

2계 선형 미분 방정식의 일반 form

$$\frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = Q(x)$$

If $Q(x) = 0$, Homogeneous differential Equation (제차 미분 방정식)

If $Q(x) \neq 0$, non-homogeneous differential Equation (비제차 미분 방정식)

제차 미분 방정식

$$y'' + p(x)y' + q(x)y = 0$$

Theorem. 1

If $y_1(x)$ and $y_2(x)$ are solutions, $C_1 y_1(x) + C_2 y_2(x)$ is also solution

$$(PS) \quad \left. \begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \end{aligned} \right\} \text{--- ①}$$

$$y_2'' + p(x)y_2' + q(x)y_2 = 0$$

$$(C_1 y_1 + C_2 y_2)'' = C_1 y_1'' + C_2 y_2''$$

$$(C_1 y_1' + C_2 y_2') = C_1 y_1' + C_2 y_2'$$

$$\Rightarrow (C_1 y_1 + C_2 y_2)'' + p(x)(C_1 y_1 + C_2 y_2)' + q(x)(C_1 y_1 + C_2 y_2)$$

$$= C_1 \underbrace{(y_1'' + p(x)y_1' + q(x)y_1)}_{=0} + C_2 \underbrace{(y_2'' + p(x)y_2' + q(x)y_2)}_{=0}$$

$$= 0$$

Theorem. =

If $y_1(x)$ and $y_2(x)$ are solutions and they are linearly independent with each other, the general solution is

$$y = C_1 y_1(x) + C_2 y_2(x)$$

* $f(x)$ and $g(x)$ are linearly independent if

$$W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \neq 0.$$

$$(Ex) \quad \frac{d^2 y}{dx^2} + y = 0$$

$$y_1 = \sin x$$

$$y_2 = \cos x$$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1 \neq 0.$$

y_1 and y_2 are linearly independent

\Rightarrow 일반해

$$y = C_1 \sin x + C_2 \cos x$$

pt2

(문제 2)

$$y'' + 11y' + 24y = 0 : y(0) = 1 \quad y'(0) = 4$$

(첫 번째)

$$y_1 = e^{-3x} : \text{solution}$$

$$\left(\begin{array}{l} y_1'' + 11y_1' + 24y_1 \\ = 9e^{-3x} - 33e^{-3x} + 24e^{-3x} \\ = 0 \end{array} \right)$$

$$y_2 = e^{-8x} : \text{solution}$$

$$\left(\begin{array}{l} y_2'' + 11y_2' + 24y_2 \\ = 64e^{-8x} - 88e^{-8x} + 24e^{-8x} \\ = 0 \end{array} \right)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-3x} & e^{-8x} \\ -3e^{-3x} & -8e^{-8x} \end{vmatrix} = -5e^{-11x} \neq 0$$

\Rightarrow linearly independent

$$\Rightarrow \text{일반해 } \underline{\underline{y = C_1 e^{-3x} + C_2 e^{-8x}}}$$

$$y(0) = C_1 + C_2 = 1$$

$$y'(0) = -3C_1 - 8C_2 = 4$$

$$y'(0) = -3C_1 - 8C_2 = 4$$

$$\Rightarrow C_1 = \frac{12}{5}, C_2 = -\frac{17}{5}$$

$$\Rightarrow y = \frac{12}{5} e^{-3x} - \frac{17}{5} e^{-8x}$$

*.

Theorem 3

$$y'' + p(x)y' + q(x)y = f(x)$$

일반해: $y = y_g + y_p$

y_g : general solution of homogeneous equation

y_p : particular solution of non-homogeneous equation

(Ex) $y'' - y = 4$

(i) y_g

$$y'' - y = 0$$

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$

\Rightarrow linearly independent

$$y_g = C_1 e^x + C_2 e^{-x}$$

(ii) $y_p = -4$

$$\Rightarrow y = C_1 e^x + C_2 e^{-x} - 4 \quad *$$

$$y'' + Ay' + By = 0 \quad - (1)$$

Put

$$y = e^{\lambda x} \quad - (2)$$

(2) \rightarrow (1)

$$\lambda^2 + A\lambda + B = 0 \quad : \text{characteristic equation (제2차 방정식)}$$

$$\lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2} \quad - (3)$$

(i) $A^2 - 4B > 0$

Put $a = \frac{-A + \sqrt{A^2 - 4B}}{2}, \quad b = \frac{-A - \sqrt{A^2 - 4B}}{2} \quad (4)$

Then

$$\underline{y = C_1 e^{ax} + C_2 e^{bx}}$$

pt4 (예제 2.3)

$$y'' - y' - 6y = 0$$

\Rightarrow characteristic equation

$$\lambda^2 - \lambda - 6 = 0$$

$$\rightarrow A = -1, B = -6$$

$$A^2 - 4B = 1 + 24 = 25 > 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = 3 \text{ or } -2$$

$$y = C_1 e^{3x} + C_2 e^{-2x}$$

※

$$(ii) \quad A^2 - 4B = 0$$

27

$$\underline{\underline{y = (C_1 + C_2 x) e^{-\frac{A}{2} x}}}$$

(Pf) put

$$y = u(x) e^{-\frac{A}{2} x} \quad \text{--- ①}$$

$$y' = (u' - \frac{A}{2} u) e^{-\frac{A}{2} x}$$

$$y'' = (u'' - Au' + \frac{A^2}{4} u) e^{-\frac{A}{2} x} \quad \left. \begin{array}{l} \text{--- ②} \end{array} \right\}$$

Then

$$y'' + Ay' + By = 0$$

$$\Rightarrow e^{-\frac{A}{2} x} \left[u'' + \underbrace{\left(B - \frac{A^2}{4} \right)}_{=0} u \right] = 0$$

$$\Rightarrow u'' = 0$$

$$\Rightarrow u = C_1 + C_2 x$$

#

pt 4 (ex 2.4)

$$y'' - 6y' + 9y = 0$$

\Rightarrow characteristic Eq.

$$x^2 - 6x + 9 = 0$$

$$(x-3)^2 = 0$$

$$\Rightarrow A = -6, \quad B = 9$$

$$A^2 - 4B = 36 - 36 = 0$$

$$y = (C_1 + C_2 x) e^{3x}$$

#

$$(iii) A^2 - 4B < 0$$

28

$$\lambda = \frac{-A \pm \sqrt{4B - A^2} i}{2}$$

$$y = C_1 e^{\frac{-A + \sqrt{4B - A^2} i}{2} x} + C_2 e^{\frac{-A - \sqrt{4B - A^2} i}{2} x}$$

$$= e^{-\frac{A}{2}x} \left[C_1 e^{i \frac{\sqrt{4B - A^2}}{2} x} + C_2 e^{-i \frac{\sqrt{4B - A^2}}{2} x} \right]$$

$$= e^{-\frac{A}{2}x} \left[C_1 \left(\cos \frac{\sqrt{4B - A^2}}{2} x + i \sin \frac{\sqrt{4B - A^2}}{2} x \right) + C_2 \left(\cos \frac{\sqrt{4B - A^2}}{2} x - i \sin \frac{\sqrt{4B - A^2}}{2} x \right) \right]$$

$$= e^{-\frac{A}{2}x} \left[(C_1 + C_2) \cos \frac{\sqrt{4B - A^2}}{2} x + i (C_1 - C_2) \sin \frac{\sqrt{4B - A^2}}{2} x \right]$$

$$= e^{-\frac{A}{2}x} \left[C_1 \cos \frac{\sqrt{4B - A^2}}{2} x + C_2 \sin \frac{\sqrt{4B - A^2}}{2} x \right]$$

$$\Rightarrow y = e^{-\frac{A}{2}x} \left[C_1 e^{i \frac{\sqrt{4B - A^2}}{2} x} + C_2 e^{-i \frac{\sqrt{4B - A^2}}{2} x} \right]$$

or

$$y = e^{-\frac{A}{2}x} \left[C_1 \sin \frac{\sqrt{4B - A^2}}{2} x + C_2 \cos \frac{\sqrt{4B - A^2}}{2} x \right]$$

pst

(9/21/2.5)

$$y'' + 2y' + 6y = 0$$

⇒ characteristic equation

$$\lambda^2 + 2\lambda + 6 = 0$$

$$A = 2, \quad B = 6$$

$$A^2 - 4B = 4 - 24 = -20 < 0$$

$$\Rightarrow \frac{\sqrt{4B - A^2}}{2} = \sqrt{5}$$

$$y = e^{-x} [c_1 \sin \sqrt{5} x + c_2 \cos \sqrt{5} x]$$

or

$$y = e^{-x} [c_1 e^{i\sqrt{5}x} + c_2 e^{-i\sqrt{5}x}] \quad \times$$

p57

(문제 2.7)

$$y'' - 4y' + 53y = 0$$

$$y(\pi) = -3$$

$$y'(\pi) = 2$$

⇒ characteristic Eq.

$$\lambda^2 - 4\lambda + 53 = 0$$

$$A = -4 \quad B = 53$$

$$A^2 - 4B = -196 < 0$$

$$\frac{\sqrt{4B - A^2}}{2} = 7$$

⇒ 일반해

$$y = e^{2x} [C_1 \sin 7x + C_2 \cos 7x]$$

$$y(\pi) = e^{2\pi} [C_1 \overset{=0}{\sin 7\pi} + C_2 \overset{=-1}{\cos 7\pi}] = -C_2 e^{2\pi} = -3$$

$$C_2 = 3e^{-2\pi}$$

$$y'(x) = 2e^{2x} [C_1 \sin 7x + C_2 \cos 7x]$$

$$+ 7e^{2x} [C_1 \cos 7x - C_2 \sin 7x]$$

$$= e^{2x} [(2C_1 - 7C_2) \sin 7x + (2C_2 + 7C_1) \cos 7x]$$

$$y'(\pi) = e^{2\pi} [(2C_1 - 7C_2) \overset{=0}{\sin 7\pi} + (7C_1 + 2C_2) \overset{=-1}{\cos 7\pi}]$$

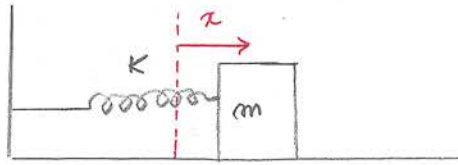
$$= -(7C_1 + 2C_2) e^{2\pi} = 2$$

$$\Rightarrow C_1 = -\frac{8}{7} e^{-2\pi}$$

$$y = e^{2x} \left[-\frac{8}{7} e^{-2\pi} \sin 7x + 3e^{-2\pi} \cos 7x \right] = e^{2(x-\pi)} \left[3 \cos 7x - \frac{8}{7} \sin 7x \right] \quad \times$$

(Ex) Harmonic Oscillator without friction

30-1



$$F = m \frac{d^2 x}{dt^2} = -Kx$$

$$\Rightarrow \frac{d^2 x}{dt^2} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{K}{m}}$$

$$\frac{d^2 y}{dx^2} + A \frac{dy}{dx} + By = 0$$

$$\begin{pmatrix} A=0 \\ B=\omega^2 \end{pmatrix}$$

$$x = C_1 \cos \omega t + C_2 \sin \omega t \quad - \textcircled{1}$$

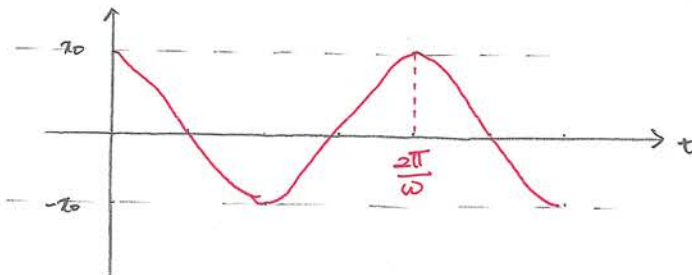
Let

$$x(0) = x_0 \quad \frac{dx}{dt}(0) = 0 \quad - \textcircled{2}$$

$$\Rightarrow \left. \begin{array}{l} C_1 = x_0 \\ C_2 = 0 \end{array} \right\} - \textcircled{3}$$

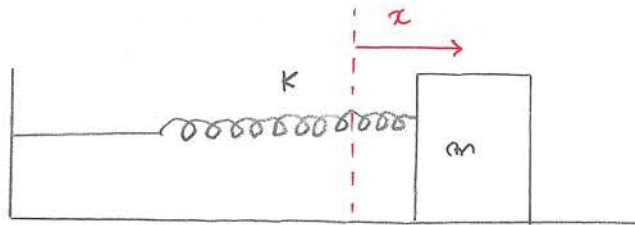
$$\textcircled{3} \rightarrow \textcircled{1}$$

$$x = x_0 \cos \omega t$$



(Ex) Harmonic Oscillator with Friction

30-2



$$F = m \frac{d^2x}{dt^2} = -kx - c \frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{c}{m} \frac{dx}{dt} + \omega^2 x = 0$$

$$\frac{dy}{dx^2} + A \frac{dy}{dx} + By = 0$$

$$A = \frac{c}{m}, B = \omega^2$$

(i) $c < 2m\omega$

$$x = e^{-\frac{c}{2m}t} \left[C_1 \cos \left[\omega^2 - \frac{c^2}{4m^2} t \right] + C_2 \sin \left[\omega^2 - \frac{c^2}{4m^2} t \right] \right] \quad - (1)$$

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0$$

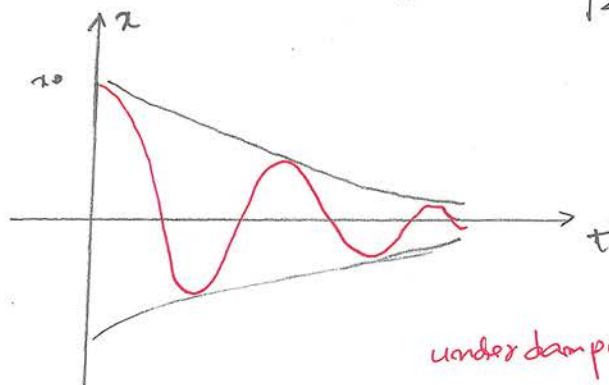
\Rightarrow

$$C_1 = x_0$$

$$C_2 = \frac{cx_0}{\sqrt{4m^2\omega^2 - c^2}} \quad \} \quad - (2)$$

$(2) \rightarrow (1)$

$$x = e^{-\frac{c}{2m}t} \left[x_0 \cos \left[\omega^2 - \frac{c^2}{4m^2} t \right] + \frac{cx_0}{\sqrt{4m^2\omega^2 - c^2}} \sin \left[\omega^2 - \frac{c^2}{4m^2} t \right] \right]$$



(ii) $c > 2m\omega$

20-3

$$x = C_1 e^{-at} + C_2 e^{-bt}$$

$$a = \frac{1}{2m} [c - \sqrt{c^2 - 4m^2\omega^2}]$$

$$b = \frac{1}{2m} [c + \sqrt{c^2 - 4m^2\omega^2}]$$

- ②

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0$$

$$C_1 = \frac{x_0 b}{b-a}$$

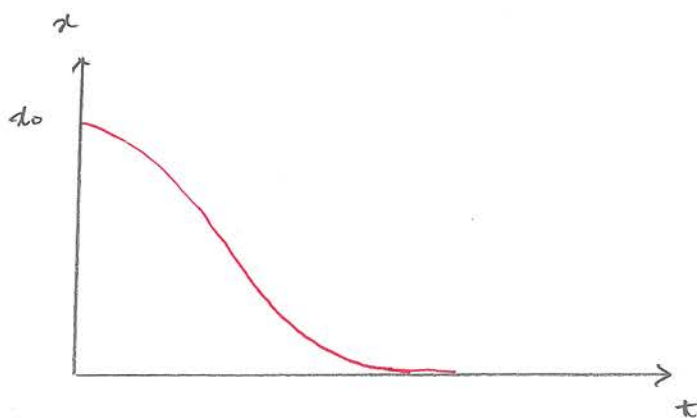
$$C_2 = \frac{-x_0 a}{b-a}$$

- ②

④ → ②

$$x = \frac{x_0}{b-a} [b e^{-at} - a e^{-bt}]$$

- ②



overdamping

$$(iii) \quad c = 2m\omega$$

30-4

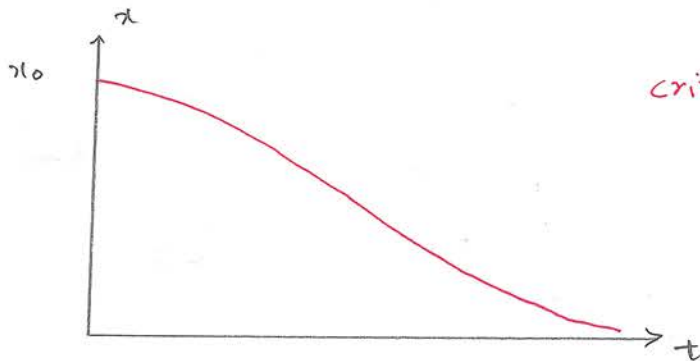
$$x = (C_1 + C_2 t) e^{-\frac{c}{2m} t} \quad - \textcircled{a}$$

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = 0$$

$$\Rightarrow C_1 = x_0, \quad C_2 = \frac{cx_0}{2m} \quad - \textcircled{b}$$

$\textcircled{a} \rightarrow \textcircled{b}$

$$x = x_0 \left(1 + \frac{c}{2m} t \right) e^{-\frac{c}{2m} t} \quad - \textcircled{c}$$



critical damping

§ Euler 방정식

$$y'' + \frac{A}{x} y' + \frac{B}{x^2} y = 0 \quad (\text{Euler Equation})$$

put

$$x = e^t$$

$$\frac{dx}{dt} = e^t$$

$$\frac{dt}{dx} = e^{-t}$$

— ①

$$y' = \frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = e^{-t} \frac{dy}{dt} \quad \text{— ②}$$

$$y'' = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d}{dx} \left(e^{-t} \frac{dy}{dt} \right)$$

$$= \frac{dt}{dx} \frac{d}{dt} \left(e^{-t} \frac{dy}{dt} \right)$$

$$= e^{-t} \left[\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right] \quad \text{— ③}$$

①, ②, ③ \Rightarrow Euler Equation

$$e^{-t} \left[\frac{d^2 y}{dt^2} + (A-1) \frac{dy}{dt} + B y \right] = 0$$

$$\Rightarrow \frac{d^2 y}{dt^2} + (A-1) \frac{dy}{dt} + B y = 0$$

5학년 수학 제1차 시험 9월 26일

pt 9

20

(Q11 2.8)

$$x^2 y'' + 2xy' - 6y = 0 \quad || \times \frac{1}{x^2}$$

$$\Rightarrow y'' + \frac{2}{x} y' - \frac{6}{x^2} = 0 \quad \text{Euler Equation } (A=2, B=-6)$$

Put

$$x = e^t \quad - (1)$$

Then

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} - 6y = 0 \quad - (2)$$

Characteristic Eq.

$$\lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 2) = 0$$

$$y = C_1 e^{3t} + C_2 e^{-2t}$$

$$= C_1 e^{3 \ln x} + C_2 e^{-2 \ln x}$$

$$= C_1 x^3 + \frac{C_2}{x^2}$$

x

$$(a(2) = .9)$$

$$x^2 y'' - 5x y' + 10y = 0 \quad y(1) = 4 \quad y'(1) = -6$$

$$\Rightarrow y'' - \frac{5}{x} y' + \frac{10}{x^2} y = 0 \quad \text{Euler Eq. } (A = -5, B = 10)$$

Put $x = e^t \quad (t = \ln x)$

Then

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 10y = 0 \quad - (2)$$

characteristic Eq

$$\lambda^2 - 6\lambda + 10 = 0 \quad - (3) \quad \left(\begin{array}{l} A^2 - 4B = 36 - 40 < 0 \\ A = -6, B = 10 \end{array} \right)$$

$$\frac{\sqrt{4B - A^2}}{2} = 1$$

general solution

$$\begin{aligned} y &= e^{3t} [C_1 \sin t + C_2 \cos t] \\ &= e^{3 \ln x} [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \\ &= x^3 [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \end{aligned}$$

$$y(1) = C_2 = 4$$

$$\begin{aligned} y'(x) &= 3x^2 [C_1 \sin(\ln x) + C_2 \cos(\ln x)] \\ &\quad + x^3 \left[C_1 \frac{1}{x} \cos(\ln x) - C_2 \frac{1}{x} \sin(\ln x) \right] \\ &= x^2 [(3C_1 - C_2) \sin(\ln x) + (C_1 + 3C_2) \cos(\ln x)] \end{aligned}$$

$$y'(1) = C_1 + 3C_2 = -6 \quad \Rightarrow C_1 = -18$$

$$\Rightarrow y = x^3 [-18 \sin(\ln x) + 4 \cos(\ln x)] \quad \times$$

$$y'' + p(x)y' + q(x)y = f(x) \quad - \textcircled{1}$$

$$y = y_h + y_p \quad - \textcircled{2}$$

$$y_h = C_1 y_1 + C_2 y_2$$

How to derive y_p ?

[1] 매개변수 방법

put

$$y_p = u(x)y_1(x) + v(x)y_2(x) \quad - \textcircled{3}$$

Then

$$y_p' = u y_1' + v y_2' + (u' y_1 + v' y_2) \quad - \textcircled{4}$$

put

$$\underline{u' y_1 + v' y_2 = 0} \quad - \textcircled{5}$$

first condition

$\textcircled{5} \rightarrow \textcircled{4}$

$$y_p' = u y_1' + v y_2' \quad - \textcircled{6}$$

$$y_p'' = u y_1'' + u' y_1' + v y_2'' + v' y_2' \quad - \textcircled{7}$$

$\textcircled{6}, \textcircled{7}, \textcircled{5} \rightarrow \textcircled{1}$

$$(u y_1'' + u' y_1' + v y_2'' + v' y_2') + p(x)(u y_1' + v y_2') + q(x)(u y_1 + v y_2) = f$$

$$\Rightarrow u(x) \underbrace{[y_1'' + p y_1' + q y_1]}_{=0} + v(x) \underbrace{[y_2'' + p y_2' + q y_2]}_{=0} + (u' y_1' + v' y_2') = f(x)$$

$$\underline{u' y_1' + v' y_2' = f(x)} \quad - \textcircled{8}$$

second condition

From ① & ②

$$u' = \frac{\begin{vmatrix} 0 & g_2 \\ f & g_1' \end{vmatrix}}{W} = \frac{-g_2 f}{W}$$

$$w' = \frac{\begin{vmatrix} g_1 & 0 \\ g_1' & f \end{vmatrix}}{W} = \frac{g_1 f}{W}$$

} - ③

$$W = \begin{vmatrix} g_1 & g_2 \\ g_1' & g_2' \end{vmatrix}$$

From ③ one can obtain u and w.

⇒ Eq. ③ g_p can be obtained!!

(제 21.10)

$$y'' + 4y = \sec x$$

(i) Homogeneous equation

$$y'' + 4y = 0$$

$$y_1 = \cos 2x \quad \left. \vphantom{y_1} \right\} - \textcircled{1}$$

$$y_2 = \sin 2x$$

$$\Rightarrow y_h = C_1 \cos 2x + C_2 \sin 2x \quad - \textcircled{2}$$

(ii) Non-Homogeneous equation.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$f(x) = \sec x = \frac{1}{\cos x}$$

$$\Rightarrow u' = \frac{-y_2 f}{2} = -\frac{1}{2} \sin 2x \cdot \frac{1}{\cos x} = -\sin x$$

$$v' = \frac{y_1 f}{2} = \frac{1}{2} \cos 2x \cdot \frac{1}{\cos x} = \frac{1}{2} (2\cos^2 x - 1) \cdot \frac{1}{\cos x} = \cos x - \frac{1}{2} \sec x$$

$$\Rightarrow u = \cos x$$

$$v = \sin x - \int \sec x dx = \sin x - \frac{1}{2} \ln |\sec x + \tan x|$$

$$\Rightarrow y_p(x) = u y_1 + v y_2$$

$$= \cos x \cos 2x + \left(\sin x - \frac{1}{2} \ln |\sec x + \tan x| \right) \sin 2x$$

(iii) 일반해

$$y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x + \cos x \cos 2x + \left(\sin x - \frac{1}{2} \ln |\sec x + \tan x| \right) \sin 2x$$

$$y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1 \quad (x > 0)$$

(i) Homogeneous Eq.

$$y'' - \frac{4}{x} y' + \frac{4}{x^2} y = 0 \quad \text{Euler Equation } (A = -4, B = 4)$$

Put

$$x = e^t \quad - (1) \quad (t = \ln x)$$

Then

$$\frac{d^2 y}{dx^2} - 5 \frac{dy}{dt} + 4y = 0 \quad - (2)$$

\Rightarrow characteristic Eq.

$$m^2 - 5m + 4 = 0$$

$$(m-4)(m-1) = 0$$

$$y_1 = e^{4t} = e^{4 \ln x} = x^4$$

$$y_2 = e^t = e^{\ln x} = x$$

$$y_g = C_1 y_1 + C_2 y_2 = C_1 x^4 + C_2 x$$

(ii) non-homogeneous Eq.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^4 & x \\ 4x^3 & 1 \end{vmatrix} = -3x^4$$

$$\Rightarrow u' = - \frac{\int f}{W} = - \frac{x(x^2+1)}{-3x^4} = \frac{1}{3} \left(\frac{1}{x} + \frac{1}{x^3} \right)$$

$$u = \frac{1}{3} \left[\ln x - \frac{1}{2} \frac{1}{x^2} \right] = \frac{1}{3} \ln x - \frac{1}{6} \frac{1}{x^2}$$

$$\Rightarrow v' = \frac{y_1 f}{W} = \frac{x^4 \cdot (x^2 + 1)}{-3x^4} = -\frac{1}{3}(x^2 + 1)$$

$$v = -\frac{1}{3} \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) = -\frac{1}{9}x^3 - \frac{1}{6}x^2$$

$$\begin{aligned} y_p &= uy_1 + vx_2 = \left(\frac{1}{2} \ln x - \frac{1}{6} \frac{1}{x^2} \right) x^4 + \left(-\frac{1}{9}x^3 - \frac{1}{6}x^2 \right) x \\ &= \frac{x^2}{2} \left[x^2 \ln x - \frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{2} \right] \end{aligned}$$

(iv) general solution

$$y = y_g + y_p$$

$$= C_1 x^4 + C_2 x + \frac{x^2}{2} \left[x^2 \ln x - \frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{2} \right]$$

[2] 미분방정식

p65

(0121 2.12)

$$y'' - 4y = 8x^2 - 2x$$

(i) homogeneous Eq.

$$y'' - 4y = 0$$

$$y_g = C_1 e^{2x} + C_2 e^{-2x}$$

(ii) non-homogeneous Eq.

$$y_p = ax^2 + bx + c$$

$$y_p'' = 2a$$

$$\Rightarrow y_p'' - 4y_p = -4ax^2 - 4bx + (2a - 4c) = 8x^2 - 2x$$

$$a = -2, \quad b = \frac{1}{2}, \quad c = -1$$

$$\Rightarrow y_p = -2x^2 + \frac{1}{2}x - 1$$

(iii) general solution

$$y = y_g + y_p = C_1 e^{2x} + C_2 e^{-2x} - 2x^2 + \frac{1}{2}x - 1 \quad *$$

p66

(09/21/2.13)

$$y'' + 2y' - 3y = 4e^{2x}$$

(i) Homogeneous Eq.

$$y'' + 2y' - 3y = 0$$

$$y_h = c_1 e^{-3x} + c_2 e^x$$

(ii) non-homogeneous Eq.

Put

$$y_p = a e^{2x}$$

$$y_p'' + 2y_p' - 3y_p = 5a e^{2x} = 4e^{2x}$$

$$\Rightarrow 5a = 4$$

$$a = \frac{4}{5}$$

$$\Rightarrow y_p = \frac{4}{5} e^{2x}$$

(iii) general solution

$$y = c_1 e^{-3x} + c_2 e^x + \frac{4}{5} e^{2x}$$

p66

(Ques 2.14)

$$y'' - 5y' + 6y = -3 \sin 2x$$

(i) Homogeneous Eq.

$$y'' - 5y' + 6y = 0$$

$$y = C_1 e^{2x} + C_2 e^{3x}$$

(ii) non-homogeneous Eq.

$$y_p = a \sin 2x + b \cos 2x$$

$$y_p' = 2a \cos 2x - 2b \sin 2x$$

$$y_p'' = -4a \sin 2x - 4b \cos 2x$$

$$\Rightarrow y_p'' - 5y_p' + 6y_p$$

$$= (2a + 10b) \sin 2x + (-10a + 2b) \cos 2x$$

$$= -3 \sin 2x$$

$$2a + 10b = -3$$

$$-10a + 2b = 0$$

$$\Rightarrow a = -\frac{3}{52}, \quad b = -\frac{15}{52}$$

$$\Rightarrow y_p = -\frac{3}{52} \sin 2x - \frac{15}{52} \cos 2x$$

(iii) general solution

$$y = C_1 e^{2x} + C_2 e^{3x} - \frac{3}{52} \sin 2x - \frac{15}{52} \cos 2x$$

$$y'' + 2y' - 3y = 8e^x$$

(i) Homogeneous Eq.

$$y'' + 2y' - 3y = 0$$

$$y_g = C_1 e^{-3x} + C_2 e^x$$

(ii) non-homogeneous Eq.

(A) 첫 시도

$y_g + y_p$

$$= C_1 e^{-3x} + C_2 e^x + a e^x$$

$$= C_1 e^{-3x} + (C_2 + a) e^x$$

$$= C_1 e^{-3x} + C_2 e^x$$

$$y_p = a e^x \Rightarrow y_g \text{ 가 항상 0}$$

$$y_p'' + 2y_p' - 3y_p = 0 \neq 8e^x$$

(B) Another Try.

$$y_p = a x e^x$$

$$y_p' = a(x+1)e^x$$

$$y_p'' = a(x+2)e^x$$

$$y_p'' + 2y_p' - 3y_p = 4a e^x = 8e^x$$

$$a = 2$$

$$y_p = 2x e^x$$

(iii) general solution

$$y = y_g + y_p = C_1 e^{-3x} + C_2 e^x + 2x e^x$$

$$y_1 = e^{-3x} \quad y_2 = e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 4e^{-2x}$$

$$f = 8e^x$$

$$u' = -\frac{y_2 f}{W} = -2e^{4x}$$

$$\Rightarrow u = -\frac{1}{2} e^{4x}$$

$$v' = \frac{y_1 f}{W} = 2$$

$$\Rightarrow v = 2x$$

$$y_p = u y_1 + v y_2 = -\frac{1}{2} e^x + 2x e^x$$

$$y = y_g + y_p$$

$$= C_1 e^{-3x} + C_2 e^x - \frac{1}{2} e^x + 2x e^x$$

$$= C_1 e^{-3x} + (C_2 - \frac{1}{2}) e^x + 2x e^x$$

$$= C_1 e^{-3x} + C_2 e^x + 2x e^x \quad \times$$

(09/21/2017)

$$y'' - 6y' + 9y = 5e^{3x}$$

(i) homogeneous Eq

$$y'' - 6y' + 9y = 0$$

$$y_3 = (C_1 + C_2 x) e^{3x}$$

(ii) $y_p = a x^2 e^{3x}$

$$y_p' = a(3x^2 + 2x) e^{3x}$$

$$y_p'' = a(9x^2 + 12x + 2) e^{3x}$$

$$y_p'' - 6y_p' + 9y_p = 2a e^{3x} = 5e^{3x}$$

$$a = \frac{5}{2}$$

$$y_p = \frac{5}{2} x^2 e^{3x}$$

(iii) general solution

$$y = y_3 + y_p = (C_1 + C_2 x) e^{3x} + \frac{5}{2} x^2 e^{3x}$$

$$y_1 = e^{3x} \quad y_2 = x e^{3x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{6x}$$

$$f = 5e^{3x}$$

$$\Rightarrow u' = -\frac{y_2 f}{W} = -5x$$

$$u = -\frac{5}{2} x^2$$

$$\Rightarrow v' = \frac{y_1 f}{W} = 5$$

$$v = 5x$$

$$y_p = u y_1 + v y_2 = \frac{5}{2} x^2 e^{3x}$$

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

Let

$$y_{p1}'' + p(x)y_{p1}' + q(x)y_{p1} = f_1(x)$$

$$y_{p2}'' + p(x)y_{p2}' + q(x)y_{p2} = f_2(x)$$

Put

$$y = y_{p1} + y_{p2}$$

$$\Rightarrow y'' + p(x)y' + q(x)y$$

$$= (y_{p1}'' + y_{p2}'') + p(x)(y_{p1}' + y_{p2}') + q(x)(y_{p1} + y_{p2})$$

$$= \underbrace{(y_{p1}'' + p(x)y_{p1}' + q(x)y_{p1})}_{f_1(x)} + \underbrace{(y_{p2}'' + p(x)y_{p2}' + q(x)y_{p2})}_{f_2(x)}$$

$$= f_1(x) + f_2(x)$$

(Q11 2.19)

$$y'' + 4y = x + 2e^{-2x}$$

(i) homogeneous equation

$$y = C_1 \sin 2x + C_2 \cos 2x$$

(ii) non-homogeneous Eq.

① first term

$$y'' + 4y = x$$

$$y_{p1} = \frac{x}{4}$$

② second term

$$y'' + 4y = 2e^{-2x}$$

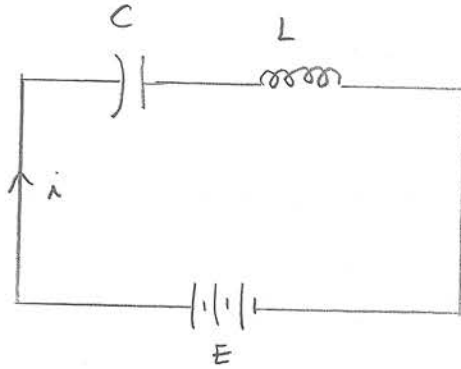
$$y_{p2} = \frac{1}{4} e^{-2x}$$

$$\Rightarrow y_p = y_{p1} + y_{p2} = \frac{1}{4} (x + e^{-2x})$$

(iii) general solution

$$y = C_1 \sin 2x + C_2 \cos 2x + \frac{1}{4} (x + e^{-2x}) \quad \#$$

(i) 직류 전원 나 LC 회로



$$E = \frac{q}{C} + L \frac{di}{dt} \quad - ①$$

$$i = \frac{dq}{dt}$$

$$L \frac{d^2q}{dt^2} + \frac{q}{C} = E$$

$$\leftrightarrow L \rightarrow m$$

Harmonic Oscillator

$$\frac{1}{C} \rightarrow k$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{1}{LC} q = \frac{E}{L}$$

$$\left(\begin{array}{l} q(0) = q_0 \\ \frac{dq}{dt}(0) = 0 \end{array} \right)$$

$$q(t) = A \cos \omega_0 t + B \sin \omega_0 t + CE \quad - ②$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$A = q_0 - CE$$

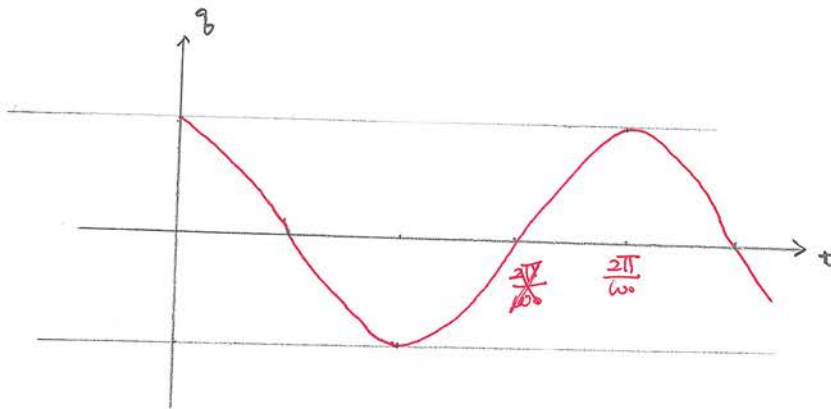
$$B = 0$$

② → ②

$$q(t) = (q_0 - CE) \cos \omega_0 t + CE \quad - ③$$

If $E=0$,

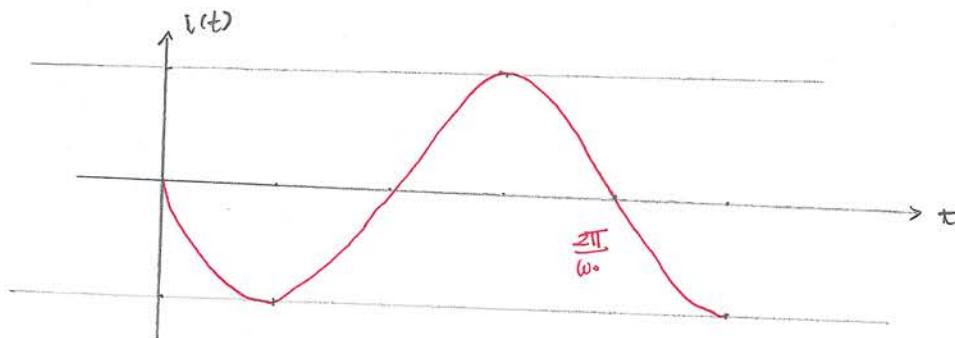
$$q(t) = q_0 \cos \omega_0 t$$



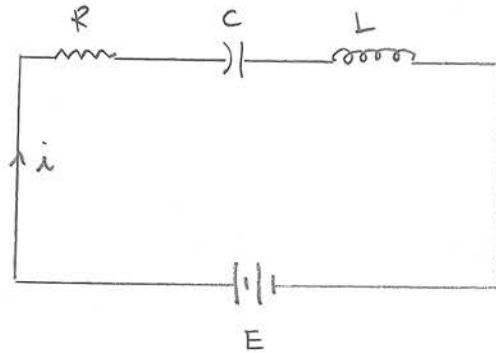
$$i = \frac{dq}{dt} = -(q_0 - CE) \omega_0 \sin \omega_0 t \quad - \textcircled{D}$$

If $E=0$,

$$i = -q_0 \omega_0 \sin \omega_0 t$$



(ii) 직류 전압과 RLC 회로



$$E = iR + \frac{q}{C} + L \frac{di}{dt} \quad - (1)$$

$$i = \frac{dq}{dt}$$

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = \frac{E}{L} \quad - (2)$$

non-homogeneous Eq.

(A) $E = 0 \cos$

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0 \quad - (2) \Leftrightarrow m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + \omega^2 x = 0$$

$$m=1$$

$$c = \frac{R}{L}$$

$$\omega^2 = \frac{1}{LC}$$

$$* \frac{d^2y}{dx^2} + A \frac{dy}{dx} + By = 0$$

$$y = e^{-\frac{A}{2}x} \left[C_1 \sin \frac{\sqrt{4B-A^2}}{2} x + C_2 \cos \frac{\sqrt{4B-A^2}}{2} x \right] \quad \text{if } A^2 < 4B$$

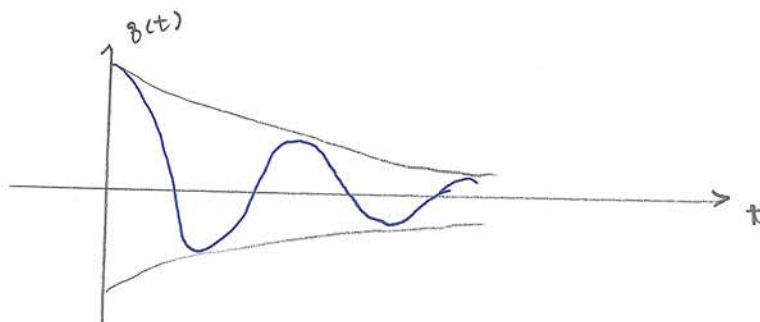
Put

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad - (3)$$

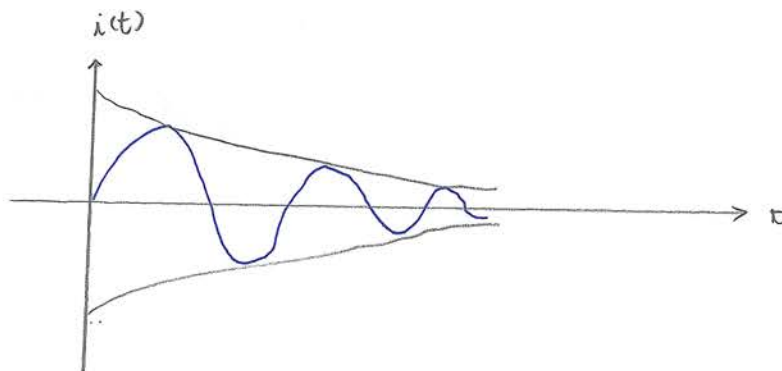
If $\frac{R^2}{L^2} < 4\omega_0^2$,

$$q(t) = e^{-\frac{R}{2L}t} \left[c_1 \sin \omega t + c_2 \cos \omega t \right]$$

$$\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$$



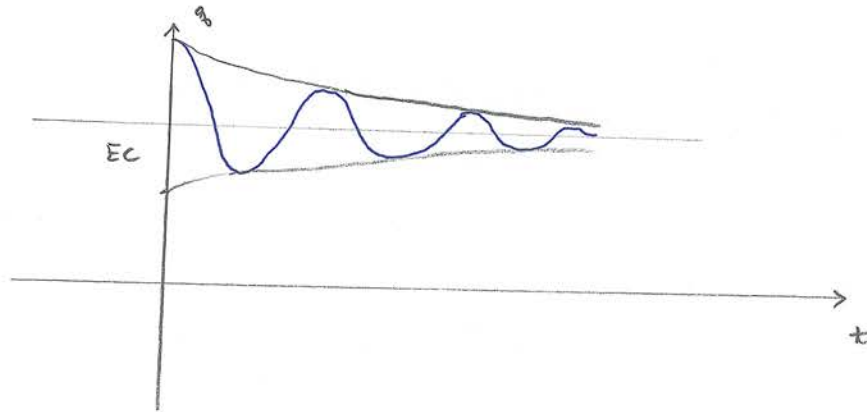
$$i(t) = \frac{dq}{dt} = e^{-\frac{R}{2L}t} \left[\left(\omega c_1 - \frac{R}{2L} c_2 \right) \cos \omega t - \left(\omega c_2 + \frac{R}{2L} c_1 \right) \sin \omega t \right]$$



(B) $E \neq 0$ and $(\frac{R^2}{L^2} < 4\omega^2)$

50

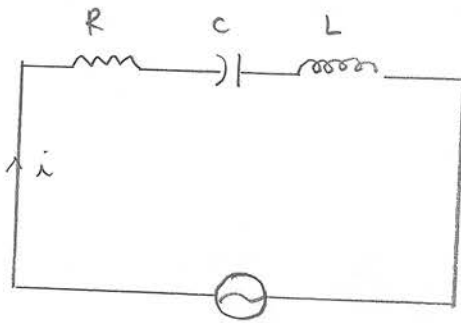
$$q = e^{-\frac{R}{2L}t} \left[C_1 \sin \omega t + C_2 \cos \omega t \right] + E C$$



$$i = \frac{dq}{dt} = e^{-\frac{R}{2L}t} \left[(\omega C_1 - \frac{R}{2L} C_2) \cos \omega t - (\omega C_2 + \frac{R}{2L} C_1) \sin \omega t \right]$$

(iii) 과감 진동인 RLC 회로 ($\frac{R^2}{L^2} < 4\omega_0^2$)

21



$$E = V_0 \sin \omega t$$

$$\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \omega_0^2 q = \frac{V_0}{L} \sin \omega t \quad) - ①$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$\frac{d}{dt}$ Eq. ①

$$\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \omega_0^2 i = \frac{V_0}{L} \omega \cos \omega t \quad - ②$$

non-homogeneous Eq.

$$i = i_g + i_p$$

$$i_g = e^{-\frac{R}{2L}t} \left[\left(\tilde{\omega} C_1 - \frac{R}{2L} C_2 \right) \cos \tilde{\omega} t - \left(\tilde{\omega} C_2 + \frac{R}{2L} C_1 \right) \sin \tilde{\omega} t \right]$$

$$\tilde{\omega} = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}} \quad - ③$$

($t \gg 1$ or $\dot{i}_c \rightarrow 0$)

$$i_p = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \sin(\omega t - \phi) \quad - (4)$$

$$\sqrt{R^2 + (X_L - X_C)^2} : \text{Impedance}$$

$$\left(\begin{array}{l} X_L = \omega L \\ X_C = \frac{1}{\omega C} \\ X = X_L - X_C = \omega L - \frac{1}{\omega C} : \text{Reactance} \\ \tan \phi = \frac{X}{R} \end{array} \right) \quad - (5)$$

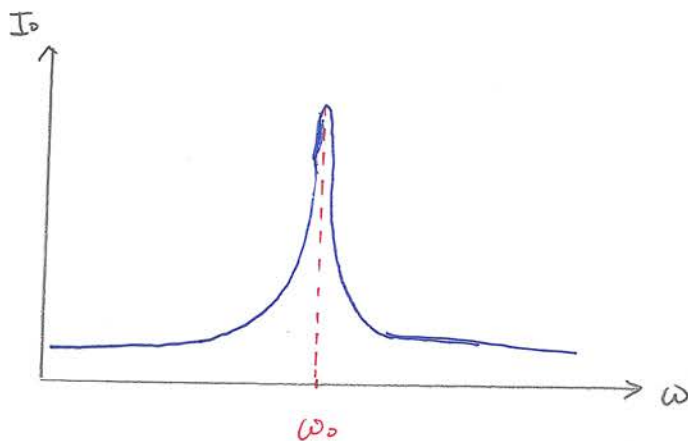
P_{avg}

$$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \quad - (6)$$

$$i_p = I_0 \sin(\omega t - \phi) \quad - (7)$$

If $\omega = \omega_0 = \frac{1}{\sqrt{LC}}, \quad X_L - X_C = 0.$

$$I = \frac{V_0}{R} \Rightarrow Z_0 \text{ or } Z$$



* i_p $\frac{1}{\pi} E$

53

Let

$$i_p = I_0 \sin(\omega t - \phi)$$

$$\frac{di_p}{dt} = I_0 \omega \cos(\omega t - \phi)$$

$$\frac{d^2 i_p}{dt^2} = -I_0 \omega^2 \sin(\omega t - \phi)$$

— (★1)

(★1) \rightarrow ②

$$\sin \omega t \left[I_0 (\omega_0^2 - \omega^2) \cos \phi + \frac{R}{L} I_0 \omega \sin \phi \right]$$

$$+ \cos \omega t \left[\frac{R}{L} I_0 \omega \cos \phi - I_0 (\omega_0^2 - \omega^2) \sin \phi - \frac{V_0}{L} \omega \right] = 0 \quad (\star 2)$$

From Eq. (★2)

$$(\omega_0^2 - \omega^2) \cos \phi + \frac{R}{L} \omega \sin \phi = 0 \quad (\star 3)$$

$$I_0 \left[\frac{R}{L} \omega \cos \phi - (\omega_0^2 - \omega^2) \sin \phi \right] = \frac{V_0}{L} \omega \quad (\star 4)$$

From (★3)

$$\tan \phi = - \frac{\omega_0^2 - \omega^2}{\frac{R}{L} \omega}$$

$$= - \frac{1}{R} \left[\frac{L}{\omega} \frac{1}{L C} - \frac{1}{\omega} \omega^2 \right]$$

$$= \frac{1}{R} \left[\omega L - \frac{1}{\omega C} \right]$$

(★5)

Let

$$\left. \begin{aligned} X_L &\equiv \omega L \\ X_C &\equiv \frac{1}{\omega C} \end{aligned} \right\} \quad (46)$$

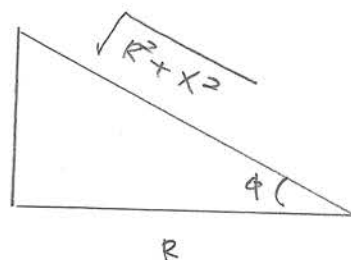
and

$$X \equiv X_L - X_C = \omega L - \frac{1}{\omega C} \quad (47)$$

reactance

Then Eq. (45) becomes

$$\tan \phi = \frac{X}{R} \quad (48)$$



Eq. (44) becomes

$$I_0 \left[\frac{R}{L} \omega \frac{R}{\sqrt{R^2 + X^2}} - (\omega_0^2 - \omega^2) \frac{X}{\sqrt{R^2 + X^2}} \right] = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \left[\frac{R^2}{L} \omega - (\omega_0^2 - \omega^2) X \right] = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \frac{\omega}{L} \left[R^2 - \frac{L}{\omega} (\omega_0^2 - \omega^2) X \right] = \frac{V_0}{L} \omega$$

$$\frac{L}{\omega} \left[\frac{1}{L C} - \omega^2 \right]$$

$$= \frac{1}{\omega C} - \omega L$$

$$= -(X_L - X_C)$$

$$= -X$$

$$\Rightarrow I_0 \frac{1}{\sqrt{R^2 + X^2}} \frac{\omega}{L} (R^2 + X^2) = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 \frac{\omega}{L} \sqrt{R^2 + X^2} = \frac{V_0}{L} \omega$$

$$\Rightarrow I_0 = \frac{V_0}{\sqrt{R^2 + X^2}} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \quad *$$