Quantum Information induced by Negativity in Random Pure State

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Abstract

The average negativity of the random bipartite pure state $|\psi\rangle_{AB}$ is explicitly computed when the Hilbert space dimensions of the party A and B are m and n respectively with assuming $m \leq n$. It is shown that for large n the difference between the maximum and average negativities is roughly $(m^2 - 1)/8n$ while corresponding value for the entanglement entropy is $m/2n$.The implication in the information loss problem is briefly discussed.

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Few decades ago the average of entanglement for bipartite random pure state $|\psi\rangle_{AB}$ was investigated[1–3]. In particular, Page in Ref.[3] conjectured the average of entanglement entropy (EE)[1] for $|\psi\rangle_{AB}$. If the Hilbert space dimensions of the parties A and B are m and n ($m \leq n$) respectively, his conjecture is

$$S_{EE}(m,n) \equiv \langle S \rangle = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \sim \ln m - \frac{m}{2n} \tag{1}$$

where S is a von Neumann entropy of subsystem A and the last equation is valid for $1 \ll m \leq n$. The last term $m/2n$ implies that the EE obeys a volume law[4]. The conjecture was rigorously proven in Ref.[5–7].

Subsequently, in Ref.[8] Page applied Eq. (1) to the Hawking radiation[9, 10] of black hole. The main interest of Ref.[8] was an information loss paradox of black hole[11, 12]. He assumed that the whole random pure state $|\psi\rangle_{AB}$ is a composite state of the Hawking radiation (ρ_A) and the remaining black hole (ρ_B) states, where

$$\rho_A = \mathrm{tr}_B |\psi\rangle_{AB}\langle\psi| \qquad \rho_B = \mathrm{tr}_A |\psi\rangle_{AB}\langle\psi|. \tag{2}$$

He argued that the information may come out so slowly at the early stage of Hawking radiation. In order to obtain a sufficient information it may take at least the time necessary to radiate half the entropy of the whole black hole[13, 14].

Few years ago the variance of the von Neumann entropy of a subsystem A was conjectured[15] in the form

$$\begin{aligned} V_A(m,n) &\equiv \langle S^2 \rangle - \langle S \rangle^2 \\ &= -\psi_1(mn+1) + \frac{m+n}{mn+1}\psi_1(n) - \frac{(m+1)(m+2n+1)}{4n^2(mn+1)} \sim \frac{2n-m}{4n^3} \end{aligned} \tag{3}$$

where $\psi_1(z)$ is the trigamma function. It was also rigorously proven in Ref. [16]. Furthermore, the third statistical moment M_3, called skewness, is computed in Ref.[17], whose

[1] This is a von Neumann entropy of the substate.

explicit expression is

$$\begin{aligned}
M_3(m,n) &\equiv \langle S^3\rangle - \langle S\rangle^3 - 3\langle S\rangle M_2 \qquad (4)\\
&= \psi_2(mn+1) - \frac{m^2+3mn+n^2+1}{(mn+1)(mn+2)}\psi_2(n+1) + \frac{(m^2-1)(mn-3n^2+1)}{n(mn+1)^2(mn+2)}\psi_1(n+1)\\
&\quad - \frac{(m-1)\left\{(2m^3n+4mn^3+3m^2n^2)-(4m^2n+3mn^2)\right\}}{4n^3(mn+1)^2(mn+2)}\\
&\quad - \frac{(m-1)\left\{(2m^2+10n^2+8mn)-(4m+6n)+2\right\}}{4n^3(mn+1)^2(mn+2)}\\
&\sim -\frac{4n^2-5mn+2m^2}{4mn^5}
\end{aligned}$$

where $\psi_2(z)$ is a tetra-gamma function. It is worthwhile noting that $M_3 < 0$ at the large m and n region. This implies that the distribution of the EE for the random pure state has a left tail longer than a right tail.

Besides EE, other quantum information quantities were considered in the random state. When $m = n$, the average and its variance for the negativity defined as

$$\mathcal{N}(\rho_{AB}) = \frac{||\rho_{AB}^{T_A}|| - 1}{2} \qquad (5)$$

are explicitly computed in Ref.[18], where T_A is a partial transposition and $||A||$ is a trace norm of A. In Ref. [19] and [20] the authors considered two random states and computed the average relative entropy and trace distance between them in the large n regime respectively. In Ref.[21] the average Rényi entropy was computed and its implication to the information loss problem was discussed. Extension to the multipartite and random mixed cases were discussed in Ref.[22, 23] and Ref.[24] respectively.

In this paper we will compute the average negativity of $|\psi\rangle_{AB}$ when $m \leq n$. It was shown that for large n the difference between maximum negativity and the average is $(m^2-1)/8n$, which approaches to zero when $n \gg m$. It is interesting to compare it with the EE case, where the difference is $m/2n$. Since the denominator is proportional to m^2 in the case of negativity, the difference for negativity is much larger than that for EE case when n is fixed.

The main result of this paper is that the average of the negativity is given by

$$\langle \mathcal{N}\rangle \equiv \mathcal{N}_{m,n} = \frac{1}{2mn}\sum_{k,\ell=0}^{m-1}\gamma_{k,\ell}\left(J_{kk}^{(1/2)}J_{\ell\ell}^{(1/2)} - J_{k\ell}^{(1/2)}J_{\ell k}^{(1/2)}\right) \qquad (6)$$

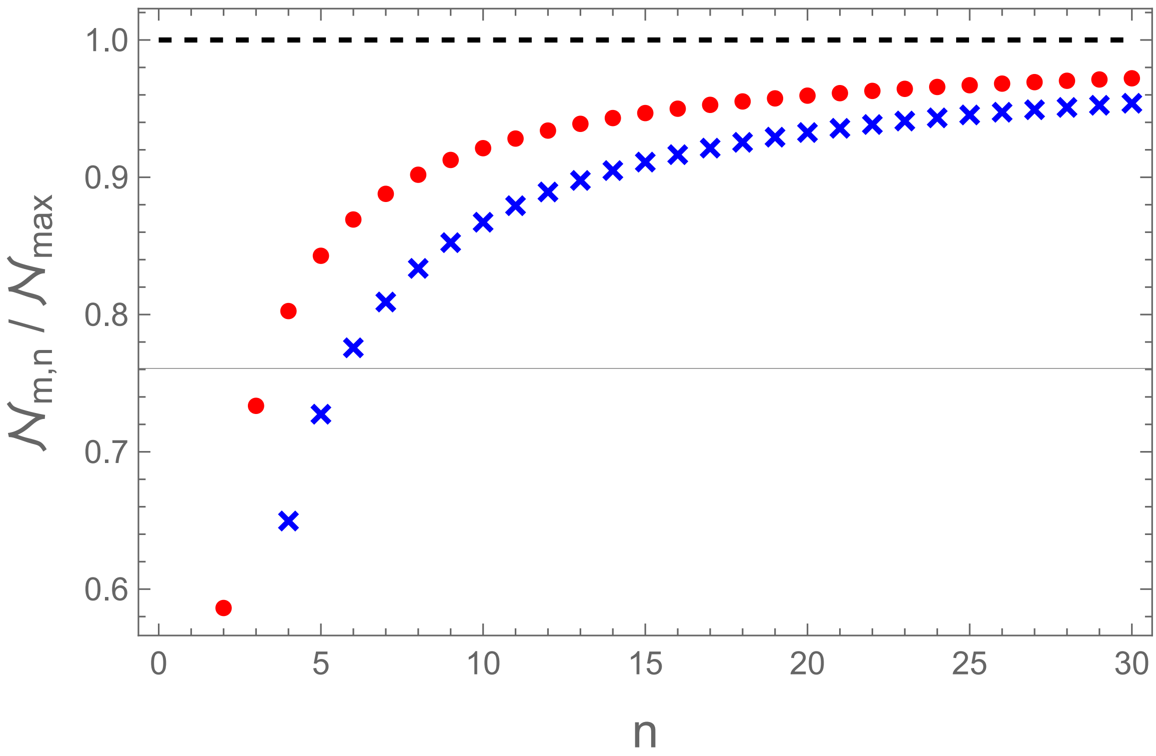

FIG. 1: (Color online) The n-dependence of $\langle \mathcal{N} \rangle / \mathcal{N}_{max}$, where $\mathcal{N}_{max} = (m-1)/2$. The red dot and blue cross correspond to $m = 2$ and $m = 4$ respectively.

where

$$\gamma_{k,\ell} = \frac{1}{k!\ell!(k+n-m)!(\ell+n-m)!} \tag{7}$$
$$J_{k\ell}^{(\beta)} = (-1)^k \frac{\Gamma(n-m+k+1)\Gamma(\beta+1)\Gamma(n-m+\beta+1)}{\Gamma(n-m+1)\Gamma(\beta-\ell+1)}$$
$$\times {}_3F_2\left(\beta+1, n-m+\beta+1, -k; n-m+1, \beta+1-\ell; 1\right).$$

In Eq. (7) ${}_pF_q$ is a hypergeometric function. It is easy to show that when $m = n$, Eq. (6) reduces to the result of Ref. [18], where the average negativity and its variance were derived when $m = n$. The n-dependence of f $\langle \mathcal{N} \rangle / \mathcal{N}_{max}$ is plotted in Fig. 1, where $\mathcal{N}_{max} = (m-1)/2$ is a maximum of the negativity. The red dot and blue cross in the figure correspond to $m = 2$ and $m = 4$ respectively. Both exhibit monotonic increasing behaviors and approach to 1 in large n regime.

Using

$$\lim_{z\to\infty}\Gamma(1+z)\sim e^{-z}z^z\sqrt{2\pi z}\left[1+\frac{1}{12z}+\mathcal{O}(z^{-2})\right] \tag{8}$$
$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x\sim e^a\left[1-\frac{a^2}{2x}+\mathcal{O}(x^{-2})\right],$$

and

$$_2F_1(a,b;c:1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{9}$$

One can show straightforwardly that for large n $\mathcal{N}_{m,n}$ reduces to

$$\mathcal{N}_{m,n}\sim\mathcal{N}_{max}-\frac{m^2-1}{8n}. \tag{10}$$

As Page defined in Ref. [3], it is convenient to define the average information of the subsystem induced by negativity as a form

$$I_{m,n}=\mathcal{N}_{max}-\langle\mathcal{N}\rangle=\frac{m-1}{2}-\mathcal{N}_{m,n}. \tag{11}$$

Thus, for large n $I_{m,n}$ reduces to $(m^2-1)/(8n)$ while the quantum information induced from the entanglement entropy reduces to $m/(2n)$[3].

The $\ln\mathcal{N}_{max}$ - dependence of $\mathcal{N}_{m,n}$ and $I_{m,n}$ when $mn=2^4 3^6 5^2=291600$ are plotted in Fig. 2 . For comparison we plot the entanglement entropy and corresponding quantum information in Fig. 2(c), which was firstly plotted in Ref. [8]. The behaviors between negativity and entanglement entropy are very similar even though the scale is different. Thus, if the parties A and B are Hawking radiation and the remaining states respectively, the information induced by negativity also suggests that in order to obtain a sufficient information from Hawking radiation it takes at least the time necessary to radiate half the entropy of the black hole[13, 14].

Now, we explain how to derive Eq. (6). Let $|\psi\rangle_{AB}$ be a bipartite random pure state whose Hilbert space dimension is mn with $m\le n$. We define the substates as

$$\rho_A=\mathrm{Tr}_B|\psi\rangle_{AB}\langle\psi| \qquad \rho_B=\mathrm{Tr}_A|\psi\rangle_{AB}\langle\psi|, \tag{12}$$

whose Hilbert space dimension is m and n respectively. Then, the negativity for ρ_A is

$$\mathcal{N}=\frac{1}{2}\left[\left(\sum_{i=1}^m\sqrt{p_i}\right)^2-1\right]=\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^m\sqrt{p_ip_j}. \tag{13}$$

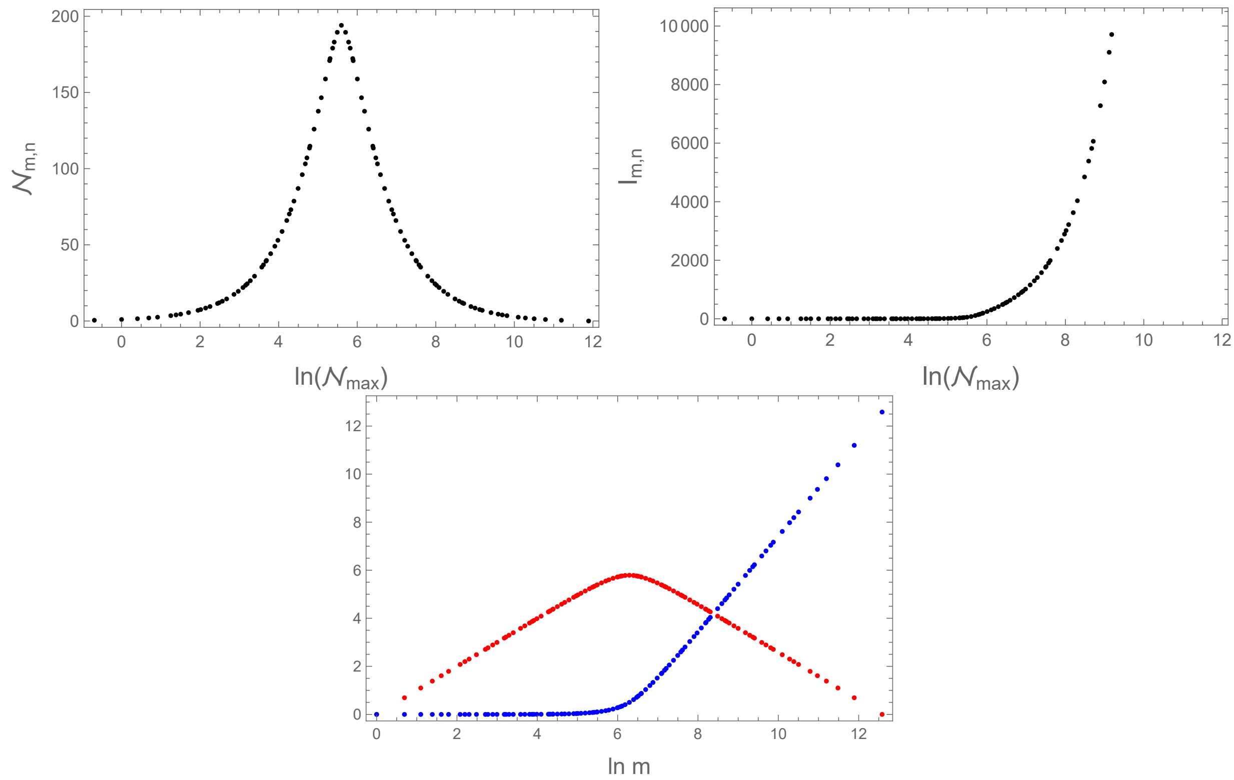

FIG. 2: (Color online) The $\ln \mathcal{N}_{max}$ - dependence of (a) $\mathcal{N}_{m,n}$ and (b) $I_{m,n}$ when $mn = 2^4 3^6 5^2 = 291600$. For comparison we plot the entanglement entropy and corresponding quantum information in (c), which was plotted in Ref. [8]. The behaviors between negativity and entanglement entropy are very similar even though the scale is different.

Thus, the average of it is given by

$$\langle \mathcal{N} \rangle = \frac{1}{2} \int \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \sqrt{p_i p_j} P(\mathbf{p}) d\mathbf{p} \tag{14}$$

where $P(\mathbf{p})d\mathbf{p}$ is a probability distribution of the eigenvalues p_i[1–3]

$$P(\mathbf{p})d\mathbf{p} = \mathcal{C} \delta \left(1 - \sum_{i=1}^{m} p_i \right) \Delta_m(\mathbf{p}) \prod_{k=1}^{m} \left(p_k^{n-m} dp_k \right). \tag{15}$$

In Eq. (15) $\Delta_m(\mathbf{p})$ and the normalization constant $\mathcal{C}$[5] are

$$\Delta_m(\mathbf{p}) = \prod_{1 \leq i < j \leq m} (p_i - p_j)^2 \qquad \mathcal{C} = \frac{(mn-1)!}{\prod_{k=1}^{m} [k!(n-k)!]}. \tag{16}$$

Defining $q_j = r p_j \;\; (j = 1, 2, \cdots, m)$, one can show straightforwardly

$$\int \sqrt{p_i p_j} P(\mathbf{p}) d\mathbf{p} = \widetilde{\mathcal{C}} \int \sqrt{q_i q_j} Q(\mathbf{q}) d\mathbf{q} \tag{17}$$

where

$$\widetilde{\mathcal{C}} = \frac{\Gamma(mn)}{\Gamma(mn+1)} \left[\prod_{k=1}^{m} \{k!(n-k)!\} \right]^{-1} \qquad Q(\mathbf{q})\mathrm{d}\mathbf{q} = \Delta_m(\mathbf{q}) \prod_{k=1}^{m} \left(e^{-q_k} q_k^{n-m} dq_k \right). \tag{18}$$

Therefore, $\langle N \rangle$ can be written as a form

$$\langle N \rangle = \frac{\widetilde{\mathcal{C}}}{2} \int \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \sqrt{q_i q_j} Q(\mathbf{q}) \mathrm{d}\mathbf{q}. \tag{19}$$

As Ref.[6, 7] stressed, $\Delta_m(\mathbf{q})$ can be written in a form:

$$\Delta_m(\mathbf{q}) = \begin{vmatrix} p_0^\beta(q_1) & \cdots & p_0^\beta(q_m) \\ p_1^\beta(q_1) & \cdots & p_1^\beta(q_m) \\ \vdots & \ddots & \vdots \\ p_{m-1}^\beta(q_1) & \cdots & p_{m-1}^\beta(q_m) \end{vmatrix}^2 \tag{20}$$

where

$$p_k^\beta(q) = \sum_{r=0}^{k} \begin{pmatrix} k \\ r \end{pmatrix} (-1)^r \frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta-r+1)} q^{k-r} = (-1)^k k! L_k^\beta(q). \tag{21}$$

In Eq. (21) $L_k^\beta(q)$ is a generalized Laguerre polynomial. It is worthwhile noting that Eq. (20) is valid for any real β. Thus, we can choose β freely for convenience. Using the properties of the generalized Laguerre polynomial, one can show[25, 26]

$$\int_0^\infty dq e^{-q} q^\beta p_{k_1}^\beta(q) p_{k_2}^\beta(q) = \Gamma(k_1+1)\Gamma(k_1+\beta+1)\delta_{k_1,k_2} \tag{22}$$

and

$$\int_0^\infty dq e^{-q} q^{a-1} p_k^b(q) = (1-a+b)_k \Gamma(a)(-1)^k \tag{23}$$

where $(a)_k = a(a+1)\cdots(a+k-1)$.

Now, let us consider a permutation group S_m and let us define

$$R = \begin{pmatrix} 0 & 1 & \cdots & m-1 \\ r(0) & r(1) & \cdots & r(m-1) \end{pmatrix} = \begin{cases} 2 & \text{for even permutation} \\ 1 & \text{for odd permutation.} \end{cases} \tag{24}$$

Then, it is possible to show

$$\Delta_m(\mathbf{q}) = \sum_{R,S \in S_m} (-1)^{R+S} \prod_{k=1}^{m} \left(p_{r(k-1)}^\beta(q_k) p_{s(k-1)}^\beta(q_k) \right). \tag{25}$$

Employing Eq. (25) and using Eq. (22) one can compute

$$\overline{Q} \equiv \int Q(\mathbf{q}) \mathrm{d}\mathbf{q} = \prod_{k=1}^{m} \left[k!(n-k)! \right]. \tag{26}$$

When deriving Eq. (26) we should choose β as a $\beta = n - m$. Following similar, but long and tedious calculation on can derive

$$\int \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \sqrt{q_i q_j} Q(\mathbf{q}) \mathrm{d}\mathbf{q} = \overline{Q} \sum_{k,\ell=0}^{m-1} \gamma_{k,\ell} \begin{vmatrix} J_{kk}^{(1/2)} & J_{k\ell}^{(1/2)} \\ J_{\ell k}^{(1/2)} & J_{\ell\ell}^{(1/2)} \end{vmatrix} \tag{27}$$

where $\gamma_{k,\ell}$ is given in Eq. (7) and

$$J_{k\ell}^{(\beta)} = \int_0^\infty e^{-q} q^{n-m+\beta} p_k^{n-m}(q) p_\ell^{n-m}(q) dq. \tag{28}$$

Inserting Eq. (27) into Eq. (19) one can derive our main result (6). Finally, using an integral formula[25]

$$\int_0^\infty x^{\alpha-1} e^{-cx} L_m^\gamma(bx) L_n^\lambda(cx) dx = \frac{(1+\gamma)_m (1-\alpha+\lambda)_n \Gamma(\alpha)}{m! n! b^\alpha} \times {}_3F_2 \left(-m, \alpha, \alpha-\lambda; 1+\gamma, \alpha-\lambda-n; \frac{b}{c} \right) \tag{29}$$

and $p_k^\beta(q) = (-1)^k k! L_k^\beta(q)$, one can derive $J_{k\ell}^{(\beta)}$ analytically, which is given in Eq. (7).

In this paper we compute the average of negativity $\mathcal{N}_{m,n}$ analytically in the random bipartite pure state $|\psi\rangle_{AB}$, whose Hilbert space dimension is mn. For large n it turns out $\mathcal{N}_{max} - \mathcal{N}_{m,n} \sim (m^2-1)/8n$. Similar equation for EE is $\ln m - S_{EE}(m,n) \sim m/2n$. Thus, both the negativity and EE of the random pure state approach to their maximum with increasing n. If $1 \ll m \ll n$, the m^2-dependence in the denominator of $\mathcal{N}_{max} - \mathcal{N}_{m,n}$ implies $\mathcal{N}_{max} - \mathcal{N}_{m,n} \gg \ln m - S_{EE}(m,n)$. Thus, with increasing n the approach to the maximum value for the negativity is much slower than that for the EE. The $ln\mathcal{N}_{max}$-dependence of the negativity and its information is plotted in Fig. 2 (a) and (b). Comparison between them and Fig. 2 (c) shows that the only difference between negativity and EE is only scale difference.

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