

CH.2 2계 미분 방정식

2계 선형 미분 방정식의 일반 form

$$\frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x)y = Q(x)$$

If $Q(x) = 0$, homogeneous differential Equation (제자 미분 방정식)

If $Q(x) \neq 0$, non-homogeneous differential Equation (제자 미분 방정식)

• 제자 미분 방정식

$$y'' + p(x)y' + q(x)y = 0$$

Theorem. 1

If $y_1(x)$ and $y_2(x)$ are solutions, $C_1 y_1(x) + C_2 y_2(x)$ is also solution

$$(PF) \quad y'' + p(x)y' + q(x)y = 0 \quad \left\{ \begin{array}{l} \\ \end{array} \right. - ①$$

$$y_1'' + p(x)y_1' + q(x)y_1 = 0$$

$$(C_1 y_1 + C_2 y_2)'' = C_1 y_1'' + C_2 y_2''$$

$$(C_1 y_1' + C_2 y_2') = C_1 y_1' + C_2 y_2'$$

$$\Rightarrow (C_1 y_1 + C_2 y_2)'' + p(x)(C_1 y_1 + C_2 y_2)' + q(x)(C_1 y_1 + C_2 y_2) =$$

$$= C_1 \underline{\underline{y_1'' + p(x)y_1' + q(x)y_1}} + C_2 \underline{\underline{y_2'' + p(x)y_2' + q(x)y_2}} = 0$$

$$= 0$$

x.

Theorem. =

If $y_1(x)$ and $y_2(x)$ are solutions and they are linearly independent with each other, the general solution is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

* $f(x)$ and $g(x)$ are linearly independent if

$$W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \neq 0.$$

$$(Ex) \quad \frac{d^2y}{dx^2} + y = 0$$

$$y_1 = \sin x$$

$$y_2 = \cos x$$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0.$$

y_1 and y_2 are linearly independent.

\Rightarrow 일반식

$$y = C_1 \sin x + C_2 \cos x$$

p50

(Ex 2)

$$y'' + 11y' + 24y = 0 : \quad y(0) = 1 \quad y'(0) = 4$$

(Ansatz)

$$y_1 = e^{-3x} : \text{solution}$$

$$\left. \begin{aligned} & y_1'' + 11y_1' + 24y_1 \\ &= 9e^{-3x} - 33e^{-3x} + 24e^{-3x} \\ &= 0 \end{aligned} \right\}$$

$$y_2 = e^{-8x} : \text{solution}$$

$$\left. \begin{aligned} & y_2'' + 11y_2' + 24y_2 \\ &= 64e^{-8x} - 88e^{-8x} + 24e^{-8x} \\ &= 0 \end{aligned} \right\}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix} = -5e^{-11x} \neq 0$$

\Rightarrow linearly independent

$$\Rightarrow \text{allgemeine} \quad \underline{y = C_1 e^{-3x} + C_2 e^{-8x}}$$

$$y(0) = C_1 + C_2 = 1$$

$$y'(0) = -3C_1 e^{-3x} - 8C_2 e^{-8x}$$

$$y'(0) = -3C_1 - 8C_2 = 4$$

$$\Rightarrow C_1 = \frac{12}{5}, \quad C_2 = -\frac{11}{5}$$

$$\Rightarrow y = \frac{12}{5} e^{3x} - \frac{11}{5} e^{-8x}$$

✓

Theorem 3

$$y'' + p(x)y' + q(x)y = f(x)$$

$$\text{일반해: } y = y_g + y_p$$

y_g : general solution of homogeneous equation

y_p : particular solution of non-homogeneous equation

$$(Ex) \quad y'' - y = 4$$

(i) y_g

$$y'' - y = 0$$

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$

\Rightarrow linearly independent

$$y_g = C_1 e^x + C_2 e^{-x}$$

$$(ii) \quad y_p = -4$$

$$\Rightarrow y = C_1 e^x + C_2 e^{-x} - 4$$

상수계수 미분 방정식

$$y'' + Ay' + By = 0 \quad - (1)$$

Put

$$y = e^{\lambda x} \quad - (2)$$

(2) \rightarrow (1)

$$\lambda^2 + A\lambda + B = 0 \quad : \text{characteristic equation (특성방정식)}$$

$$\lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2} \quad - (3)$$

$$(i) A^2 - 4B > 0$$

Put

$$a = \frac{-A + \sqrt{A^2 - 4B}}{2}, \quad b = \frac{-A - \sqrt{A^2 - 4B}}{2} \quad (4)$$

Then

$$\underline{y = C_1 e^{ax} + C_2 e^{bx}}$$

pt 4 (ex 2.3)

$$y'' - y' - 6y = 0$$

\Rightarrow characteristic equation

$$\lambda^2 - \lambda - 6 = 0$$

$$\rightarrow A = 1, B = -6$$

$$A^2 - 4B = 1 + 24 = 25 > 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = 3 \text{ or } -2$$

$$y = C_1 e^{3x} + C_2 e^{-2x}$$

*.

$$(ii) A^2 - 4B = 0$$

$$\underline{y = (C_1 + C_2 x) e^{-\frac{A}{2}x}}$$

(pf) put

$$y = u(x) e^{-\frac{A}{2}x} \quad \text{--- } ①$$

$$y' = (u' - \frac{A}{2}u) e^{-\frac{A}{2}x} \quad \left. \begin{array}{l} \\ \end{array} \right\} - ②$$

$$y'' = (u'' - Au' + \frac{A^2}{4}u) e^{-\frac{A}{2}x}$$

Theorem

$$y'' + Ay' + By = 0$$

$$\Rightarrow e^{-\frac{A}{2}x} \left[u'' + \underbrace{\left(B - \frac{A^2}{4} \right) u'}_{''} \right] = 0$$

$$\Rightarrow u'' = 0$$

$$\Rightarrow u = C_1 + C_2 x$$

pt4 (11/21/2.4)

$$y'' - 6y' + 9y = 0$$

\Rightarrow characteristic Eq.

$$\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \quad A = -6, \quad B = 9$$

$$(\lambda - 3)^2 = 0 \quad A^2 - 4B = 36 - 36 = 0$$

$$\underline{y = (C_1 + C_2 x) e^{3x}}$$

$$(iii) A^2 - 4B < 0$$

$$\tau = \frac{-A \pm \sqrt{4B-A^2} i}{2}$$

$$y = C_1 e^{\frac{-A + \sqrt{4B-A^2} i}{2} x} + C_2 e^{\frac{-A - \sqrt{4B-A^2} i}{2} x}$$

$$= e^{-\frac{A}{2}x} \left[C_1 e^{i \frac{\sqrt{4B-A^2}}{2} x} + C_2 e^{-i \frac{\sqrt{4B-A^2}}{2} x} \right]$$

$$= e^{-\frac{A}{2}x} \left[C_1 \left(\cos \frac{\sqrt{4B-A^2}}{2} x + i \sin \frac{\sqrt{4B-A^2}}{2} x \right) + C_2 \left(\cos \frac{\sqrt{4B-A^2}}{2} x - i \sin \frac{\sqrt{4B-A^2}}{2} x \right) \right]$$

$$= e^{-\frac{A}{2}x} \left[(C_1 + C_2) \cos \frac{\sqrt{4B-A^2}}{2} x + i (C_1 - C_2) \sin \frac{\sqrt{4B-A^2}}{2} x \right]$$

$$= e^{-\frac{A}{2}x} \left[C_1 \cos \frac{\sqrt{4B-A^2}}{2} x + C_2 \sin \frac{\sqrt{4B-A^2}}{2} x \right]$$

$$\Rightarrow y = e^{-\frac{A}{2}x} \left[C_1 e^{i \frac{\sqrt{4B-A^2}}{2} x} + C_2 e^{-i \frac{\sqrt{4B-A^2}}{2} x} \right]$$

or

$$y = e^{-\frac{A}{2}x} \left[C_1 \sin \frac{\sqrt{4B-A^2}}{2} x + C_2 \cos \frac{\sqrt{4B-A^2}}{2} x \right]$$

PDE

(9/21/2015)

$$y'' + 2y' + 6y = 0$$

\Rightarrow characteristic equation

$$\lambda^2 + 2\lambda + 6 = 0$$

$$A = 2, \quad B = 6$$

$$A^2 - 4B = 4 - 24 = -20 < 0$$

$$\Rightarrow \frac{\sqrt{4B-A^2}}{2} = \sqrt{5}$$

$$y = e^{-x} [C_1 \sin \sqrt{5} x + C_2 \cos \sqrt{5} x]$$

or

$$y = e^{-x} [C_1 e^{i\sqrt{5}x} + C_2 e^{-i\sqrt{5}x}]$$

p57

(문제 2.7)

$$y'' - 4y' + 5y = 0 \quad y(\pi) = -3 \quad y(\tau) = -2$$

\Rightarrow characteristic Eq.

$$\lambda^2 - 4\lambda + 5 = 0 \quad A = -4 \quad B = 5$$

$$A^2 - 4B = -196 < 0$$

$$\frac{\sqrt{4B-A^2}}{2} = 7$$

\Rightarrow 일 반 해

$$y = e^{z\pi} [C_1 \sin \eta x + C_2 \cos \eta x]$$

$$y(\pi) = e^{z\pi} [C_1 \overset{''}{\underset{=}{\sin}} \pi + C_2 \overset{''}{\underset{=}{\cos}} \pi] = -C_2 e^{z\pi} = -3$$

$$C_2 = 3e^{-z\pi}$$

$$y'(x) = z e^{z\pi} [C_1 \sin \eta x + C_2 \cos \eta x]$$

$$+ \eta e^{z\pi} [C_1 \cos \eta x - C_2 \sin \eta x]$$

$$= e^{z\pi} [(zC_1 - \eta C_2) \sin \eta x + (zC_2 + \eta C_1) \cos \eta x]$$

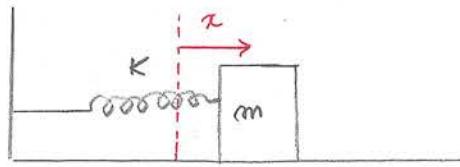
$$y'(\pi) = e^{z\pi} [(zC_1 - \eta C_2) \overset{''}{\underset{=}{\sin}} \pi + (\eta C_1 + zC_2) \overset{''}{\underset{=}{\cos}} \pi]$$

$$= -(\eta C_1 + zC_2) e^{z\pi} = -2$$

$$\Rightarrow C_1 = -\frac{8}{\eta} e^{-z\pi}$$

$$y = e^{z\pi} \left[-\frac{8}{\eta} e^{-z\pi} \sin \eta x + 3e^{-z\pi} \cos \eta x \right] = e^{(z-\pi)} \left[3 \cos \eta x - \frac{8}{\eta} \sin \eta x \right]$$

(Ex) Harmonic Oscillator without friction



$$F = -m \frac{d^2x}{dt^2} = -Kx$$

$$\Rightarrow \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{K}{m}}$$

$$\frac{dy}{dx} + A \frac{dy}{dx} + By = 0$$

$$\begin{pmatrix} A=0 \\ B=\omega^2 \end{pmatrix}$$

$$x = C_1 \cos \omega t + C_2 \sin \omega t \quad \text{--- ①}$$

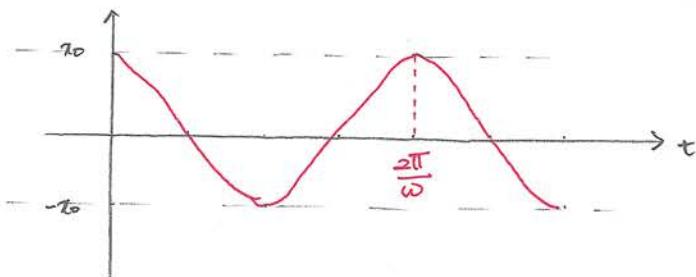
Let

$$x(0) = x_0 \quad \frac{dx}{dt}(0) = 0 \quad \text{--- ②}$$

$$\Rightarrow \begin{cases} C_1 = x_0 \\ C_2 = 0 \end{cases} \quad \text{--- ③}$$

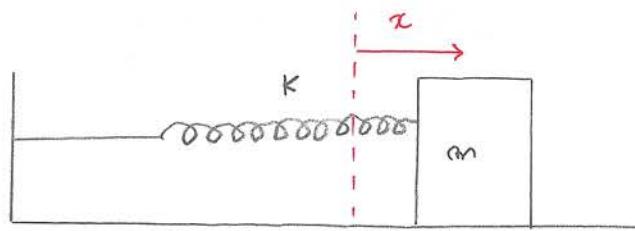
③ → ①

$$x = x_0 \cos \omega t$$



(Ex) Harmonic Oscillator with Friction

30-2



$$F = -m \frac{d^2x}{dt^2} = -Kx - C \frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{C}{m} \frac{dx}{dt} + \omega^2 x = 0$$

$$\frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = 0$$

$$A = \frac{C}{m}, \quad B = \omega^2$$

$$(i) \quad C < 2m\omega$$

$$x = e^{-\frac{C}{2m}t} \left[C_1 \cos \left[\sqrt{\omega^2 - \frac{C^2}{4m^2}} t + C_2 \sin \left[\sqrt{\omega^2 - \frac{C^2}{4m^2}} t \right] \right] \quad \textcircled{1}$$

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0$$

\Rightarrow

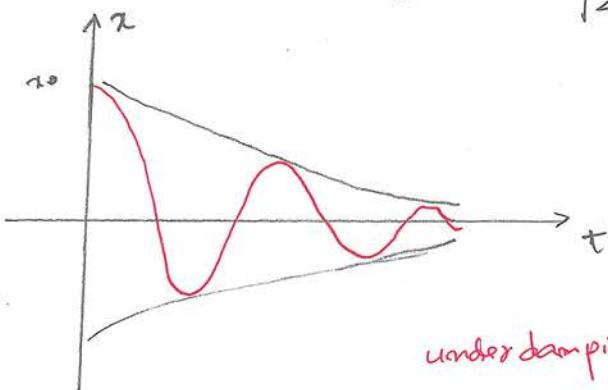
$$C_1 = x_0$$

$$C_2 = \frac{C x_0}{\sqrt{4m^2\omega^2 - C^2}}$$

? $\quad \textcircled{2}$

$\textcircled{2} \rightarrow \textcircled{1}$

$$x = e^{-\frac{C}{2m}t} \left[x_0 \cos \left[\sqrt{\omega^2 - \frac{C^2}{4m^2}} t + \frac{Cx_0}{\sqrt{4m^2\omega^2 - C^2}} \sin \left[\sqrt{\omega^2 - \frac{C^2}{4m^2}} t \right] \right]$$



(ii) $c > 2m\omega$

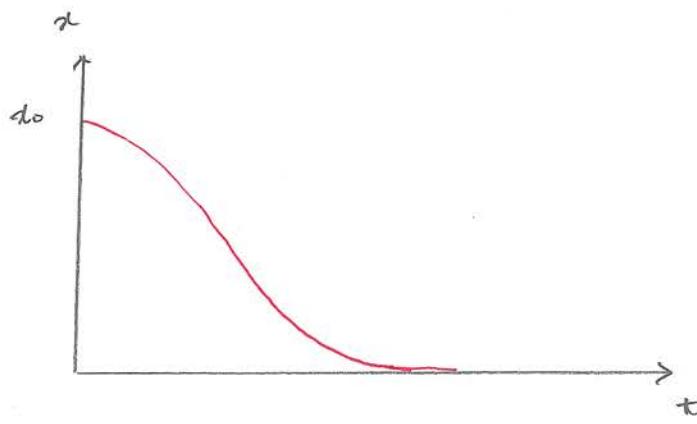
$$\left. \begin{aligned} x &= C_1 e^{-at} + C_2 e^{-bt} \\ a &= \frac{1}{2m} [c - \sqrt{c^2 - 4m^2\omega^2}] \\ b &= \frac{1}{2m} [c + \sqrt{c^2 - 4m^2\omega^2}] \end{aligned} \right\} - \textcircled{2}$$

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0$$

$$\left. \begin{aligned} C_1 &= \frac{x_0 b}{b-a} \\ C_2 &= \frac{-x_0 a}{b-a} \end{aligned} \right\} - \textcircled{2}$$

 $\textcircled{4} \rightarrow \textcircled{2}$

$$x = \frac{x_0}{b-a} [b e^{-at} - a e^{-bt}] \quad - \textcircled{4}$$



overdamping

$$(ii) \quad c = z\omega\omega$$

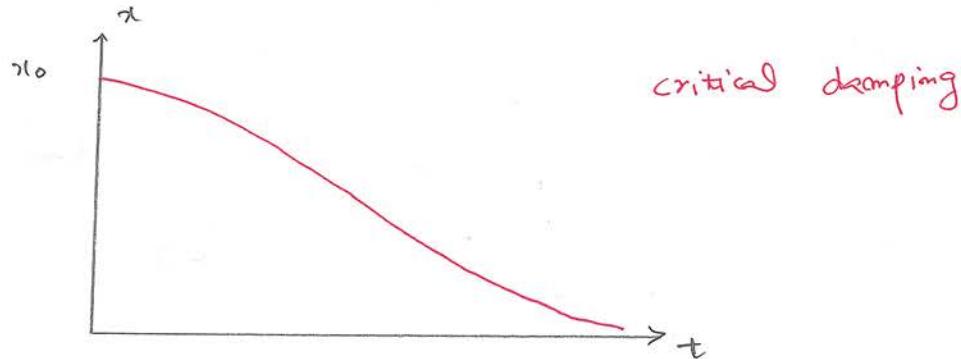
$$x = (C_1 + C_2 t) e^{-\frac{c}{2m}t} \quad - \textcircled{4}$$

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0$$

$$\Rightarrow C_1 = x_0, \quad C_2 = \frac{c x_0}{2m} \quad - \textcircled{5}$$

$\textcircled{4} \rightarrow \textcircled{5}$

$$x = x_0 \left(1 + \frac{c}{2m} t \right) e^{-\frac{c}{2m}t} \quad - \textcircled{6}$$



Euler 方程식

$$y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0 \quad (\text{Euler Equation})$$

put

$$\left. \begin{array}{l} x = e^t \\ \frac{dx}{dt} = e^t \\ \frac{dt}{dx} = e^{-t} \end{array} \right\} - ①$$

$$y' = \frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = e^{-t} \frac{dy}{dt} \quad - ②$$

$$y'' = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dx} \left(e^{-t} \frac{dy}{dt} \right)$$

$$= \frac{dt}{dx} \frac{d}{dt} \left(e^{-t} \frac{dy}{dt} \right)$$

$$= e^{-2t} \left[\frac{d^2y}{dt^2} - \frac{dy}{dt} \right] \quad - ③$$

①, ②, ③ \Rightarrow Euler Equation

$$e^{-2t} \left[\frac{d^2y}{dt^2} + (A-1) \frac{dy}{dt} + By \right] = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + (A-1) \frac{dy}{dt} + By = 0$$

수학적 문제 해결

P59

(Ex 2.8)

$$x^2 y'' + 2xy' - 6y = 0 \quad || \times \frac{1}{x^2}$$

$$\Rightarrow y'' + \frac{2}{x} y' - \frac{6}{x^2} = 0 \quad \text{Euler Equation } (A=2, B=-6)$$

Put

$$x = e^t \quad \text{--- ①}$$

Then

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0 \quad \text{--- ②}$$

Characteristic Eq.

$$\lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow (\lambda-3)(\lambda+2) = 0$$

$$y = C_1 e^{3t} + C_2 e^{-2t}$$

$$= C_1 e^{3\ln x} + C_2 e^{-2\ln x}$$

$$= C_1 x^3 + \frac{C_2}{x^2}$$

X

P 60

(Q(2)=9)

$$x^2 y'' - 5x y' + 10y = 0 \quad y(1) = 4 \quad y'(1) = -6$$

$$\Rightarrow y'' - \frac{5}{x} y' + \frac{10}{x^2} y = 0 \quad \text{Euler Eq. } (A = -5, B = 10)$$

Put $x = e^t - 0 \quad (t = \ln x)$

Then

$$\frac{dy}{dt} - 6 \frac{dy}{dt} + 10y = 0 \quad -\textcircled{2}$$

characteristic Eq

$$t^2 - 6t + 10 = 0 \quad -\textcircled{3} \quad \left(A^2 - 4B = 36 - 40 < 0 \right)$$

$$A = -6, B = 10$$

$$\frac{\sqrt{4B-A^2}}{2} = 1$$

general solution

$$\begin{aligned} y &= e^{3t} [C_1 \sin t + C_2 \cos t] \\ &= e^{3\ln x} [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \\ &= x^3 [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \end{aligned}$$

$$y(1) = C_2 = 4$$

$$\begin{aligned} y'(1) &= 3x^2 [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \\ &\quad + x^3 \left[C_1 \frac{1}{x} \cos(\ln x) - C_2 \frac{1}{x} \sin(\ln x) \right] \\ &= x^2 [(3C_1 - C_2) \sin(\ln x) + (C_1 + 3C_2) \cos(\ln x)] \end{aligned}$$

$$y'(1) = C_1 + 3C_2 = -6 \quad \Rightarrow \quad C_1 = -18$$

$$\Rightarrow y = x^3 [-18 \sin(\ln x) + 4 \cos(\ln x)] \quad *$$

을 이해하니 매우 어렵다

$$y'' + p(x)y' + q(x)y = f(x) \quad -\textcircled{1}$$

$$\begin{aligned} y &= y_g + y_p \\ y_g &= C_1 y_1 + C_2 y_2 \end{aligned} \quad -\textcircled{2}$$

How to derive y_p ?

[1] 매개 변수 정학법

Put

$$y_p = u(x)y_1(x) + v(x)y_2(x) \quad -\textcircled{3}$$

Then

$$y'_p = u'y_1' + v'y_2' + (u'y_1 + v'y_2) \quad -\textcircled{4}$$

Put

$$\underline{u'y_1 + v'y_2 = 0} \quad -\textcircled{5}$$

first condition

$\textcircled{3} \rightarrow \textcircled{4}$

$$y'_p = u'y_1' + v'y_2' \quad -\textcircled{4}$$

$$y''_p = u'y''_1 + u'y'_1 + v'y''_2 + v'y'_2 \quad -\textcircled{5}$$

$\textcircled{2}, \textcircled{4}, \textcircled{5} \rightarrow \textcircled{1}$

$$(u'y''_1 + u'y'_1 + v'y''_2 + v'y'_2) + p(x)(u'y_1' + v'y_2') + q(x)(u'y_1 + v'y_2) = f$$

$$\Rightarrow u(x) \underline{[y''_1 + p y'_1 + q y_1]} + v(x) \underline{[y''_2 + p y'_2 + q y_2]} + (u'y_1' + v'y_2') = f(x)$$

$$\underline{u'y_1' + v'y_2'} = f(x) \quad -\textcircled{6}$$

second condition

From ② & ③

$$U' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2 \end{vmatrix}}{w} = \frac{-y_2 f}{w}$$

$$W' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{w} = \frac{y_1 f}{w}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

From ④ one can obtain U and W .

\Rightarrow Eq. ② y_p can be obtained!!

(여기서 $\omega = 1$)

$$y'' + 4y = \sec x$$

(i) Homogeneous equation

$$y'' + 4y = 0$$

$$y_1 = \cos 2x \quad) - \theta$$

$$y_2 = \sin 2x$$

$$\Rightarrow y_g = C_1 \cos 2x + C_2 \sin 2x - \theta$$

(ii) non-Homogeneous equation.

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$f(x) = \sec x = \frac{1}{\cos x}$$

$$\Rightarrow u' = \frac{-y_2 f}{2} = -\frac{1}{2} \sin 2x \cdot \frac{1}{\cos x} = -\sin x$$

$$v' = \frac{y_1 f}{2} = \frac{1}{2} \cos 2x \cdot \frac{1}{\cos x} = \frac{1}{2}(2\cos^2 x - 1) \frac{1}{\cos x} = \cos x - \frac{1}{2} \sec x$$

$$\Rightarrow u = \cos x.$$

$$v = \sin x - \int \sec x dx = \sin x - \frac{1}{2} \operatorname{Im}(\sec x + \tan x)$$

$$\Rightarrow y_p(x) = u y_1 + v y_2$$

$$= \cos x \cos 2x + \left(\sin x - \frac{1}{2} \operatorname{Im}(\sec x + \tan x) \right) \sin 2x$$

(iii) 일반해

$$y = y_g + y_p = C_1 \cos 2x + C_2 \sin 2x + \cos x \cos 2x + \left(\sin x - \frac{1}{2} \operatorname{Im}(\sec x + \tan x) \right) \sin 2x$$

X

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = x^2 + 1 \quad (x > 0)$$

(i) homogeneous Eq.

$$y'' - \frac{4}{x}y' + \frac{4}{x^2}y = 0 \quad \text{Euler Equation } (A = -4, B = 4)$$

Put

$$x = e^t \quad (t = \ln x)$$

Then

$$\frac{d^2y}{dt^2} - 5 \frac{dy}{dt} + 4y = 0 \quad \text{--- (2)}$$

\Rightarrow characteristic Eq.

$$\lambda^2 - 5\lambda + 4 = 0$$

$$(\lambda - 4)(\lambda - 1) = 0$$

$$y_1 = e^{4t} = e^{4\ln x} = x^4$$

$$y_2 = e^t = e^{\ln x} = x$$

$$y_g = C_1 y_1 + C_2 y_2 = C_1 x^4 + C_2 x$$

(ii) non-homogeneous Eq.

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} x^4 & x \\ 4x^3 & 1 \end{vmatrix} = -3x^4$$

$$\Rightarrow u' = -\frac{f}{W} = -\frac{x(x^2+1)}{-3x^4} = \frac{1}{3} \left(\frac{1}{x} + \frac{1}{x^3} \right)$$

$$u = \frac{1}{3} \left[\ln x - \frac{1}{2} \frac{1}{x^2} \right] = \frac{1}{3} \ln x - \frac{1}{6} \frac{1}{x^2}$$

$$\Rightarrow v' = \frac{y_1 f}{W} = \frac{x^4 \cdot (x^2+1)}{-3x^4} = -\frac{1}{3} (x^2 + 1)$$

$$v = -\frac{1}{3} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) = -\frac{1}{9}x^3 - \frac{1}{6}x^2$$

$$\begin{aligned} y_p &= u y_1 + v y_2 = \left(\frac{1}{2} \ln x - \frac{1}{6} \frac{1}{x^2} \right) x^4 + \left(-\frac{1}{9}x^3 - \frac{1}{6}x^2 \right) x \\ &= \frac{x^2}{3} \left[x^2 \ln x - \frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{2} \right] \end{aligned}$$

iii) general solution

$$\begin{aligned} y &= y_g + y_p \\ &= C_1 x^4 + C_2 x + \frac{x^2}{3} \left[x^2 \ln x - \frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{2} \right] \end{aligned}$$

[2] 미적분학 예제

p65

($m_1 = -1$)

$$y'' - 4y = 8x^2 - 2x$$

(i) Homogeneous Eq.

$$y'' - 4y = 0$$

$$y_g = C_1 e^{2x} + C_2 e^{-2x}$$

(ii) non-homogeneous Eq.

$$y_p = ax^2 + bx + c$$

$$y_p'' = 2a$$

$$\Rightarrow y_p'' - 4y_p = -4ax^2 - 4bx + (2a - 4c) = 8x^2 - 2x$$

$$a = -2, \quad b = \frac{1}{2}, \quad c = -1$$

$$\Rightarrow y_p = -2x^2 + \frac{1}{2}x - 1$$

(iii) general solution

$$y = y_g + y_p = C_1 e^{2x} + C_2 e^{-2x} - 2x^2 + \frac{1}{2}x - 1 \quad *$$

p66

(Ex 1.3)

$$y'' + 2y' - 3y = 4e^{2x}$$

(i) Homogeneous Eq.

$$y'' + 2y' - 3y = 0$$

$$y_h = C_1 e^{-3x} + C_2 e^x$$

(ii) non-Homogeneous Eq.

Put

$$y_p = a e^{2x}$$

$$y_p'' + 2y_p' - 3y_p = 5a e^{2x} = 4e^{2x}$$

$$\Rightarrow 5a = 4$$

$$a = \frac{4}{5}$$

$$\Rightarrow y_p = \frac{4}{5} e^{2x}$$

(iii) general solution

$$y = C_1 e^{-3x} + C_2 e^x + \frac{4}{5} e^{2x}$$

p66

(or $\omega = 1.4$)

$$y'' - 5y' + 6y = -3 \sin 2x$$

(i) Homogeneous Eq.

$$y'' - 5y' + 6y = 0$$

$$y = C_1 e^{2x} + C_2 e^{3x}$$

(ii) non-homogeneous Eq.

$$y_p = a \sin 2x + b \cos 2x$$

$$y'_p = 2a \cos 2x - 2b \sin 2x$$

$$y''_p = -4a \sin 2x - 4b \cos 2x$$

$$\Rightarrow y''_p - 5y'_p + 6y_p$$

$$= (2a + 10b) \sin 2x + (-10a + 2b) \cos 2x$$

$$= -3 \sin 2x$$

$$2a + 10b = -3$$

$$-10a + 2b = 0$$

$$\Rightarrow a = -\frac{3}{52}, \quad b = -\frac{15}{52}$$

$$\Rightarrow y_p = -\frac{3}{52} \sin 2x - \frac{15}{52} \cos 2x$$

(iii) general solution

$$y = C_1 e^{2x} + C_2 e^{3x} - \frac{3}{52} \sin 2x - \frac{15}{52} \cos 2x$$

P67 (문제 2.15) (7월 2.16)

$$y'' + 2y' - 3y = 8e^x$$

(i) homogeneous Eq.

$$y'' + 2y' - 3y = 0$$

$$y_g = C_1 e^{-3x} + C_2 e^x$$

(ii) non-homogeneous Eq.

(A) 설 II

$$y_g + y_p \quad y_p = Ae^x \Rightarrow y_g \text{의 충돌}$$

$$= C_1 e^{-3x} + C_2 e^x + Ae^x$$

$$= C_1 e^{-3x} + (C_2 + A)e^x$$

$$= C_1 e^{-3x} + C_2 e^x \quad \downarrow \quad (B) \text{ Another Try.}$$

$$y_p = Ax e^x$$

$$y'_p = A(x+1)e^x$$

$$y''_p = A(x+2)e^x$$

$$y''_p + 2y'_p - 3y_p = 4Ax e^x = 8e^x$$

$$A = 2$$

$$y_p = 2x e^x$$

(iii) general solution

$$y = y_g + y_p = C_1 e^{-3x} + C_2 e^x + 2x e^x$$

$$y_1 = e^{-3x} \quad y_2 = e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = 4e^{-2x}$$

$$f = 8e^x$$

$$u' = -\frac{y_2 f}{W} = -2e^{4x}$$

$$\Rightarrow u = -\frac{1}{2} e^{4x}$$

$$w' = \frac{y_1 f}{W} = 2$$

$$\Rightarrow w = 2x$$

$$y_p = u y_1 + w y_2 = -\frac{1}{2} e^{4x} + 2x e^x$$

$$y = y_g + y_p$$

$$= C_1 e^{-3x} + C_2 e^x - \frac{1}{2} e^x + 2x e^x$$

$$= C_1 e^{-3x} + (C_2 - \frac{1}{2}) e^x + 2x e^x$$

$$= C_1 e^{-3x} + C_2 e^x + 2x e^x \quad *$$

(ex 2.17)

$$y'' - 6y' + 9y = 5e^{3x}$$

(i) Homogeneous Eq

$$y'' - 6y' + 9y = 0$$

$$y_h = (C_1 + C_2 x) e^{3x}$$

$$(ii) \quad y_p = a x^2 e^{3x}$$

$$y'_p = a (3x^2 + 2x) e^{3x}$$

$$y''_p = a (9x^2 + 12x + 2) e^{3x}$$

$$y''_p - 6y'_p + 9y_p = 2a e^{3x} = 5e^{3x}$$

$$a = \frac{5}{2}$$

$$y_p = \frac{5}{2} x^2 e^{3x}$$

(iii) general solution

$$y = y_h + y_p = (C_1 + C_2 x) e^{3x} + \frac{5}{2} x^2 e^{3x}$$

$$y_1 = e^{3x} \quad y_2 = x e^{3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{6x}$$

$$f = 5e^{3x}$$

$$\Rightarrow u' = - \frac{y_2 f}{W} = -5x$$

$$u = -\frac{5}{2} x^2$$

$$\Rightarrow v' = \frac{y_1 f}{W} = 5$$

$$v = 5x$$

$$y_p = u y_1 + v y_2 = \frac{5}{2} x^2 e^{3x}$$

8 중점분석

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

Let

$$y_{p,1}'' + p(x)y_{p,1}' + q(x)y_{p,1} = f_1(x)$$

$$y_{p,2}'' + p(x)y_{p,2}' + q(x)y_{p,2} = f_2(x)$$

Put

$$y = y_{p,1} + y_{p,2}$$

$$\Rightarrow y'' + p(x)y' + q(x)y$$

$$= (y_{p,1}'' + y_{p,2}'') + p(x)(y_{p,1}' + y_{p,2}') + q(x)(y_{p,1} + y_{p,2})$$

$$f_2(x)$$

$$= \underline{(y_{p,1}'' + p(x)y_{p,1}' + q(x)y_{p,1})} + \underline{(y_{p,2}'' + p(x)y_{p,2}' + q(x)y_{p,2})}$$

$$= f_1(x) + f_2(x)$$

PTI

(01/11/2019)

$$y'' + 4y = x + 2e^{-2x}$$

(i) homogeneous equation

$$y = C_1 \sin 2x + C_2 \cos 2x$$

(ii) non-homogeneous Eq.

① First term

$$y'' + 4y = x$$

$$y_{p1} = \frac{x}{4}$$

② second term

$$y'' + 4y = 2e^{-2x}$$

$$y_{p2} = \frac{1}{4} e^{-2x}$$

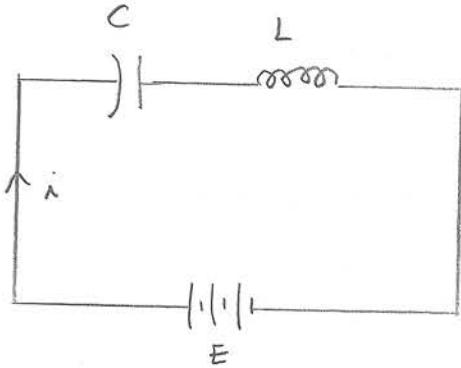
$$\Rightarrow y_p = y_{p1} + y_{p2} = \frac{1}{4} (x + e^{-2x})$$

(iii) general solution

$$y = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4} (x + e^{-2x})$$

卷之三

(i) 측류 전원과 LC 회로



$$E = \frac{q}{c} + L \frac{di}{dt} \quad) - \textcircled{1}$$

$$i = \frac{d\phi}{dt}$$

$$L \frac{d^2q}{dt^2} + \frac{q}{c} = E \quad \Leftrightarrow \quad L \rightarrow m \quad \text{Harmonic Oscillator}$$

$\frac{1}{c} \rightarrow K$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{1}{LC} q = \frac{E}{L}$$

$$\left(\begin{array}{l} q(0) = q_0 \\ \frac{dq}{dt}(0) = 0 \end{array} \right)$$

$$g(t) = A \cos \omega_0 t + B \sin \omega_0 t + C E \quad \left. \right\} - \textcircled{2}$$

$$\omega_0 = \frac{1}{\sqrt{Lc}}$$

$$A = g_0 - CE \quad) - ②$$

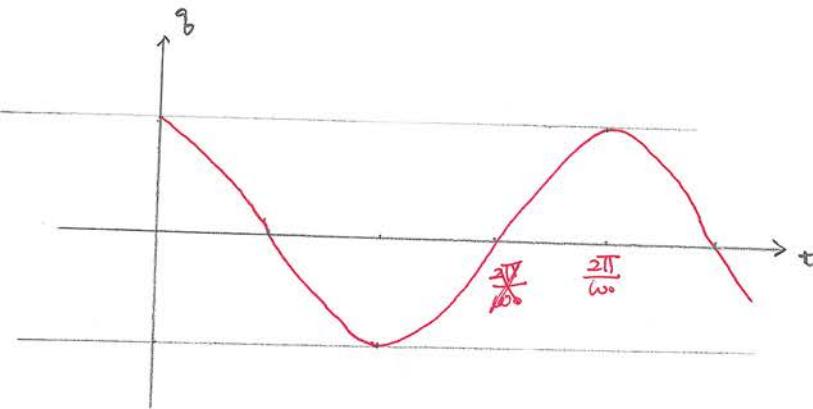
$$-\beta = 0$$

③ → ②

$$q(t) = (q_0 - CE) \cos \omega_0 t + CE. \quad \text{--- ④}$$

If $E = 0$,

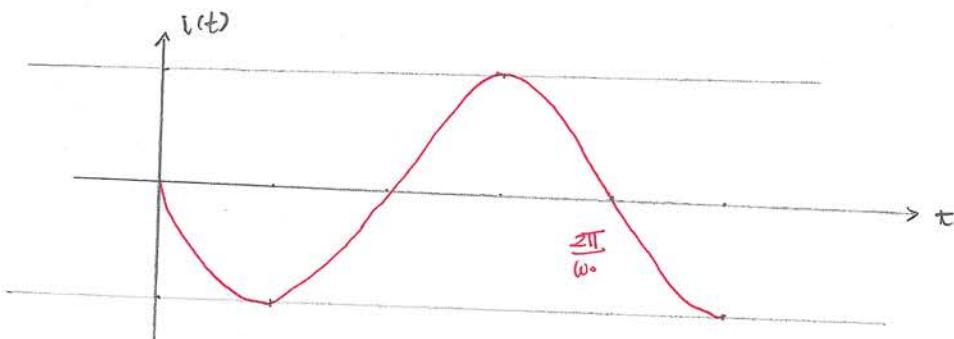
$$q(t) = q_0 \cos \omega_0 t$$



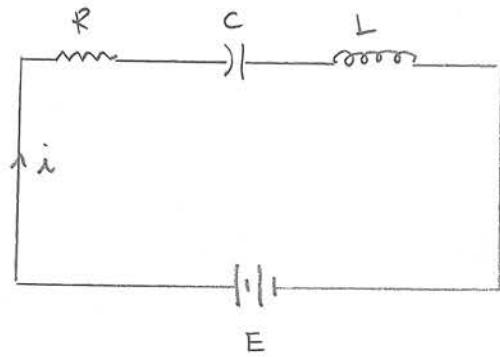
$$i = \frac{dq}{dt} = -(q_0 - CE) \omega_0 \sin \omega_0 t \quad \text{--- (4)}$$

If $E = 0$,

$$i = -q_0 \omega_0 \sin \omega_0 t$$



(ii) 직류 전流通과 RLC 회로



$$E = iR + \frac{q}{C} + L \frac{di}{dt}$$

}

$$i = \frac{dq}{dt} \quad -\textcircled{1}$$

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = \frac{E}{L} \quad -\textcircled{2}$$

non-homogeneous Eq.

(A) $E = 0$ case

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0 \quad -\textcircled{3} \Leftrightarrow m \frac{dx^2}{dt^2} + c \frac{dx}{dt} + \omega^2 x = 0$$

$m=1$

$$\cancel{\frac{R}{L}},$$

$$c = \frac{R}{L}$$

$$\omega^2 = \frac{1}{LC}$$

$$* \frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = 0$$

$$y = e^{-\frac{A}{2}x} \left[C_1 \sin \frac{\sqrt{4B-A^2}}{2} x + C_2 \cos \frac{\sqrt{4B-A^2}}{2} x \right] \quad \text{if } A^2 < 4B$$

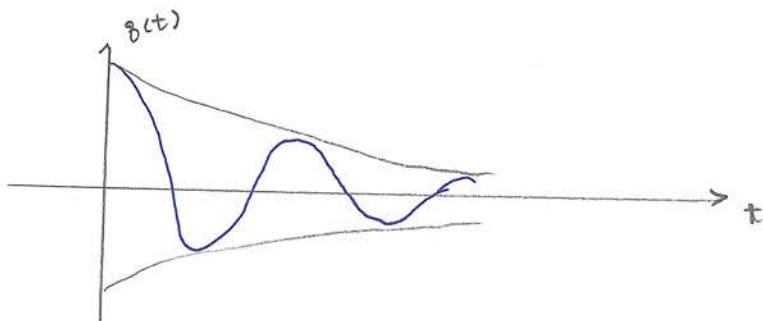
P.W

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad -\textcircled{4}$$

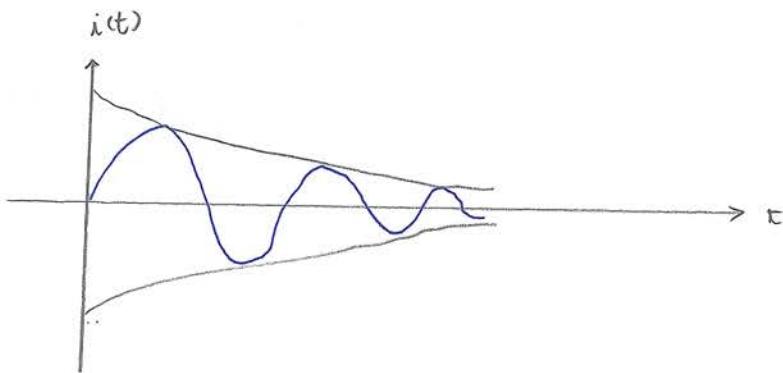
$$\text{If } \frac{R^2}{L} < 4\omega_0^2,$$

$$q(t) = C e^{-\frac{R}{2L}t} [C_1 \sin \omega t + C_2 \cos \omega t]$$

$$\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$$



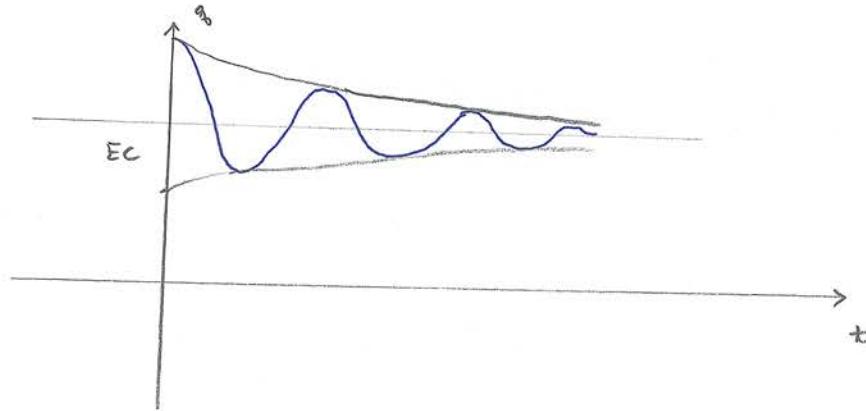
$$i(t) = \frac{dq}{dt} = C e^{-\frac{R}{2L}t} \left[(\omega C_1 - \frac{R}{2L} C_2) \cos \omega t - (\omega C_2 + \frac{R}{2L} C_1) \sin \omega t \right]$$



$$(B) E \neq 0 \text{ case } \left(\frac{R^2}{L} < 4\omega^2 \right)$$

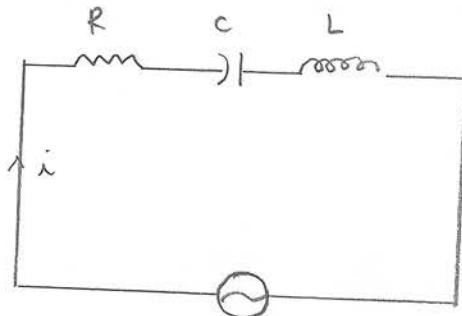
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$$q = e^{-\frac{R}{2L}t} \left[C_1 \sin \omega t + C_2 \cos \omega t \right] + E_C$$



$$i = \frac{dq}{dt} = e^{-\frac{R}{2L}t} \left[(\omega C_1 - \frac{R}{2L} C_2) \cos \omega t - (\omega C_2 + \frac{R}{2L} C_1) \sin \omega t \right]$$

(iii) 고주전원의 RLC 회로 ($\frac{R^2}{L^2} < \omega_0^2$)



$$E = V_0 \sin \omega t$$

$$\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \omega_0^2 \cdot q = \frac{V_0}{L} \sin \omega t \quad) - \textcircled{1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\frac{d}{dt} \quad \text{Eq. } \textcircled{1}$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \omega_0^2 i = \frac{V_0}{L} \cos \omega t \quad - \textcircled{2}$$

non-homogeneous Eq.

$$i = i_g + i_p$$

$$i_g = e^{-\frac{R}{2L}t} \left[\left(\tilde{\omega} C_1 - \frac{R}{2L} C_2 \right) \cos \tilde{\omega} t - \left(\tilde{\omega} C_2 + \frac{R}{2L} C_1 \right) \sin \tilde{\omega} t \right]$$

$$\tilde{\omega} = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}} \quad - \textcircled{3}$$

$$(t \gg 1 \text{ when } i_c \rightarrow 0)$$

$$i_p = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \sin(\omega t - \phi) \quad - \textcircled{4}$$

$\sqrt{R^2 + (X_L - X_C)^2}$: Impedance

$$\left. \begin{array}{l} X_L = \omega L \\ X_C = \frac{1}{\omega C} \\ X = X_L - X_C = \omega L - \frac{1}{\omega C} : \text{reactance} \\ \tan \phi = \frac{X}{R} \end{array} \right\} - \textcircled{5}$$

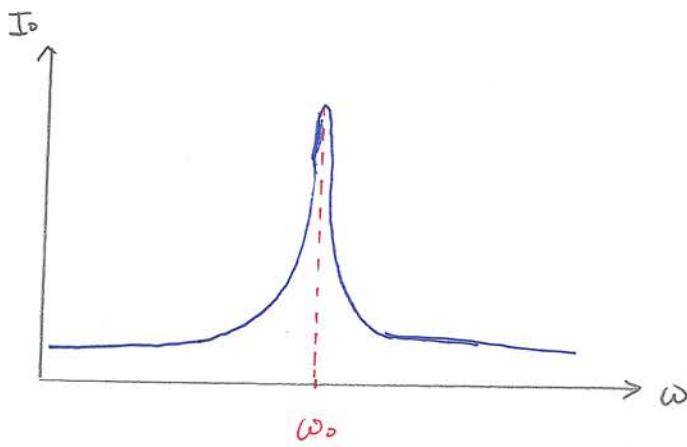
P_{WT}

$$I_o = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \quad - \textcircled{6}$$

$$i_p = I_o \sin(\omega t - \phi) \quad - \textcircled{7}$$

If $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$, $X_L - X_C = 0$.

$$I = \frac{V_0}{R} \Rightarrow \text{Resonance}$$



* ip $\frac{\partial \psi}{\partial z}$

Let

$$\left. \begin{aligned} i_p &= I_0 \sin(\omega t - \phi) \\ \frac{di_p}{dt} &= I_0 \omega \cos(\omega t - \phi) \\ \frac{d^2 i_p}{dt^2} &= -I_0 \omega^2 \sin(\omega t - \phi) \end{aligned} \right\} \quad - (\star 1)$$

($\star 1$) \rightarrow ②

$$\begin{aligned} \sin \omega t & \left[I_0 (\omega_0^2 - \omega^2) \cos \phi + \frac{R}{L} I_0 \omega \sin \phi \right] \\ + \cos \omega t & \left[\frac{R}{L} I_0 \omega \cos \phi - I_0 (\omega_0^2 - \omega^2) \sin \phi - \frac{V_0}{L} \omega \right] = 0 \end{aligned} \quad (\star 2)$$

From Eq. ($\star 2$)

$$(\omega_0^2 - \omega^2) \cos \phi + \frac{R}{L} \omega \sin \phi = 0 \quad (\star 3)$$

$$I_0 \left[\frac{R}{L} \omega \cos \phi - (\omega_0^2 - \omega^2) \sin \phi \right] = \frac{V_0}{L} \omega \quad (\star 4)$$

From ($\star 3$)

$$\tan \phi = - \frac{\omega_0^2 - \omega^2}{\frac{R}{L} \omega}$$

$$= - \frac{1}{R} \left[\frac{L}{\omega} \frac{1}{LC} - \frac{L}{\omega} \omega^2 \right]$$

$$= \frac{1}{R} \left[\omega L - \frac{1}{\omega C} \right] \quad (\star 5)$$

Let

$$\begin{aligned} X_L &= \omega L \\ X_C &= \frac{1}{\omega C} \end{aligned}$$

(*)

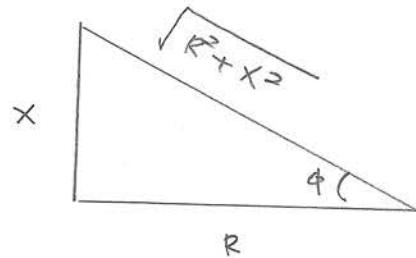
and

$$X = X_L - X_C = \omega L - \frac{1}{\omega C} \quad (*)$$

Reactance

From Eq. (*) becomes

$$\tan \phi = \frac{X}{R} \quad (*)$$



Eg. (*) becomes

$$I_0 \left[\frac{R}{L} \omega \cdot \frac{R}{\sqrt{R^2 + X^2}} - (\omega_0^2 - \omega^2) \frac{X}{\sqrt{R^2 + X^2}} \right] = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \left[\frac{R^2}{L} \omega - (\omega_0^2 - \omega^2) X \right] = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \frac{\omega}{L} \left[R^2 - \frac{L}{\omega} (\omega_0^2 - \omega^2) X \right] = \frac{V_0}{L} \omega$$

$$\frac{L}{\omega} \left[\frac{1}{L} - \omega^2 \right]$$

$$= \frac{1}{\omega C} - \omega L$$

$$= -(X_L - X_C)$$

$$= -X$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \frac{\omega}{L} (R^2 + X^2) = \frac{V_0}{L} \omega$$

$$\Rightarrow I_o \frac{\omega}{L} \sqrt{R^2 + x^2} = \frac{V_o}{L} \omega$$

$$\Rightarrow I_o = \frac{V_o}{\sqrt{R^2 + x^2}} = \frac{V_o}{\sqrt{R^2 + (x_L - x_c)^2}} *$$