CH 7 벡터 해석 (Vector analysis)

※ 벡터함수 와 곡률 (curvature)

$$\vec{F}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$$: vector function

$$\vec{F}'(t) = \frac{dF}{dt}(t) \equiv x'(t)\,\hat{x} + y'(t)\,\hat{y} + z'(t)\,\hat{z}$$

: derivative of vector function

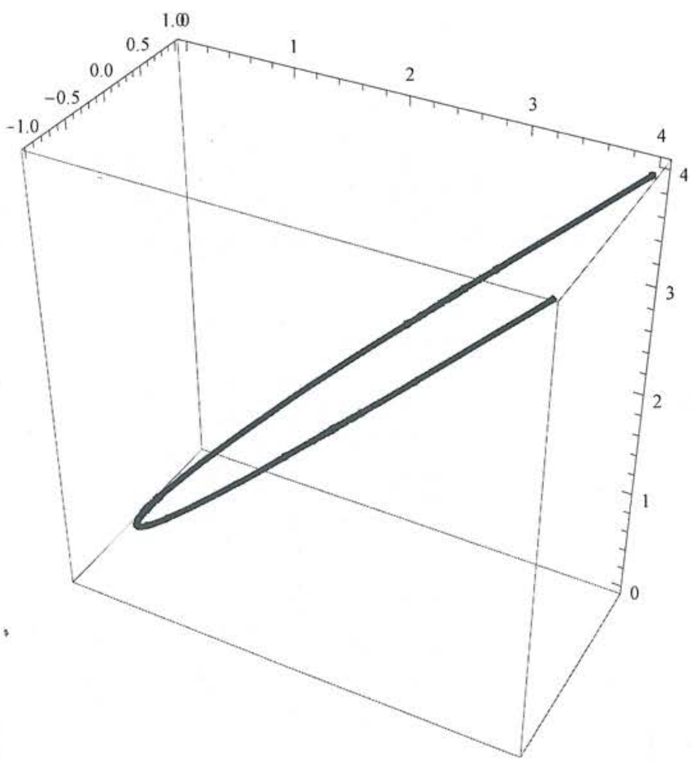
p271

(예제 7.1)

$$\vec{H}(t) = t^2\,\hat{x} + \sin t\,\hat{y} - t^2\,\hat{z}$$

```
In[7]:= ParametricPlot3D[{t^2, Sin[t], t^2}, {t, -2, 2},
          Boxed → True, PlotStyle → {Thickness[0.01], Red}]
```

Out[7]=



$$H'(t) = 2t\,\hat{x} + \cos t\,\hat{y} - 2t\,\hat{z}$$

※

* 구간 $a \le t \le b$ 에서 $\vec{F}(t)$ 의 length: L

$$L = \int dL$$

$$= \int \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$\Rightarrow \boxed{L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_a^b \|\vec{F}'(t)\|\, dt}$$

p273

(예제 7.2)

$$\vec{F}(t) = \cos t\,\hat{x} + \sin t\,\hat{y} + \frac{t}{3}\hat{z}$$

$$\vec{F}'(t) = -\sin t\,\hat{x} + \cos t\,\hat{y} + \frac{1}{3}\hat{z}$$

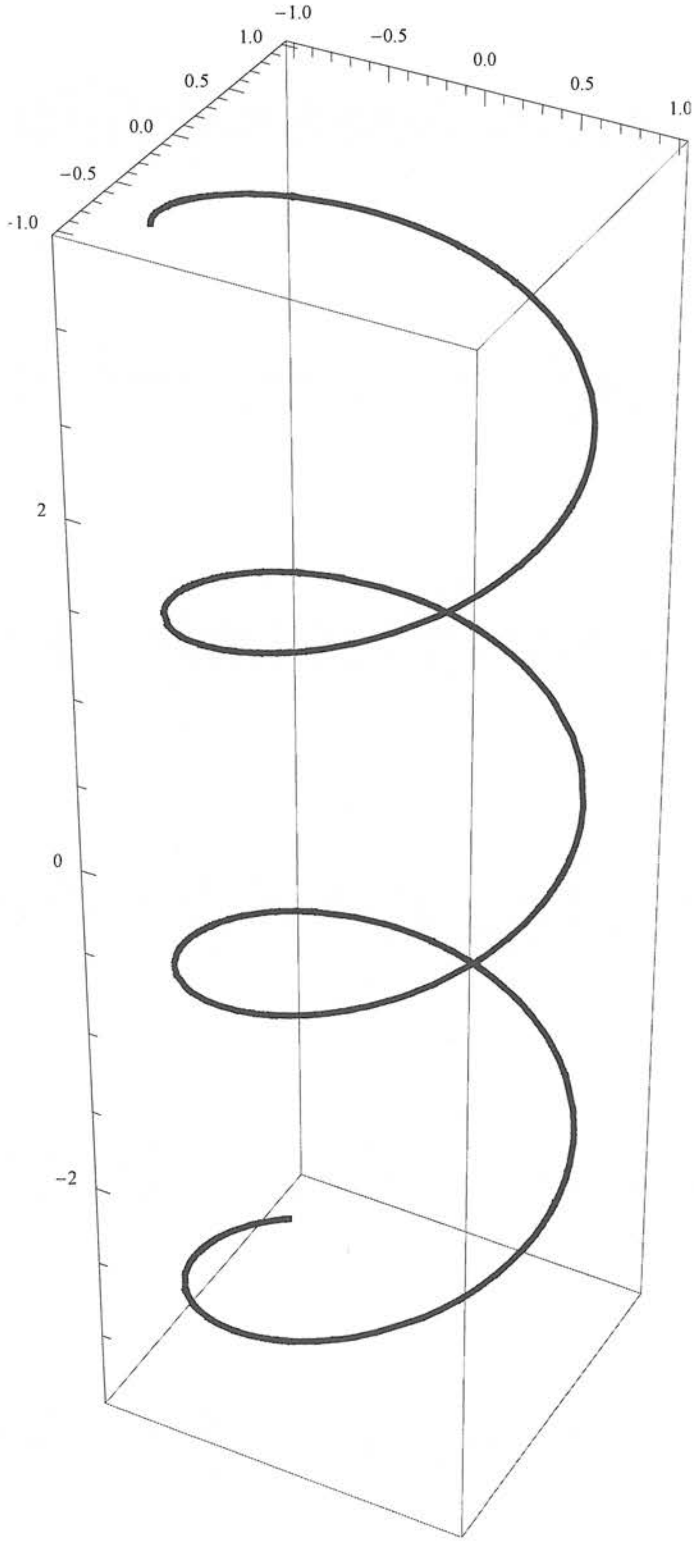
$$\|\vec{F}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \frac{1}{9}} = \frac{\sqrt{10}}{3}$$

$-4\pi \le t \le 4\pi$ 사이의 length

$$L = \int_{-4\pi}^{4\pi} \|\vec{F}'(t)\|\, dt = \frac{8\pi\sqrt{10}}{3}$$

```
In[15]:= ParametricPlot3D [{Cos [t], Sin[t], t / 3},
   {t, -10, 10}, Boxed → True, PlotStyle → {Thickness [0.01], Red}]
```

Out[15]=



* 단위 접선 vector (unit tangent vector)

$$\boxed{\frac{ds}{dt} = \|\vec{F}'(t)\|}$$

$$\vec{F}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$$

$$s(t) = \int_a^t \|\vec{F}'(\xi)\|\, d\xi$$ $t=a$ 부터 $t=t$ 까지의 거리 (length)

: monotonically increasing function

$\Rightarrow \quad t = t(s)$

$$\Rightarrow \quad \underline{G(s) = \vec{F}(t(s)) = x(t(s))\,\hat{x} + y(t(s))\,\hat{y} + z(t(s))\,\hat{z}}$$

unit tangent vector 를 갖고

$\vec{F}(t)$ 와 같은 궤적을 그리는 vector

pf)

$$\vec{G}'(s) = \frac{d}{ds}\vec{F}(t(s))$$

$$= \frac{dt}{ds}\frac{d}{dt}\vec{F}(t) \quad -①$$

Since

$$s(t) \equiv \int_a^t \|\vec{F}'(\xi)\|\, d\xi,$$

$$\frac{ds}{dt} = \|\vec{F}'(t)\| \quad -②$$

② → ①

$$\vec{G}'(s) = \frac{1}{\|\vec{F}'(t)\|}\vec{F}'(t)$$

$$\Rightarrow \|\vec{G}'(s)\| = 1$$

※

1. $\frac{d}{dt}\left(\vec{F}(t)+\vec{G}(t)\right)=\vec{F}'(t)+\vec{G}'(t)$

2. $\frac{d}{dt}\left[f(t)\,\vec{F}(t)\right]=f'(t)\,\vec{F}(t)+f(t)\,\vec{F}'(t)$

3. $\frac{d}{dt}\left[\vec{F}(t)\cdot\vec{G}(t)\right]=\vec{F}'(t)\cdot\vec{G}(t)+\vec{F}(t)\cdot\vec{G}'(t)$

4. $\frac{d}{dt}\left[\vec{F}(t)\times\vec{G}(t)\right]=\vec{F}'(t)\times\vec{G}(t)+\vec{F}(t)\times\vec{G}'(t)$

5 $\frac{d}{dt}\,\vec{F}(f(t))=f'(t)\,\vec{F}'(f(t))$

(pf)

③ Let $\vec{F}(t)=f_1(t)\,\hat{x}+f_2(t)\,\hat{y}+f_3(t)\,\hat{z}$

$\vec{G}(t)=g_1(t)\,\hat{x}+g_2(t)\,\hat{y}+g_3(t)\,\hat{z}$

$\Rightarrow\ \vec{F}(t)\cdot\vec{G}(t)=f_1g_1+f_2g_2+f_3g_3$

$$\frac{d}{dt}\left[\vec{F}(t)\cdot\vec{G}(t)\right]=f_1'g_1+f_1g_1'+f_2'g_2+f_2g_2'+f_3'g_3+f_3g_3'$$

$$=\underbrace{(f_1'g_1+f_2'g_2+f_3'g_3)}_{\vec{F}'(t)\cdot\vec{G}(t)}+\underbrace{(f_1g_1'+f_2g_2'+f_3g_3')}_{\vec{F}(t)\cdot\vec{G}'(t)}$$

$$=\vec{F}'(t)\cdot\vec{G}(t)+\vec{F}(t)\cdot\vec{G}'(t)$$

(예제 7.3)

$$\vec{F}(t) = \cos t\ \hat{x} + \sin t\ \hat{y} + \frac{t}{3}\hat{z} \qquad -4\pi \le t \le 4\pi$$

$$\|\vec{F}'(t)\| = \frac{\sqrt{10}}{3}$$

$$s(t) = \int_{-4\pi}^{t} \|\vec{F}'(\xi)\|\, d\xi$$

$$= \frac{\sqrt{10}}{3}(t + 4\pi)$$

$$\Rightarrow \quad t = \frac{3}{\sqrt{10}}s - 4\pi$$

$$\Rightarrow \vec{G}(s) = \cos\left(\frac{3}{\sqrt{10}}s - 4\pi\right)\hat{x} + \sin\left(\frac{3}{\sqrt{10}}s - 4\pi\right)\hat{y} + \left(\frac{s}{\sqrt{10}} - \frac{4\pi}{3}\right)\hat{z}$$

$$= \cos\left(\frac{3}{\sqrt{10}}s\right)\hat{x} + \sin\left(\frac{3}{\sqrt{10}}s\right)\hat{y} + \left(\frac{s}{\sqrt{10}} - \frac{4\pi}{3}\right)\hat{z}$$

$$\vec{G}'(s) = -\frac{3}{\sqrt{10}}\sin\left(\frac{3}{\sqrt{10}}s\right)\hat{x} + \frac{3}{\sqrt{10}}\cos\left(\frac{3}{\sqrt{10}}s\right)\hat{y} + \frac{1}{\sqrt{10}}\hat{z}$$

$$\|\vec{G}'(s)\|^2 = \frac{9}{10} + \frac{1}{10} = 1$$

※

$$\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$$

If $\vec{r}(t)$ is position vector (위치 vector) and t is time,

$\vec{v}(t) = \vec{r}\,'(t)$: velocity vector (속도 벡터)

$v(t) = \|\vec{v}(t)\|$: 속력

$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$: acceleration vector (가속도 벡터)

p-76

(예제 7.4)

$$\vec{r}(t) = \sin t\,\hat{x} + 2e^{-t}\,\hat{y} + t^2\,\hat{z}$$

$$\vec{v}(t) = \cos t\,\hat{x} - 2e^{-t}\,\hat{y} + 2t\,\hat{z}$$

$$v(t) = \sqrt{\cos^2 t + 4e^{-2t} + 4t^2}$$

$$\vec{a}(t) = -\sin t\,\hat{x} + 2e^{-t}\,\hat{y} + 2\,\hat{z}$$ ※

정의 : 곡률 (curvature)

Let

$$\vec{T}(t) = \frac{1}{v(t)} \vec{v}(t)$$ 단위 접선 벡터

$$\Rightarrow \quad \kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\|$$: curvature

(의미) $\vec{r}(t)$을 따라 $\vec{T}(t)$의 방향이 많이 바뀌면 curvature 가 크다.

예를 들어 직선에서는 $\kappa(s) = 0$ 이다.

(note) $\kappa(s)$를 쉽게 구하는 법

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \frac{dt}{ds} = \frac{1}{\|\vec{r}'(t)\|} \vec{T}'(t)$$

$$\Rightarrow \boxed{\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}}$$

p278

(예제 7.5)

$$\vec{r}(t) = (a+bt)\hat{x} + (c+dt)\hat{y} + (e+ht)\hat{z} \quad \Rightarrow \text{직선} \quad \kappa(s) = 0$$

$$\vec{v}(t) = \vec{r}'(t) = b\hat{x} + d\hat{y} + h\hat{z}$$

$$v(t) = \sqrt{b^2 + d^2 + h^2} = \|\vec{r}'(t)\|$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{1}{\sqrt{b^2+d^2+h^2}} (b\hat{x} + d\hat{y} + h\hat{z})$$

$$\vec{T}'(t) = 0$$

$$\Rightarrow \|\vec{T}(t)\| = 0$$

$$\kappa(t) = 0$$

#

p.778

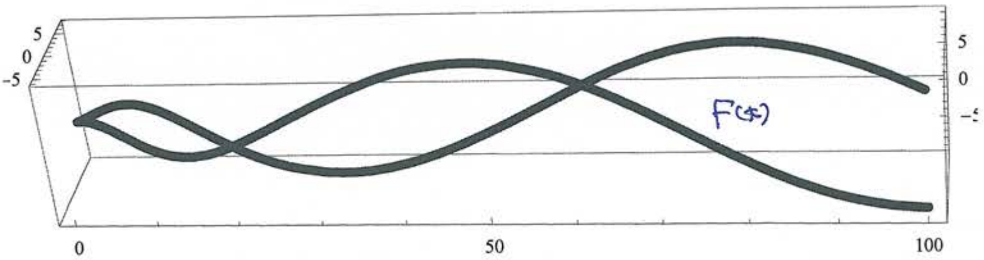
(예제 7.6)

$$\vec{r}(t) = [\cos t + t\sin t]\,\hat{x} + [\sin t - t\cos t]\,\hat{y} + t^2\,\hat{z}$$

$$\vec{v}(t) = t\cos t\,\hat{x} + t\sin t\,\hat{y} + 2t\,\hat{z}$$

```
In[7]:= ParametricPlot3D[{Cos[t] + t Sin[t], Sin[t] - t Cos[t], t^2},
          {t, -10, 10}, PlotStyle → {Thickness[0.01], Red}]
```

Out[7]=



$$v(t) = \|\vec{v}(t)\| = \sqrt{5}\,t$$

$$\vec{T}(t) = \frac{\vec{v}(t)}{v(t)} = \frac{1}{\sqrt{5}}\cos t\,\hat{x} + \frac{1}{\sqrt{5}}\sin t\,\hat{y} + \frac{2}{\sqrt{5}}\,\hat{z}$$

$$\vec{T}'(t) = -\frac{1}{\sqrt{5}}\sin t\,\hat{x} + \frac{1}{\sqrt{5}}\cos t\,\hat{y}$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{5}}$$

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{v(t)} = \frac{1}{5t}$$

※

정의: unit normal vector (단위 법선 벡터)

$$\vec{N}(s) = \frac{1}{\kappa(s)} \vec{T}'(s)$$

note) Since $\kappa(s) = \|\vec{T}'(s)\|$, $\|\vec{N}(s)\| = 1$

note) Since $\|\vec{T}(s)\|^2 = \vec{T}(s) \cdot \vec{T}(s) = 1$,

$$\vec{T}(s) \cdot \vec{T}'(s) = 0$$

$$\Rightarrow \vec{N}(s) \perp \vec{T}(s)$$

p281

(예제 9.9)

$$\vec{F}(t) = [\cos t + t \sin t]\,\hat{x} + [\sin t - t\cos t]\,\hat{y} + t^2 \hat{z} \qquad (t > 0)$$

$$\vec{V}(t) = t\cos t\,\hat{x} + t \sin t\,\hat{y} + 2t\,\hat{z}$$

$$v(t) = \|\vec{V}(t)\| = \sqrt{5}\,t$$

$$s = \int_0^t v(\xi)\,d\xi = \frac{\sqrt{5}}{2} t^2$$

$$\left.\begin{aligned} \Rightarrow t &= \sqrt{\frac{2}{\sqrt{5}}}\,\sqrt{s} = \alpha\sqrt{s} \\ \alpha &= \sqrt{\frac{2}{\sqrt{5}}} \end{aligned}\right) \quad -①$$

$$\vec{G}(s) = \vec{F}(t(s)) = [\cos(\alpha\sqrt{s}) + \alpha\sqrt{s}\,\sin(\alpha\sqrt{s})]\,\hat{x} + [\sin(\alpha\sqrt{s}) - \alpha\sqrt{s}\,\cos(\alpha\sqrt{s})]\,\hat{y} + \alpha^2 s\,\hat{z} \quad -②$$

~~unit tangent vector~~

$$\vec{T}(s) = \vec{G}'(s) = \frac{\alpha^2}{2}\cos(\alpha\sqrt{s})\,\hat{x} + \frac{\alpha^2}{2}\sin(\alpha\sqrt{s})\,\hat{y} + \alpha^2\,\hat{z} \quad -③$$

note)

$\vec{T}(s)$ can be obtained by $\left.\frac{\vec{V}(t)}{v(t)}\right|_{t\to\alpha\sqrt{s}}$.

note)

$$\|\vec{T}(s)\|^2 = \frac{5}{4}\alpha^4 = 1$$

$$\vec{T}'(s) = -\frac{\alpha^3}{4\sqrt{s}} \sin(\alpha\sqrt{s})\,\hat{x} + \frac{\alpha^3}{4\sqrt{s}} \cos(\alpha\sqrt{s})\,\hat{y} \quad - ④$$

$$\kappa(s) = \|\vec{T}'(s)\| = \frac{\alpha^3}{4\sqrt{s}} = \frac{1}{5^{\frac{3}{4}}\sqrt{2s}} \quad - ⑤$$

note)

$$\kappa(t) = \frac{1}{5t}$$

$$\Rightarrow \kappa(s) = \frac{1}{5\,\alpha\sqrt{s}} = \frac{1}{5^{\frac{3}{4}}\sqrt{2s}}$$

Therefore

$$\vec{N}(s) = \frac{1}{\kappa(s)}\,\vec{T}'(s) = -\sin(\alpha\sqrt{s})\,\hat{x} + \cos(\alpha\sqrt{s})\,\hat{y} \quad - ⑥$$

note)

$$\|\vec{N}(s)\| = 1$$

$$\vec{N}(s)\cdot\vec{T}(s) = 0$$

※

p282

정리 7.1

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{T} + v^2 \kappa \vec{N}}$$

(pf) Since $\vec{T}(t) = \dfrac{\vec{V}(t)}{v(t)}$,

$$\vec{V}(t) = v(t)\vec{T}(t) \quad - ①$$

Differentiate with respect to t :

$$\vec{a}(t) = \frac{d}{dt}\left[v(t)\vec{T}(t)\right]$$

$$= \frac{dv}{dt}\vec{T}(t) + v(t)\frac{d\vec{T}}{dt}$$

$$= \frac{dv}{dt}\vec{T}(t) + v(t)\frac{ds}{dt}\frac{d}{ds}\vec{T} \qquad \frac{ds}{dt} = v$$

$$= \frac{dv}{dt}\vec{T}(t) + v^2 \underline{\frac{d\vec{T}}{ds}} \quad \kappa(s)\vec{N}(s)$$

$$= \frac{dv}{dt}\vec{T} + v^2 \kappa \vec{N} \qquad \divideontimes$$

note) $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$$a_T = \frac{dv}{dt}$$

$$a_N = v^2\kappa$$

$$\|\vec{a}\|^2 = a_T^2 + a_N^2$$

p283

(예제 7.8)

$$\vec{r}(t) = (\cos t + t\sin t)\hat{x} + (\sin t - t\cos t)\hat{y} + t^2\hat{z} \qquad (t>0)$$

$$\vec{v}(t) = t\cos t\,\hat{x} + t\sin t\,\hat{y} + 2t\,\hat{z}$$

$$v(t) = \sqrt{5}\,t$$

$$a_T = \frac{dv}{dt} = \sqrt{5} \quad -①$$

$$\vec{a}(t) = (\cos t - t\sin t)\hat{x} + (\sin t + t\cos t)\hat{y} + 2\hat{z} \quad -②$$

$$\|\vec{a}(t)\| = \sqrt{5+t^2} \quad -③$$

Since $\|\vec{a}(t)\|^2 = a_T^2 + a_N^2$,

$$a_N = t = \kappa v^2 \quad -④$$

$$\kappa = \frac{t}{v^2} = \frac{1}{5t} \quad -⑤$$

$$\vec{a} = \sqrt{5}\,\vec{T} + t\,\vec{N} \quad -⑥$$

$$\vec{T} = \frac{1}{v}\vec{v} = \frac{1}{\sqrt{5}}\cos t\,\hat{x} + \frac{1}{\sqrt{5}}\sin t\,\hat{y} + \frac{2}{\sqrt{5}}\hat{z} \quad -⑦$$

$$\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$$

$$= \frac{1}{\kappa}\frac{dt}{ds}\frac{d\vec{T}}{dt} \qquad \frac{ds}{dt} = v$$

$$= \frac{1}{\kappa v}\frac{d\vec{T}}{dt}$$

$$= \sqrt{5}\left[-\frac{1}{\sqrt{5}}\sin t\,\hat{x} + \frac{1}{\sqrt{5}}\cos t\,\hat{y}\right]$$

$$= -\sin t\,\hat{x} + \cos t\,\hat{y} \quad -⑧$$

//

p284

Theorem 7.2

$$\boxed{\kappa = \frac{\|\vec{F}' \times \vec{F}''\|}{\|\vec{F}'\|^3}}$$

pf)

$$\vec{F}''(t) = \vec{a}(t) = \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N} \quad \Big) \; - ①$$

$$\vec{F}' = v\vec{T}$$

$$\Rightarrow \vec{F}' \times \vec{F}'' = v\frac{dv}{dt}\underbrace{\vec{T}\times\vec{T}}_{=0} + \kappa v^3 \vec{T}\times\vec{N} = \kappa v^3 \vec{T}\times\vec{N}$$

$$\Rightarrow \|\vec{F}' \times \vec{F}''\| = \kappa v^3 \underbrace{\|\vec{T}\times\vec{N}\|}_{=1} = \kappa v^3$$

$$\Rightarrow \quad \kappa = \frac{\|\vec{F}' \times \vec{F}''\|}{v^3} = \frac{\|\vec{F}' \times \vec{F}''\|}{\|\vec{F}'\|^3}$$

※

p284

(예제 7.9)

$$\vec{F}(t) = t^2\hat{x} - t^3\hat{y} + t\hat{z}$$

$$\vec{F}'(t) = 2t\hat{x} - 3t^2\hat{y} + \hat{z}$$

$$\vec{F}''(t) = 2\hat{x} - 6t\hat{y}$$

$$\vec{F}'(t) \times \vec{F}''(t) = 6t\hat{x} + 2\hat{y} - 6t^2\hat{z}$$

$$\|\vec{F}'(t) \times \vec{F}''(t)\| = \sqrt{4 + 36t^2 + 36t^4}$$

$$\|\vec{F}'(t)\| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\Rightarrow \kappa = \frac{\|\vec{F}' \times \vec{F}''\|}{\|\vec{F}'(t)\|^3} = \frac{\sqrt{4 + 36t^2 + 36t^4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

※

p285

definition: binormal vector (종법선 vector)

$$\vec{B} = \vec{T} \times \vec{N}$$

* Frenet formula

Since $\vec{N} = \frac{1}{\kappa}\frac{d\vec{T}}{ds}$,

$$\frac{d\vec{T}}{ds} = \kappa \vec{N} \quad - ①$$

Let

$$\frac{d\vec{N}}{ds} = \alpha \vec{T} + \tau \vec{B} \quad - ②$$

From $\vec{B} = \vec{T} \times \vec{N}$,

$$\frac{d\vec{B}}{ds} = \underbrace{\frac{d\vec{T}}{ds} \times \vec{N}}_{\kappa \vec{N} \times \vec{N} = 0} + \vec{T} \times \frac{d\vec{N}}{ds}$$

$\vec{A} \times (\vec{B} \times \vec{C})$

$= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

$$= \vec{T} \times \frac{d\vec{N}}{ds}$$

$$= \vec{T} \times [\alpha \vec{T} + \tau \vec{B}]$$

$$= \alpha \underbrace{\vec{T} \times \vec{T}}_{=0} + \tau \underbrace{\vec{T} \times \vec{B}}_{= -\vec{N}}$$

$$= -\tau \vec{N}$$

Thus, we get

$$\frac{d\vec{B}}{ds} = -\tau \vec{N} \quad - ③$$

From $\vec{B} = \vec{T} \times \vec{N}$,

$$\vec{N} = \vec{B} \times \vec{T} \quad - ④$$

Thus

$$\frac{d\vec{N}}{ds} = \underbrace{\frac{d\vec{B}}{ds}}_{=-\tau\vec{N}} \times \vec{T} + \vec{B} \times \underbrace{\frac{d\vec{T}}{ds}}_{\kappa\vec{N}}$$

$$= -\tau \underbrace{\vec{N} \times \vec{T}}_{-\vec{B}} + \kappa \underbrace{\vec{B} \times \vec{N}}_{-\vec{T}}$$

$$= -\kappa \vec{T} + \tau \vec{B} \quad - ⑤$$

From ①, ③ and ⑤

$$\boxed{\begin{aligned} \frac{d\vec{T}}{ds} &= \kappa \vec{N} \\ \frac{d\vec{N}}{ds} &= -\kappa \vec{T} + \tau \vec{B} \\ \frac{d\vec{B}}{ds} &= -\tau \vec{N} \end{aligned}}$$

Frenet formula

$\tau(s)$: torsion (비틀림)

$\boxed{\text{p286 : 10}}$

5 벡터장 (vector field) and del-operator

Vector field

$$\vec{G}(x,y,z) = g_1(x,y,z)\,\hat{x} + g_2(x,y,z)\,\hat{y} + g_3(x,y,z)\,\hat{z}$$

: 3-dimensional vector field

$$\vec{F}(x,y) = f_1(x,y)\,\hat{x} + f_2(x,y)\,\hat{y}$$

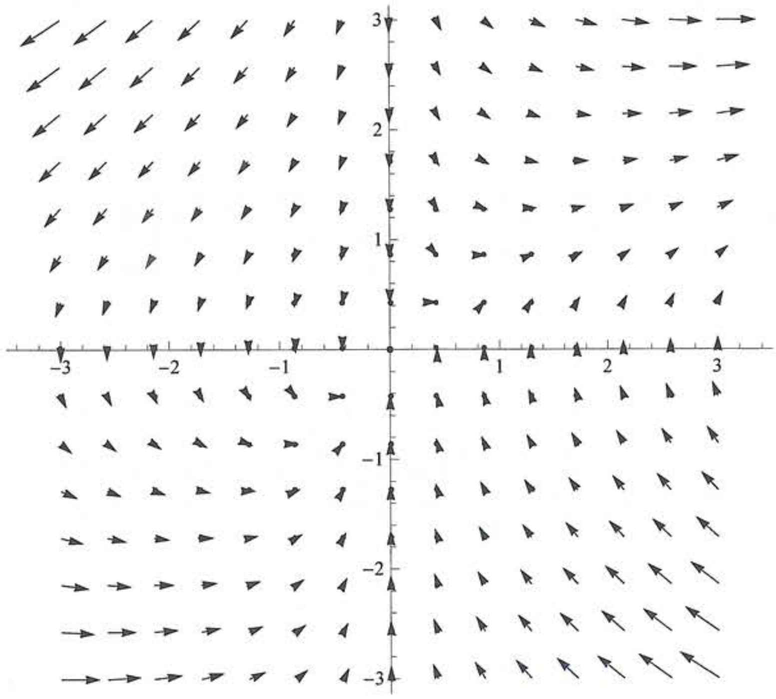
: 2-dimensional vector field

(Ex) $\vec{G}(x,y) = xy\,\hat{x} + (x-y)\,\hat{y}$

In[11]:=

```
VectorFieldPlot[{x y, x - y}, {x, -3, 3}, {y, -3, 3}, Axes → True]
```

Out[11]=



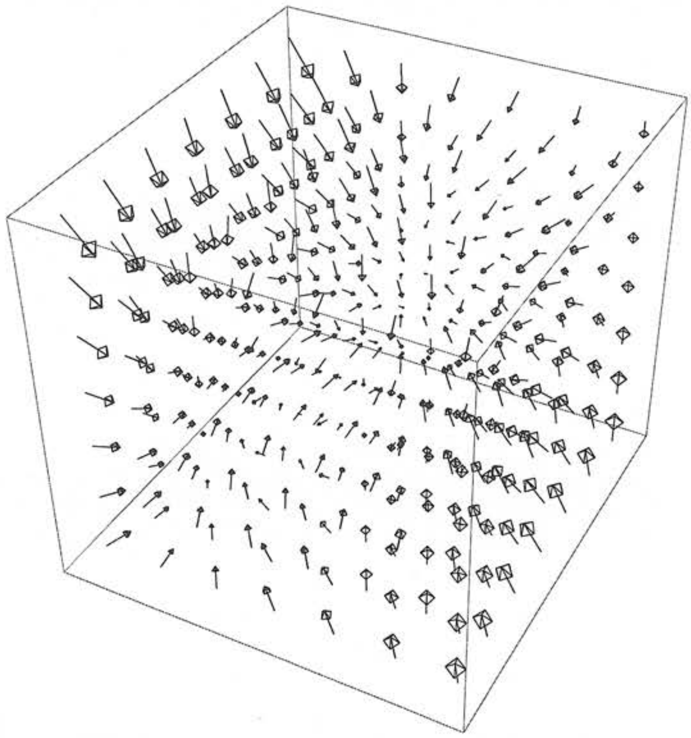
※

(Ex)

$$\vec{F}(x,y,z) = \cos(x+y)\,\hat{x} - x\,\hat{y} + (x-z)\,\hat{z}$$

```
In[17]:= VectorFieldPlot3D[{Cos[x + y], -x, x - z},
          {x, -3, 3}, {y, -3, 3}, {z, -3, 3}, VectorHeads → True]
```

Out[17]=



*

* Partial derivative

Let $\vec{F}(x,y,z) = f_1(x,y,z)\,\hat{x} + f_2(x,y,z)\,\hat{y} + f_3(x,y,z)\,\hat{z}$

Then

$$\frac{\partial \vec{F}}{\partial x} \equiv \vec{F}_x = \frac{\partial f_1}{\partial x}\hat{x} + \frac{\partial f_2}{\partial x}\hat{y} + \frac{\partial f_3}{\partial x}\hat{z}$$

$$\frac{\partial \vec{F}}{\partial y} \equiv \vec{F}_y = \frac{\partial f_1}{\partial y}\hat{x} + \frac{\partial f_2}{\partial y}\hat{y} + \frac{\partial f_3}{\partial y}\hat{z}$$

$$\frac{\partial \vec{F}}{\partial z} \equiv \vec{F}_z = \frac{\partial f_1}{\partial z}\hat{x} + \frac{\partial f_2}{\partial z}\hat{y} + \frac{\partial f_3}{\partial z}\hat{z}$$

streamline (흐름선)

Let us consider a set of curves C.

If the tangent direction of the curves in C is proportional to a vector field $\vec{F}(x, y, z)$, we call those curves by streamlines of vector field $\vec{F}(x, y, z)$

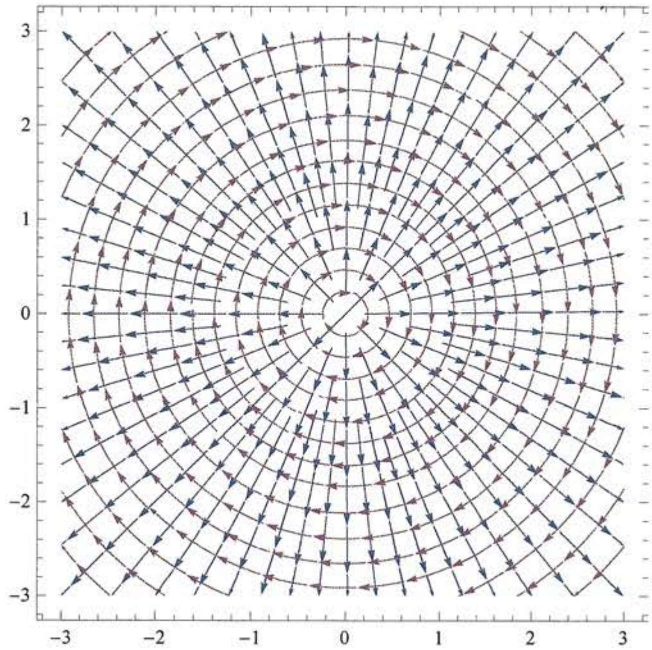
Ex)

note)

streamline (흐름선) = flow line (유선) = line of force (역선)

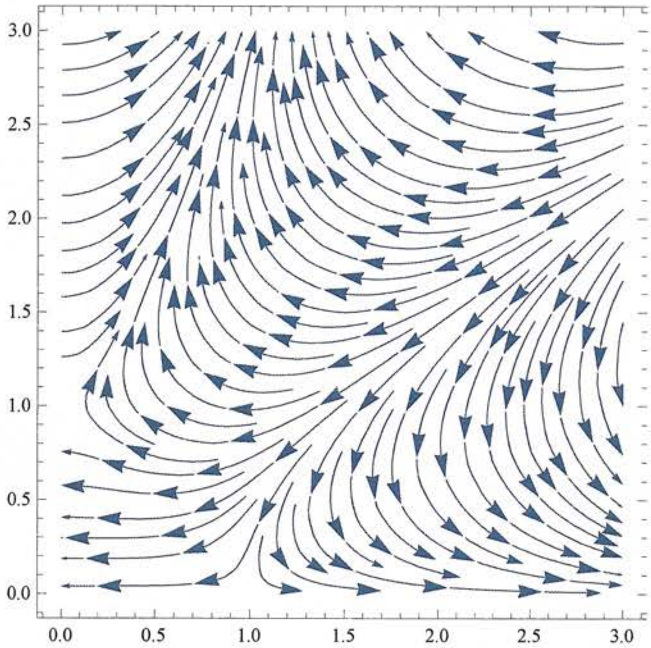
In[1]:= `StreamPlot[{{x, y}, {y, -x}}, {x, -3, 3}, {y, -3, 3}]`

Out[1]=



In[2]:= `StreamPlot[Evaluate@{Re[(x + I * y) ^ 4 - 1], -Im[(x + I * y) ^ 4 - 1]},`
`{x, 0, 3}, {y, 0, 3}, StreamScale → Large]`

Out[2]=



* 주어진 vector Field 의 streamline 구하는 법

Consider a vector field

$$\vec{F}(x,y,z) = f(x,y,z)\,\hat{x} + g(x,y,z)\,\hat{y} + h(x,y,z)\,\hat{z} \quad — ①$$

Consider a curve which is parametrized by

$$x = x(\xi), \quad y = y(\xi), \quad z = z(\xi) \quad — ②$$

Then the position vector of the curve is

$$\vec{R}(\xi) = x(\xi)\,\hat{x} + y(\xi)\,\hat{y} + z(\xi)\,\hat{z} \quad — ③$$

Therefore, its tangential direction is

$$\vec{R}'(\xi) = \frac{dx(\xi)}{d\xi}\,\hat{x} + \frac{dy(\xi)}{d\xi}\,\hat{y} + \frac{dz(\xi)}{d\xi}\,\hat{z} \quad — ④$$

Therefore the streamline satisfies

$$\vec{R}'(\xi) \propto \vec{F}(x(\xi), y(\xi), z(\xi))$$

$$\Rightarrow \quad \vec{R}'(\xi) = t\,\vec{F}(x(\xi), y(\xi), z(\xi)) \quad — ⑤$$

The component equations of Eq. ⑤ are

$$\left.\begin{aligned} \frac{dx(\xi)}{d\xi} &= t\, f(x(\xi), y(\xi), z(\xi)) \\ \frac{dy(\xi)}{d\xi} &= t\, g(x(\xi), y(\xi), z(\xi)) \\ \frac{dz(\xi)}{d\xi} &= t\, h(x(\xi), y(\xi), z(\xi)) \end{aligned}\right\} \quad — ⑥$$

From ⑥ we get

$$\frac{dx}{f} = \frac{dy}{g} = \frac{dz}{h} \quad — ⑦$$

p289

(예제 7.10)

$$\vec{F} = x^2\,\hat{x} + 2y\,\hat{y} - \hat{z} \quad — ①$$

Then streamline satisfies

$$\left.\begin{aligned} \frac{dx}{d\xi} &= k\,x^2 \\ \frac{dy}{d\xi} &= 2k\,y \\ \frac{dz}{d\xi} &= -k \end{aligned}\right\} \quad — ②$$

From ② we have

$$\frac{dx}{x^2} = \frac{dy}{2y} = -dz \quad — ③$$

First, let us consider

$$\frac{dx}{x^2} = -dz$$

$$\Rightarrow\ -\frac{1}{x} = -z + c_1 \quad — ④$$

Next, let us consider

$$\frac{dy}{2y} = -dz$$

$$\Rightarrow\ \frac{1}{2}\ln|y| = -z + c_2 \quad — ⑤$$

From ④ and ⑤

$$x = \frac{1}{z - c_1}, \quad y = e^{2c_2}\,e^{-2z} = a\,e^{-2z} \quad — ⑥$$

Thus stream line is defined by

$$x = \frac{1}{z-c}, \quad y = a e^{-2z}, \quad z = z$$

If we want to find a streamline at $(-1, 6, 2)$, we have

$$-1 = \frac{1}{2-c}, \quad 6 = a e^{-4}$$

$$\Rightarrow c = 3 \quad \text{and} \quad a = 6e^{4}$$

Thus the streamline is defined by

$$x = \frac{1}{z-3}, \quad y = 6e^{4-2z}, \quad z = z$$

※

$\varphi(x, y, z)$: scalar field

p290

(경도)

definition : gradient (기울기)

$$\boxed{\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} + \frac{\partial \varphi}{\partial z}\hat{z}}$$

$\vec{\nabla}$: del - operator

note) φ : scalar field

$\vec{\nabla}\varphi$: vector field

(Ex)

$$\varphi = x^2 y \cos(yz)$$

$$\vec{\nabla}\varphi = 2xy\cos(yz)\,\hat{x} + x^2\left[\cos(yz) - yz\sin(yz)\right]\hat{y} - x^2y^2\sin(yz)\,\hat{z} \quad //$$

p292

* directional derivative

Let

$$P_0 = (x_0, y_0, z_0)$$

$$\vec{u} = a\,\hat{x} + b\,\hat{y} + c\,\hat{z} \qquad \left(\|\vec{u}\| = \sqrt{a^2 + b^2 + c^2} = 1\right)$$

Then $\vec{u}$-directional derivative of scalar field $\varphi(x, y, z)$

at P_0 is defined as

$$\boxed{D_{\vec{u}}\varphi(P_0) = \frac{d}{dt}\varphi(x_0 + at,\ y_0 + bt,\ z_0 + ct)\Big|_{t=0}}$$

p=9=

Theorem 7.3

$$\boxed{D_{\vec{u}} \varphi(P_0) = \vec{\nabla} \varphi(P_0) \cdot \vec{u}}$$

pf)

$$\frac{d}{dt} \varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

$$= \frac{\partial(x_0+at)}{\partial t} \frac{\partial}{\partial(x_0+at)} \varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

$$+ \frac{\partial(y_0+bt)}{\partial t} \frac{\partial}{\partial(y_0+bt)} \varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

$$+ \frac{\partial(z_0+ct)}{\partial t} \frac{\partial}{\partial(z_0+ct)} \varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

$$= \vec{u} \cdot \vec{\nabla} \varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

Therefore

$$D_u \varphi(P_0)$$

$$\equiv \frac{d}{dt} \varphi(x_0+at,\ y_0+bt,\ z_0+ct) \Big|_{t=0}$$

$$= \vec{u} \cdot \vec{\nabla} \varphi(x_0,\ y_0,\ z_0)$$

$$= \vec{u} \cdot \vec{\nabla} \varphi(P_0)$$

※

p292

(예제 7.11)

$$\varphi(x,y,z) = x^2 y - x e^{z}, \quad P_0 = (2, -1, \pi), \quad \vec{u} = \frac{1}{\sqrt{6}}(\hat{x} - 2\hat{y} + \hat{z})$$

(i) $$D_{\vec{u}}\varphi(P_0) = \frac{d}{dt}\varphi(x_0+at,\ y_0+bt,\ z_0+ct)\Big|_{t=0}$$

$$\varphi(x_0+at,\ y_0+bt,\ z_0+ct)$$

$$= \varphi\left(2+\frac{t}{\sqrt{6}},\ -1-\frac{2}{\sqrt{6}}t,\ \pi+\frac{1}{\sqrt{6}}t\right)$$

$$= -\left(2+\frac{t}{\sqrt{6}}\right)\left[\left(2+\frac{t}{\sqrt{6}}\right)\left(1+\frac{2}{\sqrt{6}}t\right) + e^{\pi+\frac{1}{\sqrt{6}}t}\right]$$

$$\Rightarrow \frac{d}{dt}\varphi(x_0+at,\ y_0+bt,\ z_0+ct)\Big|_{t=0}$$

$$= -\frac{3}{\sqrt{6}}\left(e^{\pi}+4\right)$$

(ii) $$D_{\vec{u}}\varphi(P_0) = \vec{\nabla}\varphi(P_0)\cdot\vec{u}$$

$$\vec{\nabla}\varphi(x,y,z) = (2xy - e^{z})\hat{x} + x^2\hat{y} - xe^{z}\hat{z}$$

$$\Rightarrow \vec{\nabla}\varphi(P_0) = (-4-e^{\pi})\hat{x} + 4\hat{y} - 2e^{\pi}\hat{z}$$

$$\Rightarrow \vec{\nabla}\varphi(P_0)\cdot\vec{u} = -\frac{3}{\sqrt{6}}\left(e^{\pi}+4\right)$$

※

(정리 7.4)

(i) The maximum directional derivative of $\varphi(x, y, z)$ at P_0 is

$$D_{\frac{\vec{\nabla}\varphi(P_0)}{\|\vec{\nabla}\varphi(P_0)\|}} \varphi(P_0) = \|\nabla\varphi(P_0)\|$$

(ii) The minimum directional derivative of $\varphi(x, y, z)$ at P_0 is

$$D_{-\frac{\vec{\nabla}\varphi(P_0)}{\|\vec{\nabla}\varphi(P_0)\|}} \varphi(P_0) = -\|\nabla\varphi(P_0)\|$$

p=93

(예제 7.12)

$\varphi(x, y, z) = 2xz + e^y z^2$ $\quad P_0 = (2, 1, 1)$

$\vec{\nabla}\varphi = 2z\hat{x} + e^y z^2 \hat{y} + (2x + 2ze^y)\hat{z}$

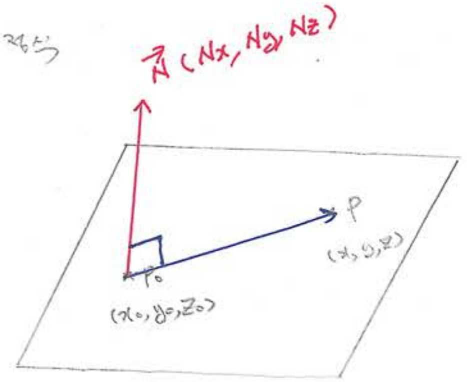
$\vec{\nabla}\varphi(P_0) = 2\hat{x} + e\hat{y} + (4 + 2e)\hat{z}$

$\|\vec{\nabla}\varphi(P_0)\| = \sqrt{4 + e^2 + (4+2e)^2}$

maximum directional derivative $= \sqrt{4 + e^2 + (4+2e)^2}$

minimum directional derivative $= -\sqrt{4 + e^2 + (4+2e)^2}$ ※

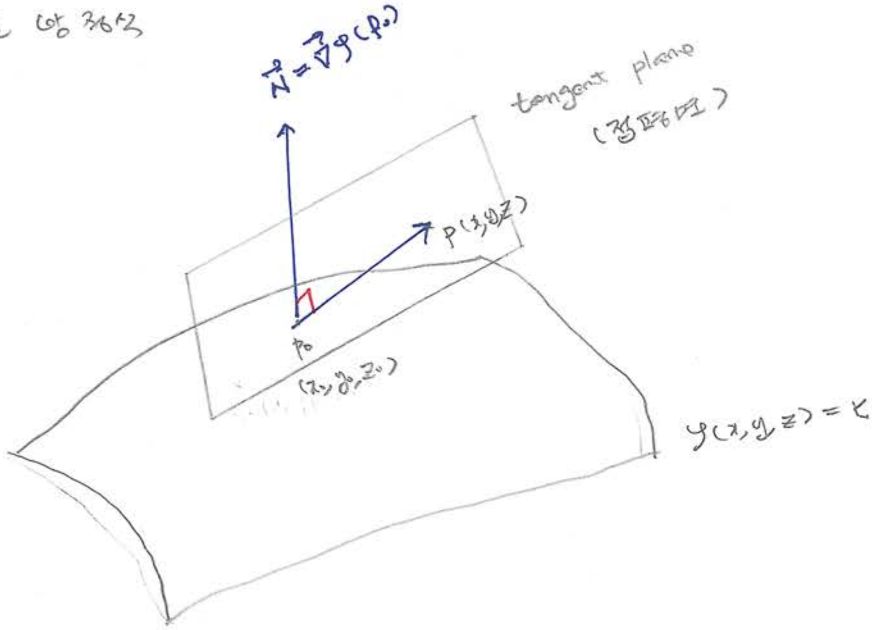
* 평면의 방정식



$$\vec{N} \cdot \overrightarrow{P_0P} = 0$$

$$\Rightarrow N_x (x - x_0) + N_y (y - y_0) + N_z (z - z_0) = 0$$

* 접평면 방정식



$$\vec{\nabla}\varphi(P_0) \cdot \overrightarrow{P_0P} = 0$$

$$\frac{\partial \varphi}{\partial x}(P_0)\,(x - x_0) + \frac{\partial \varphi}{\partial y}(P_0)\,(y - y_0) + \frac{\partial \varphi}{\partial z}(P_0)\,(z - z_0) = 0$$

접평면 방정식

p-96 (예제 7.13)

$\varphi(x,y,z) = 0$

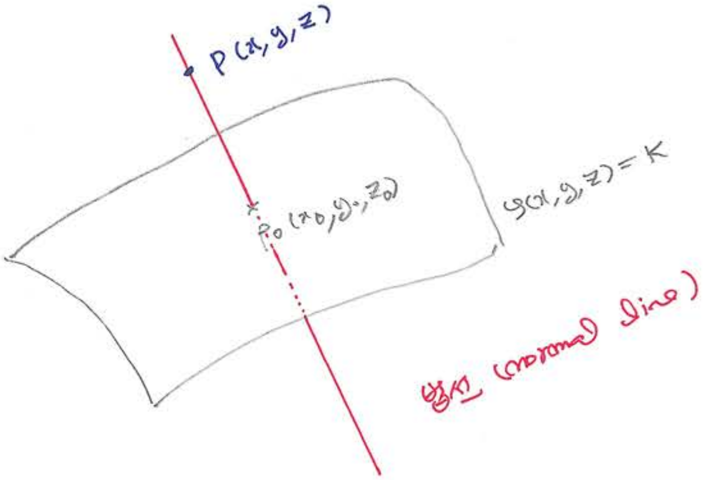
$\varphi(x,y,z) = \sin(xy) - z$

$\vec{\nabla}\varphi = y\cos(xy)\,\hat{x} + x\cos(xy)\,\hat{y} - \hat{z}$

If $P_0 = (x_0, y_0, z_0)$, equation of the tangent plane is

$$y_0\cos(x_0y_0)(x-x_0) + x_0\cos(x_0y_0)(y-y_0) - (z-z_0) = 0$$ ※

* 법선 (normal line) 방정식



$\overrightarrow{P_0P} \propto \nabla\varphi(P_0)$

$$\Rightarrow \boxed{\begin{aligned} x - x_0 &= t\,\frac{\partial\varphi}{\partial x}(P_0) \\ y - y_0 &= t\,\frac{\partial\varphi}{\partial y}(P_0) \\ z - z_0 &= t\,\frac{\partial\varphi}{\partial z}(P_0) \end{aligned}}$$

법선 방정식

p298

(예제 7.14)

$$\varphi(x,y,z) = x^2 + y^2 - z = 0$$

$$P_0 = (2, -2, 8)$$

$$\vec{\nabla}\varphi = 2x\,\hat{x} + 2y\,\hat{y} - \hat{z}$$

$$\vec{\nabla}\varphi(P_0) = 4\hat{x} - 4\hat{y} - \hat{z}$$

tangent plane

$$4(x-2) - 4(y+2) - (z-8) = 0$$

or

$$4x - 4y - z = 8$$

normal line

$$x-2 = 4t \qquad\qquad x = 4t+2$$

$$y+2 = -4t \quad \text{or} \quad y = -4t-2$$

$$z-8 = -t \qquad\qquad z = -t+8 \quad ※$$

p298

definition: 7.9 : divergence (발산)

If $\vec{F}(x,y,z) = f(x,y,z)\,\hat{x} + g(x,y,z)\,\hat{y} + h(x,y,z)\,\hat{z}$,

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$: divergence of $\vec{F}$

note) Divergence is map from vector field to scalar field

(Ex)

$$\vec{F} = 2xy\,\hat{x} + [xyz^2 - \sin(yz)]\,\hat{y} + z e^{x+y}\,\hat{z}$$

$$\vec{\nabla}\cdot\vec{F} = 2y + xz^2 - z\cos(yz) + e^{x+y} \quad \divideontimes$$

p299

definition : 7.10 curl (회전)

If $\vec{F}(x,y,z) = f(x,y,z)\,\hat{x} + g(x,y,z)\,\hat{y} + h(x,y,z)\,\hat{z}$,

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \hat{x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \hat{y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \hat{z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

note) curl is mapping from vector field to vector field.

(Ex)

$$\vec{F} = y\,\hat{x} + 2xz\,\hat{y} + z e^{x}\,\hat{k}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 2xz & ze^{x} \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}(ze^{x}) - \frac{\partial}{\partial z}(2xz)\right] + \hat{y}\left[\frac{\partial}{\partial z}y - \frac{\partial}{\partial x}(ze^{x})\right]$$

$$+ \hat{z}\left[\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial y}y\right]$$

$$= -2x\,\hat{x} - ze^{x}\,\hat{y} + (2z-1)\,\hat{z} \quad \divideontimes$$

p300

Theorem 7.6 and 7.7

$$\vec{\nabla} \times (\vec{\nabla} \varphi) = 0$$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

pf)

① $\vec{\nabla} \times (\vec{\nabla} \varphi)$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) + \hat{y}\left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) + \hat{z}\left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) = 0$$

② $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F})$ $\quad (\vec{F} = f\,\hat{x} + g\,\hat{y} + h\,\hat{z})$

$$= \vec{\nabla} \cdot \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)$$

$$= 0$$

※

8 선적분

curve (선)

$a \le t \le b$ $x = x(t)$, $y = y(t)$, $z = z(t)$

$(x(a), y(a), z(a))$: initial point

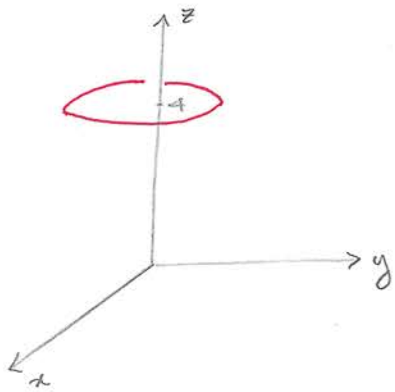
$(x(b), y(b), z(b))$: end point

If $(x(a), y(a), z(a)) \neq (x(b), y(b), z(b))$, open curve

If $(x(a), y(a), z(a)) = (x(b), y(b), z(b))$, closed curve

Ex)

$0 \le t \le 2\pi$ $x = 2\cos t$, $y = 2\sin t$, $z = 4$



at $t=0$, $(x, y, z) = (2, 0, 4)$

at $t=2\pi$, $(x, y, z) = (2, 0, 4)$ closed curve !!

※.

p303

definition : 선적분

$C: \quad a \le t \le b \qquad x = x(t), \quad y = y(t), \quad z = z(t)$

$$\int_C \left[f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz \right]$$

$$\equiv \int_a^b \left[f(x(t), y(t), z(t)) \frac{dx}{dt} + g(x(t), y(t), z(t)) \frac{dy}{dt} + h(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt$$

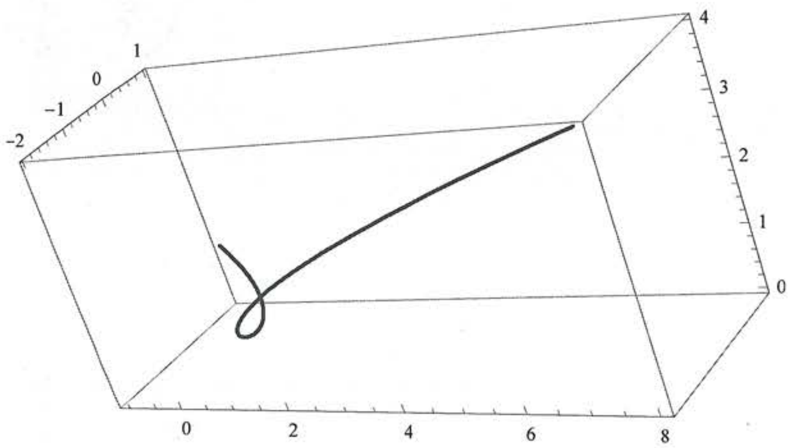
p303

(예제 7.16)

$C: \quad -1 \le t \le 2 \qquad x = t^3, \quad y = -t, \quad z = t^2$

```
In[122]:= ParametricPlot3D[{t^3, -t, t^2}, {t, -1, 2}, PlotStyle → {Thickness[0.005], Red}]
```

Out[122]=



$$\int_C \left[x\,dx - yz\,dy + e^z\,dz \right]$$

$$= \int_{-1}^{2} \left[t^3 \cdot 3t^2 - (-t^3)(-1) + e^{t^2} \cdot 2t \right] dt$$

$$= \int_{-1}^{2} \left[3t^5 - t^3 + 2t e^{t^2} \right] dt$$

$$= \frac{111}{4} + e^4 - e \qquad \divideontimes$$

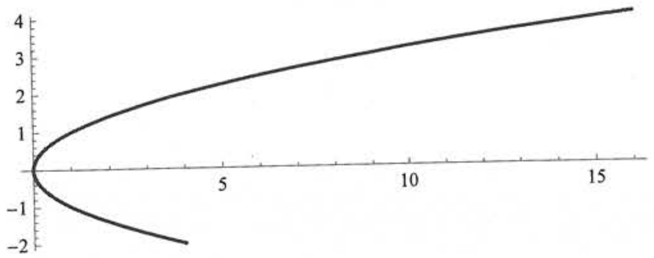
p.304

예제 7.17

$C : -1 \le t \le 4 \qquad x(t) = t^2, \quad y(t) = t$

```
In[123]:= ParametricPlot[{t^2, t}, {t, -2, 4}, PlotStyle → {Thickness[0.005], Red}]
```

Out[123]=



$$\int_C \left[(xy)\, dx - (y \sin x)\, dy \right]$$

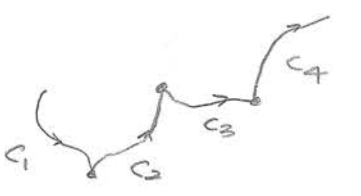
$$= \int_{-1}^{4} dt \left[t^3 \frac{dx}{dt} - \left(t \sin t^2 \right) \frac{dy}{dt} \right]$$

$$= \int_{-1}^{4} dt \left[2t^4 - t \sin t^2 \right]$$

$$= 410 + \frac{1}{2} \cos(16) - \frac{1}{2} \cos(1) \quad ※$$

* When curve is not smooth

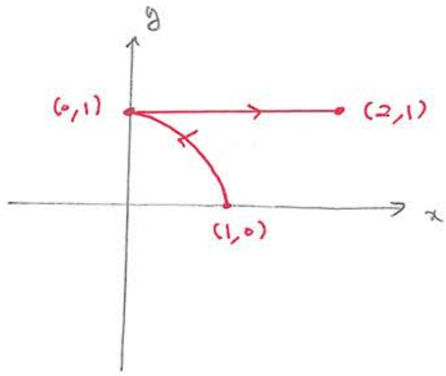
$C =$



$$\int_C \Rightarrow \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

p3-4

(예제 9.18)



$C = C_1 + C_2$

$C_1 : \quad x = \cos t, \quad y = \sin t \qquad (0 \le t \le \frac{\pi}{2})$

$C_2 : \quad x = s, \quad y = 1 \qquad (0 \le s \le 2)$

$$\int_C (x^2 y\, dx + y^2 dy)$$

$$= \int_{C_1} (x^2 y\, dx + y^2 dx) + \int_{C_2} (x^2 y\, dx + y^2 dx) \quad -①$$

$$\int_{C_1} (x^2 y\, dx + y^2 dy)$$

$$= \int_0^{\frac{\pi}{2}} dt \left[\cos^2 t \sin t (-\sin t) + \sin^2 t \cos t \right]$$

$$= \int_0^{\frac{\pi}{2}} dt \left[-\sin^2 t \cos^2 t + \sin^2 t \cos t \right]$$

$$= -\frac{\pi}{16} + \frac{1}{3} \quad -②$$

$$\int_{C_2} (x^2 y\, dx + y^2 dy)$$

$$= \int_0^2 ds \left[s^2 \frac{dx}{ds} + \frac{dy}{ds} \right]$$

$$= \int_0^2 s^2\, ds = \frac{8}{3} \quad -②$$

②, ③ → ①

$$\int_C (x^2 y\,dx + y^2\,dy) = 3 - \frac{\pi}{16}$$ ※

If we define

$$d\vec{s} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\vec{F} = f(x,y,z)\,\hat{x} + g(x,y,z)\,\hat{y} + h(x,y,z)\,\hat{z},$$

$$\int_C [f\,dx + g\,dy + h\,dz] = \int_C \vec{F}\cdot d\vec{s}$$

note) If $\vec{F}$ is force and C is path of particle,

$\int_C \vec{F}\cdot d\vec{s}$ is work performed by force $\vec{F}$.

p206

(예제 7.19)

C: $0 \le t \le 1$ $\quad x = t, \quad y = -t^2, \quad z = t$

force: $\vec{F} = \hat{x} - y\,\hat{y} + xyz\,\hat{z}$

$$W = \int \vec{F}\cdot d\vec{s}$$

$$= \int [dx - y\,dy + (xyz)\,dz]$$

$$= \int_0^1 dt\,[1 - (-t^2)(-2t) + (-t^4)]$$

$$= \int_0^1 dt\,[1 - 2t^3 - t^4]$$

$$= \frac{3}{10}$$

※

P307

(Theorem 7.9)

$$\int_{-c} [f\,dx + g\,dy + h\,dz] = -\int_c [f\,dx + g\,dy + h\,dz]$$

P308

Theorem 7.10 : Curve 위에서 scalar 함수 적분

C: $a \le t \le b$ $\quad x = x(t)$, $y = y(t)$, $z = z(t)$

Then

$$\int_c \varphi(x,y,z)\,dl = \int_a^b dt\ \varphi(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Pf)

$$dl = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$\Rightarrow \int_c \varphi(x,y,z)\,dl$$

$$= \int_a^b dt\ \varphi(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

※

p308

(예제 7.21)

C: $0 \le t \le \frac{\pi}{2}$ $x = 4\cos t$, $y = 4\sin t$, $z = -3$

$$\int xy \, ds$$

$$= \int_0^{\frac{\pi}{2}} 16 \sin t \cos t \sqrt{16\sin^2 t + 16\cos^2 t} \, dt$$

$$= 64 \int_0^{\frac{\pi}{2}} \sin t \cos t \, dt$$

$$= 32$$

//

p311

§ Green Theorem

C =  양의 방향으로 닫힌 curve (counterclockwise closed curve)

C =  음의 방향으로 닫힌 curve (clockwise closed curve)

C = 

외부 (exterior region)

p312

Theorem 7.11 Green Theorem

C : counterclockwise closed curve

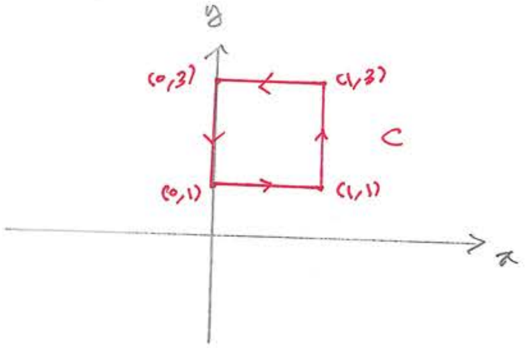
D : interior region of C

If $f, g, \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}$ are continuous in D,

$$\oint_C [f(x,y)\,dx + g(x,y)\,dy] = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

P312

(예제 7.23)



C: counterclockwise closed curve

Let

$\vec{F} = (y - x^2 e^x)\hat{x} + (\cos(2y^2) - x)\hat{y}$: force

$W = \oint_C \vec{F} \cdot d\vec{s}$

$= \iint_D dA \left[\frac{\partial}{\partial x}\left(\cos(2y^2) - x \right) - \frac{\partial}{\partial y}\left(y - x^2 e^x \right) \right]$

$= \iint_D dA \left[-1 - 1 \right]$

$= -2 \underline{\iint_D dA}^{=2}$

$= -4$ ※

(예제 7.24)

C: counterclockwise closed curve

$\oint \left[2x \cos(2y)\, dx - 2x^2 \sin 2y \, dy \right]$

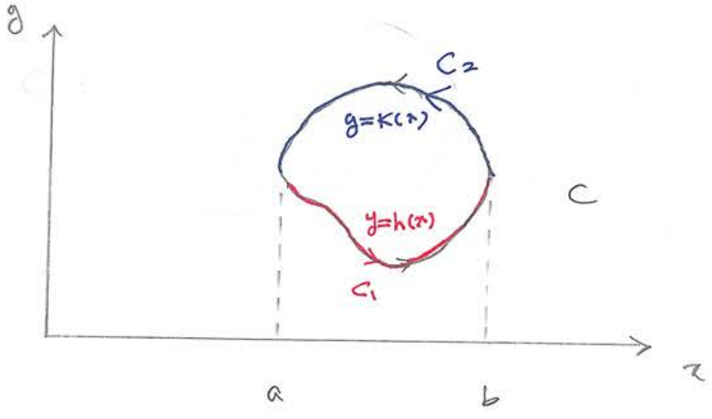
$= \iint_D \left[\frac{\partial}{\partial x}\left(-2x^2 \sin 2y \right) - \frac{\partial}{\partial y}\left(2x \cos(2y) \right) \right] dA$

$= \iint_D \left[-4x \sin 2y + 4x \sin 2y \right] dA$

$= 0$

※.

p213 (Green theorem 증명)



$C = C_1 + C_2$

$\oint_C f(x,y)\,dx$

$= \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx$

$= \int_a^b f(x, h(x))\,dx + \int_b^a f(x, k(x))\,dx$

$= -\int_a^b [\, f(x, k(x)) - f(x, h(x)) \,]\,dx$ — ①

$\iint_D \frac{\partial f}{\partial y}\,dA$

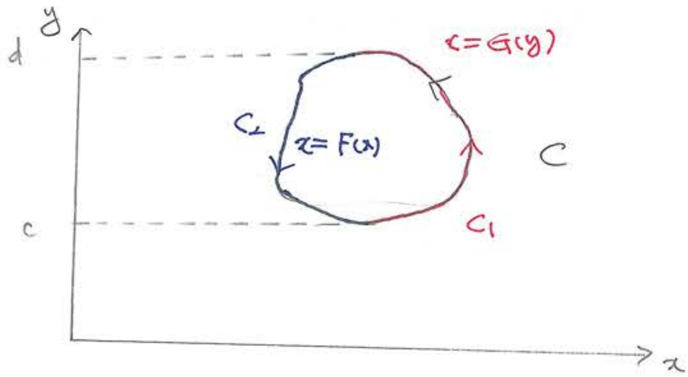
$= \int_a^b dx \int_{h(x)}^{k(x)} dy\, \frac{\partial f}{\partial y}$

$= \int_a^b dx\, \left[f(x,y) \right]_{y=h(x)}^{y=k(x)}$

$= \int_a^b dx \left[f(x, k(x)) - f(x, h(x)) \right]$ — ②

From ① and ②

$\oint f(x,y)\,dx = -\iint_D \frac{\partial f}{\partial y}\,dA$ — ③



$C = C_1 + C_2$

$\oint_C g(x,y)\, dy$

$= \int_{C_1} g(x,y)\, dy + \int_{C_2} g(x,y)\, dy$

$= \int_c^d g(G(y), y)\, dy + \int_d^c g(F(x), y)\, dy$

$= \int_c^d [g(G(y), y) - g(F(x), y)]\, dy$ — ④

$\iint_D \frac{\partial g}{\partial x}\, dA$

$= \int_c^d dy \int_{F(x)}^{G(x)} dx \quad \frac{\partial g}{\partial x}$

$= \int_c^d dy \quad g(x,y) \Big|_{x=F(x)}^{x=G(x)}$

$= \int_c^d dy\, [g(G(x), y) - g(F(x), y)]$ — ⑤

From ④ and ⑤

$\oint_C g(x,y)\, dy = \iint_D \frac{\partial g}{\partial x}\, dA$ — ⑥

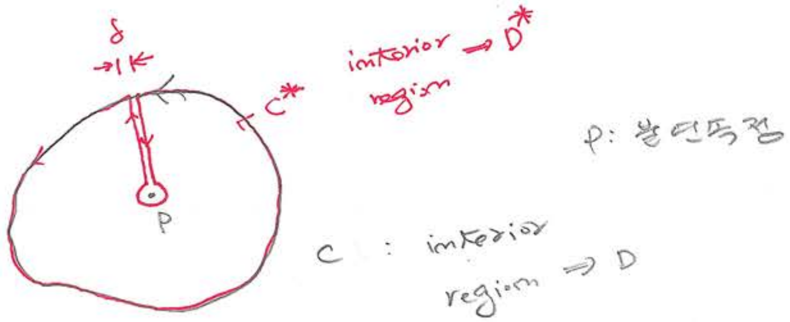
From ② and ③

$$\oint_C [f(x,y)\,dx + g(x,y)\,dy]$$

$$= -\iint_D \frac{\partial f}{\partial y}\,dA + \iint_D \frac{\partial g}{\partial x}\,dA$$

$$= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \qquad \#$$

만약 C의 내부에서 f, g, $\frac{\partial g}{\partial x}$, $\frac{\partial f}{\partial y}$ 가 불연속하면 어떨까?



$$\oint_{C^*} [f\,dx + g\,dy] = \iint_{D^*} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA \qquad -①$$

If we take $\delta \to 0$ limit,

$$C^* = C - C_1 \qquad -②$$

⊖ → ◯

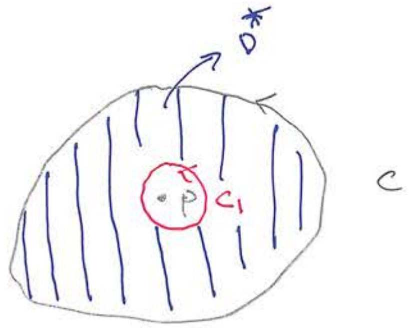
$$\oint_{C^*} [f\,dx + g\,dy]$$

$$= \oint_C [f\,dx + g\,dy] - \oint_{C_1} [f\,dx + g\,dy]$$

$$= \iint_{D^*} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

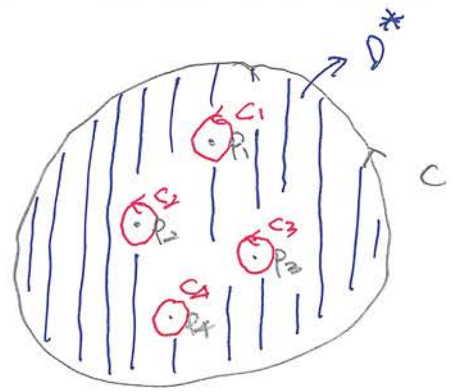
$$\Rightarrow \oint_C [f\,dx + g\,dy]$$

$$= \iint_{D^*} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA + \oint_{C_1} [f\,dx + g\,dy]$$



If there are many points, where $f, g, \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}$ are discontinuous,

$$\boxed{\oint_C [f\,dx + g\,dy] = \iint_{D^*} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA + \sum_n \oint_{C_n} [f\,dx + g\,dy]}$$



p216

(예제 7. 22)

$$I = \oint_C [f\,dx + g\,dy] \qquad -①$$

C: counterclockwise closed curve

$$f(x,y) = \frac{-y}{x^2+y^2}, \qquad g(x,y) = \frac{x}{x^2+y^2} \qquad -②$$

Then we have

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad -③$$

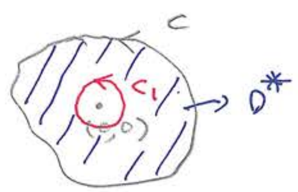
Thus f, g, $\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous except origin (0,0).

(i) If origin is outside of C,

we apply Green theorem

$$I = \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = 0$$

(ii) If origin is inside of C,



$$I = \underbrace{\iint_{D^*} \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right] dA}_{=0} + \oint_{C_1} [f\,dx + g\,dy]$$

$$= \oint_{C_1} [f\,dx + g\,dy] \qquad -④$$

C_1 can be written as

$$C_1 : \quad 0 \le \theta \le 2\pi, \qquad x = r\cos\theta \quad , \quad y = r\sin\theta \qquad - ⑥$$

Thus

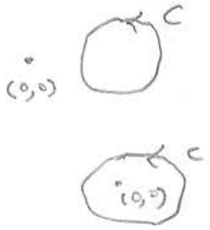
$$\oint_{C_1} [f\,dx + g\,dy]$$

$$= \int_0^{2\pi} d\theta \left[\frac{-r\sin\theta}{r^2} \frac{dx}{d\theta} + \frac{r\cos\theta}{r^2} \frac{dy}{d\theta} \right]$$

$$= \int_0^{2\pi} d\theta \left[-\frac{\sin\theta}{r} (-r\sin\theta) + \frac{\cos\theta}{r} r\cos\theta \right]$$

$$= 2\pi$$

$$\Rightarrow \qquad I = \begin{cases} 0 \\ \\ 2\pi \end{cases}$$



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§ 경로의 독립성과 potential

Consider a work performed by force

$$\vec{F} = f(x,y)\,\hat{x} + g(x,y)\,\hat{y} \qquad (1)$$

along a counterclockwise closed curve C.

If f, g, $\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial x}$ are continuous in interior region D,

$$W = \oint_C \vec{F}\cdot d\vec{s}$$

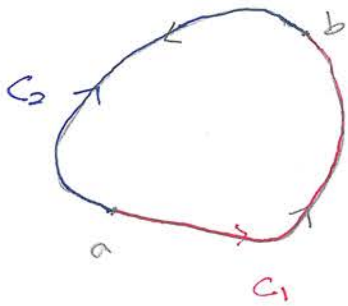
$$= \oint_C \left[f(x,y)\,dx + g(x,y)\,dy \right] \qquad (2)$$

$$= \iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

If

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}, \qquad (3)$$

$$W = \oint_C \vec{F}\cdot d\vec{s} = 0.$$



Since

$$C = C_1 - C_2 \qquad (4)$$

$$\oint_C \vec{F}\cdot d\vec{s} = \int_{C_1} \vec{F}\cdot d\vec{s} - \int_{C_2} \vec{F}\cdot d\vec{s} = 0$$

$$\Rightarrow \int_{a,C_1}^{b} \vec{F}\cdot d\vec{s} = \int_{a,C_2}^{b} \vec{F}\cdot d\vec{s} \qquad (5)$$

경로에 independent 하다.

⇒ 이경우 $\vec{F}$를 "conservative force (보존력)" 이라 함

* $\vec{F} = f(x,y)\,\hat{x} + g(x,y)\,\hat{y}$

$$W = \int_a^b \vec{F} \cdot d\vec{r}$$

W의 경로-independence 판정법

[1] If $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$, path-independent

If $\dfrac{\partial g}{\partial x} \neq \dfrac{\partial f}{\partial y}$, path-dependent

[2] If $\oint_C \vec{F} \cdot d\vec{s} = 0$, path-independent

If $\oint_C \vec{F} \cdot d\vec{s} \neq 0$, path-dependent

[3] If $\vec{F} = \vec{\nabla} \varphi(x,y)$, path-independent

$$\left(\begin{aligned} \because\ & \vec{F} = \vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\hat{x} + \frac{\partial \varphi}{\partial y}\hat{y} \\ \Rightarrow\ & f(x,y) = \frac{\partial \varphi}{\partial x}, \quad g(x,y) = \frac{\partial \varphi}{\partial y} \\ \Rightarrow\ & \frac{\partial f}{\partial y} = \frac{\partial^2 \varphi}{\partial x \partial y}, \quad \frac{\partial g}{\partial x} = \frac{\partial^2 \varphi}{\partial x \partial y} \\ \Rightarrow\ & \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \end{aligned} \right)$$

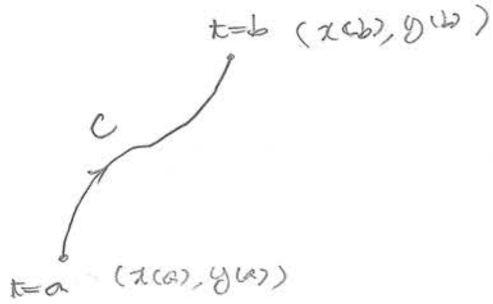
$\varphi(x,y)$: potential of $\vec{F}(x,y)$

If $\vec{F} = \nabla \varphi$,

$$W = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \left[\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \right]$$

$$= \int_a^b \left[\frac{\partial \varphi}{\partial x} \frac{dx}{dt} + \frac{\partial \varphi}{\partial y} \frac{dy}{dt} \right] dt$$



$$= \int_a^b \frac{d\varphi(x,y)}{dt} dt$$

$$= \varphi(x(b), y(b)) - \varphi(x(a), y(a))$$

$$\boxed{\begin{aligned} * \quad & W = \int_C \vec{F} \cdot d\vec{r} = \varphi(\text{end point}) - \varphi(\text{initial point}) \\ & \vec{F} = \vec{\nabla} \varphi \end{aligned}}$$

p319

(예제 7.26)

$$\vec{F}(x, y) = f(x, y)\,\hat{x} + g(x, y)\,\hat{y}$$

$$f(x, y) = 2x \cos(2y), \quad g(x, y) = -(2x^2 \sin(2y) + 4y^2)$$

$$\frac{\partial g}{\partial x} = -4x \sin 2y \qquad \frac{\partial f}{\partial y} = -4x \sin 2y$$

$$\Rightarrow \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

$\vec{F}$: conservative force

$\Rightarrow$ potential 존재함

Let $\varphi(x, y)$ be potential of $\vec{F}$.

$$\frac{\partial \varphi}{\partial x} = 2x \cos 2y$$

$$\frac{\partial \varphi}{\partial y} = -(2x^2 \sin 2y + 4y^2)$$

$$\Rightarrow \varphi(x, y) = x^2 \cos 2y - \frac{4}{3} y^3$$ ※

p220

(예제 7.27)

$$\vec{F}(x, y) = f(x, y)\,\hat{x} + g(x, y)\,\hat{y}$$

$$f(x, y) = 2xy^2 + y$$

$$g(x, y) = 2x^2 y + e^x y$$

$$\frac{\partial f}{\partial y} = 4xy + 1 \quad , \quad \frac{\partial g}{\partial x} = 4xy + e^x y$$

$$\Rightarrow \frac{\partial f}{\partial y} \neq \frac{\partial g}{\partial x}$$

$\Rightarrow$ $\vec{F}$: non-conservative force

$\Rightarrow$ "potential 은 존재하지 않는다" ※

§ 접평면과 법정선

Surface is defined as $\{x(u,v),\ y(u,v),\ z(u,v)\}$.

If, for example, we consider a surface

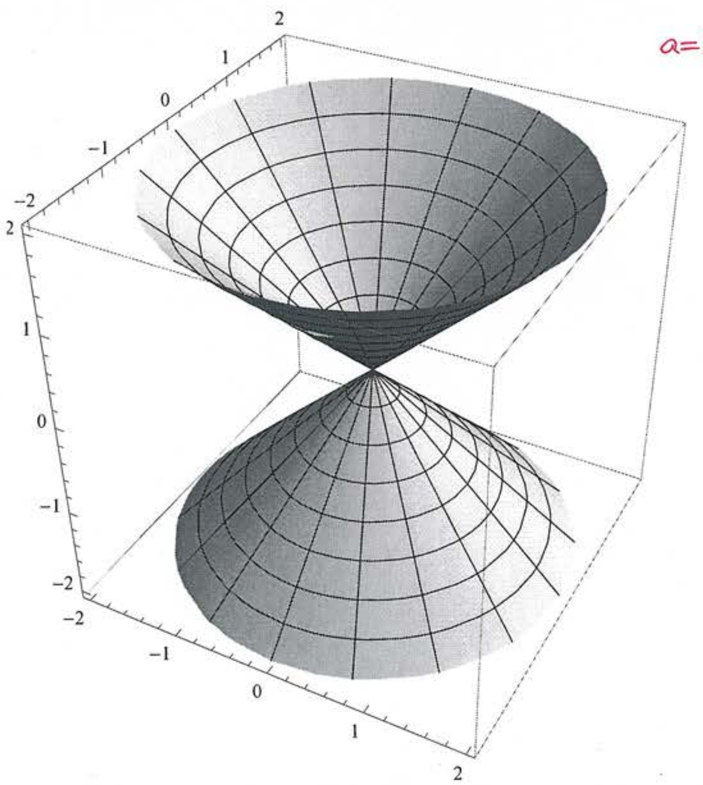
$$x(u,v) = a\,u\cos v, \quad y(u,v) = b\,u\sin v, \quad z(u,v) = u,$$

this is a surface

$$z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{(elliptical cone)}$$

In[126]:= `ParametricPlot3D[{u Cos[v], u Sin[v], u}, {u, -2, 2}, {v, 0, 2 Pi}]`

Out[126]=



a=b=1

Consider a surface Σ parametrized by

$$x(u,v), \quad y(u,v) \quad \text{and} \quad z(u,v)$$

and a point $P_0 = (x(u_0,v_0), y(u_0,v_0), z(u_0,v_0))$ on Σ.

Then we consider a line C_1 which is parametrized by

$$x(u,v_0), \quad y(u,v_0), \quad z(u,v_0)$$

Then the tangent vector of C_1 at P_0 is given by

$$\vec{T}_{v_0} = \frac{\partial x}{\partial u}(u_0,v_0)\,\hat{x} + \frac{\partial y}{\partial u}(u_0,v_0)\,\hat{y} + \frac{\partial z}{\partial u}(u_0,v_0)\,\hat{z} \qquad (1)$$

If we consider another curve C_2, which is parametrized by

$$x(u_0,v), \quad y(u_0,v), \quad z(u_0,v),$$

the tangent vector of C_2 at P_0 is given by

$$\vec{T}_{u_0} = \frac{\partial x}{\partial v}(u_0,v_0)\,\hat{x} + \frac{\partial y}{\partial v}(u_0,v_0)\,\hat{y} + \frac{\partial z}{\partial v}(u_0,v_0)\,\hat{z} \qquad (2)$$

$\vec{T}_{v_0}$ and $\vec{T}_{u_0}$ are on the tangent surface.

* definition of normal vector
$\vec{N}(P_0) = \vec{T}_{v_0} \times \vec{T}_{u_0}$

Then

$$\vec{N}(P_0) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial x}{\partial u}(P_0) & \frac{\partial y}{\partial u}(P_0) & \frac{\partial z}{\partial u}(P_0) \\ \frac{\partial x}{\partial v}(P_0) & \frac{\partial y}{\partial v}(P_0) & \frac{\partial z}{\partial v}(P_0) \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial y}{\partial u}(P_0)\frac{\partial z}{\partial v}(P_0) - \frac{\partial y}{\partial v}(P_0)\frac{\partial z}{\partial u}(P_0)\right]$$

$$+ \hat{y}\left[\frac{\partial z}{\partial u}(P_0)\frac{\partial x}{\partial v}(P_0) - \frac{\partial z}{\partial v}(P_0)\frac{\partial x}{\partial u}(P_0)\right] \qquad (3)$$

$$+ \hat{z}\left[\frac{\partial x}{\partial u}(P_0)\frac{\partial y}{\partial v}(P_0) - \frac{\partial x}{\partial v}(P_0)\frac{\partial y}{\partial u}(P_0)\right]$$

Now we define Jacobian

$$\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{vmatrix} = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \qquad (4)$$

Then

$$\boxed{\vec{N}(P_0) = \frac{\partial(y,z)}{\partial(u,v)}\hat{x} + \frac{\partial(z,x)}{\partial(u,v)}\hat{y} + \frac{\partial(x,y)}{\partial(u,v)}\hat{z}} \qquad (5)$$

p224

(예제 7.28)

$x = a u \cos v, \quad y = b u \sin v, \quad z = u$

Let us choose $u_0 = \frac{1}{2}$, $v_0 = \frac{\pi}{6}$. Then

$P_0 = \left(\frac{\sqrt{3}}{4} a, \ \frac{b}{4}, \ \frac{1}{2} \right)$.

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} b \sin v & b u \cos v \\ 1 & 0 \end{vmatrix} = -b u \cos v$$

$$\left. \frac{\partial(y,z)}{\partial(u,v)} \right|_{P_0} = -\frac{\sqrt{3}}{4} b$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ a \cos v & -a u \sin v \end{vmatrix} = -a u \sin v$$

$$\left. \frac{\partial(z,x)}{\partial(u,v)} \right|_{P_0} = -\frac{a}{4}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a \cos v & -a u \sin v \\ b \sin v & b u \cos v \end{vmatrix} = a b u$$

$$\left. \frac{\partial(x,y)}{\partial(u,v)} \right|_{P_0} = \frac{1}{2} a b$$

$$\vec{N}(P_0) = -\frac{\sqrt{3}}{4} b \, \hat{x} - \frac{a}{4} \hat{y} + \frac{ab}{2} \hat{z}$$

※

If curved surface is given by

$$z = S(x, y),$$

this surface is parametrized by

$$x = u, \quad y = v, \quad z = S(u, v)$$

$$\frac{\partial(y,z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{\partial S}{\partial u} & \frac{\partial S}{\partial v} \end{vmatrix} = -\frac{\partial S}{\partial u} = -\frac{\partial S(x,y)}{\partial x}$$

$$\frac{\partial(z,x)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{\partial S}{\partial u} & \frac{\partial S}{\partial v} \\ 1 & 0 \end{vmatrix} = -\frac{\partial S}{\partial v} = -\frac{\partial S(x,y)}{\partial y}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Thus normal vector at P_0 is given by

$$\boxed{\begin{aligned} \vec{N}(P_0) &= -\frac{\partial S}{\partial x}(P_0)\,\hat{x} - \frac{\partial S}{\partial y}(P_0)\,\hat{y} + \hat{z} \\ &= -\frac{\partial z}{\partial x}(P_0)\,\hat{x} - \frac{\partial z}{\partial y}(P_0)\,\hat{y} + \hat{z} \end{aligned}}$$

P 325

(예제 7.29)

$$z = \sqrt{x^2+y^2}$$

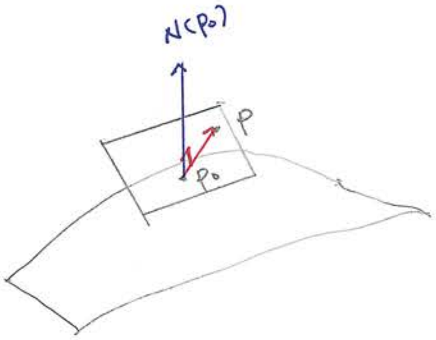
$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} , \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\vec{N}(P_0) = -\left.\frac{x}{\sqrt{x^2+y^2}}\right|_{P_0} \hat{x} - \left.\frac{y}{\sqrt{x^2+y^2}}\right|_{P_0} \hat{y} + \hat{z}$$

If $P_0 = (3, 1, \sqrt{10})$,

$$\vec{N}(P_0) = -\frac{3}{\sqrt{10}}\hat{x} - \frac{1}{\sqrt{10}}\hat{y} + \hat{z}$$ ※

* Equation of tangent plane



$$\vec{N} \cdot \overrightarrow{P_0P} = 0$$

$$\frac{\partial(y,z)}{\partial(u,v)}(P_0)(x-x_0) + \frac{\partial(z,x)}{\partial(u,v)}(P_0)(y-y_0) + \frac{\partial(x,y)}{\partial(u,v)}(P_0)(z-z_0) = 0$$

접평면 방정식

p3-7

(예제 7.20)

Consider a surface parametrized by

$$x = au\cos v, \quad y = bu\sin v, \quad z = u$$

and $\left(u_0 = \frac{1}{2},\ v_0 = \frac{\pi}{6}\right)$

Then

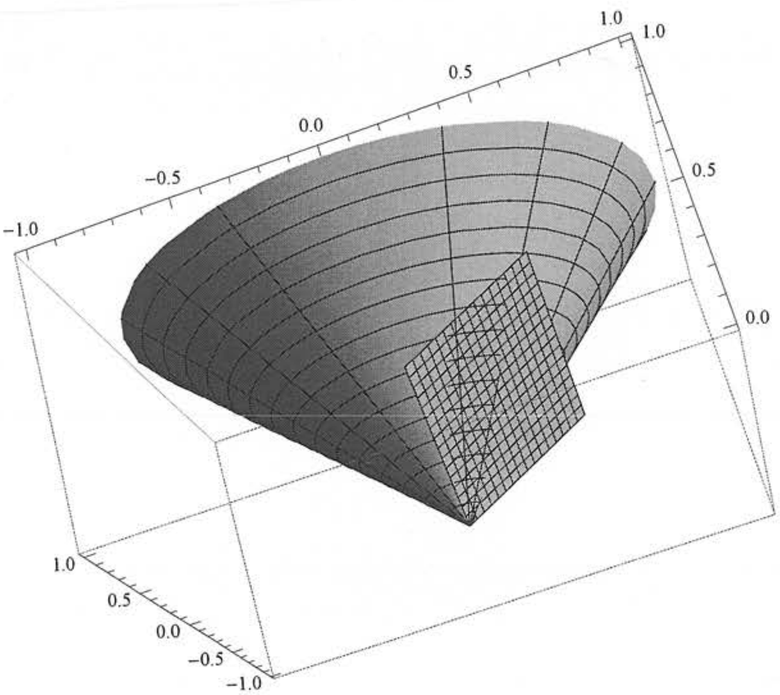
$$P_0 = \left(\frac{\sqrt{3}}{4}a,\ \frac{b}{4},\ \frac{1}{2}\right)$$

$$\vec{N}(P_0) = -\frac{\sqrt{3}b}{4}\hat{x} - \frac{a}{4}\hat{y} + \frac{ab}{2}\hat{z}$$

접평면 방정식

$$-\frac{\sqrt{3}b}{4}\left(x - \frac{\sqrt{3}}{4}a\right) - \frac{a}{4}\left(y - \frac{b}{4}\right) + \frac{ab}{2}\left(z - \frac{1}{2}\right) = 0$$

Out[133]=



If curved surface is given by $z = S(x, y)$,

$$\vec{N} = -\frac{\partial S}{\partial x}(P_0)\,\hat{x} - \frac{\partial S}{\partial y}(P_0)\,\hat{y} + \hat{z}$$

⇒ 접평면 방정식

$$-\frac{\partial S}{\partial x}(P_0)(x - x_0) - \frac{\partial S}{\partial y}(P_0)(y - y_0) + (z - z_0) = 0$$

$$\Rightarrow \boxed{(z - z_0) = \frac{\partial S}{\partial x}(P_0)(x - x_0) + \frac{\partial S}{\partial y}(P_0)(y - y_0)}$$

If the curved surface is given by

$$z = S(x, y),$$

its area is

$$\Sigma \text{ 의 표면적} = \iint_D \sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2}\, dA \quad - ①$$

(D: S가 정의되는 영역)

Since

$$\vec{N} = -\frac{\partial S}{\partial x}\,\hat{x} - \frac{\partial S}{\partial y}\,\hat{y} + \hat{z},$$

$$\|\vec{N}\| = \sqrt{1 + \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2} \quad - ②$$

② → ①

$$\boxed{\Sigma \text{ 의 표면적} = \iint_D \|\vec{N}\|\, dA}$$

If curved surface Σ is defined by

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

$$\boxed{\Sigma \text{ 의 표면적} = \iint_D \| N(u, v) \| \, du \, dv}$$

p328

(예제 7.31)

$$z = s(x, y) = \sqrt{9 - x^2 - y^2}$$

영역: $D = \{ (x, y) \mid x^2 + y^2 \le 9 \}$

$$\frac{\partial s}{\partial x} = -\frac{x}{\sqrt{9 - x^2 - y^2}}, \quad \frac{\partial s}{\partial y} = -\frac{y}{\sqrt{9 - x^2 - y^2}}$$

Σ 의 표면적

$$= \iint_D dA \sqrt{1 + \frac{x^2 + y^2}{9 - x^2 - y^2}}$$

$$= 3 \iint_D dA \frac{1}{\sqrt{9 - (x^2 + y^2)}}$$

$$(x = r\cos\theta, \quad y = r\sin\theta)$$

$$= 3 \int_0^3 dr \int_0^{2\pi} d\theta \, r \frac{1}{\sqrt{9 - r^2}}$$

$$= 18\pi$$

※

p329

definition : 면적분

The curved surface Σ is defined by

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v)$$

Then the integration of scalar function $f(x,y,z)$ on Σ is

defined as

$$\iint_\Sigma f(x,y,z)\,d\sigma = \iint_D f(x(u,v), y(u,v), z(u,v))\,\|N(u,v)\|\,du\,dv$$

If curved surface Σ is defied by

$$z = S(x,y),$$

$$\iint_\Sigma f(x,y,z)\,d\sigma = \iint_D f(x, y, S(x,y))\sqrt{1+\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2}\,dx\,dy$$

p330

(예제 9.32)

$$D = \{(x,y) \mid 4 \le x^2+y^2 \le 9,\ x>0,\ y>0\}$$

$$z = x^2+y^2$$

$$1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 1+4(x^2+y^2)$$

$$\iint_\Sigma \frac{xy}{z}\,d\sigma = \iint_\Sigma \frac{xy}{x^2+y^2}\sqrt{1+4(x^2+y^2)}\,dx\,dy \qquad \left(\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}\right)$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_2^3 dr\, r\, \frac{r^2\sin\theta\cos\theta}{r^2}\sqrt{1+4r^2}$$

$$= \int_0^{\frac{\pi}{2}} d\theta\, \sin\theta\cos\theta \int_2^3 dr\, r\sqrt{1+4r^2}$$

$$= \frac{1}{24}\left[(37)^{3/2} - (17)^{3/2}\right]$$

※

§ Divergence Theorem and Stock Theorem

p320

Theorem 7.14 : Divergence theorem

Σ : closed surface

$\vec{N}$: normal unit vector from interior to exterior regions

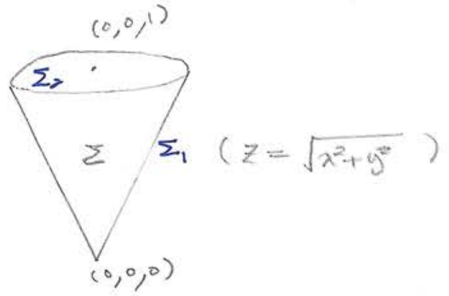
$\vec{F}$: Vector field

$$\iint_{\Sigma} \vec{F}\cdot\vec{N}\, ds = \iiint_{M} (\vec{\nabla}\cdot\vec{F})\, dV$$

M: volume enclosed by Σ

p322

(예제 7.33)



$$\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\iiint_{M} (\vec{\nabla}\cdot\vec{F})\, dV = 3\cdot\frac{1}{3}(\pi 1^2)\cdot 1 = \pi \quad -①$$

Now let us calculate $\iint_{\Sigma} \vec{F}\cdot\vec{N}\, ds$. Since $\Sigma = \Sigma_1 + \Sigma_2$,

$$\iint_{\Sigma} \vec{F}\cdot\vec{N}\, ds = \iint_{\Sigma_1} \vec{F}\cdot\vec{N}\, ds + \iint_{\Sigma_2} \vec{F}\cdot\vec{N}\, ds \quad -②$$

at Σ_1, $\quad \vec{N} = \frac{1}{\sqrt{2}}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right)$

$$= \frac{1}{\sqrt{2}}\left[\frac{x}{\sqrt{x^2+y^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\hat{y} - \hat{z}\right] \quad -③$$

Therefore

$$\vec{F}\cdot\vec{N} = \frac{1}{\sqrt{2}}\left(\frac{x^2+y^2}{\sqrt{x^2+y^2}} - z \right)$$

$$= \frac{1}{\sqrt{2}}\left[\sqrt{x^2+y^2} - \sqrt{x^2+y^2} \right]$$

$$= 0 \qquad - ④$$

Therefore

$$\iint_{\Sigma_1} \vec{F}\cdot\vec{N}\, ds = 0 \qquad - ⑤$$

at Σ_2, $\vec{N} = \hat{z}$.

Thus $\vec{F}\cdot\vec{N} = z = 1 \qquad - ⑥$

Therefore,

$$\iint_{\Sigma_2} \vec{F}\cdot\vec{N}\, ds = \iint_{\Sigma_2} ds = \pi \qquad - ⑦$$

⑤, ⑦ → ③

$$\iint_{\Sigma} \vec{F}\cdot\vec{N}\, ds = \pi \qquad - ⑧$$

From ① and ⑧

$$\iint_{\Sigma} \vec{F}\cdot\vec{N}\, ds = \iiint_M (\vec{\nabla}\cdot\vec{F})\, dV \qquad ※$$

Theorem 7.15 : Stock Theorem

C : closed curve

Σ: surface enclosed by C

If $\vec{F}$ is vector field,

$$\oint_C \vec{F} \cdot d\vec{s} = \iint_\Sigma (\vec{\nabla} \times \vec{F}) \cdot \vec{n} \, d\sigma$$

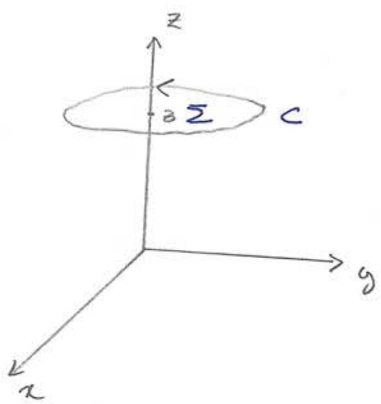
(note) $\vec{n}$ 은 오른나사법칙으로 결정된다

(Ex)

$C: \; z = 3, \; x^2 + y^2 = 9$

$\vec{F} = -y\hat{x} + x\hat{y} - xyz\,\hat{z}$

$\Sigma: \; z = 3, \; x^2 + y^2 \le 9$



(i) $\oint_C \vec{F} \cdot d\vec{s}$

C is parametrized by

$0 \le t \le 2\pi \; : \; x = 3\cos t \quad y = 3\sin t, \quad z = 3$

$$\oint_C \vec{F} \cdot d\vec{s}$$

$$= \oint_C [-y\,dx + x\,dy - xyz\,dz]$$

$$= \int_0^{2\pi} dt \left[-3\sin t \frac{dx}{dt} + 3\cos t \frac{dy}{dt} - 27 \sin t \cos t \frac{dz}{dt} \right]$$

$$= 9 \int_0^{2\pi} dt$$

$$= 18\pi$$

(ii) $\iint_{\Sigma} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -xyz \end{vmatrix} = -xz\,\hat{x} + yz\,\hat{y} + 2\hat{z}$$

at Σ, $\vec{\nabla}\times\vec{F} = -3x\,\hat{x} + 3y\,\hat{y} + 2\hat{z}$

$$\vec{n} = \hat{z}$$

$$\Rightarrow (\vec{\nabla}\times\vec{F})\cdot\vec{n} = 2$$

$$\Rightarrow \iint_{\Sigma} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma$$

$$= 2\iint_{\Sigma} d\sigma$$

$$= 2\cdot\pi\cdot 3^2$$

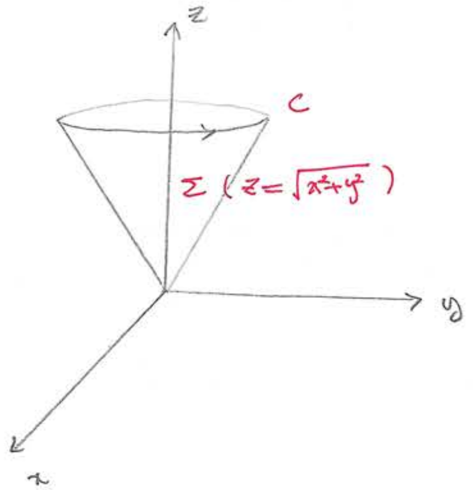
$$= 18\pi$$

$$\Rightarrow \oint_{C} \vec{F}\cdot d\vec{s} = \iint_{\Sigma} (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma \quad \text{※}$$

Σ can be chosed any surface whose boundary is C.

(iii) (p336 예제 7.34)

We choose Σ as cone :



$$\vec{N} = -\frac{x}{\sqrt{x^2+y^2}}\hat{x} - \frac{y}{\sqrt{x^2+y^2}}\hat{y} + \hat{z} \quad — ①$$

$$\|\vec{N}\| = \sqrt{2} \quad — ②$$

$$\vec{n} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{1}{\sqrt{2}}\left[-\frac{x}{\sqrt{x^2+y^2}}\hat{x} - \frac{y}{\sqrt{x^2+y^2}}\hat{y} + \hat{z}\right] \quad — ③$$

$$\vec{\nabla}\times\vec{F} = -xz\,\hat{x} + yz\,\hat{y} + 2\hat{z} \quad — ④$$

$$(\vec{\nabla}\times\vec{F})\cdot\vec{n} = \frac{1}{\sqrt{2}}\left[\frac{z}{\sqrt{x^2+y^2}}(x^2-y^2)+2\right] \quad — ⑤$$

at Σ ($z=\sqrt{x^2+y^2}$),

$$(\vec{\nabla}\times\vec{F})\cdot\vec{n} = \frac{1}{\sqrt{2}}\left[(x^2-y^2)+2\right] \quad — ⑥$$

$$\iint_\Sigma (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma$$

$$= \iint_D dx\,dy\ \frac{1}{\sqrt{2}}\left[(x^2-y^2)+2\right]\|\vec{N}\|$$

$$= \iint_D dx\,dy\ (x^2-y^2+2) \qquad \begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$$

$$= \int_0^3 dr \int_0^{2\pi} d\theta\ r\left[r^2\cos 2\theta + 2\right]$$

$$= 18\pi$$

Thus

$$\oint_C \vec{F}\cdot d\vec{s} = \iint_\Sigma (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, d\sigma$$

※

* $\vec{F} = f(x,y)\,\hat{x} + g(x,y)\,\hat{y}$

If $\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial y}$, $\quad \vec{F} = \nabla \varphi \qquad \varphi$: potential

$\Rightarrow$ 3-dimension 으로 확장

$$\boxed{\text{If } \vec{\nabla} \times \vec{F} = 0, \quad \vec{F} = \nabla \varphi \qquad \varphi : \text{potential}}$$

P337

(예제 7.35)

$$\vec{F} = (yz\,e^{xyz} - 4x)\,\hat{x} + (xz\,e^{xyz} + z + \cos y)\,\hat{y} + (xy\,e^{xyz} + y)\,\hat{z}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz\,e^{xyz} - 4x & xz\,e^{xyz} + z + \cos y & xy\,e^{xyz} + y \end{vmatrix} = 0$$

Thus potential 존재 !!

$$\vec{F} = \vec{\nabla} \varphi$$

$$\frac{\partial \varphi}{\partial x} = yz\,e^{xyz} - 4x$$

$$\frac{\partial \varphi}{\partial y} = xz\,e^{xyz} + z + \cos y$$

$$\frac{\partial \varphi}{\partial z} = xy\,e^{xyz} + y$$

$$\Rightarrow \varphi(x,y,z) = e^{xyz} - 2x^2 + yz + \sin y$$

*