§ complex plane

$$z = x + iy = \underline{re^{i\theta}} \qquad (i = \sqrt{-1}) \quad : \text{complex number}$$

polar form (극형식)

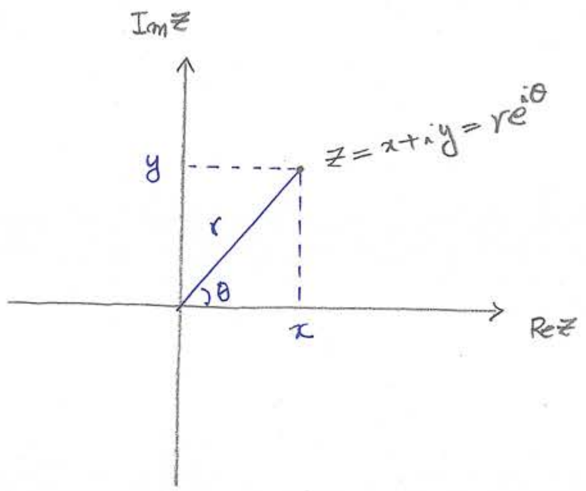
$x = \mathrm{Re}\, z$: real part of z

$y = \mathrm{Im}\, z$: Imaginary part of z

$r = |z| = \mathrm{Mod}(z)$: absolute value of z or modulus of z

$\theta = \mathrm{Arg}(z)$: $\begin{cases} \text{argument of } z \\ \text{angle of } z \\ \text{phase of } z \end{cases}$

$$\boxed{\begin{array}{ll} r = \sqrt{x^2 + y^2} & \theta = \tan^{-1}\dfrac{y}{x} \\ x = r\cos\theta & y = r\sin\theta \end{array}}$$



(complex plane)

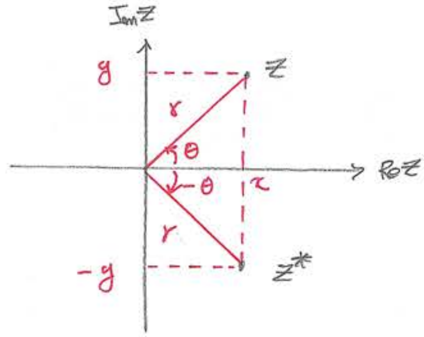
정의: Complex conjugate (공액 복소수)

If $z = x + iy$, its complex conjugate z^* is defined as

$$z^* = x - iy = re^{-i\theta}$$

note)

z	z^*
x	x
y	$-y$
r	r
θ	$-\theta$



note) $|z|^2 = zz^*$

p503

(예제 10.1)

$z = 3 + 3i$

$r = \sqrt{3^2 + 3^2} = \sqrt{18}$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \cdots)$

$z = \sqrt{18}\, e^{i(\frac{\pi}{4} + 2n\pi)}$

If we choose $0 \le \theta < 2\pi$, $\theta = \frac{\pi}{4}$. Then

$z = \sqrt{18}\, e^{i\frac{\pi}{4}}$ ※

* Complex algebra

(A) Simplification

(Ex) $(1+i)^2 = 2i$

(Ex) $\dfrac{2+i}{3-i} = \dfrac{1}{2}(1+i)$

(B) Finding complex conjugate

$$(z_1+z_2)^* = z_1^* + z_2^*$$

$$(z_1 z_2)^* = z_1^* z_2^*$$

$$\left(\frac{z_2}{z_1}\right)^* = \frac{z_2^*}{z_1^*}$$

Ex) $z = \dfrac{2-3i}{i+4}$ $\qquad z^* = \dfrac{2+3i}{-i+4}$

(c) Finding $|z|$

$$|z| = \sqrt{z z^*}$$

Ex) $z = \dfrac{\sqrt{5}+3i}{1-i}$

$$|z|^2 = z z^* = \frac{\sqrt{5}+3i}{1-i}\;\frac{\sqrt{5}-3i}{1+i} = \frac{14}{2} = 7$$

$$|z| = \sqrt{7}$$

(D) Complex equation

Ex) $z^2 = 2i$

Put $z = x + iy$

$$(x^2 - y^2) + 2xyi = 2i$$

$$x^2 - y^2 = 0$$

$$xy = 1$$

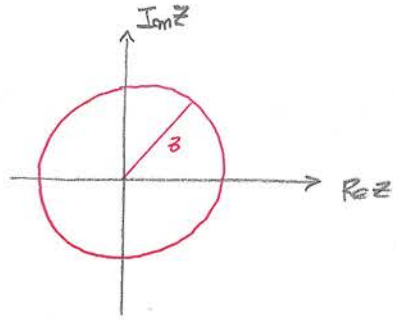
$x = y = 1$ or $x = y = -1$

$z = 1 + i$ or $-1 - i$

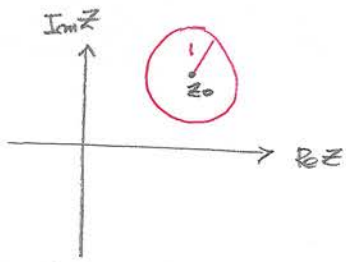
(E) Graphs

(Ex) $|z| = 3$

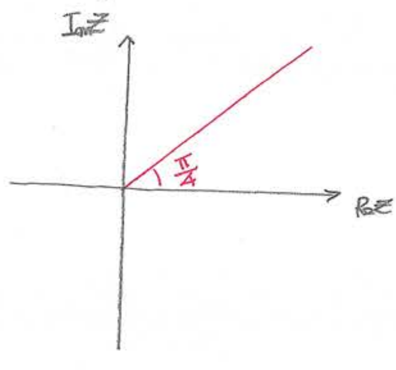
$z = 3e^{i\theta}$



(Ex) $|z - z_0| = 1$

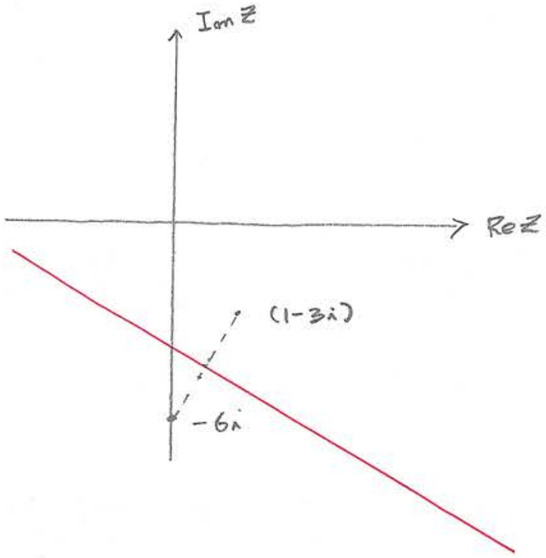


(Ex) $\arg z = \frac{\pi}{4}$



(예제 10.3)

$$|z+6i| = |z-(1-3i)|$$



$$|z+6i|^2 = |z-(1-3i)|^2$$

$$(z+6i)(z^*-6i) = (z-1+3i)(z^*-1-3i)$$

$$zz^*+36-6i(z-z^*) = zz^*+10-(z+z^*)-3i(z-z^*)$$

$$\Rightarrow \quad 26 = -(z+z^*)+3i(z-z^*) \qquad -①$$

Put

$$\left.\begin{aligned} z &= x+iy \\ z^* &= x-iy \end{aligned}\right) \quad -②$$

Then Eq. ① becomes

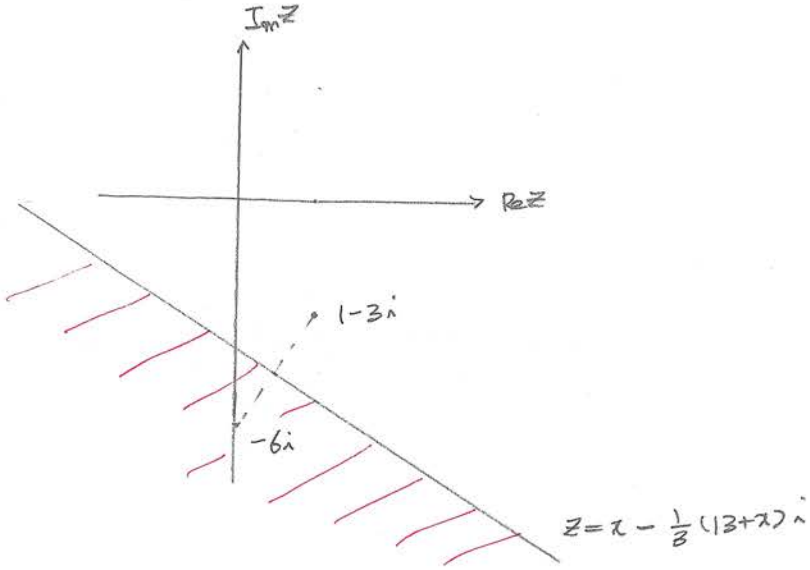
$$13 = -x-3y$$

$$\Rightarrow y = -\frac{1}{3}(13+x)$$

$$\Rightarrow z = x-\frac{1}{3}(13+x)i$$

//

Ex) $|z+6i| < |z-(1-3i)|$



p506

(예제 10.4)

$$|z|^2 + 3\,\mathrm{Re}\,z^2 = 4 \qquad - ①$$

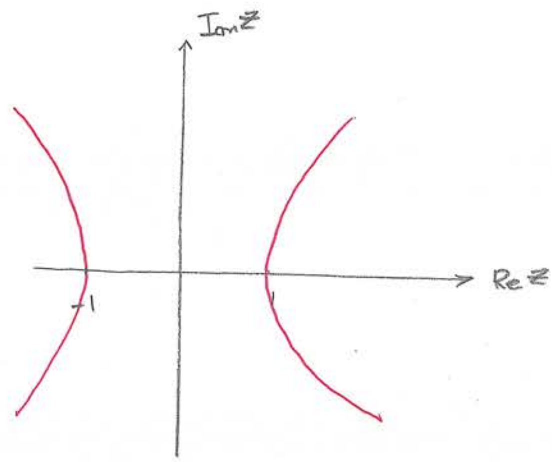
Put $z = x + iy$

$z^2 = (x^2 - y^2) + 2ixy$) $-$ ②

② → ①

$$x^2 + y^2 + 3(x^2 - y^2) = 4$$

$$2x^2 - y^2 = 2$$



수열 (Series)

$$\boxed{\{z_0, z_1, z_2 \cdots\} = \{z_n \mid n = 0, 1, 2, \cdots\}}$$

If $\lim_{n\to\infty} z_n = z_0$ (fixed), we say "Series $\{z_n\}$ is convergent to z_0".

If $\lim_{n\to\infty} z_n$ is not fixed, we say "Series $\{z_n\}$ is divergent".

예7

(예제 10.5)

$$\{i^n \mid n = 1, 2, \cdots\}$$

$$\lim_{n\to\infty} i^n = \begin{cases} i & n = 4m+1 \\ -1 & n = 4m+2 \\ -i & n = 4m+3 \\ 1 & n = 4m \end{cases}$$

$\Rightarrow$ divergent series

(예제 10.6)

$$\{z_n = 1 + \frac{i}{n} \mid n = 1, 2, \cdots\}$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \left(1 + \frac{i}{n}\right) = 1$$

$\Rightarrow$ Convergent series

(정리 10.3)

$$\{z_n = x_n + i y_n \mid n = 0, 1, 2, \cdots\}$$

If $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$, $\{z_n\}$ is convergent to $a+ib$.

Ex) $\{z_n = (1 + \frac{1}{n})^n + \frac{n+2}{n} i\}$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e, \quad \lim_{n\to\infty} \frac{n+2}{n} i = i$$

$\Rightarrow$ Convergent to $e+i$ ※

p510

§ Derivative

Derivative of $f(z)$

$$\boxed{f'(z) \equiv \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[f(z+\Delta z) - f(z) \right]}$$

note) If $\lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[f(z+\Delta z) - f(z) \right]$ is dependent on the approaching direction, we say $f'(z)$ does not exist !!

p510
(예제 10.9)

$$\frac{d}{dz} z^2 = \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = 2z$$

$$\frac{d}{dz} z^n = n z^{n-1}$$

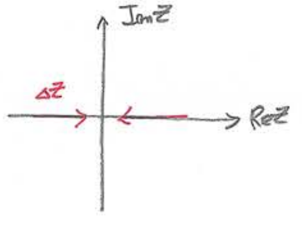
ex) $f(z) = |z|^2$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[|z+\Delta z|^2 - |z|^2 \right] \quad — ①$$

Put
$$\left.\begin{aligned} z &= x+iy \\ \Delta z &= \Delta x + i\Delta y \end{aligned}\right\} — ②$$

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{1}{\Delta z}\left[(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2) \right] = \lim_{\Delta z \to 0} \frac{2x\Delta x + 2y\Delta y}{\Delta x + i\Delta y} \quad — ③$$

(i)

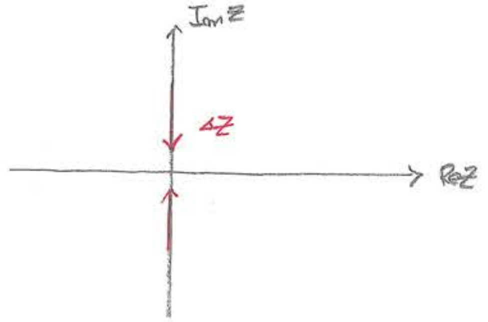


$\Delta y = 0$

$\Delta x \to 0$

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{2x\Delta x}{\Delta x} = 2x$$

(ii)



$\Delta x = 0$

$\Delta y \to 0$

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{2y\,\Delta y}{i\,\Delta y} = -2iy$$

$\frac{d}{dz}|z|^2$ does not exist !! ※

definition : analytic function

A function $f(z)$ is analytic (or regular, or holomorphic, or monogenic)

if it has a unique derivative at every point of the complex plane.

Ex)

$f(z) = z^n$: analytic function

$f(z) = |z|^2$: non-analytic function

p514

Theorem 10.7 : Cauchy - Riemann condition

If $f(z) = u(x,y) + i\,v(x,y)$ is analytic in region R,

then in that region

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Cauchy - Riemann Condition

Ex) $f(z) = z^2 = u(x,y) + i\,v(x,y)$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x \qquad \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$f(z)$: analytic in whole complex plane

p514
(예제 10.12)

$f(z) = z^* = u(x,y) + i\,v(x,y)$

$u(x,y) = x \quad , \quad v(x,y) = -y$

$\frac{\partial u}{\partial x} = 1 \quad , \quad \frac{\partial v}{\partial y} = -1 \qquad \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 0 \quad , \quad \frac{\partial v}{\partial x} = 0 \qquad \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow f(z) = z^*$ is not analytic in whole complex plane

$\Rightarrow f(z) = z^*$ 는 whole complex plane 에서 미분 불가능

(예제 10.13)

$f(z) = z\,\mathrm{Re}\,z = (x+iy)x = x^2 + ixy = u(x,y) + i\,v(x,y)$

$u(x,y) = x^2, \quad v(x,y) = xy$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = x \qquad \Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ except $x = 0$

$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = y \qquad \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ except $y = 0$

$\Rightarrow f(z) = z\,\mathrm{Re}\,z$ is not analytic except $z = 0$.

$\Rightarrow f(z) = z\,\mathrm{Re}\,z$ 는 $z = 0$ 을 제외하고는 미분 불가능 ※

note)

Cauchy-Riemann condition is necessary condition for differentiability of complex function.

If Cauchy-Riemann condition is not satisfied at $z=z_0$ of $f(z)$, $f(z)$ is not differentiable at $z=z_0$.

If Cauchy-Riemann condition is satisfied at $z=z_0$ of $f(z)$, we can not say anything on the differentiability of $f(z)$ at $z=z_0$.

p515

Theorem 10.8

If $u(x,y)$ and $v(x,y)$ and their partial derivatives with respect to x and y are continuous and $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann conditions at $z=z_0$,

$f(z)$ is analytic at $z=z_0$

Ex) $f(z) = z^2 = u(x,y) + i\,v(x,y)$

$u(x,y) = x^2-y^2$, $v(x,y) = 2xy$ $\Rightarrow$ satisfy Cauchy-Riemann condition

$u(x,y)$, $v(x,y)$: continuous

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = -2y$, $\frac{\partial v}{\partial x} = 2y$, $\frac{\partial v}{\partial y} = 2x$ $\Rightarrow$ all continuous

$\Rightarrow f(z) = z^2$ is analytic function in whole complex plane. ※

p516

§ Power Series

* Convergence of Series

① $S = \sum_{n=0}^{\infty} f_n$

$$\rho = \lim_{n \to \infty} \left| \frac{f_{n+1}}{f_n} \right|$$

If $\rho < 1$, S is convergent

If $\rho > 1$, S is divergent

If $\rho = 1$, we do not know !! : ratio test

② $S = \sum_{n=0}^{\infty} (-1)^n f_n = f_0 - f_1 + f_2 - f_3 + \cdots$: alternating series

If $\lim_{n \to \infty} f_n = 0$, S is convergent !!

Ex) $S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$

p517

(예제 10.14)

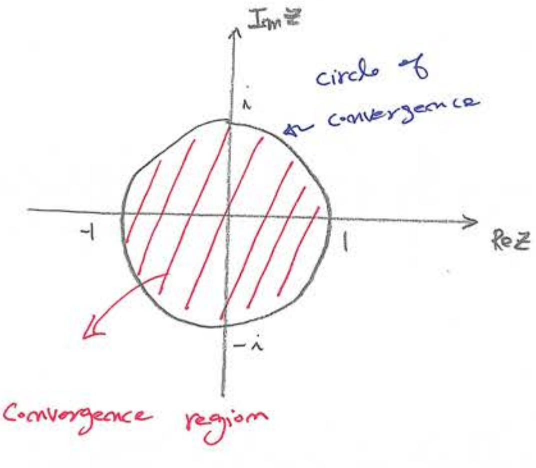
$$S = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

$$\rho = \lim_{n \to \infty} \left| \frac{z^{n+1}}{z^n} \right| = |z|$$

If $|z| < 1$, $S = \frac{1}{1-z}$

If $|z| > 1$, $S \rightarrow$ divergent

※



p518

(예제 10.15)

$$S = \sum_{n=1}^{\infty} (-1)^n \frac{2-i}{(1+i)^n}$$

$$\rho = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} \dfrac{2-i}{(1+i)^{n+1}}}{(-1)^n \dfrac{2-i}{(1+i)^n}} \right|$$

$$= \left| \frac{1}{1+i} \right|$$

$$= \frac{1}{|1+i|}$$

$$= \frac{1}{\sqrt{2}} < 1$$

S is convergent !! *

p518

definition : Power Series

$\sum_{n=0}^{\infty} C_n (z-z_0)^n$ is called Power Series (멱급수).

z_0 : center of series

C_n : coefficient

p519

(예제 10. 16)

$$S = \sum_{n=1}^{\infty} \left(\frac{2}{3i}\right)^n (z-i)^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{\left(\frac{2}{3i}\right)^{n+1} (z-i)^{n+1}}{\left(\frac{2}{3i}\right)^n (z-i)^n} \right|$$

$$= \left| \frac{2}{3i} (z-i) \right|$$

$\downarrow \; |z_1 z_2| = |z_1| \, |z_2|$

$$= \left| \frac{2}{3i} \right| |z-i|$$

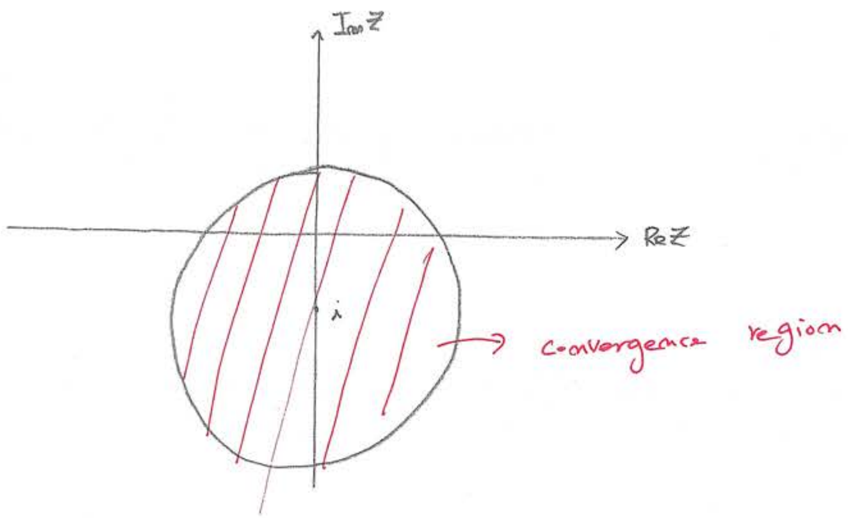
$\downarrow \; \left| \frac{z_2}{z_1} \right| = \frac{|z_2|}{|z_1|}$

$$= \frac{|2|}{|3i|} |z-i|$$

$$= \frac{2}{3} |z-i|$$

$|z-i| < \frac{3}{2} \Rightarrow S$ is convergent

$|z-i| > \frac{3}{2} \Rightarrow S$ is divergent



✗

p. 22

§ 지수함수 와 삼각함수

definition 10.12

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

* checking convergence

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \lim_{n\to\infty} \frac{|z|}{n+1} = 0$$

$\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is convergent in whole complex plane.

Theorem 10.13 & 10.14 & 10.15

① $\frac{d}{dz} e^z = e^z$

② $e^{z+w} = e^z \cdot e^w$

③ If $e^z = 1$, $z = 2n\pi i$ $(n = 0, \pm1, \pm2, \cdots)$

④ If $e^z = -1$, $z = (2n+1)\pi i$ $(n = 0, \pm1, \pm2, \cdots)$

p. 24

definition 10.13

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$\frac{d}{dz} \sin z = \cos z$, $\frac{d}{dz} \cos z = -\sin z$

Theorem 10.16

$$\boxed{\begin{aligned} e^{iz} &= \cos z + i\sin z \\ e^{-iz} &= \cos z - i\sin z \end{aligned}}$$

Euler formula

pf)

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \cdots$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$

$$= \cos z + i\sin z$$

※

note) $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$

p ㅏ 23

(예제 10.17)

$$e^z = 1 + 2i \quad -①$$

put $z = x + iy$

Then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y) \quad -②$$

From ① and ②

$$e^x \cos y = 1 \quad -③$$

$$e^x \sin y = 2 \quad -④$$

$$(e^x\cos y)^2 + (e^x \sin y)^2 = e^{2x} = 5$$

$$x = \frac{1}{2}\ln 5 \quad -⑤$$

④ ÷ ③

$$\tan y = 2$$

$$y = \tan^{-1} 2 \quad -⑥ \quad \Rightarrow \quad z = \frac{1}{2}\ln 5 + i(\tan^{-1} 2)$$

※

p526

(예제 10.18)

$$\cos z = i$$

put

$$z = x + iy \quad -①$$

Then

$$\cos z = \cos(x+iy)$$

$$= \cos x \cos iy - \sin x \sin iy \qquad (\because \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta)$$

$$= \cos x \frac{e^{i(iy)}+e^{-i(iy)}}{2} - \sin x \frac{e^{i(iy)}-e^{-i(iy)}}{2i} \qquad \left(\because \begin{array}{l} \sin z = \frac{1}{2i}(e^{iz}-e^{-iz}) \\ \cos z = \frac{1}{2}(e^{iz}+e^{-iz}) \end{array}\right)$$

$$= \cos x \frac{e^{y}+e^{-y}}{2} + \sin x \frac{e^{y}-e^{-y}}{2i}$$

$$= \cos x \cosh y - i \sin x \sinh y \qquad \left(\because \begin{array}{l} \sinh y = \frac{1}{2}(e^{y}-e^{-y}) \\ \cosh y = \frac{1}{2}(e^{y}+e^{-y}) \end{array}\right)$$

$$\llcorner ②$$

Then $\cos z = i$ becomes

$$\cos x \cosh y = 0 \quad -③$$

$$\sin x \sinh y = -1 \quad -④$$

Since $\cosh y \neq 0$, Eq. ③ implies

$$x = (n+\tfrac{1}{2})\pi \quad -⑤ \qquad (n = 0, \pm1, \pm2, \cdots)$$

(i) $n = 0, \pm2, \pm4, \cdots$

$\sin x = 1$

$\sinh y = -1$

$y = \sinh^{-1}(-1)$

$$z = x + iy = (n+\tfrac{1}{2})\pi + i \sinh^{-1}(-1)$$

(ii) $n = \pm1, \pm3, \cdots$

$\sin x = -1$, $\sinh y = 1 \Rightarrow y = \sinh^{-1}(1)$

$$z = x + iy = (n+\tfrac{1}{2})\pi + i \sinh(1)$$

//

definition : logarithm function

$$W = \ln z \equiv \ln|z| + i \arg z$$

note)

Let $z = re^{i\theta}$

$$\ln z = \ln(re^{i\theta}) = \ln r + \ln e^{i\theta} = \ln|z| + i\theta = \ln|z| + i \arg z$$

p518

(예제 10.19)

$z = 1+i$, what is $\ln z$?

$z = 1+i = r e^{i\theta}$

$r = \sqrt{2}$, $\theta = \frac{\pi}{4} + 2n\pi$

$\Rightarrow \ln(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$ $(n = 0, \pm1, \pm2, \cdots)$

(예제 10.20)

$z = -3$, what is $\ln z$?

$z = -3 = re^{i\theta}$

$r = 3$, $\theta = (2n+1)\pi$

$\Rightarrow \ln(-3) = \ln 3 + i(2n+1)\pi$ $(n = 0, \pm1, \pm2, \cdots)$ ※.

Theorem 10.20 & 10.21

① $z = e^{\ln z}$

② $\ln(zw) = \ln z + \ln w$

(예제 10.21)

$$e^z = 1+2i$$

$$1+2i = \sqrt{5}\, e^{i\theta} \qquad \theta = \tan^{-1} 2 + 2n\pi$$

$$e^z = \sqrt{5}\, e^{i\theta} \quad - ①$$

Taking ln in Eq. ①

$$z = \ln(\sqrt{5}\, e^{i\theta}) = \ln\sqrt{5} + \ln e^{i\theta} = \ln\sqrt{5} + i\theta$$

$\Rightarrow$

$$z = \ln\sqrt{5} + i\theta$$

$$\theta = \tan^{-1} 2 + 2n\pi$$

※

Since $\ln z$ is not single valued function, we define single-valued logarithmic function as follows

$$\boxed{\begin{aligned} \mathrm{Ln}\, z &= \ln|z| + i \arg z \\ 0 \le \arg &z < 2\pi \end{aligned}}$$

Ex) $\mathrm{Ln}(1+i) = \ln\sqrt{2} + i\cdot\frac{\pi}{4}$

$\mathrm{Ln}(-3) = \ln 3 + i\pi$ ※

p530

등 복소수의 거듭제곱

$$\boxed{\begin{aligned} z &= re^{i\theta} \\ z^n &= r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta) \\ z^{\frac{1}{n}} &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right) \end{aligned}}$$

p531

(예제 10.22)

$$8^{\frac{1}{3}} = ?$$

Put

$$z = 8^{\frac{1}{3}}$$

$$z^3 = 8 \quad - ①$$

Put

$$z = re^{i\theta} \quad - ②$$

② → ①

$$r^3 e^{i3\theta} = 8 = 2^3 e^{i2n\pi}$$

$$\Rightarrow r = 2$$

$$3\theta = 2n\pi$$

$$\theta = \frac{2n}{3}\pi \Rightarrow \theta = 0, \ \frac{2}{3}\pi, \ \frac{4}{3}\pi$$

$$z = 2$$

$$z = 2e^{i\frac{2}{3}\pi} = -1 + \sqrt{3}\,i$$

$$z = 2e^{i\frac{4}{3}\pi} = -1 - \sqrt{3}\,i$$

※

p532

(예제 10.24)

$$(2-2i)^{\frac{3}{5}} = ?$$

$$(2-2i)^3 = -16-16i = \sqrt{512}\, e^{i(\frac{5}{4}\pi + 2m\pi)}$$

Put

$$z = (2-2i)^{\frac{3}{5}}$$

$$\Rightarrow \quad z^5 = (2-2i)^3 = \sqrt{512}\, e^{i(\frac{5}{4}\pi + 2m\pi)} \qquad -①$$

$$z = re^{i\theta} \qquad -②$$

② → ①

$$r^5 e^{i5\theta} = \sqrt{512}\, e^{i(\frac{5}{4}\pi + 2m\pi)} \qquad -③$$

$$r = 512^{\frac{1}{10}}$$

$$\theta = \frac{\pi}{4} + \frac{2}{5}m\pi \Rightarrow \frac{\pi}{4},\ \frac{13}{20}\pi,\ \frac{21}{20}\pi,\ \frac{29}{20}\pi,\ \frac{37}{20}\pi$$

$$z = (512)^{\frac{1}{10}}\, e^{i\frac{\pi}{4}}$$

$$z = (512)^{\frac{1}{10}}\, e^{i\frac{13}{20}\pi}$$

$$z = (512)^{\frac{1}{10}}\, e^{i\frac{21}{20}\pi}$$

$$z = (512)^{\frac{1}{10}}\, e^{i\frac{29}{20}\pi}$$

$$z = (512)^{\frac{1}{10}}\, e^{i\frac{37}{20}\pi} \qquad ※$$

(예제 10.25)

$$(1-i)^{1+i} = ?$$

Put

$$z = (1-i)^{1+i} = e^{\ln (1-i)^{1+i}} = e^{(1+i)\ln(1-i)} \quad — ①$$

$$1-i = \sqrt{2}\, e^{i(-\frac{\pi}{4}+2m\pi)}$$

$$\Rightarrow \ln(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4}+2m\pi\right) \quad —②$$

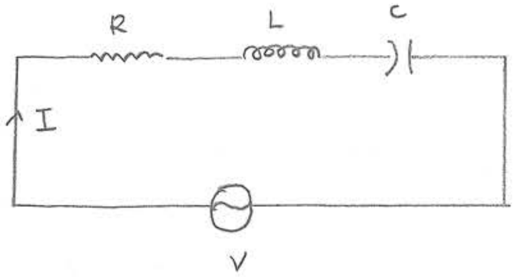
② → ①

$$z = e^{(1+i)\left[\ln\sqrt{2} + i\left(-\frac{\pi}{4}+2m\pi\right)\right]}$$

$$= e^{\left\{\ln\sqrt{2}+\frac{\pi}{4}-2m\pi\right\} + i\left\{\ln\sqrt{2}-\frac{\pi}{4}+2m\pi\right\}}$$

$$= \sqrt{2}\, e^{\frac{\pi}{4}-2m\pi}\left[\cos\left(\ln\sqrt{2}-\frac{\pi}{4}\right) + i\sin\left(\ln\sqrt{2}-\frac{\pi}{4}\right)\right] \quad ※$$

§ Application : RLC 회로



(i) When $I = I_0 \sin\omega t$, what is V ?

Let $$I_z = I_0 e^{i\omega t} \qquad - ①$$

Then $$I = \mathrm{Im}\, I_z \qquad - ②$$

Then
$$V_z = V_R + V_L + V_c \qquad - ③$$

where
$$\left.\begin{aligned} V_R &= I_z R = I_0 R e^{i\omega t} \\ V_L &= L \frac{dI_z}{dt} = i\omega L I_0 e^{i\omega t} \\ V_c &= \frac{q_z}{C} = \frac{I_0}{C}\int e^{i\omega t} dt = \frac{I_0}{i\omega C} e^{i\omega t} \end{aligned}\right) \qquad - ④$$

④ → ③

$$V_z = I_0 \left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] e^{i\omega t} \qquad - ⑤$$

$$\begin{aligned} V &= \mathrm{Im}\, V_z \\ &= I_0\, \mathrm{Im}\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] e^{i\omega t} \\ &= I_0\, \mathrm{Im}\left[\left\{ R + i\left(\omega L - \frac{1}{\omega C}\right)\right\} (\cos\omega t + i \sin\omega t) \right] \\ &= I_0 \left[R \sin\omega t + \left(\omega L - \frac{1}{\omega C}\right) \cos\omega t \right] \end{aligned}$$
※

(ii) When $V = V_0 \sin\omega t$, what is I ?

Put
$$V_z = V_0 e^{i\omega t} \qquad - (6)$$

Then
$$V = \mathrm{Im}\, V_z \qquad - (7)$$

From Eq. (1) and (2) we get

$$\left.\begin{aligned} V_z &= I_z Z \\ Z &= R + i\left(\omega L - \frac{1}{\omega C}\right) \quad : \text{complex impedance} \end{aligned}\right) - (8)$$

Then

$$I_z = \frac{V_z}{Z}$$
$$= \frac{V_0 e^{i\omega t}}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$
$$= \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \left[R - i\left(\omega L - \frac{1}{\omega C}\right)\right] e^{i\omega t} \qquad - (9)$$

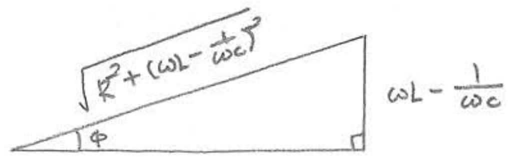
$$I = \mathrm{Im}\, I_z$$
$$= \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \mathrm{Im}\left[\left\{R - i\left(\omega L - \frac{1}{\omega C}\right)\right\}(\cos\omega t + i\sin\omega t)\right]$$

$$= \frac{V_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left[R\sin\omega t - \left(\omega L - \frac{1}{\omega C}\right)\cos\omega t\right]$$

$$= \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\left[\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin\omega t - \frac{\omega L - \frac{1}{\omega C}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos\omega t\right] \qquad - (10)$$

Let

$$\tan\phi = \frac{\omega L - \frac{1}{\omega C}}{R} \qquad -\text{⑪}$$

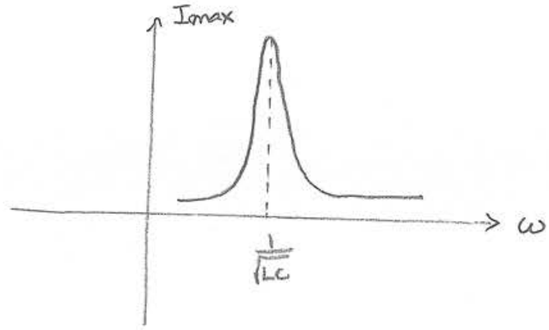


Then Eq. ⑩ becomes

$$I = \frac{V_0}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}}\left[\cos\phi \sin\omega t - \sin\phi \cos\omega t\right]$$

$$= \frac{V_0}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}} \sin(\omega t - \phi)$$

$$I_{max} = \frac{V_0}{\sqrt{R^2+(\omega L - \frac{1}{\omega C})^2}} \Rightarrow \text{Resonance occurs at } \omega = \frac{1}{\sqrt{LC}}$$



※

3. Taylor Expansion

Theorem 10.22

Taylor expansion of $f(z)$ at $z=z_0$ is defined as

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$

note)
Taylor Expansion at $z_0=0$ is called Maclaurin Expansion

p525

(예제 10.26)

e^z : Taylor Expansion at $z=i$.

$$f(z)=e^z$$

$$f^{(n)}(z)=e^z \Rightarrow f^{(n)}(i)=e^i$$

$$\Rightarrow e^z=\sum_{n=0}^{\infty}\frac{e^i}{n!}(z-i)^n \quad -①$$

Taylor Expansion

⟨convergence⟩

$$\rho=\lim_{n\to\infty}\left|\frac{\frac{e^i}{(n+1)!}(z-i)^{n+1}}{\frac{e^i}{n!}(z-i)^n}\right| = \lim_{n\to\infty}\frac{|z-i|}{n+1}=0$$

Eq. ① is convergent in whole complex plane !! ※

(예제 10.27)

$\cos z^3$: Maclaurin Expansion

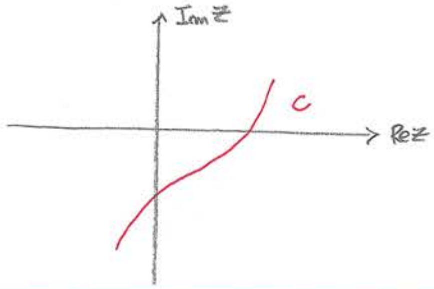
$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$$

$$\Rightarrow \cos z^3 = 1-\frac{z^6}{2!}+\frac{z^{12}}{4!}-\frac{z^{18}}{6!}+\cdots \quad ※$$

§ Cauchy Theorem

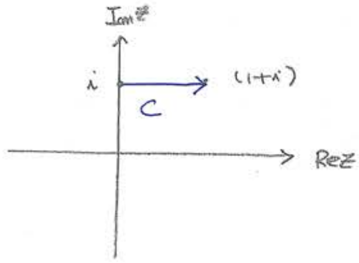
Contour Integral (경로 적분)

$\int_C f(z)\,dz$: integral along the contour C



(Ex)

① $\displaystyle\int_{i,\,C}^{1+i} z\,dz$



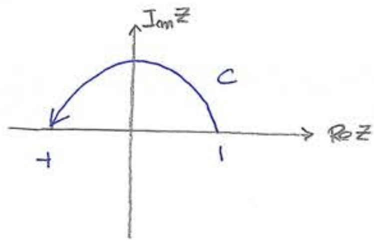
at contour C $\quad y=1,\ dy=0$

$$\Rightarrow \begin{pmatrix} z = x+iy = x+i \\ dz = dx + i\,dy = dx \end{pmatrix}$$

$$\int_C z\,dz = \int_0^1 dx\,(x+i) = \frac{1}{2} + i$$

※

② $\displaystyle\int_C z\,dz$



at contour C , $z = e^{i\theta}$, $dz = i e^{i\theta}\,d\theta$

$$\Rightarrow \int_C z\,dz$$

$$= \int_0^{\pi} e^{i\theta}\, i e^{i\theta}\, d\theta$$

$$= i\int_0^{\pi} e^{2i\theta}\, d\theta$$

$$= i \cdot \frac{1}{2i}\, e^{2i\theta}\Big|_0^{\pi}$$

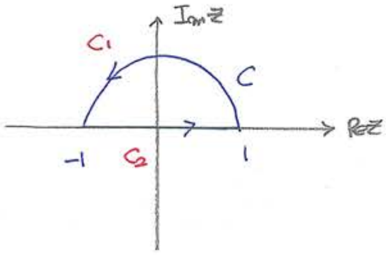
$$= \frac{1}{2}\left(e^{2i\pi} - e^{0}\right)$$

$$= \frac{1}{2}(1-1)$$

$$= 0$$

※.

②



$$\oint_C z^2 \, dz$$

$$= \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \quad — ①$$

at contour C_1 $\quad z = e^{i\theta}$, $dz = i e^{i\theta} d\theta$

$$\int_{C_1} z^2 dz = \int_0^{\pi} e^{2i\theta} \, i e^{i\theta} d\theta$$

$$= i \int_0^{\pi} e^{3i\theta} d\theta$$

$$= i \, \frac{1}{3i} \, e^{3i\theta} \Big|_0^{\pi}$$

$$= \frac{1}{3} \left[e^{3i\pi} - 1 \right]$$

$$= -\frac{2}{3} \quad — ②$$

at contour C_2 $\quad y = 0$, $dy = 0$

$$\left(\begin{array}{l} z = x + iy = x \\ dz = dx \end{array} \right)$$

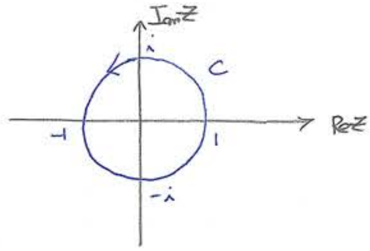
$$\int_{C_2} z^2 dz = \int_{-1}^{1} x^2 dx = \frac{2}{3} \quad — ③$$

②, ③ → ①

$$\oint_C z^2 dz = 0$$

※

④ $\oint_c z^2 dz$



At contour C : $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$

$$\oint_c z^2 dz$$

$$= \int_0^{2\pi} e^{2i\theta} \, ie^{i\theta} d\theta$$

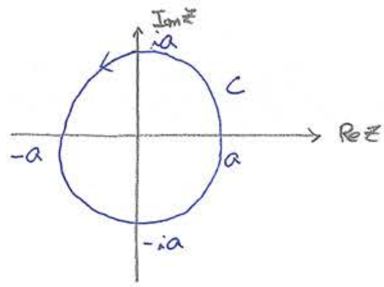
$$= i \int_0^{2\pi} e^{3i\theta} d\theta$$

$$= i \frac{1}{3i} e^{3i\theta} \Big|_0^{2\pi}$$

$$= \frac{1}{3} \left[e^{6i\pi} - 1 \right]$$

$$= 0 \quad \divideontimes$$

⑤ $\oint_c \frac{dz}{z}$



At contour C

$$z = ae^{i\theta}, \quad dz = iae^{i\theta} d\theta$$

$$\Rightarrow \oint_c \frac{dz}{z}$$

$$= \int_0^{2\pi} \frac{iae^{i\theta}}{ae^{i\theta}} d\theta$$

$$= 2\pi i \quad \divideontimes$$

p558

정리 11.7 : Cauchy Theorem

Let C be a simple closed curve.

If $f(z)$ is analytic on and inside C, then

$$\oint f(z)dz = 0$$

Pf)

$$\oint_C f(z)dz$$

$$= \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (udx - vdy) + i\oint_C (udy + vdx) \qquad - ①$$

Stock Theorem

$$\oint_C \vec{F}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, da \qquad - ②$$

Let $\vec{F} = (u(x,y), -v(x,y), 0)$

Then

$$\oint_C \vec{F}\cdot d\vec{s} = \oint_C (udx - vdy) \qquad - ③$$

$$\vec{\nabla}\times\vec{F} = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\hat{z}$$

$$\Rightarrow \int (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, da = -\int \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dxdy \qquad - ④$$

From ③ and ④

$$\oint_C (udx - vdy) = -\int \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dxdy \qquad - ⑤$$

By same way

$$\oint_C (udy + vdx) = \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \qquad - ⑥$$

Since $f(z) = u + iv$ is analytic, we get Cauchy-Riemann condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad - ⑦$$

⑦ → ⑤, ⑥

$$\oint_C (udx - vdy) = \oint_C (udy + vdx) = 0 \qquad - ⑧$$

⑧ → ①

$$\oint_C f(z)\,dz = 0$$

※

Ex) $\oint_C z^n dz = 0$ for any closed contour C.

($\because$ z^n is analytic function in the whole complex plane)

p558

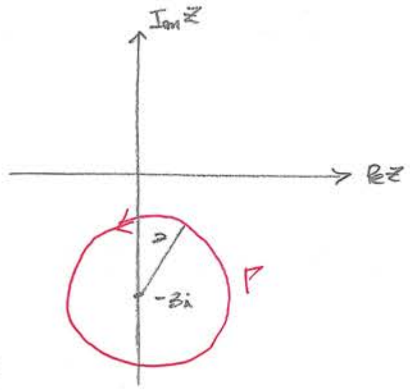
(예제 11.13)

$\oint_C e^{z^2} dz = 0$ for any closed contour C.

($\because$ e^{z^2} is analytic function in the whole complex plane) ※

(예제 11.14)

$$\oint_\Gamma \frac{2z+1}{z^2+3iz}\,dz \quad -①$$



Since $\dfrac{2z+1}{z^2+3iz} = \dfrac{2z+1}{z(z+3i)}$ is not analytic at $z=-3i$, we cannot use Cauchy theorem in this case.

at Γ : $z = -3i + 2e^{i\theta}$, $dz = 2ie^{i\theta}d\theta$ $\quad -②$

② → ①

$$\oint_\Gamma \frac{2z+1}{z(z+3i)}\,dz$$

$$= \oint_\Gamma \left[\frac{1}{3i}\,\frac{1}{z} + \frac{6+i}{3}\,\frac{1}{z+3i} \right] dz$$

$$= \frac{1}{3i}\oint_\Gamma \frac{1}{z}\,dz + \frac{6+i}{3}\oint_\Gamma \frac{1}{z+3i}\,dz \quad -③$$

Since $\frac{1}{z}$ is analytic on and inside Γ, Cauchy theorem implies

$$\oint_\Gamma \frac{dz}{z} = 0 \qquad — ④$$

Using Eq. ③,

$$\oint_\Gamma \frac{1}{z+3i}\,dz$$

$$= \int_0^{2\pi} \frac{2ie^{i\theta}}{2e^{i\theta}}\,d\theta$$

$$= 2\pi i \qquad — ⑤$$

④, ⑤ → ②

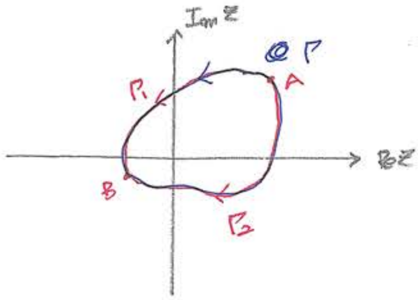
$$\oint_\Gamma \frac{2z+1}{z(z+3i)}\,dz = \frac{2\pi i}{3}(6+i) = \pi\left(-\frac{2}{3}+4i\right)$$ ※

p560

8. Cauchy Theorem 의 결과

If $f(z)$ is analytic on and inside a contour Γ, Cauchy theorem tells

$$\oint_{\Gamma} f(z)\,dz = 0$$



Since $\Gamma = \Gamma_1 - \Gamma_2$, This implies

$$\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$

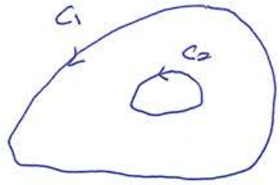
independent of path

p561

정리 11.8

Consider two contours C_1 and C_2 shown in Figure.

If $f(z)$ is analytic on and between C_1 and C_2, we get

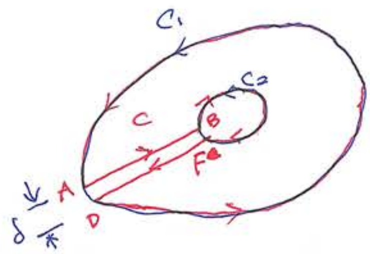


$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Pf) Consider a Contour C

Then Cauchy Theorem tells

$$\oint_C f(z)\,dz = 0$$



Thus we have

$$\int_D^A f(z)dz + \int_A^B f(z)dz + \int_B^F f(z)dz + \int_F^D f(z)\,dz = 0 \qquad - ①$$

Now we take $\delta \to 0$ limit. In this limit

$$\left.\begin{aligned} \int_D^A f(z)\,dz &= \oint_{C_1} f(z)\,dz \\ \int_A^B f(z)\,dz &= -\int_F^D f(z)\,dz \\ \int_B^F f(z)\,dz &= -\oint_{C_2} f(z)\,dz \end{aligned}\right\} \qquad - ②$$

② → ①

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

※

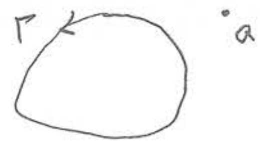
p560

(예제 11. 15)

$$\oint_{\Gamma} \frac{1}{z-a}\,dz$$

$\frac{1}{z-a}$ is analytic except $z=a$.

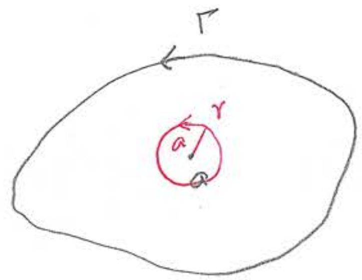
① If $z=a$ is outside Γ,



Cauchy theorem tells

$$\oint_{\Gamma} \frac{1}{z-a}\,dz = 0$$

② Let us consider the case where $z=a$ is



Since

$$\oint_{\Gamma} \frac{1}{z-a}\,dz = \oint_{\gamma} \frac{1}{z-a}\,dz ,$$

it is easier to compute with contour γ.

At contour γ,

$$z = a + e^{i\theta} \qquad dz = ie^{i\theta}\,d\theta \quad —①$$

Thus

$$\oint_{\gamma} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = 2\pi i \quad —②$$

Therefore

$$\oint_{\Gamma} \frac{1}{z-a}\,dz = \begin{cases} 0 & \text{when } z=a \text{ is outside } \Gamma. \\ 2\pi i & \text{when } z=a \text{ is inside } \Gamma. \end{cases}$$

※

p562

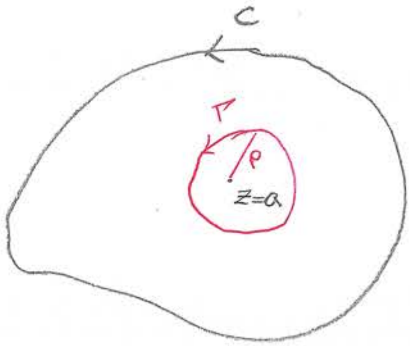
정리 11.9 : Cauchy Integral Theorem

If $f(z)$ is analytic on and inside a simple closed contour C and $z=a$ is inside C, then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = f(a)$$

Pf)

Since $f(z)$ is analytic on and inside C, $\frac{f(z)}{z-a}$ is analytic on and inside C except only $z=a$.



Thus

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_\Gamma \frac{f(z)}{z-a}\,dz.$$

We will take $\rho \to 0$ after calculation.

At contour Γ

$$z = a + \rho e^{i\theta}, \quad dz = i\rho e^{i\theta}\,d\theta \qquad -①$$

Then

$$\lim_{\rho\to 0}\oint_\Gamma \frac{f(z)}{z-a}\,dz$$

$$= \lim_{\rho\to 0}\int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$= i\lim_{\rho\to 0}\int_0^{2\pi} f(a+\rho e^{i\theta})\,d\theta$$

$$= i\int_0^{2\pi} d\theta \left[\lim_{\rho\to 0} f(a+\rho e^{i\theta})\right]$$

$$= i f(a)\int_0^{2\pi} d\theta$$

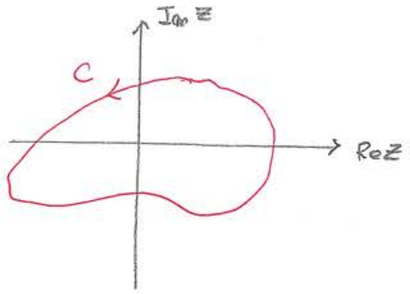
$$= 2\pi i f(a) \qquad -②$$

Thus we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = f(a) \qquad ※$$

(Ex)

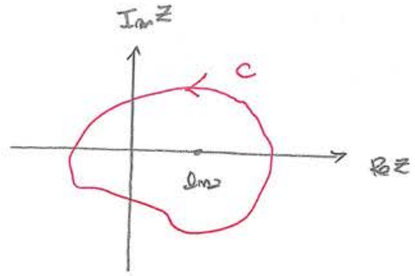
① $\oint_C z^2 dz$



Since z^2 is analytic in the whole complex plane,

$$\oint_C z^2 dz = 0.$$

② $\oint_C \frac{e^{3z}}{z-\ln 2} dz$



Cauchy-integral theorem tells

$$\oint_C \frac{e^{3z}}{z-\ln 2} dz = 2\pi i \, e^{3\ln 2} = 16\pi i$$ ※

p563

(예제 11.16)

$$\oint_\Gamma \frac{e^{z^2}}{z-i} dz$$

① If $z=i$ is outside Γ, Cauchy theorem implies

$$\oint_\Gamma \frac{e^{z^2}}{z-i} dz = 0$$

② If $z=i$ is inside Γ, Cauchy integral theorem implies

$$\oint_\Gamma \frac{e^{z^2}}{z-i} dz = 2\pi i \, e^{i^2} = 2\pi i \, e^{-1}$$ ※

P563

정리 11.10 : 고계 도함수에 대한 Cauchy integral formula

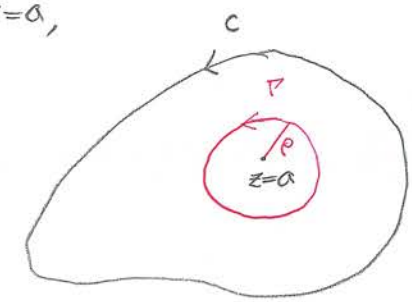
If $f(z)$ is analytic on and inside a simple closed contour C and $z=a$ is inside C, then

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Pf) Now we will prove $n=2$ case. Other case can be proven similarly.

Since $\dfrac{f(z)}{(z-a)^2}$ is not analytic only at $z=a$,

we can change a contour



$$\oint_C \frac{f(z)}{(z-a)^2}\,dz = \oint_\Gamma \frac{f(z)}{(z-a)^2}\,dz \quad -①$$

at Γ

$$z = a+\rho e^{i\theta} \qquad dz = i\rho e^{i\theta}\,d\theta \quad -②$$

Thus,

$$\oint_\Gamma \frac{f(z)}{(z-a)^2}\,dz$$

$$= \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho^2 e^{2i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$= \frac{i}{\rho}\int_0^{2\pi} f(a+\rho e^{i\theta})\, e^{-i\theta}\,d\theta \quad -③$$

Now we assume ρ is small quantity. Then we can use Taylor expansion;

$$f(a+\rho e^{i\theta}) = f(a) + \frac{f'(a)}{1!}\rho e^{i\theta} + \frac{f''(a)}{2!}(\rho e^{i\theta})^2 + \cdots \quad -④$$

② → ⑤

$$\oint_\Gamma \frac{f(z)}{(z-a)^2}\, dz$$

$$= \frac{i}{\rho}\int_0^{2\pi} d\theta\, e^{-i\theta}\left[f(a) + \frac{f'(a)}{1!}\rho e^{i\theta} + \frac{f''(a)}{2!}\rho^2 e^{2i\theta} + \cdots \right]$$

$$= \frac{i}{\rho}\int_0^{2\pi} d\theta \left[f(a) e^{-i\theta} + \frac{f'(a)}{1!}\rho + \frac{f''(a)}{2!}\rho^2 e^{i\theta} + \cdots \right]$$

$$= \frac{i}{\rho}\left[f(a) \underbrace{\int_0^{2\pi} d\theta\, e^{-i\theta}}_{=0} + \frac{f'(a)}{1!}\rho \underbrace{\int_0^{2\pi} d\theta}_{=2\pi} + \frac{f''(a)}{2!}\rho^2 \underbrace{\int_0^{2\pi} d\theta\, e^{i\theta}}_{=0} + \cdots \right]$$

$$= 2\pi i\, f'(a)$$

Therefore

$$\oint_c \frac{f(z)}{(z-a)^2}\, dz = \frac{2\pi i}{1!} f'(a)$$

※

p564

(예제 11.19)

$$\oint_\Gamma \frac{e^{z^3}}{(z-i)^3}\, dz$$

① If $z=i$ is outside Γ, $\oint_\Gamma \frac{e^{z^3}}{(z-i)^3}\, dz = 0$

② If $z=i$ is inside Γ, Cauch integral theorem implies

$$\oint_\Gamma \frac{e^{z^3}}{(z-i)^3}\, dz = \frac{2\pi i}{2!}\left(\frac{d^2}{dz^2} e^{z^3}\right)_{z=i}$$

$$= \pi i \left[(6z + 9z^4)\, e^{z^3} \right]_{z=i}$$

$$= \pi i \left[(9 + 6i)\, e^{-i} \right]$$

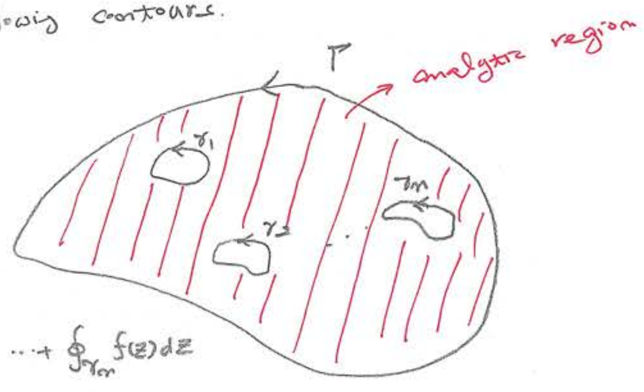
$$= \pi(-6 + 9i)\, e^{-i}$$

※

p565

정리 11.13 확장 변형 정리

Let $\Gamma, \gamma_1, \gamma_2, \cdots, \gamma_m$ are following contours.

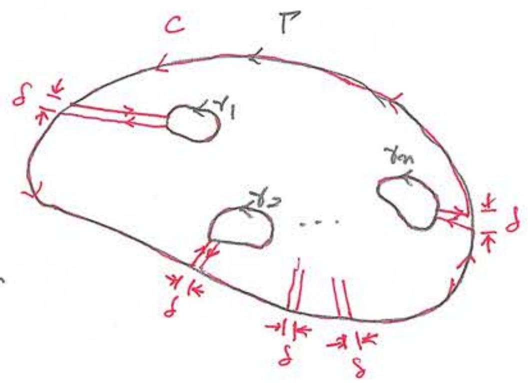


If $f(z)$ is analytic inside Γ and outside $\{\gamma_1, \gamma_2, \cdots, \gamma_m\}$,

then

$$\oint_\Gamma f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \cdots + \oint_{\gamma_m} f(z)dz$$

Pf) Consider the following contour C.



Then Cauchy theorem implies

$$\oint_C f(z)dz = 0.$$

Taking $\delta \to 0$ limit, we can easily show

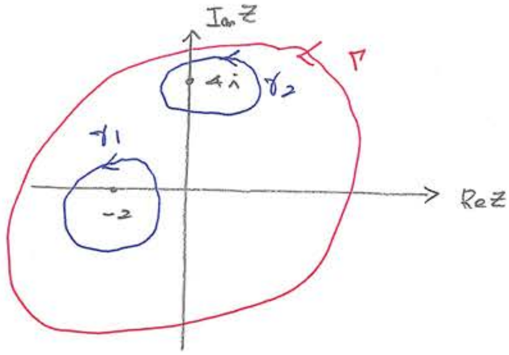
$$\oint_\Gamma f(z)dz = \oint_{\gamma_1} f(z)dz + \oint_{\gamma_2} f(z)dz + \cdots + \oint_{\gamma_m} f(z)dz$$

※

P565

(예제 11.18)

$$\oint_\Gamma \frac{z}{(z+2)(z-4i)}\,dz$$



Then we can write

$$\oint_\Gamma \frac{z}{(z+2)(z-4i)}\,dz$$

$$= \oint_{\gamma_1} \frac{\left(\frac{z}{z-4i}\right)}{z+2}\,dz + \oint_{\gamma_2} \frac{\left(\frac{z}{z+2}\right)}{(z-4i)}\,dz \qquad -①$$

Using Cauchy integral theorem, we can easily show

$$\left.\begin{aligned} \oint_{\gamma_1} \frac{\left(\frac{z}{z-4i}\right)}{z+2}\,dz &= 2\pi i \left(\frac{z}{z-4i}\right)_{z=-2} = \frac{2\pi i}{1+2i} \\ \oint_{\gamma_2} \frac{\left(\frac{z}{z+2}\right)}{z-4i}\,dz &= 2\pi i \left(\frac{z}{z+2}\right)_{z=4i} = \frac{-4\pi}{1+2i} \end{aligned}\right\} \qquad -②$$

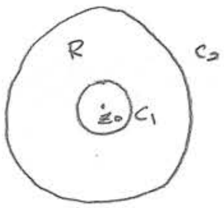
② → ①

$$\oint_\Gamma \frac{z}{(z+2)(z-4i)}\,dz = \frac{2\pi i - 4\pi}{1+2i} = 2\pi i \qquad //$$

§ Laurent Expansion

정리 10.29-I : Laurent's theorem

Let C_1 and C_2 be two circles with center at z_0.



Let $f(z)$ be analytic in the region R between the circles.

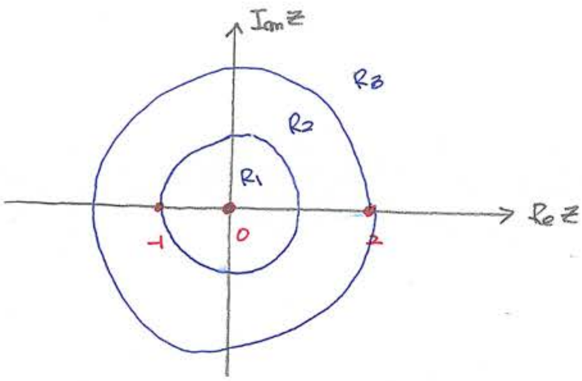
Then $f(z)$ can be expanded as

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Such a series is called a Laurent series

Ex) $f(z) = \dfrac{12}{z(2-z)(1+z)}$ 의 Laurent expansion 을 구하라

$f(z)$ is not analytic at $z=0$, $z=2$, and $z=-1$.



(i) R_1 region ($|z|<1$)

$$f(z) = \frac{1}{z}\left[\frac{12}{(2-z)(1+z)}\right]$$

$$= \frac{4}{z}\left[\frac{1}{1+z} + \frac{1}{2-z}\right] \quad —①$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \quad \Big\} - ②$$

$$\frac{1}{2-z} = \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

② → ①

$$f(z) = \frac{6}{z} - 3 + \frac{9}{2}z - \frac{15}{4}z^2 + \frac{33}{8}z^3 + \cdots$$

Laurent expansion at R_1

(ii) R_3 region ($|z| > 2$)

$$\frac{1}{1+z} = \frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] \quad \Big\} - ③$$

$$\frac{1}{2-z} = -\frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = -\frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right]$$

③ → ①

$$f(z) = -\frac{12}{z^2}\left[1 + \frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{11}{z^4} + \cdots\right]$$

Laurent expansion at R_3

(iii) R_2 region ($1 < |z| < 2$)

$$\frac{1}{1+z} = \frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right] \quad \Big\} - ④$$

$$\frac{1}{2-z} = \frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

④ → ①

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - \cdots\right)$$

Laurent expansion at R_2

※

(예제 10.30)

$e^{\frac{1}{z}}$: analytic except $z=0$.

Since $e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$,

$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$ Laurent Expansion around $z=0$.

(예제 10.31)

$\frac{\cos z}{z^5}$: analytic except $z=0$

$\frac{1}{z^5}\cos z$

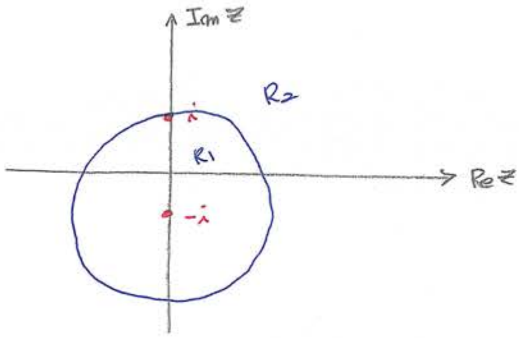
$= \frac{1}{z^5}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots \right]$

$= \frac{1}{z^5} - \frac{1}{2!\,z^3} + \frac{1}{4!\,z} - \frac{z}{6!} + \frac{z^3}{8!} - \cdots$ Laurent Expansion around $z=0$

(예제 10.32)

$\frac{1}{1+z^2}$: Laurent expansion at $z=-i$

$\frac{1}{1+z^2}$ is not analytic at $z=\pm i$.



$\frac{1}{1+z^2} = \frac{i}{2}\left[\frac{1}{z+i} - \frac{1}{z-i} \right]$ — ①

(i) Region R_1 ($|z+i|<2$)

$$\frac{1}{z-i} = \frac{1}{(z+i)-2i}$$

$$= -\frac{1}{2i}\frac{1}{1-\frac{z+i}{2i}}$$

$$= -\frac{1}{2i}\left[1+\frac{z+i}{2i}+\left(\frac{z+i}{2i}\right)^2+\left(\frac{z+i}{2i}\right)^3+\cdots\right] \quad -②$$

② → ①

$$\frac{1}{1+z^2} = \frac{i}{2}\frac{1}{z+i} - \frac{i}{2}\left(-\frac{1}{2i}\right)\left[1+\frac{z+i}{2i}+\left(\frac{z+i}{2i}\right)^2+\left(\frac{z+i}{2i}\right)^3+\cdots\right]$$

$$= \underline{\frac{i}{2}\frac{1}{z+i} + \frac{1}{4}\left[1+\frac{z+i}{2i}+\left(\frac{z+i}{2i}\right)^2+\left(\frac{z+i}{2i}\right)^3+\cdots\right]}$$

Laurent Expansion at R_1 around $z=-i$.

(ii) Region R_2 ($|z+i|>2$)

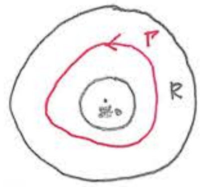
각자 계산 할것 !!

정리 10.29-II : Laurent Theorem

Let $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ be Laurent Series

around z_0 at region R. Let P be a

simple closed contour in R.



Then Laurent coefficient b_n is given

by

$$b_n = \frac{1}{2\pi i} \oint_P \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

pf)

$$\oint_P \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$= \oint_P \frac{1}{(z-z_0)^{n+1}} \sum_{m=-\infty}^{\infty} b_m (z-z_0)^m \, dz$$

$$= \sum_{m=-\infty}^{\infty} b_m \oint_P (z-z_0)^{m-n-1}\, dz$$

$$= \sum_{m=-\infty}^{n-1} b_m \oint_P (z-z_0)^{m-n-1} dz + b_n \oint_P \frac{dz}{z-z_0} + \underbrace{\sum_{m=n+1}^{\infty} b_m \oint_P (z-z_0)^{m-n-1}\, dz}_{①}$$

Since $(z-z_0)^{m-n-1}$ is analytic in whole complex plane when $m \geq n+1$,

Cauchy theorem gives

$$\sum_{m=n+1}^{\infty} b_m \oint_P (z-z_0)^{m-n-1}\, dz = 0 \qquad -②$$

Since Cauchy integral formula is

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a), \qquad -③$$

we get

$$\oint_P \frac{1}{(z-z_0)^p}\, dz = 0 \qquad (p = 2, 3, \cdots) \qquad -④$$

Eq. ④ implies

$$\sum_{m=-\infty}^{n-1} b_m \oint_\Gamma (z-z_0)^{m-n-1}\, dz = 0 \qquad - ⑤$$

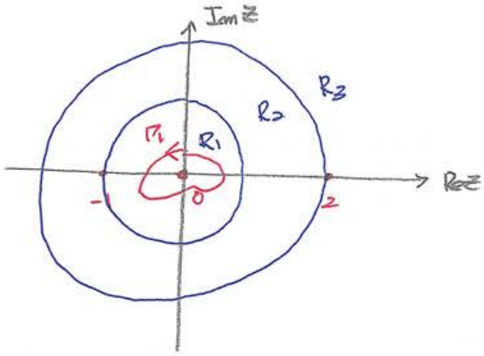
③, ⑤ $\rightarrow$ ①

$$\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz = 2\pi i\, b_n$$

$$\Rightarrow \quad b_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz \qquad \#$$

Ex)

$$f(z) = \frac{12}{z(2-z)(1+z)}$$



(i) Laurent Expansion at R_1

$$f(z) = \sum_{n=-\infty}^{\infty} b_n z^n$$

$$b_n = \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{f(z)}{z^{n+1}} dz$$

$$= \frac{1}{2\pi i} \oint_{\Gamma_1} \frac{12}{z^{n+2}(2-z)(1+z)} dz$$

$$= \frac{12}{2\pi i} \oint_{\Gamma_1} \frac{\left(\frac{1}{(2-z)(1+z)}\right)}{z^{n+2}} dz \qquad \text{— ①}$$

(A) $n \leq -2$

Since $\dfrac{\left(\frac{1}{(2-z)(1+z)}\right)}{z^{n+2}}$ is analytic when $n \leq -2$,

Cauchy theorem gives

$$\oint_{\Gamma_1} \frac{\left(\frac{1}{(2-z)(1+z)}\right)}{z^{n+2}} dz = 0 \qquad (n = -2, -3, -4, \cdots)$$

└②

Thus we have

$$b_{-2} = b_{-3} = \cdots = 0 \qquad \text{— ②}$$

(B) $n \geq -1$

Since Cauchy integral theorem is

$$\oint \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a) , \quad - ⑤$$

$$b_n = \frac{12}{2\pi i} \frac{2\pi i}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left[\frac{1}{(2-z)(1+z)} \right] \Bigg|_{z=0}$$

$$= \frac{4}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left[\frac{1}{1+z} + \frac{1}{2-z} \right] \Bigg|_{z=0} \quad - ⑤$$

Since

$$\left. \begin{aligned} \frac{d^{n+1}}{dz^{n+1}} \frac{1}{1+z} &= (-1)^{n+1} (n+1)! \frac{1}{(1+z)^{n+2}} \\ \frac{d^{n+1}}{dz^{n+1}} \frac{1}{2-z} &= (n+1)! \frac{1}{(2-z)^{n+2}} \end{aligned} \right) \quad - ⑥$$

$$\frac{d^{n+1}}{dz^{n+1}} \left[\frac{1}{1+z} + \frac{1}{2-z} \right] \Bigg|_{z=0} = (n+1)! \left[2^{-(n+2)} + (-1)^{n+1} \right] \quad - ⑦$$

⑦ → ⑤

$$b_n = 4 \left[2^{-(n+2)} + (-1)^{n+1} \right] \quad - ⑧$$

$b_{-1} = 6$, $b_0 = -3$, $b_1 = \frac{9}{2}$, $b_2 = -\frac{15}{4}$, $b_3 = \frac{33}{8}$ ...

Exactly same with previous result !!

(ii) Laurent Expansion at R_2

⇒ 각자 해볼 것

(iii) Laurent Expansion at R_3

⇒ 각자 해볼 것

p570

3 Pole and Residue

definition

Let $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n$ be Laurent expansion near $z=z_0$.

[1] If $b_{-1} = b_{-2} = \cdots = 0$, $f(z)$ is analytic at $z=z_0$.

In this case $z=z_0$ is called "regular point of $f(z)$".

[2] If $b_{-m} \neq 0$ and $b_{-m-1} = b_{-m-2} = \cdots = 0$, we say "$f(z)$ has a pole of order m at $z=z_0$". If $m=1$, we say "$f(z)$ has a simple pole at $z=z_0$".

[3] The coefficient b_{-1} is called "residue of $f(z)$ at $z=z_0$".

Ex)

① $e^z = 1 + z + \frac{z^2}{2!} + \cdots$

no pole, $R(0) = 0$ (R: residue)

② $\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z} + \frac{1}{3!} + \frac{1}{4!} z + \cdots$

$\Rightarrow$ pole of order $m=3$ at $z=0$

$\Rightarrow$ $R(0) = \frac{1}{2}$

③ $\frac{z+3}{z^2(z-1)^3(z+1)}$

$z=0$: pole of order 2

$z=1$: pole of order 3

$z=-1$: simple pole

p570

(예제12.1)

$$f(z) = \frac{1}{z}(1 - \cos z)$$

$$= \frac{1}{z}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right]$$

$$= \frac{1}{2!}z - \frac{1}{4!}z^3 + \frac{1}{6!}z^5 - \cdots$$

no pole at $z=0$

(예제 12.2)

$$f(z) = \frac{1}{z+i}$$

simple pole at $z = -i$

p572

(예제12.4)

$$f(z) = \frac{\sin z}{z^3}$$

$$= \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{1}{5!}z^2 - \frac{z^4}{7!} + \cdots$$

" pole of order 2 at $z=0$ $\Rightarrow$ double pole at $z=0$ "

§ Finding Residue

[1] Simple Pole (P. 577 정리 12.6)

$$\boxed{\text{If } f(z) \text{ has simple pole at } z = z_0, \quad R(z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

p577

(예제 12.10)

$$f(z) = \frac{\sin z}{z^2} \quad : \text{simple pole at } z = 0$$

$$R(0) = \lim_{z \to 0} z\, f(z)$$

$$= \lim_{z \to 0} \frac{\sin z}{z}$$

$$= 1$$

(예제 12.11)

$$f(z) = \frac{z - 6i}{(z-2)^2 (z + 4i)}$$

$z = 2$: double pole

$z = -4i$: simple pole

$$R(-4i) = \lim_{z \to -4i} (z + 4i) f(z)$$

$$= \lim_{z \to -4i} \frac{z - 6i}{(z-2)^2}$$

$$= -\frac{2}{5} + \frac{3}{10} i$$

p579

정리 12.7

If $f(z)$ has a pole of order n at $z = z_0$, its residue is

$$R(z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$$

pf) 생략 (매우 쉬움)

(예제 12.13)

$$f(z) = \frac{\cos z}{(z+i)^3}$$

at $z = -i$: pole of order 3.

$$R(-i) = \frac{1}{2!} \lim_{z \to -i} \frac{d^2}{dz^2} \left[(z+i)^3 f(z) \right]$$

$$= \frac{1}{2} \lim_{z \to -i} \frac{d^2}{dz^2} \cos z$$

$$= \frac{1}{2} \lim_{z \to -i} \left[-\cos z \right]$$

$$= -\frac{1}{2} \cos(-i)$$

$$= -\frac{1}{2} \cos(i)$$

※

P576

정리 12.5 : Residue Theorem

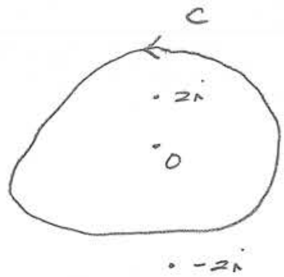
$$\oint_C f(z)\,dz = 2\pi i \times [\text{ sum of the residues of } f(z) \text{ inside } C\ ]$$

where C is the counterclockwise closed contour

(증명) 생략 (매우 쉬움)

P580

(예제 12.14)



$$f(z) = \frac{\sin z}{z^2\,(z^2+4)}$$

$$= \frac{\sin z}{z^2\,(z+2i)\,(z-2i)}$$

$$R(2i) = \lim_{z \to 2i} (z-2i)\,f(z) = \frac{\sin(2i)}{(-4)(4i)} = \frac{i}{16}\sin(2i)$$

Since $z=0$ is simple pole ,

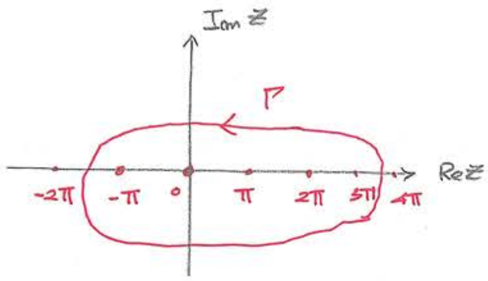
$$R(0) = \lim_{z \to 0} \frac{\sin z}{z\,(z^2+4)} = \frac{1}{4}$$

$$\Rightarrow \oint_C f(z)\,dz = 2\pi i \left[\frac{1}{4} + \frac{i}{16}\sin(2i)\right]$$

※

p580

(예제 12.15)



$$\oint_\Gamma \cot z\, dz = \oint_\Gamma \frac{\cos z}{\sin z} dz$$

$$R(-\pi) = \lim_{z\to-\pi} (z+\pi)\frac{\cos z}{\sin z}$$

$$= \cos(-\pi) \lim_{z\to-\pi} \frac{1}{\cos z}$$

$$= 1$$

$$R(0) = \lim_{z\to 0} z \frac{\cos z}{\sin z} = 1$$

$$R(\pi) = \lim_{z\to\pi} (z-\pi)\frac{\cos z}{\sin z} = \cos\pi \lim_{z\to\pi}\frac{1}{\cos z} = 1$$

$$R(2\pi) = \lim_{z\to 2\pi}(z-2\pi)\frac{\cos z}{\sin z} = \lim_{z\to 2\pi}\frac{1}{\cos z} = 1$$

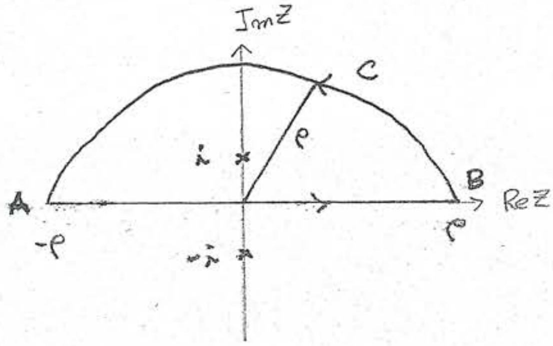
$$R(3\pi) = \lim_{z\to 3\pi}(z-3\pi)\frac{\cos z}{\sin z} = \cos(3\pi)\lim_{z\to 3\pi}\frac{1}{\cos z} = 1$$

$$\Rightarrow \oint_\Gamma \cot z\, dz = 2\pi i \times 5 = 10\pi i$$

※

Ex)

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$



$$I_Z = \oint_C \frac{dZ}{1+Z^2}$$

$$= \oint \frac{dZ}{(Z+i)(Z-i)}$$

$$= 2\pi i \frac{1}{2i}$$

$$= \pi$$

$$\equiv \int_A^B \frac{dZ}{1+Z^2} + \int_B^A \frac{dZ}{1+Z^2}$$

$$\int_A^B \frac{dZ}{1+Z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} \qquad (Z = x)$$

$$\int_B^A \frac{dZ}{1+Z^2} = \int_0^{\pi} \frac{i\rho e^{i\theta}}{1+\rho^2 e^{2i\theta}} d\theta$$

Now we take $\rho \to \infty$ limit.

Then

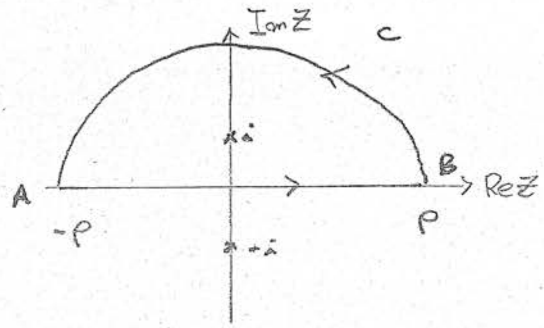
$$I_Z = \pi = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = I$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

*

예) 숙제!!

Ex) $I = \int_0^\infty \frac{\cos x}{1+x^2}\,dx$



Consider

$$I_Z = \oint_C \frac{e^{iZ}}{1+Z^2}\,dZ$$

$$= 2\pi i \, \frac{e^{-1}}{2i}$$

$$= \frac{\pi}{e}$$

$$\equiv \int_A^B \frac{e^{iZ}}{1+Z^2}\,dZ + \int_B^A \frac{e^{iZ}}{1+Z^2}\,dZ$$

$$\int_A^B \frac{e^{iZ}}{1+Z^2}\,dZ = \int_{-\rho}^{\rho} \frac{e^{ix}}{1+x^2}\,dx \qquad (Z = x)$$

$$\int_B^A \frac{e^{iZ}\,dZ}{1+Z^2} = \int_0^\pi \frac{e^{iZ}}{1+\rho^2 e^{2i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$|e^{iZ}| = |e^{i(x+iy)}| = |e^{ix}|\,|e^{-y}| = |e^{-y}|$$

Since $y > 0$ in contour C, $|e^{iZ}| = |e^{-y}| < 1$.

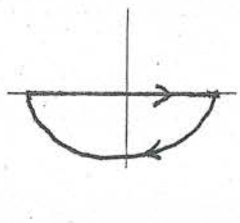
Taking $\rho \to \infty$ limit yields

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e} = 2I$$

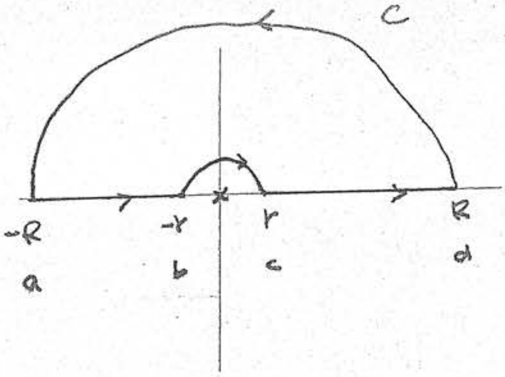
$$\Rightarrow I = \int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e} \quad \#$$

$$I_Z = \oint \frac{e^{-iZ}}{1+Z^2}\,dZ$$



~~Impossible!!~~

Ex) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$



$$I_z \equiv \oint_C \frac{e^{iz}}{z}\, dz$$

$$= 0$$

$$\equiv \int_a^b \frac{e^{iz}}{z} dz + \int_b^c \frac{e^{iz}}{z} dz + \int_c^d \frac{e^{iz}}{z} dz + \int_d^a \frac{e^{iz}}{z} dz$$

$$\int_a^b \frac{e^{iz}}{z} dz = \int_{-R}^{-r} \frac{e^{ix}}{x} dx \quad (z=x) \Rightarrow \int_{-\infty}^{0} \frac{e^{ix}}{x} dx \quad \left(\begin{matrix} r \to 0 \text{ limit} \\ R \to \infty \text{ limit} \end{matrix}\right)$$

$$\int_b^c \frac{e^{iz}}{z} dz = \int_\pi^0 \frac{e^{iz}}{re^{i\theta}}\, ire^{i\theta} d\theta = i\int_\pi^0 e^{iz}\, d\theta \Rightarrow i\int_\pi^0 d\theta \quad (r \to 0 \text{ limit})$$

$$= -i\pi$$

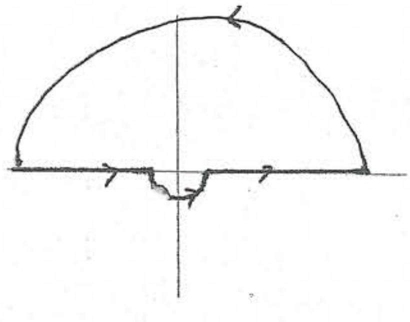
$$\int_c^d \frac{e^{iz}}{z} dz = \int_r^R \frac{e^{ix}}{x} dx \Rightarrow \int_0^\infty \frac{e^{ix}}{x} dx$$

$$\int_d^a \frac{e^{iz}}{z} dz = 0 \quad \left(\because \left|\frac{e^{iz}}{z}\right| = \frac{e^{-y}}{R} \sim 0\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

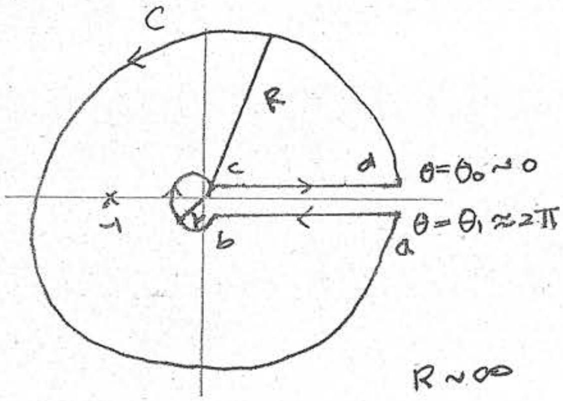
※



[illegible]!!

Ex)

$$I = \int_0^\infty \frac{r^{p-1}}{1+r}\, dr \qquad (0<p<1)$$



$R \sim \infty$

$r \sim 0$

$$I_z \equiv \oint_C \frac{z^{p-1}}{1+z}\, dz$$

$$= 2\pi i\, (-1)^{p-1}$$

$$= 2\pi i\, (e^{i\pi})^{p-1}$$

$$= -2\pi i\, e^{i\pi p}$$

$$\equiv \int_a^b \frac{z^{p-1}}{1+z}\, dz + \int_b^c \frac{z^{p-1}}{1+z}\, dz + \int_c^d \frac{z^{p-1}}{1+z}\, dz + \int_d^a \frac{z^{p-1}}{1+z}\, dz$$

$$\int_d^a \frac{z^{p-1}}{1+z}\, dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}}\, iRe^{i\theta} \propto \frac{1}{R^{1-p}} = 0 \quad (\text{when } R\to\infty \text{ limit})$$

$$\int_b^c \frac{z^{p-1}}{1+z}\, dz = \int_{2\pi}^0 \frac{(re^{i\theta})^{p-1}}{1+re^{i\theta}}\, ire^{i\theta} \propto r^p = 0 \quad (\text{when } r\to 0 \text{ limit})$$

$$\int_c^d \frac{z^{p-1}}{1+z}\, dz = \int_0^\infty \frac{r^{p-1}}{1+r}\, dr \qquad (z = re^{i\theta_0} \sim r)$$

$$\int_a^b \frac{z^{p-1}}{1+z}\, dz = \int_\infty^0 \frac{(re^{i2\pi})^{p-1}}{1+re^{i2\pi}}\, e^{i2\pi} dr \qquad (z = re^{i\theta_1} \sim re^{i2\pi})$$

$$= -\underline{e^{i2\pi(p-1)}} \int_0^\infty \frac{r^{p-1}}{1+r}\, dr$$

$$e^{2\pi ip}\cdot e^{-2\pi i} = e^{2\pi ip}$$

$$= -e^{2\pi ip} \int_0^\infty \frac{r^{p-1}}{1+r}\, dr$$

$\Rightarrow$

$$-2\pi i\, e^{i\pi p} = (1 - e^{2\pi i p}) \int_0^\infty \frac{r^{p-1}}{1+r}\, dr$$

$$\Rightarrow \int_0^\infty \frac{r^{p-1}}{1+r}\, dr = -2\pi i \, \frac{e^{i\pi p} \;\|\; e^{-i\pi p}}{1 - e^{2\pi i p} \;\|\; e^{-i\pi p}}$$

$$= 2\pi i \, \frac{1}{e^{i\pi p} - e^{-i\pi p}}$$

$$= \frac{2\pi i}{2i \sin \pi p}$$

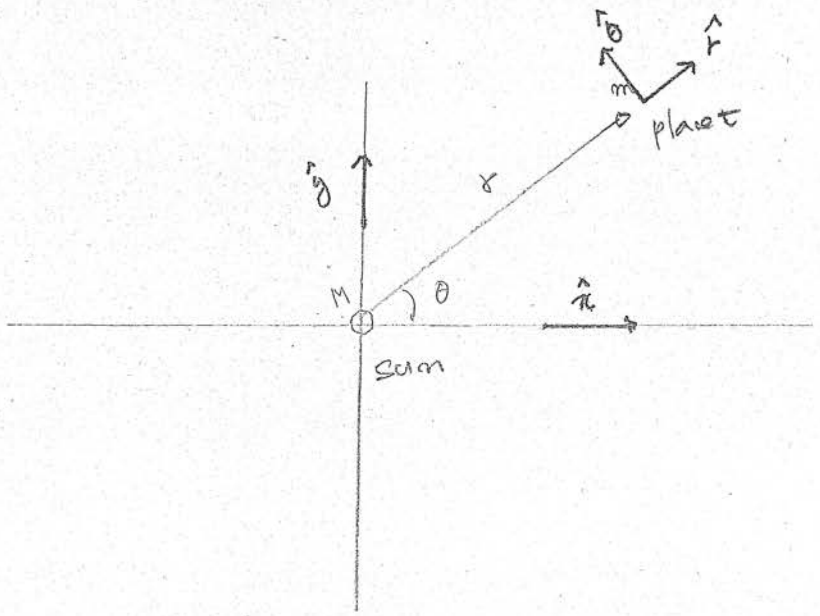
$$= \frac{\pi}{\sin \pi p}$$

$$= \Gamma(p)\,\Gamma(1-p)$$

$$\Gamma(p) \equiv \int_0^\infty t^{p-1} e^{-t}\, dt \quad \Rightarrow \text{p 462 Eq. (5.4)}$$

*

§ Physical Application : Planet Motion in Central force



$$\vec{F} = -G\frac{Mm}{r^2}\,\hat{r} \quad — ①$$

$$\left.\begin{aligned} \hat{r} &= \cos\theta\ \hat{x} + \sin\theta\ \hat{y} \\ \hat{\theta} &= -\sin\theta\ \hat{x} + \cos\theta\ \hat{y} \\ \frac{d\hat{r}}{dt} &= \dot{\theta}\,\hat{\theta}, \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r} \end{aligned}\right\} \quad — ②$$

$$\left.\begin{aligned} \vec{r} &= r\,\hat{r} \\ \frac{d\vec{r}}{dt} &= \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} \\ \frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta} \end{aligned}\right\} \quad — ③$$

From Newton law

$$\vec{F} = -G\frac{Mm}{r^2}\,\hat{r} = m\frac{d^2\vec{r}}{dt^2} = m\left[(\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}\right]$$

$$\Rightarrow \quad m(\ddot{r} - r\dot{\theta}^2) = -G\frac{Mm}{r^2} \quad — ④$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad — ⑤$$

From ⑤

$$mr^2\dot{\theta} = l = \text{const} \quad - ⑥$$

: angular momentum conservation

$$\dot{\theta} = \frac{l}{mr^2} \Rightarrow \text{Eq. ④}$$

$$m\ddot{r} = \frac{l^2}{mr^3} - G\frac{Mm}{r^2} \quad \| \times \dot{r}$$

$$\Rightarrow m\ddot{r}\,\dot{r} = \left[\frac{l^2}{mr^3} - G\frac{Mm}{r^2}\right]\frac{dr}{dt}$$

$$\Rightarrow m\dot{r}\frac{d\dot{r}}{dt} = \left[\frac{l^2}{mr^3} - G\frac{Mm}{r^2}\right]\frac{dr}{dt}$$

$$\Rightarrow \int m\dot{r}\,d\dot{r} = \int\left[\frac{l^2}{mr^3} - G\frac{Mm}{r^2}\right]dr$$

$$\frac{1}{2}m\dot{r}^2 = -\frac{l^2}{2mr^2} + \frac{GMm}{r} + C$$

C is total energy

$$\Big(\because\ E = \frac{1}{2}m\dot{\vec{r}}^2 + V(r)$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\underbrace{\dot{\theta}^2}_{\frac{l^2}{m^2r^4}} - \frac{GMm}{r}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}$$

$$= C \Big)$$

$$\frac{1}{2}m\dot{r}^2 = -\frac{l^2}{2mr^2} + \frac{GMm}{r} + E \quad - ⑦$$

Now we eliminate time t as following:

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{l}{mr^2}\cdot\frac{dr}{d\theta} \quad - ⑧$$

⑧ → ⑦

$$\frac{1}{r^2}\frac{dr}{d\theta} = \sqrt{-\frac{1}{r^2} + \frac{2GMm^2}{l^2 r} + \frac{2mE}{l^2}} \quad - ⑨$$

put

$$\left.\begin{aligned} u &\equiv \frac{1}{r} \\ \frac{du}{d\theta} &= -\frac{1}{r^2}\frac{dr}{d\theta} \end{aligned}\right) - ⑩$$

$$\frac{du}{d\theta} = -\sqrt{\frac{2mE}{l^2} + \frac{2GMm^2}{l^2}u - u^2}$$

$$-\int d\theta = \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2GMm^2}{l^2}u - u^2}} \quad - ⑪$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-c}}\sin^{-1}\frac{2cx+b}{\sqrt{-\Delta}} \quad \left(\begin{aligned} &c<0 \\ &\Delta \equiv 4ac-b^2<0 \end{aligned}\right)$$

$$a = \frac{2mE}{l^2}, \quad b = \frac{2GMm^2}{l^2}, \quad c = -1$$

$$\Delta = -\left[\frac{8mE}{l^2} + \frac{4G^2M^2m^4}{l^4}\right] < 0$$

$$-\theta = \sin^{-1} \frac{-2u + \frac{2GMm^2}{l^2}}{\sqrt{\frac{2mE}{l^2} + \frac{4G^2M^2m^4}{l^4}}}$$

$$\Rightarrow u = \frac{1}{r} = \frac{GMm^2}{l^2} + \sqrt{\frac{2mE}{l^2} + \frac{G^2M^2m^4}{l^4}} \sin\theta \qquad - (12)$$

Define

$$\left.\begin{aligned} \epsilon &= \frac{l^2}{GMm^2} \sqrt{\frac{G^2M^2m^4}{l^4} + \frac{2mE}{l^2}} \\ r_0 &= \frac{1}{1+\epsilon} \frac{l^2}{GMm^2} \end{aligned}\right) \qquad - (13)$$

Then

$$r = \frac{r_0(1+\epsilon)}{1+\epsilon\sin\theta} \qquad - (13)$$

ϵ : eccentricity

$\epsilon = 0$: circle

$0 < \epsilon < 1$: ellipse

$\epsilon = 1$: parabola

$\epsilon > 1$: hyperbola

For ellipse $(0 < \epsilon < 1)$, put

$$a = \frac{r_0}{1-\epsilon}, \quad b = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\, r_0 \qquad \text{— (14)}$$

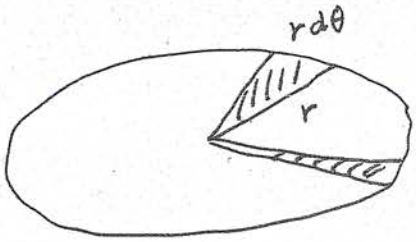
a: 장반경

b: 단반경

Kepler 제 1 법칙

Then

$$\frac{1}{r} = \frac{a}{b^2} + \frac{\sqrt{a^2-b^2}}{b^2}\sin\theta \qquad \text{— (15)}$$



주기

$$T = \int_0^T dt$$

$$= \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{d\theta}{\dot{\theta}}$$

$$= \frac{m}{l}\int_0^{2\pi} r^2\, d\theta$$

$$= \frac{m}{l}\int_0^{2\pi} d\theta \quad \frac{d\theta}{[A + B\sin\theta]^2} \qquad \text{— (16)}$$

$$S = \frac{1}{2} r^2 d\theta$$

$$\frac{dS}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2m} = \text{const}$$

Kepler 제 2 법칙

where

$$\left.\begin{aligned} A &= \frac{a}{b^2} \\ B &= \frac{\sqrt{a^2-b^2}}{b^2} \end{aligned}\right) \qquad \text{— (17)}$$

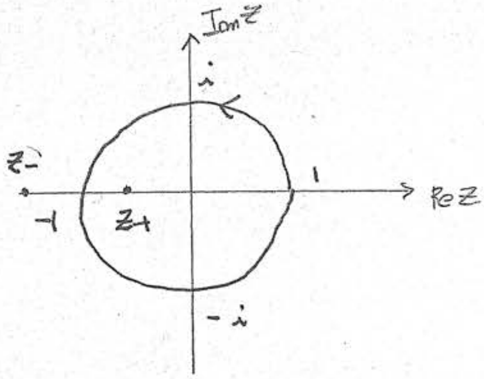
Let

$$J \equiv \int_0^{2\pi} d\theta \; \frac{1}{[A + B\sin\theta]^2} = \int_0^{2\pi} d\theta \; \frac{1}{[A + B\cos\theta]^2} \quad — (18)$$

Let

$$z = e^{i\theta}$$

$$d\theta = \frac{1}{iz}\, dz$$



$$J = \oint_C \frac{dz}{iz} \; \frac{1}{[A + \frac{B}{2}(z + \frac{1}{z})]^2}$$

$$= \oint_C \frac{dz}{iz} \; \frac{1}{[\frac{B}{2}z + A + \frac{B}{2z}]^2}$$

$$= \frac{1}{i}\oint_C \frac{z\,dz}{[\frac{B}{2}z^2 + Az + \frac{B}{2}]^2}$$

$$= \frac{1}{i}\left(\frac{2}{B}\right)^2 \oint_C \frac{z\,dz}{[z^2 + \frac{2A}{B}z + 1]^2}$$

$$= \frac{1}{i}\left(\frac{2}{B}\right)^2 \oint_C \frac{z\,dz}{(z - z_+)^2 (z - z_-)^2} \quad — (19)$$

$$z_\pm = -\frac{A}{B} \pm \sqrt{\frac{A^2}{B^2} - 1} \quad — (20)$$

Since $A > B$, $z_- < -1$.

Since $z_+ = -\sqrt{\dfrac{a-b}{a+b}}$, $-1 < z_+ < 0$.

$$J = \frac{1}{i}\left(\frac{2}{B}\right)^2 2\pi i \; R(z = z_+) \qquad \text{double pole}$$

$$= \frac{8\pi}{B^2} \frac{d}{dz}\left(\frac{z}{(z-z_-)^2}\right)\Bigg|_{z=z_+}$$

$$= \frac{8\pi}{B^2}\left[\frac{1}{(z-z_-)^2} - 2\frac{z}{(z-z_-)^3}\right]\Bigg|_{z=z_+}$$

$$= \frac{8\pi}{B^2}\left[\frac{1}{(z_+-z_-)^2} - \frac{2z_+}{(z_+-z_-)^3}\right]$$

$$= \frac{8\pi}{B^2}\frac{-(z_++z_-)}{(z_+-z_-)^3} \qquad \left(\begin{array}{l} z_+ + z_- = -\frac{2A}{B} \\ z_+ - z_- = 2\sqrt{\frac{A^2}{B^2} - 1} \end{array}\right)$$

$$= \frac{\cancel{8}\pi}{B^2}\frac{\frac{2A}{B}}{\cancel{8}\left(\frac{A^2}{B^2}-1\right)^{\frac{3}{2}}}$$

$$= \frac{2\pi A}{\cancel{B^3}}\frac{\cancel{B^3}}{(A^2-B^2)^{\frac{3}{2}}} \qquad \left(A^2 - B^2 = \frac{1}{b^2}\right)$$

$$= 2\pi \frac{a}{b^2} b^3$$

$$= 2\pi ab \qquad - (21)$$

(21) → (18)

$$\underline{T = \frac{m}{l} 2\pi ab} \qquad - (22)$$

Since

$$\frac{b}{a} = \sqrt{1-\epsilon^2} \qquad - (23)$$

$$T = 2\pi\sqrt{1-\epsilon^2}\,\frac{m}{l}\,a^2 \qquad - (24)$$

4·8

Since

$$a = \frac{1}{1-\epsilon}\,\frac{r_0}{}\,\frac{1}{1+\epsilon}\,\frac{\ell^2}{GMm^2}$$

$$= \frac{1}{1-\epsilon^2}\,\frac{\ell^2}{m^2}\,\frac{1}{GM}$$

$$\Rightarrow (1-\epsilon^2)\frac{m^2}{\ell^2} = \frac{1}{GMa}$$

$$\Rightarrow \sqrt{1-\epsilon^2}\,\frac{m}{\ell} = \frac{1}{\sqrt{GMa}} \qquad - (25)$$

(25) → (24)

$$\underline{T = \frac{1}{\sqrt{GM}}\,a^{\frac{3}{2}}} \qquad - (26)$$

$T^2 \propto a^3 \Rightarrow$ Kepler's third law !!

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