \\ = L\, J(0,t)}} + \underbrace{\overbrace{\int_D^A \vec{J}\cdot d\vec{l}}^{=0}} \Big]$$

$$= \rho L \left(J(0,t) - J(r,t) \right) \qquad - ⑥$$

From ⑤ and ⑥ we have

$$\rho L \left[J(0,t) - J(r,t) \right] = -L \frac{\partial}{\partial t} \int_0^r B(\xi,t)\, d\xi$$

$$\Rightarrow \quad \rho \left[J(0,t) - J(r,t) \right] = -\frac{\partial}{\partial t} \int_0^r B(\xi,t)\, d\xi \qquad - ⑦$$

Differentiate Eq. ⑦ with respect to r

$$-\rho \frac{\partial}{\partial r} J(r,t) = -\frac{\partial}{\partial t} \underbrace{\frac{\partial}{\partial r} \int_0^r B(\xi,t)\, d\xi}^{B(r,t)}$$

$$\Rightarrow \quad \rho \frac{\partial}{\partial r} J(r,t) = \frac{\partial}{\partial t} B(r,t) \qquad - ⑧$$

Multiplying r to Eq. ⑧ yields

$$\rho r \frac{\partial}{\partial r} J(r,t) = r \frac{\partial}{\partial t} B(r,t) = \frac{\partial}{\partial t}\left[r B(r,t) \right] \quad - ⑨$$

Differentiate Eq. ⑨ with respect to r

$$\rho \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r} J(r,t) \right] = \frac{\partial}{\partial t} \frac{\partial}{\partial r}\left[r B(r,t) \right] \quad - ⑩$$

From ⑦ and ⑩ one can eliminate the magnetic field;

$$\rho \frac{\partial}{\partial r}\left[r \frac{\partial}{\partial r} J(r,t) \right] = \frac{\partial}{\partial t}\left[\mu r J(r,t) \right]$$

$$\Rightarrow \quad \rho r \frac{\partial^2 J}{\partial r^2} + \rho \frac{\partial J}{\partial r} = \mu r \frac{\partial}{\partial t} J \quad - ⑪$$

Put

$$J(r,t) = f(r)\, e^{i\omega t} \quad - ⑫$$

$(J(r,t) = \mathrm{Re}\, f(r)\, e^{i\omega t})$

Inserting ⑫ into ⑪, we get

$$\left.\begin{aligned} &\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - b^2 f = 0 \\ &b^2 = \frac{i\mu\omega}{\rho} \end{aligned}\right\} \quad - ⑬$$

Modified Bessel Function

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + \nu^2)\, y = 0$$

$$y = A\, I_\nu(x) + B\, K_\nu(x)$$

Put

$$x = br$$

$$\frac{\partial}{\partial r} = b \frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial r^2} = b^2 \frac{\partial^2}{\partial x^2}$$

— (14)

(14) → (13)

$$b^2 \frac{d^2 f}{dx^2} + \frac{b}{x} b \frac{\partial f}{\partial x} - b^2 f = 0$$

$$\Rightarrow \quad \underline{\frac{d^2 f}{dx^2} + \frac{1}{x} \frac{df}{dx} - f = 0} \qquad \text{— (15)}$$

Modified Bessel function with $\nu = 0$.

$$f(r) = A I_0 (br) + B K_0 (br) \qquad \text{— (16)}$$

Since $\lim_{r \to 0} f(r)$ should be finite, we have to choose

$$B = 0 \qquad \text{— (17)}$$

Thus

$$f(r) = A I_0 (br) \qquad \text{— (18)}$$

Inserting (18) into (12) we have

$$J(r, t) = A I_0 (br) e^{i\omega t} \qquad \text{— (19)}$$

Now, let us compute

$$\gamma(r) = \frac{I(r)}{I_0} \quad -(20)$$

$$I_0 = \int_0^a J(\xi, t)\, 2\pi \xi \, d\xi$$

$$= 2\pi \int_0^a A\, I_0(b\xi)\, e^{i\omega t}\, \xi \, d\xi$$

$$= 2\pi A\, e^{i\omega t} \int_0^a I_0(b\xi)\, \xi \, d\xi \qquad y = b\xi$$

$$= \frac{2\pi A\, e^{i\omega t}}{b^2} \int_0^{ab} I_0(y)\, y\, dy \quad -(21)$$

Integral formula

$$\boxed{\int x\, I_0(x)\, dx = x\, I_1(x) + C} \quad -(22)$$

Using (22)

$$I_0 = \frac{2\pi A\, e^{i\omega t}}{b^2}\, ab\, I_1(ab) = 2\pi A \frac{a}{b} e^{i\omega t} I_1(ab) \quad -(23)$$

By same way

$$I(r) = \int_0^r J(\xi, t)\, 2\pi\xi\, d\xi = 2\pi A \frac{r}{b} e^{i\omega t} I_1(br) \quad -(24)$$

Therefore

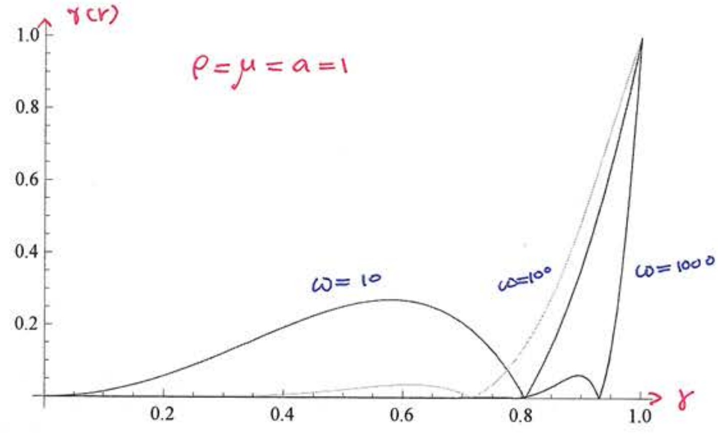
$$\gamma(r) = \frac{Re[r\, I_1(br)]}{Re[a\, I_1(ba)]} \quad -(25)$$

with

$$b = \sqrt{\frac{\mu\omega}{\rho}} \frac{1+i}{\sqrt{2}} \quad -(26)$$

```
In[27]:= rho = 1; mu = 1;  a = 1;
         b[w_] := Exp[I Pi / 4] Sqrt[mu w / rho];
         ratio[r_, w_] := Abs[Re[r BesselI[1, b[w] r]] / Re[a BesselI[1, b[w] a]]];
         Plot[{ratio[r, 10], ratio[r, 100], ratio[r, 1000]},
          {r, 0, 1}, PlotRange → All, PlotStyle → {Red, Green, Blue}]
```

Out[30]=



When ω is large, b is also large.

Since

$$\lim_{z \to \infty} I_\nu(z) \sim \frac{e^{z}}{\sqrt{2\pi z}} \qquad -(27)$$

$$\gamma(r) \sim \sqrt{\frac{r}{a}}\, e^{-b(a-r)} \sim \sqrt{\frac{r}{a}}\, e^{-\alpha\sqrt{\omega}\,(a-r)} \qquad -(28)$$

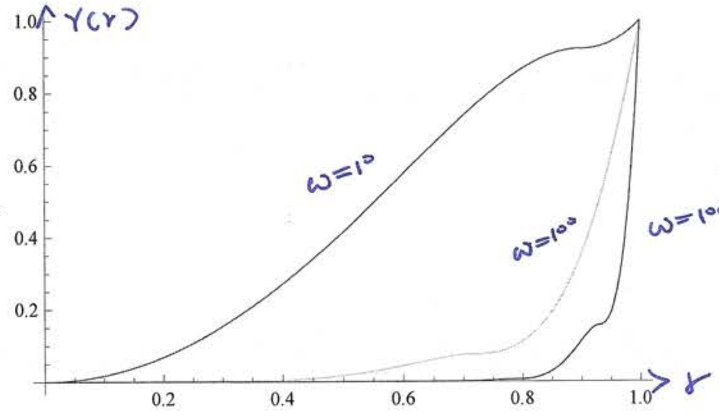
$\Rightarrow \quad r \neq a \qquad \gamma(r) \sim 0$

$\quad\;\; r = a \qquad \gamma(r) = 1$

This is a skin effect !!

```
In[43]:= rho = 1; mu = 1; a = 1;
      b[w_] := Exp[I Pi / 4] Sqrt[mu w / rho];
      currentd[w_, r_] := Abs[Re[BesselI[0, b[w] r]]] ;
      current[w_, r_] := NIntegrate[x currentd[w, x], {x, 0, r}];
      ratio[w_, r_] := current[w, r] / current[w, a];
      Plot[{ratio[10, r], ratio[100, r], ratio[1000, r]},
       {r, 0, 1}, PlotRange → All, PlotStyle → {Red, Green, Blue}]
```

Out[48]=



Skin effect

From Wikipedia, the free encyclopedia

Skin effect is the tendency of an alternating electric current (AC) to distribute itself within a conductor so that the current density near the surface of the conductor is greater than that at its core. That is, the electric current tends to flow at the "skin" of the conductor, at an average depth called the **skin depth**. The skin effect causes the effective resistance of the conductor to increase with the frequency of the current because much of the conductor carries little current. Skin effect is due to eddy currents set up by the AC current. At 60 Hz in copper, skin depth is about 8.5 mm. At high frequencies skin depth is much smaller.

Methods to minimise skin effect include using specially woven wire and using hollow pipe-shaped conductors.

Contents

Introduction

When an electromagnetic wave interacts with a conductive material, mobile charges within the material are made to oscillate back and forth with the same frequency as the impinging fields. The movement of these charges, usually electrons, constitutes an alternating electric current, the magnitude of which is greatest at the conductor's surface. The decline in current density versus depth is known as the *skin effect* and the *skin depth* is a measure of the distance over which the current density falls to 1/e of its original value. A gradual change in phase accompanies the change in magnitude, so that, at a given time and at appropriate depths, the current can be flowing in the opposite direction to that at the surface.

The effect was first described in a paper by Horace Lamb in 1883 for the case of spherical conductors, and was generalised to conductors of any shape by Oliver Heaviside in 1885. The skin effect has practical consequences in the design of radio-frequency and microwave circuits and to some extent in AC electrical power transmission and distribution systems. Also, it is of considerable importance when designing discharge tube circuits.

The current density J in an *infinitely thick* plane conductor decreases exponentially with depth d from the surface, as follows:

$$J = J_S e^{-d/\delta}$$

where δ is a quantity called the *skin depth* and J_S is the current density at the surface. The skin depth is defined as the depth below the surface of the conductor at which the current density decays to 1/e (about 0.37) of J_S. It can be calculated as follows:

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}}$$

where

ρ = resistivity of conductor
ω = angular frequency of current = $2\pi \times$ frequency
μ = absolute magnetic permeability of conductor $= \mu_0 \cdot \mu_r$, where μ_0 is the permeability of free space ($4\pi\times10^{-7}$ $N\cdot A^{-2}$) and μ_r is the relative permeability of the conductor.

The resistance of a flat slab (much thicker than δ) to alternating current is exactly equal to the resistance of a plate of thickness δ to direct current. For long, cylindrical conductors such as wires, with diameter D large compared to δ, the resistance is *approximately* that of a hollow tube with wall thickness δ carrying direct current. That is, the AC resistance is approximately:

$$R \approx \frac{\rho}{\delta}\left(\frac{L}{\pi(D-\delta)}\right) \approx \frac{\rho}{\delta}\left(\frac{L}{\pi D}\right)$$

where

L = length of conductor
D = diameter of conductor

The final approximation above is accurate if $D >> \delta$.

A convenient formula (attributed to F.E. Terman) for the diameter D_W of a wire of circular cross-section whose resistance will increase by 10% at frequency f is:

$$D_{\mathrm{W}} = \frac{200\ \mathrm{mm}}{\sqrt{f/\mathrm{Hz}}}$$

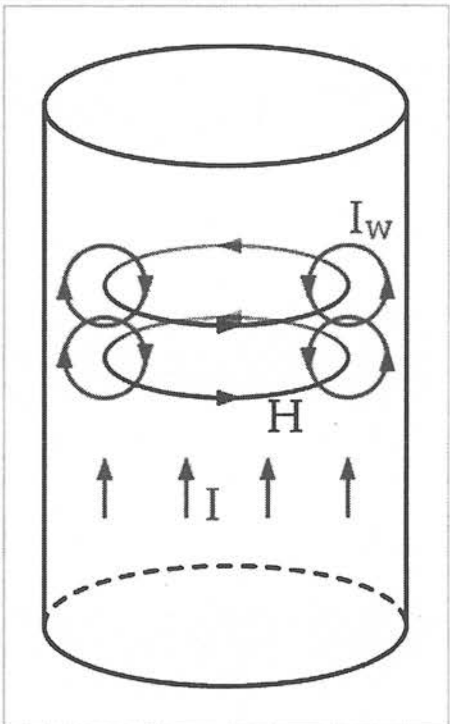
The increase in AC resistance described above is accurate only for an isolated wire. For a wire close to other wires, e.g. in a cable or a coil, the ac resistance is also affected by proximity effect, which often causes a much more severe increase in ac resistance.

Material effect on skin depth

Skin depth varies as the inverse square root of the conductivity. This means that better conductors have a reduced skin depth. The overall resistance of the better conductor is lower even though the skin depth is less. This tends to reduce the difference in high frequency resistance between metals of different conductivity.

Skin depth also varies as the inverse square root of the permeability of the conductor. In the case of iron, its conductivity is about 1/7 that of copper. Its permeability is about 10,000 times greater, however. The skin depth for iron is about 1/38 that of copper or about 220 micrometres at 60 Hz. Iron wire is worthless as a conductor at power line frequencies. Skin effect reduces both the effective thickness of laminations in power transformers and their losses.

Iron rods work well for direct-current (DC) welding but it is impossible to use them at frequencies much higher than 60 Hz. At a few kilohertz, the welding rod will glow red hot from skin effect losses but will barely have enough power available to sustain an arc. Only non-magnetic rods can be used for high-frequency welding.



Skin depth is due to the circulating eddy currents (arising from a changing H field) cancelling the current flow in the center of a conductor and reinforcing it in the skin.

Mitigation

A type of cable called litz wire (from the German *Litzendraht*, braided wire) is used to mitigate the skin effect for frequencies of a few kilohertz to about one megahertz. It consists of a number of insulated wire strands woven together in a carefully designed pattern, so that the overall magnetic field acts equally on all the wires and causes the total current to be distributed equally among them. This has the effect of reducing the effective permeability and increasing the skin depth.[1]

Litz wire is often used in the windings of high-frequency transformers, to increase their efficiency by mitigating both skin effect and, more importantly, proximity effect.

Large power transformers are wound with stranded conductors of similar construction to litz wire, but of larger cross-section.[2]

High-voltage, high-current overhead power transmission lines often use aluminum cable with a steel reinforcing core, where the higher resistivity of the steel core is largely immaterial.

In other applications, solid conductors are replaced by tubes, which have the same resistance at high frequencies but lighter weight. Very recently, researchers have been able to create extremely light cell-phone antennas using carbon-nanotubes,[3] their performance attributed to skin effect.

Solid or tubular conductors may also be silver-plated providing a better conductor (the best possible conductor except for superconductors) than copper on the "skin" of the conductor. Silver-plating is most effective at VHF and microwave frequencies, because the very thin skin depth (conduction layer) at those frequencies means that the silver plating can economically be applied at thicknesses greater than the skin depth.

Examples

If the electrical resistivity of a material is equal to $1/\sigma$ and its relative permeability is defined as μ / μ_0, where μ_0 is the magnetic permeability of free space.

$$\delta = \frac{1}{\sqrt{\pi\mu_o}}\sqrt{\frac{\rho}{\mu_r f}} \approx 503\sqrt{\frac{\rho}{\mu_r f}}$$

where

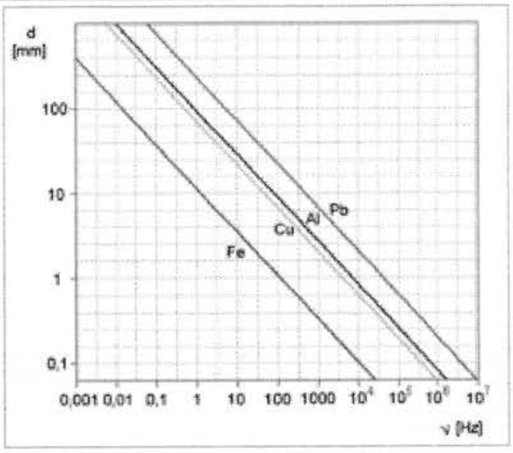
δ = the skin depth in metres

$\mu_0 = 4\pi \times 10^{-7}$ H/m

μ_r = the relative permeability of the medium

ρ = the resistivity of the medium in Ω·m (for Cu = 1.68×10^{-8} Ω·m)

f = the frequency of the wave in Hz



Skin depths for some metals

If the resistivity of aluminum is taken as 2.8×10^{-8} Ω·m and its relative permeability is 1, then the skin depth at a frequency of 50 Hz is given by

$$\delta = 503\sqrt{\frac{2.82\cdot10^{-8}}{1\cdot50}} = 11.9\,\mathrm{mm}$$

Iron has a higher resistivity, 1.0×10^{-7} Ω·m, and this will increase the skin depth. However, its relative permeability is typically 90, which will have the opposite effect. At 50 Hz the skin depth in iron is given by

$$\delta = 503\sqrt{\frac{1.0\cdot10^{-7}}{90\cdot50}} = 2.4\,\mathrm{mm}$$

Hence, the higher magnetic permeability of iron more than compensates for the lower resistivity of aluminium and the skin depth in iron is therefore one-fifth that of aluminium. This will be true whatever the frequency, assuming the material properties are not themselves frequency-dependent.

Skin depth values for some common good conductors at a frequency of 10 GHz (microwave region) are indicated below.

Conductor	Skin depth (μm)
Aluminum	0.80
Copper	0.65
Gold	0.79
Silver	0.64

At microwave frequencies, most of the current in a good conductor flows in an extremely thin region near the surface. The extremely short skin depth at microwave frequencies shows that only surface coating of guiding conductor is important. A piece of glass with an evaporated silver surface 3 μm thick is an excellent conductor at these frequencies.

In copper, the skin depth at various frequencies is shown below.

Frequency	Skin depth (μm)
60 Hz	8470
10 kHz	660
100 kHz	210
1 MHz	66

10 MHz	21

In *Engineering Electromagnetics*, Hayt points out that in a power station a bus bar for alternating current at 60 Hz with a radius larger than one-third of an inch (8 mm) is a waste of copper, and in practice bus bars for heavy AC current are rarely more than half an inch (12 mm) thick except for mechanical reasons.

See also

- Proximity effect (electromagnetism)
- Tesla coil: The 'skin effect'
- Surface wave
- Litz wire

References

1. ^ [1] (http://ieeexplore.ieee.org/Xplore/login.jsp?url=http%3A%2F%2Fieeexplore.ieee.org%2Fiel5%2F10182%2F32507%2F01518758.pdf%3Farnumber%3D1518758&authDecision=-203)
2. ^ Central Electricity Generating Board (1982). *Modern Power Station Practice*. Pergamon Press.
3. ^ Spinning Carbon Nanotubes Spawns New Wireless Applications (http://www.sciencedaily.com/releases/2009/03/090309121941.htm)

- Hayt, William Hart. *Engineering Electromagnetics Seventh Edition*. New York: McGraw Hill, 2006. ISBN 0-07-310463-9.
- Nahin, Paul J. *Oliver Heaviside: Sage in Solitude*. New York: IEEE Press, 1988. ISBN 0-87942-238-6.
- Ramo, S., J. R. Whinnery, and T. Van Duzer. *Fields and Waves in Communication Electronics*. New York: John Wiley & Sons, Inc., 1965.
- Terman, F. E. *Radio Engineers' Handbook*. New York: McGraw-Hill, 1943. For the Terman formula mentioned above.

- Ramo, Whinnery, Van Duzer (1994). *Fields and Waves in Communications Electronics*. John Wiley and Sons.

External links

- Skin Effect in HiFi Cables (http://www.st-andrews.ac.uk/~jcgl/Scots_Guide/audio/skineffect/page1.html)
- Skin Effect Relevance in Speaker Cables (http://www.audioholics.com/education/cables/skin-effect-relevance-in-speaker-cables)