Thermal entanglement phase transition in coupled harmonic oscillators with arbitrary time-dependent frequencies

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Abstract

We derive explicitly the thermal state of the two-coupled harmonic oscillator system when the spring and coupling constants are arbitrarily time-dependent. In particular, we focus on the case of sudden change of frequencies. In this case we compute purity function, Rényi and von Neumann entropies, and mutual information analytically and examine their temperature dependence. We also discuss on the thermal entanglement phase transition by making use of the negativity-like quantity. Our calculation shows that the critical temperature T_c increases with increasing the difference between the initial and final frequencies. In this way we can protect the entanglement against the external temperature by introducing large difference of initial and final frequencies.

Keywords Two-coupled harmonic oscillator system · Thermal entanglement phase transition · Arbitrary time-dependent frequencies

1 Introduction

Entanglement [1–3] is a key physical resource in quantum information processing. For example, it plays crucial role in quantum teleportation [4], superdense coding [5], quantum cloning [6], quantum cryptography [7,8], quantum metrology [9], and quantum computer [10,11]. In particular, physical realization of quantum cryptography and quantum computer seems to be accomplished in the near future.[1]

   Although entanglement is highly useful property of quantum state, it is normally fragile when quantum system interacts with its surroundings. Interaction with the

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[1] See Ref. [12] and web page https://www.computing.co.uk/ctg/news/3065541/european-union-reveals-test-projects-for-first-tranche-of-eur1bn-quantum-computing-fund.

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environments makes the given quantum system undergo decoherence [13], and as a result, it loses its quantum properties. Thus, decoherence significantly changes the quantum entanglement. Sometimes entanglement exhibits an exponential decay in time by successive halves. Sometimes, however, entanglement sudden death (ESD) occurs when the entangled multipartite quantum system is embedded in Markovian environments [14–19]. This means that the entanglement is completely disentangled at finite times.

Most typical surrounding is external temperature. At finite temperature quantum mechanics the external temperature is introduced via imaginary time at zero temperature quantum mechanics. Thus, the exponential decay or ESD-like phenomenon can occur in external temperature. If external temperature induces the ESD-like phenomenon in temperature, this means there exists a critical temperature T_c, below or above which the entanglement of a system is nonzero or completely zero. We will call this phenomenon thermal entanglement phase transition (TEPT) between nonzero entanglement phase and zero entanglement phase. The TEPT and the critical temperature T_c were explored [20] recently by making use of concurrence [21,22] in anisotropic Heisenberg XYZ spin model with Dzyaloshinskii–Moriya interaction [23,24].

The purpose of this paper is to study on the TEPT phenomenon in continuous variable system. Most simple continuous variable system seems to be two-coupled harmonic oscillator system. In this reason we will choose this system to explore the TEPT when the spring constant k_0 and coupling constant J are arbitrarily time-dependent. Another reason we choose this system is because of the fact that the thermal state of this system is Gaussian. It is known that the Peres–Horodecki positive partial transposition (PPT) criterion [25–27] provides a necessary and sufficient condition for separability of Gaussian continuous variable states [28,29]. Thus, the temperature dependence of entanglement can be roughly deduced by considering the negativity-like quantity [30]. What we are interested in is to examine how the arbitrarily time-dependent parameters affect the critical temperature. In particular, we focus in this paper on the sudden quenched model, where the system parameters abruptly change at $t = 0$.

The paper is organized as follows. In Sect. 2 we derive the thermal state of single harmonic oscillator system when the frequency is arbitrarily time-dependent. We focus on the case of sudden quenched model (SQM). For SQM we derive the purity function and von Neumann entropy of the thermal state analytically. In Sect. 3 we derive explicitly the thermal state of two-coupled harmonic oscillator system when the spring constant k_0 and coupling constant J are arbitrarily time-dependent. In Sect. 4 we compute the purity function, Rényi and von Neumann entropies, and mutual information analytically for the thermal state of two-coupled harmonic oscillator system in the case of SQM. It is shown that the thermal state is less mixed with increasing the difference between initial and final frequencies at the given external temperature. The mutual information shows that the common information parties A and B share does not completely vanish even in the infinity temperature limit. In Sect. 5 the TEPT is discussed for the case of SQM by making use of the negativity-like quantity. It is shown that the critical temperature T_c increases with increasing the frequency difference. Thus, using SQM with large difference of initial and final frequencies it seems to be possible to protect entanglement against external temperature. In Sect. 6 a brief

conclusion is given. In appendix A the eigenvalue equation for the thermal state of the coupled harmonic oscillator system is explicitly solved.

2 Thermal state for single harmonic oscillator with arbitrary time-dependent frequency

Let us consider a single harmonic oscillator with time-dependent frequency, whose Hamiltonian is

$$H_1 = \frac{1}{2}p^2 + \frac{1}{2}\omega^2(t)x^2. \tag{2.1}$$

Then, the action function of this system is given by

$$S[x] = \int_0^t \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2(t)x^2 \right]. \tag{2.2}$$

Usually Kernel for any quantum system can be derived by computing the path integral [31]

$$K[x', x : t] = \int_{(0,x)}^{(t,x')} \mathcal{D}x e^{iS[x]}. \tag{2.3}$$

Although the path integral with constant frequency can be computed [31,32], it does not seem to be simple matter to compute the path integral explicitly when ω is arbitrary time-dependent. However, it is possible to derive the Kernel without computing the path integral if one uses the Schrödinger description of Kernel

$$K[x', t_2 : x, t_1] = \sum_n \psi_n \left(x', t_2 \right) \psi_n^* \left(x, t_1 \right) \tag{2.4}$$

where n represents all possible quantum numbers and $\psi_n \left(x, t \right)$ is linearly independent solution of time-dependent Schrödinger equation (TDSE).

The TDSE of our system was exactly solved in Ref. [33,34]. The linearly independent solutions $\psi_n(x, t)$ $(n = 0, 1, \cdots)$ are expressed in a form

$$\psi_n(x, t) = e^{-iE_n\tau(t)} e^{\frac{i}{2}\left(\frac{\dot{b}}{b}\right)x^2} \phi_n \left(\frac{x}{b} \right) \tag{2.5}$$

where

$$E_n = \left(n + \frac{1}{2} \right) \omega(0), \qquad \tau(t) = \int_0^t \frac{ds}{b^2(s)}$$

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\omega(0)}{\pi b^2} \right)^{1/4} H_n \left(\sqrt{\omega(0)}x \right) e^{-\frac{\omega(0)}{2}x^2}. \tag{2.6}$$

In Eq. (2.6), $H_n(z)$ is nth-order Hermite polynomial and $b(t)$ satisfies the Ermakov equation

$$\ddot{b} + \omega^2(t)b = \frac{\omega^2(0)}{b^3} \tag{2.7}$$

with $b(0) = 1$ and $\dot{b}(0) = 0$.

Solutions of the Ermakov equation were discussed in Ref. [35]. If $\omega(t)$ is time-independent, $b(t)$ is simply one. If $\omega(t)$ is instantly changed as

$$\omega(t) = \begin{cases} \omega_0 & t = 0 \\ \omega & t > 0, \end{cases} \tag{2.8}$$

then $b(t)$ becomes

$$b(t) = \sqrt{\frac{\omega^2 - \omega_0^2}{2\omega^2} \cos(2\omega t) + \frac{\omega^2 + \omega_0^2}{2\omega^2}}. \tag{2.9}$$

For more general time-dependent case the Ermakov equation should be solved numerically.

Inserting Eq. (2.5) into Eq. (2.4) and using

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) H_n(y) = (1 - 4t^2)^{-1/2} \exp\left[\frac{4txy - 4t^2(x^2 + y^2)}{1 - 4t^2}\right], \tag{2.10}$$

Kernel for this system becomes

$$K[x', x : t] = e^{\frac{i}{2}\left(\frac{\dot{b}}{b}\right)x'^2} \frac{(\omega_0 \omega')^{1/4}}{\sqrt{2\pi i \sin \Gamma(t)}}$$
$$\exp\left[\frac{i}{2 \sin \Gamma(t)} \left\{(\omega_0 x^2 + \omega' x'^2) \cos \Gamma(t) - 2\sqrt{\omega_0 \omega'} xx'\right\}\right] \tag{2.11}$$

where

$$\omega_0 = \omega(t = 0), \qquad \omega'(t) = \frac{\omega_0}{b^2(t)}, \qquad \Gamma(t) = \int_0^t \omega'(s)ds. \tag{2.12}$$

For time-independent case $b(t) = 1$, $\omega_0 = \omega' = \omega$, and $\Gamma(t) = \omega t$. Then, the Kernel in Eq. (2.11) reduces to usual well-known harmonic oscillator Kernel

$$K[x', x : t] = \sqrt{\frac{\omega}{2\pi i \sin \omega t}} \exp\left[\frac{i\omega}{2 \sin \omega t} \left\{(x^2 + x'^2) \cos \omega t - 2xx'\right\}\right]. \tag{2.13}$$

It is remarkable to note that the $x \leftrightarrow x'$ symmetry in Eq. (2.13) is broken in Eq. (2.11) due to the time-dependence of frequency. In fact, it is manifest due to the fact that the system parameters at $t = 0$ are different from those at $t > 0$.

From now on in this section we consider only the case of SQM given in Eq. (2.8). In this case the Kernel becomes

$$K[x', x : t] = e^{-\left(\frac{i(\omega^2 - \omega_0^2)}{4\omega b^2} \sin 2\omega t\right)x'^2} \sqrt{\frac{\omega_0}{2\pi i b \sin \Gamma(t)}}$$
$$\exp\left[\frac{i\omega_0}{2\sin\Gamma(t)}\left\{\left(x^2 + \frac{x'^2}{b^2}\right)\cos\Gamma(t) - \frac{2xx'}{b}\right\}\right] \quad (2.14)$$

where $b(t)$ is given in Eq. (2.9) and $\Gamma(t)$ becomes

$$\Gamma(t) = \tan^{-1}\left(\frac{\omega_0}{\omega}\tan\omega t\right) = \frac{1}{2i}\ln\frac{\omega + i\omega_0 \tan\omega t}{\omega - i\omega_0 \tan\omega t}. \quad (2.15)$$

In quantum mechanics the inverse temperature $\beta = 1/k_B T$ is introduced as a Euclidean time $\beta = it$ (see Ch. 10 of Ref. [31]), where k_B is a Boltzmann constant. Then, the thermal density matrix is defined as

$$\rho_T[x', x : \beta] = \frac{1}{\mathcal{Z}(\beta)}G[x', x : \beta] \quad (2.16)$$

where $\beta = 1/k_B T$, $G[x', x : \beta] = K[x', x : -i\beta]$, and $\mathcal{Z}(\beta) = \text{tr}G[x', x : \beta]$ is a partition function. Throughout this paper we use $k_B = 1$ for convenience. For SQM case $b(t)$ and $\Gamma(t)$ are changed into $b(\beta)$ and $\Gamma(\beta)$, whose explicit expressions are

$$b(\beta) = \sqrt{\frac{\omega^2 - \omega_0^2}{2\omega^2}\cosh(2\omega\beta) + \frac{\omega^2 + \omega_0^2}{2\omega^2}}, \quad \Gamma(\beta) = -i\Gamma_0(\beta) \quad (2.17)$$

where[2]

$$\Gamma_0(\beta) = \frac{1}{2}\ln\left(\frac{\omega + \omega_0 \tanh\omega\beta}{\omega - \omega_0 \tanh\omega\beta}\right). \quad (2.18)$$

Then, the partition function of this system becomes

$$\mathcal{Z}(\beta) = \sqrt{\frac{\omega_0}{2\pi b \sinh\Gamma_0}}\sqrt{\frac{\pi}{a_{0,-}}} \quad (2.19)$$

where

$$a_{0,\pm}(\beta) = A_0(\beta) + \frac{\omega_0}{2\sinh\Gamma_0}\left[\left(1 + \frac{1}{b^2}\right)\cosh\Gamma_0 \pm \frac{2}{b}\right] \quad (2.20)$$

with

$$A_0(\beta) = \frac{\omega^2 - \omega_0^2}{4\omega b^2}\sinh(2\omega\beta). \quad (2.21)$$

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[2] In fact, one can show that $b(\beta)$ in Eq. (2.17) is a solution of $\frac{d^2b}{d\beta^2} - \omega^2 b = -\frac{\omega_0^2}{b^3}$.

Using the partition function one can derive the thermal density matrix in a form

$$\rho_0[x', x : \beta] = \sqrt{\frac{a_{0,-}}{\pi}} e^{-A_0(\beta)x'^2} \exp\left[-\frac{\omega_0}{2\sinh\Gamma_0}\left\{\left(x^2 + \frac{x'^2}{b^2}\right)\cosh\Gamma_0 - \frac{2xx'}{b}\right\}\right].$$
(2.22)

The thermal density matrix is in general mixed state. In order to explore how much it is mixed we first compute the purity function $P_0(\beta) = \text{tr}(\rho_0)^2$. If it is one, this means that ρ_0 is pure state. If it is zero, this means ρ_0 is completely mixed state. If $0 < P_0(\beta) < 1$, this means that ρ_0 is partially mixed state. It is not difficult to show that the purity function of this system is

$$P_0(\beta) \equiv \int dx dx' \rho_0[x', x : \beta]\rho_0[x, x' : \beta] = \sqrt{\frac{a_{0,-}}{a_{0,+}}}.$$
(2.23)

Another quantity we want to compute is a von Neumann entropy $S[\rho_0]$ of ρ_0. If ρ_0 is pure state, $S[\rho_0]$ is zero. If its mixedness increases, $S[\rho_0]$ also increases from zero and eventually goes to infinity for completely mixed state in this continuum case. In order to compute the von Neumann entropy we should solve the eigenvalue equation

$$\int dx \rho_0[x', x : \beta] f_n(x) = \lambda_n(\beta) f_n(x').$$
(2.24)

One can show that the eigenvalue equation

$$\int dx \left(A e^{-a_1 x^2 - a_2 x'^2 + 2bxx'}\right) f_n(x) = \lambda_n f_n(x')$$
(2.25)

can be solved, and the eigenfunction and corresponding eigenvalue are

$$f_n(x) = C_n^{-1} H_n(\sqrt{\epsilon_0}x) e^{-\frac{\alpha_0}{2}x^2}$$
$$\lambda_n = A\sqrt{\frac{2\pi}{(a_1 + a_2) + \epsilon_0}}\left[\frac{(a_1 + a_2) - \epsilon_0}{(a_1 + a_2) + \epsilon_0}\right]^{n/2}$$
(2.26)

where $\epsilon_0 = \sqrt{(a_1 + a_2)^2 - 4b^2}$ and $\alpha_0 = \epsilon_0 - (a_1 - a_2)$. By making use of integral formula [36]

$$\int_{-\infty}^{\infty} e^{-(x-y)^2} H_m(cx) H_n(cx) dx$$
$$= \sqrt{\pi} \sum_{k=0}^{\min(m,n)} 2^k k! \binom{m}{k}\binom{n}{k}(1 - c^2)^{\frac{m+n}{2}-k} H_{m+n-2k}\left(\frac{cy}{\sqrt{1-c^2}}\right)$$
(2.27)

and various properties of gamma function [37], the normalization constant $\mathcal{C}_n$ can be written in a form

$$\mathcal{C}_n^2 = \frac{1}{\sqrt{\alpha_0}} \sum_{k=0}^{n} 2^{2n-k} \left(\frac{\epsilon_0}{\alpha_0} - 1\right)^{n-k} \frac{\Gamma^2(n+1)\Gamma\left(n-k+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma^2(n-k+1)}. \tag{2.28}$$

If $a_1 = a_2$, $\alpha_0 = \epsilon_0$, which makes nonzero in k-summation of Eq. (2.28) only when $k = n$. Then, $\mathcal{C}_n$ becomes usual harmonic oscillator normalization constant

$$\mathcal{C}_n^{-1} = \frac{1}{\sqrt{2^n n!}} \left(\frac{\epsilon_0}{\pi}\right)^{1/4}. \tag{2.29}$$

Using Eqs. (2.25) and (2.26) the eigenvalue of Eq. (2.24) becomes

$$\lambda_n(\beta) = (1 - \xi_0)\xi_0^n \tag{2.30}$$

where

$$\xi_0 = \frac{\sqrt{a_{0,+}} - \sqrt{a_{0,-}}}{\sqrt{a_{0,+}} + \sqrt{a_{0,-}}} = \frac{1 - P_0(\beta)}{1 + P_0(\beta)}. \tag{2.31}$$

Thus, the spectral decomposition of ρ_0 can be written as

$$\rho_0[x', x : \beta] = \sum_n \lambda_n(\beta) f_n(x' : \beta) f_n^*(x : \beta), \tag{2.32}$$

where $f_n(x : \beta)$ is given by Eq. (2.26) with $\epsilon_0 = \sqrt{a_{0,+} a_{0,-}}$ and $\alpha_0 = \epsilon_0 + A_0(\beta) - \frac{\omega_0 \cosh \Gamma_0}{2 \sinh \Gamma_0}\left(1 - \frac{1}{b^2}\right)$. Equation (2.30) implies $\sum_n \lambda_n(\beta) = 1$, which is consistent with $\mathrm{tr}\rho_0 = 1$. Then the von Neumann entropy of ρ_0 becomes

$$S[\rho_0] \equiv -\sum_n \lambda_n(\beta) \ln \lambda_n(\beta) = -\ln(1 - \xi_0) - \frac{\xi_0}{1 - \xi_0} \ln \xi_0. \tag{2.33}$$

For constant frequency, *i.e.*, $\omega = \omega_0$, $A_0 = 0$, $a_{0,+} = \omega \coth \frac{\omega\beta}{2}$, $a_{0,-} = \omega \tanh \frac{\omega\beta}{2}$, and $\xi_0 = e^{-\omega\beta}$. For the case of SQM A_0 and $a_{0,\pm}$ become larger than those in constant frequency case in the entire range of temperature. As a result, $P_0(\beta)$ and ξ_0 become larger and smaller compared to the constant frequency case. Since $-\ln(1-x) - \frac{x}{1-x} \ln x$ is monotonically increasing function in the range $0 \leq x \leq 1$, this fact decreases the von Neumann entropy.

The temperature dependence of the purity function and von Neumann entropy is plotted in Fig. 1a, b when $\omega = 3$ (black line), 5 (red line), 7 (blue line) with $\omega_0 = 3$. All figures show that ρ_0 becomes more and more mixed with increasing the external temperature. Both figures also show that ρ_0 becomes less mixed with increasing $|\omega - \omega_0|$ at the given temperature. Thus, we can use SQM model to protect the purity against the external temperature.
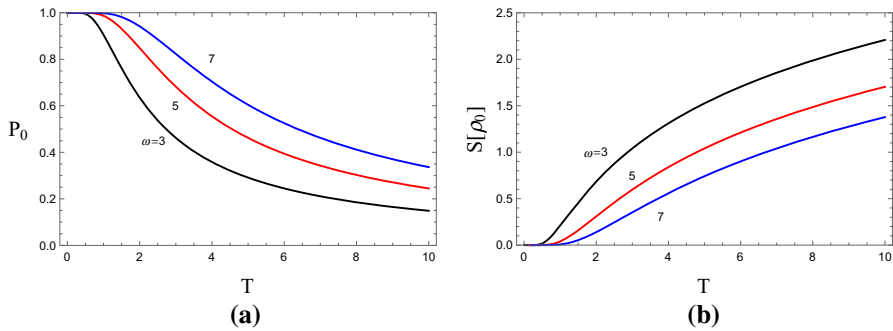
Fig. 1 (Color online) The T-dependence of **a** purity function and **b** von Neumann entropy when $\omega = 3$ (black line), 5 (red line), 7 (blue line) with $\omega_0 = 3$. Both figures show that the mixedness of thermal state (2.22) decreases with increasing $|\omega - \omega_0|$ at the given temperature. All figures show that ρ_0 becomes more and more mixed with increasing the external temperature

3 Thermal state for two-coupled harmonic oscillators with arbitrary time-dependent frequencies

In this section we will derive the thermal state for two-coupled harmonic oscillator system, whose Hamiltonian is

$$H_2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2). \tag{3.1}$$

We choose the potential $V(x_1, x_2)$ as a quadratic function with arbitrary time-dependent spring and coupling parameters. The explicit expression of the potential is chosen in a form

$$V(x_1, x_2) = \frac{1}{2}\left[k_0(t)(x_1^2 + x_2^2) + J(t)(x_1 - x_2)^2\right]. \tag{3.2}$$

Then, the action function of this system is

$$S[x_1, x_2] = \int_0^t dt \left[\frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) - V(x_1, x_2)\right]. \tag{3.3}$$

It is easy to show that the potential is diagonalized by introducing $y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$ and $y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$. In terms of new canonical variables the action becomes that of two noninteracting harmonic oscillators in a form

$$S[x_1, x_2] = \int_0^t dt \left[\frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2\right) + \frac{1}{2}\left\{\omega_1^2(t)y_1^2 + \omega_2^2(t)y_2^2\right\}\right] \tag{3.4}$$

where $\omega_1(t) = \sqrt{k_0(t)}$ and $\omega_2(t) = \sqrt{k_0(t) + 2J(t)}$. Thus, the Kernel for this system is

$$K[x_1', x_2' : x_1, x_2 : t]$$

$$= \prod_{j=1}^{2} \left[e^{\frac{i}{2}\left(\frac{\dot{b}_j}{b_j}\right)y_j'^2} \frac{\left(\omega_{j,0}\omega_j'\right)^{1/4}}{\sqrt{2\pi i \sin \Gamma_j(t)}} \right.$$

$$\left. \exp\left[\frac{i}{2 \sin \Gamma_j(t)} \left\{ (\omega_{j,0}y_j^2 + \omega_j'y_j'^2)\cos \Gamma_j(t) - 2\sqrt{\omega_{j,0}\omega_j'}y_j y_j' \right\} \right] \right]$$

(3.5)

where $\omega_{j,0} = \omega_j(t = 0)$, $\omega_j' = \frac{\omega_{j,0}}{b_j^2(t)}$, and $\Gamma_j(t) = \int_0^t \omega_j'(s)ds$. Of course, $b_1(t)$ and $b_2(t)$ satisfy the Ermakov equation

$$\ddot{b}_j + \omega_j^2(t)b_j = \frac{\omega_{j,0}}{b_j^3} \qquad (j = 1, 2)$$

(3.6)

with $\dot{b}_j(0) = 0$ and $b_j(0) = 1$. Then the thermal density matrix of this system is given by

$$\rho_T[x_1', x_2' : x_1, x_2 : \beta] = \frac{1}{\mathcal{Z}(\beta)} K[x_1', x_2' : x_1, x_2 : -i\beta]$$

(3.7)

where $\beta = 1/k_B T$ and $\mathcal{Z}(\beta) = \mathrm{tr}K[x_1', x_2' : x_1, x_2 : -i\beta]$.

In this paper we will examine only the case of SQM. More general time-dependent cases will be explored elsewhere. If spring and coupling constants are abruptly changed as

$$k_0 = \begin{cases} k_{0,i} & t = 0 \\ k_{0,f} & t > 0 \end{cases} \qquad J = \begin{cases} J_i & t = 0 \\ J_f & t > 0, \end{cases}$$

(3.8)

ω_1 and ω_2 become

$$\omega_1 = \begin{cases} \omega_{1,0} \equiv \omega_{1,i} = \sqrt{k_{0,i}} & t = 0 \\ \omega_{1,f} = \sqrt{k_{0,f}} & t > 0, \end{cases} \qquad \omega_2 = \begin{cases} \omega_{2,0} \equiv \omega_{2,i} = \sqrt{k_{0,i} + 2J_i} & t = 0 \\ \omega_{2,f} = \sqrt{k_{0,f} + 2J_f} & t > 0. \end{cases}$$

(3.9)

Then, the thermal density matrix of this system is given by

$$\rho_T[x_1', x_2' : x_1, x_2 : \beta]$$

$$= \prod_{j=1}^{2} \sqrt{\frac{a_{j,-}}{\pi}} \exp\left[-A_j y_j'^2 - \frac{\omega_{j,i}}{2 \sinh \Gamma_{E,j}} \left\{ \left(y_j^2 + \frac{y_j'^2}{b_j^2} \right)\cosh \Gamma_{E,j} - \frac{2y_j y_j'}{b_j} \right\} \right]$$

(3.10)

where[3]

$$b_j = \sqrt{\frac{\omega_{j,f}^2 - \omega_{j,i}^2}{2\omega_{j,f}^2} \cosh(2\omega_{j,f}\beta) + \frac{\omega_{j,f}^2 + \omega_{j,i}^2}{2\omega_{j,f}^2}},$$

$$\Gamma_{E,j} = \frac{1}{2} \ln \frac{\omega_{j,f} + \omega_{j,i} \tanh(\omega_{j,f}\beta)}{\omega_{j,f} - \omega_{j,i} \tanh(\omega_{j,f}\beta)}$$

$$A_j = \frac{\omega_{j,f}^2 - \omega_{j,i}^2}{2\omega_{j,f} b_j^2} \sinh(2\omega_{j,f}\beta),$$

$$a_{j,\pm} = A_j + \frac{\omega_{j,i}}{2\sinh\Gamma_{E,j}} \left[\left(1 + \frac{1}{b_j^2}\right) \cosh\Gamma_{E,j} \pm \frac{2}{b_j} \right] \qquad (3.11)$$

with $j = 1, 2$. For the limit of $\omega_{j,i} = \omega_{j,f} \equiv \omega_j$, we have $A_j = 0$, $b_j = 1$, $\Gamma_{E,j} = \omega_j\beta$, $a_{j,+} = \omega_j \coth(\omega_j\beta/2)$, and $a_{j,-} = \omega_j \tanh(\omega_j\beta/2)$. In terms of x_j-coordinates the thermal state reduces to

$$\rho_T[x_1', x_2' : x_1, x_2 : \beta]$$
$$= \frac{\sqrt{a_{1,-}a_{2,-}}}{\pi} \exp\left[-\alpha_1(x_1'^2 + x_2'^2) - \alpha_2(x_1^2 + x_2^2) \right.$$
$$\left. + 2\alpha_3 x_1' x_2' + 2\alpha_4 x_1 x_2 + 2\alpha_5(x_1 x_1' + x_2 x_2') + 2\alpha_6(x_1 x_2' + x_2 x_1') \right]$$
$$(3.12)$$

where

$$\alpha_1 = \sum_{j=1}^{2} \left[\frac{A_j}{2} + \frac{\omega_{j,i}\cosh\Gamma_{E,j}}{4b_j^2 \sinh\Gamma_{E,j}} \right], \quad \alpha_2 = \sum_{j=1}^{2} \frac{\omega_{j,i}\cosh\Gamma_{E,j}}{4\sinh\Gamma_{E,j}}$$

$$\alpha_3 = \sum_{j=1}^{2}(-1)^j \left[\frac{A_j}{2} + \frac{\omega_{j,i}\cosh\Gamma_{E,j}}{4b_j^2 \sinh\Gamma_{E,j}} \right], \quad \alpha_4 = \sum_{j=1}^{2}(-1)^j \frac{\omega_{j,i}\cosh\Gamma_{E,j}}{4\sinh\Gamma_{E,j}}$$

$$\alpha_5 = \sum_{j=1}^{2} \frac{\omega_{j,i}}{4b_j \sinh\Gamma_{E,j}}, \quad \alpha_6 = \sum_{j=1}^{2}(-1)^{j-1} \frac{\omega_{j,i}}{4b_j \sinh\Gamma_{E,j}}. \qquad (3.13)$$

It is worthwhile noting that α_j satisfy

$$\alpha_1 + \alpha_2 = \frac{(a_{1,+} + a_{1,-}) + (a_{2,+} + a_{2,-})}{4}$$
$$\alpha_3 + \alpha_4 = -\frac{(a_{1,+} + a_{1,-}) - (a_{2,+} + a_{2,-})}{4}$$

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[3] The subscript E in $\Gamma_{E,j}$ stands for "Euclidean." This subscript is attached to stress the point that the inverse temperature β is introduced as a Euclidean time.

$$\alpha_5 = \frac{1}{8} \left[(a_{1,+} - a_{1,-}) + (a_{2,+} - a_{2,-}) \right], \quad \alpha_6 = \frac{1}{8} \left[(a_{1,+} - a_{1,-}) - (a_{2,+} - a_{2,-}) \right].$$

$$(3.14)$$

Using Eq. (3.14) it is straightforward to show $\mathrm{tr}\rho_T = 1$. In the next section we compute several quantum information quantities analytically, which measure how much ρ_T is mixed.

4 Various quantities of thermal state: case of SQM

In this section we will compute purity function, Rényi and von Neumann entropies, and mutual information of $\rho_T[x'_1, x'_2 : x_1, x_2 : \beta]$ given in Eq. (3.10) or equivalently Eq. (3.12). As a by-product we derive the spectral decomposition of $\rho_T[x'_1, x'_2 : x_1, x_2 : \beta]$.

4.1 Purity function

The purity function is defined as

$$P(\beta) = \mathrm{tr}\rho_T^2 \equiv \int dx'_1 dx'_2 dx_1 dx_2 \rho_T[x'_1, x'_2 : x_1, x_2 : \beta]\rho_T[x_1, x_2 : x'_1, x'_2 : \beta].$$

$$(4.1)$$

If $P(\beta) = 1$ or 0, this means that ρ_T is pure or completely mixed state. Direct calculation shows

$$P(\beta) = \sqrt{\frac{a_{1,-}a_{2,-}}{a_{1,+}a_{2,+}}}.$$

$$(4.2)$$

For the case of constant frequencies $\omega_{j,i} = \omega_{j,f} \equiv \omega_j$ it reduces to

$$P(\beta) = \tanh\frac{\omega_1\beta}{2} \tanh\frac{\omega_2\beta}{2}.$$

$$(4.3)$$

The temperature dependence of the purity function is plotted in Fig. 2a when $k_{0,f} = J_f = 6$ (red line) and $k_{0,f} = J_f = 9$ (blue line). The $k_{0,i}$ and J_i are fixed as $k_{0,i} = J_i = 3$. The black dashed line corresponds to constant frequencies $k_0 = J = 3$. As expected ρ_T becomes more and more mixed with increasing temperature. Figure 2a also shows that ρ_T is less mixed when $|k_{0,f} - k_{0,i}|$ and $|J_f - J_i|$ increase.

4.2 Rényi and von Neumann entropies

In order to solve the Rényi and von Neumann entropies of ρ_T we should solve the eigenvalue equation

$$\int dx_1 dx_2 \rho_T[x'_1, x'_2 : x_1, x_2 : \beta]u_{mn}(x_1, x_2 : \beta) = p_{mn}(\beta)u_{mn}(x'_1, x'_2 : \beta). \quad (4.4)$$
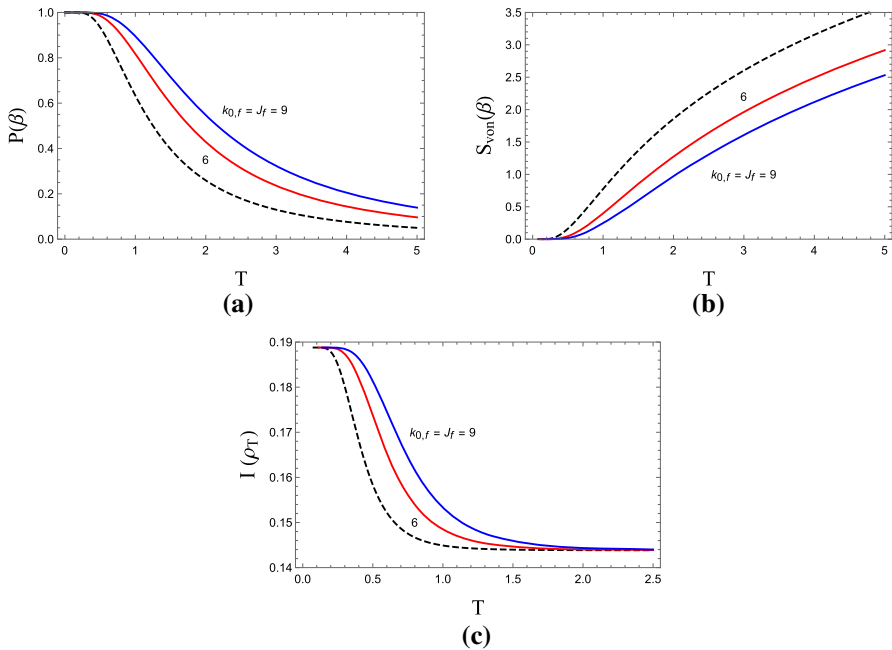
Fig. 2 (Color online) The T-dependence of **a** $P(\beta)$ **b** S_{von}, and **c** $I(\rho_T)$ when $k_{0,f} = J_f = 6$ (red line) and $k_{0,f} = J_f = 9$ (blue line) when $k_{0,i} = J_i = 3$. The black dashed line corresponds to constant frequencies $k_{0,i} = J_i = k_{0,f} = J_f = 3$

Equation (4.4) is solved in appendix A, and the eigenvalue $p_{mn}(\beta)$ is

$$p_{mn}(\beta) = (1 - \xi_1)\xi_1^m (1 - \xi_2)\xi_2^n \tag{4.5}$$

where

$$\xi_1 = \frac{\sqrt{a_{1,+}} - \sqrt{a_{1,-}}}{\sqrt{a_{1,+}} + \sqrt{a_{1,-}}}, \qquad \xi_2 = \frac{\sqrt{a_{2,+}} - \sqrt{a_{2,-}}}{\sqrt{a_{2,+}} + \sqrt{a_{2,-}}}. \tag{4.6}$$

In terms of ξ_1 and ξ_2 the purity function $P(\beta)$ in Eq. (4.2) can be written as

$$P(\beta) = \frac{1 - \xi_1}{1 + \xi_1} \frac{1 - \xi_2}{1 + \xi_2}. \tag{4.7}$$

Then, the Rényi and von Neumann entropies of ρ_T reduce to

$$S_\alpha = S_{1,\alpha} + S_{2,\alpha} \qquad S_{von} = S_{1,von} + S_{2,von} \tag{4.8}$$

where

$$S_{j,\alpha} = \frac{1}{1 - \alpha} \ln \frac{(1 - \xi_j)^\alpha}{1 - \xi_j}, \qquad S_{j,von} = -\ln(1 - \xi_j) - \frac{\xi_j}{1 - \xi_j} \ln \xi_j \tag{4.9}$$

with $j = 1, 2$.

One can show also that the normalized eigenfunction $u_{mn}(x_1, x_2 : \beta)$ is

$$u_{mn}(x_1, x_2 : \beta) = \left(\frac{1}{\mathcal{C}_{1,m}} H_m(\sqrt{\epsilon_1} y_1) e^{-\mu_1 y_1^2} \right) \left(\frac{1}{\mathcal{C}_{2,n}} H_n(\sqrt{\epsilon_2} y_2) e^{-\mu_2 y_2^2} \right) \quad (4.10)$$

where

$$\epsilon_1 = \sqrt{a_{1,+} a_{1,-}} \qquad \epsilon_2 = \sqrt{a_{2,+} a_{2,-}}$$

$$\mu_1 = \frac{1}{2} [\epsilon_1 + (\alpha_1 - \alpha_2) - (\alpha_3 - \alpha_4)], \quad \mu_2 = \frac{1}{2} [\epsilon_2 + (\alpha_1 - \alpha_2) + (\alpha_3 - \alpha_4)]$$

$$\mathcal{C}_{1,m}^2 = \frac{1}{\sqrt{2\mu_1}} \sum_{k=0}^{m} 2^{2m-k} \left(\frac{\epsilon_1}{2\mu_1} - 1 \right)^{m-k} \frac{\Gamma^2(m+1)\Gamma(m-k+1/2)}{\Gamma(k+1)\Gamma^2(m-k+1)}$$

$$\mathcal{C}_{2,n}^2 = \frac{1}{\sqrt{2\mu_2}} \sum_{k=0}^{n} 2^{2n-k} \left(\frac{\epsilon_2}{2\mu_2} - 1 \right)^{n-k} \frac{\Gamma^2(n+1)\Gamma(n-k+1/2)}{\Gamma(k+1)\Gamma^2(n-k+1)}. \quad (4.11)$$

For the case of constant frequencies $\alpha_1 - \alpha_2 = \alpha_3 - \alpha_4 = 0$, which results in $\mu_j = \frac{\epsilon_j}{2}$. In this case the sum in $\mathcal{C}_{1,m}^2$ or $\mathcal{C}_{2,n}^2$ is nonzero only when $k = m$ or $k = n$, and this fact yields well-known quantities $\mathcal{C}_{1,m}^{-1} = \frac{1}{\sqrt{2^m m!}} \left(\frac{\epsilon_1}{\pi} \right)^{1/4}$ and $\mathcal{C}_{2,n}^{-1} = \frac{1}{\sqrt{2^n n!}} \left(\frac{\epsilon_2}{\pi} \right)^{1/4}$. Thus, the spectral decomposition of ρ_T can be written as

$$\rho_T[x_1', x_2' : x_1, x_1 : \beta] = \sum_{m,n} p_{mn}(\beta) u_{mn}(x_1', x_2' : \beta) u_{mn}^*(x_1, x_2 : \beta). \quad (4.12)$$

The temperature dependence of the von Neumann entropy is plotted in Fig. 2b when $k_{0,f} = J_f = 6$ (red line) and $k_{0,f} = J_f = 9$ (blue line). The $k_{0,i}$ and J_i are fixed as $k_{0,i} = J_i = 3$. The black dashed line corresponds to constant frequencies $k_0 = J = 3$. As expected ρ_T becomes more and more mixed with increasing temperature. Figure 2b also shows that ρ_T is less entangled when $|k_{0,f} - k_{0,i}|$ and $|J_f - J_i|$ increase as purity function exhibits.

4.3 Mutual information

From ρ_T in Eq. (3.12) one can derive the substates $\rho_{T,A} = \text{tr}_B \rho_T$ and $\rho_{T,B} = \text{tr}_A \rho_T$ by performing partial trace appropriately. Then, the substates become

$$\rho_{T,A}[x', x : \beta] = \rho_{T,B}[x', x : \beta] = \sqrt{\frac{a_{1,-} a_{2,-}}{\pi(\alpha_1 + \alpha_2 - 2\alpha_5)}} e^{-B_1 x^2 - B_2 x'^2 + 2B_3 xx'}$$

$$(4.13)$$

where

$$B_1 = \frac{\alpha_2(\alpha_1 + \alpha_2 - 2\alpha_5) - (\alpha_4 + \alpha_6)^2}{\alpha_1 + \alpha_2 - 2\alpha_5}, \quad B_2 = \frac{\alpha_1(\alpha_1 + \alpha_2 - 2\alpha_5) - (\alpha_3 + \alpha_6)^2}{\alpha_1 + \alpha_2 - 2\alpha_5}$$

$$B_3 = \frac{\alpha_5(\alpha_1 + \alpha_2 - 2\alpha_5) + (\alpha_3 + \alpha_6)(\alpha_4 + \alpha_6)}{\alpha_1 + \alpha_2 - 2\alpha_5}. \tag{4.14}$$

It is not difficult to show that the eigenvalues of $\rho_{T,A}$ or $\rho_{T,B}$ are $(1 - \zeta)\zeta^n$, where

$$\zeta = \frac{2B_3}{(B_1 + B_2) + \nu} \tag{4.15}$$

with $\nu = \sqrt{(B_1 + B_2)^2 - 4B_3^2}$. Using the eigenvalues the Rényi and von Neumann entropies of $\rho_{T,A}$ and $\rho_{T,B}$ can be obtained as

$$S_{A,\alpha} = S_{B,\alpha} = \frac{1}{1-\alpha} \ln \frac{(1-\zeta)^\alpha}{1-\zeta^\alpha}, \quad S_{A,von} = S_{B,von} = -\ln(1-\zeta) - \frac{\zeta}{1-\zeta} \ln \zeta. \tag{4.16}$$

Therefore, the mutual information of ρ_T is given by

$$I(\rho_T) = S_{A,von} + S_{B,von} - S_{von}. \tag{4.17}$$

The temperature dependence of the mutual information is plotted in Fig. 2c when $k_{0,f} = J_f = 6$ (red line) and $k_{0,f} = J_f = 9$ (blue line). The $k_{0,i}$ and J_i are fixed as $k_{0,i} = J_i = 3$. The black dashed line corresponds to constant frequencies $k_0 = J = 3$. Like other quantities mutual information also decreases with increasing temperature. However, it does not completely vanish at $T = \infty$. Figure 2c shows that the mutual information seems to approach 0.144 at the large temperature limit. This implies that the common information parties A and B share does not completely vanish even in the infinity temperature limit.

5 Thermal entanglement phase transition: case of SQM

Since the thermal state ρ_T given in Eq. (3.12) is mixed state, its entanglement is in general defined via the convex-roof method [38,39];

$$\mathcal{E}(\rho_T) = \min \sum_j p_j \mathcal{E}(\psi_j), \tag{5.1}$$

where minimum is taken over all possible pure state decompositions, i.e., $\rho_T = \sum_j p_j |\psi_j\rangle\langle\psi_j|$, with $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$. The decomposition which yields minimum value is called the optimal decomposition. However, it seems to be highly difficult problem to derive the optimal decomposition in the continuous variable system.

Because of this difficulty, we will consider the negativity-like quantity [30] of ρ_T. Let σ_T be a partial transpose of ρ_T, i.e.,

$$
\begin{aligned}
\sigma_T[x_1', x_2' : x_1, x_2 : \beta] &\equiv \rho_T[x_1, x_2' : x_1', x_2 : \beta] \\
&= \frac{\sqrt{a_{1,-}a_{2,-}}}{\pi} \exp\Big[-\alpha_1(x_1^2 + x_2'^2) - \alpha_2(x_1'^2 + x_2^2) + 2\alpha_3 x_1 x_2' + 2\alpha_4 x_1' x_2 \\
&\quad + 2\alpha_5(x_1 x_1' + x_2 x_2') + 2\alpha_6(x_1 x_2 + x_1' x_2') \Big].
\end{aligned}
\tag{5.2}
$$

Then, the negativity-like quantity $\mathcal{N}(\rho_T)$ is defined as

$$
\mathcal{N}(\rho_T) = \sum_{m,n} |\Lambda_{mn}| - 1,
\tag{5.3}
$$

where Λ_{mn} is eigenvalue of σ_T, i.e.,

$$
\int dx_1 dx_2 \sigma_T[x_1', x_2' : x_1, x_2 : \beta] f_{mn}(x_1, x_2) = \Lambda_{mn}(\beta) f_{mn}(x_1', x_2' : \beta).
\tag{5.4}
$$

One may wonder why the negativity-like quantity is introduced, because the PPT is known as necessary and sufficient criterion of separability for only 2×2 qubit–qubit and 2×3 qubit–qudit states [25–27]. However, as Ref. [28,29] have shown, PPT also provides a necessary and sufficient criterion of the separability for Gaussian continuous variable quantum states. Furthermore, the distillation protocols to maximally entangled state have been already suggested in Ref. [40,41] in the Gaussian states. Thus, our negativity-like quantity is valid at least to determine whether the given Gaussian state is entangled or not. Since $\mathcal{N}(\rho_T)$ is proportional to $\mathcal{E}(\rho_T)$, $\mathcal{N}(\rho_T) = 0$ at the critical temperature $T = T_c$ of the TEPT if the external temperature induces the ESD phenomenon. Thus, if eigenvalue equation (5.4) is solved, it is possible to compute T_c.

As we will show in the following, however, it seems to be very difficult to solve Eq. (5.4) directly. In order to solve Eq. (5.4) we define

$$
f_{mn}(x_1, x_2 : \beta) = e^{\frac{\alpha_1 - \alpha_2}{2}(x_1^2 - x_2^2)} g_{mn}(x_1, x_2, \beta).
\tag{5.5}
$$

Then, Eq. (5.4) can be written as

$$
\begin{aligned}
&\frac{\sqrt{a_{1,-}a_{2,-}}}{\pi} \exp\left[-\frac{\alpha_1 + \alpha_2}{2}\left(x_1'^2 + x_2'^2\right) + 2\alpha_6 x_1' x_2' \right] \\
&\times \int dx_1 dx_2 \exp\left[-\frac{\alpha_1 + \alpha_2}{2}(x_1^2 + x_2^2) + 2\alpha_6 x_1 x_2 + 2\left(x_1', x_2'\right) \begin{pmatrix} \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \\
&\times g_{mn}(x_1, x_2 : \beta) = \Lambda_{mn}(\beta) g_{mn}(x_1', x_2' : \beta).
\end{aligned}
\tag{5.6}
$$

If one changes the variables as $y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2)$ and $y_2 = \frac{1}{\sqrt{2}}(-x_1 + x_2)$, Eq. (5.6) reduces to

$$\frac{\sqrt{a_1, -a_2, -}}{\pi} e^{-\mu_- y_1'^2 - \mu_+ y_2'^2} \int dy_1 dy_2 \exp\left[-\mu_- y_1^2 - \mu_+ y_2^2 + (y_1', \ y_2') A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right]$$

$$\times g_{mn}(y_1, y_2 : \beta) = \Lambda_{mn}(\beta) g_{mn}(y_1', y_2' : \beta) \tag{5.7}$$

where

$$\mu_\pm = \frac{\alpha_1 + \alpha_2}{2} \pm \alpha_6, \qquad A = \begin{pmatrix} 2\alpha_5 + (\alpha_3 + \alpha_4) & -(\alpha_3 - \alpha_4) \\ \alpha_3 - \alpha_4 & 2\alpha_5 - (\alpha_3 + \alpha_4) \end{pmatrix}. \tag{5.8}$$

The difficulty arises because of the fact that A is not symmetric matrix if $\alpha_3 \neq \alpha_4$. Due to this fact it seems to be impossible to factorize Eq. (5.7) into two single-party eigenvalue equations as we did in appendix A.

However, Eq. (5.7) can be solved for the case of constant frequencies, i.e., $\omega_{1,i} = \omega_{1,f} \equiv \omega_1$ and $\omega_{2,i} = \omega_{2,f} \equiv \omega_2$, because in this case α_3 is exactly equal to α_4. Furthermore, in this case we get

$$\mu_+ = \frac{1}{4}\left[\omega_1 \coth \frac{\omega_1 \beta}{2} + \omega_2 \tanh \frac{\omega_2 \beta}{2} \right], \quad \mu_- = \frac{1}{4}\left[\omega_1 \tanh \frac{\omega_1 \beta}{2} + \omega_2 \coth \frac{\omega_2 \beta}{2} \right]$$

$$\nu_+ \equiv \alpha_5 - \frac{\alpha_3 + \alpha_4}{2} = \frac{1}{4}\left[\omega_1 \coth \frac{\omega_1 \beta}{2} - \omega_2 \tanh \frac{\omega_2 \beta}{2} \right]$$

$$\nu_- \equiv \alpha_5 + \frac{\alpha_3 + \alpha_4}{2} = -\frac{1}{4}\left[\omega_1 \tanh \frac{\omega_1 \beta}{2} - \omega_2 \coth \frac{\omega_2 \beta}{2} \right]. \tag{5.9}$$

Since $\alpha_3 = \alpha_4$ in this case, Eq. (5.7) is factorized into the following two single-party eigenvalue equations:

$$e^{-\mu_- y_1'^2} \int dy_1 e^{-\mu_- y_1^2 + 2\nu_- y_1' y_1} g_{1,m}(y_1 : \beta) = p_m(\beta) g_{1,m}(y_1' : \beta)$$

$$e^{-\mu_+ y_2'^2} \int dy_2 e^{-\mu_+ y_2^2 + 2\nu_+ y_2' y_2} g_{2,n}(y_2 : \beta) = q_n(\beta) g_{2,n}(y_2' : \beta). \tag{5.10}$$

Then, the total eigenvalue Λ_{mn} and the normalized eigenfunction $f_{mn}(x_1, x_2 : \beta)$ are expressed as

$$\Lambda_{mn} = \frac{1}{\pi} \sqrt{\omega_1 \omega_2 \tanh \frac{\omega_1 \beta}{2} \tanh \frac{\omega_2 \beta}{2}} p_m(\beta) q_n(\beta)$$

$$f_{mn}(x_1, x_2 : \beta) = g_{1,m}(y_1 : \beta) g_{2,n}(y_2 : \beta), \tag{5.11}$$

where $g_{1,m}(y_1 : \beta)$ and $g_{2,n}(y_2 : \beta)$ are normalized eigenfunctions of Eq. (5.10). Solving Eq. (5.10) it is straightforward to show that the normalized eigenfunctions are

$$g_{1,m}(y_1 : \beta) = \frac{1}{\sqrt{2^m m!}} \left(\frac{\epsilon_1}{\pi}\right)^{1/4} H_m \left(\sqrt{\epsilon_1} y_1\right) e^{-\frac{\epsilon_1}{2} y_1^2}$$

$$g_{2,n}(y_2 : \beta) = \frac{1}{\sqrt{2^n n!}} \left(\frac{\epsilon_2}{\pi}\right)^{1/4} H_n \left(\sqrt{\epsilon_2} y_2\right) e^{-\frac{\epsilon_2}{2} y_2^2}, \tag{5.12}$$

where

$$\epsilon_1 = 2\sqrt{\mu_-^2 - \nu_-^2} = \sqrt{\omega_1 \omega_2 \tanh \frac{\omega_1 \beta}{2} \coth \frac{\omega_2 \beta}{2}}$$

$$\epsilon_2 = 2\sqrt{\mu_+^2 - \nu_+^2} = \sqrt{\omega_1 \omega_2 \coth \frac{\omega_1 \beta}{2} \tanh \frac{\omega_2 \beta}{2}}. \tag{5.13}$$

One can also show that the eigenvalue Λ_{mn} is

$$\Lambda_{mn} = (1 - \zeta_1)(1 - \zeta_2)\zeta_1^m \zeta_2^n \tag{5.14}$$

where

$$\zeta_1 = \frac{\nu_-}{\mu_- + \frac{\epsilon_1}{2}} = \frac{\sqrt{\mu_- + \nu_-} - \sqrt{\mu_- - \nu_-}}{\sqrt{\mu_- + \nu_-} + \sqrt{\mu_- - \nu_-}} = -\frac{\sqrt{\omega_1 \tanh \frac{\omega_1 \beta}{2}} - \sqrt{\omega_2 \coth \frac{\omega_2 \beta}{2}}}{\sqrt{\omega_1 \tanh \frac{\omega_1 \beta}{2}} + \sqrt{\omega_2 \coth \frac{\omega_2 \beta}{2}}}$$

$$\zeta_2 = \frac{\nu_+}{\mu_+ + \frac{\epsilon_2}{2}} = \frac{\sqrt{\mu_+ + \nu_+} - \sqrt{\mu_+ - \nu_+}}{\sqrt{\mu_+ + \nu_+} + \sqrt{\mu_+ - \nu_+}} = \frac{\sqrt{\omega_1 \coth \frac{\omega_1 \beta}{2}} - \sqrt{\omega_2 \tanh \frac{\omega_2 \beta}{2}}}{\sqrt{\omega_1 \coth \frac{\omega_1 \beta}{2}} + \sqrt{\omega_2 \tanh \frac{\omega_2 \beta}{2}}}.$$

$$\tag{5.15}$$

One can compute $\pm 1 - \zeta_1$ and $\pm 1 - \zeta_2$ explicitly, which result in $-1 < \zeta_1, \zeta_2 \leq 1$ for arbitrary temperature. Thus, it is easy to show $\sum_{m,n} \Lambda_{mn}(\beta) = 1$ as expected. Equations (5.3) and (5.14) make $\mathcal{N}(\beta)$ to be

$$\mathcal{N}(\beta) = \frac{(1 - \zeta_1)(1 - \zeta_2)}{(1 - |\zeta_1|)(1 - |\zeta_2|)} - 1. \tag{5.16}$$

The T-dependence of $\mathcal{N}(\beta)$ is plotted in Fig. 3 for (a) positive and (b) negative J with fixing $k_0 = 1$. Both figures show $\mathcal{N}(\beta)$ is zero at $T \geq T_c$. Similar results were obtained for general bosonic harmonic lattice systems [25,42]. Since $\mathcal{N}(\beta)$ is proportional to entanglement of ρ_T, this fact implies that ρ_T is entangled (or separable) state at $T < T_c$ (or $T \geq T_c$). The critical temperature T_c increases with increasing $|J|$.

From Eq. (5.16) it is evident that ρ_T is separable when $\zeta_1 \geq 0$ and $\zeta_2 \geq 0$. Equation (5.15) implies that this separability criteria can be rewritten in a form

$$x \tanh x - y \coth y \leq 0, \qquad x \coth x - y \tanh y \geq 0 \tag{5.17}$$

where $x = \omega_1 \beta / 2$ and $y = \omega_2 \beta / 2$. If $J \geq 0$, first equation of Eq. (5.17) is automatically satisfied. Hence, the second equation plays a role as a genuine separability
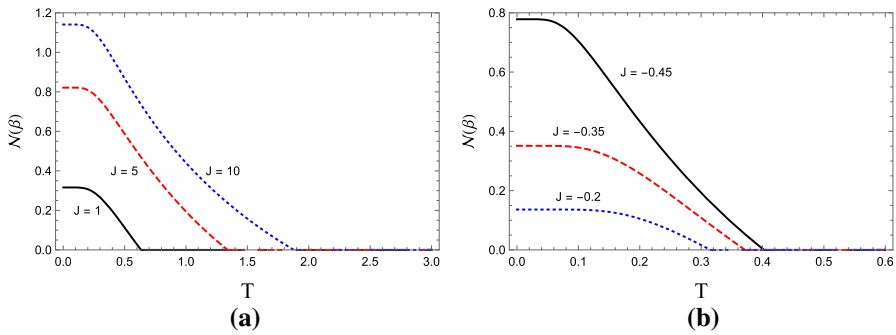
Fig. 3 (Color online) The T-dependence of $\mathcal{N}(\beta)$ for **a** $J = 1$ (black line), 5 (red dashed line), 10 (blue dotted line) and **b** $J = -0.45$ (black line), -0.35 (red dashed line), -0.2 (blue dotted line) with fixing $k_0 = 1$. Both figures show $\mathcal{N}(\beta)$ is zero at $T \geq T_c$. Since $\mathcal{N}(\beta)$ is proportional to entanglement of ρ_T, this fact implies that ρ_T is entangled (or separable) state at $T < T_c$ (or $T \geq T_c$). The critical temperature T_c increases with increasing $|J|$
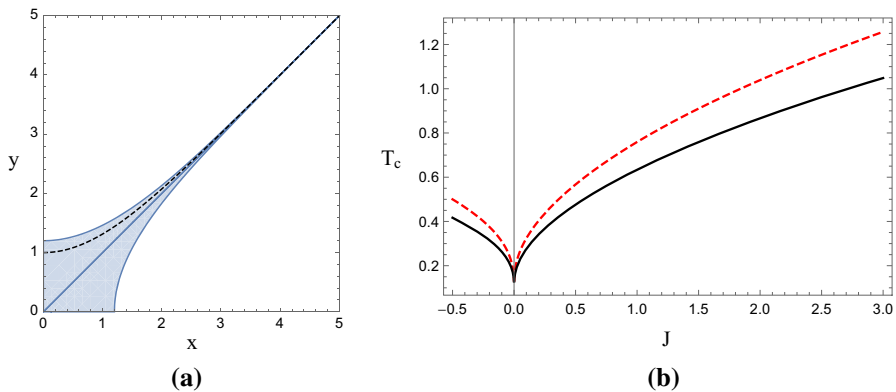

Fig. 4 (Color online) **a** The shaded region in x-y plane is a region where the separable states of ρ_T reside. The boundary contains an information on the critical temperature T_c. The black dashed line in the shaded region is $y = x \coth x$. This is used to compute T_c approximately. **b** The J-dependence of T_c when $k_0 = 1$. The black solid line and red dashed line correspond to exact and approximate T_c, respectively

criterion. If $J < 0$, first equation is true criterion. It is worthwhile noting that two equations in Eq. (5.17) can be transformed into each other by interchanging x and y. This fact implies that the region in x-y plane, where the separable states reside, is symmetric with respect to $y = x$.

The shaded region in Fig. 4a is a region where the separable states of ρ_T reside in x-y plane. As expected, the region is symmetric with respect to $y = x$. It is shown that most separable states are accumulated in $0 \leq x, y \leq 1$. The boundary of the region contains an information about the critical temperature T_c. The black dashed line in the region is $y = x \coth x$. Since this is very close to upper boundary, this can be used to compute T_c approximately.

Let the upper boundary of Fig. 4a be expressed by $y_c = x_c g(x_c)$, where x_c and y_c are x and y at $T = T_c$. Then the low boundary should be $x_c = y_c g(y_c)$. The function

$g(z)$ can be derived numerically by using Eq. (5.17) after changing the inequality into equality. Then, T_c can be computed by

$$T_c = \frac{\omega_{min}}{2g^{-1}\left(\frac{\omega_{max}}{\omega_{min}}\right)} \tag{5.18}$$

where $\omega_{min} = \min(\omega_1, \omega_2)$ and $\omega_{max} = \max(\omega_1, \omega_2)$. If one uses $g(z) \approx \coth z$, the critical temperature is approximately

$$T_c \approx \frac{\omega_{min}}{\ln\left(\frac{\omega_{max}+\omega_{min}}{\omega_{max}-\omega_{min}}\right)}. \tag{5.19}$$

In Fig. 4b the J-dependence of T_c is plotted when $k_0 = 1$. The black solid line and red dashed line correspond to Eqs. (5.18) and (5.19), respectively. It is shown that T_c increases with increasing $|J|$ as expected from Fig. 3.

As we commented earlier, for the case of SQM it seems to be highly difficult problem to solve eigenvalue equation (5.4) directly. However, we can conjecture the eigenvalue $\Lambda_{mn}(\beta)$ without deriving the eigenfunction $f_{mn}[x_1, x_2 : \beta]$ as follows. Since $\sum_{m,n} \Lambda_{mn}(\beta) = 1$, $\Lambda_{mn}(\beta)$ might be represented as Eq. (5.14). If this is right, we can compute ζ_1 and ζ_2 by making use of the Rényi entropy. If the eigenvalue is represented as Eq. (5.14), the Rényi entropy of σ_T can be written as

$$S_\alpha[\sigma_T] \equiv \frac{1}{1-\alpha} \ln \text{tr}\left[(\sigma_T)^\alpha\right] = \frac{1}{1-\alpha}\left[\ln \frac{(1-\zeta_1)^\alpha}{1-\zeta_1^\alpha} + \ln \frac{(1-\zeta_2)^\alpha}{1-\zeta_2^\alpha}\right]. \tag{5.20}$$

Putting $\alpha = 2$ or 3 in Eq. (5.20), it is possible to derive

$$\frac{(1-\zeta_1)(1-\zeta_2)}{(1+\zeta_1)(1+\zeta_2)} = \beta_1, \qquad \frac{(1-\zeta_1)^2(1-\zeta_2)^2}{(1+\zeta_1+\zeta_1^2)(1+\zeta_2+\zeta_2^2)} = \beta_2 \tag{5.21}$$

where

$$\beta_1 \equiv \text{tr}\sigma_T^2 = \sqrt{\frac{X_1}{X_2}}, \qquad \beta_2 \equiv \text{tr}\sigma_T^3 = \frac{4X_1}{X_1 + 3X_2 - 12(\alpha_5^2 - \alpha_3\alpha_4)} \tag{5.22}$$

with

$$X_1 = (\alpha_1+\alpha_2-2\alpha_5)^2-(\alpha_3+\alpha_4+2\alpha_6)^2, \quad X_2 = (\alpha_1+\alpha_2+2\alpha_5)^2-(\alpha_3+\alpha_4-2\alpha_6)^2. \tag{5.23}$$
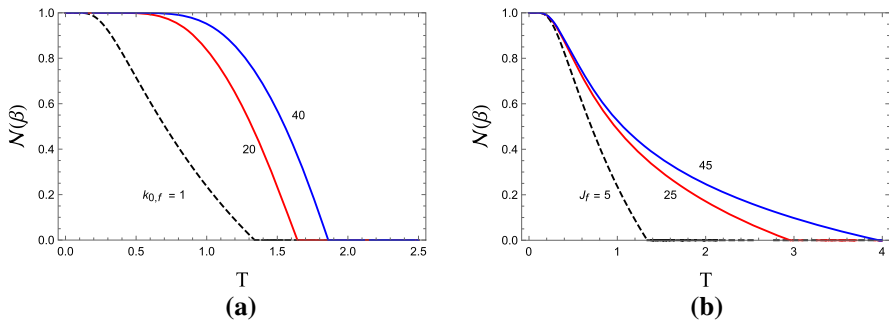
Fig. 5 (Color online) **a** The temperature dependence of $\mathcal{N}(\beta)/\mathcal{N}(\infty)$ when $k_{0,f} = 1$ (black dashed line), $k_{0,f} = 20$ (red line), and $k_{0,f} = 40$ (blue line) with fixing $k_{0,i} = 1$ and $J_i = J_f = 5$. **b** The temperature dependence of $\mathcal{N}(\beta)/\mathcal{N}(\infty)$ when $J_f = 5$ (black dashed line), $J_f = 25$ (red line), and $J_f = 45$ (blue line) with fixing $k_{0,i} = k_{0,f} = 1$ and $J_i = 5$. Both figures exhibit that the critical temperature T_c increases with increasing $|k_{0,f} - k_{0,i}|$ and $|J_f - J_i|$

Solving Eq. (5.21), we get

$$u = \zeta_1 + \zeta_2 = \frac{1 - \beta_1}{2(4\beta_1^2 - \beta_1^2\beta_2 - 3\beta_2)}\left[-3\beta_2(1 + \beta_1) + \sqrt{3\beta_2\left[16\beta_1^2 - \beta_2(3 - \beta_1)^2\right]}\right]$$

$$v = \zeta_1\zeta_2 = -1 + \frac{1 + \beta_1}{2(4\beta_1^2 - \beta_1^2\beta_2 - 3\beta_2)}\left[-3\beta_2(1 + \beta_1) + \sqrt{3\beta_2\left[16\beta_1^2 - \beta_2(3 - \beta_1)^2\right]}\right]. \quad (5.24)$$

Thus, ζ_1 and ζ_2 for the case of SQM become

$$\zeta_1 = \frac{u + \sqrt{u^2 - 4v}}{2}, \qquad \zeta_2 = \frac{u - \sqrt{u^2 - 4v}}{2}. \quad (5.25)$$

At the case of constant frequency β_1 and β_2 reduce to

$$\beta_1 = \frac{\epsilon_1\epsilon_2}{4(\mu_+ + \nu_+)(\mu_- + \nu_-)}, \qquad \beta_2 = \frac{4(\mu_+ - \nu_+)(\mu_- - \nu_-)}{(2\mu_+ + \nu_+)(2\mu_- + \nu_-)}. \quad (5.26)$$

Using Eq. (5.26) and after tedious calculation, one can show that ζ_1 and ζ_2 in Eq. (5.25) exactly coincide with those in Eq. (5.15) when $\omega_{1,i} = \omega_{1,f} = \omega_1$ and $\omega_{2,i} = \omega_{2,f} = \omega_2$. Then, the negativity-like quantity can be written in a form

$$\mathcal{N}(\beta) = \frac{1 - u + v}{1 + |v| - (|\zeta_1| + |\zeta_2|)} - 1. \quad (5.27)$$

The temperature dependence of $\mathcal{N}(\beta)/\mathcal{N}(\infty)$ is plotted in Fig. 5. In Fig. 5a we choose $k_{0,f} = 1$ (black dashed line), $k_{0,f} = 20$ (red line), and $k_{0,f} = 40$ (blue line) when $k_{0,i} = 1$ and $J_i = J_f = 5$. As this figure exhibits, the critical temperature T_c increases with increasing $|k_{0,f} - k_{0,i}|$. In Fig. 5b we choose $J_f = 5$ (black dashed line), $J_f = 25$ (red line), and $J_f = 45$ (blue line) when $k_{0,i} = k_{0,f} = 1$ and $J_i = 5$. This figure also shows that T_c increases with increasing $|J_f - J_i|$.

6 Conclusions

In this paper we derive explicitly the thermal state of the two-coupled harmonic oscillator system when the spring and coupling constants are arbitrarily time-dependent. In particular, we focus on the SQM model (see Eqs. (3.8) and (3.9)). In this model we compute purity function, Rényi and von Neumann entropies, and mutual information analytically and examine their temperature dependence. We also discuss on the TEPT by making use of the negativity-like quantity. Our calculation shows that the critical temperature T_c increases with increasing the difference between the initial and final frequencies. In this way we can use the SQM model to protect the entanglement against the external temperature by introducing a large difference of frequencies, i.e., $|\omega_f - \omega_i| \gg 1$.

There are several issues related to our paper. Since the SQM model we consider involves a discontinuity at $t = 0$, it is unrealistic in some sense. In order to escape this fact we can introduce the time-dependence of frequencies as a form $\omega = \omega_i + (\omega_f - \omega_i) \sin \Omega t$. Then, we have to solve the Ermakov equation numerically. In this case the critical temperature T_c might be dependent on Ω and $|\omega_f - \omega_i|$. Then, it may be possible to protect the entanglement in the thermal bath by adjusting Ω and $|\omega_f - \omega_i|$ appropriately.

In this paper we introduce the negativity-like quantity to examine the thermal entanglement, because we do not know how to derive the optimal decomposition of Eq. (5.1). Recently, the upper and lower bounds of entanglement of formation (EoF) are examined for arbitrary two-mode Gaussian state [43]. It seems to be of interest to examine the TEPT with EoF.

Appendix A

In this section we examine the eigenvalue equation of the following bipartite Gaussian state:

$$
\begin{aligned}
\rho_2 [x_1', x_2' &: x_1, x_2] \\
&= A \exp \Big[-a_1 (x_1'^2 + x_2'^2) - a_2 (x_1^2 + x_2^2) + 2b_1 x_1' x_2' + 2b_2 x_1 x_2 \\
&\quad + 2c (x_1 x_1' + x_2 x_2') + 2f (x_1 x_2' + x_2 x_1') \Big]
\end{aligned}
\tag{A.1}
$$

where $A = \sqrt{(a_1 + a_2 - 2c)^2 - (b_1 + b_2 + 2f)^2} / \pi$. If $a_1 = \alpha_1$, $a_2 = \alpha_2$, $b_1 = \alpha_3$, $b_2 = \alpha_4$, $c = \alpha_5$, and $f = \alpha_6$, ρ_2 is exactly the same with the thermal state ρ_T given in Eq. (3.12). Now let us consider the eigenvalue equation

$$
\int dx_1 dx_2 \rho_2 [x_1', x_2' : x_1, x_2] f_{mn}(x_1, x_2) = \lambda_{mn} f_{mn}(x_1', x_2').
\tag{A.2}
$$

First we change the variables as

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \qquad y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2). \tag{A.3}$$

Then Eq. (A.2) is simplified as

$$A\mathrm{e}^{-(a_1-b_1)y_1'^2-(a_1+b_1)y_2'^2} \int \mathrm{d}y_1\mathrm{d}y_2 \mathrm{e}^{-(a_2-b_2)y_1^2-(a_2+b_2)y_2^2+2(c+f)y_1'y_1+2(c-f)y_2'y_2} f_{mn}(y_1, y_2)$$

$$= \lambda_{mn} f(y_1', y_2'). \tag{A.4}$$

Now, we define

$$f_{mn}(y_1, y_2) = g_m(y_1)h_n(y_2). \tag{A.5}$$

Then, Eq. (A.4) is solved if one solves the following two single-party eigenvalue equations:

$$\mathrm{e}^{-(a_1-b_1)y_1'^2} \int \mathrm{d}y_1 \mathrm{e}^{-(a_2-b_2)y_1^2+2(c+f)y_1'y_1} g_m(y_1) = p_m g_m(y_1')$$

$$\mathrm{e}^{-(a_1+b_1)y_2'^2} \int \mathrm{d}y_2 \mathrm{e}^{-(a_2+b_2)y_2^2+2(c-f)y_2'y_2} h_n(y_2) = q_n h_n(y_2'). \tag{A.6}$$

The eigenvalue of Eq. (A.2) can be computed as $\lambda_{mn} = Ap_m q_n$.

By making use of Eqs. (2.25) and (2.26) one can show $\lambda_{mn} = (1-\xi_1)\xi_1^m(1-\xi_2)\xi_2^n$, where

$$\begin{aligned}
\xi_1 &= \frac{2(c+f)}{(a_1+a_2-b_1-b_2)+\epsilon_1} \\
&= \frac{\sqrt{(a_1+a_2-b_1-b_2)+2(c+f)} - \sqrt{(a_1+a_2-b_1-b_2)-2(c+f)}}{\sqrt{(a_1+a_2-b_1-b_2)+2(c+f)} + \sqrt{(a_1+a_2-b_1-b_2)-2(c+f)}} \\
\xi_2 &= \frac{2(c-f)}{(a_1+a_2+b_1+b_2)+\epsilon_2} \\
&= \frac{\sqrt{(a_1+a_2+b_1+b_2)+2(c-f)} - \sqrt{(a_1+a_2+b_1+b_2)-2(c-f)}}{\sqrt{(a_1+a_2+b_1+b_2)+2(c-f)} + \sqrt{(a_1+a_2+b_1+b_2)-2(c-f)}}
\end{aligned} \tag{A.7}$$

with

$$\epsilon_1 = \sqrt{(a_1+a_2-b_1-b_2)^2-4(c+f)^2}, \qquad \epsilon_2 = \sqrt{(a_1+a_2+b_1+b_2)^2-4(c-f)^2}. \tag{A.8}$$

We can also use Eqs. (2.25) and (2.26) to derive the normalized eigenfunction, whose explicit expression is

$$f_{mn}(x_1, x_2) = \left(\frac{1}{C_{1,m}} H_m(\sqrt{\epsilon_1}y_1)\mathrm{e}^{-\frac{\alpha_1}{2}y_1^2}\right) \left(\frac{1}{C_{2,n}} H_n(\sqrt{\epsilon_2}y_2)\mathrm{e}^{-\frac{\alpha_2}{2}y_2^2}\right) \tag{A.9}$$

where

$$\alpha_1 = \epsilon_1 + (a_1 - a_2) - (b_1 - b_2), \qquad \alpha_2 = \epsilon_2 + (a_1 - a_2) + (b_1 - b_2) \quad (A.10)$$

and the normalization constants $\mathcal{C}_{1,m}$ and $\mathcal{C}_{2,n}$ are

$$\mathcal{C}_{1,m}^2 = \frac{1}{\sqrt{\alpha_1}} \sum_{k=0}^{m} 2^{2m-k} \left(\frac{\epsilon_1}{\alpha_1} - 1 \right)^{m-k} \frac{\Gamma^2(m+1)\Gamma(m-k+1/2)}{\Gamma(k+1)\Gamma^2(m-k+1)}$$

$$\mathcal{C}_{2,n}^2 = \frac{1}{\sqrt{\alpha_2}} \sum_{k=0}^{n} 2^{2n-k} \left(\frac{\epsilon_2}{\alpha_2} - 1 \right)^{n-k} \frac{\Gamma^2(n+1)\Gamma(n-k+1/2)}{\Gamma(k+1)\Gamma^2(n-k+1)}. \quad (A.11)$$

Thus, the spectral decomposition of ρ_2 is

$$\rho_2[x_1', x_2' : x_1, x_2] = \sum_{m,n} \lambda_{mn} f_{mn}(x_1', x_2') f_{mn}^*(x_1, x_2), \qquad (A.12)$$

where λ_{mn} and f_{mn} are given in Eqs. (A.7) and (A.9), respectively.

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