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ON A p-ADIC ANALOGUE OF k-PLE RIEMANN ZETA FUNCTION

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Abstract. In this paper, we construct a p-adic analogue of multiple Riemann zeta values and express their values at non-positive integers. In particular, we obtain a new congruence of the higher order Frobenius-Euler numbers and the Kummer congruences for the Bernoulli numbers as a corollary.

1. Introduction

Let ε be a root of unity of order relatively prime with p and $\varepsilon \neq 1$. We consider the Frobenius-Euler numbers $H_m(\varepsilon)$ defined by

$$\frac{\varepsilon - 1}{\varepsilon e^t - 1} = \sum_{m=0}^{\infty} H_m(\varepsilon)\frac{t^m}{m!}, \tag{1.1}$$

which can be written symbolically as $e^{H(\varepsilon)t} = (\varepsilon - 1)/(\varepsilon e^t - 1)$, interpreted to mean that $(H(\varepsilon))^m$ must be replaced by $H_m(\varepsilon)$ when expand on the left (cf. [9, 13]). This relation can also be written $\varepsilon e^{(H(\varepsilon)+1)t} - e^{H(\varepsilon)t} = \varepsilon - 1$, or, if we equate powers of t,

$$H_0(\varepsilon) = 1, \quad \varepsilon(H(\varepsilon) + 1)^m - H_m(\varepsilon) = 0 \quad \text{if } m \geq 1, \tag{1.2}$$

where again we must first expand and then replace $(H(\varepsilon))^i$ by $H_i(\varepsilon)$. We note that

$$H_m(-1) = E_m, \tag{1.3}$$

where E_m denotes the so-called Euler numbers (cf. [8, 9]). The Frobenius-Euler polynomials $H_m(x, \varepsilon)$ are defined by

$$H_m(x, \varepsilon) = \sum_{i=0}^{m} \binom{m}{i} x^{m-i} H_i(\varepsilon). \tag{1.4}$$

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We easily see that

$$H_{m-1}(-1) = \frac{2}{m}(1-2^m)B_m, \quad m \geq 1. \tag{1.5}$$

Here the Bernoulli numbers are defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \tag{1.6}$$

The Bernoulli polynomials $B_m(x)$ are also defined by $B_m(x)=\sum_{i=0}^{m}\binom{m}{i}x^{m-i}B_i$.

Among many properties of Bernoulli numbers the Kummer congruences for Bernoulli numbers are widely known [2, 5, 19, 20]. Kummer congruences of Bernoulli numbers were first known to us by Kummer [12] a century ago, but their interpretation in terms of p-adic interpolation of the Riemann zeta function was only discovered in 1964 by Kubota and Leopoldt [11]. In 1910, Frobenius [4] gave a generalization of the Kummer congruence. Vandiver [19] obtained the complementary congruences, which were extended by Carlitz [2] in many directions. Congruences for higher order Bernoulli numbers have been studied by many authors, Adelberg [1], Carlitz [3], Howard [5], etc.

In [13], Osipov's congruences are the generalization of the Kummer congruences for ordinary Bernoulli numbers. He also obtained the Witt's formula of the numbers $H_m(\varepsilon)$, which of the similar kinds are given in [6, 8, 10, 11, 14, 15, 16, 17, 18]. Recently, Kim and Lee [9] obtained some interesting identities related to the Frobenius-Euler polynomials $H_m(x,\varepsilon)$ by using the ordinary fermionic p-adic invariant integral on $\mathbb{Z}_p$.

In this paper we construct a p-adic analogue of k-ple Riemann zeta function and express their values at non-positive integers. Also, we obtain a new congruence of the higher order Frobenius-Euler numbers and the Kummer congruences for the Bernoulli numbers as a corollary.

2. The values of k-ple Riemann zeta function at non-positive integers

Let ε be roots of unity of order relatively prime with p and $\varepsilon \neq 1$. Then the higher order Frobenius-Euler numbers are defined by means of the following generating function

$$g_\varepsilon(t) = \left(\frac{1-\varepsilon}{1-\varepsilon e^t}\right)^k = \sum_{m=0}^{\infty} H_m^{(k)}(\varepsilon)\frac{t^m}{m!}. \tag{2.1}$$

The higher order Frobenius-Euler polynomials are also defined by means of the following generating function

$$g_\varepsilon(x,t) = g_\varepsilon(t)e^{xt} = \sum_{m=0}^{\infty} H_m^{(k)}(x,\varepsilon)\frac{t^m}{m!}. \tag{2.2}$$

Setting $x=0$ in (2.2), $H_m^{(k)}(0,\varepsilon)=H_m^{(k)}(\varepsilon)$. If $k=1$, it is less well known that the explicit representations for the Frobenius-Euler numbers and polynomials, complementing those given in [8, 9, 14]. Setting $\varepsilon=-1$ in (2.2), $H_m^{(k)}(x,-1)=E_m^{(k)}(x)$ are called the higher order Euler polynomials; setting $k=1$ and $\varepsilon=-1$ in (2.2), $H_m^{(1)}(x,-1)=E_m(x)$ are called the classical Euler polynomials.

Let x be a positive real number and let $|\varepsilon|\le 1$. The k-ple Riemann zeta function $\zeta_k(s,x,\varepsilon)$ is defined by

$$\zeta_k(s,x,\varepsilon)=\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{\varepsilon^{n_1+\cdots+n_k}}{(x+n_1+\cdots+n_k)^s}. \tag{2.3}$$

In practice, the k-ple Riemann zeta function $\zeta_k(s,x,\varepsilon)$ for $s=0,-1,-2,\ldots$ are of particular interest. We shall discuss these matters as follows.

The k-ple Riemann zeta function $\zeta_k(s,x,\varepsilon)$ is expressed as an integral,

$$\Gamma(s)\zeta_k(s,x,\varepsilon)=\int_0^\infty\frac{e^{-xt}t^{s-1}}{(1-\varepsilon e^{-t})^k}dt, \tag{2.4}$$

where $\Gamma(s)$ is the gamma function, which satisfies $\Gamma(s+1)=s\Gamma(s),\Gamma(1)=1$, so that, in particular, $\Gamma(m)=(m-1)!$ for positive integers m. Let C denote the contour which starts from $+\infty$, runs on the real axis, encircling the origin once counter-clockwise on the circle of small radius with the center at 0, runs the real axis and returns to $+\infty$. Since

$$\int_C\frac{e^{-xz}z^{s-1}}{(1-\varepsilon e^{-z})^k}dz=(e^{2\pi is}-1)\int_0^\infty\frac{e^{-xt}t^{s-1}}{(1-\varepsilon e^{-t})^k}dt,$$

we have

$$\zeta_k(s,x,\varepsilon)=\frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\int_C\frac{e^{-xz}z^{s-1}}{(1-\varepsilon e^{-z})^k}dz. \tag{2.5}$$

This is the main virtue to obtain a contour integral representation for an analytic function. In particular, we see that $\zeta_k(s,x,\varepsilon)$ can be continued analytically to the whole s-plane (cf. [16, 20]). Furthermore, by (2.2) and (2.4), sufficiently large N we have

$$\begin{aligned}(1-\varepsilon)^k\zeta_k(s,x,\varepsilon)&=\sum_{m=0}^{N}\frac{H_m^{(k)}(x,\varepsilon)}{m!\Gamma(s)}\frac{(-1)^m}{s+m}+\frac{1}{\Gamma(s)}H_N(s)\\&\quad+\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}g_\varepsilon(x,-t)dt,\end{aligned} \tag{2.6}$$

where $H_N(s)$ is entire. For an integer $m\ge 0$, we have

$$(1-\varepsilon)^k\lim_{s\to -m}(s+m)\Gamma(s)\zeta_k(s,x,\varepsilon)=H_m^{(k)}(x,\varepsilon)\frac{(-1)^m}{m!}. \tag{2.7}$$

If $m\ge 0$, we have $\lim_{s\to -m}(s+m)\Gamma(s)=(-1)^m m!$ and thus we obtain the following lemma.

Lemma 2.1. *For $m \geq 0$ and $\varepsilon \neq 1$,*

$$\zeta_k(-m, x, \varepsilon) = \frac{H_m^{(k)}(x, \varepsilon)}{(1-\varepsilon)^k}.$$

Define

$$\tilde{\zeta}_k(s, x, \varepsilon) = \sum_{\substack{n_1, \ldots, n_k = 0 \\ p \nmid (n_1 + \cdots + n_k)}}^{\infty} \frac{\varepsilon^{n_1 + \cdots + n_k}}{(x + n_1 + \cdots + n_k)^s}. \tag{2.8}$$

For the special case of $\tilde{\zeta}_k(s, x, \varepsilon)$, i.e., when $s = 0, -1, -2, \ldots$, it is clear that from (2.3) and (2.8)

$$\begin{aligned}
\tilde{\zeta}_k(-m, x, \varepsilon) &= \zeta_k(-m, x, \varepsilon) - \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \sum_{n_1, \ldots, n_k = 0}^{\infty} \frac{\varepsilon^{|a| + p(n_1 + \cdots + n_k)}}{(x + |a| + p(n_1 + \cdots + n_k))^{-m}} \\
&= \zeta_k(-m, x, \varepsilon) - p^m \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} \zeta_k\left(-m, \frac{x + |a|}{p}, \varepsilon^p\right),
\end{aligned}$$

where $m \geq 0$ and $|a| = a_1 + \cdots + a_k$ (cf. [10]). It follows from this and Lemma 2.1 that

$$\begin{aligned}
&H_m^{(k)}(x, \varepsilon) - p^m \left(\frac{1}{[p]_\varepsilon}\right)^k \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x + |a|}{p}, \varepsilon^p\right) \\
&= (1-\varepsilon)^k \left(\zeta_k(-m, x, \varepsilon) - p^m \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} \zeta_k\left(-m, \frac{x + |a|}{p}, \varepsilon^p\right)\right) \\
&= (1-\varepsilon)^k \tilde{\zeta}_k(-m, x, \varepsilon).
\end{aligned}$$

Lemma 2.2. *Let $m \geq 0$ and $|a| = a_1 + \cdots + a_k$. Then*

$$\tilde{\zeta}_k(-m, x, \varepsilon) = \frac{1}{(1-\varepsilon)^k} \left(H_m^{(k)}(x, \varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1, \ldots, a_k = 0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x + |a|}{p}, \varepsilon^p\right)\right).$$

3. *p*-adic *k*-ple Riemann zeta function and Kummer-type congruences

In this section, let p be an odd prime number. The symbol $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ denote the rings of p-adic integers, the field of p-adic numbers and the field of p-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic absolute value in $\mathbb{C}_p$ is normalized in such way that $|p|_p = 1/p$.

We denote two particular subrings of $\mathbb{C}_p$ in the following manner

$$\mathfrak{o}_p = \{s \in \mathbb{C}_p \mid |s|_p \le 1\}, \quad \mathfrak{m}_p = \{s \in \mathbb{C}_p \mid |s|_p < 1\}.$$

Then $\mathfrak{m}_p$ is a maximal ideal of $\mathfrak{o}_p$. If $s \in \mathbb{C}_p$ such that $|s|_p \le |p|_p^r$, where $r \in \mathbb{Q}$, then $s \in p^r\mathfrak{o}_p$, and so we shall also write this as $s \equiv 0 \pmod{p^r\mathfrak{o}_p}$ (cf. [6, 10, 20]).

Note that the two fields $\mathbb{C}$ and $\mathbb{C}_p$ are algebraically isomorphic, and any one of the two can be embedded in the other.

We begin with the following result.

Lemma 3.1. *Let* $\varepsilon^r = 1, \varepsilon \ne 1$ *and* $(r,p) = 1$. *Then there exists* h *such that* $r \mid (p^h - 1)$, *and*

$$H_0(\varepsilon) = 1, \quad \lim_{n\to\infty} \sum_{a=0}^{p^{hn}-1} a^m \varepsilon^a = H_m(\varepsilon), \quad m \ge 1.$$

Proof. Put $h = \varphi(r)$, where φ is the Euler function. Then $p^{\varphi(r)} \equiv 1 \pmod r$ since $(r,p) = 1$. This gives $p^{\varphi(r)n} \equiv 1 \pmod r, n \ge 0$ and so $r \mid (p^{\varphi(r)n} - 1)$. That is $\varepsilon^{p^{hn}} = \varepsilon$. Thus we have

$$\begin{aligned}\sum_{m=0}^{\infty}\left(\lim_{n\to\infty}\sum_{a=0}^{p^{hn}-1} a^m\varepsilon^a\right)\frac{t^m}{m!} &= \lim_{n\to\infty}\sum_{a=0}^{p^{hn}-1} e^{at}\varepsilon^a = \lim_{n\to\infty}\frac{\varepsilon^{p^{hn}}e^{tp^{hn}}-1}{\varepsilon e^t - 1}\\ &= \frac{\varepsilon-1}{\varepsilon e^t-1} = \sum_{m=0}^{\infty} H_m(\varepsilon)\frac{t^m}{m!},\end{aligned}$$

where $|t|_p < p^{-1/(p-1)}$. The result follows at once. □

Let r be a positive integer prime to p, and $\varepsilon \in \mathbb{C}_p$ a r-th root of unity different from 1. Let $f : \mathbb{Z}_p^k \to \mathbb{C}_p$ be any continuous function and let $a = (a_1, \dots, a_k)$ be a variable on $\mathbb{Z}_p^k$. We define the p-adic integration of f on $\mathbb{Z}_p^k$, if it exists, by the formula

$$\int_{\mathbb{Z}_p^k} f(a)d\mu_\varepsilon(a) = \lim_{\substack{n_1\to\infty\\ \cdots \\ n_k\to\infty}} \sum_{a_1=0}^{p^{hn_1}-1}\cdots\sum_{a_k=0}^{p^{hn_k}-1} f(a_1,\dots,a_k)\varepsilon^{a_1}\cdots\varepsilon^{a_k}, \tag{3.1}$$

where h is a positive integer such that $r \mid (p^h - 1)$ (cf. [13]).

Lemma 3.2. *For integer* $m \ge 0$ *and* $x \in \mathbb{C}_p$,

$$H_m^{(k)}(x,\varepsilon) = \int_{\mathbb{Z}_p^k} (x + |a|)^m d\mu_\varepsilon(a),$$

where $a = (a_1, \dots, a_k) \in \mathbb{Z}_p^k$ *and* $|a| = a_1 + \cdots + a_n$.

Proof. Note that

$$\begin{aligned}\int_{\mathbb{Z}_p^k} e^{t(x+|a|)} d\mu_\varepsilon(a) &= \lim_{\substack{n_1\to\infty \\ \cdots \\ n_k\to\infty}} \sum_{a_1=0}^{p^{hn_1}-1} \cdots \sum_{a_k=0}^{p^{hn_k}-1} e^{t(x+a_1+\cdots+a_k)} \varepsilon^{a_1} \cdots \varepsilon^{a_k} \\ &= e^{tx} \prod_{i=1}^{k} \left(\lim_{n_i\to\infty} \frac{1-\varepsilon^{p^{hn_i}} e^{tp^{hn_i}}}{1-\varepsilon e^t} \right) = g_\varepsilon(x,t)\end{aligned}$$

(cf. [10]). Taking the coefficient of the terms $t^m/m!$ in the above formula, we obtain the lemma. □

Put $|a| = a_1 + \cdots + a_n$. Let $a = (a_1, \ldots, a_k)$ be a variable on $\mathbb{Z}_p^k$ and let $\mathbb{Z}_p^\times$ be the group of p-adic units. It is easy to see that

$$(3.2)\quad \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) = \int_{\mathbb{Z}_p^k} (x+|a|)^m d\mu_\varepsilon(a) - \int_{\substack{\mathbb{Z}_p^k \\ |a|\in p\mathbb{Z}_p}} (x+|a|)^m d\mu_\varepsilon(a)$$

(cf. [10]). We use the notation

$$[n]_\varepsilon = \frac{1-\varepsilon^n}{1-\varepsilon}.$$

Now, we need to compute (3.2). The following lemma deals with the second integral in (3.2).

Lemma 3.3. *For integer $m \geq 0$ and $x \in \mathbb{C}_p$,*

$$\int_{\substack{\mathbb{Z}_p^k \\ |a|\in p\mathbb{Z}_p}} (x+|a|)^m d\mu_\varepsilon(a) = \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. Note that

$$\begin{aligned}&\int_{\substack{\mathbb{Z}_p^k \\ |a|\in p\mathbb{Z}_p}} e^{t(x+|a|)} d\mu_\varepsilon(a) \\ &= e^{tx} \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \sum_{b_1=0}^{p^{hn_1-1}-1} \cdots \sum_{b_k=0}^{p^{hn_k-1}-1} \prod_{i=1}^{k} (\varepsilon e^t)^{a_i+pb_i} \\ &= e^{tx} \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \prod_{i=1}^{k} (\varepsilon a^t)^{a_i} \sum_{b_1=0}^{p^{hn_1-1}-1} \cdots \sum_{b_1=0}^{p^{hn_k-1}-1} \prod_{i=1}^{k} (\varepsilon e^t)^{pb_i} \\ &= \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{a_1+\cdots+a_k} e^{t(x+a_1+\cdots+a_k)} \lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \prod_{i=1}^{k} \left(\frac{1-\varepsilon e^{tp^{hn_i}}}{1-\varepsilon^p e^{tp}} \right)\end{aligned}$$

$$= \frac{1}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{a_1+\cdots+a_k} g_{\varepsilon^p}(tp) e^{t(x+a_1+\cdots+a_k)}.$$

Taking the coefficient of the terms $t^m/m!$ in the above formula, we obtain the lemma. □

Lemma 3.4. *For integer $m \geq 0$ and $x \in \mathbb{C}_p$,*

$$\int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) = H_m^{(k)}(x,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. By (3.2), Lemmas 3.2 and 3.3 we obtain the desired identity. □

Lemma 3.5. *Let $x \in \mathfrak{m}_p$. The function*

$$-m \longmapsto H_m^{(k)}(x,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\ldots,a_k=0 \\ p\nmid|a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(\frac{x+|a|}{p}, \varepsilon^p\right)$$

admits a continuation from the dense subset $\{0, -1, \ldots\} \subset \mathbb{Z}_p$ to a continuous function

$$\zeta_{p,k}(\cdot, x, \varepsilon) : \mathbb{Z}_p \to \mathbb{C}_p$$

and

$$\zeta_{p,k}(s, x, \varepsilon) = \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^{-s} d\mu_\varepsilon(a),$$

where $a = (a_1, \ldots, a_k) \in \mathbb{Z}_p^k$ and $|a| = a_1 + \cdots + a_n$.

Proof. Let $|a| \in \mathbb{Z}_p^\times, x \in \mathfrak{m}_p$ and let $m \equiv m' \pmod{(p-1)p^n}$. It is easy to see that $(x+|a|)^m \equiv (x+|a|)^{m'} \pmod{p^{n+1}\mathfrak{o}_p}$. Therefore we have

$$(3.3) \qquad \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^m d\mu_\varepsilon(a) \equiv \int_{\substack{\mathbb{Z}_p^k \\ |a|\in\mathbb{Z}_p^\times}} (x+|a|)^{m'} d\mu_\varepsilon(a) \pmod{p^{n+1}\mathfrak{o}_p}$$

and they would also belong to a continuous p-adic function on $\mathbb{Z}_p$. The result now follows from Lemma 3.4. □

If $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbb{Z}_p^\times$, $a + pt \equiv a \pmod{p\mathfrak{o}_p}$. Thus we define

$$\omega(a+pt) = \omega(a)$$

for these values of t and the Teichmüller character ω. We also define

$$\langle a+pt \rangle = \omega^{-1}(a)(a+pt)$$

for $t \in \mathbb{C}_p$ such that $|t|_p \le 1$ (cf. [16, 20]). We define a function $\zeta_{p,k}(s,t,\varepsilon)$ on $\mathbb{Z}_p$ by

$$\zeta_{p,k}(s,t,\varepsilon) = \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} \langle |a| + pt \rangle^{-s} d\mu_\varepsilon(a), \tag{3.4}$$

where $|a| = a_1 + \cdots + a_k$ and $t \in \mathbb{C}_p$ such that $|t|_p \le 1$.

Theorem 3.6 (p-adic k-ple Riemann zeta function)**.** *For $t \in \mathbb{C}_p$ such that $|t|_p \le 1$, the function $\zeta_{p,k}(s,t,\varepsilon)$ is analytic on $\mathbb{Z}_p$ and*

$$\zeta_{p,k}(s,t,\varepsilon) = \sum_{n=0}^{\infty} \binom{-s}{n} \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a)$$

holds, which interpolates $(1-\varepsilon)^k \tilde{\zeta}_k(-m, pt, \varepsilon)$ in the sense that

$$\zeta_{p,k}(-m,t,\varepsilon) = H_m^{(k)}(pt,\varepsilon) - p^m \left(\frac{1}{[p]_\varepsilon}\right)^k \sum_{\substack{a_1,\ldots,a_k=0 \\ p \nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(t + \frac{|a|}{p}, \varepsilon^p\right)$$

for integers $m \ge 0$ with $m \equiv 0 \pmod{p-1}$ and $|a| = a_1 + \cdots + a_k$.

Proof. From Lemma 3.5, $\zeta_{p,k}(-s,t,\varepsilon)$ can be written uniquely as the Mahler expansion (cf. [20])

$$\zeta_{p,k}(-s,t,\varepsilon) = \sum_{n=0}^{\infty} a_n \binom{s}{n}, \quad a_n = \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a)$$

and

$$\begin{aligned} |a_n|_p &= \left| \int_{\substack{\mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} (\langle |a| + pt \rangle - 1)^n d\mu_\varepsilon(a) \right|_p \\ &\le \sup_{\substack{a \in \mathbb{Z}_p^k \\ |a| \in \mathbb{Z}_p^\times}} |\langle |a| + pt \rangle - 1|_p^n \\ &= p^{-n} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Note that the coefficients a_n are given by

$$a_n = \Delta^n \zeta_{p,k}(-s,t,\varepsilon)|_{s=0},$$

where $\Delta f(x) = f(x+1) - f(x)$. Moreover we have

$$\frac{1}{n!} a_n \to 0 \quad \text{as} \quad n \to \infty,$$

so that $\zeta_{p,k}(s,t,\varepsilon)$ is analytic. Therefore the result follows from Lemma 2.2 and Lemma 3.5. $\square$

From Lemma 3.4 and (3.3), we also have:

Corollary 3.7. *Let* $m \equiv m' \pmod{p^n(p-1)}$ *and let* $t \in \mathbb{C}_p$ *such that* $|t|_p \le 1$. *Then*

$$\begin{aligned} & H_m^{(k)}(pt,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\dots,a_k=0 \\ p\nmid |a|}}^{p-1} \varepsilon^{|a|} H_m^{(k)}\left(t+\frac{|a|}{p},\varepsilon^p\right) \\ \equiv\ & H_{m'}^{(k)}(pt,\varepsilon) - \frac{p^m}{[p]_\varepsilon^k} \sum_{\substack{a_1,\dots,a_k=0 \\ p\nmid |a|}}^{p-1} \varepsilon^{|a|} H_{m'}^{(k)}\left(t+\frac{|a|}{p},\varepsilon^p\right) \pmod{p^{n+1}\mathfrak{o}_p}. \end{aligned}$$

In particular if $t=0$ and $k=1$, we an rewrite Corollary 3.7 as

$$H_m(\varepsilon) - \frac{p^m}{[p]_\varepsilon} H_m(\varepsilon^p) \equiv H_{m'}(\varepsilon) - \frac{p^{m'}}{[p]_\varepsilon} H_{m'}(\varepsilon^p) \pmod{p^{n+1}\mathfrak{o}_p}, \tag{3.5}$$

which is the same as (23) in [13]. If $\varepsilon=-1$ in (3.5), then we have the following corollary.

Corollary 3.8. *If* $m \equiv m' \pmod{p^n(p-1)}$, *then*

$$(1-p^m)H_m(-1) \equiv (1-p^{m'})H_{m'}(-1) \pmod{p^{n+1}\mathbb{Z}_p}.$$

By (1.5) and Corollary 3.8, it is easy to see that

$$(1-p^m)(1-2^{m+1})\frac{B_{m+1}}{m+1} \equiv (1-p^{m'})(1-2^{m'+1})\frac{B_{m'+1}}{m'+1} \pmod{p^{n+1}\mathbb{Z}_p}. \tag{3.6}$$

If we further assume that $m+1 \not\equiv 0 \pmod{p-1}$, then we have $1/(1-2^{m+1}) \equiv 1/(1-2^{m'+1}) \pmod{p^{n+1}\mathbb{Z}_p}$. Multiplying these two congruences, we obtain the Kummer congruences for the Bernoulli numbers (see [13, 20]):

Corollary 3.9 (Kummer congruences)**.** *If* $m+1 \not\equiv 0 \pmod{p-1}$ *and if* $m \equiv m' \pmod{p^n(p-1)}$, *then*

$$(1-p^m)\frac{B_{m+1}}{m+1} \equiv (1-p^{m'})\frac{B_{m'+1}}{m'+1} \pmod{p^{n+1}\mathbb{Z}_p}.$$

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References

[1] A. Adelberg, *Arithmetic properties of the Norlund polynomial* $B_n^{(x)}$, Discrete Math. **204** (1999), no. 1-3, 5–13.
[2] L. Carlitz, *Some congruences for the Bernoulli numbers*, Amer. J. Math. **75** (1953), 163–172.
[3] ———, *Some properties of the Norlund polynomial* $B_n^{(x)}$, Math. Nachr. **33** (1967), 297–311.
[4] G. Frobenius, *Über die Bernoullischen Zahlen un die Eulerschen Polynome*, Sitz. Preuss. Akad. Wiss. (1910), 809–847.
[5] F. T. Howard, *Congruences and recurrences for Bernoulli numbers of higher order*, Fibonacci Quart. **32** (1994), no. 4, 316–328.

[6] K. Iwasawa, *Lectures on p-adic L-functions*, Annals of Mathematics Studies, No. 74, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972.
[7] M.-S. Kim and J.-W. Son, *On a multidimensional Volkenborn integral and higher order Bernoulli numbers*, Bull. Austral. Math. Soc. **65** (2002), no. 1, 59–71.
[8] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on* $\mathbb{Z}_p$ *at* $q = -1$, J. Math. Anal. Appl. **331** (2007), no. 2, 779–792.
[9] T. Kim and B. Lee, *Some identities of the Frobenius-Euler polynomials*, Abstr. Appl. Anal. **2009** (2009), Art. ID 639439, 7 pp.
[10] N. Koblitz, *p-Adic Analysis: a Short Course on Recent Work*, Cambridge University Press, Mathematical Society Lecture Notes Series 46, 1980.
[11] T. Kubota and H. W. Leopoldt, *Eine p-adische Theorie der Zetawerte. I. Einfuhrung der p-adischen Dirichletschen L-Funktionen*, J. Reine Angew. Math. **214/215** (1964), 328–339.
[12] E. E. Kummer, *Über eine allgemeine Eigenschaft der rationalen Entwickelungs-coëficienten einer bestimmten Gattung analytischer Funktionen*, J. Reine Angew. Math. **41** (1851), 368–372.
[13] Yu. V. Osipov, *p-adic zeta functions and Bernoulli numbers*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **93** (1980), 192–203; English transl. in Journal of Mathematical Sciences **19** (1982), 1186–1194.
[14] K. Shiratani, *On Euler numbers*, Mem. Fac. Sci. Kyushu Univ. Ser. A **27** (1973), 1–5.
[15] Y. Simsek, *on twisted generalized Euler numbers*, Bull. Korean Math. Soc. **41** (2004), no. 2, 299–306.
[16] ———, *Twisted* (h, q)*-Bernoulli numbers and polynomials related to twisted* (h, q)*-zeta function and L-function*, J. Math. Anal. Appl. **324** (2006), no. 2, 790–804.
[17] ———, *q-analogue of twisted l-series and q-twisted Euler numbers*, J. Number Theory **110** (2005), no. 2, 267–278.
[18] Y. Simsek, V. Kurt, and O. Yurekli, *on interpolation functions of the twisted generalized Frobenius-Euler numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **15** (2007), no. 2, 187–194.
[19] H. S. Vandiver, *Certain congruences involving the Bernoulli numbers*, Duke Math. J. **5** (1939), 548–551.
[20] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., Graduate Texts in Mathematics 83, Springer-Verlag, New York, 1997.

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