Quantum Information Processing (2024) 23:37
https://doi.org/10.1007/s11128-023-04249-x

Average Rényi entropy of a subsystem in random pure state

MuSeong Kim[1] · Mi-Ra Hwang[2] · Eylee Jung[2] · DaeKil Park[2,3]

Received: 4 June 2023 / Accepted: 28 December 2023

Abstract
In this paper, we examine the average Rényi entropy S_α of a subsystem A when the whole composite system AB is a random pure state. We assume that the Hilbert space dimensions of A and AB are m and mn, respectively. First, we compute the average Rényi entropy analytically for $m = \alpha = 2$. We compare this analytical result with the approximate average Rényi entropy, which is shown to be very close. For general case, we compute the average of the approximate Rényi entropy $\widetilde{S}_\alpha(m,n)$ analytically. When $1 \ll n$, $\widetilde{S}_\alpha(m,n)$ reduces to $\ln m - \frac{\alpha}{2n}(m - m^{-1})$, which is in agreement with the asymptotic expression of the average von Neumann entropy. Based on the analytic result of $\widetilde{S}_\alpha(m,n)$, we plot the $\ln m$-dependence of the Rényi information derived from $\widetilde{S}_\alpha(m,n)$. It is remarkable to note that the nearly vanishing region of the information becomes shorten with increasing α and eventually disappears in the limit of $\alpha \to \infty$. The physical implication of the result is briefly discussed.

Keywords Random pure state · Average Renyi entropy

1 Introduction

Although their motivations are different, the authors of Ref. [1–3] considered a similar problem: the average von Neumann entropy of a subsystem ρ_A whose Hilbert space dimension is m when the whole system is a mn-dimensional random bipartite pure state $\rho = |\psi\rangle_{AB}\langle\psi|$ with a condition $m \leq n$. Of course, $\rho_A = \mathrm{Tr}_B \rho$ and $\rho_B = \mathrm{Tr}_A \rho$.

✉ DaeKil Park
dkpark@kyungnam.ac.kr

1 Pharos iBio Co., Ltd., Head Office: #1408, 38, Heungan-daero 427beon-gil, Dongan-gu, Anyang 14059, Korea

2 Department of Electronic Engineering, Kyungnam University, Changwon 631-701, Korea

3 Department of Physics, Kyungnam University, Changwon 631-701, Korea

Published online: 27 January 2024

In particular, Ref. [3] introduced the probability distribution

$$P\left(p_1, \cdots, p_m\right) dp_1 \cdots dp_m \propto \delta\left(1-\sum_{i=1}^m p_i\right) \prod_{1 \le i<j \le m}\left(p_1-p_j\right)^2 \prod_{k=1}^m\left(p_k^{n-m} dp_k\right) \tag{1.1}$$

where $\{p_1, \ldots, p_m\}$ are eigenvalues of ρ_A. Thus, the problem can be summarized as a computation of the following quantity:

$$S_{von}(m, n) \equiv \langle S_A \rangle = -\int\left(\sum_{i=1}^m p_i \ln p_i\right) P\left(p_1, \cdots, p_m\right) dp_1 \cdots dp_m. \tag{1.2}$$

Page in Ref. [3] computed $S_{von}(2, n)$ and $S_{von}(3, n)$ analytically, and $S_{von}(4, n)$ and $S_{von}(5, n)$ with the aid of MATHEMATICA 2.0. Finally, he conjectured that $S_{von}(m, n)$ is

$$S_{von}(m, n) = \sum_{n=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \sim \ln m - \frac{m}{2n} \tag{1.3}$$

where the last equation is valid only for $1 \ll m \le n$. The last term $\frac{m}{2n}$ indicates that the entanglement entropy obeys a volume-law [4]. The Page's conjecture was rigorously proven in Ref. [5–7]. In particular, authors in Ref. [6, 7] change the multiple integral of Eq. (1.2) into a single integral by using a generalized Laguerre polynomial [8].

In Ref. [9] Page applied Eq. (1.3) to the information loss problem [10, 11] in the Hawking radiation [12, 13]. He assumed that the whole random pure state $|\psi\rangle_{AB}$ represents the Hawking radiation (ρ_A) and the remaining black hole (ρ_B) states. The reason why the random state is chosen is that the composite state is assumed to be highly complicate and hence, we do not know the state $|\psi\rangle_{AB}$ exactly. Defining the quantum information $I_{von}(m, n) = \ln m - S_{von}(m, n)$, he plots the $\ln m$-dependence of $I_{von}(m, n)$ (see Fig. 2b) and claimed that the information may come out initially so slowly. His calculation suggests that in order to obtain a sufficient information from Hawking radiation it takes at least the time necessary to radiate half the entropy of the black hole [14, 15]. Research on this issue is not concluded and is still ongoing.

The Page curve (1.3) is extended to the multipartite case [16, 17] and random mixed states [18]. Besides black hole, it is also applied to many different fields such as fermion systems [19, 20], random spin chain [21], bosonic [22] and fermion [23, 24] Gaussian states, quantum thermalization [25–28], and quantum chaos [26, 29–33]. It is also applied to the quantum information theories like random quantum circuits [34–36] and random quantum channels [37–39].

In this paper, we will extend Ref. [3] to the average Rényi entropy defined as

$$S_\alpha(m, n) \equiv \langle S_{A,\alpha} \rangle = \frac{1}{1-\alpha} \int \ln\left(\sum_{i=1}^m p_i^\alpha\right) P\left(p_1, \ldots, p_m\right) dp_1 \cdots dp_m. \tag{1.4}$$

Even though we apply the method of Ref. [7], it is impossible to convert the multiple integral of Eq. (1.4) into a single integral. Thus, it seems to be impossible to compute $S_\alpha(m,n)$ analytically. In the next section, however, we compute $S_{\alpha=2}(2,n)$ analytically. It was shown in this section that the analytical result of $S_{\alpha=2}(2,n)$ is very close to the approximate Rényi entropy defined by

$$\widetilde{S}_\alpha(m,n) = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^{m} \langle p_i^\alpha \rangle \right) \tag{1.5}$$

when $m = \alpha = 2$. In Eq. (1.5) $\sum_{i=1}^{m} \langle p_i^\alpha \rangle$ is defined as

$$\sum_{i=1}^{m} \langle p_i^\alpha \rangle \equiv Z_\alpha = \int \left(\sum_{i=1}^{m} p_i^\alpha \right) P(p_1, \ldots, p_m)\, dp_1 \cdots dp_m. \tag{1.6}$$

In Sect. 3 we will compute Z_α explicitly for any positive real α. It is represented as double summations. In Sect. 4 we compute the approximate Rényi entropy $\widetilde{S}_\alpha(m,n)$ analytically. Defining the quantum information $I_\alpha(m,n) = \ln m - \widetilde{S}_\alpha(m,n)$ and using various asymptotic formula, we show that for large n $I_\alpha(m,n)$ reduces to $\frac{\alpha}{2n}(m - m^{-1})$, which is in agreement with Eq. (1.3) when $\alpha = 1$ and $m \gg 1$. We plot $I_\alpha(m,n)$ with varying α in this section and compare it to the case of von Neumann entropy presented in Ref. [9]. With increasing α, the region for the almost vanishing information of $I_\alpha(m,n)$ becomes shorten and eventually disappears at $\alpha = \infty$. This means that in the application to the black hole radiation the quantum information of the Rényi entropy comes out more earlier than that of von Neumann entropy. In Sect. 5 a brief conclusion is given.

2 Computation of $S_{\alpha=2}(2,n)$

Defining $q_i = rp_i$, one can show from Eq. (1.4) that $S_\alpha(m,n)$ can be expressed as

$$S_\alpha(m,n) = \frac{\alpha}{\alpha-1}\psi(mn) - \frac{1}{\alpha-1} \frac{\int \ln\left(\sum_{i=1}^{m} q_i^\alpha\right) Q dq_1 \cdots dq_m}{\int Q dq_1 \cdots dq_m} \tag{2.1}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is a digamma function and[1]

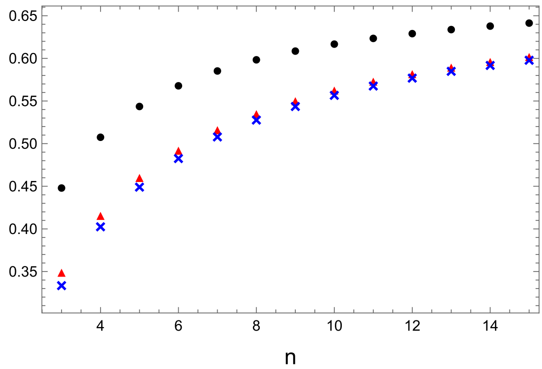
$$Q(q_1, \cdots, q_m) dq_1 \cdots dq_m = \prod_{1 \le i < j \le m} (q_1 - q_j)^2 \prod_{k=1}^{m} \left(e^{-q_k} q_k^{n-m} dq_k \right). \tag{2.2}$$

Now, we put $m = \alpha = 2$. In this case, it is easy to show

$$\int Q dq_1 dq_2 = \frac{2}{n-1} \Gamma^2(n). \tag{2.3}$$

[1] Eq. (2.2) is called a density of the eigenvalues of the Wishart matrix.

Fig. 1 (Color online) The n-dependence of exact Rényi entropy (red triangle) given in Eq. (2.8) and approximate Rényi entropy (blue cross) given in Eq. (2.9) when $m = \alpha = 2$. The black dot represents the exact von Neumann entropy given in Eq. (1.3) with $m = 2$



Also the numerator in Eq. (2.1) can be written as

$$\int \ln(q_1^2 + q_2^2) Q dq_1 dq_2$$
$$= \int_0^\infty dq_1 \int_0^\infty dq_2 e^{-(q_1+q_2)} \left[q_1^n q_2^{n-2} + q_1^{n-2} q_2^n - 2 \left(q_1^{n-1} \right)^2 \right] \ln(q_1^2 + q_2^2). \tag{2.4}$$

In order to compute Eq. (2.4) analytically, we use the following double integral formula [40]

$$\int_0^\infty dx \int_0^\infty dy \ln(x^2 + y^2) e^{-px-qy} = -\frac{2}{pq} \left[\gamma + \frac{2p^2 \ln q + 2q^2 \ln p - \pi pq}{2(p^2 + q^2)} \right] \tag{2.5}$$

where $\gamma = 0.5772$ is Euler's constant. Applying $\left(-\frac{\partial}{\partial p}\right)^m \left(-\frac{\partial}{\partial q}\right)^n$ to both sides of Eq. (2.5) and putting $p = q = 1$ at the final stage of calculation, one can compute

$$F(m, n) = \int_0^\infty dx \int_0^\infty dy x^m y^n \ln(x^2 + y^2) e^{-x-y} \tag{2.6}$$

analytically. For example, $F(2, 3) = F(3, 2) = -24\gamma + 21\pi - 14$. In principle, the general expression of $F(m, n)$ for arbitrary integers m and n can be derived with the aid of MATHEMATICA 13.1. Since, however, it is very lengthy and complicated[2], we will not present the explicit expression in this paper.

[2] Furthermore, $F(m, n)$ depends on the j^{th} term of some recurrence relations, where j is a function of m and n. This term is expressed with the aid of few special functions such as Lerch transcendent

$$\Phi(z, s, a) = \sum_{k=0}^\infty \frac{z^k}{(k + a)^s}.$$

Using Eq. (2.6) it is easy to show

$$\int \ln(q_1^2 + q_2^2) Q dq_1 dq_2 = 2F(n, n-2) - 2F(n-1, n-1). \tag{2.7}$$

Inserting Eqs. (2.3) and (2.7) into Eq. (2.1) with assuming $m = \alpha = 2$, the average Rényi entropy becomes

$$S_{\alpha=2}(2, n) = 2\psi(2n) - (n-1)\frac{F(n, n-2) - F(n-1, n-1)}{\Gamma^2(n)}. \tag{2.8}$$

As we will show later, one can show $\langle p_1^2 + p_2^2 \rangle = (n-2)/(2n+1)$. Therefore, Eq. (1.5) reduces to

$$\widetilde{S}_{\alpha=2}(2, n) = -\ln\left(\frac{n+2}{2n+1}\right). \tag{2.9}$$

In Fig. 1 we plot the n-dependence of $S_{\alpha=2}(2, n)$ and $\widetilde{S}_{\alpha=2}(2, n)$ as red triangle and blue cross, respectively. The black dot represent the average von Neumann entropy given in Eq. (1.3) with $m = 2$. As expected, the Rényi entropy is less than the von Neumann entropy. As the figure shows, the exact and approximate Rényi entropies are very close to each other.

3 Computation of $Z_\alpha = \sum_{j=1}^m \langle p_j^\alpha \rangle$

In this section, we will compute $\sum_{j=1}^m \langle p_j^\alpha \rangle$ analytically. First, we assume that α is integer for simplicity. Later, we will derive the expression of Z_α for any positive real α. This will be used later to compute $\widetilde{S}_\alpha(m, n)$ presented in Eq. (1.5).

Introducing $q_i = r p_i$ again, one can show

$$\sum_{j=1}^m \langle p_j^\alpha \rangle = \frac{\Gamma(mn)}{\Gamma(mn+\alpha)} \frac{\int \left(\sum_{i=1}^m q_i^\alpha\right) Q dq_1 \cdots dq_m}{\int Q dq_1 \cdots dq_m}. \tag{3.1}$$

As Refs. [6, 7] shows, we first note

$$\prod_{1 \le i < j \le m} (q_1 - q_j)^2 = \begin{vmatrix} p_0^\beta(q_1) & \cdots & p_0^\beta(q_m) \\ p_1^\beta(q_1) & \cdots & p_1^\beta(q_m) \\ \vdots & \ddots & \vdots \\ p_{m-1}^\beta(q_1) & \cdots & p_{m-1}^\beta(q_m) \end{vmatrix}^2 \tag{3.2}$$

where

$$p_k^\beta(q) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r \frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-r+1)} q^{k-r} = (-1)^k k! L_k^\beta(q). \tag{3.3}$$

In Eq. (3.3) $L_k^\beta(q)$ is a generalized Laguerre polynomial. It is worthwhile noting that Eq. (3.2) is valid for any real β. Thus, we can choose β freely for convenience. Using the properties of the generalized Laguerre polynomial, one can show [40, 41]

$$\int_0^\infty dq e^{-q} q^\beta p_{k_1}^\beta(q) p_{k_2}^\beta(q) = \Gamma(k_1+1)\Gamma(k_1+\beta+1)\delta_{k_1,k_2} \tag{3.4}$$

and

$$\int_0^\infty dq e^{-q} q^{a-1} p_k^b(q) = (1-a+b)_k \Gamma(a)(-1)^k \tag{3.5}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

Now, let us define

$$J_m \equiv \frac{\int \left(\sum_{i=1}^m q_i^\alpha\right) Q dq_1 \cdots dq_m}{\int Q dq_1 \cdots dq_m}. \tag{3.6}$$

First, we consider the case of $m = 2$ for simplicity. In this case, we choose $\beta = n-2$. Using Eq. (3.2) and orthogonality condition (3.4) it is easy to show

$$\int Q dq_1 dq_2 = 2! \left[\int dq_1 e^{-q_1} q_1^{n-2} \left(p_0^{n-2}(q_1)\right)^2\right] \left[\int dq_2 e^{-q_2} q_2^{n-2} \left(p_1^{n-2}(q_2)\right)^2\right]. \tag{3.7}$$

Similarly, it is straightforward to show

$$\begin{aligned} &\int \left(\sum_{i=1}^2 q_i^\alpha\right) Q dq_1 dq_2 \\ &= 2! \Bigg[\left\{\int dq_1 e^{-q_1} q_1^{n-2} \left(p_0^{n-2}(q_1)\right)^2\right\} \left\{\int dq_2 e^{-q_2} q_2^{n+\alpha-2} \left(p_1^{n-2}(q_2)\right)^2\right\} \\ &\quad + \left\{\int dq_1 e^{-q_1} q_1^{n+\alpha-2} \left(p_0^{n-2}(q_1)\right)^2\right\} \left\{\int dq_2 e^{-q_2} q_2^{n-2} \left(p_1^{n-2}(q_2)\right)^2\right\} \Bigg]. \end{aligned} \tag{3.8}$$

Inserting Eqs. (3.7) and (3.8) into Eq. (3.6) with $m = 2$ and using Eqs. (3.4) and (3.5), one can show

$$J_2 = \sum_{k=0}^{1} \frac{I_{k,\alpha}(n-2)}{\Gamma(k+1)\Gamma(k+n-1)} \tag{3.9}$$

where

$$I_{k,\alpha}(x) = \int dq e^{-q} q^{\alpha+x} \left(p_k^x(q)\right)^2. \tag{3.10}$$

Therefore, the multiple integral in Eq. (3.6) is changed into a single integral.

Now, we consider the general case. In this case, we choose $\beta = n - m$. Similar calculation leads

$$J_m = \sum_{k=0}^{m-1} \frac{I_{k,\alpha}(n-m)}{\Gamma(k+1)\Gamma(k+n-m+1)}. \tag{3.11}$$

Finally, we should compute $I_{k,\alpha}(x)$ analytically. First, we note the recursion relation $p_k^x(q) = p_k^{x+1}(q) + kp_{k-1}^{x+1}(q)$. Applying this recursion relation iteratively, one can derive

$$p_k^x(q) = \sum_{i=0}^{\ell} \binom{\ell}{i} (k-i+1)_i \, p_{k-i}^{x+\ell}(q) \tag{3.12}$$

for all nonnegative integer ℓ. Choosing $\ell = \alpha$ and using the orthogonality condition (3.4), one can compute $I_{k,\alpha}(x)$, whose explicit expression is

$$I_{k,\alpha}(x) = \Gamma^2(k+1) \sum_{i=0}^{\alpha} \binom{\alpha}{i}^2 \frac{\Gamma(k+\alpha+x-i+1)}{\Gamma(k-i+1)}. \tag{3.13}$$

Thus, inserting Eq. (3.13) into Eq. (3.11) one can derive J_m as double summations. Then, Eq. (3.1) is expressed as

$$\begin{aligned} Z_\alpha \equiv \sum_{i=1}^{m} \langle p_i^\alpha \rangle = \frac{\Gamma(mn)}{\Gamma(mn+\alpha)} \sum_{k=0}^{m-1} \frac{\Gamma(k+1)}{\Gamma(k+n-m+1)} \\ \sum_{i=0}^{\alpha} \binom{\alpha}{i}^2 \frac{\Gamma(k+n-m+\alpha-i+1)}{\Gamma(k-i+1)}. \end{aligned} \tag{3.14}$$

Equation (3.14) can be used to prove the Page's conjecture (1.3). From Eq. (3.14) one can differentiate Z_α with respect to α. Using $\Gamma'(z) = \Gamma(z)\psi(z)$ and $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$, it is straightforward to show that $-\frac{\partial}{\partial\alpha} Z_\alpha|_{\alpha=1}$ coincides with Eq. (1.3) exactly.

Finally, let me derive a different expression of Z_α, which is valid for any positive real α. In the second summation of Eq. (3.14), the actual upper bound of the parameter i is $\min(\alpha, k)$ because if $k < \alpha$, $\Gamma(k-i+1)$ located in denominator diverges when $k+1 \leq i \leq \alpha$. In order to avoid this inconvenience, we introduce a new variable $j = k - i$, which runs from $-\alpha$ to $m-1$. In this case, $\Gamma(k-i+1)$ is changed into $\Gamma(j+1)$, which goes to infinity for $j \leq -1$. In this reason, negative j does not contribute to Z_α. As a result, Z_α is expressed in the form:

$$Z_\alpha = \frac{\Gamma(mn)\Gamma^2(\alpha+1)}{\Gamma(mn+\alpha)} \sum_{k=0}^{m-1} \frac{\Gamma(k+1)}{\Gamma(k+n-m+1)} \sum_{j=0}^{m-1} \frac{\Gamma(n-m+\alpha+1+j)}{\Gamma^2(k-j+1)\Gamma^2(\alpha-k+j+1)\Gamma(j+1)}. \tag{3.15}$$

Although this expression has similar problem when $j > k+1$, α is not involved in the summation upper bound. Thus, Eq. (3.15) is valid for any positive real α. The exactly same expression was derived in Ref. [42]. However, Eq. (3.14) is more convenient if α is integer and $\alpha \ll m$ because number of summation is very small compared to that of Eq. (3.15).

4 Rényi Information from $\widetilde{S}_\alpha(m, n)$

From the previous sections the approximate Rényi entropy $\widetilde{S}_\alpha(m, n)$ defined in Eq. (1.5) is given by

$$\widetilde{S}_\alpha(m, n) = \frac{1}{1-\alpha} \ln Z_\alpha \tag{4.1}$$

where Z_α is presented in Eq. (3.14). Now, we assume $n \gg 1$. Using

$$\begin{aligned} &\lim_{z\to\infty} \Gamma(1+z) \sim e^{-z} z^z \sqrt{2\pi z}\left[1 + \frac{1}{12z} + \mathcal{O}(z^{-2})\right] \\ &\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^x \sim e^a \left[1 - \frac{a^2}{2x} + \mathcal{O}(x^{-2})\right], \end{aligned} \tag{4.2}$$

one can show

$$\begin{aligned} &\frac{\Gamma(mn)}{\Gamma(mn+\alpha)} \sim (mn)^{-\alpha}\left[1 - \frac{\alpha(\alpha-1)}{2mn} + \mathcal{O}(n^{-2})\right] \\ &\frac{\Gamma(n+k-m+1+\alpha-i)}{\Gamma(n+k-m+1)} \\ &\sim n^{\alpha-i}\left[1 + \frac{1}{2n}\left\{i^2 + i(2m-2k-2\alpha-1) - \alpha(2m-2k-\alpha-1)\right\} + \mathcal{O}(n^{-2})\right]. \end{aligned} \tag{4.3}$$

Then, for large n Z_α reduces to

$$Z_\alpha \sim m^{1-\alpha}\left[1 + \frac{\alpha(\alpha-1)}{2n}(m - m^{-1}) + \mathcal{O}(n^{-2})\right]. \tag{4.4}$$

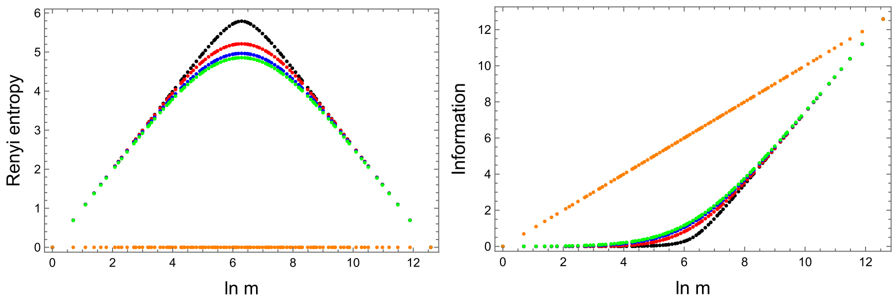


Fig. 2 (Color online) $\ln m$-dependence of **a** $\widetilde{S}_\alpha(m,n)$ and **b** $I_\alpha(m,n)$. Here, we take $mn = 2^4 3^6 5^2 = 291,600$. In both figures, the black, red, blue, green, and orange dots correspond to $\alpha = 1, 10, 100, 1000$, and ∞, respectively

As expected, Z_α becomes m or 1 when $\alpha = 0$ or 1.

Combining Eqs. (4.1) and (4.4), for large n $\widetilde{S}_\alpha(m,n)$ behaves as follows:

$$\widetilde{S}_\alpha(m,n) \approx \ln m - \frac{1}{\alpha-1}\ln\left[1+\frac{\alpha(\alpha-1)}{2n}(m-m^{-1})\right] \sim \ln m - \frac{\alpha}{2n}(m-m^{-1}). \tag{4.5}$$

If $1 \ll m$ and $\alpha = 1$, this equation reduces to $\widetilde{S}_\alpha(m,n) \sim \ln m - m/(2n)$, which coincides with Eq. (1.3).

The quantum information can be defined as the deficit of the average Rényi entropy from the maximum:

$$I_\alpha(m,n) = \ln m - \widetilde{S}_\alpha(m,n) \sim \frac{1}{\alpha-1}\ln\left[1+\frac{\alpha(\alpha-1)}{2n}(m-m^{-1})\right] \tag{4.6}$$

where last equation is valid for $n \gg 1$.

Now, let us consider $m > n$ case. From the Schmidt decomposition we know that the eigenvalues of the density operators of systems A and B are the same. Thus, the approximate Rényi entropy becomes $\widetilde{S}_\alpha(n,m)$. If $m \gg 1$, the information reduces to

$$I_\alpha(m,n) \sim \ln m - \ln n + \frac{1}{\alpha-1}\ln\left[1+\frac{\alpha(\alpha-1)}{2m}(n-n^{-1})\right]. \tag{4.7}$$

Finally, let us consider $\alpha \to \infty$ limit. Equation (3.14) implies that the leading term of $Z_{\alpha=\infty}$ is

$$Z_{\alpha=\infty} \sim \frac{\Gamma(mn)}{\Gamma(m)\Gamma(n)}\alpha^{-(m-1)(n-1)}. \tag{4.8}$$

Therefore, $\widetilde{S}_{\alpha\to\infty} = 0$ and $I_{\alpha=\infty} = \ln m$.

The $\ln m$-dependence of $\widetilde{S}_\alpha(m,n)$ and $I_\alpha(m,n)$ is plotted in Fig. 2 for $\alpha = 1$ (black), 10 (red), 100 (blue), 1000 (green) and ∞ (orange). As Fig. 2a exhibits, the

Table 1 The α-dependence of m_*

α	1	10	100	1000	∞
m_*	243	90	40	27	2

$\ln m$-dependence of the average Rényi entropy $\widetilde{S}_\alpha(m, n)$ decreases with increasing α, and eventually approaches to zero at $\alpha = \infty$. As Fig. 2b exhibits, the nearly vanishing region of $I_\alpha(m, n)$ is shorten with increasing α and eventually disappears at $\alpha = \infty$.

For example, let us define m_*, which is the smallest m with satisfying $I_\alpha(m, n) > 0.1$. The α-dependence of m_* is summarized at Table 1. As expected, m_* is decreasing with increasing α.

5 Conclusions

In this paper we examine the average Rényi entropy S_α of a subsystem A when the whole composite system AB is a random pure state. We assume that the Hilbert space dimensions of A and AB are m and mn, respectively, with $m \leq n$. If $m \geq n$, the Schmidt decomposition guarantees that the average value is obtained by simply interchanging m and n. First, we compute the average Rényi entropy analytically for $m = \alpha = 2$. We compare this analytical result with the approximate average Rényi entropy $\widetilde{S}_{\alpha=2}(2, n)$. As Fig. 1 shows, these two results are very close to each other, especially when n is large. For general case, we compute $\widetilde{S}_\alpha(m, n)$ analytically. When $1 \ll n$, $\widetilde{S}_\alpha(m, n)$ reduces to $\ln m - \frac{\alpha}{2n}(m - m^{-1})$, which is in agreement with the asymptotic expression of the average von Neumann entropy given in Ref. [3]. Defining the information by Eq. (4.6), we plot the $\ln m$ dependence of the information $I_\alpha(m, n)$ in Fig. 2b. It is remarkable to note that the nearly vanishing region of $I_\alpha(m, n)$ becomes shorten with increasing α, and eventually disappears in the limit of $\alpha \to \infty$.

This result has important implication in the application of information loss problem. If we assume that A and B are the radiation and remaining black hole states, the information derived from the Rényi entropy can be obtained from Hawking radiation more and more earlier to that of von Neumann entropy with increasing α, and in the limit of $\alpha = \infty$ the information is radiated as soon as Hawking radiation starts. If this is right, we should re-consider the "Alice and Bob" thought-experiment described in Ref. [43, 44] on no-cloning theorem more carefully. Besides black hole physics, we want to examine the effect of our result in the quantum information theories like random quantum circuit and random quantum channel.

The defect of our result is a fact that our calculation is based on $\widetilde{S}_\alpha(m, n)$. Although we guess $S_\alpha(m, n)$ also exhibits a similar behavior in Fig. 2, we cannot prove it on the analytical ground. Numerical calculation is also very difficult when m is large, because the calculation requires m-multiple integration. Probably, we may need a new idea to explore this issue.

Acknowledgements This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1A2C1094580).

Data Availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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