CH8 Fourier 해석

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※ Fourier Series

$f(x)$: defined at $-L \le x \le L$

$\Rightarrow$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad (n = 0, 1, 2, \cdots)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad (n = 1, 2, \cdots)$$

Fourier Series

Pf)

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases} \qquad \text{— ①}$$

Then one can show

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0$$

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn}$$

└ ②

Put

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \qquad \text{— ③}$$

Then

$$\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2}a_0 \underbrace{\int_{-L}^{L} dx}_{=2L} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx}_{=0} + b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx}_{=0} \right]$$

$$= L a_0$$

$$\Rightarrow \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad — ④$$

Consider

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) f(x)\,dx$$

$$= \frac{a_0}{2} \underbrace{\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx}_{L\,\delta_{mn}}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx}_{=0}$$

$$= L a_m$$

$$\Rightarrow \quad a_m = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) f(x)\,dx \qquad — ⑤$$

$$(m = 1, 2, \cdots)$$

From ④ and ⑤

$$a_n = \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x)\,dx \qquad — ⑥$$

$$(n = 0, 1, 2, \cdots)$$

Consider

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) f(x)\, dx$$

$$= \frac{a_0}{2} \underbrace{\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) dx}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx}_{=0}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx}_{L\,\delta_{mn}}$$

$$= L\, b_m$$

$$\Rightarrow \quad b_m = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) f(x)\, dx \qquad (m = 1, 2, \cdots)$$

$$\Rightarrow \quad b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x)\, dx \quad - ⑦$$

Eq. ⑥ and Eq. ⑦ complete the proof. ※

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(예제 2.1)

$$f(x) = x \qquad (-\pi \le x \le \pi)$$

Fourier series of $f(x)$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx$$

$$= 0 \qquad - ①$$

$$(\because \int_{-\pi}^{\pi} g(x)dx = 0 \quad \text{if} \quad g(-x) = -g(x))$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx$$

$$= -\frac{2}{n} \underbrace{\cos(n\pi)}_{=(-1)^n}$$

$$= \frac{2}{n}(-1)^{n+1} \qquad - ②$$

$$b_1 = 2, \quad b_2 = -1, \quad b_3 = \frac{2}{3}, \quad b_4 = -\frac{2}{4}, \quad \cdots$$

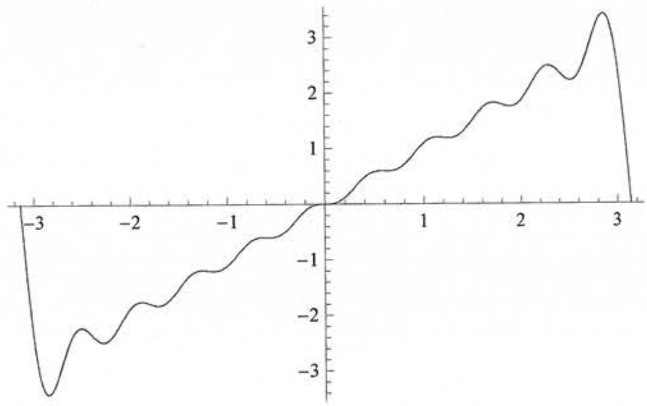
$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx$$

$$= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \cdots$$

```
In[1]:= Plot[Sum[(-1)^(n + 1) (2 / n) Sin[n x], {n, 1, 10}], {x, - Pi, Pi}]
```

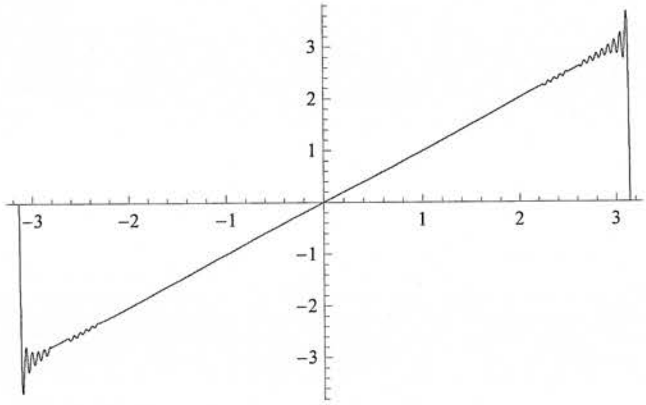
Out[1]=



$\sum_{n=1}^{10} b_n \sin nx$

```
In[2]:= Plot[Sum[(-1)^(n + 1) (2 / n) Sin[n x], {n, 1, 100}], {x, - Pi, Pi}]
```

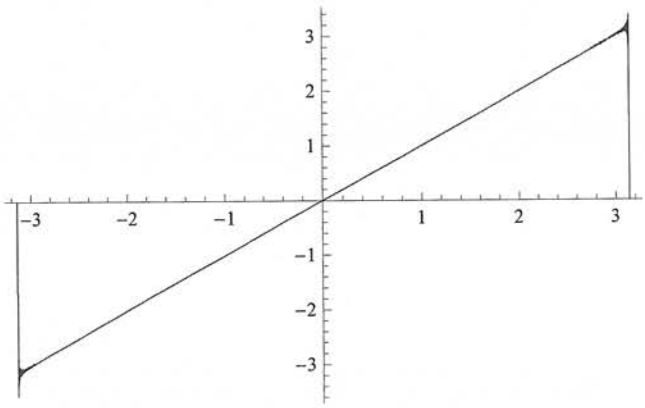
Out[2]=



$\sum_{n=1}^{100} b_n \sin nx$

```
In[3]:= Plot[Sum[(-1)^(n + 1) (2 / n) Sin[n x], {n, 1, 1000}], {x, - Pi, Pi}]
```

Out[3]=



$\sum_{n=1}^{1000} b_n \sin nx$

(예제 8.2)

$$f(x) = \begin{cases} 0 & -3 \le x \le 0 \\ x & 0 \le x \le 3 \end{cases}$$

$$a_n = \frac{1}{3} \int_{-3}^{3} f(x) \cos\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{1}{3} \int_{0}^{3} x \cos\left(\frac{n\pi x}{3}\right) dx$$

$$a_0 = \frac{3}{2}$$

$$a_n = \frac{3}{n^2\pi^2} \left[(-1)^n - 1 \right] \quad (n=1,2,\cdots)$$

) — ①

$$b_n = \frac{1}{3} \int_0^3 x \sin\left(\frac{n\pi x}{3}\right) dx = \frac{3}{n\pi} (-1)^{n+1} \quad \text{— ②}$$

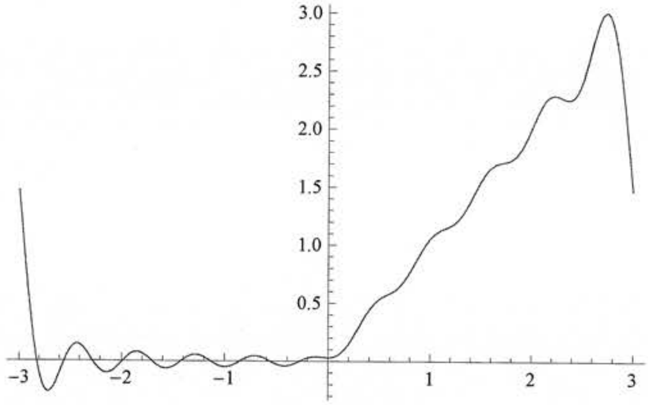
$(n=1,2,\cdots)$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{3}\right) + b_n \sin\left(\frac{n\pi x}{3}\right) \right]$$

$$= \frac{3}{4} + \sum_{n=1}^{\infty} \frac{3}{n^2\pi^2} \left[(-1)^n - 1 \right] \cos\left(\frac{n\pi x}{3}\right) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n\pi} \sin \frac{n\pi x}{3}$$

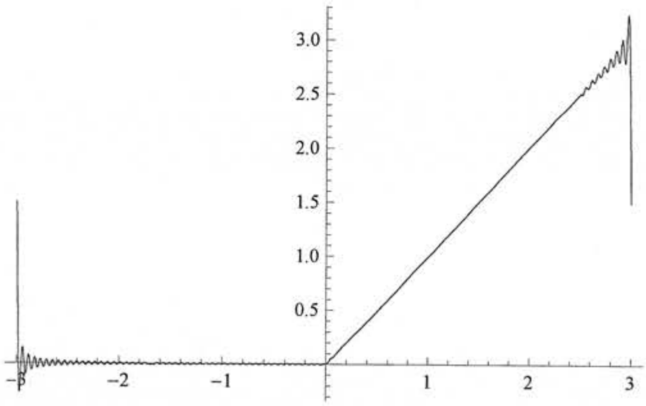
```
In[4]:= a[n_] := 3 ((-1)^n - 1) / (n^2 Pi^2);
        b[n_] := (-1)^(n + 1) 3 / (n Pi);
        Plot[3 / 4 + Sum[a[n] Cos[n Pi x / 3] + b[n] Sin[n Pi x / 3], {n, 1, 10}], {x, -3, 3}]
```

Out[6]=



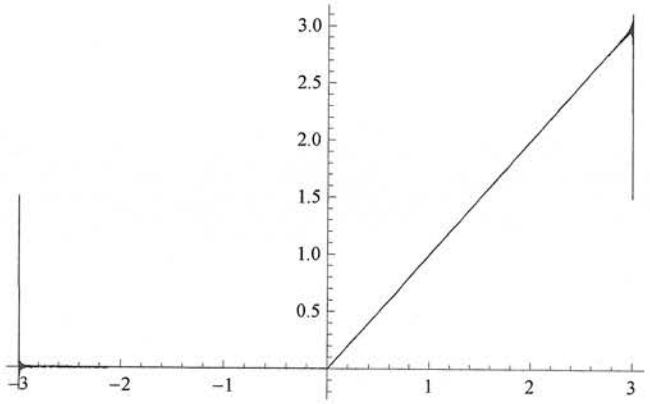

```
In[7]:= Plot[3 / 4 + Sum[a[n] Cos[n Pi x / 3] + b[n] Sin[n Pi x / 3], {n, 1, 100}], {x, -3, 3}]
```

Out[7]=




```
In[8]:= Plot[3 / 4 + Sum[a[n] Cos[n Pi x / 3] + b[n] Sin[n Pi x / 3], {n, 1, 1000}], {x, -3, 3}]
```

Out[8]=



§ Convergence of Fourier Series

Let $f(x)$ be function defined at $-L \le x \le L$ and $\tilde{f}(x)$ be Fourier series of $f(x)$. If $f(x)$ is not continuous function, generally

$$f(x) \ne \tilde{f}(x).$$

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정의 8.2 ; 구분적 연속함수 (piecewise continuous function)

$[a, b]$ 의 유한개의 점을 제외한 곳에서 정의되는 함수 $f(x)$ 가 다음조건을 만족하면 $f(x)$ 는 구간 $[a, b]$ 에서 piecewise continuous function 이라 한다.

1. $f(x)$ 가 유한개의 점을 제외한 구간 $[a, b]$ 에서 연속
2. $\lim_{x \to a^+} f(x)$ 와 $\lim_{x \to b^-} f(x)$ 가 모두 존재하고 유한
3. $f(x)$ 가 구간 (a, b) 내의 임의의 점 x_0 에서 불연속이면 $\lim_{x \to x_0^+} f(x)$ 와 $\lim_{x \to x_0^-} f(x)$ 가 모두 존재하고 유한

note)

$f(x)$ 가 piecewise continuous function 인 경우 불연속점을 도약 불연속점 (jump discontinuity) 이라 부른다.

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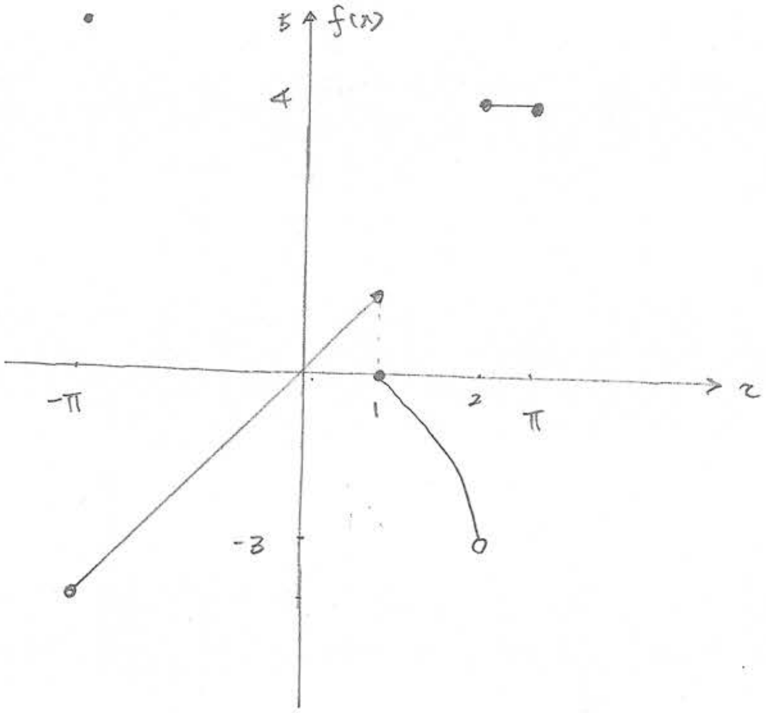
(예제 8.3) f(x) is defined at $[-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} 5 & x = -\pi \\ x & -\pi < x < 1 \\ 1 - x^2 & 1 \le x < 2 \\ 4 & 2 \le x \le \pi \end{cases}$$



도약불연속점: $x = 1, 2$

$\lim_{x \to -\pi^+} f(x) = -\pi$, $\lim_{x \to \pi^-} f(x) = 4$

$\lim_{x \to 1^-} f(x) = 1$ $\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to 2^-} f(x) = -3$ $\lim_{x \to 2^+} f(x) = 4$

$\Rightarrow$ f(x): piecewise continuous

※

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정의 8.2 구분적으로 매끄러운 함수 (piecewise smooth function)

If $f(x)$ and $f'(x)$ are piecewise continuous functions at $[a, b]$,

$f(x)$ is piecewise smooth function at $[a, b]$.

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정리 8.1

Let $f(x)$ be piecewise smooth function at $[-L, L]$,

and $\tilde{f}(x)$ be Fourier series of $f(x)$ at $-L < x < L$

Then

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right]$$

note)

$$f(x^+) = \lim_{y \to x^+} f(y), \quad f(x^-) = \lim_{y \to x^-} f(y)$$

note) If $f(x)$ is continuous at $x = x_0$, i.e.

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = f(x_0),$$

$$\tilde{f}(x_0) = f(x_0)$$

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(예제 8.4)

$f(x)$: piecewise smooth function at $[-2\pi, 2\pi]$ defined as

$$f(x) = \begin{cases} 5\sin x & -2\pi \le x < -\frac{\pi}{2} \\ 4 & x = -\frac{\pi}{2} \\ x^2 & -\frac{\pi}{2} < x < 2 \\ 8\cos x & 2 \le x < \pi \\ 4x & \pi \le x \le 2\pi \end{cases}$$

jump discontinuity $x = -\frac{\pi}{2}$, $x=2$, $x=\pi$

$\lim_{x \to -\frac{\pi}{2}^-} f(x) = 5\sin(-\frac{\pi}{2}) = -5$

$\lim_{x \to -\frac{\pi}{2}^+} f(x) = \frac{\pi^2}{4}$

$\tilde{f}(-\frac{\pi}{2}) = \frac{1}{2}\left[\frac{\pi^2}{4} - 5\right]$

$\lim_{x \to 2^-} f(x) = 4$, $\lim_{x \to 2^+} f(x) = 8\cos 2$: $\tilde{f}(2) = 2 + 4\cos 2$

$\lim_{x \to \pi^-} f(x) = 8\cos\pi = -8$, $\lim_{x \to \pi^+} f(x) = 4\pi$: $\tilde{f}(\pi) = -4 + 2\pi$

$$\Rightarrow \tilde{f}(x) = \begin{cases} 5\sin x & -2\pi < x < -\frac{\pi}{2} \\ \frac{1}{2}(\frac{\pi^2}{4} - 5) & x = -\frac{\pi}{2} \\ x^2 & -\frac{\pi}{2} < x < 2 \\ 2 + 4\cos 2 & x = 2 \\ 8\cos x & 2 < x < \pi \\ -4 + 2\pi & x = \pi \\ 4x & \pi < x < 2\pi \end{cases}$$

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정의 8.4 우도함수

If $f(x)$ is defined at $c < x < c+\gamma$ $(\gamma > 0)$ and

$$f(c^+) \equiv \lim_{x \to c^+} f(x) < \infty,$$

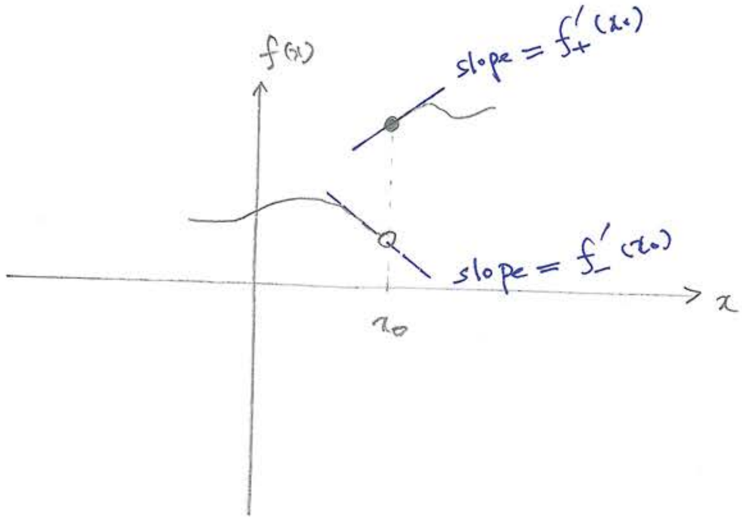
$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c^+)}{h} \quad : \text{우도함수}$$

정의 8.5 좌도함수

If $f(x)$ is defined at $c-\gamma < x < c$ $(\gamma > 0)$ and

$$f(c^-) \equiv \lim_{x \to c^-} f(x) < \infty,$$

$$f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c^-)}{h} \quad : \text{좌도함수}$$



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정리 8.2

$f(x)$: piecewise continuous function at $[-L, L]$

$\tilde{f}(x)$: Fourier series defined at $[-L, L]$.

(1) If $f(x)$ has left and right derivatives at $-L < x < L$,

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-) \right] \quad \text{at} \quad -L < x < L$$

(2) If $f_+'(-L)$ and $f_-'(L)$ exist,

$$\tilde{f}(L) = \tilde{f}(-L) = \frac{1}{2}\left[f(-L^+) + f(L^-) \right]$$

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(예제 8.5)

$f(x)$: piecewise continuous function at $[-2, 2]$ defined as

$$f(x) = \begin{cases} e^{x} & -2 \le x < 1 \\ -2x^2 & 1 \le x < 2 \\ 4 & x = 2 \end{cases}$$

jump discontinuity $x = 1$

$$\lim_{x \to 1^-} f(x) = \frac{1}{e} \qquad \lim_{x \to 1^+} f(x) = -2 \quad \Rightarrow \quad \tilde{f}(1) = \frac{1}{2}\left(\frac{1}{e} - 2 \right)$$

$$\left.\begin{aligned} f_+'(-2) &= \lim_{h \to 0^+} \frac{f(-2+h) - f(-2^+)}{h} = -e^{-2} \\ f_-'(2) &= \lim_{h \to 0^-} \frac{f(2+h) - f(2^-)}{h} = -8 \end{aligned}\right\} \text{존재}$$

$$\tilde{f}(2) = \tilde{f}(-2) = \frac{1}{2}\left[f(-2^+) + f(2^-) \right] = \frac{1}{2}\left[e^{-2} - 8 \right]$$

$$\tilde{f}(x) = \begin{cases} \frac{1}{2}(e^{2}-8) & x=-2 \\ e^{-x} & -2<x<1 \\ \frac{1}{2}(\frac{1}{e}-2) & x=1 \\ -2x^{2} & 1<x<2 \\ \frac{1}{e}(e^{2}-8) & x=2 \end{cases}$$

※

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(예제 2.6) $\quad f(x) = x \quad (-\pi \le x \le \pi)$

$$f_+'(-\pi) = f_-'(\pi) = 1 \ : \ \text{존재}$$

$$\tilde{f}(-\pi) = \tilde{f}(\pi) = \frac{1}{2}\left[f(-\pi^{+}) + f(\pi^{-}) \right] = \frac{1}{2}(-\pi+\pi) = 0$$

$$\tilde{f}(x) = \begin{cases} 0 & x=-\pi \\ x & -\pi<x<\pi \\ 0 & x=\pi \end{cases}$$

See page 217 !!

※

§ Fourier 사인 및 코사인 급수

[i] Fourier cosine series

$f(x)$: defined at $[0, L]$

⇒ Define

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

Then

$$f_e(-x) = f_e(x) \quad : \text{even function}$$

$f_e(x)$: $[-L, L]$ 에서 $f(x)$ 의 even extension

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(예제 8.7)

$$f(x) = e^{x} \quad at \quad 0 \le x \le 2$$

Let

$$f_e(x) = \begin{cases} e^{x} & 0 \le x \le 2 \\ e^{-x} & -2 \le x < 0 \end{cases}$$

$$a_n = \frac{1}{2} \int_{-2}^{2} f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_{0}^{2} f_e(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_{0}^{2} e^{x} \cos \frac{n\pi x}{2} dx$$

$$\int e^{x} \cos ax\, dx = \frac{e^{x}}{1+a^2} (\cos ax + a \sin ax)$$

$$a_0 = e^{2} - 1$$

$$a_n = \frac{1}{1 + \frac{n^2\pi^2}{4}} \left[e^{2} (-1)^{n} - 1 \right] \qquad (n = 1, 2, \cdots)$$

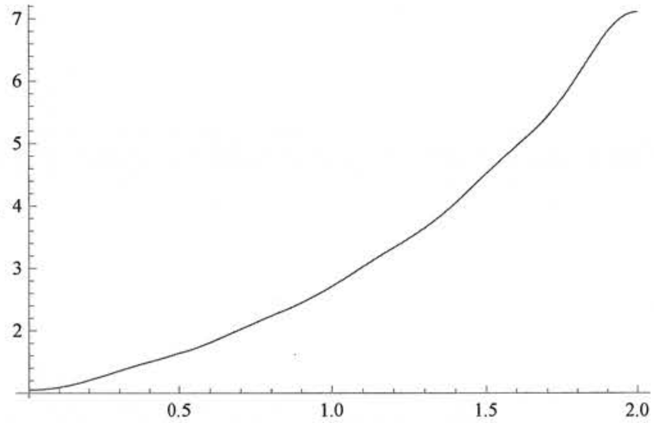
$$b_n = 0 \qquad (n = 1, 2, \cdots)$$

$\tilde{f}(x)$: Fourier Series of $f(x)$

$$\tilde{f}(x) = \frac{e^{2}-1}{2} + \sum_{n=1}^{\infty} \frac{e^{2}(-1)^{n} - 1}{1 + \frac{n^2\pi^2}{4}} \cos \frac{n\pi x}{2}$$

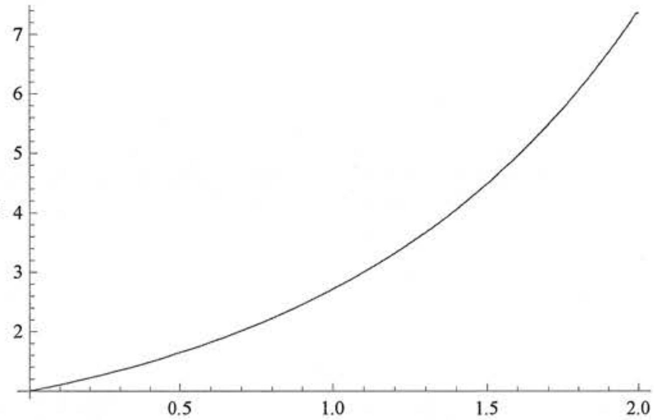
```
In[17]:= a[n_] := (E^2 (-1)^n - 1) / (1 + (n^2 Pi^2 / 4));
         Plot[{(E^2 - 1) / 2 + Sum[ a[n] Cos[n Pi x / 2], {n, 1, 10}]}, {x, 0, 2}]
```

Out[18]=



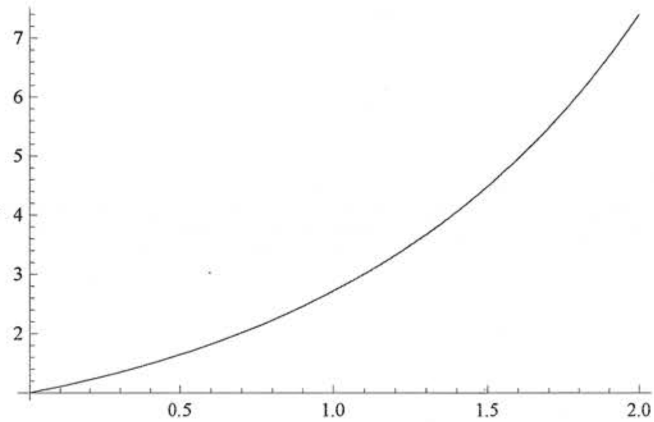
```
In[19]:= Plot[{(E^2 - 1) / 2 + Sum[ a[n] Cos[n Pi x / 2], {n, 1, 100}]}, {x, 0, 2}]
```

Out[19]=



```
In[20]:= Plot[{(E^2 - 1) / 2 + Sum[ a[n] Cos[n Pi x / 2], {n, 1, 1000}]}, {x, 0, 2}]
```

Out[20]=



$\not\times$

* Fourier cosine series

$f(x)$: defined at $[0, L]$

$$f_e(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x) \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f_e(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(x) \sin\frac{n\pi x}{L}\,dx = 0$$

$$\tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Fourier cosine series of $f(x)$

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정리 8.3 : Convergence of Fourier cosine series

$f(x)$: piecewise continuous function at $[0, L]$

$\tilde{f}(x)$: Fourier cosine series

(i) If $f(x)$ has left and right derivatives at $0 < x < L$,

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] \qquad (0 < x < L)$$

(ii) If $f'_+(0)$ exists, $\tilde{f}(0) = f(0^+)$

(iii) If $f'_-(L)$ exists, $\tilde{f}(L) = f(L^-)$

[2] Fourier Sine series

$f(x)$: defined at $[0, L]$

$\Rightarrow$ Define

$$f_o(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

Then

$$f_o(-x) = -f_o(x) \quad : \text{odd function}$$

$f_o(x)$: $[-L, L]$ 에서 $f(x)$ 의 odd extension

* Fourier sine series

$$a_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\cos\frac{n\pi x}{L}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_o(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$ Fourier sine series

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정리 8.3 : Convergence of Fourier sine series

$f(x)$: piecewise continuous function at $[0, L]$

$\tilde{f}(x)$: Fourier sine series

(1) If $f(x)$ has left and right derivative at $0<x<L$

$$\tilde{f}(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

(2) $\tilde{f}(0) = \tilde{f}(L) = 0$

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(예제 8.8)

$f(x)$: defined at $0 \le x \le \pi$ as following

$$f(x) = \begin{cases} 1 & 0 \le x \le \frac{\pi}{2} \\ 2 & \frac{\pi}{2} < x \le \pi \end{cases}$$

i) Fourier cosine series

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \cos nx \, dx + 2 \int_{\frac{\pi}{2}}^{\pi} \cos nx \, dx \right]$$

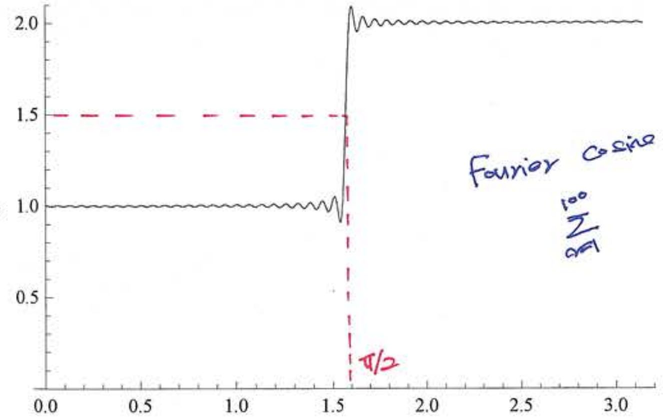
$$a_0 = 3$$

$$a_n = -\frac{2}{\pi n} \sin \frac{n\pi}{2} \qquad (n = 1, 2, \cdots)$$

$$f(x) = \frac{3}{2} - \sum_{n=1}^{\infty} \left(\frac{2}{\pi n} \sin \frac{n\pi}{2} \right) \cos nx$$ Fourier cosine series

```
In[25]:= Plot[3 / 2 - Sum[(2 Sin[n Pi / 2] / (Pi n)) Cos[n x], {n, 1, 100}],
          {x, 0, Pi}, PlotRange → {0, 2.1}]
```

Out[25]=



(ii) Fourier sine series

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \sin nx\,dx + 2\int_{\frac{\pi}{2}}^{\pi} \sin nx\,dx\right]$$

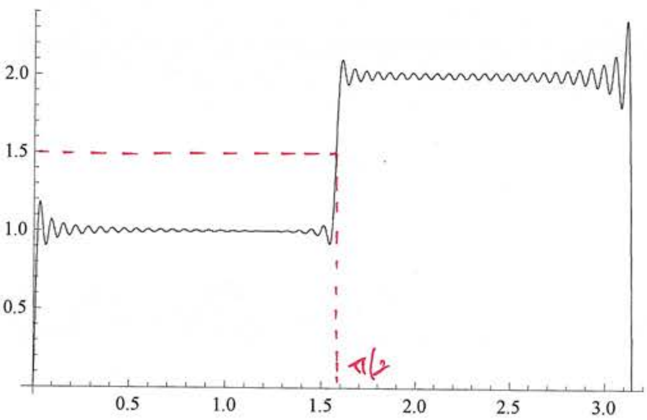
$$= \frac{2}{n\pi}\left[1 + \cos\frac{n\pi}{2} - 2(-1)^n\right]$$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1+\cos\frac{n\pi}{2} - 2(-1)^n\right]\sin nx$$

Fourier sine series

```
In[29]:= Plot[Sum[(2 / (n Pi)) (1 + Cos[n Pi / 2] - 2 (-1)^n) Sin[n x], {n, 1, 100}], {x, 0, Pi}]
```

Out[29]=



Fourier sine series

$\sum_{n=1}^{100}$

#

§ Integration and Differentiation of Fourier series

Let

$$f(x) = x \qquad -\pi \leq x \leq \pi$$

Then its Fourier series is

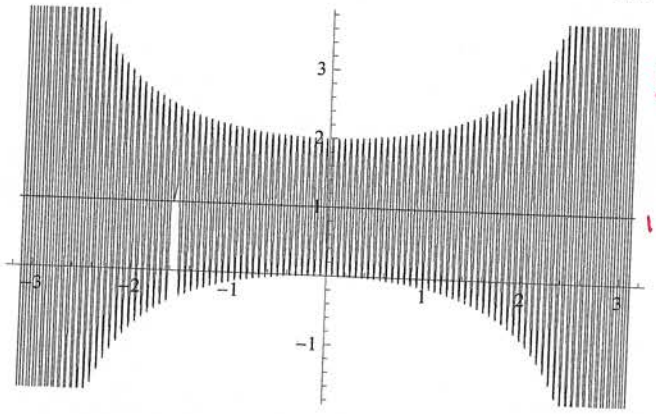
$$\tilde{f}(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

$$f'(x) = 1$$

$$\tilde{f}'(x) = \sum_{n=1}^{\infty} 2 (-1)^{n+1} \cos nx$$

In[33]:= `Plot[{Sum[2 (-1) ^ (n + 1) Cos[n x], {n, 1, 100}], 1}, {x, -Pi, Pi}]`

Out[33]=



$\sum_{n=1}^{100} 2 (-1)^{n+1} \cos nx$

1

Therefore, generally

$$f'(x) \neq \tilde{f}'(x)$$

정리 8.5 : Integration of Fourier Series

$f(x)$: piecewise continuous function defined at $[-L, L]$

$\tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$: Fourier Series of $f(x)$

Then

$$\int_{-L}^{x} f(t)\,dt = \int_{-L}^{x} \tilde{f}(t)\,dt = \frac{a_0}{2}(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin\frac{n\pi x}{L} - b_n\left\{\cos\frac{n\pi}{L}x - (-1)^n\right\}\right]$$

(예제 8.9)

$$f(x) = x \qquad -\pi \le x \le \pi$$

$$\tilde{f}(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$\int_{-\pi}^{x} t\,dx = \frac{1}{2}(x^2 - \pi^2)$$

$$\Rightarrow \frac{1}{2}(x^2-\pi^2) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\int_{-\pi}^{x} \sin nt\,dt$$

$$= \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^{n}\left[\cos nx - (-1)^n\right]$$

$\#$

P360

(정리 8.6)

$f(x)$: continuous function at $[-L, L]$ and $f(L) = f(-L)$

If $f'(x)$ is piecewise continuous function at $[-L, L]$,

$$f(x) = \tilde{f}(x) \quad \text{at} \quad [-L, L]$$

where $\tilde{f}(x)$ is Fourier series of $f(x)$.

If $f''(x)$ exists,

$$f'(x) = \tilde{f}'(x)$$

(예제 8.10)

$$f(x) = x^2 \qquad -2 \leq x \leq 2$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2 \cos\frac{n\pi x}{2}\,dx \qquad \int x^2 \cos ax\,dx = \frac{2ax\cos ax + (-2 + a^2x^2)\sin ax}{a^3}$$

$$a_0 = \frac{8}{3}$$

$$a_n = \frac{16}{n^2\pi^2}(-1)^n \qquad (n = 1, 2, \cdots)$$

$$b_n = 0$$

$$\tilde{f}(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2} \qquad \text{— ①}$$

Since $f(2) = f(-2) = 4$,

$$f(x) = \tilde{f}(x) \qquad \text{— ②}$$

Since $f''(x) = 2$,

$$2x = -\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2} \qquad \text{— ③}$$

put $x = 1$ in Eq. ② we get

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$\divideontimes$

8. Phase Angle form of Fourier series

$f(x)$: periodic function with period $p \Rightarrow f(x+p) = f(x)$

Then its Fourier series is

$$\tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{p} + b_n \sin\frac{2n\pi x}{p}\right]$$

$$a_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\cos\frac{2n\pi x}{p}\,dx \qquad (n = 0, 1, 2, \cdots)$$

$$b_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\sin\frac{2n\pi x}{p}\,dx \qquad (n = 1, 2, \cdots)$$

Since

$$a_n \cos\frac{2n\pi x}{p} + b_n \sin\frac{2n\pi x}{p}$$

$$= \sqrt{a_n^2 + b_n^2}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos\frac{2n\pi x}{p} + \frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin\frac{2n\pi x}{p}\right]$$

$$\left(\cos\delta_n = \frac{a_n}{\sqrt{a_n^2+b_n^2}}, \quad \sin\delta_n = -\frac{b_n}{\sqrt{a_n^2+b_n^2}}\right)$$

$$= \sqrt{a_n^2+b_n^2}\cos\left(\frac{2\pi n x}{p} + \delta_n\right)$$

$$\left(c_n = \sqrt{a_n^2+b_n^2}, \quad \omega_0 = \frac{2\pi}{p}\right)$$

$$= c_n \cos(n\omega_0 x + \delta_n)$$

$$\Rightarrow \boxed{\begin{gathered}\tilde{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 x + \delta_n) \\ \omega_0 = \frac{2\pi}{p}, \quad c_n = \sqrt{a_n^2+b_n^2}, \quad \delta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)\end{gathered}}$$

phase angle form of Fourier Series

$\cos(n\omega_0 x + \delta_n)$: $f(x)$의 n차 조화

c_n : n차 조화 amplitude

δ_n : $f(x)$의 n차 phase

p365

(예제 8.13)

$$f(x) = x^2 \qquad 0 \le x < 3 \qquad p = 3$$

$$a_n = \frac{2}{3}\int_0^3 f(x) \cos\frac{2n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 x^2 \cos\frac{2n\pi x}{3}\,dx$$

$$\left(\int x^2 \cos ax\,dx = \frac{2ax\cos ax + (-2+a^2x^2)\sin ax}{a^3} \right)$$

$$a_0 = 6$$

$$a_n = \frac{9}{n^2\pi^2} \quad (n=1,2,\cdots)$$ — ①

$$b_n = \frac{2}{3}\int_0^3 x^2 \sin\frac{2n\pi x}{3}\,dx$$

$$\left(\int x^2 \sin ax\,dx = \frac{(2-a^2x^2)\cos ax + 2ax\sin ax}{a^3} \right)$$

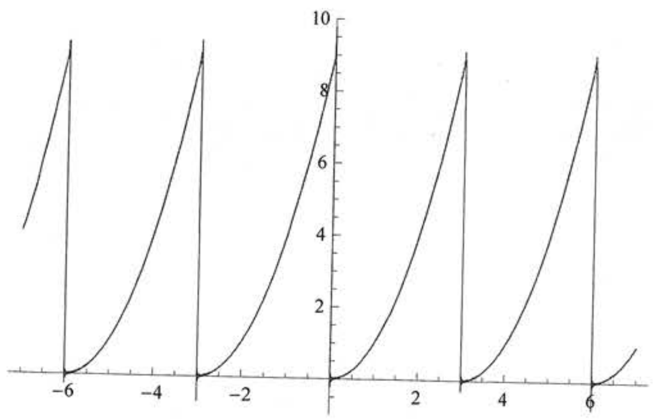
$$b_n = -\frac{9}{n\pi}$$ — ②

Fourier series

$$\tilde{f}(x) = 3 + \sum_{n=1}^{\infty}\left[\frac{9}{n^2\pi^2}\cos\frac{2n\pi x}{3} - \frac{9}{n\pi}\sin\frac{2n\pi x}{3} \right]$$ — ③

```
In[55]:= Plot[
    3 + Sum[(9 / (n^2 Pi^2)) Cos[2 n Pi x / 3] - (9 / (n Pi)) Sin[2 n Pi x / 3], {n, 1, 500}],
    {x, -7, 7}]
```

Out[55]=



$$C_n = \sqrt{a_n^2 + b_n^2} = \frac{9}{n^2\pi^2}\sqrt{1+n^2\pi^2}$$

$$\omega_0 = \frac{2\pi}{p} = \frac{2\pi}{3}$$

$$\delta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) = \tan^{-1}(n\pi)$$

— ⑥

Then the phase angle expression of the Fourier Series is

$$\tilde{f}(x) = 3 + \sum_{n=1}^{\infty} \frac{9}{n^2\pi^2}\sqrt{1+n^2\pi^2}\,\cos\left(\frac{2n\pi x}{3} + \tan^{-1}(n\pi)\right)$$ ※

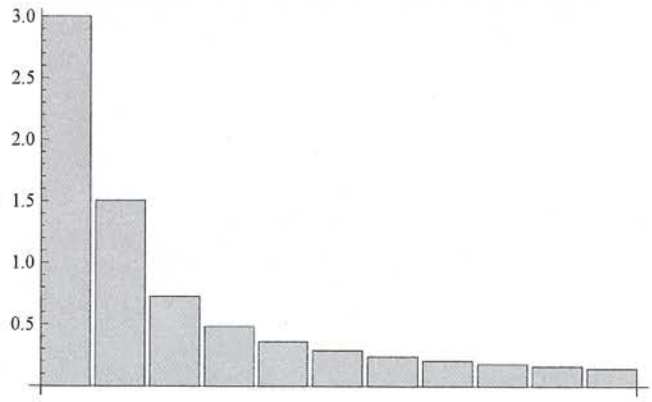
$$\boxed{\begin{array}{l}\text{* Amplitude Spectrum}\\ \text{Set of points } \left\{\left(0, \frac{|a_0|}{2}\right), \left(n\omega_0, \frac{C_n}{2}\right)\right\}\end{array}}$$

(Ex) $f(x) = x^2$ $\quad 0 \le x < 3 \quad p = 3$

Amplitude spectrum $= \left\{ (0, 3),\ \left(n\omega_0, \frac{9}{2n^2\pi^2}\sqrt{1+n^2\pi^2}\right) \ (n = 1, 2, \cdots) \right\}$

```
In[71]:= z = {3}; omega0 = 2 Pi / 3;
         For[n = 1, n ≤ 10, ++n,
             z = Append[z, 9 Sqrt[1 + n^2 Pi^2] / (2 n^2 Pi^2)]];
         BarChart[z]
```

Out[73]=



※

* Complex Form of Fourier Series

$$\tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 x + b_n \sin n\omega_0 x\right]$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \frac{e^{in\omega_0 x} + e^{-in\omega_0 x}}{2} + b_n \frac{e^{in\omega_0 x} - e^{-in\omega_0 x}}{2i}\right]$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} + \frac{1}{2}(a_n + ib_n)e^{-in\omega_0 x}\right]$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} + \sum_{n=-\infty}^{-1}\frac{1}{2}(a_{-n} + ib_{-n})e^{in\omega_0 x} \qquad (1)$$

Since

$$a_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\cos\frac{2n\pi x}{p}\,dx \qquad (2)$$

$$b_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\sin\frac{2n\pi x}{p}\,dx \,,$$

$a_{-n} = a_n$ and $b_{-n} = -b_n$. Thus Eq. (1) becomes

$$\tilde{f}(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} + \sum_{n=-\infty}^{-1}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x}$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} - \frac{1}{2}(a_0 - ib_0) + \frac{1}{2}a_0$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{2}(a_n - ib_n)e^{in\omega_0 x} \qquad (b_0 = 0)$$

$\llcorner$ (3)

$$\frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\cos\frac{2n\pi x}{p}\,dx - i\frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\sin\frac{2n\pi x}{p}\,dx\right]$$

$$= \frac{1}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\left[\cos\frac{2n\pi x}{p} - i\sin\frac{2n\pi x}{p}\right]dx$$

$$= \frac{1}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(x)\, e^{-in\omega_0 x}\,dx. \qquad (4)$$

Complex form of Fourier Series

$$\tilde{f}(x) = \sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x}$$

$$d_n = \frac{1}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(x) e^{-in\omega_0 x}$$

* Frequency spectrum $= \{(n\omega_0, |d_n|) \mid n = 0, \pm1, \pm2, \cdots\}$

P369

(예제 8.14)

$f(x) = \frac{3}{4}x \qquad 0 \le x < 8 \qquad p = 8$

$\omega_0 = \frac{2\pi}{p} = \frac{\pi}{4}$

$d_n = \frac{1}{8}\int_0^8 f(x) e^{-in\frac{\pi}{4}x} dx$

$= \frac{3}{32}\int_0^8 x e^{-in\frac{\pi}{4}x} dx \qquad \left(\int x e^{ax} dx = \frac{e^{ax}(ax-1)}{a^2} \right)$

$d_0 = 3$

$d_n = \frac{3i}{n\pi} \quad (n \neq 0)$ — ①

$\Rightarrow \tilde{f}(x) = 3 + \frac{3i}{\pi} \sum_{n=-\infty, n\neq 0}^{\infty} \frac{1}{n} e^{\frac{in\pi x}{4}}$ — ②

Frequency spectrum $= \{ (0, 3), (\frac{n\pi}{4}, \frac{3}{|n|\pi}) \mid n = \pm1, \pm2, \cdots \}$

p371

3 Fourier Integral

$\tilde{f}(x)$: $f(x)$의 Fourier integral

$$\tilde{f}(x) \equiv \frac{1}{\pi}\int_0^\infty d\omega \left[A_\omega \cos\omega x + B_\omega \sin\omega x \right]$$

$$A_\omega = \int_{-\infty}^{\infty} f(x) \cos\omega x \, dx$$

$$B_\omega = \int_{-\infty}^{\infty} f(\omega) \sin\omega x \, dx.$$

note)

If $f(x)$ is piecewise smooth and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$,

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-) \right]$$

note) If $f(x)$ is continuous and $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, $f(x) = \tilde{f}(x)$

p372

(예제 8.15)

$f(x) = x e^{-|x|}$: continuous function

$\int_{-\infty}^{\infty} |f(x)| dx = 2\int_0^\infty x e^{-x} dx = 2 < \infty$ $\quad \left(\int_0^\infty x^n e^{-\mu x} dx = n! \, \mu^{-n-1} \right)$

$$\Rightarrow \quad f(x) = \tilde{f}(x)$$

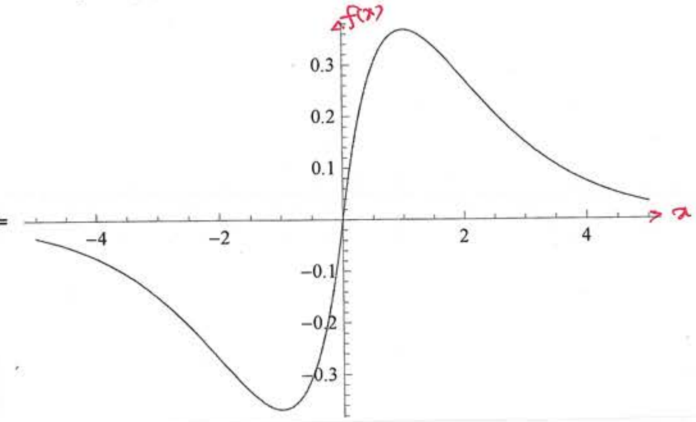
$$A_\omega = \int_{-\infty}^{\infty} f(x) \cos\omega x \, dx = \int_{-\infty}^{\infty} x e^{-|x|} \cos\omega x \, dx = 0$$

$$B_\omega = \int_{-\infty}^{\infty} f(x) \sin\omega x \, dx = 2\int_0^\infty x e^{-x} \sin\omega x \, dx = \frac{4\omega}{(1+\omega^2)^2}$$

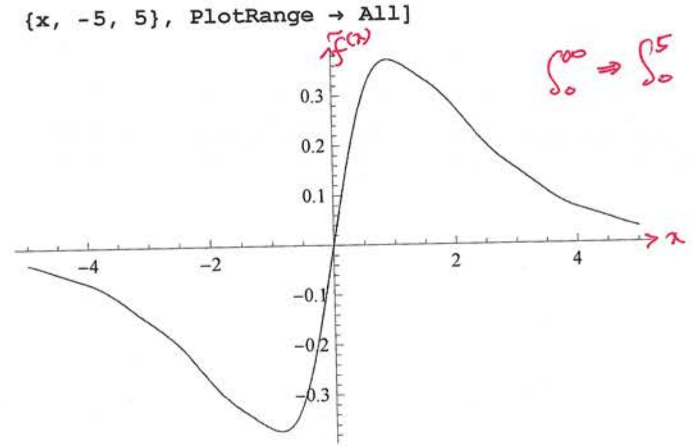
$$\left(\int_0^\infty x^{\mu-1} e^{-\beta x} \sin\delta x \, dx = \frac{\Gamma(\mu)}{(\beta^2+\delta^2)^{\frac{\mu}{2}}} \sin\left(\mu \tan^{-1}\frac{\delta}{\beta}\right) \right)$$

$$\Rightarrow \tilde{f}(x) = \frac{4}{\pi}\int_0^\infty d\omega \, \frac{\omega}{(1+\omega^2)^2} \sin\omega x$$

In[80]:= `Plot[x Exp[-Abs[x]], {x, -5, 5}]`

Out[80]=


In[88]:= `Plot[(4 / Pi) Integrate[(w / (1 + w^2)^2) Sin[w x], {w, 0, 5}], {x, -5, 5}, PlotRange → All]`

Out[88]=


$$\Rightarrow \quad x e^{-|x|} = \frac{4}{\pi} \int_0^\infty d\omega \, \frac{\omega}{(1+\omega^2)^2} \sin \omega x \qquad \#$$

* different Expression of Fourier integral

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^\infty d\omega \left[\left(\int_{-\infty}^{\infty} f(\xi)\cos\omega\xi\, d\xi\right)\cos\omega x + \left(\int_{-\infty}^{\infty} f(\xi)\sin\omega\xi\, d\xi\right)\sin\omega x \right]$$

$$= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} d\xi\, f(\xi)\left[\cos\omega\xi\cos\omega x + \sin\omega\xi\sin\omega x\right]$$

$$= \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} d\xi\, f(\xi)\cos\omega(\xi - x)$$

$$\Rightarrow \quad \tilde{f}(x) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} d\xi\, f(\xi)\cos\omega(\xi - x)$$

P373

4. $$\tilde{f}(x) = \frac{1}{\pi}\int_0^\infty d\omega \int_{-\infty}^{\infty} dt\, f(t)\cos\omega(t-x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\int_0^\infty d\Omega\,\cos\Omega(t-x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\lim_{\omega\to\infty}\int_0^\omega d\Omega\,\cos\Omega(t-x)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\lim_{\omega\to\infty}\frac{1}{t-x}\sin\Omega(t-x)\Big|_{\Omega=0}^{\Omega=\omega}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\lim_{\omega\to\infty}\frac{\sin\omega(t-x)}{t-x}$$

$$= \lim_{\omega\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} dt\, f(t)\frac{\sin\omega(t-x)}{t-x} \qquad \divideontimes$$

§ Fourier sine and cosine Integral.

Let $f(x)$ be defined at $[0, \infty)$.

Then we define

$$f_e(x) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$$

Then

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty} d\omega \left[A_\omega \cos\omega x + B_\omega \sin\omega x \right]$$

$$A_\omega = \int_{-\infty}^{\infty} f_e(x) \cos\omega x \, dx = 2\int_0^{\infty} f(x) \cos\omega x \, dx$$

$$B_\omega = \int_{-\infty}^{\infty} f_e(x) \sin\omega x \, dx = 0$$

$$\Rightarrow \boxed{\begin{aligned} \tilde{f}(x) &= \frac{1}{\pi}\int_0^{\infty} d\omega \, A_\omega \cos\omega x \\ A_\omega &= 2\int_0^{\infty} f(x) \cos\omega x \, dx \end{aligned}}$$

Fourier cosine integral

note)

If $f(x)$ is piecewise continuous at $[0, \infty)$,

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-) \right]$$

$$\tilde{f}(0) = f(0)$$

If $f(x)$ is continuous,

$$\tilde{f}(x) = f(x)$$

Define

$$f_o(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x \leq 0 \end{cases}$$

Then

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^\infty d\omega \left[A_\omega \cos\omega x + B_\omega \sin\omega x \right]$$

$$A_\omega = \int_{-\infty}^{\infty} f_o(x)\cos\omega x\, dx = 0$$

$$B_\omega = \int_{-\infty}^{\infty} f_o(x)\sin\omega x\, dx = 2\int_0^\infty f(x)\sin\omega x\, dx$$

$$\Rightarrow \quad \boxed{\begin{aligned} \tilde{f}(x) &= \frac{1}{\pi}\int_0^\infty d\omega\, B_\omega \sin\omega x \\ B_\omega &= 2\int_0^\infty f(x)\sin\omega x\, dx \end{aligned}}$$

Fourier sine integral

note) If $f(x)$ is piecewise smooth,

$$\tilde{f}(x) = \frac{1}{2}\left[f(x^+) + f(x^-) \right] \qquad 0 < x < \infty$$

$$\tilde{f}(0) = 0$$

p279

(예제 8.16)

$$f(x) = e^{-kx} \quad (k>0) \quad \text{at} \quad x \geq 0$$

$$\int_0^\infty |f(x)|\,dx = \int_0^\infty e^{-kx}\,dx = \frac{1}{k} < \infty$$

* Fourier cosine integral

$$A_\omega = 2\int_0^\infty f(x)\cos\omega x\,dx$$

$$= 2\int_0^\infty e^{-kx}\cos\omega x\,dx$$

$$= \frac{2k}{k^2+\omega^2}$$

$$\boxed{\int_0^\infty e^{-px}\cos(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left[p\cos\lambda - q\sin\lambda\right]}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty d\omega\, A_\omega \cos\omega x$$

$$\Rightarrow \underline{e^{-kx} = \frac{1}{\pi}\int_0^\infty d\omega\, \frac{2k}{k^2+\omega^2}\cos\omega x = \frac{2k}{\pi}\int_0^\infty d\omega\, \frac{1}{k^2+\omega^2}\cos\omega x}$$

Laplace integral

* Fourier sine integral.

$$B_\omega = 2\int_0^\infty f(x)\sin\omega x\,dx$$

$$= 2\int_0^\infty e^{-kx}\sin\omega x\,dx$$

$$= \frac{2\omega}{k^2+\omega^2}$$

$$\boxed{\int_0^\infty e^{-px}\sin(qx+\lambda)\,dx = \frac{1}{p^2+q^2}\left[q\cos\lambda + p\sin\lambda\right]}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty d\omega\, B_\omega \sin\omega x$$

$$\underline{e^{-kx} = \frac{2}{\pi}\int_0^\infty d\omega\, \frac{\omega}{k^2+\omega^2}\sin\omega x}$$

Laplace integral

※

P275

§. Complex Fourier Integral and Fourier Transform

Fourier integral

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) \cos\omega(\xi - x)$$

$$= \frac{1}{\pi}\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) \; \frac{e^{i\omega(\xi-x)} + e^{-i\omega(\xi-x)}}{2}$$

$$= \frac{1}{2\pi}\Big[\underbrace{\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) e^{i\omega(\xi-x)}}_{\omega \to -\omega} + \underline{\int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) e^{-i\omega(\xi-x)}} \Big]$$

$$= \frac{1}{2\pi}\Big[\int_{-\infty}^{0} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) e^{-i\omega(\xi-x)} + \int_0^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) e^{-i\omega(\xi-x)} \Big]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\xi \; f(\xi) \; e^{-i\omega(\xi-x)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \Big[\int_{-\infty}^{\infty} d\xi \; f(\xi) e^{-i\omega\xi} \Big] e^{+i\omega x}$$

Put

$$C_\omega \equiv \int_{-\infty}^{\infty} dx \; f(x) \; e^{-i\omega x}$$

Then

$$\boxed{\begin{aligned} \tilde{f}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; C_\omega e^{+i\omega x} \\ C_\omega &\equiv \int_{-\infty}^{\infty} dx \; f(x) \; e^{-i\omega x} \end{aligned}}$$

Complex Fourier integral

p276

(예제8.17)

$$f(x) = e^{-a|x|} \qquad (a>0)$$

$$C_\omega = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-i\omega x}$$

$$= \int_{-\infty}^{0} dx\, e^{ax} e^{-i\omega x} + \int_{0}^{\infty} e^{-ax} e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{0} dx\, e^{(a-i\omega)x} + \int_{0}^{\infty} e^{-(a+i\omega)x}\, dx$$

$$= \frac{1}{a-i\omega} e^{(a-i\omega)x}\Big|_{x=-\infty}^{x=0} - \frac{1}{a+i\omega} e^{-(a+i\omega)x}\Big|_{x=0}^{x=\infty}$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega}$$

$$= \frac{2a}{a^2+\omega^2} \qquad \text{— (1)}$$

$$\Rightarrow \quad e^{-a|x|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{2a}{a^2+\omega^2}\, e^{i\omega x} \qquad *$$

p277

정의8.10 : Fourier Transform

$$\mathcal{F}[f](\omega) \equiv \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

: Fourier transform of $f(t)$

note) $$\tilde{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \hat{f}(\omega)\, e^{i\omega x}$$

(예제 8.18)

$$f(t) = \begin{cases} 0 & t<0 \\ e^{-at} & t>0 \end{cases} \qquad (a>0)$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+i\omega)t}\, dt$$

$$= -\frac{1}{a+i\omega}\, e^{-(a+i\omega)t} \Big|_{t=0}^{t=\infty}$$

$$= \frac{1}{a+i\omega}$$

$$\mathcal{F}[f](\omega) = \frac{1}{a+i\omega} \quad \text{or} \quad \mathcal{F}[f(t)](\omega) = \frac{1}{a+i\omega}$$ ※

P378

(예제 8.19)

$$f(t) = \begin{cases} k & -a \le t < a \\ 0 & t<-a,\ t \ge a \end{cases} \qquad (a, k>0)$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-a}^{a} k\, e^{-i\omega t}\, dt$$

$$= -\frac{k}{i\omega}\, e^{-i\omega t} \Big|_{t=-a}^{t=a}$$

$$= -\frac{k}{i\omega}\left(e^{-i\omega a} - e^{i\omega a}\right) = -2i\sin\omega a$$

$$= \frac{2k}{\omega}\sin\omega a$$

※

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\ \hat{f}(\omega)\, e^{i\omega t}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}$$

Fourier transform pair

p279

정리 8.9 : time translation

$$\mathcal{F}\left[f(t-t_0)\right](\omega) = e^{-i\omega t_0}\,\hat{f}(\omega)$$

pf)

$\mathcal{F}\left[f(t-t_0)\right](\omega)$

$$= \int_{-\infty}^{\infty} dt\ f(t-t_0)\, e^{-i\omega t} \qquad (s = t - t_0)$$

$$= \int_{-\infty}^{\infty} ds\ f(s)\, e^{-i\omega(s+t_0)}$$

$$= e^{-i\omega t_0}\int_{-\infty}^{\infty} ds\ f(s)\, e^{-i\omega s}$$

$$= e^{-i\omega t_0}\underbrace{\int_{-\infty}^{\infty} dt\ f(t)\, e^{-i\omega t}}_{} \qquad \hat{f}(\omega)$$

$$= e^{-i\omega t_0}\,\hat{f}(\omega)$$

※

note)

$$\mathcal{F}^{-1}\left[e^{-i\omega t_0}\,\hat{f}(\omega)\right] = f(t-t_0)$$

p379

(예제 8.20)

$$g(t) = \begin{cases} 0 & t<3, \quad t \geq 7 \\ 6 & 3 \leq t < 7 \end{cases}$$

Define

$$f(t) = \begin{cases} 0 & t<-2, \ t \geq 2 \\ 6 & -2 \leq t < 2 \end{cases}$$

Then

$$g(t) = f(t-5)$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t} = \frac{12}{\omega} \sin(2\omega)$$

Then

$$\hat{g}(\omega) = e^{-5i\omega} \frac{12}{\omega} \sin 2\omega$$ ※

p280

(예제 8.21)

$$\mathcal{F}^{-1}\left[\frac{e^{2i\omega}}{5+i\omega}\right]$$

Let

$$f(t) = \mathcal{F}^{-1}\left[\frac{1}{5+i\omega}\right] \quad -①$$

See (예제 8.18)

when $f(t) = \begin{cases} 0 & t<0 \\ e^{-at} & t>0 \end{cases} = H(t)\,e^{-at}$

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{(Heaviside function)},$$

$$\hat{f}(\omega) = \frac{1}{a+i\omega}$$

$$\Rightarrow \underline{\underline{\mathcal{F}^{-1}\left[\frac{1}{a+i\omega}\right] = H(t)\,e^{-at}}} \quad -②$$

Therefore

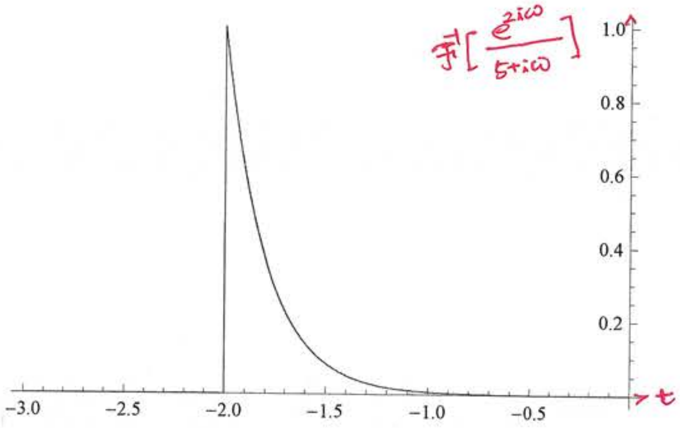
$$f(t) = \mathcal{F}^{-1}\left[\frac{1}{5+i\omega}\right] = H(t)\,e^{-5t} \quad -③$$

Then

$$\mathcal{F}^{-1}\left[\frac{e^{2i\omega}}{5+i\omega}\right] = f(t+2) = H(t+2)\,e^{-5(t+2)}$$

```
In[96]:= H[t_] := If[t ≥ 0, 1, 0];
         Plot[H[t + 2] Exp[-5 (t + 2)], {t, -3, 0}, PlotRange → All]
```

Out[97]=



※

p301

정리 8.10 : frequency translation

$$\boxed{\mathcal{F}\left[e^{i\omega_0 t} f(t)\right] = \hat{f}(\omega-\omega_0)}$$

pf)

$$\mathcal{F}\left[e^{i\omega_0 t} f(t)\right]$$

$$= \int_{-\infty}^{\infty} dt\, e^{i\omega_0 t} f(t)\, e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i(\omega-\omega_0)t}$$

$$= \hat{f}(\omega-\omega_0) \qquad //$$

note)

$$\mathcal{F}^{-1}\left[\hat{f}(\omega-\omega_0)\right] = e^{i\omega_0 t}\, \mathcal{F}^{-1}\left[\hat{f}(\omega)\right]$$

p281

정리 8.11 : scaling theorem

$$\boxed{\mathcal{F}[f(at)](\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right)}$$

pf)

$\mathcal{F}[f(at)](\omega)$

$$= \int_{-\infty}^{\infty} dt\, f(at)\, e^{-i\omega t} \qquad s = at \qquad (a>0)$$

$$= \int_{-\infty}^{\infty} \frac{ds}{a} f(s)\, e^{-i\frac{\omega}{a}s}$$

$$= \frac{1}{a}\underbrace{\int_{-\infty}^{\infty} ds\, f(s)\, e^{-i\frac{\omega}{a}s}}_{\hat{f}\left(\frac{\omega}{a}\right)}$$

$$= \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right) \qquad — ①$$

if $a<0$,

$\mathcal{F}[f(at)](\omega)$

$$= \int_{-\infty}^{\infty} dt\, f(at)\, e^{-i\omega t} \qquad s = at \qquad (a<0)$$

$$= \int_{\infty}^{-\infty} \frac{ds}{a} f(s)\, e^{-i\frac{\omega}{a}s}$$

$$= -\frac{1}{a}\int_{-\infty}^{\infty} ds\, f(s)\, e^{-i\frac{\omega}{a}s}$$

$$= -\frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right) \qquad — ②$$

From ① and ② we have

$$\mathcal{F}[f(at)](\omega) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right) \qquad ※$$

note)

$$\mathcal{F}^{-1}\left[\hat{f}\left(\frac{\omega}{a}\right)\right](t) = |a|\, f(at)$$

p281

(예제8.22)

$$f(t) = \begin{cases} 1-|t| & -1 \le t \le 1 \\ 0 & |t| > 1 \end{cases}$$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t}$$

$$= \int_{-1}^{1} dt\, (1-|t|)\, e^{-i\omega t}$$

$$= \int_{-1}^{1} dt\, e^{-i\omega t} - \int_{-1}^{1} dt\, |t| e^{-i\omega t} \quad -①$$

$$\int_{-1}^{1} dt\, e^{-i\omega t} = \frac{2}{\omega} \sin\omega \quad -②$$

$$* \int t\, e^{at}\, dt = \frac{e^{at}(at-1)}{a^2}$$

$$\int_{-1}^{1} dt\, |t| e^{-i\omega t} = \int_{0}^{1} dt\, t\, e^{-i\omega t} - \int_{-1}^{0} t\, e^{-i\omega t}\, dt = \frac{2}{\omega^2}\left[-1 + \omega\sin\omega + \cos\omega\right] \quad -③$$

②, ③ → ①

$$\hat{f}(\omega) = \frac{2(1-\cos\omega)}{\omega^2} \quad -④$$

Let

$$g(t) = f(\eta t) = \begin{cases} 1-\eta|t| & -\frac{1}{\eta} \le t \le \frac{1}{\eta} \\ 0 & |t| > \frac{1}{\eta} \end{cases}$$

Then

$$\hat{g}(\omega) = \frac{1}{\eta}\hat{f}\left(\frac{\omega}{\eta}\right) = \frac{1}{\eta}\,\frac{2(1-\cos\frac{\omega}{\eta})}{(\frac{\omega}{\eta})^2} = 2\eta\,\frac{1-\cos\frac{\omega}{\eta}}{\omega^2}$$

※

Summary

$$\mathcal{F}\left[H(t)\, e^{-at} \right] = \frac{1}{a+i\omega} \qquad \mathcal{F}^{-1}\left[\frac{1}{a+i\omega} \right] = H(t)\, e^{-at}$$

$$\mathcal{F}\left[f(t-t_0) \right](\omega) = e^{-i\omega t_0}\, \hat{f}(\omega) \qquad \mathcal{F}^{-1}\left[e^{-i\omega t_0}\, \hat{f}(\omega) \right] = f(t-t_0)$$

$$\mathcal{F}\left[e^{i\omega_0 t} f(t) \right](\omega) = \hat{f}(\omega-\omega_0) \qquad \mathcal{F}^{-1}\left[\hat{f}(\omega-\omega_0) \right] = e^{i\omega_0 t} f(t)$$

$$\mathcal{F}\left[f(at) \right](\omega) = \frac{1}{|a|}\, \hat{f}\left(\frac{\omega}{a}\right) \qquad \mathcal{F}^{-1}\left[\hat{f}\left(\frac{\omega}{a}\right) \right] = |a|\, f(at)$$

§ Application

[1] 도함수의 Fourier transform

* If $\lim_{t \to \pm\infty} f(t) = 0$,

$$\mathcal{F}[f'(t)] = i\omega \hat{f}(\omega)$$

(pf)

$$\mathcal{F}[f'(t)](\omega)$$

$$= \int_{-\infty}^{\infty} dt \, f'(t) \, e^{-i\omega t}$$

$$= f(t) e^{-i\omega t} \Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} dt \, f(t) \frac{d}{dt} e^{-i\omega t}$$

$$= i\omega \int_{-\infty}^{\infty} dt \, f(t) \, e^{-i\omega t}$$

$$= i\omega \hat{f}(\omega) \quad *$$

p282

(정리 8.12)

If

$$\lim_{t \to \pm\infty} f^{(k)}(t) = 0 \qquad (k = 0, 1, 2, \cdots, n-1),$$

$$\mathcal{F}[f^{(n)}(t)](\omega) = (i\omega)^n \hat{f}(\omega)$$

p383

(예제 8.23)

$$y' - 4y = H(t)e^{-4t} = \begin{cases} 0 & t<0 \\ e^{-4t} & t \ge 0 \end{cases}$$

Taking Fourier transform

$$i\omega \hat{y}(\omega) - 4\hat{y}(\omega) = \mathcal{F}[H(t)e^{-4t}] = \frac{1}{4+i\omega}$$

$$\hat{y}(\omega) = \frac{1}{(i\omega-4)(i\omega+4)} = -\frac{1}{16+\omega^2} \quad -①$$

$$y(t) = \mathcal{F}^{-1}\left[-\frac{1}{16+\omega^2}\right] = -\mathcal{F}^{-1}\left[\frac{1}{16+\omega^2}\right] \quad -②$$

See 표 8.1 (p392)

$$\mathcal{F}^{-1}\left[\frac{2a}{a^2+\omega^2}\right] = e^{-a|t|}$$

$$\Rightarrow \mathcal{F}^{-1}\left[\frac{1}{a^2+\omega^2}\right] = \frac{1}{2a}e^{-a|t|} \quad -③$$

③ → ②

$$y(t) = -\frac{1}{8}e^{-4|t|}$$

※

[3] 주파수 미분

p384

정리8.14

$$\boxed{\mathcal{F}\left[t^n f(t)\right](\omega) = i^n \frac{d^n}{d\omega^n} \hat{f}(\omega)}$$

pf)

$\mathcal{F}\left[t^n f(t)\right](\omega)$

$$= \int_{-\infty}^{\infty} dt\, t^n f(t)\, e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} dt\, f(t) \left(+i \frac{d}{d\omega}\right)^n e^{-i\omega t}$$

$$= \left(i \frac{d}{d\omega}\right)^n \int_{-\infty}^{\infty} dt\, f(t)\, e^{-i\omega t}$$

$$= i^n \frac{d^n}{d\omega^n} \hat{f}(\omega) \quad ※$$

p385

(예제8.24)

$$\mathcal{F}\left[t^2 e^{-5|t|}\right]$$

$$\hat{f}(\omega) \equiv \mathcal{F}\left[e^{-5|t|}\right] = \frac{10}{25+\omega^2} \quad -① \qquad \text{(see Table 8.1 p. 390)}$$

Then

$$\mathcal{F}\left[t^2 e^{-5|t|}\right] = -\frac{d^2}{d\omega^2}\left[\frac{10}{25+\omega^2}\right] = 20\, \frac{25-3\omega^2}{(25+\omega^2)^3} \quad ※$$

[3] 적분함수의 Fourier 변환

$$\boxed{\mathcal{F}\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right] = \frac{1}{i\omega}\hat{f}(\omega)}$$

[4] Convolution

정의 : convolution

$$(f*g)(t) = \int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau$$

P386

정리 8.16

[1] $f*g = g*f$

[2] $(\alpha f+\beta g)*h = \alpha\, f*h + \beta\, g*h$

(pf)

[1] $(f*g)(t)$

$$= \int_{-\infty}^{\infty} d\tau\ f(t-\tau)\,g(\tau) \qquad (\tau = t-s)$$

$$= \int_{\infty}^{-\infty} (-ds)\ f(s)\,g(t-s)$$

$$= \int_{-\infty}^{\infty} ds\ g(t-s)\,f(s)$$

$$= (g*f)(t)$$

XX.

p306

정리 8.17

(1) $\int_{-\infty}^{\infty} dt\, (f * g)(t) = \int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} g(t)\, dt$

(2) $\mathcal{F}\left[(f*g)(t)\right] = \hat{f}(\omega)\, \hat{g}(\omega)$

$\mathcal{F}^{-1}\left[\hat{f}(\omega)\, \hat{g}(\omega)\right] = (f*g)(t)$

(3) $\mathcal{F}\left[f(t)\, g(t)\right] = \frac{1}{2\pi}(\hat{f} * \hat{g})(\omega)$

$\mathcal{F}^{-1}\left[(\hat{f} * \hat{g})(\omega)\right] = 2\pi\, f(t)\, g(t)$

(예제 8.25)

$$\mathcal{F}^{-1}\left[\frac{1}{(4+\omega^2)(9+\omega^2)}\right]$$

$$\left.\begin{array}{l}\mathcal{F}^{-1}\left[\dfrac{1}{4+\omega^2}\right] = f(t) = \dfrac{1}{4}e^{-2|t|} \\ \mathcal{F}^{-1}\left[\dfrac{1}{9+\omega^2}\right] = g(t) = \dfrac{1}{6}e^{-3|t|}\end{array}\right\} - ①$$ (See Table 8.1)

Since $\mathcal{F}^{-1}[\hat{f}(\omega)\hat{g}(\omega)] = (f*g)(t)$,

$$\mathcal{F}^{-1}\left[\frac{1}{(4+\omega^2)(9+\omega^2)}\right]$$

$$= (f*g)(t)$$

$$= \int_{-\infty}^{\infty} d\tau\, f(t-\tau)\, g(\tau)$$

$$= \int_{-\infty}^{\infty} d\tau\, \frac{1}{4}e^{-2|t-\tau|} \cdot \frac{1}{6}e^{-3|\tau|}$$

$$= \frac{1}{24}\int_{-\infty}^{\infty} d\tau\, e^{-2|t-\tau|}\, e^{-3|\tau|}$$

$$= \frac{1}{24} J(t) \qquad - ②$$

where

$$J(t) = \int_{-\infty}^{\infty} d\tau\, e^{-2|t-\tau|}\, e^{-3|\tau|} \qquad - ③$$

(i) $t > 0$

$$J(t) = \int_{-\infty}^{0} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|} + \int_{0}^{t} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|} + \int_{t}^{\infty} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|}$$

$$= \int_{-\infty}^{0} d\tau\, e^{-2(t-\tau)} e^{3\tau} + \int_{0}^{t} d\tau\, e^{-2(t-\tau)} e^{-3\tau} + \int_{t}^{\infty} d\tau\, e^{2(t-\tau)} e^{-3\tau}$$

$$= e^{-2t} \underbrace{\int_{-\infty}^{0} d\tau\, e^{5\tau}}_{=\frac{1}{5}} + e^{-2t} \underbrace{\int_{0}^{t} d\tau\, e^{-\tau}}_{-(e^{-t}-1)} + e^{2t} \underbrace{\int_{t}^{\infty} d\tau\, e^{-5\tau}}_{\frac{1}{5}e^{-5t}}$$

$$= \frac{6}{5} e^{-2t} - \frac{4}{5} e^{-3t}$$

(ii) $t < 0$

$$J(t) = \int_{-\infty}^{t} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|} + \int_{t}^{0} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|} + \int_{0}^{\infty} d\tau\, e^{-2|t-\tau|} e^{-3|\tau|}$$

$$= \int_{-\infty}^{t} d\tau\, e^{-2(t-\tau)} e^{3\tau} + \int_{t}^{0} d\tau\, e^{2(t-\tau)} e^{3\tau} + \int_{0}^{\infty} d\tau\, e^{2(t-\tau)} e^{-3\tau}$$

$$= e^{-2t} \underbrace{\int_{-\infty}^{t} d\tau\, e^{5\tau}}_{\frac{1}{5}e^{5t}} + e^{2t} \underbrace{\int_{t}^{0} d\tau\, e^{\tau}}_{1-e^{t}} + e^{2t} \underbrace{\int_{0}^{\infty} d\tau\, e^{-5\tau}}_{\frac{1}{5}}$$

$$= \frac{6}{5} e^{2t} - \frac{4}{5} e^{3t}$$

(iii) $t = 0$

$$J(t) = \int_{-\infty}^{\infty} d\tau\, e^{-5|\tau|}$$

$$= \int_{-\infty}^{0} d\tau\, e^{5\tau} + \int_{0}^{\infty} d\tau\, e^{-5\tau}$$

$$= \frac{2}{5}$$

$$J(t) = \begin{cases} \frac{6}{5} e^{-2t} - \frac{4}{5} e^{-3t} & (t > 0) \\ \frac{2}{5} & (t = 0) \\ \frac{6}{5} e^{2t} - \frac{4}{5} e^{3t} & (t < 0) \end{cases}$$

$$J(t) = \frac{6}{5} e^{-2|t|} - \frac{4}{5} e^{-3|t|} \qquad - B$$

A → B

$$\mathcal{F}^{-1}\left[\frac{1}{(4+\omega^2)(9+\omega^2)} \right] = \frac{1}{120}\left[6e^{-2|t|} - 4e^{-3|t|} \right]$$

$$= \frac{1}{20} e^{-2|t|} - \frac{1}{30} e^{-3|t|} \qquad ※$$

§ Fourier sine and cosine transform

Let $f(t)$ be defined at $[0, \infty)$

Define

$$f_e(t) = \begin{cases} f(t) & t \geq 0 \\ f(-t) & t < 0 \end{cases}$$

$$\hat{f}_e(\omega) = \int_{-\infty}^{\infty} f_e(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f_e(t) \cos\omega t\, dt$$

$$= 2\int_0^{\infty} f(t) \cos\omega t\, dt \equiv 2\hat{f}_c(\omega) \qquad (\hat{f}_c(-\omega) = \hat{f}_c(\omega))$$

$$\Rightarrow \quad f_e(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \tilde{f}_e(\omega)\, e^{i\omega t}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, \hat{f}_c(\omega)\, e^{i\omega t}$$

$$= \frac{2}{\pi}\int_0^{\infty} d\omega\, \hat{f}_c(\omega) \cos\omega t$$

* Fourier cosine transform

$$\mathcal{F}_c[f](\omega) \equiv \hat{f}_c(\omega) = \int_0^{\infty} dt\, f(t) \cos\omega t$$

$$f(t) = \frac{2}{\pi}\int_0^{\infty} d\omega\, \hat{f}_c(\omega) \cos\omega t$$

(예제 8.06)

$$f(t) = \begin{cases} 1 & 0 \le t \le K \\ 0 & t > K \end{cases}$$

$$\hat{f}_c(\omega) = \int_0^\infty dt\, f(t) \cos\omega t$$

$$= \int_0^K dt \cos\omega t$$

$$= \frac{\sin\omega K}{\omega} \quad ※$$

Let $f(t)$ be defined at $[0, \infty]$.

Then we define

$$f_o(t) = \begin{cases} f(t) & t \ge 0 \\ -f(-t) & t < 0 \end{cases}$$

Then

$$\hat{f}_o(\omega) = \int_{-\infty}^{\infty} f_o(t)\, e^{-i\omega t}\, dt$$

$$= -2i \int_0^\infty dt\, f(t) \sin\omega t$$

$$\equiv -2i\, \hat{f}_s(\omega)$$

$$\hat{f}_s(-\omega) = -\hat{f}_s(\omega)$$

$$\Rightarrow \quad f_o(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \hat{f}_o(\omega)\, e^{i\omega t}$$

$$= -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega\, \hat{f}_s(\omega)\, e^{i\omega t}$$

$$= \frac{2}{\pi} \int_0^\infty d\omega\, \hat{f}_s(\omega) \sin\omega t$$

$$\boxed{\begin{aligned} \mathcal{F}_s[f] &\equiv \hat{f}_s(\omega) = \int_0^\infty dt\, f(t) \sin\omega t \\ f(t) &= \frac{2}{\pi} \int_0^\infty d\omega\, \hat{f}_s(\omega) \sin\omega t \end{aligned}}$$

Fourier sine transform

P390

(예제 8.27)

$$f(t) = \begin{cases} 1 & 0 \le t \le K \\ 0 & K < t \end{cases}$$

$$\hat{f}_s(\omega) = \int_0^\infty dt\, f(t) \sin \omega t$$

$$= \int_0^K dt\, \sin \omega t$$

$$= \frac{1}{\omega}\left[1 - \cos \omega K\right] \quad ※$$

P390

정리 8.18

If $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = 0$,

$$\mathcal{F}_c[f''(t)](\omega) = -\omega^2 \hat{f}_c(\omega) - f'(0)$$

$$\mathcal{F}_s[f''(t)](\omega) = -\omega^2 \hat{f}_s(\omega) + \omega f(0)$$

pf)

① $\mathcal{F}_c[f''(t)](\omega)$

$$= \int_0^\infty dt\, f''(t) \cos \omega t$$

$$= f'(t) \cos \omega t \Big|_{t=0}^{t=\infty} - \int_0^\infty dt\, f'(t) \frac{d}{dt} \cos \omega t$$

$$= -f'(0) + \omega \int_0^\infty dt\, f'(t) \sin \omega t$$

$$= -f'(0) + \omega \left[f(t) \sin \omega t \Big|_{t=0}^{t=\infty} - \int_0^\infty dt\, f(t) \frac{d}{dt} \sin \omega t \right]$$

$$= -f'(0) - \omega^2 \int_0^\infty dt\, f(t) \cos \omega t$$

$$= -f'(0) - \omega^2 \hat{f}_c(\omega)$$

②

$$\mathcal{F}_s[f''(t)](\omega)$$

$$= \int_0^\infty dt\ f''(t)\sin\omega t$$

$$= f'(t)\sin\omega t\Big|_{t=0}^{t=\infty} - \int_0^\infty dt\ f'(t)\frac{d}{dt}\sin\omega t$$

$$= -\omega\int_0^\infty dt\ f'(t)\cos\omega t$$

$$= -\omega\left[\ f(t)\cos\omega t\Big|_{t=0}^{t=\infty} - \int_0^\infty dt\ f(t)\frac{d}{dt}\cos\omega t\ \right]$$

$$= -\omega\left[-f(0) + \omega\int_0^\infty dt\ f(t)\sin\omega t\right]$$

$$= -\omega^2\hat{f}_s(\omega) + \omega f(0) \qquad ※$$

Let $f(t)$ be piecewise smooth at $[0, \pi]$

* 유한 cosine transform

$$\tilde{f}_c(n) = \int_0^{\pi} f(t) \cos nt \, dt$$

$$f(t) = \frac{1}{\pi} \tilde{f}_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} \tilde{f}_c(n) \cos nt$$

pf) Let us define

$$f_e(t) = \begin{cases} f(t) & 0 \le t \le \pi \\ f(-t) & -\pi \le t < 0 \end{cases}$$

Then

$$f_e(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f_e(t) \cos nt \, dt = \frac{2}{\pi} \tilde{f}_c(n)$$

$$\Rightarrow \quad f(t) = \frac{1}{\pi} \tilde{f}_c(0) + \sum_{n=1}^{\infty} \frac{2}{\pi} \tilde{f}_c(n) \cos nt \qquad \divideontimes$$

* 유한 sine transform

$$\tilde{f}_s(n) = \int_0^{\pi} f(t) \sin nt \, dt$$

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \tilde{f}_s(n) \sin nx$$