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Quantum entanglement with generalized uncertainty principle

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Abstract

We explore how the quantum entanglement is modified in the generalized uncertainty principle (GUP)-corrected quantum mechanics by introducing the coupled harmonic oscillator system. Constructing the ground state ρ_0 and its reduced substate $\rho_A = \text{Tr}_B \rho_0$, we compute two entanglement measures of ρ_0, i.e. $\mathcal{E}_{EoF}(\rho_0) = S_{von}(\rho_A)$ and $\mathcal{E}_\gamma(\rho_0) = S_\gamma(\rho_A)$, where S_{von} and S_γ are the von Neumann and Rényi entropies, up to the first order of the GUP parameter α. It is shown that $\mathcal{E}_\gamma(\rho_0)$ increases with increasing α when $\gamma = 2, 3, \cdots$. The remarkable fact is that $\mathcal{E}_{EoF}(\rho_0)$ does not have first-order of α. Based on there results we conjecture that $\mathcal{E}_\gamma(\rho_0)$ increases or decreases with increasing α when $\gamma > 1$ or $\gamma < 1$ respectively for nonnegative real γ.

1. Introduction

As IC (integrated circuit) becomes smaller and smaller in modern classical technology, the effect of quantum mechanics becomes prominent more and more. As a result, quantum technology (technology based on quantum mechanics and quantum information theories [1]) becomes important more and more recently. The representative constructed by quantum technology is a

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quantum computer [2], which was realized recently by making use of superconducting qubits. In quantum information processing quantum entanglement [1,3,4] plays an important role as a physical resource. It is used in various quantum information processing, such as quantum teleportation [5,6], superdense coding [7], quantum cloning [8], quantum cryptography [9,10], quantum metrology [11], and quantum computer [2,12,13]. Furthermore, with many researchers trying to realize such quantum information processing in the laboratory for the last few decades, quantum cryptography and quantum computer seem to approaching the commercial level [14,15].

Physics at the Planck scale suggests the existence of the minimal length (ML). The existence of the ML at this scale seems to be a universal characteristic of quantum gravity [16–18]. It appears in loop quantum gravity [19–22], string theory [23–25], path-integral quantum gravity [26–30], and black hole physics [31]. ML also appeared in some microscope thought-experiment [32]. From an aspect of quantum mechanics the existence of ML modifies the uncertainty principle from Heisenberg uncertainty principle (HUP) [33,34] $\Delta P \Delta Q \geq \frac{\hbar}{2}$ to generalized uncertainty principle (GUP) [35,36]. This is because of the fact that the uncertainty of the position ΔQ should be larger than the ML.

Then, it is natural to ask how the quantum entanglement is modified at the Planck scale. This question might be important to unveil the role of the quantum information at the Planck scale or early universe. In order to explore this issue we choose the GUP as the simplest form proposed in Ref. [36]:

$$\Delta P_i \Delta Q_i \geq \frac{\hbar}{2} \left[1 + \alpha \left\{ (\Delta \mathbf{P})^2 + \langle \widehat{\mathbf{P}} \rangle^2 \right\} + 2\alpha \left\{ (\Delta P_i)^2 + \langle \widehat{P}_i \rangle^2 \right\} \right] \qquad (i = 1, 2, \cdots, d) \tag{1.1}$$

where α is a GUP parameter, which has a dimension (momentum)$^{-2}$. Using $\Delta A \Delta B \geq \frac{1}{2} |\langle [\widehat{A}, \widehat{B}] \rangle|$, Eq. (1.1) induces the modification of the commutation relation as

$$\begin{aligned} \left[\widehat{Q}_i, \widehat{P}_j \right] &= i\hbar \left(\delta_{ij} + \alpha \delta_{ij} \widehat{\mathbf{P}}^2 + 2\alpha \widehat{P}_i \widehat{P}_j \right) \\ \left[\widehat{Q}_i, \widehat{Q}_j \right] &= \left[\widehat{P}_i, \widehat{P}_j \right] = 0. \end{aligned} \tag{1.2}$$

The existence of the ML can be seen in Eq. (1.1). If $\langle \widehat{\boldsymbol{P}} \rangle = 0$ for simplicity, the equality of Eq. (1.1) yields

$$\Delta Q_i^2 \geq \Delta Q_{i,min}^2 = 3\alpha\hbar^2 \qquad (i = 1, 2, \cdots, d) \tag{1.3}$$

which arises when $\Delta P_j = 0 \;\; (j \neq i)$. If α is small, Eq. (1.2) can be solved as

$$\widehat{P}_i = \widehat{p}_i \left(1 + \alpha \widehat{\mathbf{p}}^2 + \alpha^2 \widehat{\mathbf{p}}^4 \right) + \mathcal{O}(\alpha^3) \qquad \widehat{Q}_i = \widehat{q}_i \left(1 + \alpha^2 \widehat{\mathbf{p}}^4 \right) + \mathcal{O}(\alpha^3) \tag{1.4}$$

where p_i and q_i obey the usual Heisenberg algebra $[q_i, p_j] = i\hbar\delta_{ij}$. Thus, the ordering ambiguity occurs at $\mathcal{O}(\alpha^2)$. We will use Eq. (1.4) in the following to compute the quantum entanglement within the first order of α.

As commented before the purpose of this paper is to examine how the quantum entanglement is modified in the GUP-corrected quantum mechanics. In order to explore the issue we consider the two harmonic oscillator systems, which are coupled with each other via the quadratic term. The Hamiltonian of the system is presented in section 2. In section 3 we derive the vacuum state ρ_0 and its reduced substate ρ_A. In this paper we adopt the entanglement measure for ρ_0 as von Neumann and Rényi entropies of the substate;

$$\mathcal{E}_{EoF}(\rho_0) = S_{von}(\rho_A) = -\text{Tr}(\rho_A \ln \rho_A) \tag{1.5}$$
$$\mathcal{E}_\gamma(\rho_0) = S_\gamma(\rho_A) = \frac{1}{1-\gamma} \ln\left(\text{Tr}\rho_A^\gamma\right),$$

where S_{von} and S_γ denote the von Neumann and Rényi entropies. It is easy to show that these entanglement measures are invariant in the choice of the substate due to the Schmidt decomposition [1]. The first entanglement measure is the most popular one called "entanglement of formation (EoF)" [37]. The second measure was used in Ref. [38,39] to explore the entanglement of the anisotropic XY spin chain with a transverse magnetic field in the various phases. In order to compute the entanglement measures in our system we compute $\text{Tr}\rho_A^n$ up to $\mathcal{O}(\alpha)$ in section 4. In section 5 we compute the entanglement of formation $\mathcal{E}_{EoF}(\rho_0)$ and the second entanglement measure $\mathcal{E}_\gamma(\rho_0)$ within the first order of α when γ is positive integer. In this section it is shown that $\mathcal{E}_\gamma(\rho_0)$ increases with increasing α when $\gamma = 2, 3, \cdots$. However, it is also shown that the first-order term of α in $\mathcal{E}_{EoF}(\rho_0)$ is exactly zero. In section 6 a brief conclusion is given.

2. Hamiltonian

Let us consider the two coupled harmonic oscillator system, whose Hamiltonian is

$$\widehat{H}_2 = \frac{1}{2m}\left(\widehat{P}_1^2 + \widehat{P}_2^2\right) + \frac{1}{2}\left[k_0\left(\widehat{X}_1^2 + \widehat{X}_2^2\right) + J\left(\widehat{X}_1 - \widehat{X}_2\right)^2\right], \tag{2.1}$$

where $\left(\widehat{X}_i, \widehat{P}_i\right)$ obeys the GUP (1.2). If we set

$$\widehat{X}_j = \widehat{x}_j \qquad \widehat{P}_j = \widehat{p}_j(1+\alpha\widehat{p}_j^2) \qquad (j=1,2), \tag{2.2}$$

where $\left(\widehat{x}_j, \widehat{p}_j\right)$ obeys the HUP, $\widehat{H}_2$ becomes

$$\widehat{H}_2 = \widehat{h}_1 + \widehat{h}_2 + \frac{J}{2}(\widehat{x}_1 - \widehat{x}_2)^2 + \mathcal{O}(\alpha^2) \tag{2.3}$$

where

$$\widehat{h}_j = \frac{1}{2m}\left(\widehat{p}_j^2 + 2\alpha\widehat{p}_j^4\right) + \frac{1}{2}k_0\widehat{x}_j^2 \qquad (j=1,2). \tag{2.4}$$

Now, we introduce the new coordinates

$$\widehat{y}_1 = \frac{1}{\sqrt{2}}(\widehat{x}_1 + \widehat{x}_2) \qquad \widehat{y}_2 = \frac{1}{\sqrt{2}}(-\widehat{x}_1 + \widehat{x}_2). \tag{2.5}$$

Then, the Hamiltonian $\widehat{H}_2$ reduces to

$$\widehat{H}_2 = \widehat{H}_0 + \Delta\widehat{H} \tag{2.6}$$

where

$$\widehat{H}_0 = \sum_{j=1}^{2}\left[\frac{1}{2m}\left(\widehat{\pi}_j^2 + \alpha\widehat{\pi}_j^4\right) + \frac{1}{2}m\omega_j^2\widehat{y}_j^2\right] + \mathcal{O}(\alpha^2) \qquad \Delta\widehat{H} = \frac{3\alpha}{m}\widehat{\pi}_1^2\widehat{\pi}_2^2. \tag{2.7}$$

In eq. (2.7) $\widehat{\pi}_1$ and $\widehat{\pi}_2$ are the canonical momenta of $\widehat{y}_1$ and $\widehat{y}_2$, and the frequencies are

$$\omega_1 = \sqrt{\frac{k_0}{m}} \qquad \omega_2 = \sqrt{\frac{k_0 + 2J}{m}}. \tag{2.8}$$

In next section we will derive the ground state for $\widehat{H}_2$ up to the order of α by treating $\Delta\widehat{H}$ as a small perturbation.

3. Ground and its reduced states for $\widehat{H}_2$

Before we solve the Schrödinger equation for $\widehat{H}_2$, let us consider the one oscillator problem, whose Hamiltonian is $\widehat{H}_1 = \frac{\widehat{p}^2}{2m} + \frac{\alpha}{m}\widehat{p}^4 + \frac{1}{2}m\omega^2\widehat{x}^2 + \mathcal{O}(\alpha^2)$. In Ref. [40] the Schrödinger equation for $\widehat{H}_1$ is solved up to $\mathcal{O}(\alpha)$. For example, the nth eigenstate and the corresponding eigenvalue are

$$\begin{aligned}
&\psi_n(x:\alpha,\omega) = \phi_n(x:\omega) \qquad (3.1)\\
&+(\alpha m\hbar\omega)\Bigg[\frac{(2n+3)\sqrt{(n+1)(n+2)}}{4}\phi_{n+2}(x:\omega) - \frac{(2n-1)\sqrt{n(n-1)}}{4}\phi_{n-2}(x:\omega)\\
&+\frac{\sqrt{n(n-1)(n-2)(n-3)}}{16}\phi_{n-4}(x:\omega) - \frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{16}\phi_{n+4}(x:\omega)\Bigg]\\
&+\mathcal{O}(\alpha^2)\\
&\mathcal{E}_n(\alpha,\omega) = \left(n+\frac{1}{2}\right)\hbar\omega\left[1+\frac{3(2n^2+2n+1)}{2(2n+1)}(\alpha m\hbar\omega)\right] + \mathcal{O}(\alpha^2),
\end{aligned}$$

where $n = 0, 1, 2, \cdots$ and

$$\phi_n(x:\omega) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left[-\frac{m\omega}{2\hbar}x^2\right]. \qquad (3.2)$$

In Eq. (3.2) $H_n(z)$ is a nth-order Hermite polynomial. We assume $\phi_m(z:\omega) = 0$ for $m < 0$.

Now, let us consider the Schrödinger equation for $\widehat{H}_0$:

$$\widehat{H}_0\phi^{(0)}_{n_1,n_2}(x_1,x_2) = E^{(0)}_{n_1,n_2}\phi^{(0)}_{n_1,n_2}(x_1,x_2). \qquad (3.3)$$

Since $\widehat{H}_0$ is diagonalized, the eigenvalue and the corresponding eigenfunction are

$$\begin{aligned}
&E^{(0)}_{n_1,n_2} = \mathcal{E}_{n_1}\left(\frac{\alpha}{2},\omega_1\right) + \mathcal{E}_{n_2}\left(\frac{\alpha}{2},\omega_2\right) \qquad (3.4)\\
&\phi^{(0)}_{n_1,n_2}(x_1,x_2) = \psi_{n_1}\left(y_1:\frac{\alpha}{2},\omega_1\right)\psi_{n_2}\left(y_2:\frac{\alpha}{2},\omega_2\right).
\end{aligned}$$

If we treat $\Delta\widehat{H}$ as a small perturbation, the ground state $\Phi_{0,0}$ and its eigenvalue $E_{0,0}$ for $\widehat{H}_2$ become

$$\begin{aligned}
&E_{0,0} = \frac{\hbar}{2}(\omega_1+\omega_2) + \frac{3}{8}(\alpha m\hbar^2)(\omega_1+\omega_2)^2 + \mathcal{O}(\alpha^2) \qquad (3.5)\\
&\Phi_{0,0}(x_1,x_2) = \phi_0(y_1:\omega_1)\phi_0(y_2:\omega_2)\\
&+(\alpha m\hbar)\Bigg[\frac{3\sqrt{2}}{8}(\omega_1+\omega_2)\{\phi_0(y_1:\omega_1)\phi_2(y_2:\omega_2) + \phi_2(y_1:\omega_1)\phi_0(y_2:\omega_2)\}\\
&\qquad -\frac{\sqrt{6}}{16}\{\omega_1\phi_4(y_1:\omega_1)\phi_0(y_2:\omega_2) + \omega_2\phi_0(y_1:\omega_1)\phi_4(y_2:\omega_2)\}\\
&\qquad -\frac{3}{4}\frac{\omega_1\omega_2}{\omega_1+\omega_2}\phi_2(y_1:\omega_1)\phi_2(y_2:\omega_2)\Bigg] + \mathcal{O}(\alpha^2).
\end{aligned}$$

Thus, the density matrix for the ground state is

$$
\begin{aligned}
&\rho_0[x_1, x_2 : x_1', x_2'] \\
&= \Phi_{0,0}(x_1, x_2)\Phi^*_{0,0}(x_1', x_2') \\
&= \phi_0(y_1 : \omega_1)\phi_0(y_2, \omega_2)\phi_0(y_1' : \omega_1)\phi_0(y_2' : \omega_2) \\
&+(\alpha m \hbar)\Bigg\{\phi_0(y_1 : \omega_1)\phi_0(y_2, \omega_2) \\
&\qquad \times\Bigg[\frac{3\sqrt{2}}{8}(\omega_1+\omega_2)\left\{\phi_0(y_1' : \omega_1)\phi_2(y_2' : \omega_2) + \phi_2(y_1' : \omega_1)\phi_0(y_2' : \omega_2)\right\} \\
&\qquad\quad -\frac{\sqrt{6}}{16}\Big\{\omega_1\phi_4(y_1' : \omega_1)\phi_0(y_2' : \omega_2) + \omega_2\phi_0(y_1' : \omega_1)\phi_4(y_2' : \omega_2)\Big\} \\
&\qquad\quad -\frac{3}{4}\frac{\omega_1\omega_2}{\omega_1+\omega_2}\phi_2(y_1' : \omega_1)\phi_2(y_2' : \omega_2)\Bigg] \\
&\qquad\quad +(y_j \leftrightarrow y_j')\Bigg\} + \mathcal{O}(\alpha^2)
\end{aligned}
\tag{3.6}
$$

where

$$
y_1' = \frac{1}{\sqrt{2}}(x_1' + x_2') \qquad\qquad y_2' = \frac{1}{\sqrt{2}}(-x_1' + x_2'). \tag{3.7}
$$

In order to examine the entanglement of ρ_0 we should derive its reduced substate. After long and tedious calculation one can derive the reduced state $\rho_A = \mathrm{Tr}_B\,\rho_0$ in a form

$$
\begin{aligned}
&\rho_A[x_1, x_1'] \\
&\equiv \int dx_2 \rho_0[x_1, x_2 : x_1', x_2] \\
&= \sqrt{\frac{2m\omega_1\omega_2}{\pi\hbar(\omega_1+\omega_2)}}\, e^{-a(x_1^2 + x_1'^2) + 2bx_1x_1'} \\
&\quad \times\Bigg[1 + \frac{\alpha m}{256\hbar(\omega_1+\omega_2)^5}\Bigg\{g_1(x_1^4 + x_1'^4) + g_2(x_1^3x_1' + x_1x_1'^3) \\
&\qquad\qquad + g_3x_1^2x_1'^2 + g_4(x_1^2 + x_1'^2) + g_5x_1x_1' + g_6\Bigg\}\Bigg] + \mathcal{O}(\alpha^2)
\end{aligned}
\tag{3.8}
$$

where

$$
a = \frac{m}{8\hbar(\omega_1+\omega_2)}(\omega_1^2 + \omega_2^2 + 6\omega_1\omega_2) \qquad\qquad b = \frac{m(\omega_1-\omega_2)^2}{8\hbar(\omega_1+\omega_2)} \tag{3.9}
$$

and

$$
\begin{aligned}
g_1 &= -m^2(\omega_1^8 + 5\omega_1^7\omega_2 + 94\omega_1^6\omega_2^2 + 459\omega_1^5\omega_2^3 + 930\omega_1^4\omega_2^4 + 459\omega_1^3\omega_2^5 + 94\omega_1^2\omega_2^6 \\
&\qquad + 5\omega_1\omega_2^7 + \omega_2^8) \\
g_2 &= 4m^2(\omega_1-\omega_2)^2(\omega_1^6 + 7\omega_1^5\omega_2 + 35\omega_1^4\omega_2^2 + 106\omega_1^3\omega_2^3 + 35\omega_1^2\omega_2^4 + 7\omega_1\omega_2^5 + \omega_2^6) \\
g_3 &= -6m^2(\omega_1-\omega_2)^4(\omega_1^4 + 9\omega_1^3\omega_2 + 28\omega_1^2\omega_2^2 + 9\omega_1\omega_2^3 + \omega_2^4)
\end{aligned}
\tag{3.10}
$$

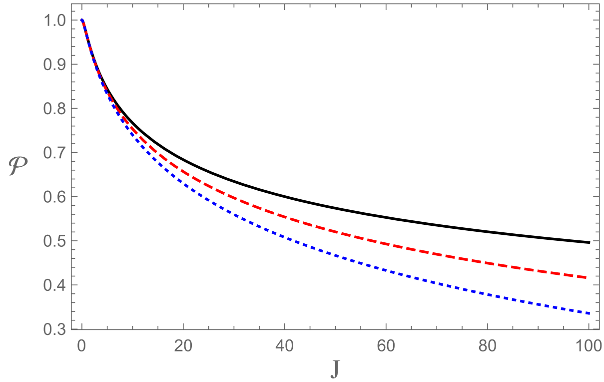

Fig. 1. The J-dependence of the purity function when $k_0 = m = \hbar = 1$. The black solid, red dashed, and blue dotted lines correspond to $\alpha = 0$, 0.2, and 0.4 respectively. The figure shows that the reduced state ρ_A becomes more mixed with increasing the GUP parameter α.

$$g_4 = 24\hbar m(\omega_1 + \omega_2)(2\omega_1^6 + 23\omega_1^5\omega_2 + 82\omega_1^4\omega_2^2 + 170\omega_1^3\omega_2^3 + 82\omega_1^2\omega_2^4 + 23\omega_1\omega_2^5 + 2\omega_2^6)$$

$$g_5 = -48\hbar m(\omega_1 + \omega_2)(\omega_1 - \omega_2)^2(2\omega_1^4 + 11\omega_1^3\omega_2 + 30\omega_1^2\omega_2^2 + 11\omega_1\omega_2^3 + 2\omega_2^4)$$

$$g_6 = -48\hbar^2(\omega_1 + \omega_2)^2(4\omega_1^4 + 17\omega_1^3\omega_2 + 38\omega_1^2\omega_2^2 + 17\omega_1\omega_2^3 + 4\omega_2^4).$$

It is useful to note

$$a + b = \frac{m(\omega_1 + \omega_2)}{4\hbar} \qquad a - b = \frac{m\omega_1\omega_2}{\hbar(\omega_1 + \omega_2)}. \tag{3.11}$$

Also, one can show

$$\frac{3}{16(a-b)^2}(2g_1 + 2g_2 + g_3) + \frac{1}{4(a-b)}(2g_4 + g_5) + g_6 = 0. \tag{3.12}$$

Using Eq. (3.12) one can explicitly show $\mathrm{Tr}\rho_A = 1$ within the leading order of α, which guarantees that ρ_A is a mixed quantum state.

In order to quantify how much ρ_A is mixed we compute the purity function, whose expression is

$$\mathcal{P}(\rho_A) \equiv \mathrm{Tr}\rho_A^2 = \frac{2\sqrt{\omega_1\omega_2}}{\omega_1 + \omega_2}\left[1 - \frac{3\alpha m\hbar}{32}\frac{(\omega_1 - \omega_2)^4}{(\omega_1 + \omega_2)^3} + \mathcal{O}(\alpha^2)\right]. \tag{3.13}$$

Fig. 1 shows the J-dependence of the purity function when $k_0 = m = \hbar = 1$. The black solid, red dashed, and blue dotted lines correspond to $\alpha = 0$, 0.2, and 0.4 respectively. When $J = 0$, ρ_A is a pure state regardless of α. With increasing J, ρ_A becomes more and more mixed. The remarkable fact is that at fixed J the GUP parameter α makes ρ_A to be more mixed. This is due to the minus sign in the bracket of Eq. (3.13).

4. Calculation of $\mathrm{Tr}\rho_A^n$

The most typical way for computing the Rényi and von Neumann entropies of ρ_A is to solve the eigenvalue equation

$$\int dx_1' \rho_A[x_1, x_1'] f_n(x_1') = \lambda_n f_n(x_1). \tag{4.1}$$

If ρ_A is a Gaussian state, the eigenvalue equation (4.1) can be solved straightforwardly [41]. Then, the Rényi entropy of order γ and von Neumann entropy can be computed by making use of the eigenvalue λ_n as follows:

$$\mathcal{E}_\gamma(\rho_0) = \frac{1}{1-\gamma} \ln \sum_n (\lambda_n)^\gamma \qquad \mathcal{E}_{EoF}(\rho_0) = -\sum_n \lambda_n \ln \lambda_n, \tag{4.2}$$

where γ is arbitrary nonnegative real. The problem is that ρ_A is not Gaussian state if $\alpha \neq 0$ as Eq. (3.8) shows. Thus, it seems to be extremely difficult to solve Eq. (4.1) directly.

Although we cannot solve the eigenvalue equation (4.1) explicitly, we can compute $\mathcal{E}_{\gamma=n}(\rho_0)$ and $\mathcal{E}_{EoF}(\rho_0)$ at least up to the $\mathcal{O}(\alpha)$ by computing $\text{Tr}\rho_A^n$ [42]. In this case $\mathcal{E}_{\gamma=n}(\rho_0)$ can be computed by

$$\mathcal{E}_{\gamma=n}(\rho_0) = \frac{1}{1-n} \ln \text{Tr}\rho_A^n. \tag{4.3}$$

Then, $\mathcal{E}_{EoF}(\rho_0)$ also can be computed from Eq. (4.3) by taking $n \to 1$ limit. In this reason we will compute $\text{Tr}\rho_A^n$ in this section within $\mathcal{O}(\alpha)$.

From Eq. (3.8) one can show

$$\begin{aligned}
\text{Tr}\rho_A^n &\equiv \int dx_1 \cdots dx_n \rho_A[x_1, x_2]\rho_A[x_2, x_3] \cdots \rho_A[x_{n-1}, x_n]\rho_A[x_n, x_1] \\
&= \left(\frac{2m\omega_1\omega_2}{\pi\hbar(\omega_1+\omega_2)} \right)^{n/2} \int dx_1 \cdots dx_n \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger \right] \\
&\times \Bigg\{ 1 + \frac{\alpha m}{256\hbar(\omega_1+\omega_2)^5} \Bigg[2g_1 \left(x_1^4 + \cdots + x_n^4 \right) \\
&\qquad + g_2 \left[x_1 x_2 (x_1^2 + x_2^2) + \cdots + x_{n-1} x_n (x_{n-1}^2 + x_n^2) + x_n x_1 (x_n^2 + x_1^2) \right] \\
&\qquad + g_3 (x_1^2 x_2^2 + \cdots + x_{n-1}^2 x_n^2 + x_n^2 x_1^2) + 2g_4 (x_1^2 + \cdots + x_n^2) \\
&\qquad + g_5 (x_1 x_2 + \cdots + x_{n-1} x_n + x_n x_1) + n g_6 \Bigg] + \mathcal{O}(\alpha^2) \Bigg\},
\end{aligned} \tag{4.4}$$

where X is a n-dimensional row vector defined by $\boldsymbol{X} = (x_1, x_2, \cdots, x_n)$ and G_n is a $n \times n$ matrix given by

$$G_n = \begin{pmatrix}
2a & -b & & & & -b \\
-b & 2a & \bullet & & & \\
 & \bullet & \bullet & \bullet & & \\
 & & \bullet & \bullet & \bullet & \\
 & & & \bullet & \bullet & -b \\
-b & & & & -b & 2a
\end{pmatrix}. \tag{4.5}$$

In Eq. (4.5) the matrix components in the empty space are all zero. As shown in Ref. [42], the determinant of G_n is

$$\det G_n = 2^{-n} \left[\left(\sqrt{a+b} + \sqrt{a-b} \right)^n - \left(\sqrt{a+b} - \sqrt{a-b} \right)^n \right]^2$$

$$= \left(\frac{m}{8\hbar(\omega_1 + \omega_2)} \right)^n \mathcal{Z}^2_{-,2n} \tag{4.6}$$

where

$$\mathcal{Z}_{\pm,\ell} = \left(\sqrt{\omega_2} + \sqrt{\omega_1}\right)^\ell \pm \left(\sqrt{\omega_2} - \sqrt{\omega_1}\right)^\ell . \tag{4.7}$$

Then, it is possible to show

$$\int dx_1 \cdots dx_n \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] = \frac{\pi^{n/2}}{\sqrt{\det G_n}} \equiv h_n \tag{4.8}$$

$$\begin{aligned}
&\int dx_1 \cdots dx_n (x_1^2 + \cdots + x_n^2) \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] \\
&= h_n \frac{n}{4\sqrt{a^2 - b^2}} \frac{(\sqrt{a+b} + \sqrt{a-b})^n + (\sqrt{a+b} - \sqrt{a-b})^n}{(\sqrt{a+b} + \sqrt{a-b})^n - (\sqrt{a+b} - \sqrt{a-b})^n} \\
&= h_n \frac{n\hbar}{2m\sqrt{\omega_1\omega_2}} \frac{\mathcal{Z}_{+,2n}}{\mathcal{Z}_{-,2n}} \\
&\int dx_1 \cdots dx_n (x_1 x_2 + \cdots + x_{n-1}x_n + x_n x_1) \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] \\
&= h_n \frac{nb}{2\sqrt{a^2 - b^2}} \frac{(\sqrt{a+b} + \sqrt{a-b})^{n-2} + (\sqrt{a+b} - \sqrt{a-b})^{n-2}}{(\sqrt{a+b} + \sqrt{a-b})^n - (\sqrt{a+b} - \sqrt{a-b})^n} \\
&= h_n \frac{n\hbar(\omega_1 - \omega_2)^2}{2m\sqrt{\omega_1\omega_2}} \frac{\mathcal{Z}_{+,2n-4}}{\mathcal{Z}_{-,2n}}.
\end{aligned}$$

Also, one can show [42]

$$\begin{aligned}
&\int dx_1 \cdots dx_n (x_1^4 + \cdots + x_n^4) \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] \\
&= h_n \frac{3n}{4} \frac{(\det H_{n-1})^2}{(\det G_n)^2} = h_n \frac{3n\hbar^2}{4m^2\omega_1\omega_2} \left(\frac{\mathcal{Z}_{+,2n}}{\mathcal{Z}_{-,2n}}\right)^2 \\
&\int dx_1 \cdots dx_n (x_1^2 x_2^2 + \cdots + x_{n-1}^2 x_n^2 + x_n^2 x_1^2) \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] \\
&= h_n \frac{n}{4(\det G_n)^2} \Big[12a^2 (\det H_{n-2})^2 - 12ab^2 (\det H_{n-2})(\det H_{n-3}) \\
&\qquad\qquad + 3b^4 (\det H_{n-3})^2 - 2(\det H_{n-2})(\det G_n) \Big] \\
&= h_n \frac{n\hbar^2}{4m^2\omega_1\omega_2 \mathcal{Z}^2_{-,2n}} \left[3\mathcal{Z}^2_{+,2n} - 16(\omega_1 + \omega_2)\sqrt{\omega_1\omega_2}\,\mathcal{Z}_{-,4n-4} \right] \\
&\int dx_1 \cdots dx_n \left[x_1 x_2 (x_1^2 + x_2^2) + \cdots + x_{n-1}x_n (x_{n-1}^2 + x_n^2) + x_n x_1 (x_n^2 + x_1^2) \right] \\
&\qquad\qquad \times \exp\left[-\boldsymbol{X} G_n \boldsymbol{X}^\dagger\right] \\
&= h_n \frac{3n}{2} \frac{\left[b^{n-1} + b(\det H_{n-2})\right]\left[2a(\det H_{n-2}) - b^2(\det H_{n-3})\right]}{(\det G_n)^2}
\end{aligned} \tag{4.9}$$

$$= h_n \frac{3n\hbar^2(\omega_1-\omega_2)^2}{2m^2\omega_1\omega_2} \frac{\mathcal{Z}_{+,2n}}{\mathcal{Z}^3_{-,2n}} \left[8(\omega_1+\omega_2)\sqrt{\omega_1\omega_2}(\omega_1-\omega_2)^{2(n-2)} + \mathcal{Z}_{-,4n-4} \right],$$

where H_n is a $n \times n$ tridiagonal matrix given by

$$H_n = \begin{pmatrix} 2a & -b & & & & \\ -b & 2a & \bullet & & & \\ & \bullet & \bullet & \bullet & & \\ & & \bullet & \bullet & \bullet & \\ & & & \bullet & \bullet & -b \\ & & & & -b & 2a \end{pmatrix}. \tag{4.10}$$

It is straightforward to show

$$\det H_n = \frac{1}{\sqrt{a^2-b^2}} \left[a\sqrt{\det G_{2n}} - \frac{b^2}{2}\sqrt{\det G_{2n-2}} \right] \tag{4.11}$$
$$= \frac{1}{8(\omega_1+\omega_2)\sqrt{\omega_1\omega_2}} \left(\frac{m}{8\hbar(\omega_1+\omega_2)} \right)^n \mathcal{Z}_{-,4n+4}.$$

Eq. (4.11) is valid for any nonnegative integer n.

Inserting Eqs. (4.8) and (4.9) into Eq (4.4), one can show that $\mathrm{Tr}\rho_A^n$ can be written as a form

$$\mathrm{Tr}\rho_A^n = \frac{(1-\xi)^n}{1-\xi^n} \left[1 - \frac{3n(\alpha m\hbar)}{4096} \frac{(\omega_1-\omega_2)^4}{\omega_1\omega_2(\omega_1+\omega_2)^5} \frac{\mathcal{Z}_{-,4}}{\mathcal{Z}^2_{-,2n}} \mathcal{J}_n(\omega_1,\omega_2) + \mathcal{O}(\alpha^2) \right] \tag{4.12}$$

where

$$\mathcal{J}_n(\omega_1,\omega_2) = \mathcal{Z}_{-,2n} \left[\mathcal{Z}_{+,2n+4} + 2(\omega_1-\omega_2)^2 \mathcal{Z}_{+,2n} - 2(\omega_1-\omega_2)^6 \mathcal{Z}_{+,2n-8} \right] \tag{4.13}$$
$$-3(\omega_1-\omega_2)^{2n}\mathcal{Z}_{-,4} - (\omega_1-\omega_2)^8 \mathcal{Z}_{-,4n-12}$$

and $\xi = \left[(\sqrt{\omega_2}-\sqrt{\omega_1})/(\sqrt{\omega_2}+\sqrt{\omega_1}) \right]^2$.

It is easy to show that when $n=2$, Eq. (4.12) reproduces the purity function in Eq. (3.13). When $n=3$, Eq. (4.12) yields

$$\mathrm{Tr}\rho_A^3 = \frac{16\omega_1\omega_2}{(3\omega_1+\omega_2)(\omega_1+3\omega_2)} \left[1 - \frac{9(\alpha m\hbar)}{4} \frac{(\omega_1+\omega_2)(\omega_1-\omega_2)^4}{(3\omega_1+\omega_2)^2(\omega_1+3\omega_2)^2} + \mathcal{O}(\alpha^2) \right]. \tag{4.14}$$

It is not difficult to show that as expected, Eq. (4.14) exactly coincides with $\int dx_1 dx_2 dx_3 \rho_A[x_1, x_2]\rho_A[x_2,x_3]\rho_A[x_3,x_1]$. In next section we will discuss the entanglement of ρ_0 by making use of Eq. (4.12).

5. Entanglement for ρ_0

The second entanglement measure $\mathcal{E}_{\gamma=n}(\rho_0)$ can be derived by inserting Eq. (4.12) into Eq. (4.3), which is

$$\mathcal{E}_{\gamma=n} = \frac{1}{1-n} \Bigg[\ln \frac{(1-\xi)^n}{1-\xi^n}$$
$$+ \ln \left\{ 1 - \frac{3n(\alpha m\hbar)}{4096} \frac{(\omega_1-\omega_2)^4}{\omega_1\omega_2(\omega_1+\omega_2)^5} \frac{\mathcal{Z}_{-,4}}{\mathcal{Z}^2_{-,2n}} \mathcal{J}_n(\omega_1,\omega_2) + \mathcal{O}(\alpha^2) \right\} \Bigg]. \tag{5.1}$$

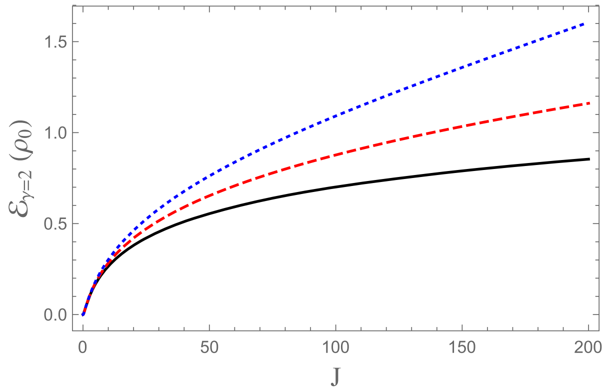

Fig. 2. The J-dependence of the $\mathcal{E}_{\gamma=2}(\rho_0)$ when $k_0 = m = \hbar = 1$. The black solid, red dashed, and blue dotted lines correspond to $\alpha = 0$, 0.2, and 0.4 respectively. This figure shows that it increases with increasing the GUP parameter α.

Now, let us compute the EoF of ρ_0. This is achieved by taking $n \to 1$ limit to Eq. (5.1). One can show that $\mathcal{J}_n(\omega_1, \omega_2)$ in Eq. (4.13) satisfies

$$\mathcal{J}_1(\omega_1, \omega_2) = \frac{d}{dn}\mathcal{J}_n(\omega_1, \omega_2)\bigg|_{n=1} = 0. \tag{5.2}$$

Eq. (5.2) implies that the EoF of ρ_0 does not involve the first order of α. Thus, it is expressed as

$$\mathcal{E}_{EoF}(\rho_0) = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi + \mathcal{O}(\alpha^2). \tag{5.3}$$

In Fig. 2 we plot the J-dependence of $\mathcal{E}_{\gamma=2}(\rho_0)$ when α is 0 (black solid line), 0.2 (red dashed line), and 0.4 (blue dotted line). We set $k_0 = m = \hbar = 1$ for simplicity. This figure shows that it increases with increasing the GUP parameter α. This can be seen from the fact that the second term in the bracket of Eq. (5.1) increases in the negative region with respect to α.

6. Conclusions

In this paper we examine how the quantum entanglement is modified in the GUP-corrected quantum mechanics. In order to explore this issue we consider the coupled harmonic oscillator system. Constructing the vacuum state ρ_0 and its substate ρ_A, we compute the entanglement by choosing the EoF $\mathcal{E}_{EoF}(\rho_0) = S_{von}(\rho_A)$ and the Rényi entropy $\mathcal{E}_\gamma(\rho_0) = S_\gamma(\rho_A)$ of the substate as entanglement measures. It is shown that the second entanglement measure increases with increasing α when $\gamma = 2, 3, \cdots$. Remarkable fact is that the EoF is invariant within the first-order of α in quantum mechanics with HUP and GUP. Since $\mathcal{E}_{EoF}(\rho_0) = \lim_{\gamma\to 1}\mathcal{E}_\gamma(\rho_0)$, we conjecture that $\mathcal{E}_\gamma(\rho_0)$ decreases with increasing α when $\gamma < 1$.

In order to compute $\mathcal{E}_\gamma(\rho_0)$ for nonnegative real γ we should derive the eigenvalue λ_n in Eq. (4.1). However, it seems to be highly difficult (or might be impossible) to derive the eigenvalue due to non-Gaussian nature of ρ_A. Thus, we cannot confirm our conjecture directly. If $\mathcal{E}_\gamma(\rho_0)$ is equal to the right-hand side of Eq. (5.1) with changing only $n \to \gamma$ for all nonnegative real γ, it is possible to show that our conjecture is right. For example, we plot the J- dependence and α-dependence of $\mathcal{E}_{\gamma=0.7}(\rho_0)$ in Fig. 3(a) and Fig. 3(b) respectively. We choose various α

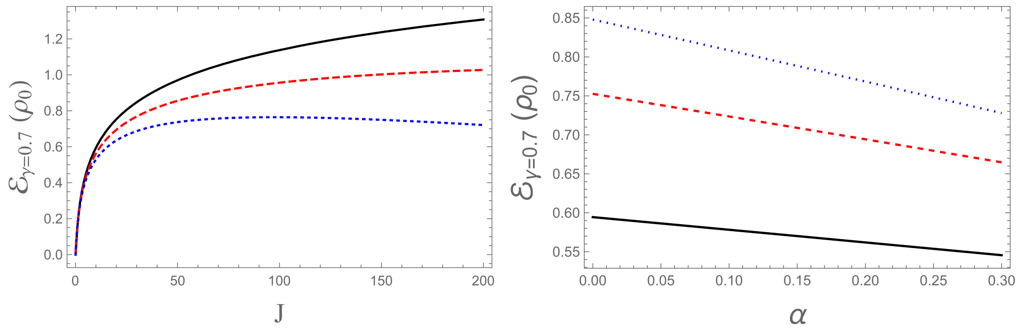

Fig. 3. (a) The J-dependence of the $\mathcal{E}_{\gamma=0.7}(\rho_0)$ when $k_0 = m = \hbar = 1$. The black solid, red dashed, and blue dotted lines correspond to $\alpha = 0$, 0.2, and 0.4 respectively. (b) The α-dependence of the $\mathcal{E}_{\gamma=0.7}(\rho_0)$ when $k_0 = m = \hbar = 1$. The black solid, red dashed, and blue dotted lines correspond to $J = 10$, 20, and 30 respectively. Both figures shows that it decreases with increasing the GUP parameter α.

in Fig. 3(a) and various J in Fig. 3(b). These two figures show that $\mathcal{E}_{\gamma=0.7}(\rho_0)$ decreases with increasing α, which is consistent with our conjecture.

The well-known example of the Planck scale is an early universe ($t \leq 10^{-43}(s)$ after big bang). However, we do not understand the role of quantum information at this early stage, in particular in the context of cosmology. We hope to explore this issue in the future.

CRediT authorship contribution statement

The author "DaeKil Park" performs all calculation and drawing all figures by using Mathematica.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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