CH 6. 선형 연립 미분 방정식

※ 1계 선형 연립 미분 방정식 (First-order linear coupled differential Equation)

General form

$$x_1'(t) = a_{11}(t)\,x_1(t) + a_{12}(t)\,x_2(t) + \cdots + a_{1n}(t)\,x_n(t) + g_1(t)$$

$$x_2'(t) = a_{21}(t)\,x_1(t) + a_{22}(t)\,x_2(t) + \cdots + a_{2n}(t)\,x_n(t) + g_2(t)$$

$$\vdots$$

$$x_n'(t) = a_{n1}(t)\,x_1(t) + a_{n2}(t)\,x_2(t) + \cdots + a_{nn}(t)\,x_n(t) + g_n(t)$$

Let

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & & a_{nn}(t) \end{pmatrix}$$

$$G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

⇒ General Form

$$\underline{X'(t) = A(t)\,X(t) + G(t)}$$

If $G(t) = 0$, homogeneous differential Equation

If $G(t) \neq 0$, non-homogeneous differential Equation

p227

(예제6.1)

$$x_1' = 3x_1 + 3x_2 + 8$$

$$x_2' = x_1 + 5x_2 + 4e^{3t}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

Solution

$$x_1(t) = 3e^{2t} + e^{6t} - 4e^{3t} - \frac{10}{3}$$

⇒ 확인시켜볼것 !!

$$x_2(t) = -e^{2t} + e^{6t} + \frac{2}{3}$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{2t} + e^{6t} - 4e^{3t} - \frac{10}{3} \\ -e^{2t} + e^{6t} + \frac{2}{3} \end{pmatrix}$$

solution vector

//

p228

정리 6.1

If $a_{ij}(t)$ and $g_j(t)$ are continuous, the differential equation

$$X' = AX + G \qquad X(t_0) = X_0$$

has unique solution.

정리 6.2 : homogeneous equation

If $X_1, X_2, \cdots, X_K$ are solution of the homogeneous equation

$$X' = AX,$$

$c_1X_1 + c_2X_2 + \cdots + c_KX_K$ is also solution.

p228

(예제 6.3)

$$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X$$

$$X_1 = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix} \qquad X_2 = \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$$ ⇒ 확인 시켜 줄 것!!

solution vector

Then

$$c_1X_1 + c_2X_2 = \begin{pmatrix} -[2c_2t + (2c_1 - c_2)]e^{3t} \\ (c_2t + c_1)e^{3t} \end{pmatrix}$$ ⇒ 확인 시켜줄 것!!

is also solution vector.

p230

정리 6.4 : General solution of homogeneous equation

Consider a homogeneous equation

$$X' = A X,$$

were A is $n \times n$ matrix.

Let $X_1(t), X_2(t), \cdots, X_n(t)$ are solution vectors of the homogeneous equation and linearly-independent.

Then the general solution of the homogeneous equation is

$$X = C_1 X_1(t) + C_2 X_2(t) + \cdots + C_n X_n(t)$$

* linearly-independent condition :

Let $X_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \\ \vdots \\ x_{1n}(t) \end{pmatrix}$ $\quad X_2(t) = \begin{pmatrix} x_{21}(t) \\ x_{22}(t) \\ \vdots \\ x_{2n}(t) \end{pmatrix}$ $\quad \cdots \quad X_n(t) = \begin{pmatrix} x_{n1}(t) \\ x_{n2}(t) \\ \vdots \\ x_{nn}(t) \end{pmatrix}$

Consider $n \times n$ determinant

$$W = \begin{vmatrix} x_{11}(t) & x_{21}(t) & \cdots & x_{n1}(t) \\ x_{12}(t) & x_{22}(t) & \cdots & x_{n2}(t) \\ \vdots & \vdots & & \vdots \\ x_{1n}(t) & x_{2n}(t) & \cdots & x_{nn}(t) \end{vmatrix}$$

If $W \neq 0$, $\{X_1, X_2, \cdots, X_n\}$ is linearly-independent

If $W = 0$, $\{X_1, X_2, \cdots, X_n\}$ is linearly-dependent. #

p230 (예제 6.4)

$$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X$$

$$X_1 = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix} \qquad X_2 = \begin{pmatrix} (1-2t)\,e^{3t} \\ t\,e^{3t} \end{pmatrix}$$ solution vectors

$$W = \begin{vmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & t e^{3t} \end{vmatrix} = -e^{6t} \neq 0$$

Therefore $\{X_1, X_2\}$ is linearly independent.

General solution

$$\boxed{X(t) = c_1 X_1 + c_2 X_2}$$

$$\Rightarrow X(t) = c_1 \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix} + c_2 \begin{pmatrix} (1-2t)e^{3t} \\ te^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -2c_1 e^{3t} + c_2(1-2t)e^{3t} \\ c_1 e^{3t} + c_2 t e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \Omega(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Omega(t) = \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix}$$: 기본행렬 (fundamental matrix)

※

p231

(예제6.5)

$$X' = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} X \qquad X(0) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

general solution

$$X(t) = \Omega(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{3t} & te^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$X(0) = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$-2c_1 + c_2 = -2$$

$$c_1 = 3$$

$$\Rightarrow \quad c_1 = 3$$
$$c_2 = 4$$

$$\Rightarrow X(t) = \Omega(t) \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2e^{3t} & (1-2t)e^{3t} \\ e^{2t} & te^{3t} \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -6e^{3t} + 4(1-2t)e^{2t} \\ 3e^{3t} + 4te^{3t} \end{pmatrix}$$ ※

정리 6.5 : non-homogeneous Equation

Let us consider non-homogeneous differential Equation:

$$X' = AX + G$$

Let $X = \Omega C$ be a general solution of the homogeneous equation $X' = AX$, and Ψ_p be a particular solution of the non-homogeneous equation $X' = AX + G$. Then the general solution of $X' = AX + G$ is

$$X = \Omega C + \Psi_p$$

P233

§ 상수계수 homogeneous coupled linear differential Equation

$$X' = AX \quad \text{— ①}$$

A : constant $n \times n$ matrix

Let

$$X = \xi e^{\lambda t} \quad \text{— ②}$$

ξ : $(n \times 1)$ constant matrix

② → ①

$$\lambda \xi e^{\lambda t} = A \xi e^{\lambda t}$$

$$\Rightarrow \quad A\xi = \lambda \xi \quad \text{— ③}$$

λ: eigenvalue of A

ξ: eigenvector of A

정리 6.6

$X' = AX$ 의 일반해

A의 eigenvalue $= \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$

A의 eigenvector $= \{\xi_1, \xi_2, \cdots, \xi_n\}$

$$X = c_1 \xi_1 e^{\lambda_1 t} + c_2 \xi_2 e^{\lambda_2 t} + \cdots + c_n \xi_n e^{\lambda_n t}$$

p233

(예제6.6)

$$X' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} X$$

$$A = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$$

* eigenvalue

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda = 1 \quad \text{and} \quad 6$$

* eigenvector

(i) $\lambda = 1$

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$3a + 2b = 0$$

$$a = 2, \quad b = -3 \quad \Rightarrow \xi_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

(ii) $\lambda = 6$

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 6\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a = b$$

$$\Rightarrow a = b = 1 \quad \Rightarrow \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$X(t) = c_1 \xi_1 e^{\lambda_1 t} + c_2 \xi_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t}$$

$$= \begin{pmatrix} 2c_1 e^{t} + c_2 e^{6t} \\ -3c_1 e^{t} + c_2 e^{6t} \end{pmatrix} = \Phi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where

$$\Omega(t) = \begin{pmatrix} 2e^{t} & e^{6t} \\ -3e^{t} & e^{6t} \end{pmatrix}$$

\#

fundamental matrix

p=325 (예제 6.7)

$$x' = A x$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

* eigenvalue

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & -2-\lambda & -2 \\ 0 & 2 & -\lambda \end{vmatrix} = 0$$

$$\lambda = 2, \quad -1 \pm \sqrt{3}i$$

(i) $\lambda_1 = 2$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$2a + c = 2a$$

$$-2b - 2c = 2b$$

$$\Rightarrow \quad \xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(ii) $\lambda_2 = -1 + \sqrt{3}i$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (-1 + \sqrt{3}i) \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$(-3 + \sqrt{3}i)a = c$$

$$(1 + \sqrt{3}i)b = -2c$$

$$\Rightarrow \quad \xi_2 = \begin{pmatrix} 1 \\ -2\sqrt{3}i \\ -3 + \sqrt{3}i \end{pmatrix}$$

(iii) $\lambda_3 = -1-\sqrt{3}i$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (-1-\sqrt{3}i)\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$-(3+\sqrt{3}i)a = c$$

$$2b = (-1-\sqrt{3}i)c$$

$$\Rightarrow \xi_3 = \begin{pmatrix} 1 \\ 2\sqrt{3}i \\ -3-\sqrt{3}i \end{pmatrix}$$

$$X = C_1 \xi_1 e^{\lambda_1 t} + C_2 \xi_2 e^{\lambda_2 t} + C_3 \xi_3 e^{\lambda t}$$

$$= \Omega(t) \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

where

$$\Omega(t) = \begin{pmatrix} e^{2t} & e^{(-1+\sqrt{3}i)t} & e^{-(1+\sqrt{3}i)t} \\ 0 & -2\sqrt{3}i\,e^{(-1+\sqrt{3}i)t} & 2\sqrt{3}i\,e^{-(1+\sqrt{3}i)t} \\ 0 & (-3+\sqrt{3}i)\,e^{(-1+\sqrt{3}i)t} & -(3+\sqrt{3}i)\,e^{-(1+\sqrt{3}i)t} \end{pmatrix}$$

fundamental matrix

$\Rightarrow$ different expression

$$\xi_2 e^{\lambda_2 t}$$

$$= \begin{pmatrix} 1 \\ -2\sqrt{3}\, i \\ -3+\sqrt{3}\, i \end{pmatrix} e^{(-1+\sqrt{3} i) t}$$

$$= \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \right] e^{-t} \left(\cos\sqrt{3}t + i \sin\sqrt{3}t \right)$$

$$= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cos\sqrt{3}t - \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \sin\sqrt{3}t \right]$$

$$+ i e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \sin\sqrt{3}t + \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \cos\sqrt{3}t \right]$$

By same way

$$\xi_3 e^{\lambda_3 t}$$

$$= e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \cos\sqrt{3}t - \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \sin\sqrt{3}t \right]$$

$$- i e^{-t} \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \sin\sqrt{3}t + \begin{pmatrix} 0 \\ -2\sqrt{3} \\ \sqrt{3} \end{pmatrix} \cos\sqrt{3}t \right]$$

Thus one can choose two linearly-independent solution as

$$x_1 = \xi_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

$$x_2 = \frac{1}{2}\left[\xi_2 e^{\lambda_2 t} + \xi_3 e^{\lambda_3 t} \right] = e^{-t} \begin{pmatrix} \cos\sqrt{3}t \\ 2\sqrt{3}\sin\sqrt{3}t \\ -3\cos\sqrt{3}t - \sqrt{3}\sin\sqrt{3}t \end{pmatrix}$$

$$x_3 = \frac{1}{2i}\left[\xi_2 e^{\lambda_2 t} - \xi_3 e^{\lambda_3 t} \right] = e^{-t} \begin{pmatrix} \sin\sqrt{3}t \\ -2\sqrt{3}\cos\sqrt{3}t \\ -3\sin\sqrt{3}t + \sqrt{3}\cos\sqrt{3}t \end{pmatrix}$$

general solution

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= \Omega(t) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where

$$\Omega(t) = \begin{pmatrix} e^{2t} & e^{-t} \cos\sqrt{3}t & e^{-t} \sin\sqrt{3}t \\ 0 & 2\sqrt{3}\, e^{-t} \sin\sqrt{3}t & -2\sqrt{3}\, e^{-t} \cos\sqrt{3}t \\ 0 & -e^{-t}(3\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t) & e^{-t}(-3\sin\sqrt{3}t + \sqrt{3}\cos\sqrt{3}t) \end{pmatrix}$$ #

p237

(예제6.8)

$$X' = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix} X$$

(i) eigenvalue

$$\begin{vmatrix} 5-\lambda & -4 & 4 \\ 12 & -11-\lambda & 12 \\ 4 & -4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+3)(\lambda-1)^2 = 0$$

$\lambda = -3, \quad 1$ (중근)

i) $\lambda_1 = -3$

$$\begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -3 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \quad 2a - b + c = 0$$
$$3a - 2b + 2c = 0$$

$$\Rightarrow \quad \xi_1 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

(ii) $\lambda_2 = 1$

$$\begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 10 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$a - b + c = 0$$

$$\xi_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \xi_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

General solution

$$X = c_1 \xi_1 e^{\lambda_1 t} + c_2 \xi_2 e^{\lambda_2 t} + c_3 \xi_3 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{t}$$

$$= \begin{pmatrix} e^{-3t} & e^{t} & e^{t} \\ 3e^{-3t} & e^{t} & 0 \\ e^{-3t} & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \#$$

(예제 6.9)

$$X' = AX \qquad A = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix}$$

* eigenvalue

$$\begin{vmatrix} 1-\lambda & 3 \\ -3 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda = 4 \quad (\text{중근})$$

* eigenvector

$$\begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow a = b$$

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad X_1 = \xi_1 e^{\lambda t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} \qquad - ①$$

How to obtain X_2 ?

Put

$$X_2 = \xi_1 t e^{\lambda t} + \eta e^{\lambda t} \qquad - ②$$

Then

$$X_2' = \xi_1 e^{\lambda t} + \lambda \xi_1 t e^{\lambda t} + \lambda \eta e^{\lambda t} \qquad - ③$$

$$AX_2 = \underbrace{A\xi_1}_{=\lambda\xi_1} t e^{\lambda t} + A\eta e^{\lambda t}$$

$$= \lambda \xi_1 t e^{\lambda t} + A\eta e^{\lambda t} \qquad - ④$$

If X_2 is solution vector, $X_2' = AX_2$.

$$\Rightarrow \quad \xi_1 e^{\lambda t} + \lambda \eta e^{\lambda t} = A\eta e^{\lambda t}$$

$$\Rightarrow \quad (A - \lambda I)\eta = \xi_1 \qquad - ⑤$$

For our case

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which gives

$$\eta = \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} \qquad - ⑥$$

⑥ → ②

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{4t} + \begin{pmatrix} 1 \\ \frac{4}{3} \end{pmatrix} e^{4t}$$

$$= \begin{pmatrix} t+1 \\ t+\frac{4}{3} \end{pmatrix} e^{4t} \qquad - ⑦$$

General solution

$$X = C_1 X_1 + C_2 X_2$$

$$= \begin{pmatrix} e^{4t} & (t+1)e^{4t} \\ e^{4t} & (t+\frac{4}{3})e^{4t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \qquad ※$$

(예제 6.10)

$$X' = AX \qquad A = \begin{pmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

* eigenvalue

$$\begin{vmatrix} -2-\lambda & -1 & -5 \\ 25 & -7-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+2)^3 = 0$$

$$\lambda = -2 \qquad — ①$$

* eigenvector

$$\begin{pmatrix} -2 & -1 & -5 \\ 25 & -7 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -2\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \begin{aligned} 5a - b &= 0 \\ b + 5c &= 0 \end{aligned}$$

$$\Rightarrow \xi_1 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} \qquad — ②$$

$$\Rightarrow X_1 = \xi_1 e^{\lambda t} = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} e^{-2t} \qquad — ③$$

How to obtain X_2 and X_3 ?

Put

$$X_2 = \xi_1 t e^{\lambda t} + \eta_1 e^{\lambda t} \quad - ⑤$$

Then

$$(A - \lambda I)\eta_1 = \xi_1 \quad - ⑤$$

$$\Rightarrow \begin{pmatrix} 0 & -1 & -5 \\ 25 & -5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \eta_1 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \eta_1 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \quad - ⑥$$

Thus

$$X_2 = \begin{pmatrix} -1 \\ -5 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} e^{-2t}$$

$$= \begin{pmatrix} -t-1 \\ -5t-4 \\ t+1 \end{pmatrix} e^{-2t} \quad - ⑦$$

How to compute X_3? Put

$$X_3 = \frac{1}{2}\xi_1 t^2 e^{\lambda t} + \eta_2 t e^{\lambda t} + \eta_3 e^{\lambda t} \quad — ⑧$$

$$X_3' = \frac{\lambda}{2}\xi_1 t^2 e^{\lambda t} + \xi_1 t e^{\lambda t} + \eta_2 e^{\lambda t} + \lambda\eta_2 t e^{\lambda t} + \lambda\eta_3 e^{\lambda t} \quad — ⑨$$

$$AX_3 = \frac{1}{2}\underbrace{A\xi_1}_{\lambda\xi_1} t^2 e^{\lambda t} + A\eta_2 t e^{\lambda t} + A\eta_3 e^{\lambda t}$$

$$= \frac{\lambda}{2}\xi_1 t^2 e^{\lambda t} + A\eta_2 t e^{\lambda t} + A\eta_3 e^{\lambda t} \quad — ⑩$$

Since $X_3' = AX_3$, we have

$$(A - \lambda I)\eta_2 = \xi_1 \quad — ⑪$$

$$(A - \lambda I)\eta_3 = \eta_2 \quad — ⑫$$

Comparing Eq. ⑤ with Eq. ⑪, we have

$$\eta_2 = \eta_1 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} \quad — ⑬$$

⑬ → ⑫

$$\begin{pmatrix} 0 & -1 & -5 \\ -5 & -5 & 0 \\ 0 & 1 & 5 \end{pmatrix} \eta_3 = \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \eta_3 = \begin{pmatrix} -\frac{24}{5} \\ -4 \\ 1 \end{pmatrix} \quad — ⑭$$

⑬, ⑭ → ⑧

$$X_3 = \begin{pmatrix} -\frac{1}{2}t^2 - t - \frac{24}{5} \\ -\frac{5}{2}t^2 - 4t - 4 \\ \frac{1}{2}t^2 + t + 1 \end{pmatrix} e^{-2t} \quad — ⑮$$

General solution

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3$$

$$= \begin{pmatrix} -e^{-2t} & -(t+1)e^{-2t} & -\left(\frac{1}{2}t^2 + t + \frac{24}{5}\right)e^{-2t} \\ -5e^{-2t} & -(5t+4)e^{-2t} & -\left(\frac{5}{2}t^2 + 4t + 4\right)e^{-2t} \\ e^{-2t} & (t+1)e^{-2t} & \left(\frac{1}{2}t^2 + t + 1\right)e^{-2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad /\!/$$

* eigenvalue $= \lambda$ (k번 중복)

eigenvector $= \xi_1$

$$X_1(t) = \xi_1 e^{\lambda t}$$

$$X_2(t) = \xi_1 t e^{\lambda t} + \eta_1 e^{\lambda t}$$

$$X_3(t) = \frac{1}{2}\xi_1 t^2 e^{\lambda t} + \eta_2 t e^{\lambda t} + \eta_3 e^{\lambda t}$$

$$X_4(t) = \frac{1}{3!}\xi_1 t^3 e^{\lambda t} + \frac{1}{2!}\eta_4 t^2 e^{\lambda t} + \eta_5 t e^{\lambda t} + \eta_6 e^{\lambda t}$$

$$\vdots$$

$$X_k(t) = \frac{1}{(k-1)!}\xi_1 t^{k-1} e^{\lambda t} + \frac{1}{(k-2)!}\bar{\eta}_1 t^{k-2} e^{\lambda t} + \cdots + \bar{\eta}_{k-1} e^{\lambda t}$$

* Matrix Diagonalization

$$X' = A X \quad - (1)$$

If

$$A = P D P^{-1} \quad - (2)$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & \lambda_n \end{pmatrix}, \quad - (3)$$

Eq. (1) becomes

$$X' = P D P^{-1} X \quad (4)$$

Put

$$Z = P^{-1} X \quad (5)$$

(5) → (4)

$$Z' = D Z \quad (6)$$

Thus

$$Z = \Omega_D(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad (7)$$

where

$$\Omega_D(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \quad - (8)$$

(7) → (5)

$$X = P Z = P \, \Omega_D(t) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad - (9)$$

p242

(예제 6.11)

$$X' = AX \qquad A = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$$

- diagonalization

(1) eigenvalues

$$\begin{vmatrix} 3-\lambda & 3 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

$$\lambda = 2 \ , \quad \lambda = 6$$

(2) eigenvectors

(i) $\lambda = 2$

$$\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\xi_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

(ii) $\lambda = 6$

$$\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 6\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(3) diagonalization

$$P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \qquad P^{-1} = -\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}$$

$$\Rightarrow A = P D P^{-1}$$

$$\Rightarrow \quad X' = P D P^{-1} X$$

$$\Rightarrow \quad P^{-1} X' = D \, P^{-1} X$$

$$\Rightarrow \quad (P^{-1} X)' = D \, (P^{-1} X) \qquad - ①$$

Put

$$Z = P^{-1} X \qquad - ②$$

② → ①

$$Z' = D Z$$

$$Z = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad - ③$$

③ → ②

$$X = P Z$$

$$= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad ※$$

* non-homogeneous differential Eq.

$$X' = AX + G \quad — ①$$

$$X = \Omega(t)C + \Psi_p(t) \quad — ②$$

$\Omega(t)C$: general solution of $X' = AX$

$\Psi_p(t)$: particular solution of $X' = AX + G$

How to derive $\Psi_p(t)$?

Put

$$\Psi_p(t) = \Omega(t)U(t) \quad — ③$$

③ → ①

$$\underline{\Omega'(t)}U(t) + \Omega(t)U'(t) = A\Omega(t)U(t) + G$$

$= A\Omega(t)$

$$A\Omega(t)\cancel{U(t)} + \Omega(t)U'(t) = \cancel{A\Omega(t)U(t)} + G$$

$$U'(t) = \Omega^{-1}(t)G$$

$$\underline{U(t) = \int \Omega^{-1}(t)G\,dt} \quad — ④$$

④ → ③

$$\boxed{\Psi_p(t) = \Omega(t)\int \Omega^{-1}(t)G\,dt}$$

※

p244

(예제 6.12)

$$X' = AX + B \qquad A = \begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} e^t \\ \sin t \end{pmatrix}$$

(i) General solution of homogeneous Equation

① eigenvalues

$$\begin{vmatrix} 1-\lambda & -10 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda = -1, \ 6$$

② eigenvectors

(i) $\lambda_1 = -1$

$$\begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\xi_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(ii) $\lambda_2 = 6$

$$\begin{pmatrix} 1 & -10 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 6\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\xi_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$X_g = \Omega(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad - ①$$

$$\Omega(t) = \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix} \qquad - ②$$

$$\Omega^{-1}(t) = \frac{1}{7} \begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{pmatrix} \qquad - ③$$

(ii) Particular solution

$\Omega^{-1}(t)\,B$

$$= \frac{1}{7}\begin{pmatrix} e^{t} & 2e^{t} \\ -e^{-6t} & 5e^{-6t} \end{pmatrix}\begin{pmatrix} e^{t} \\ \sin t \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} e^{2t} + 2\sin t\, e^{t} \\ -e^{-5t} + 5\sin t\, e^{-6t} \end{pmatrix} \quad - ⑤$$

$$\left.\begin{aligned} &\int (e^{2t} + 2\sin t\, e^{t})\,dt = \frac{1}{2}e^{2t} + e^{t}(\sin t - \cos t) \\ &\int (-e^{-5t} + 5\sin t\, e^{-6t})\,dt = \frac{1}{5}e^{-5t} - \frac{5}{37}e^{-6t}(6\sin t + \cos t) \end{aligned}\right\} - ⑤$$

* formula

$$\int \sin t\, e^{at}\,dt = \frac{e^{at}\left[-\cos t + a\sin t\right]}{1+a^2}$$

$$\Rightarrow \int \Omega^{-1}(t)\,B(t)\,dt = \begin{pmatrix} \frac{1}{14}e^{2t} + \frac{1}{7}e^{t}(\sin t - \cos t) \\ \frac{1}{35}e^{-5t} - \frac{5}{259}e^{-6t}(6\sin t + \cos t) \end{pmatrix} \quad - ⑥$$

Thus,

$$\Phi_p(t) = \Omega(t)\int \Omega^{-1}(t)\,B(t)\,dt$$

$$= \begin{pmatrix} \frac{3}{10}e^{t} + \frac{35}{37}\sin t - \frac{25}{37}\cos t \\ \frac{1}{10}e^{t} + \frac{1}{37}\sin t - \frac{6}{37}\cos t \end{pmatrix} \quad - ⑦$$

Thus

$$X(t) = \begin{pmatrix} 5e^{-t} & -2e^{6t} \\ e^{-t} & e^{6t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} \frac{3}{10}e^{t} + \frac{35}{37}\sin t - \frac{25}{37}\cos t \\ \frac{1}{10}e^{t} + \frac{1}{37}\sin t - \frac{6}{37}\cos t \end{pmatrix} \quad //$$

* Matrix diagonalization

$$X' = AX + G \quad -①$$

$$\left.\begin{array}{l} A = PDP^{-1} \\ D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \end{array}\right\} \quad -②$$

② → ①

$$X' = PDP^{-1}X + G$$

$P^{-1} \times ||$

$$P^{-1}X' = DP^{-1}X + P^{-1}G \quad -③$$

Put

$$Z = P^{-1}X \quad -④$$

④ → ③

$$Z' = DZ + P^{-1}G \quad -⑤$$

Let

$$Z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \qquad P^{-1}G = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Then

$$z_1' = \lambda_1 z_1 + f_1(x)$$
$$z_2' = \lambda_2 z_2 + f_2(x)$$
$$\vdots$$
$$z_n' = \lambda_n z_n + f_n(x)$$

linear Eqn

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y = e^{-\int P dx}\left[\int Q e^{\int P dx} dx + C\right]$$

(예제 6.13)

$$X' = \begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix} X + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix} \quad -①$$

$$\left.P = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \quad P^{-1} = -\frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} \right\} -②$$

$$A = PDP^{-1}$$

② → ①

$$X' = PDP^{-1}X + \begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

$$(P^{-1}X)' = D(P^{-1}X) + P^{-1}\begin{pmatrix} 8 \\ 4e^{3t} \end{pmatrix}$$

put

$$Z = P^{-1}X \quad -③$$

Then

$$Z' = DZ + \begin{pmatrix} -2 + e^{3t} \\ 2 + 3e^{3t} \end{pmatrix} \quad -④$$

put $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ $-⑤$

⑤ → ④

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} -2+e^{3t} \\ 2+3e^{3t} \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} z_1' &= 2z_1 + (-2+e^{3t}) \\ z_2' &= 6z_2 + (2+3e^{3t}) \end{aligned}\right) -⑥$$

Solution

$$\left.\begin{aligned} z_1(t) &= C_1 e^{2t} + e^{3t} + 1 \\ z_2(t) &= C_2 e^{6t} - e^{2t} - \tfrac{1}{3} \end{aligned}\right) \quad - ⑦$$

$$\Rightarrow \quad Z(t) = \begin{pmatrix} C_1 e^{2t} + e^{3t} + 1 \\ C_2 e^{6t} - e^{3t} - \tfrac{1}{3} \end{pmatrix} \quad - ⑧$$

⑧ → ②

$$\begin{aligned} X &= PZ \\ &= \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{2t} + e^{3t} + 1 \\ C_2 e^{6t} - e^{3t} - \tfrac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} -3C_1 e^{2t} + C_2 e^{6t} - 4e^{3t} - \tfrac{10}{3} \\ C_1 e^{2t} + C_2 e^{6t} + \tfrac{2}{3} \end{pmatrix} \\ &= \Omega(t) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} -4e^{3t} - \tfrac{10}{3} \\ \tfrac{2}{3} \end{pmatrix} \quad - ⑨ \end{aligned}$$

where

$$\Omega(t) = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \quad - ⑩$$

※

(예제 6.14)

$$X(0) = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{22}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

$$c_1 = -\frac{17}{4}, \quad c_2 = -\frac{41}{12}$$

$$X(t) = \begin{pmatrix} -3e^{2t} & e^{6t} \\ e^{2t} & e^{6t} \end{pmatrix} \begin{pmatrix} -\frac{17}{4} \\ -\frac{41}{12} \end{pmatrix} + \begin{pmatrix} -4e^{3t} - \frac{10}{3} \\ \frac{2}{3} \end{pmatrix} \quad /\!/$$