CH. 편미분 방정식 (partial differential Equation)

§ 파동방정식

$$\boxed{\frac{\partial^2}{\partial t^2}\psi(\vec{r},t) = c^2 \nabla^2 \psi(\vec{r},t) \qquad c : \text{velocity of wave}}$$

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§ 파동방정식의 Fourier 급수해

(1) 초기속도가 0인 진동현

1-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad 0<x<L, \ t>0 \qquad (1)$$

$$y(0,t) = y(L,t) = 0 \qquad t \ge 0 \ : \text{boundary condition} \qquad (2)$$

$$\left.\begin{aligned} y(x,0) &= f(x) && 0 \le x \le L \\ \frac{\partial y}{\partial t}(x,0) &= 0 && 0 \le x \le L \end{aligned}\right\} \text{initial condition}$$

<Fourier method>

Put

$$y(x,t) = X(x)\,T(t) \qquad (3)$$

(3) → (1) we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \qquad (4)$$

$\frac{X''}{X}$: function of only x

$\frac{T''}{T}$: function of only t

Therefore Eq. (4) gives

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda \qquad (5)$$

where λ is some constant. From (5) we get

$$X'' + \lambda X = 0 \qquad T'' + \lambda c^2 T = 0 \qquad (6)$$

Now we consider the boundary conditions:

$$y(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$y(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0$$

$$\Rightarrow X(0) = X(L) = 0 \qquad (7)$$

note) If we choose $T(t) = 0$, we get the trivial solution $y(x,t) = 0$.

First, we consider $X(x)$ which satisfies

$$X'' + \lambda X = 0 \qquad (8)$$

$$X(0) = X(L) = 0$$

(i) $\lambda = 0$ case

$X(x) = cx + d$

$X(0) = d = 0$

$X(L) = cL = 0 \Rightarrow c = 0$

$\Rightarrow X(x) = 0 \Rightarrow$ trivial solution

(ii) $\lambda < 0$ case

In this case we get

$$X(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$$

$$X(0) = c_1 + c_2 = 0$$

$$X(L) = c_1 e^{\sqrt{-\lambda}\,L} + c_2 e^{-\sqrt{-\lambda}\,L} = 0$$

$\Rightarrow$ $c_1 = c_2 = 0$ $\Rightarrow$ trivial solution

(iii) $\lambda > 0$ case

$$X(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$

$$X(0) = c_1 = 0$$

$$X(L) = c_2 \sin\sqrt{\lambda}\,L = 0$$

$$\Rightarrow \sqrt{\lambda}\,L = n\pi$$

$$\Rightarrow \underline{\lambda_n = \frac{n^2\pi^2}{L^2} \quad (n = 1, 2, 3, \cdots)} \qquad (9)$$

Therefore

$$\underline{X(x) = \sin\frac{n\pi x}{L}} \qquad (10)$$

Now, let us consider $T(t)$, which satisfies

$$T'' + \lambda c^2 T = 0$$

$$\Rightarrow T'' + \frac{n^2\pi^2c^2}{L^2} T = 0 \qquad (11)$$

From initial condition

$$\frac{\partial y}{\partial t}(x,0) = 0, \Rightarrow X(x)\,T'(0) = 0 \Rightarrow T'(0) = 0$$

$$\Rightarrow \qquad T'(0) = 0 \qquad (12)$$

The general solution of Eq. (11) is

$$T(t) = a\cos\frac{n\pi ct}{L} + b\sin\frac{n\pi ct}{L} \qquad (13)$$

From (12) and (13) we get $b = 0$. Thus $T(t)$ becomes

$$T(t) = C_n \cos\frac{n\pi ct}{L} \qquad (14)$$

From (10) and (14) $y(x,t)$ becomes

$$y(x,t) = C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \qquad (n = 1, 2, \cdots) \qquad (15)$$

Since wave equation is linear differential equation, the general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi cT}{L}} \qquad (16)$$

note) The coefficient C_n is determined by $y(x,0) = f(x)$.

(Ex1)

$$y(x,0) = 14 \sin \frac{3\pi x}{L}$$

From (16)

$$y(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = 14 \sin \frac{3\pi x}{L}$$

Therefore

$$\left.\begin{aligned} &C_1 = C_2 = C_4 = C_5 = \cdots = 0 \\ &C_3 = 14 \end{aligned}\right\} \quad -①$$

① → (16)

$$y(x,t) = 14 \sin \frac{3\pi x}{L} \cos \frac{3\pi c t}{L}$$

⇒ example1.nb

(Ex2)

$$y(x,0) = f(x) = \begin{cases} x & 0 \le x \le \frac{L}{2} \\ L-x & \frac{L}{2} \le x \le L \end{cases}$$

Then

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x)$$

This is Fourier sine series. Therefore

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\left[\int_0^{\frac{L}{2}} x \sin\frac{n\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} (L-x) \sin\frac{n\pi x}{L}\, dx\right]$$

$$= \frac{4L}{n^2\pi^2} \sin\frac{n\pi}{2} \quad \text{— ①}$$

$$* \int x \sin ax\, dx = -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax$$

Thus when $n = \text{even}$, $C_n = 0$

$n=1$: $C_1 = \frac{4L}{\pi^2}$

$n=3$: $C_3 = -\frac{4L}{9\pi^2}$

$n=5$: $C_5 = \frac{4L}{25\pi^2}$

$\vdots$

Therefore Eq. (16) becomes

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} \cos\frac{n\pi c t}{L}$$

$$= \frac{4L}{\pi^2}\sin\frac{\pi x}{L}\cos\frac{\pi c t}{L} - \frac{4L}{9\pi^2}\sin\frac{3\pi x}{L}\cos\frac{3\pi c t}{L}$$

$$+ \frac{4L}{25\pi^2}\sin\frac{5\pi x}{L}\cos\frac{5\pi c t}{L} - \frac{4L}{49\pi^2}\sin\frac{7\pi x}{L}\cos\frac{7\pi c t}{L} + \cdots$$

$$= \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi c t}{L} - \frac{1}{9}\sin\frac{3\pi x}{L}\cos\frac{3\pi c t}{L} + \frac{1}{25}\sin\frac{5\pi x}{L}\cos\frac{5\pi c t}{L} - \cdots\right]$$

$$= \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\frac{(2k-1)\pi x}{L}\cos\frac{(2k-1)\pi c t}{L}$$

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= Plot[y[x, 0.7], {x, 0, L}, PlotRange → All, Frame → True]
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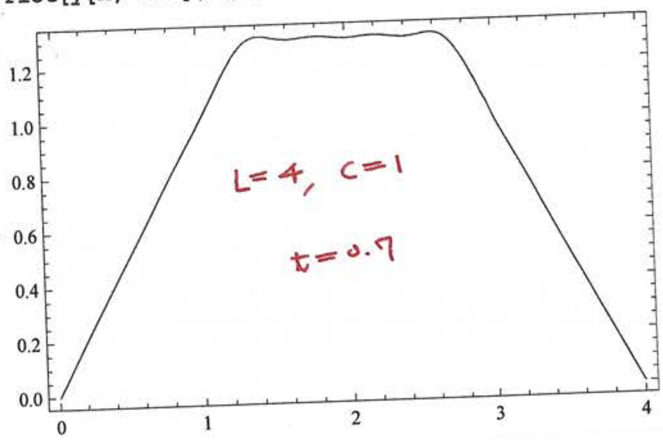
]=

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Plot[y[x, 1.3], {x, 0, L}, PlotRange → All, Frame → True]
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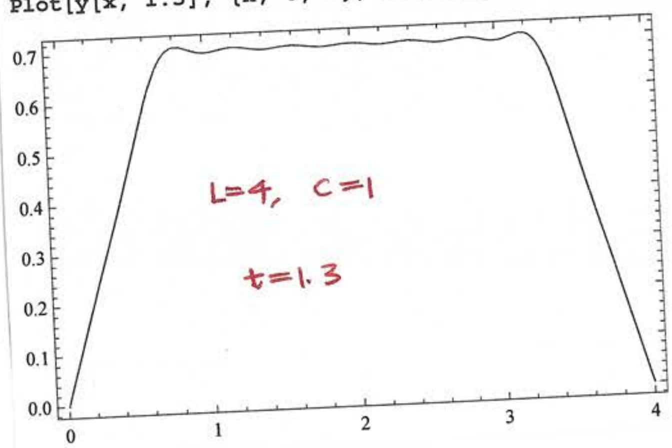

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Plot[y[x, 2.5], {x, 0, L}, PlotRange → All, Frame → True]
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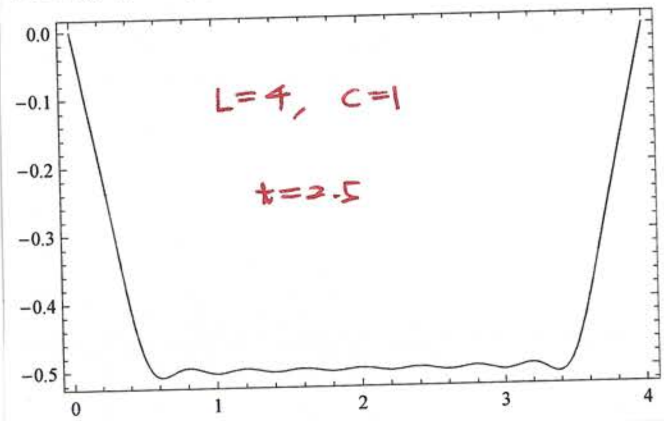


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Plot[y[x, 3.0], {x, 0, L}, PlotRange → All, Frame → True]
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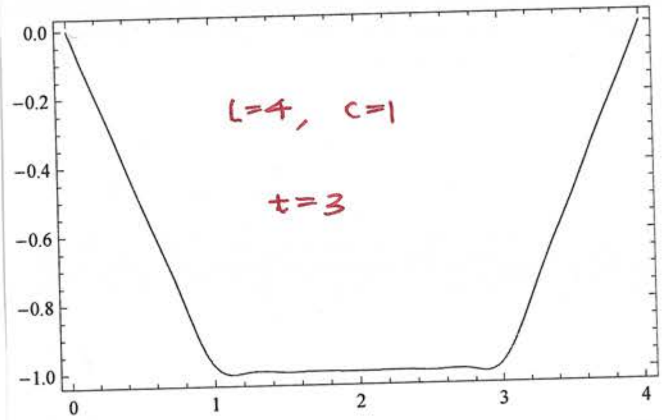


⇒ example2.nb

[2] 초기속도가 주어지고 초기변위가 0인 진동현

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad 0 < x < L, \ t > 0$$

$$y(0,t) = y(L,t) = 0 \qquad t \geq 0 \qquad \text{: Boundary condition}$$

$$\left.\begin{aligned} & y(x,0) = 0 && 0 \leq x \leq L \\ & \frac{\partial y}{\partial t}(x,0) = g(x) && 0 \leq x \leq L \end{aligned}\right\} \text{ initial condition}$$

$$y(x,t) \equiv X(x)\,T(t)$$

$$\Rightarrow \quad X'' + \lambda X = 0 \qquad X(0) = X(L) = 0$$

$$\Rightarrow \quad \left.\begin{aligned} & \lambda_n = \frac{n^2\pi^2}{L^2} \\ & X(x) = \sin\frac{n\pi x}{L} \end{aligned}\right\} \qquad (17)$$

Now we consider $T(t)$ which satisfies

$$T'' + \frac{n^2\pi^2 c^2}{L^2} T = 0 \qquad (18)$$

From $y(x,0) = 0$

$$\Rightarrow \quad X(x)\,T(0) = 0$$

$$\Rightarrow \quad T(0) = 0 \qquad (19)$$

Thus we have to solve Eq. (18) under Eq. (19).

The general solution of Eq. (18) is

$$T(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L} \qquad (20)$$

Then Eq. (19) gives $A = 0$. Thus we get

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\sin\frac{n\pi ct}{L}} \qquad (21)$$

note) The coefficient C_n is determined by $\frac{dy}{dt}(x,0) = g(x)$

From (21)

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} C_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

$$\Rightarrow \frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} C_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x) \qquad (22)$$

This is Fourier sine series with coefficient $C_n \frac{n\pi c}{L}$. Therefore

$$C_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow \quad C_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad (23)$$

(Ex) $g(x) = x\left(1 + \cos\frac{\pi x}{L}\right)$

Then

$$C_n = \frac{2}{n\pi c}\int_0^L x\left(1+\cos\frac{\pi x}{L}\right)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\pi c}\begin{cases} \dfrac{3L^2}{4\pi} & n=1 \\[2ex] \dfrac{(-1)^n L^2}{n^2(n^2-1)\pi} & n\neq 1 \end{cases}$$

$$C_1 = \frac{3L^2}{2\pi^2 c}$$

$$C_n = \frac{(-1)^n 2L^2}{\pi^2 c\, n^2(n^2-1)} \qquad (n=2,3,\cdots)$$

Therefore

$$y(x,t) = \frac{3L^2}{2\pi^2 c}\sin\frac{\pi x}{L}\sin\frac{\pi c t}{L} + \sum_{n=2}^{\infty}\frac{(-1)^n 2L^2}{\pi^2 c\, n^2(n^2-1)}\sin\frac{n\pi x}{L}\sin\frac{n\pi c t}{L}$$

$\Rightarrow$ example3.nb

[3] 초기변위와 초기속도를 갖는 진동현

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad 0<x<L, \quad t>0 \qquad (24)$$

$$y(0,t) = y(L,t) = 0 \qquad t \geq 0$$: Boundary condition

$$y(x,0) = f(x) \qquad 0 \leq x \leq L$$

$$\frac{dy}{dt}(x,0) = g(x) \qquad 0 \leq x \leq L$$

} initial condition

Then

$$y(x,t) = y_1(x,t) + y_2(x,t) \qquad (25)$$

where $y_1(x,t)$ and $y_2(x,t)$ satisfy

$$y_1(0,t) = y_1(L,t) = 0$$

$$y_1(x,0) = f(x), \quad \frac{\partial}{\partial t} y_1(x,0) = 0 \qquad (26)$$

$$y_2(0,t) = y_2(L,t) = 0$$

$$y_2(x,0) = 0, \quad \frac{\partial}{\partial t} y_2(x,0) = g(x)$$

pf)

$$y(x,0) = y_1(x,0) + y_2(x,0) = f(x)$$

$$\frac{\partial}{\partial t} y(x,0) = \frac{\partial}{\partial t} y_1(x,0) + \frac{\partial}{\partial t} y_2(x,0) = g(x) \quad *$$

Since $y_1(x,t)$ and $y_2(x,t)$ can be obtained by previous methods,

one can solve Eq. (24).

Ex)

$$f(x) = \begin{cases} x & 0 \le x \le \frac{L}{2} \\ L-x & \frac{L}{2} < x \le L \end{cases}$$

$$g(x) = x\left(1 + \cos\frac{\pi x}{L}\right)$$

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$y_1(x,t) = \frac{4L}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin\frac{(2k-1)\pi x}{L} \cos\frac{(2k-1)\pi c t}{L}$$

$$y_2(x,t) = \frac{3L^2}{2\pi^2 c} \sin\frac{\pi x}{L} \sin\frac{\pi c t}{L}$$

$$+ \sum_{n=2}^{\infty} \frac{(-1)^n 2L^2}{\pi^2 c\, n^2 (n^2-1)} \sin\frac{n\pi x}{L} \sin\frac{n\pi c t}{L}$$

⇒ example4.nb

[4] 파동방정식을 포함하는 경계값 문제의 변환

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + Ax \qquad 0<x<L,\ t>0$$

$$y(0,t) = y(L,t) = 0 \qquad t \geq 0 \qquad (27)$$

$$y(x,0) = 0, \quad \frac{\partial}{\partial t} y(x,0) = 1 \qquad 0<x<L$$

note) The term Ax can be interpreted as external force.

If we put $y(x,t) = X(x)T(t)$ again, Eq. (27) gives

$$XT'' = X''T + Ax$$

$\Rightarrow$ Thus separation of variable is impossible in this way.

Put

$$y(x,t) = Y(x,t) + \psi(x) \qquad (28)$$

(28) $\rightarrow$ (27)

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2} + \psi''(x) + Ax \qquad (29)$$

Thus we choose $\psi(x)$ as

$$\psi''(x) + Ax = 0$$

which gives

$$\psi(x) = -\frac{A}{6}x^3 + Cx + D \qquad (30)$$

Now we consider the boundary conditions

$$y(0,t) = Y(0,t) + \psi(0) = Y(0,t) + D = 0$$

If we choose $D=0$, we get $Y(0,t)=0$.

$$y(L,t) = Y(L,t) + \left(cL - \frac{A}{6}L^3\right) = 0$$

If we choose $c = \frac{A}{6}L^2$, we get $Y(L,t)=0$

Thus, all of these leads

$$\psi(x) = -\frac{A}{6}x^3 + \frac{A}{6}L^2 x \qquad (31)$$

and

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2} \qquad (32)$$

$$Y(0,t) = Y(L,t) = 0$$

Now let us derive the initial conditions.

$$y(x,0) = Y(x,0) + \psi(x)$$

$$\Rightarrow \quad Y(x,0) = y(x,0) - \psi(x) = \frac{A}{6}x(x^2 - L^2) \qquad (33)$$

$$\frac{\partial}{\partial t}y(x,0) = \frac{\partial}{\partial t}Y(x,0) = 1$$

$$\Rightarrow \quad \frac{\partial}{\partial t}Y(x,0) = 1 \qquad (34)$$

Thus $Y(x,t)$ satisfies

$$\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2}$$

$$Y(0,t) = Y(L,t) = 0 \qquad (35)$$

$$Y(x,0) = \frac{A}{6} x (x^2 - L^2)$$

$$\frac{\partial}{\partial t} Y(x,0) = 1$$

Follows previous method, one can solve Eq. (35)

$$Y(x,t) = \frac{2AL^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin\frac{n\pi x}{L} \cos\frac{n\pi t}{L} \qquad (36)$$

$$+ \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \sin\frac{n\pi x}{L} \sin\frac{n\pi t}{L}$$

Thus

$$y(x,t) = Y(x,t) + \frac{A}{6} x (L^2 - x^2)$$

§ 무한 또는 반 무한 현 파동의 움직임

[1] 무한현 파동의 움직임

① 초기속도가 0 인 경우

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad -\infty < x < \infty, \quad t > 0 \tag{37}$$

$$y(x,0) = f(x), \quad \frac{\partial}{\partial t} y(x,0) = 0 \qquad -\infty < x < \infty$$

Put $y(x,t) = X(x)\,T(t)$. Then we get

$$X'' + \lambda X = 0 \qquad T'' + \lambda c^2 T = 0 \tag{38}$$

Now we define

$$\lambda = \omega^2 \tag{39}$$

Then $X'' + \omega^2 X = 0$, which gives

$$X(x) = a_\omega \cos \omega x + b_\omega \sin \omega x \tag{40}$$

$$T(t) = A_\omega \cos \omega c t + B_\omega \sin \omega c t$$

Therefore

$$y(x,t) = [a_\omega \cos \omega x + b_\omega \sin \omega x][A_\omega \cos \omega c t + B_\omega \sin \omega c t] \tag{41}$$

Then

$$\frac{\partial}{\partial t} y(x,t) = \omega c [a_\omega \cos \omega x + b_\omega \sin \omega x][-A_\omega \sin \omega c t + B_\omega \cos \omega c t]$$

$\Rightarrow \frac{\partial}{\partial t} y(x,0) = 0$ gives

$$B_\omega = 0 \tag{42}$$

(42) → (41)

$$y(x,t) = \cos\omega c t\left[a_\omega \cos\omega x + b_\omega \sin\omega x\right] \qquad (43)$$

Since wave equation is linear differential equation, the general solution is

$$y(x,t) = \int_0^\infty d\omega \, \cos\omega c t\left[a_\omega \cos\omega x + b_\omega \sin\omega x\right] \qquad (44)$$

The coefficients a_ω and b_ω can be obtained by $y(x,0) = f(x)$.

$$y(x,0) = \int d\omega \left[a_\omega \cos\omega x + b_\omega \sin\omega x\right] = f(x)$$

This is Fourier integral. Thus

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos\omega x \, dx \qquad (45)$$

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin\omega x \, dx$$

Summary : solution of Eq. (37)

$$y(x,t) = \int_0^\infty d\omega \, \cos\omega c t\left[a_\omega \cos\omega x + b_\omega \sin\omega x\right]$$

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos\omega x \, dx$$

$$b_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin\omega x \, dx$$

Ex) $f(x) = e^{-|x|}$

Then $a_\omega = \dfrac{2}{\pi(1+\omega^2)}$, $b_\omega = 0$.

Therefore

$$y(x,t) = \frac{2}{\pi}\int_0^\infty d\omega \ \frac{1}{1+\omega^2} \cos\omega x \ \cos\omega c t$$

$$= \frac{1}{2}\left[\cosh(x-ct) + \cosh(x+ct) - \frac{|x-ct| \sinh(x-ct)}{x-ct} - \frac{|x+ct| \sinh(x+ct)}{x+ct} \right]$$

$*$ $\displaystyle\int_0^\infty \frac{\cos a x}{\beta^2 + x^2}\, dx = \frac{\pi}{2\beta} e^{-a\beta}$

$\Rightarrow$ example 5. a, b

② 초기변위가 0 인경우

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad -\infty < x < \infty,\ t > 0 \qquad (46)$$

$$y(x,0) = 0, \quad \frac{\partial}{\partial t} y(x,0) = g(x) \qquad -\infty < x < \infty$$

By same way we get

$$y(x,t) = [a_\omega \cos\omega x + b_\omega \sin\omega x][A_\omega \cos\omega ct + B_\omega \sin\omega ct]$$

Then

$$y(x,0) = A_\omega [a_\omega \cos\omega x + b_\omega \sin\omega x] = 0$$

$$\Rightarrow \quad A_\omega = 0$$

$$\Rightarrow \quad y(x,t) = \sin\omega ct [a_\omega \cos\omega x + b_\omega \sin\omega x]$$

Since wave equation is linear, the general solution is

$$y(x,t) = \int_0^\infty d\omega \sin\omega ct [a_\omega \cos\omega x + b_\omega \sin\omega x] \qquad (47)$$

To determine a_ω and b_ω, consider

$$\frac{\partial}{\partial t} y(x,t) = \omega c \int_0^\infty d\omega \cos\omega ct [a_\omega \cos\omega x + b_\omega \sin\omega x]$$

$$\Rightarrow \frac{\partial}{\partial t} y(x,0) = \omega c \int_0^\infty d\omega [a_\omega \cos\omega x + b_\omega \sin\omega x] = g(x)$$

$$\Rightarrow \quad a_\omega = \frac{1}{\omega c \pi} \int_{-\infty}^{\infty} g(x) \cos\omega x \, dx \qquad (48)$$

$$b_\omega = \frac{1}{\omega c \pi} \int_{-\infty}^{\infty} g(x) \sin\omega x \, dx$$

Summary: solution of (46)

$$\boxed{\begin{aligned} y(x,t) &\doteq \int_0^\infty d\omega \ \sin\omega ct \left[a_\omega \cos\omega x + b_\omega \sin\omega x \right] \\ a_\omega &= \frac{1}{\omega c\pi} \int_{-\infty}^{\infty} g(x) \cos\omega x \, dx \\ b_\omega &= \frac{1}{\omega c\pi} \int_{-\infty}^{\infty} g(x) \sin\omega x \, dx \end{aligned}}$$

Ex)

$$g(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

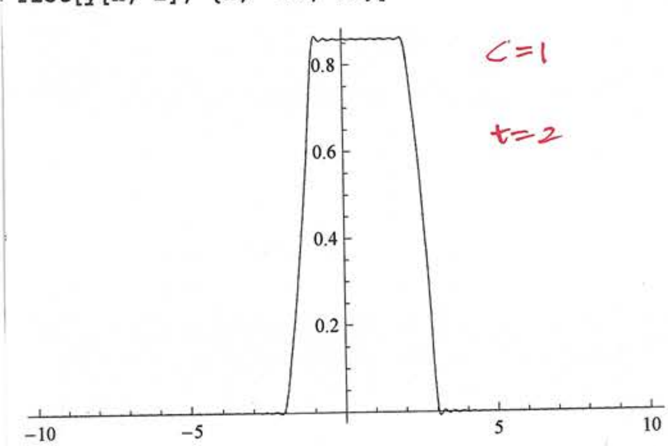
$$a_\omega = \frac{1}{\pi c\omega}\int_{-\infty}^{\infty} g(x)\cos\omega x\,dx = \frac{1}{\pi c\omega}\int_0^1 e^x \cos\omega x\,dx = \frac{e\cos\omega + e\omega\sin\omega - 1}{\pi c\omega(1+\omega^2)}$$

$$b_\omega = \frac{1}{\pi c\omega}\int_{-\infty}^{\infty} g(x)\sin\omega x\,dx = \frac{1}{\pi c\omega}\int_0^1 e^x \sin\omega x\,dx = \frac{\omega - e\omega\cos\omega + e\sin\omega}{\pi c\omega(1+\omega^2)}$$

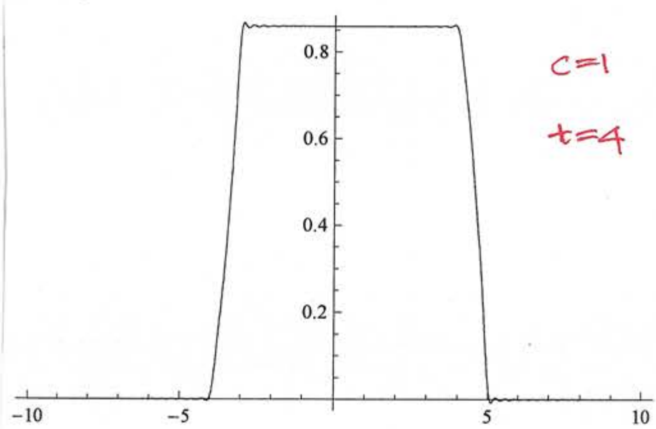
Therefore

$$y(x,t) = \int_0^{\infty} d\omega\ \sin\omega ct\left[\frac{e\cos\omega + e\omega\sin\omega - 1}{\pi c\omega(1+\omega^2)}\cos\omega x + \frac{\omega - e\omega\cos\omega + e\sin\omega}{\pi c\omega(1+\omega^2)}\sin\omega x\right]$$

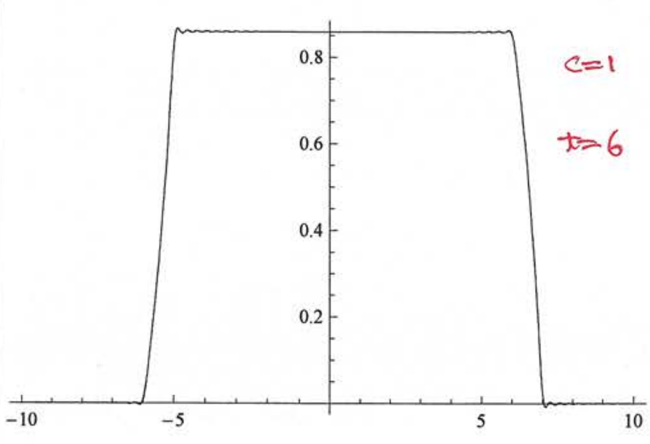
Plot[y[x, 2], {x, -10, 10}]



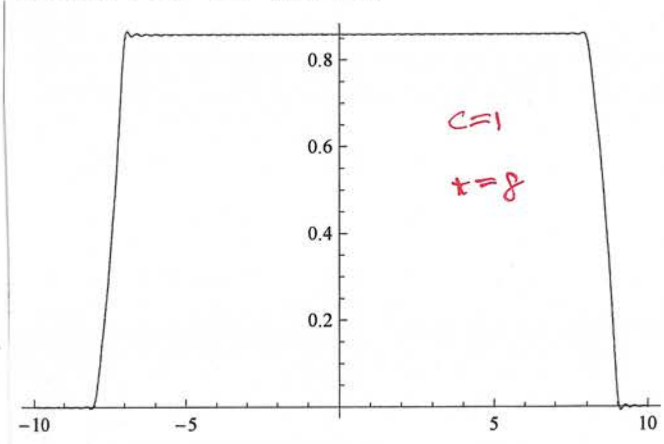
Plot[y[x, 4], {x, -10, 10}]



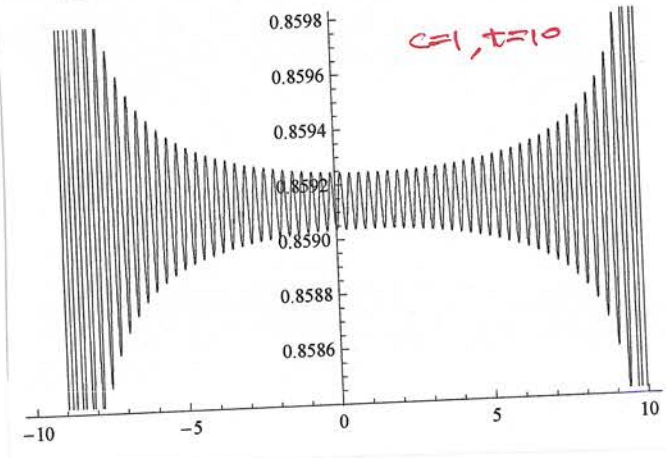
Plot[y[x, 6], {x, -10, 10}]



Plot[y[x, 8], {x, -10, 10}]



```
Plot[y[x, 10], {x, -10, 10}]
```



[2] 반 무한현 파동의 움직임

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad 0 < x < \infty, \quad t > 0 \tag{49}$$

$$y(0,t) = 0 \qquad t \geq 0 \qquad : \text{boundary conditions}$$

$$y(x,0) = f(x), \quad \frac{\partial}{\partial t} y(x,0) = g(x) \qquad 0 < x < \infty \; : \text{initial conditions}$$

Putting $y(x,t) = X(x)\,T(t)$, we get

$$X'' + \lambda X = 0 \qquad T'' + \lambda c^2 T = 0 \tag{50}$$

From $y(0,t) = X(0)\,T(t) = 0$, we also get

$$X(0) = 0 \tag{51}$$

Putting $\lambda = \omega^2$, we have

$$X(x) = a \cos\omega x + b \sin\omega x$$

Then Eq. (51) gives $a = 0$. Thus $X(x)$ becomes

$$X(x) = \sin\omega x \tag{52}$$

Eq. (50) also gives

$$T(t) = A \cos\omega c t + B \sin\omega c t \tag{53}$$

Therefore

$$y(x,t) = \sin\omega x \left[A \cos\omega c t + B \sin\omega c t \right] \tag{54}$$

(i) $g(x)=0$ case

In this case we get $B=0$, which makes Eq. (54) to be

$$y(x,t) = A \sin\omega x \cos\omega c t$$

Since wave equation is linear, the general solution is

$$y(x,t) = \int_0^\infty d\omega\, C_\omega \sin\omega x \cos\omega c t \qquad (55)$$

Then

$$y(x,0) = \int_0^\infty d\omega\, C_\omega \sin\omega x = f(x)$$: Fourier sine integral

$$\Rightarrow \quad C_\omega = \frac{2}{\pi}\int_0^\infty f(x) \sin\omega x\, dx \qquad (56)$$

Summary : solution of Eq. (49) with $g(x)=0$

$$y(x,t) = \int_0^\infty d\omega\, C_\omega \sin\omega x \cos\omega c t \equiv y_1(x,t) \qquad (57)$$

$$C_\omega = \frac{2}{\pi}\int_0^\infty f(x) \sin\omega x\, dx$$

(ii) $f(x)=0$

By similar way solution becomes

Summary : solution of Eq. (49) with $f(x)=0$

$$y(x,t) = \int_0^\infty d\omega\, d_\omega \sin\omega x \sin\omega c t \equiv y_2(x,t) \equiv y_2(x,t) \qquad (58)$$

$$d_\omega = \frac{2}{\pi c\omega}\int_0^\infty g(x) \sin\omega x\, dx$$

Then solution of Eq. (49) is

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{59}$$

Ex) $g(x) = 0$, $f(x) = \begin{cases} \sin \pi x & 0 \le x \le 4 \\ 0 & x > 4 \end{cases}$

$$C_\omega = \frac{2}{\pi}\int_0^\infty f(x)\sin\omega x\, dx$$

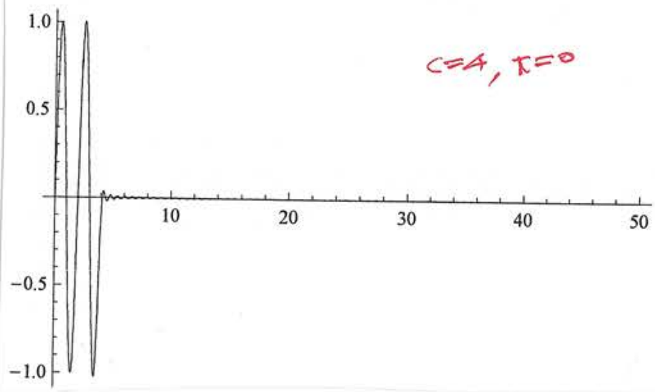
$$= \frac{2}{\pi}\int_0^4 \sin\pi x \sin\omega x\, dx$$

$$= \frac{2\sin 4\omega}{\omega^2 - \pi^2}$$

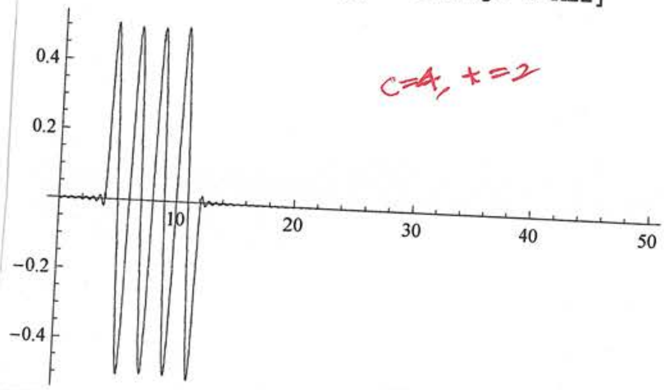
Therefore

$$y(x,t) = \int_0^\infty d\omega \frac{2\sin 4\omega}{\omega^2 - \pi^2} \sin\omega x \cos\omega c t$$

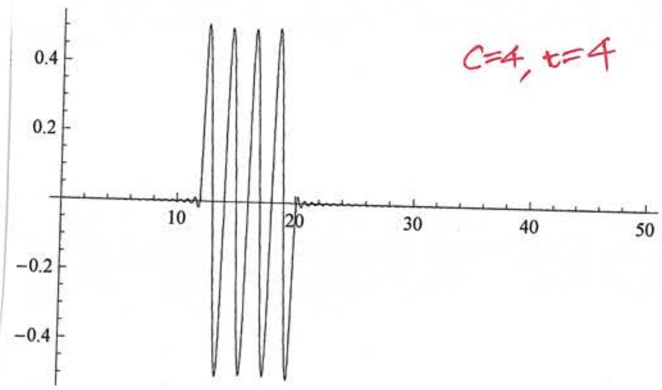
```
Plot[y[x, 0], {x, 0, 50}, PlotRange → All]
```



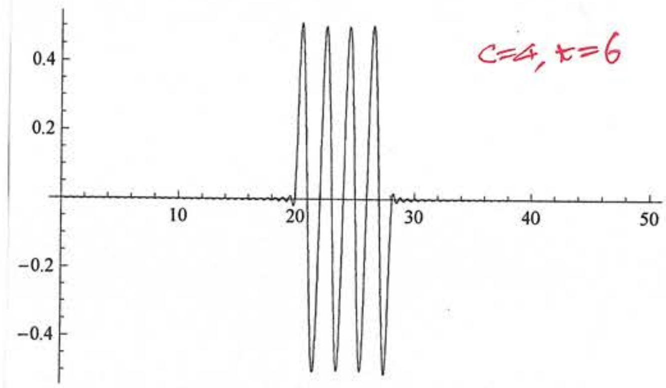

```
Plot[y[x, 2], {x, 0, 50}, PlotRange → All]
```



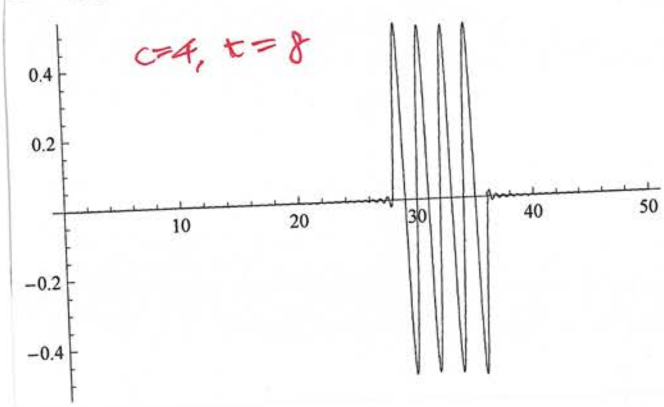

```
Plot[y[x, 4], {x, 0, 50}, PlotRange → All]
```



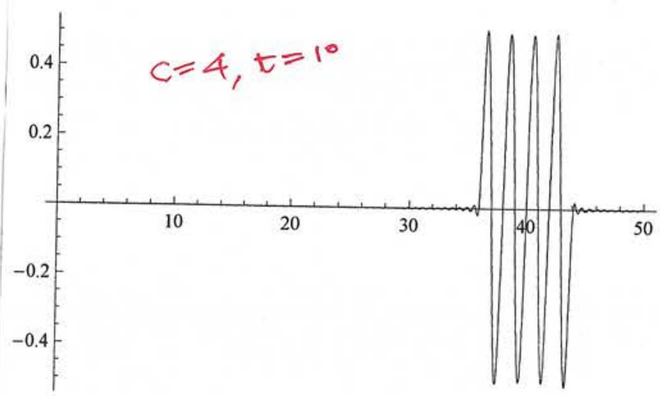

```
Plot[y[x, 6], {x, 0, 50}, PlotRange → All]
```




```
Plot[y[x, 8], {x, 0, 50}, PlotRange → All]
```




```
Plot[y[x, 10], {x, 0, 50}, PlotRange → All]
```



example 6.nb

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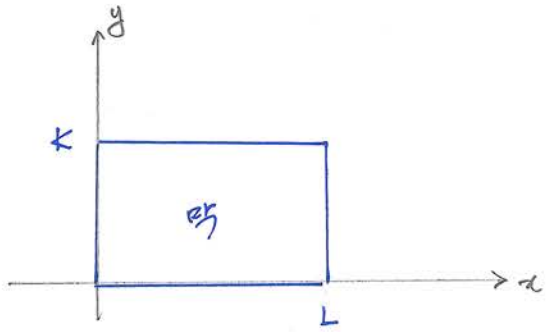
§ 직사각형 막의 진동



2-dim wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right] \qquad 0<x<L, \quad 0<y<K, \quad t>0$$

$$\left.\begin{array}{l} z(x,0,t) = z(x,K,t) = 0 \\ z(0,y,t) = z(L,y,t) = 0 \end{array}\right\} \text{ boundary condition} \qquad (60)$$

$$\left.\begin{array}{l} z(x,y,0) = f(x,y) \\ \frac{\partial}{\partial t} z(x,y,0) = g(x,y) \end{array}\right\} \text{ initial condition}$$

First we assume

$$g(x,y) = 0 \qquad (61)$$

Put

$$z(x,y,t) = X(x)\,Y(y)\,T(t) \qquad (62)$$

Then we get

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} - \frac{Y''}{Y} \qquad (63)$$

lhs of Eq. (63) : function of x

rhs of Eq. (63) : function of y and t

Thus

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} - \frac{Y''}{Y} = -\lambda$$

$\Rightarrow$

$$X'' + \lambda X = 0 \qquad (64.1)$$

$$\frac{1}{c^2}\frac{T''}{T} + \lambda = \frac{Y''}{Y} \qquad (64.2)$$

(64)

lhs of Eq. (64.2) : function of t

rhs of Eq. (64.2) : function of y

Thus

$$\frac{1}{c^2}\frac{T''}{T} + \lambda = \frac{Y''}{Y} = -\mu$$

$\Rightarrow$ $Y'' + \mu Y = 0$

$T'' + c^2(\lambda+\mu)T = 0$

$\Rightarrow$

$$X'' + \lambda X = 0 \qquad (65.1)$$

$$Y'' + \mu Y = 0 \qquad (65.2)$$

$$T'' + c^2(\lambda+\mu)T = 0 \qquad (65.3)$$

(65)

Now, let us consider boundary conditions.

$$Z(0,y,t) = X(0)Y(y)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$Z(L,y,t) = X(L)Y(y)T(t) = 0 \quad \Rightarrow \quad X(L) = 0$$

$$Z(x,0,t) = X(x)Y(0)T(t) = 0 \quad \Rightarrow \quad Y(0) = 0$$

$$Z(x,K,t) = X(x)Y(K)T(t) = 0 \quad \Rightarrow \quad Y(K) = 0$$

Therefore we get

$$X(0) = 0, \quad X(L) = 0, \quad Y(0) = 0, \quad Y(K) = 0 \tag{66}$$

Solving Eq. (65.1) and Eq. (65.2) with Eq. (66) gives

$$X(x) = \sin\frac{n\pi x}{L} \qquad (m, n : \text{integer})$$

$$Y(y) = \sin\frac{m\pi y}{K} \tag{67}$$

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad \mu = \frac{m^2\pi^2}{K^2}$$

Now, we consider Eq. (65.3), which reduces to

$$T'' + c^2\left(\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}\right)T = 0 \tag{68}$$

Eq. (68) gives

$$T(t) = A\cos\left(c\sqrt{\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}}\,t\right) + B\sin\left(c\sqrt{\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{K^2}}\,t\right) \tag{69}$$

Since

$$\frac{\partial}{\partial t} z(x,y,0) = 0, \quad \text{we get} \quad B = 0.$$

Finally, we arrive at

$$Z(x,y,t) = A \cos\left(\sqrt{\frac{n^2}{L^2}+\frac{m^2}{K^2}}\,\pi c t\right) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \qquad (70)$$

Since wave equation is linear, the general solution is

$$Z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \cos\left(\sqrt{\frac{n^2}{L^2}+\frac{m^2}{K^2}}\,\pi c t\right) \qquad (71)$$

Now, we have to compute a_{nm}. Using $Z(x,y,0) = f(x,y)$

we get

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \qquad (72)$$

Using orthogonality conditions

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{n'\pi x}{L}\,dx = \frac{L}{2}\delta_{nn'} \qquad (73)$$

$$\int_0^K \sin\frac{m\pi y}{K} \sin\frac{m'\pi y}{K}\,dy = \frac{K}{2}\delta_{mm'}$$

we get

$$a_{nm} = \frac{4}{LK}\int_0^L dx \int_0^K dy\, f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K}$$

Summary: solution of Eq. (60) with $g(x,y)=0$

$$\boxed{\begin{aligned} Z(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \cos\left(\sqrt{\frac{n^2}{L^2}+\frac{m^2}{K^2}}\,\pi c t\right) \\ a_{nm} &= \frac{4}{LK}\int_0^L dx \int_0^K dy\, f(x,y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \end{aligned}}$$

Ex)

$$f(x,y) = x(L-x)\,y(K-y)$$

Then

$$a_{nm} = \frac{4}{LK}\int_0^L dx \int_0^K dy\; x(L-x)\,y(K-y) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K}$$

$$= \frac{4}{LK}\left(\int_0^L dx\; x(L-x) \sin\frac{n\pi x}{L}\right)\left(\int_0^K dy\; y(K-y) \sin\frac{m\pi y}{K}\right)$$

$$= \frac{16L^2K^2}{(nm\pi^2)^3}\left[(-1)^n-1\right]\left[(-1)^m-1\right].$$

Therefore

$$Z(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{16L^2K^2}{(nm\pi^2)^3}\left[(-1)^n-1\right]\left[(-1)^m-1\right]$$

$$\times \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{K} \cos\left(\sqrt{\frac{n^2}{L^2}+\frac{m^2}{K^2}}\,\pi c t\right)$$

⇒ example 9. nb