CH. 1 1계 미분 방정식

§. 예비개념

$\frac{dy}{dx} = x$: 1계 미분 방정식

$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 5x^3$: 2계 미분 방정식

[1] 일반해와 특수해

$$\frac{dy}{dx} = x$$

$$\left.\begin{array}{l} y = \frac{1}{2}x^2 \\ y = \frac{1}{2}x^2 + 1 \\ y = \frac{1}{2}x^2 - 3 \\ \vdots \end{array}\right\} \text{특수해}$$

$y = \frac{1}{2}x^2 + c$: 일반해

[2] 초기값 문제

$\frac{dy}{dx} = x \qquad y(3) = 9$

$\Rightarrow y(x) = \frac{1}{2}x^2 + c$: 일반해

$$y(3) = \frac{9}{2} + c = 9$$

$$c = \frac{9}{2}$$

$$\Rightarrow y(x) = \frac{1}{2}x^2 + \frac{9}{2}$$

p7

(연습문제 1)

$2yy' = 1 \quad : \quad y(x) = \sqrt{x-1}$

$y'(x) = \frac{1}{2}\frac{1}{\sqrt{x-1}}$

$\Rightarrow 2yy' = 2\sqrt{x-1} \cdot \frac{1}{2}\frac{1}{\sqrt{x-1}} = 1$: 만족함

해이다 !!

(연습문제 8)

$y' = e^{-x} \qquad y(0) = 2$

$y(x) = -e^{-x} + C$

$y(0) = -1 + C = 2$

$C = 3$

$\Rightarrow \quad y(x) = -e^{-x} + 3$

§ 변수분리형

$$y' = A(x)\,B(y) \quad \| \times \frac{1}{B(y)}\,dx$$

$$\Rightarrow \frac{dy}{B(y)} = A(x)\,dx$$

$$\Rightarrow \text{해: } \int \frac{dy}{B(y)} = \int A(x)\,dx + C$$

pe. (예제1.5)

$$\frac{dy}{dx} = y^2 e^{-x} \quad \| \times \frac{dx}{y^2} \qquad (y \neq 0)$$

$$\Rightarrow \frac{dy}{y^2} = e^{-x}\,dx$$

$$\Rightarrow \int \frac{dy}{y^2} = \int e^{-x}\,dx$$

$$\Rightarrow -\frac{1}{y} = -e^{-x} + C$$

$$\Rightarrow \frac{1}{y} = e^{-x} + C$$

$$\Rightarrow \underline{\underline{y = \frac{1}{e^{-x} + C}}} \qquad \text{일반해}$$

$y = 0$: 특이해 (singular solution) ※

p9
(예제1.6)

$$\frac{dy}{dx} = y^2 e^{-x} \quad : \ y(1) = 4$$

$$y = \frac{1}{e^{-x} + c}$$

$$\Rightarrow \quad y(1) = \frac{1}{\frac{1}{e} + c} = 4$$

$$\Rightarrow \quad c = \frac{1}{4} - \frac{1}{e}$$

$$\Rightarrow \quad y = \frac{1}{e^{-x} + \left(\frac{1}{4} - \frac{1}{e}\right)}$$

※

p11 3

(예제1.8) 방사능 붕괴 와 탄소연대 측정

방사능 붕괴 : 질량 → 에너지

$$\frac{dm}{dt} = km \qquad m(0) = M \ , \quad m(T) = M_T$$

(k: 상수)

$$\frac{dm}{m} = k\,dt$$

$$\int \frac{dm}{m} = \int k\,dt$$

$$\ln m = kt + \ln C$$

$x = \ln e^x$

$$= \ln e^{kt} + \ln C$$

$\ln A + \ln B = \ln (AB)$

$$= \ln Ce^{kt}$$

$$\Rightarrow \quad m = Ce^{kt} \qquad — ①$$

$$m(0) = C = M \qquad —②$$

$$m(T) = Ce^{kT} = M_T$$

$$\Rightarrow \quad e^{kT} = \frac{M_T}{M}$$

$$\Rightarrow \quad kT = \ln \frac{M_T}{M}$$

$$\Rightarrow \quad k = \frac{1}{T} \ln \frac{M_T}{M} \qquad —③$$

②, ③ → ①

$$\underline{m(t) = Me^{\left(\frac{1}{T}\ln\frac{M_T}{M}\right)t}} = M\left(\frac{M_T}{M}\right)^{\frac{t}{T}}$$

If $T = H$, $M_T = \frac{M}{2}$.

$$m(t) = M e^{(\frac{1}{H}\ln\frac{1}{2})t} = M\left(\frac{1}{2}\right)^{\frac{t}{H}}$$

H: 반감기 (half-life) ※

p14

(연습문제 1)

$$3\frac{dy}{dx} = \frac{4x}{y^2} \quad \Big\| \; y^2 dx$$

$$\Rightarrow \; 3y^2 dy = 4x\,dx$$

$$\Rightarrow \; 3\int y^2 dy = 4\int x\,dx$$

$$\Rightarrow \; \underline{\underline{y^3 = 2x^2 + c}}$$

§ 완전 미분형

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

$$P(x,y) = \frac{\partial}{\partial x} u(x,y)$$

$$Q(x,y) = \frac{\partial}{\partial y} u(x,y)$$

* 완전미분형

$u(x,y)$: potential 함수

$$\Rightarrow \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow du = 0$$

$$\Rightarrow \underline{u(x,y) = C}$$

일반해

p20

(예제 1.10)

$$\frac{dy}{dx} = -\frac{2xy^3+2}{3x^2y^2+8e^{4y}} \quad \| \times (3x^2y^2+8e^{4y})\,dx$$

$$(2xy^3+2)\,dx + (3x^2y^2+8e^{4y})\,dy = 0$$

$$P(x,y) = 2xy^3+2$$

$$Q(x,y) = 3x^2y^2+8e^{4y}$$

$$u(x,y) = x^2y^3+2x+2e^{4y}$$

확인: $\frac{\partial u}{\partial x} = 2xy^3+2 = P(x,y)$

$\frac{\partial u}{\partial y} = 3x^2y^2+8e^{4y} = Q(x,y)$

$$\underline{x^2y^3+2x+2e^{4y} = C}$$

* 완전 미분형 판정법

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

If $P(x,y) = \frac{\partial u}{\partial x}$ and $Q(x,y) = \frac{\partial u}{\partial y}$,

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial Q}{\partial x}$$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ⇒ 완전 미분형

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ ⇒ 완전 미분형이 아님

p25

(예제1.14)

$$e^x \sin y - 2x + (e^x \cos y + 1)\frac{dy}{dx} = 0 \qquad \| \times dx$$

$$(e^x \sin y - 2x)\,dx + (e^x \cos y + 1)\,dy = 0$$

$$\left.\begin{aligned} P(x,y) &= e^x \sin y - 2x & \frac{\partial P}{\partial y} &= e^x \cos y \\ Q(x,y) &= e^x \cos y + 1 & \frac{\partial Q}{\partial x} &= e^x \cos y \end{aligned}\right) =$$

⇒ 완전 미분형

$$u(x,y) = \underline{e^x \sin y - x^2 + y} = c$$

p26

(연습문제) 6)

$$y + e^x + x\frac{dy}{dx} = 0 \qquad \| \times dx$$

$$(y+e^x)\,dx + x\,dy = 0$$

$$P(x,y) = y + e^x \Rightarrow \frac{\partial P}{\partial y} = 1$$

$$Q(x,y) = x \Rightarrow \frac{\partial Q}{\partial x} = 1$$

$$\Big) =$$

완전 미분형

$$u(x,y) = \underline{xy + e^x = C}$$

p15

용 선형 미분 방정식

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$ 선형 미분 방정식

(해)

$\times e^{\int P dx}$

$$e^{\int P dx}\frac{dy}{dx} + P y\, e^{\int P dx} = Q(x)\, e^{\int P dx} \quad -①$$

$$\frac{d}{dx}\left(e^{\int P dx}\, y\right)$$

Let

$$z \equiv e^{\int P(x) dx}\, y \quad -②$$

$$\frac{dz}{dx} = Q(x)\, e^{\int P(x) dx}$$: 변수분리형

$$dz = Q(x)\, e^{\int P dx}\, dx$$

$$\Rightarrow \quad z = \int\left(Q(x)\, e^{\int P dx}\right) dx + c = e^{\int P dx}\, y \qquad // \times e^{-\int P dx}$$

$$\Rightarrow \quad y = e^{-\int P dx}\left[\int\left(Q(x)\, e^{\int P dx}\right) dx + c\right]$$ 일반해

p15

(예제1.9)

$$\frac{dy}{dx} + y = \sin x$$

$$p(x) = 1, \quad Q(x) = \sin x$$

$$e^{\int p\,dx} = e^{x}$$

$$\Rightarrow y = e^{-x}\left[\underbrace{\int \sin x \, e^{x}\, dx}_{\frac{1}{2}e^{x}(\sin x - \cos x)} + c \right] = \frac{1}{2}(\sin x - \cos x) + c\,e^{-x}$$

p16

(예제 1.10)

$$\frac{dy}{dx} + \frac{1}{x} y = 3x^2 \quad : \; y(1) = 5$$

$$p(x) = \frac{1}{x}, \quad Q(x) = 3x^2$$

$$e^{\int p\,dx} = e^{\ln x} = x$$

$$\Rightarrow y = \frac{1}{x}\left[\int 3x^2 \cdot x \, dx + c \right]$$

$$= \frac{1}{x}\left[\frac{3}{4}x^4 + c \right]$$

$$= \frac{3}{4}x^3 + \frac{c}{x}$$

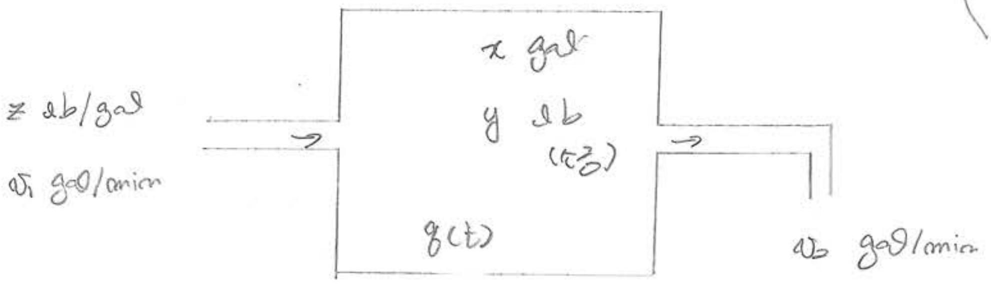
$$\Rightarrow y(1) = \frac{3}{4} + c = 5 \quad \Rightarrow c = \frac{17}{4}$$

$$\Rightarrow \underline{y(x) = \frac{3}{4}x^3 + \frac{17}{4x}}$$

p17

예제 1.11

$$\left(\begin{array}{l} 1\,gal = 3.8\,l \\ 1\,lb = 0.454\,kg \end{array} \right)$$



$$\frac{dg}{dt} = \text{유입량} - \text{유출량}$$

$$= z v_1 - \frac{g(t)}{x} v_2$$

$$\Rightarrow \quad \frac{dg}{dt} + \frac{v_2}{x} g(t) = z v_1 \quad : \text{선형}$$

$$P(t) = \frac{v_2}{x} \qquad Q(t) = z v_1$$

$$e^{\int P dt} = e^{\frac{v_2}{x} t}$$

$$g(t) = e^{-\frac{v_2}{x} t} \left[\int z v_1 \, e^{\frac{v_2}{x} t} \, dt + c \right]$$

$$= e^{-\frac{v_2}{x} t} \left[z v_1 \frac{x}{v_2} e^{\frac{v_2}{x} t} + c \right]$$

$$= \frac{z v_1 x}{v_2} + c \, e^{-\frac{v_2}{x} t}$$

$x = 200$ gal $\quad y = 100$ lb , $v_1 = v_2 = 3$ gal/min , $z = \frac{1}{8}$ lb/gal

$$g(t) = 25 + c \, e^{-\frac{3}{200} t}$$

$$g(0) = 25 + c = 100 \qquad \Rightarrow \quad c = 75$$

$$\underline{g(t) = 25 + 75 \, e^{-\frac{3}{200} t}}$$

//

p20

(연습문제 1)

$$\frac{dy}{dx} - \frac{3}{x}y = 2x^2$$: 선형

$$p(x) = -\frac{3}{x} \qquad Q(x) = 2x^2$$

$$e^{\int p\,dx} = e^{-3\ln x} = \frac{1}{x^3}$$

$$y = x^3\left[\int 2x^2 \cdot \frac{1}{x^3}\,dx + c\right]$$

$$= x^3\left[2\int \frac{1}{x}\,dx + c\right]$$

$$= x^3\left[2\ln x + c\right]$$

$$= 2x^3\ln x + cx^3$$ #

p.26

8. 제차형 미분방정식, Bernoulli 미분방정식.

[1] 제차형 미분방정식

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$: 제차형

$$u = \frac{y}{x}$$

$$\Rightarrow y = xu$$

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

$$\Rightarrow x\frac{du}{dx} + u = f(u)$$

$$\Rightarrow x\frac{du}{dx} = f(u) - u \quad \Big\| \; \frac{dx}{x(f(u)-u)}$$

$$\Rightarrow \frac{du}{f(u)-u} = \frac{dx}{x}$$: 변수분리형

$$\underline{\int \frac{du}{f(u)-u} = \ln x + C}$$ 일반해

p27
(예제1.15)

$$x\frac{dy}{dx} = \frac{y^2}{x} + y$$

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} \quad : \text{제차형}$$

$\llcorner$ ①

$$u = \frac{y}{x}$$

$$y = xu$$

$$\frac{dy}{dx} = x\frac{du}{dx} + u \quad -②$$

② → ①

$$x\frac{du}{dx} + \cancel{u} = u^2 + \cancel{u}$$

$$\Rightarrow x\frac{du}{dx} = u^2 \quad \| \times \frac{dx}{xu^2}$$

$$\Rightarrow \frac{du}{u^2} = \frac{dx}{x}$$

$$-\frac{1}{u} = \ln x + c$$

$$u = \frac{1}{-\ln x + c} = \frac{y}{x}$$

$$y = \frac{-x}{\ln x + c}$$

[5] Bernoulli 방정식

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^{\alpha} \quad -①$$

$$v = y^{1-\alpha} \quad -②$$

$$\frac{dv}{dx} = (1-\alpha)\,y^{-\alpha}\,\frac{dy}{dx}$$

$$\Rightarrow \quad y^{-\alpha}\frac{dy}{dx} = \frac{1}{1-\alpha}\,\frac{dv}{dx} \quad -③$$

$$②, ③ \rightarrow ① \times y^{-\alpha}$$

$$\frac{1}{1-\alpha}\frac{dv}{dx} + P(x)\,v = Q(x)$$

$$\frac{dv}{dx} + (1-\alpha)P(x)\,v = (1-\alpha)Q(x) \quad \Rightarrow \text{선형}$$

$$\left(\begin{aligned} &\frac{dy}{dx} + g_1(x)\,y = g_2(x) \\ &y = e^{-\int g_1 dx}\left[\int \left(g_2\, e^{\int g_1 dx} dx\right) + c\right] \end{aligned} \right)$$

$$g_1 = (1-\alpha)\,P(x)$$

$$g_2 = (1-\alpha)\,Q(x)$$

$$v = e^{-(1-\alpha)\int P dx}\left[\int (1-\alpha)Q(x)\,e^{(1-\alpha)\int P dx} dx + c\right] = y^{1-\alpha}$$

$$\Rightarrow \quad \underline{y^{1-\alpha} = e^{-(1-\alpha)\int P dx}\left[(1-\alpha)\int Q(x)\,e^{(1-\alpha)\int P dx} + c\right]}$$

P31

(예제 1.17)

$$\frac{dy}{dx} + \frac{1}{x} y = 3x^2 y^3$$

$p(x) = \frac{1}{x}$, $Q(x) = 3x^2$, $\alpha = 3$

$$y^{-2} = e^{2\ln x}\left[-2\int 3x^2 e^{-2\ln x} dx + c\right]$$

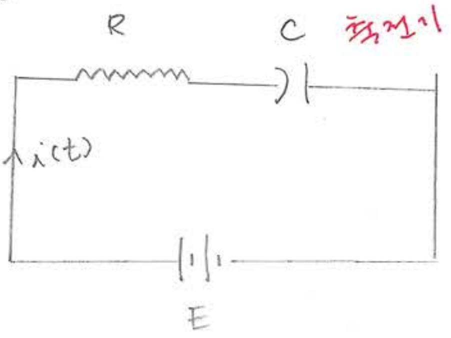
$$= x^2\left[-6\int dx + c\right]$$

$$= -6x^3 + cx^2 \quad = \frac{1}{y^2}$$

$$y(x) = \frac{1}{\sqrt{cx^2 - 6x^3}}$$ ※

§ 전기회로

[1] RC 회로



축전기 (or 콘덴서)

$$E = iR + \frac{q}{C} \qquad \frac{dq}{dt} = i$$

q(t) : 축전기의 전하량
C : 축전기의 전기용량 (Capacitance)
R : 저항
i : 전류
E : 기전력 or 전압

$$R\frac{dq}{dt} + \frac{1}{C}q = E \quad \Big\| \frac{1}{R}$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R} \quad : \text{선형}$$

$$P(t) = \frac{1}{RC}, \quad Q(t) = \frac{E}{R}$$

$$q(t) = e^{-\frac{t}{RC}}\left[\int \frac{E}{R} e^{\frac{t}{RC}} dt + c_1\right]$$

$$= e^{-\frac{t}{RC}}\left[\frac{E}{R} RC\, e^{\frac{t}{RC}} + c_1\right]$$

$$= EC + c_1 e^{-\frac{t}{RC}} \qquad - ②$$

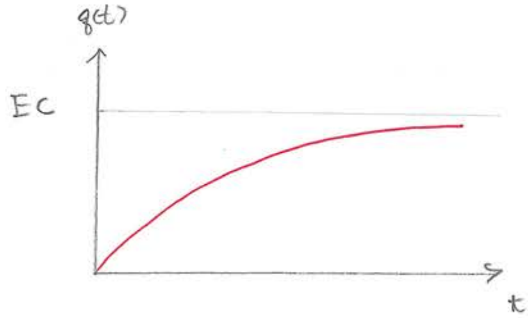
if $q(0) = 0$,

$$q(0) = EC + c_1 = 0$$

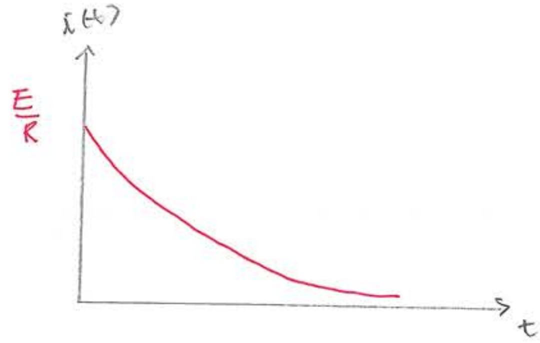
$$\Rightarrow \quad c_1 = -EC \qquad - ③$$

③ → ②

$$q(t) = Ec\left(1 - e^{-\frac{t}{RC}}\right)$$



$$i(t) = \frac{dq}{dt} = \frac{E}{R} e^{-\frac{t}{RC}}$$

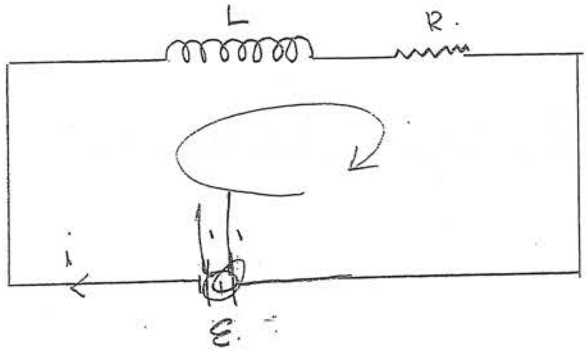


(R-L 회로)

8. inductance 와 저항이 있는 회로.

$\Phi = L i$

$V = \frac{d\Phi}{dt} = L\frac{di}{dt}$



$\Phi = L i$

$\varepsilon = -L\frac{\partial i}{\partial t}$

i가 시간적으로 변화

$$\varepsilon - L\frac{di}{dt} - iR = 0.$$

$$\Rightarrow L\frac{di}{dt} + iR = \varepsilon. \qquad t=0\,;\ i=0.$$

$\frac{di}{dt} = \frac{\varepsilon - iR}{L}$

$\frac{di}{\varepsilon - iR} = \frac{1}{L}dt$

$-\frac{1}{R}\ln(\varepsilon - iR) = \frac{t}{L} + c$

$\ln(\varepsilon - iR) = -\frac{R}{L}t + c$

$\varepsilon - iR = e^{-\frac{R}{L}t}c,$

$i = \frac{1}{R}(\varepsilon - e^{-\frac{Rt}{L}}c)$

$\boxed{c = \varepsilon}$

Solution.

$$i = \frac{\varepsilon}{R}\left[1 - e^{-Rt/L}\right]$$

$$\left(\because\ \frac{di}{dt} = -\frac{\varepsilon}{R}\left(-\frac{R}{L}\right)e^{-Rt/L} = \frac{\varepsilon}{L}e^{-Rt/L}.\right.$$

$$L\frac{di}{dt} + iR$$

$$= \varepsilon e^{-Rt/L} + \varepsilon - \varepsilon e^{-Rt/L}$$

$$\left.= \varepsilon\ \right).$$

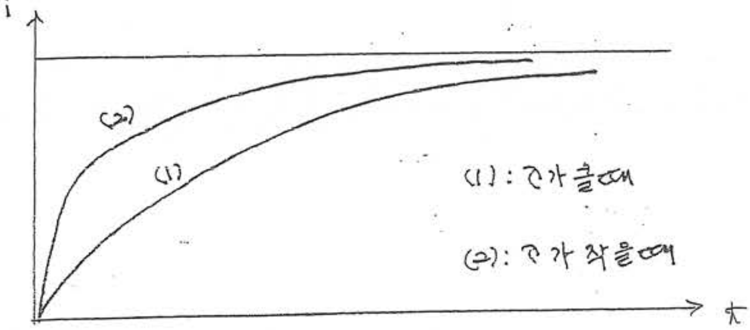
$$t \to \infty\,;\quad i = \frac{\varepsilon}{R}.$$

RL 회로에서의 time-constant

$$\boxed{\begin{array}{l}\text{정의 : } \tau \text{ 시정수 (time constant).}\\[4pt] \tau = \dfrac{L}{R}.\end{array}}$$

$$\left([\tau] = \frac{HA}{V} = \frac{Tm^2}{V} = \frac{C}{J}\cdot\frac{N}{Am}m^2 = sec\right).$$

$$i = \frac{\varepsilon}{R}\left[1 - e^{-\frac{t}{\tau}}\right]$$

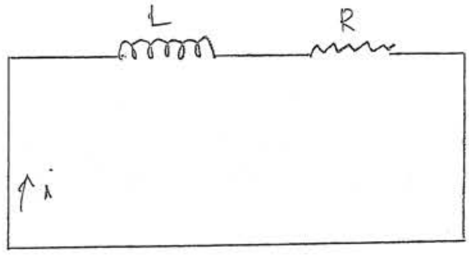


if $t = \tau$

; $i = \frac{\varepsilon}{R}\left[1 - \frac{1}{e}\right] \approx 0.63\,\frac{\varepsilon}{R}$

⇒ 시간이 한참 지난후 기전력을 제거

$t = 0$ 에서 $i = \frac{\varepsilon}{R}$

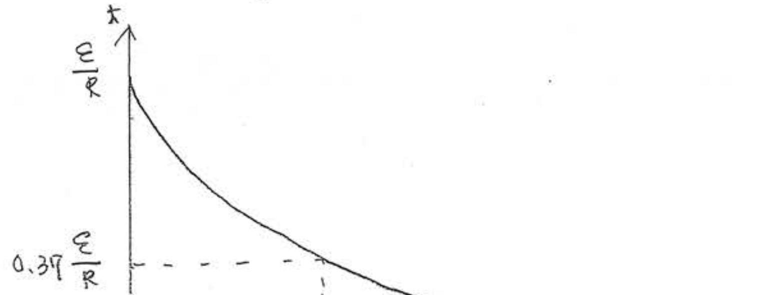


⇒ (예제 2) p160.

$$-L\frac{di}{dt} - iR = 0$$

$$\Rightarrow\ L\frac{di}{dt} + iR = 0 \qquad t=0 \text{ 에서 } i = \frac{\varepsilon}{R}$$

$$i = \frac{\varepsilon}{R}e^{-Rt/L} = \frac{\varepsilon}{R}e^{-\frac{t}{\tau}}$$



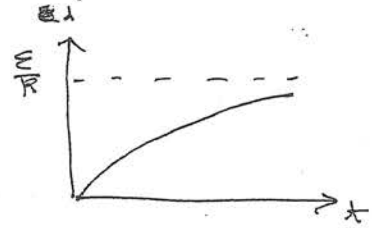
◦ 전류계산 Kirchhoff의 법칙

① 접합점 $\sum I_i = 0$

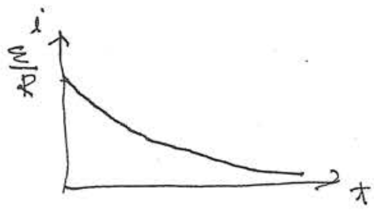
② 폐회로 $\sum \varepsilon_i = \sum R_i i_i$

◦ R-L 회로

① 기전력이 있는 경우



② 기전력을 없애면



$80 i_2 + 64 i_3 = 56$

$80 i_2 + 100 i_3 = 110$

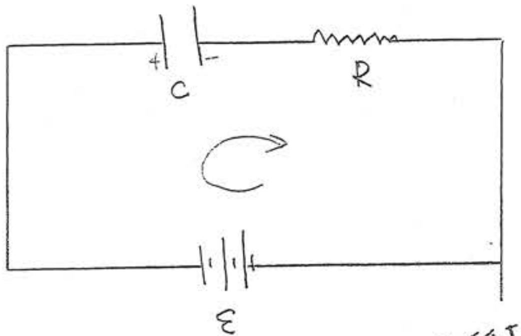
$36 i_3 = 54$

$10 i_2 + 12 = 7$

$10 i_2 = -5$

$C = \frac{Q}{V}$

8. 전기 용량과 저항이 있는 회로 (RC 회로)



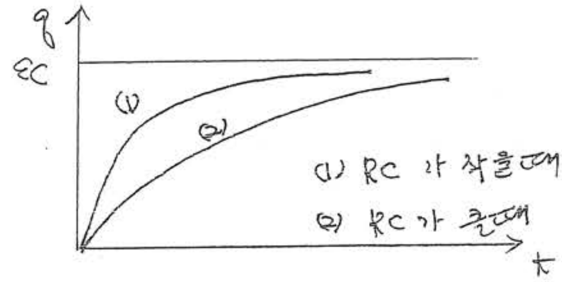
p445 문제 7

$$\varepsilon - \frac{q}{C} - iR = 0$$

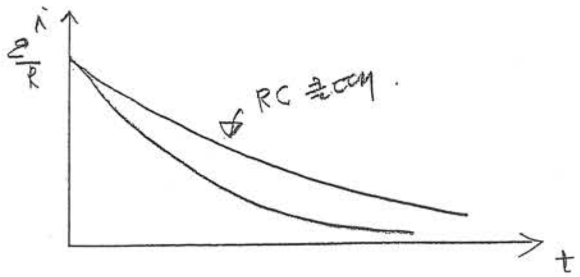
$$\varepsilon - \frac{q}{C} - R\frac{dq}{dt} = 0 \qquad t=0\ ;\ q=0$$

$$\Rightarrow \quad q = \varepsilon C(1 - e^{-t/RC})$$

RC ⇒ time-constant
(시간 상수)

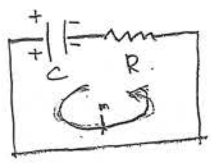


$$\Rightarrow i = \frac{dq}{dt} = \frac{\varepsilon}{R} e^{-t/RC}$$



$C = \frac{Q}{V}$

$q = q_0$ 에서 기전력을 없애면



$\boxed{i = -\frac{dq}{dt}}$ → 음의 값

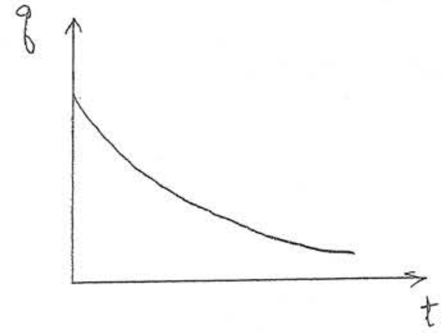
음

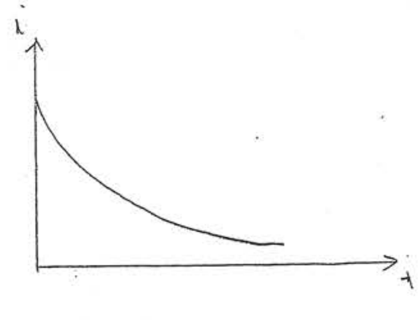
$$-iR + \frac{q}{C} = 0$$

$$R\frac{dq}{dt} + \frac{q}{C} = 0 \quad ; \; t=0 \text{ 에서 } q = q_0.$$

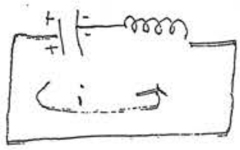
$$q = q_0 e^{-t/RC}$$

$$i = -\frac{dq}{dt} = \frac{q_0}{RC} e^{-t/RC}$$





$i = -\frac{dq}{dt}$



$$-L\frac{di}{dt} + \frac{q}{c} = 0$$

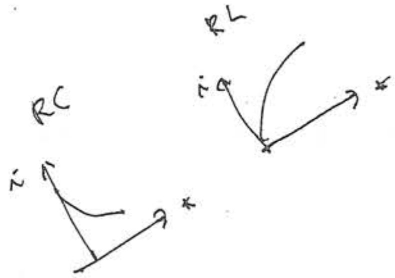
$$\Rightarrow -L\frac{d^2i}{dt^2} + \frac{1}{c}\frac{dq}{dt} = 0.$$

$$L\frac{d^2i}{dt^2} + \frac{i}{c} = 0$$

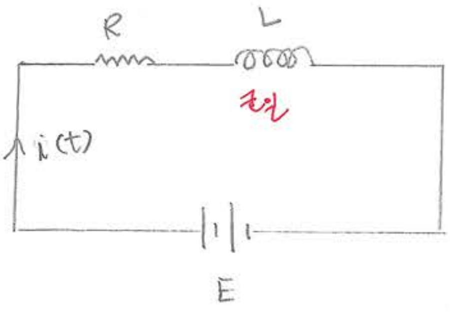
단조화 진동의 기본방정식

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x = A\cos\omega t + B\sin\omega t$$



[예] RL 회로



L : inductance

R : 저항

$$E - iR - L\frac{di}{dt} = 0$$

$$\Rightarrow L\frac{di}{dt} + iR = E \quad \| \times \frac{1}{L}$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$: 선형

$$p = \frac{R}{L}, \quad Q = \frac{E}{L}$$

$$\Rightarrow i(t) = e^{-\frac{R}{L}t}\left[\int \frac{E}{L} e^{\frac{R}{L}t} + c_1\right]$$

$$= \frac{E}{R} + c_1 e^{-\frac{R}{L}t}$$

If $i(0) = 0$,

$$c_1 = -\frac{E}{R}$$

$$\Rightarrow i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$