CH5 행렬 (Matrix)

§ 행렬

$$\begin{array}{ccc} & \text{열} & \\ A = & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} & \text{행} \end{array}$$

$n \times m$ matrix

① If $n = m$, A: 정방행렬 (square matrix)

② a_{ij} : A의 원소 (component)

(정의 5.2)

If $A = B$, $a_{ij} = b_{ij}$ $\forall i, j$

(정의 5.3)

If $A + B = C$, $c_{ij} = a_{ij} + b_{ij}$

(정의 5.4)

If $B = \alpha A$, $b_{ij} = \alpha a_{ij}$

정의 5.5

If $C = AB$ with $A = n \times r$ and $B = r \times m$,

$$C_{ij} = \sum_{m=1}^{r} A_{im} B_{mj}$$

(예제 5.1)

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 7 & 4 & 15 \\ 12 & 7 & 26 \end{pmatrix}$$ ※

p157

(예제5.2)

$$A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 4 & 1 & 6 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 8 \\ 2 & 1 \\ 1 & 1 \\ 12 & 6 \end{pmatrix}$$

$$AB = \begin{pmatrix} 15 & 17 \\ 28 & 51 \end{pmatrix}$$

$$BA = \begin{pmatrix} 31 & 7 & 46 & 15 \\ 6 & 3 & 10 & 4 \\ 5 & 2 & 8 & 3 \\ 36 & 18 & 60 & 24 \end{pmatrix}$$

$$AB \neq BA$$ ※

정리 5.1

(1) $A+B = B+A$

(2) $A(B+C) = AB+AC$

(3) $(A+B)C = AC+BC$

(4) $A(BC) = (AB)C$

(Ex) $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 2 \\ 3 & 16 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$

$$A(BC) = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 10 & 10 \\ 22 & -7 \end{pmatrix} = \begin{pmatrix} 32 & 3 \\ 96 & 9 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 7 & 18 \\ 21 & 54 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 32 & 3 \\ 96 & 9 \end{pmatrix}$$

$$A(BC) = (AB)C$$ ※

정의 5.6 zero matrix (영 행렬)

If $A_{ij} = 0$, $a_{ij} = 0$ $\forall i, j$

정의 5.7 identity matrix (단위행렬)

If $A = I$, $a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Ex)

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

정리 5.2

$A \cdot I = I \cdot A = I$

(예제5.5)

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix}$$

$$I_3 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = A$$

$$A I_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 8 \end{pmatrix} = A$$

정의 5.8 transpose matrix (전치행렬)

$$(A^t)_{ij} = (A)_{ji}$$

(예제5.6)

$$A = \begin{pmatrix} -1 & 6 & 3 & 3 \\ 0 & \pi & 12 & -5 \end{pmatrix} \quad A^t = \begin{pmatrix} -1 & 0 \\ 6 & \pi \\ 3 & 12 \\ 3 & -5 \end{pmatrix}$$

(정리 5.3)

$$\boxed{\begin{array}{l} (1)\quad (A^t)^t = A \\ (2)\quad (AB)^t = B^t A^t \end{array}}$$

(Ex)

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 7 \\ 0 & 12 \end{pmatrix} \qquad (AB)^t = \begin{pmatrix} 1 & 0 \\ 7 & 12 \end{pmatrix}$$

$$B^t A^t = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 12 \end{pmatrix}$$

$$\Rightarrow \quad (AB)^t = B^t A^t$$

- Scalar product

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$x^t y = (x_1, x_2, \cdots, x_m) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m = x \cdot y$$

5. 기본 행연산

(정의 5.9) 기본 행연산

1. 제 1 연산 : A 의 두 행을 바꾼다.
2. 제 2 연산 : A의 한행에 0 이 아닌 상수를 곱한다.
3. 제 3 연산 : A의 한행에 상수를 곱하여 다른행에 더한다.

* elementary row operation

(예제 5.7)

$$A = \begin{pmatrix} -2 & 1 & 6 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \\ 2 & -3 & 4 \end{pmatrix}$$

(i) 2행과 4행을 바꾼다.

$$\begin{pmatrix} -2 & 1 & 6 \\ 2 & -3 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

(ii) 2행에 7을 곱한다

$$\begin{pmatrix} -2 & 1 & 6 \\ 7 & 7 & 14 \\ 0 & 1 & 3 \\ 2 & -3 & 4 \end{pmatrix}$$

(iii) 1행을 두배하여 3행에 더하기

$$\begin{pmatrix} -2 & 1 & 6 \\ 1 & 1 & 2 \\ -4 & 3 & 15 \\ 2 & -3 & 4 \end{pmatrix}$$

정의 5.10 기본행렬

기본행렬 : I_m 에 기본 행연산을 하여 만들어진 행렬

(Ex)

①
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \text{기본행렬}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1행과 2행을 바꾼다.

②
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix} : \text{기본행렬}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1행에 -4를 곱하여 3행에 더한다.

(정리 5.4)

A : $m \times m$ 행렬

B : A 에 기본행연산을 하여 얻어진 행렬

E : I_m 에 같은 기본 행연산을 하여 얻어진 행렬

Then
$$B = EA$$

p165

(예제 5.8)

$$A = \begin{pmatrix} 1 & -5 \\ 9 & 4 \\ -3 & 2 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

연산 : 2행과 3행을 교환

$$B = \begin{pmatrix} 1 & -5 \\ -3 & 2 \\ 9 & 4 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ 9 & 4 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -3 & 2 \\ 9 & 4 \end{pmatrix} = B$$

연산 : 1행에 3을 곱하여 3행에 더한다

$$B = \begin{pmatrix} 1 & -5 \\ 9 & 4 \\ 0 & -13 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ 9 & 4 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 9 & 4 \\ 0 & -13 \end{pmatrix} = B$$

p165

(예제5.9)

$$A = \begin{pmatrix} -6 & 14 & 2 \\ 4 & 4 & -9 \\ -3 & 2 & 13 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

연산: 1행에 6을 곱하여 2행에 더한다.

$$B = \begin{pmatrix} -6 & 14 & 2 \\ -32 & 88 & 3 \\ -3 & 2 & 13 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -6 & 14 & 2 \\ 4 & 4 & -9 \\ -3 & 2 & 13 \end{pmatrix} = \begin{pmatrix} -6 & 14 & 2 \\ -32 & 88 & 3 \\ -3 & 2 & 13 \end{pmatrix} = B$$

$$A \xrightarrow{O_1} A_1 \xrightarrow{O_2} A_2 \rightarrow \cdots \xrightarrow{O_{r-1}} A_{r-1} \xrightarrow{O_r} A_r$$

$$A_1 = E_1 A$$

$$A_2 = E_2 A_1 = (E_2 E_1) A$$

$$\vdots$$

$$A_r = (E_r E_{r-1} \cdots E_2 E_1) A$$

(정리5.5)

A: $m \times m$ 행렬

B: A에 기본 행연산을 연달아 수행하여 얻은 행렬

$$B = \Omega A$$

Ω가 존재

p166

(예제5.10)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{pmatrix} \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

연산: O_1 : 1행과 2행을 바꾸어 A_1을 만든다.

O_2: A_1의 3행에 2를 곱하여 A_2를 만든다.

O_3: A_2의 2행에 2를 곱하여 3행에 더해서 B를 만든다.

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ -2 & 6 & 4 \end{pmatrix} \qquad E_2 E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 2 & 8 & 4 \end{pmatrix} \qquad E_3 E_2 E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} = \Omega \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\Omega A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 2 & 8 & 4 \end{pmatrix} = B$$

정의 5.11 : 행 등가 행렬 (row equivalent matrix)

B : A에 기본 행연산을 연달아 수행하여 얻어진 행렬

B : A의 행 등가 행렬

(정리 5.6)

[1] 모든 행렬은 자기 자신과 행등가 이다. (reflexive property)

[2] A가 B의 행 등가 행렬 이면, B도 A의 행등가행렬 이다. (symmetry property)

[3] A가 B의 행 등가 행렬이고, B가 C의 행 등가 행렬이면 A는 C의 행 등가 행렬이다. (transitivity)

정리 5.7

E_1: A에 기본 행연산을 한 기본행렬

⇒ $E_2(E_1A) = A$ 인 기본행렬 E_2가 존재한다.

(Ex)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

연산 : 2행 ↔ 3행

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_1 \xleftarrow{\text{-4×첫행+3행}} I_3$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$$ 기본행렬

$$E_2 \xleftarrow{\text{4×첫행+3행}} I_3$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$$ 기본행렬

$$E_2(E_1A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -4 & 0 & 1 \\ -4 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = A$$

//

*** 정의**

A: $m \times n$ matrix

모든성분이 0인행 ⇒ 영행 (zero row)

선행성분: 0행이 아닌경우 가장 왼쪽에 있는 0이 아닌 성분

Ex)

$$A = \begin{pmatrix} 0 & 2 & 7 \\ 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

선행성분 : 2

선행성분 : 1

영행

선행성분 : 9

*

정의 5.12 기약행 사다리꼴 행렬

다음 조건을 만족하는 행렬

1. 모든 선행성분은 1 이다.
2. 어떤 행의 선행성분이 j 열에 있으면 j 열의 다른 성분들은 모두 0 이다.
3. i 행이 영행이 아니고 k 행이 영행이면 $i<k$ 이다.
4. r_1 행의 선행성분이 c_1 열에 있고, r_2 행의 선행성분이 c_2 열에 있을때, $r_1 < r_2$ 이면 $c_1 < c_2$ 이다.

(Ex)

$$\begin{pmatrix} 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

② ok

선행성분 : 1

선행성분 : 1

) ① ok

1행 1열 $r_1=1$ $c_1=1$

2행 4열 $r_2=2$ $c_2=4$

) ④ ok

⇒ 기약행 사다리꼴 행렬

(Ex)

$$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

선행성분 : 1
선행성분 : 1 } ① OK
영행

②: OK

1행, 2행: 0행이 아님 } ③: OK
3행 : 0 행

1행 2열 : $r_1=1$ $c_1=2$ } ④ OK
2행 4열 : $r_2=2$ $c_2=4$

⇒ 기약행 사다리꼴 행렬

(Ex)

$$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

⇒ 기약행 사다리꼴 아님

(Ex)

$$\begin{pmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & -2 & 4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

⇒ 기약행 사다리꼴 행렬

정리 5.8

(1) 모든 행렬은 어떤 하나의 기약행 사다리꼴 행렬과 행등가이다.

(2) 모든 행렬 A의 기약행 사다리꼴 행렬 A_R은 유일하다.

정리 5.9

A: $m \times n$ 행렬

$\Omega A = A_R$ 인 $m \times m$ 행렬 Ω가 존재

(Ex)

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 4 & 3 & 4 & 0 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(-4) 3행 + 4행

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix}$$

1행 $\Leftrightarrow$ 3행

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix}$$

$\frac{1}{2} \times$ (2행)

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 \end{pmatrix}$$

(-3)(2행) + 4행

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & -4 & 1 \end{pmatrix}$$

$$-\frac{1}{4}\times 4\text{行}\quad \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & 1 & -\frac{1}{4} \end{pmatrix}$$

$$(-1)\times 4\text{行} + 1\text{行}\quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -\frac{3}{8} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{3}{8} & 1 & -\frac{1}{4} \end{pmatrix}$$

$$3\text{行} \Leftrightarrow 4\text{行}\quad \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -\frac{3}{8} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{8} & 1 & -\frac{1}{4} \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= A_R \qquad\qquad = \Omega$$

ΩA

$$= \begin{pmatrix} 0 & -\frac{3}{8} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{8} & 1 & -\frac{1}{4} \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 4 & 3 & 4 & 0 \end{pmatrix}$$

$$= A_R$$

(예제 5.14)

$$A = \begin{pmatrix} -3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$-\frac{1}{3} \times (1행)$ $$\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 4 & -2 & 1 \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}$$

$\frac{1}{4} \times (2행)$ $$\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

$(1행) - (2행)$ $$\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{6} & -\frac{1}{4} \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{4} \end{pmatrix}$$

$6 \times (2행)$ $$\begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{3}{2} \end{pmatrix} \qquad \begin{pmatrix} -\frac{1}{3} & 0 \\ -2 & -\frac{3}{2} \end{pmatrix}$$

$\frac{1}{3} \times (2행) + 1행$ $$\underbrace{\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \end{pmatrix}}_{= A_R} \qquad \underbrace{\begin{pmatrix} -1 & -\frac{1}{2} \\ -2 & -\frac{3}{2} \end{pmatrix}}_{= \Omega}$$

$$\Omega A = \begin{pmatrix} -1 & -\frac{1}{2} \\ -2 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \end{pmatrix} = A_R$$

※

§ 행공간과 위수 (rank)

A : $n \times m$ matrix

행공간 (row space) : 행 vector의 linear combination으로 만들어진 vector space

열공간 (column space) : 열 vector의 linear combination으로 만들어진 vector space

(예제 5.15)

$$B = \begin{pmatrix} -2 & 6 & 1 \\ 2 & 2 & -4 \\ 10 & -8 & 12 \\ 3 & 1 & -2 \\ 5 & -5 & 7 \end{pmatrix}$$

행 vector ; $(-2,6,1)$, $(2,2,-4)$, $(10,-8,12)$, $(3,1,-2)$, $(5,-5,7)$

If $v \in$ 행공간,

$$v = \alpha(-2,6,1) + \beta(2,2,-4) + \gamma(10,-8,12) + \delta(3,1,-2) + \epsilon(5,-5,7)$$

행공간의 차원 ? 3차원

$$(3,1,-2) = \frac{4}{101}(-2,6,1) + \frac{181}{202}(2,2,-4) + \frac{13}{101}(10,-8,12)$$

$$(5,-5,7) = -\frac{7}{101}(-2,6,1) - \frac{39}{202}(2,2,-4) + \frac{53}{101}(10,-8,12)$$

열 vector : $\begin{pmatrix} -2 \\ 2 \\ 10 \\ 3 \\ 5 \end{pmatrix}$ $\begin{pmatrix} 6 \\ 2 \\ -8 \\ 1 \\ -5 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -4 \\ 12 \\ -2 \\ 7 \end{pmatrix}$

If $w \in$ 열공간,

$$w = \alpha \begin{pmatrix} -2 \\ 2 \\ 10 \\ 3 \\ 5 \end{pmatrix} + \beta \begin{pmatrix} 6 \\ 2 \\ -8 \\ 1 \\ -5 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ -4 \\ 12 \\ -2 \\ 7 \end{pmatrix}$$

열공간의 차원 ? 3차원

$$B_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

행공간 차원 = 열공간 차원 = B_R의 영행이 아닌 행의수

(정리5.10)

실수 성분을 갖는 행렬 A에 대하여 행공간 과 열공간은 같은 차원을 갖는다.

정의5.13 : 위수 (rank)

행렬 A 의 위수(rank)는 A_R의 0행이 아닌 행의 개수이다.

* $\text{rank}(A) = \text{rank}(A_R) =$ A_R 에서 0 행이 아닌 행의 갯수

= 행공간과 열공간의 차원

p175

(예제5.16)

$$A = \begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 3 & -2 & 15 & 8 \end{pmatrix}$$

1행×(-3)+3행 $\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix}$

2행-3행 $\begin{pmatrix} 1 & -1 & 4 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2행+1행 $\begin{pmatrix} 1 & 0 & 7 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A_R$

$\text{rank}(A) = 2$ ※

◎ 선형 연립 방정식의 해

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

homogeneous coupled equation

(제차 연립 방정식)

$$\Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$$\Rightarrow \qquad AX = 0$$

(Ex)

$$x_1 - 3x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - 3x_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & -3 & 2 \\ -2 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Gauss - Jordan 방법

$$AX = 0 \Rightarrow A_R X = 0$$

$$AX = 0$$
$$(\Omega A) X = 0$$
$$A_R X = 0$$

(Ex)

$$x_1 - 3x_2 + x_3 - 7x_4 + 4x_5 = 0$$
$$x_1 + 2x_2 - 3x_3 = 0$$
$$x_2 - 4x_3 + x_5 = 0$$

$$\Rightarrow \quad AX = 0 \quad - ①$$

$$A = \begin{pmatrix} 1 & -3 & 1 & -7 & 4 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A_R = \begin{pmatrix} 1 & 0 & 0 & -\frac{35}{16} & \frac{13}{16} \\ 0 & 1 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{7}{16} & -\frac{9}{16} \end{pmatrix} \qquad \text{rank}(A) = 3$$

```
In[7]:= A = {{1, -3, 1, -7, 4}, {1, 2, -3, 0, 0}, {0, 1, -4, 0, 1}};
        MatrixForm[RowReduce[A]]
Out[8]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{35}{16} & \frac{13}{16} \\ 0 & 1 & 0 & \frac{7}{4} & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{7}{16} & -\frac{9}{16} \end{pmatrix}$$

$A_R X = 0$

$$x_1 - \frac{35}{16} x_4 + \frac{13}{16} x_5 = 0$$

$$x_2 + \frac{7}{4} x_4 - \frac{5}{4} x_5 = 0$$

$$x_3 + \frac{7}{16} x_4 - \frac{9}{16} x_5 = 0$$

If $x_4 = \alpha$ and $x_5 = \beta$,

$$x_1 = \frac{35}{16}\alpha - \frac{13}{16}\beta$$

$$x_2 = -\frac{7}{4}\alpha + \frac{5}{4}\beta$$

$$x_3 = -\frac{7}{16}\alpha + \frac{9}{16}\beta$$

$$\Rightarrow \quad X = \begin{pmatrix} \frac{35}{16}\alpha - \frac{13}{16}\beta \\ -\frac{7}{4}\alpha + \frac{5}{4}\beta \\ -\frac{7}{16}\alpha + \frac{9}{16}\beta \\ \alpha \\ \beta \end{pmatrix}$$

$m - \text{rank}(A) = 5 - 3 = 2$

⇒ 2개의 independent solution

(unknown constants)

(예제 5.8)

$$-x_1 + x_3 + x_4 + 2x_5 = 0$$

$$x_2 + 3x_3 + 4x_5 = 0$$

$$x_1 + 2x_2 + x_3 + x_4 + x_5 = 0$$

$$-3x_1 + x_2 + 4x_5 = 0$$

$$A = \begin{pmatrix} -1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \\ -3 & 1 & 0 & 0 & 4 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$A_R = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{9}{8} \\ 0 & 1 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 1 & -\frac{1}{4} \end{pmatrix} \qquad \text{rank}(A) = 4$$

```
In[9]:= A = {{-1, 0, 1, 1, 2
        MatrixForm[RowReduce
Out[10]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{9}{8} \\ 0 & 1 & 0 & 0 & \frac{5}{8} \\ 0 & 0 & 1 & 0 & \frac{9}{8} \\ 0 & 0 & 0 & 1 & -\frac{1}{4} \end{pmatrix}$$

$$A_R X = 0$$

$$x_1 - \frac{9}{8} x_5 = 0$$

$$x_2 + \frac{5}{8} x_5 = 0$$

$$x_3 + \frac{9}{8} x_5 = 0$$

$$x_4 - \frac{1}{4} x_5 = 0$$

if $x_5 = \alpha$,

$$x_1 = \frac{9}{8}\alpha, \quad x_2 = -\frac{5}{8}\alpha, \quad x_3 = -\frac{9}{8}\alpha, \quad x_4 = \frac{1}{4}\alpha$$

$$X = \begin{pmatrix} \frac{9}{8}\alpha \\ -\frac{5}{8}\alpha \\ -\frac{9}{8}\alpha \\ \frac{1}{4}\alpha \\ \alpha \end{pmatrix} = \gamma \begin{pmatrix} 9 \\ -5 \\ -9 \\ 2 \\ 8 \end{pmatrix} \qquad \left(\gamma = \frac{\alpha}{8}\right)$$

$m - \text{rank}(A) = 5 - 4 = 1$

$\Rightarrow$ one unknown constant

P179

(예제 5.19)

$$3x_1 - 11x_2 + 5x_3 = 0$$

$$4x_1 + x_2 - 10x_3 = 0$$

$$4x_1 + 9x_2 - 6x_2 = 0$$

$$A = \begin{pmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\text{rank}(A) = 3$

$$A_R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
In[11]:= A = {{3, -11, 5}, {4, 1, -10}, {4, 9, -6}};
         MatrixForm[RowReduce[A]]
Out[12]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_R X = 0$$

$$x_1 = 0, \quad x_2 = 0, \quad x_2 = 0$$

$$m - \text{rank}(A) = 3 - 3 = 0$$

$\Rightarrow$ no unknown constant !!

*

정리 5.11

$AX = O$ 의 해공간의 차원 : # of unknown constants

$\Rightarrow$ 해공간의 차원 $= n - \text{rank}(A)$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mm}x_m = b_m$$

non-homogeneous coupled Equation

$$\Rightarrow \quad AX = B$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

정리 5.13

U_p : $AX = B$ 의 특수해

H : $AX = 0$ 의 일반해

$AX = B$ 의 일반해는 $U_p + H$ 이다.

(예제 5.22)

$$-x_1 + x_2 + 3x_3 = -2$$

$$x_2 + 2x_3 = 4$$

$$\Rightarrow \quad AX = B$$

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad B = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

① U_p

$x_1 = 6, \quad x_2 = 4, \quad x_3 = 0$

$$U_p = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

② H

$$A_R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

```
In[13]:= A = {{-1, 1, 3}, {0, 1, 2}};
         MatrixForm[RowReduce[A]]
Out[14]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$x_1 = x_3$$

$$x_2 = -2x_3$$

If $x_3 = \alpha$, $x_1 = \alpha$ and $x_2 = -2\alpha$

$$H = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow U_p + H = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} \quad ※$$

* non-homogeneous coupled equation의 연해 방법

$$-3x_1 + 2x_2 + 6x_3 + x_4 = 5$$

$$3x_2 + 3x_3 - 5x_4 = 2$$

$$2x_1 + 4x_2 + 4x_3 - 6x_4 = -8$$

$\Rightarrow \quad AX = B$

$$A = \begin{pmatrix} -3 & 2 & 6 & 1 \\ 0 & 3 & 3 & -5 \\ 2 & 4 & 4 & -6 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad B = \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$$

$\Rightarrow$ 확대행렬 (augmented matrix)

$$[A:B] \equiv \left(\begin{array}{cccc|c} -3 & 2 & 6 & 1 & 5 \\ 0 & 3 & 3 & -5 & 2 \\ 2 & 4 & 4 & -6 & -8 \end{array}\right)$$

$$[A:B]_R = \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & -\frac{16}{3} \\ 0 & 1 & 0 & -3 & \frac{15}{4} \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{37}{12} \end{array}\right) = [A_R : C]$$

rank(A) = 3

In[15]:=
```
A = {{-3, 2, 6, 1, 5}, {0, 3, 3, -5, 2}, {2, 4, 4, -6, -8}};
MatrixForm[RowReduce[A]]
```

Out[16]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} & -\frac{16}{3} \\ 0 & 1 & 0 & -3 & \frac{15}{4} \\ 0 & 0 & 1 & \frac{4}{3} & -\frac{37}{12} \end{pmatrix}$$

$$x_1 + \frac{1}{3}x_4 = -\frac{16}{3}$$

$$x_2 - 3x_4 = \frac{15}{4}$$

$$x_3 + \frac{4}{3}x_4 = -\frac{37}{12}$$

① U_p

$$x_1 = -\frac{16}{3}, \quad x_2 = \frac{15}{4}, \quad x_3 = -\frac{37}{12}, \quad x_4 = 0$$

② H

if $x_4 = \alpha$,

$$x_1 = -\frac{\alpha}{3}, \quad x_2 = 3\alpha, \quad x_3 = -\frac{4}{3}\alpha$$

$$H = \begin{pmatrix} -\frac{\alpha}{3} \\ 3\alpha \\ -\frac{4}{3}\alpha \\ \alpha \end{pmatrix} = \frac{\alpha}{3}\begin{pmatrix} -1 \\ 9 \\ -4 \\ 3 \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ -9 \\ 4 \\ -3 \end{pmatrix} \qquad \left(\gamma = -\frac{\alpha}{3}\right)$$

$m - \mathrm{rank}(A) = 4 - 3 = 1$

$$H + U_p = \gamma \begin{pmatrix} 1 \\ -9 \\ 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -\frac{16}{3} \\ \frac{15}{4} \\ -\frac{37}{12} \\ 0 \end{pmatrix}$$

※

P184

(예제 5.23)

$$\begin{pmatrix} -3 & 2 & 2 \\ 1 & 4 & -6 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$$

$$[A:B] = \left(\begin{array}{ccc|c} -3 & 2 & 2 & 8 \\ 1 & 4 & -6 & 1 \\ 0 & -2 & 2 & -2 \end{array}\right)$$

```
In[17]:= A = {{-3, 2, 2, 8}, {1, 4, -6, 1}, {0, -2, 2, -2}};
         MatrixForm[RowReduce[A]]
Out[18]//MatrixForm=
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{pmatrix}$$

$$[A:B]_R = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$\mathrm{rank}(A) = 3$

$m - \mathrm{rank}(A) = 0$

① U_p

$x_1 = 0$

$x_2 = \frac{5}{2}$

$x_3 = \frac{3}{2}$

② H: $x_1 = x_2 = x_3 = 0$

$$H + U_p = \begin{pmatrix} 0 \\ \frac{5}{2} \\ \frac{3}{2} \end{pmatrix}$$

8. 행렬식 (determinant)

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$ ※

(정의 5.15)

$n \times n$ 행렬식에서 i행, j열을 제외하고 얻은 행렬식 $\det M_{ij}$ 를 소행렬식이라 한다. 또한 $(-1)^{i+j} \det M_{ij}$ 를 a_{ij} 의 여인수 (cofactor) 라 한다.

$$A_{ij} = (-1)^{i+j} \det M_{ij}$$ cofactor

(Ex)

$$A = \begin{vmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{vmatrix}$$

① $\det M_{11} = \begin{vmatrix} -5 & -9 \\ 4 & -6 \end{vmatrix} = 30 + 36 = 66$

$A_{11} = a_{11}$ 의 cofactor $= (-1)^{1+1} \det M_{11} = 66$

② $\det M_{12} = \begin{vmatrix} 12 & -9 \\ 2 & -6 \end{vmatrix} = -72 + 18 = -54$

$A_{12} = a_{12}$ 의 cofactor $= (-1)^{1+2} \det M_{12} = 54$

③ $\det M_{13} = \begin{vmatrix} 12 & -5 \\ 2 & 4 \end{vmatrix} = 58$

$A_{13} = a_{13}$ 의 cofactor $= (-1)^{1+3} \det M_{13} = 58$ ※

정의 5.16 : 여인수 전개

$$\boxed{\det A = a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n} = \sum_{k=1}^{n} a_{1k} A_{1k}}$$

$$* \quad \det A = \sum_{k=1}^{n} a_{ik} A_{ik} \qquad (i = 1, 2, \cdots, n)$$

$$= \sum_{k=1}^{n} a_{kj} A_{kj} \qquad (j = 1, 2, \cdots, n)$$

(예제 5.27)

$$A = \begin{pmatrix} -6 & 3 & 7 \\ 12 & -5 & -9 \\ 2 & 4 & -6 \end{pmatrix}$$

$A_{11} = 66$, $A_{12} = 54$, $A_{13} = 58$

$\det A = -6 \times 66 + 3 \times 54 + 7 \times 58 = 170$

(정리 5.16)

(1) A가 0행 (or 0열)을 가지면 $\det A = 0$

(2) A의 k행 (or k열)에 α를 곱한 행렬을 B라 하면, ⇒ 예로 설명

$$\det B = \alpha \det A$$

(3) A의 두행 (or 두열)을 서로 교환한 행렬을 B라 하면, ⇒ 예로 설명

$$\det B = -\det A$$

(4) A의 두행이 같거나 비례하면 $\det A = 0$ ⇒ 예로 설명

(5) $\det(AB) = \det A \cdot \det B$ ⇒ 예로 설명

(6)
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \alpha_{k1}+\beta_{k1} & \cdots & \alpha_{kn}+\beta_{kn} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \alpha_{k1} & \cdots & \alpha_{kn} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \beta_{k1} & \cdots & \beta_{kn} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$
⇒ 예로 설명

(7) A의 k행 (또는 k열)에 α를 곱하여 j행 (or j열)에 더한 행렬을

B라 하면 $\det B = \det A$ ⇒ 예로 설명

(8) $\det A^t = \det A$ ⇒ 예로 설명

(예제 5.29)

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ -1 & 2 & -1 & 4 \\ 2 & 1 & 3 & 0 \\ 1 & -3 & 3 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -1 & 7 \\ 2 & 1 & 3 & 0 \\ 1 & -3 & 3 & 2 \end{vmatrix}$$ (1행) + (2행) → 2행

$$= \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -1 & 7 \\ 0 & -3 & 3 & -6 \\ 1 & -3 & 3 & 2 \end{vmatrix}$$ −2×(1행) + (3행) → 3행

$$= \begin{vmatrix} 1 & 2 & 0 & 3 \\ 0 & 4 & -1 & 7 \\ 0 & -3 & 3 & -6 \\ 0 & -5 & 3 & -1 \end{vmatrix}$$ −1×(1행) + (4행) → 4행

$$= \begin{vmatrix} 4 & -1 & 7 \\ -3 & 3 & -6 \\ -5 & 3 & -1 \end{vmatrix}$$ 여인수 전개

$$= (-3)\begin{vmatrix} 4 & -1 & 7 \\ 1 & -1 & 2 \\ -5 & 3 & -1 \end{vmatrix}$$ 성질 (2)

$$= 3 \begin{vmatrix} 1 & -1 & 2 \\ 4 & -1 & 7 \\ -5 & 3 & -1 \end{vmatrix}$$ (1행) 과 (2행) 교환

$$= 3 \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ -5 & 3 & -1 \end{vmatrix}$$ (−4)×(1행) + 2행 → 2행

$$= 3 \begin{vmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & -2 & 9 \end{vmatrix}$$ (−5)×(1행) + 3행 → 3행

$$= 3 \begin{vmatrix} 3 & -1 \\ -2 & 9 \end{vmatrix}$$

여인수전개

$$= 3 \ (27 - 2)$$

$$= 75$$

//

§ 역행렬

정의 5.17 역행렬 (inverse matrix)

If $AB = BA = I$, $B = A^{-1}$: inverce matrix of A

(Ex) $A = \begin{pmatrix} 5 & -1 \\ 6 & 8 \end{pmatrix}$ $B = \frac{1}{46}\begin{pmatrix} 8 & 1 \\ -6 & 5 \end{pmatrix}$

$$AB = \frac{1}{46}\begin{pmatrix} 5 & -1 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 8 & 1 \\ -6 & 5 \end{pmatrix} = \frac{1}{46}\begin{pmatrix} 46 & 0 \\ 0 & 46 \end{pmatrix} = I_2$$

$$BA = \frac{1}{46}\begin{pmatrix} 8 & 1 \\ -6 & 5 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 6 & 8 \end{pmatrix} = \frac{1}{46}\begin{pmatrix} 46 & 0 \\ 0 & 46 \end{pmatrix} = I_2$$

$B = A^{-1}$ //

정의 5.18 정칙행렬 (nonsingular matrix), 특이행렬 (singular matrix)

A의 역행렬 A^{-1}가 존재하면 : $A \Rightarrow$ 정칙행렬

A의 역행렬 A^{-1}가 존재하지 않으면 : $A \Rightarrow$ 특이행렬

(정리 5.17)

(1) $I^{-1} = I$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) $(A^{-1})^{-1} = A$

(4) $(A^t)^{-1} = (A^{-1})^t$

p198

정리 5.20

$$A^{-1} = \frac{1}{\det A} \left(A_{ij} \right)^t$$

A_{ij} : cofactor

Ex)

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 6 \end{pmatrix}$$

$\det A = 13$

$A_{11} = (-1)^2 \cdot 6 = 6$

$A_{12} = (-1)^3 \; 1 = -1$

$A_{21} = (-1)^3 (-1) = 1$

$A_{22} = (-1)^4 \; 2 = 2$

$$A^{-1} = \frac{1}{13} \begin{pmatrix} 6 & -1 \\ 1 & 2 \end{pmatrix}^t = \frac{1}{13} \begin{pmatrix} 6 & 1 \\ -1 & 2 \end{pmatrix}$$

p199 (예제5.33)

$$A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & 3 & -3 \\ 2 & 9 & -5 \end{pmatrix}$$

⇒ 생략

p198 (예제5.22) 연립방정식 solution

$$2x_1 - x_2 + 3x_3 = 4$$

$$x_1 + 9x_2 - 2x_3 = -8$$

$$4x_1 - 8x_2 + 11x_3 = 15$$

$$AX = B \quad -①$$

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 9 & -2 \\ 4 & -8 & 11 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad B = \begin{pmatrix} 4 \\ -8 \\ 15 \end{pmatrix}$$

$$A^{-1} = \frac{1}{53} \begin{pmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{pmatrix} \quad -②$$

$A^{-1} \times$ Eq. ①

$$X = A^{-1}B = \frac{1}{53} \begin{pmatrix} 83 & -13 & -25 \\ -19 & 10 & 7 \\ -44 & 12 & 19 \end{pmatrix} \begin{pmatrix} 4 \\ -8 \\ 15 \end{pmatrix} = \begin{pmatrix} \frac{61}{53} \\ -\frac{51}{53} \\ \frac{13}{53} \end{pmatrix}$$

$$\Rightarrow x_1 = \frac{61}{53}, \quad x_2 = -\frac{51}{53}, \quad x_3 = \frac{13}{53}$$

※

§ Cramer 법칙

(Ex)

$$2x + 3y = 1$$

$$-x + 2y = 2$$

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}} = \frac{-4}{7}$$

$$y = \frac{\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}} = \frac{5}{7}$$

(Ex)

$$x_1 - 3x_2 - 4x_3 = 1$$

$$-x_1 + x_2 - 3x_3 = 14$$

$$x_2 - 3x_3 = 5$$

$$x_1 = \frac{\begin{vmatrix} 1 & -3 & -4 \\ 14 & 1 & -3 \\ 5 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & -4 \\ -1 & 1 & -3 \\ 0 & 1 & -3 \end{vmatrix}} = -9$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -4 \\ -1 & 14 & -3 \\ 0 & 5 & -3 \end{vmatrix}}{\begin{vmatrix} \quad \text{〃} \quad \end{vmatrix}} = -\frac{10}{13}$$

$$x_3 = \frac{\begin{vmatrix} 1 & -3 & 1 \\ -1 & 1 & 14 \\ 0 & 1 & 5 \end{vmatrix}}{\begin{vmatrix} \quad \text{〃} \quad \end{vmatrix}} = -\frac{25}{13}$$

※

§ 고유값 (eigenvalue)과 고유 vector (eigen vector)

$$AX = \lambda X$$

λ : eigenvalue

X : eigen vector

λ 계산 $\Rightarrow$ $\det(A - \lambda I) = 0$

(Ex)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

(i) eigenvalue

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$-(1-\lambda)^2(1+\lambda) = 0$$

$\lambda = 1$ (중근) $\lambda = -1$

(ii) eigen vector

ⓐ $\lambda = 1$ (중근) ~~⇒ 2개조~~

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$X - Y = X \Rightarrow Y = 0$

$Y + Z = Y \Rightarrow Z = 0$

$-Z = Z \Rightarrow Z = 0$

$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}$: eigenvector

보통은 단위vector 선택 : $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ⓑ $\lambda = -1$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = -\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$X - Y = -X \quad \Rightarrow \quad 2X = Y$$

$$Y + Z = -Y \quad \Rightarrow \quad 2Y = -Z$$

$$-Z = -Z$$

$X = \beta, \quad Y = 2\beta, \quad Z = -4\beta$

eigenvector $= \begin{pmatrix} \beta \\ 2\beta \\ -4\beta \end{pmatrix}$

$\Rightarrow$ 보통은 단위vector 선택 : $\frac{1}{\sqrt{21}} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ ※

※ 행렬의 대각화 (diagonalization)

diagonal matrix
$$A = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & d_n \end{pmatrix}$$

① $\det A = d_1 d_2 \cdots d_n$

② eigenvalue of $A = \{d_1, d_2, \cdots, d_n\}$

③ eigenvector of $A = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$

④ $A^{-1} = \begin{pmatrix} \frac{1}{d_1}, & 0, & \cdots, & 0 \\ 0, & \frac{1}{d_2}, & \cdots, & 0 \\ \vdots & \vdots & & \\ 0, & 0, & \cdots, & \frac{1}{d_n} \end{pmatrix}$

행렬 A의 diagonalization

(i) A의 eigenvalue $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ 와 eigenvector $\{v_1, v_2, \cdots, v_n\}$ 을 구한다.

(ii) $v_1, v_2, \cdots, v_n$ 을 열벡터로 하는 행렬 P를 구한다.

(iii) $P^{-1} A P = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & \cdots & \lambda_n \end{pmatrix}$

p207

(예제 5.37)

$$A = \begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}$$

eigenvalue $= \{-1,\ 3\}$

* eigenvector

(A) $\lambda = -1$

$$\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = -\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$-X + 4Y = -X \quad \Rightarrow \quad Y = 0$$

$$3Y = -Y \quad \Rightarrow \quad Y = 0$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(B) $\lambda = 3$

$$\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 3\begin{pmatrix} X \\ Y \end{pmatrix}$$

$$-X + 4Y = 3X \quad \Rightarrow \quad X = Y$$

$$3Y = 3Y$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

//

note)

대각화는 꼭 unit eigen vector 를 사용할 필요는 없다

(예제 5.38)

$$A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$$

eigenvalue = { 1 (중근), -3 }

① $\lambda = 1$

$$\begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$5X - 4Y + 4Z = X \quad \Rightarrow \quad X - Y + Z = 0$$

$$12X - 11Y + 12Z = Y \quad \Rightarrow \quad X - Y + Z = 0$$

$$4X - 4Y + 5Z = Z \quad \Rightarrow \quad X - Y + Z = 0$$

$X=1,\ Y=1,\ Z=0$

$X=1,\ Y=0,\ Z=-1$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

② $\lambda = -3$

$$5X - 4Y + 4Z = -3X \quad \Rightarrow \quad 2X - Y + Z = 0 \quad — ①$$

$$12X - 11Y + 12Z = -3Y \quad \Rightarrow \quad 3X - 2Y + 3Z = 0 \quad — ②$$

$$4X - 4Y + 5Z = -3Z \quad \Rightarrow \quad X - Y + 2Z = 0 \quad — ③$$

② − ① = ③

$$\begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0$$

homogeneous coupled equation

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -2 & 3 \\ 1 & -1 & 2 \end{pmatrix}$$

$$A_R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$ (기약행 사다리꼴)

$$X = Z$$
$$Y = 3Z$$

if $Z=1$, $X=1$ and $Y=3$

$$v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & -1 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 3 & -2 & 3 \\ -1 & 1 & -2 \\ -1 & 1 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} ※$$

정의 : 직교행렬 (orthogonal matrix)

If $AA^t = A^tA = I$, A : orthogonal matrix

* If A is orthogonal matrix, $\det A = \pm 1$

* If A is orthogonal matrix, $A^{-1} = A^t$

$\left(\because \ \det(AA^t) = \det A \det A^t = (\det A)^2 = 1 \Rightarrow \det A = \pm 1 \right)$

p209

(예제 5.39)

$$A = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

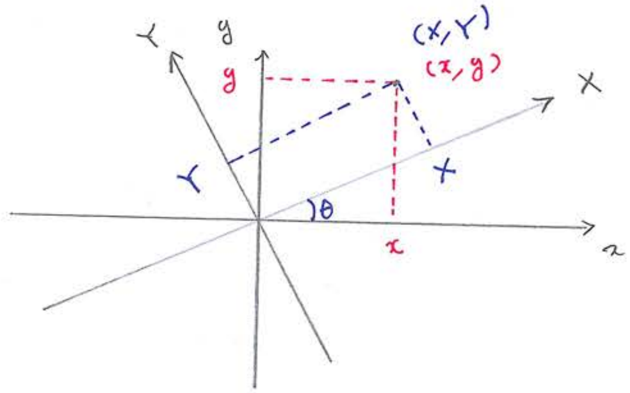
$$AA^t = A^tA = I$$

A: orthogonal matrix

(Ex)

$$Q = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad Q^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$QQ^t = Q^tQ = I_2$ $\qquad$ Q: orthogonal matrix



$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

정의 5.21 symmetric matrix

If $A = A^t$, A : symmetric matrix (대칭행렬)

(정리 5.27)

[1] If A is symmetric matrix with real components, its eigenvalues are real.

[2] If $\lambda_i \neq \lambda_j$, $v_i \perp v_j$

[3] diagonalization: $T^{-1}AT = T^tAT = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

T : orthogonal matrix

ex)

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{pmatrix}$$ symmetric matrix

eigenvalue	eigenvector
$\lambda_1 = 2$	$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
$\lambda_2 = -1$	$v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$
$\lambda_3 = 4$	$v_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

* note
$\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3 = 0$

$$P = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{pmatrix}$$

check ⇒ $PP^t = P^tP = I_3$

⇒ orthogonal matrix

$$P^{-1} = P^{T} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$P^{T} A P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$ diagonalization

※

$ax^2 + bxy + cy^2$: x, y 의 2차형식

(Ex) $x^2 + 7xy + 2y^2 = (x, y)\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = X^t A X$

A를 symmetric matrix 로 만들수 있다.

$$x^2 + 7xy + 2y = (x, y)\begin{pmatrix} 1 & \frac{7}{2} \\ \frac{7}{2} & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & \frac{7}{2} \\ \frac{7}{2} & 2 \end{pmatrix}$$

(Ex)

$$x^2 - 2xy + y^2 = (x, y) A \begin{pmatrix} x \\ y \end{pmatrix} \quad — ①$$

$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$: symmetric matrix

eigenvalue	eigenvector
0	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
2	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Let $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $Q^{-1} = Q^T = Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$Q^T A Q = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow A = Q\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} Q^T$$

Then

$$x^2 - 2xy + y^2 = (x, y)\, Q \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} Q^T \begin{pmatrix} x \\ y \end{pmatrix} \quad - ②$$

Let

$$Q^T \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x+y \\ x-y \end{pmatrix} \equiv \begin{pmatrix} X \\ Y \end{pmatrix} \quad - ③$$

$$X \equiv \frac{1}{\sqrt{2}}(x+y)$$

주축 (principal axis)

$$Y \equiv \frac{1}{\sqrt{2}}(x-y)$$

$$x^2 - 2xy + y^2 = (X, Y) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 2Y^2 \quad *$$

p224

8. $4x_1^2 - 4x_2^2 + 6x_1x_2 = 8$

$$(x_1, x_2)\, A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8 \quad —①$$

$$A = \begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \quad —②$$

eigenvalue | eigenvector

$\lambda_1 = -5$ $\quad v_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ $\quad \vec{v}_1 \cdot \vec{v}_2 = 0$

$\lambda_2 = 5$ $\quad v_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$Q = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \qquad Q^{-1} = Q^t = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

$$Q^t A Q = \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\Rightarrow \quad A = Q \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} Q^t \quad —③$$

③ → ①

$$(x_1, x_2)\, Q \begin{pmatrix} -5 & 0 \\ 0 & 5 \end{pmatrix} Q^t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 8 \quad —④$$

Let

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = Q^t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

L⑤

$$\theta \simeq -71^\circ \quad —⑥$$

principal axis

⑥ → ④

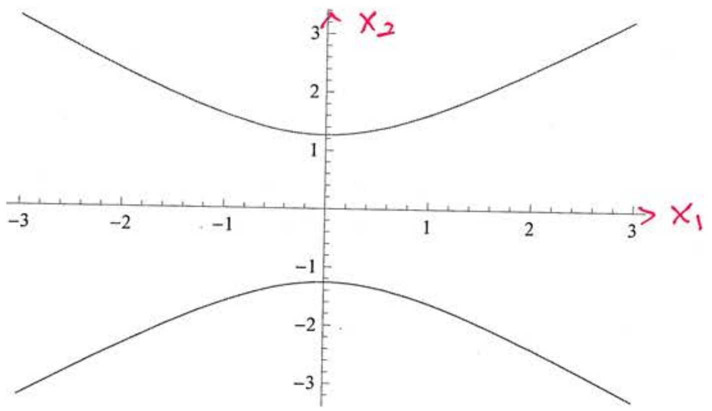
$$(x_1,\ x_2)\begin{pmatrix}-5 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = 8$$

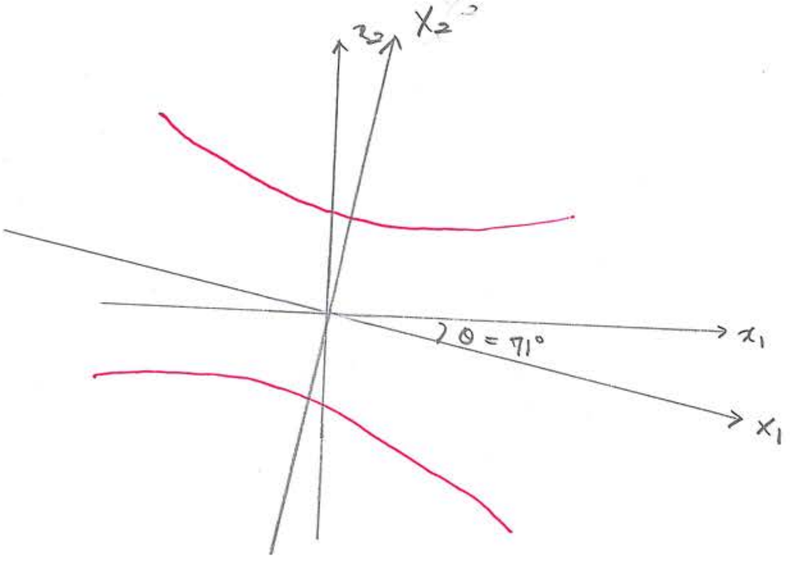
$$-5x_1^2 + 5x_2^2 = 8$$

$$x_1^2 - x_2^2 = -\frac{8}{5}$$

```
In[59]:= Plot[{-Sqrt[x^2 + 8 / 5], Sqrt[x^2 + 8 / 5]}, {x, -3, 3}]
```

Out[59]=





#

복소행렬

a matrix whose components are complex

Hermitian conjugate matrix

$A^{\dagger} = (A^{*})^{t}$: Hermitian conjugate matrix of A

(Ex)

$$A = \begin{pmatrix} 1+i & 3-2i \\ 5+3i & 1+2i \end{pmatrix}$$

$$A^{*} = \begin{pmatrix} 1-i & 3+2i \\ 5-3i & 1-2i \end{pmatrix}$$

$$A^{\dagger} = (A^{*})^{t} = \begin{pmatrix} 1-i & 5-3i \\ 3+2i & 1-2i \end{pmatrix}$$ Hermitian conjugate matrix of A

Hermitian matrix

If $A = A^{\dagger}$, A : Hermitian matrix

Ex)

$$A = \begin{pmatrix} 15 & 8i & 6-2i \\ -8i & 0 & -4+i \\ 6+2i & -4-i & -3 \end{pmatrix}$$

$$A^{\dagger} = \begin{pmatrix} 15 & 8i & 6-2i \\ -8i & 0 & -4+i \\ 6+2i & -4-i & -3 \end{pmatrix} = A$$ Hermitian matrix

p. 223 정리 5.33

Eigenvalues of Hermitian matrix are real.

unitary matrix

If $UU^{\dagger} = U^{\dagger}U = I$, I : unitary matrix

(Ex)

$$U = \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad U^{\dagger} = \begin{pmatrix} -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$UU^{\dagger} = U^{\dagger}U = I_2$$

U: unitary matrix