Comment on "Path-integral action of a particle with the generalized uncertainty principle and correspondence with noncommutativity"

DaeKil Park[1,2,*] and Eylee Jung[1]

[1]*Department of Electronic Engineering, Kyungnam University, Changwon 631-701, Korea*
[2]*Department of Physics, Kyungnam University, Changwon 631-701, Korea*

(Received 25 January 2020; accepted 18 February 2020; published 9 March 2020)

Recently in Phys. Rev. D **99**, 104010 (2019), the nonrelativistic Feynman propagator for a harmonic oscillator system was presented when the generalized uncertainty principle is employed. In this short Comment we show that the expression is incorrect, and we derive its correct form.

DOI: 10.1103/PhysRevD.101.068501

Recently, S. Gangopadhyay and S. Bhattacharyya [1] derived the nonrelativistic Feynman propagators for free and harmonic oscillator systems when the generalized uncertainty principle (GUP) is employed and discussed their correspondence with noncommutativity. The GUP [2] they used can be summarized as a modified Heisenberg algebra,

$$[Q_i, P_j] = i\hbar(\delta_{ij} + \beta\delta_{ij}\mathrm{P}^2 + 2\beta P_i P_j], \tag{1}$$

where β is a GUP parameter, which has the dimensions $(\text{momentum})^{-2}$. The modified Heisenberg algebra can be readily represented up to first order of β as $Q_i = q_i$, $P_i = p_i(1+\beta\mathrm{p}^2)$, where $\{p_i, q_i\}$ satisfies the usual Heisenberg algebra $[q_i, p_j] = i\hbar\delta_{ij}$.

The authors of Ref. [1] considered the harmonic oscillator system, whose Hamiltonian is

$$\hat{H} = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2 = \frac{p^2}{2m} + \frac{\beta}{m}p^4 + \frac{1}{2}m\omega^2 x^2 + \mathcal{O}(\beta^2). \tag{2}$$

Without any explicit explanation, they presented the Feynman propagator [see Eq. (41) of Ref. [1]] of this system in the form

$$\begin{aligned}\langle q_f, t_f | q_0, t_0\rangle &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\left[1 + \frac{3i\beta\hbar m}{T} - 6\beta m^2\left(\frac{q_f - q_0}{T}\right)^2\right.\\ &\quad\left. - \frac{3}{4}\beta m\hbar\omega^2 T\cot\omega T + \mathcal{O}(\beta^2)\right]e^{\frac{i}{\hbar}S_{cl}},\end{aligned} \tag{3}$$

where $T = t_f - t_0$ and S_{cl} is a classical action. The only comment the authors presented is that the $\omega \to 0$ limit of Eq. (3) is

$$\begin{aligned}K_F[q_f, t_f : q_0, t_0] &= \sqrt{\frac{m}{2\pi i\hbar T}}\left(1 + \frac{3i\beta\hbar m}{T} - \frac{6\beta m^2(q_0 - q_f)^2}{T^2} + \mathcal{O}(\beta^2)\right)\\ &\quad\times\exp\left[\frac{im}{2\hbar T}(q_0 - q_f)^2\left\{1 - 2\beta m^2\left(\frac{q_0 - q_f}{T}\right)^2\right\}\right],\end{aligned} \tag{4}$$

which is the Feynman propagator for free particle. Since Eq. (3) is one of the main results of Ref. [1] and it can be used in other GUP-related issues, it is worthwhile to check the validity of Eq. (3) more carefully. Unfortunately, it is incorrect, although it approaches the correct $\omega \to 0$ limit. As we will show, this should be changed to

$$\begin{aligned}\langle q_f, t_f | q_0, t_0\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}&[1 + \beta f(q_0, q_f : T)\\ &+ \mathcal{O}(\beta^2)]e^{\frac{i}{\hbar}(S_0 + \beta S_1)},\end{aligned} \tag{5}$$

where

$$\begin{aligned}S_0 &= \frac{m\omega}{2\sin\omega T}[(q_0^2 + q_f^2)\cos\omega T - 2q_0 q_f],\\ S_1 &= -\frac{m^3\omega^3}{32\sin^4\omega T}[\{12\omega T + 8\sin 2\omega T + \sin 4\omega T\}(q_0^4 + q_f^4)\\ &\quad - 4\{12\omega T\cos\omega T + 11\sin\omega T + 3\sin 3\omega T\}\\ &\quad\times q_0 q_f(q_0^2 + q_f^2)\\ &\quad + 12\{4\omega T + 2\omega T\cos 2\omega T + 5\sin 2\omega T\}q_0^2 q_f^2],\end{aligned}$$

*dkpark@kyungnam.ac.kr

$$
\begin{aligned}
f(q_0, q_f : T) = {} & \frac{3i\hbar m\omega}{8\sin^2\omega T}(2\omega T + 5\sin\omega T\cos\omega T \\
& + \omega T\cos 2\omega T) - \frac{3m^2\omega^2}{8\sin^3\omega T} \\
& \times [2\omega T\{3\cos\omega T(q_0^2 + q_f^2) \\
& - 2(2+\cos 2\omega T)q_0 q_f\} \\
& + 10\sin\omega T(q_0^2 + q_f^2 - 2q_0 q_f\cos\omega T) \\
& - 6\sin^3\omega T(q_0^2 + q_f^2)]. \qquad (6)
\end{aligned}
$$

Of course, $S_0 + \beta S_1$ is a classical action. It is straightforward to show that the $\omega \to 0$ limit of Eq. (5) also goes to $K_F[q_f, t_f : q_0, t_0]$.

In order to show Eq. (5) explicitly we note that the Feynman propagator $\langle q_f, t_f | q_0, t_0\rangle$ can be derived from the Schrödinger equation as

$$\langle q_f, t_f | q_0, t_0\rangle = \sum_n \psi_n(q_f)\psi_n^*(q_0)e^{-(i/\hbar)E_n(t_f - t_0)}, \quad (7)$$

where $\psi_n(q)$ and E_n are the nth-order eigenfunction and eigenvalue of the Schrödinger equation. The Schrödinger equation for the harmonic oscillator system is given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{\beta\hbar^4}{m}\frac{\partial^4}{\partial x^4} + \frac{1}{2}m\omega^2 x^2 + \mathcal{O}(\beta^2)\right]\psi_n(x) = E_n\psi_n(x). \quad (8)$$

If we treat the GUP term $\frac{\beta\hbar^4}{m}\frac{\partial^4}{\partial x^4}$ as a small perturbation, one can derive $\psi_n(x)$ and E_n as

$$
\begin{aligned}
\psi_n(x) = {} & \phi_n(x) + (\beta m\hbar\omega)\left[\frac{(2n+3)\sqrt{(n+1)(n+2)}}{4}\phi_{n+2}(x) - \frac{(2n-1)\sqrt{n(n-1)}}{4}\phi_{n-2}(x)\right. \\
& \left. + \frac{\sqrt{n(n-1)(n-2)(n-3)}}{16}\phi_{n-4}(x) - \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{16}\phi_{n+4}(x)\right] + \mathcal{O}(\beta^2), \\
E_n = {} & \left(n + \frac{1}{2}\right)\hbar\omega\left[1 + \frac{3(2n^2+2n+1)}{2(2n+1)}(\beta m\hbar\omega)\right] + \mathcal{O}(\beta^2), \qquad (9)
\end{aligned}
$$

where $n = 0, 1, 2, \ldots$ and

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left[-\frac{m\omega}{2\hbar}x^2\right]. \quad (10)$$

In Eq. (10) $H_n(z)$ is a nth-order Hermite polynomial. We assume $\phi_m(z) = 0$ for $m < 0$. Inserting Eq. (9) into Eq. (7), one can express $\langle q_f, t_f | q_0, t_0\rangle$ as

$$\langle q_f, t_f | q_0, t_0\rangle = J + (\beta m\hbar\omega)(K_1 + K_2) + \mathcal{O}(\beta^2), \quad (11)$$

where

$$
\begin{aligned}
J = {} & \sum_{n=0}^{\infty}\phi_n(q_f)\phi_n(q_0)\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\left\{1 + \frac{3(2n^2+2n+1)}{2(2n+1)}(\beta m\hbar\omega)\right\}\right], \\
K_1 = {} & \left[\sum_{n=0}^{\infty}\frac{(2n+3)\sqrt{(n+1)(n+2)}}{4}[\phi_n(q_f)\phi_{n+2}(q_0) + \phi_n(q_0)\phi_{n+2}(q_f)]\right. \\
& \left. - \sum_{n=2}^{\infty}\frac{(2n-1)\sqrt{n(n-1)}}{4}[\phi_n(q_f)\phi_{n-2}(q_0) + \phi_n(q_0)\phi_{n-2}(q_f)]\right]\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right], \\
K_2 = {} & \left[\sum_{n=4}^{\infty}\frac{\sqrt{n(n-1)(n-2)(n-3)}}{16}[\phi_n(q_f)\phi_{n-4}(q_0) + \phi_n(q_0)\phi_{n-4}(q_f)]\right. \\
& \left. - \sum_{n=0}^{\infty}\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{16}[\phi_n(q_f)\phi_{n+4}(q_0) + \phi_n(q_0)\phi_{n+4}(q_f)]\right]\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right]. \qquad (12)
\end{aligned}
$$

Using the extended Mehler formula [3]

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H_{k+m}(x)H_{k+n}(y)=(1-4t^2)^{-(m+n+1)/2}\exp\left[\frac{4txy-4t^2(x^2+y^2)}{1-4t^2}\right]\times\sum_{k=0}^{\min(m,n)}2^{2k}k!\binom{m}{k}\binom{n}{k}t^kH_{m-k}\left(\frac{x-2ty}{\sqrt{1-4t^2}}\right)H_{n-k}\left(\frac{y-2tx}{\sqrt{1-4t^2}}\right), \tag{13}$$

one can show that

$$J=\sqrt{\frac{m\omega}{\pi\hbar}}e^{-\frac{i}{2}\omega T}\left[\left(1-\frac{3i}{4}(\beta m\hbar\omega^2T)\right)+\frac{3i}{2}(\beta m\hbar\omega^2T)\frac{\partial^2}{\partial\mu^2}+\frac{3}{2}(\beta m\hbar\omega^2T)\frac{\partial}{\partial\mu}\right]F(\mu)\Big|_{\mu=\omega T},$$
$$K_1=-\frac{1}{2}\sqrt{\frac{m\omega}{2\pi i\hbar}}e^{-\frac{3}{2}i\omega T}\sin\omega T\left(2i\frac{\partial}{\partial\mu}+3\right)G(\mu)\Big|_{\mu=\omega T}, \tag{14}$$

where

$$F(\mu)=\frac{e^{\frac{i}{2}\mu}}{\sqrt{2i\sin\mu}}\exp\left[\frac{im\omega}{2\hbar\sin\mu}\{(q_0^2+q_f^2)\cos\mu-2q_0q_f\}\right],$$
$$G(\mu)=\frac{\sqrt{2i}e^{i\mu}}{\sin\mu}\left[\frac{im\omega}{\hbar\sin\mu}\{(q_0^2+q_f^2)\cos\mu-2q_0q_f\}+1\right]F(\mu). \tag{15}$$

Computing Eq. (14) explicitly, one can show that

$$J=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{J},\qquad K_1=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{K}_1, \tag{16}$$

where

$$\tilde{J}=1-\frac{3i\beta m\omega^2T}{8\hbar\sin^4\omega T}[-3i\hbar m\omega(q_0^2+q_f^2)\sin2\omega T+m^2\omega^2(q_0^2+q_f^2-2q_0q_f\cos\omega T)^2+4i\hbar m\omega\sin\omega T(2+\cos2\omega T)q_0q_f-\hbar^2\sin^2\omega T(2+\cos2\omega T)],$$
$$\tilde{K}_1=-\frac{i}{8\hbar^2\sin^3\omega T}[-4m^2\omega^2q_0q_f(q_0^2+q_f^2)(3+\cos2\omega T)+3\hbar^2(\cos3\omega T-\cos\omega T)+4m\omega\cos\omega T\{m\omega(q_0^4+6q_0^2q_f^2+q_f^4)+12i\hbar q_0q_f\sin\omega T\}-3i\hbar m\omega(q_0^2+q_f^2)(5\sin\omega T+\sin3\omega T)]. \tag{17}$$

Using Eq. (13) again and $H_4(z)=16z^4-48z^2+12$, one can show again that $K_2=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{K}_2$, where

$$\tilde{K}_2=-\frac{i\cos\omega T}{16\hbar^2\sin^3\omega T}[12m^2\omega^2q_0^2q_f^2-3\hbar^2(1-\cos2\omega T)+2m\omega\{m\omega\cos2\omega T(q_0^4+q_f^4)-4m\omega q_0q_f(q_0^2+q_f^2)\cos\omega T-6i\hbar\sin\omega T\{(q_0^2+q_f^2)\cos\omega T-2q_0q_f\}\}]. \tag{18}$$

Inserting J, K_1, and K_2 into Eq. (11), it is possible to show that the Feynman propagator becomes Eq. (5).

[1] S. Gangopadhyay and S. Bhattacharyya, Path-integral action of a particle with the generalized uncertainty principle and correspondence with noncommutativity, Phys. Rev. D **99**, 104010 (2019).

[2] A. Kempf, G. Mangano, and R. B. Mann, Hilbert space representation of the minimal length uncertainty relation, Phys. Rev. D **52**, 1108 (1995).

[3] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach Science Publishers, New York, 1983), Vol. 2.