§ Some More Mathematics

[1] Gamma function

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx \qquad (p>0)$$

properties

(i) $\Gamma(p+1) = p\,\Gamma(p)$

(ii) $\Gamma(N+1) = N!$ $\qquad$ (N: positive integer)

(iii) $\Gamma(p)\,\Gamma(1-p) = \dfrac{\pi}{\sin(\pi p)}$

Pf)

(i) $\Gamma(p+1) = \int_0^\infty \underbrace{x^p}_{u}\ \underbrace{e^{-x}}_{v'}\, dx$ $\qquad$ $u = x^p \quad v = -e^{-x}$

$u' = p x^{p-1} \quad v' = e^{-x}$

$$= -x^p e^{-x}\Big|_{x=0}^{x=\infty} + \underbrace{p\int_0^\infty x^{p-1} e^{-x}\, dx}_{=\Gamma(p)}$$

$$= \underbrace{-\lim_{x\to\infty} \frac{x^p}{e^x}}_{=0} + p\,\Gamma(p)$$

$$= p\,\Gamma(p)$$

(ii) $\Gamma(N+1) = N(N-1)\cdots 2\times 1\ \Gamma(1)$

$$= N!\,\Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_{x=0}^{x=\infty} = 1$$

$$\Gamma(N+1) = N! \qquad /\!/$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Pf)

(i) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x}\, dx$

$$\left(\begin{aligned} y &= \sqrt{x} \\ dy &= \frac{1}{2\sqrt{x}}\, dx \end{aligned} \right)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-y^2}\,(2\,dy)$$

$$= 2\int_0^\infty e^{-y^2}\,dy \qquad — ①$$

$$\int_0^\infty e^{-y^2}\,dy = \sqrt{\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy}$$

$$= \sqrt{\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy}$$

$$\left(\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}\left(\tfrac{y}{x}\right) \\ dx\,dy &= r\,dr\,d\theta \end{aligned} \right)$$

$$= \sqrt{\int_0^\infty dr\; r\, e^{-r^2} \int_0^{\frac{\pi}{2}} d\theta}$$

$$= \sqrt{\frac{\pi}{2}\left(-\frac{1}{2}\right) e^{-r^2}\Big|_{r=0}^{r=\infty}}$$

$$= \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2} \qquad — ②$$

② → ① $\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) put $p = \frac{1}{2}$ in (iii)

$$\Gamma^2\left(\tfrac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi \quad \Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

※

Ex)

$$\Gamma(\tfrac{7}{2}) = \tfrac{5}{2}\,\Gamma(\tfrac{5}{2})$$

$$= \tfrac{5}{2}\,\tfrac{3}{2}\,\Gamma(\tfrac{3}{2})$$

$$= \tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})$$

$$= \tfrac{15}{8}\sqrt{\pi} \quad ※$$

* Using $\Gamma(p) = \dfrac{\Gamma(p+1)}{p}$, one can define $\Gamma(p)$ for negative p.

Ex) $\Gamma(-\tfrac{1}{2}) = \dfrac{\Gamma(\tfrac{1}{2})}{-\tfrac{1}{2}} = -2\sqrt{\pi}$ ※

* Gaussian Integral

$$\int_{-\infty}^{\infty} e^{ax^2+bx}\,dx = \sqrt{\frac{\pi}{-a}}\; e^{-\frac{b^2}{4a}} \qquad (a<0)$$

Pf)

$$\int_{-\infty}^{\infty} e^{ax^2+bx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right)}\,dx$$

$$= e^{-\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{a\left(x+\frac{b}{2a}\right)^2}\,dx \qquad y = x + \frac{b}{2a}$$

$$= e^{-\frac{b^2}{4a}} \int_{-\infty}^{\infty} e^{ay^2}\,dy$$

$$= 2e^{-\frac{b^2}{4a}} \int_{0}^{\infty} e^{ay^2}\,dy \qquad \left(z = -ay^2,\ dy = \frac{dz}{2\sqrt{-a}\sqrt{z}}\right)$$

$$= 2e^{-\frac{b^2}{4a}} \int_{0}^{\infty} e^{-z}\,\frac{dz}{2\sqrt{-a}\sqrt{z}}$$

$$= \frac{1}{\sqrt{-a}}\,e^{-\frac{b^2}{4a}} \underline{\int_{0}^{\infty} \frac{1}{\sqrt{x}}\,e^{-x}\,dx} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \quad = \sqrt{\frac{\pi}{-a}}\; e^{-\frac{b^2}{4a}} \quad ※$$

Ex) 정규분포 (p296)

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx \qquad y = x - m$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}}\,dy \qquad \left(a = -\frac{1}{2\sigma^2},\ b = 0\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \sqrt{\frac{\pi}{\frac{1}{2\sigma^2}}}$$

$$= 1$$

※

Ex) Beta function

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad (p>0,\ q>0)$$

properties

(i) $B(p,q) = B(q,p)$

(ii) $B(p,q) = 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}\,d\theta$

(iii) $B(p,q) = \int_0^{\infty} \frac{y^{p-1}}{(1+y)^{p+q}}\,dy$

(iv) $B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$

(pf)

(i) $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ $\quad (y = 1-x)$

$= \int_1^0 (1-y)^{p-1} y^{q-1} (-dy)$

$= \int_0^1 y^{q-1}(1-y)^{p-1}\,dy$

$= B(q,p)$

(ii) $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

put

$x = \sin^2\theta \qquad dx = 2\sin\theta\cos\theta\,d\theta$

$B(p,q) = \int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-2}(\cos\theta)^{2q-2}\, 2\sin\theta\cos\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta$ ※

(iii) put $x = \dfrac{y}{1+y}$ $\Rightarrow$ 생략

(iv) $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,dt$ $\quad (t = y^2,\ dt = 2y\,dy)$

$= 2\int_0^\infty y^{2p-1} e^{-y^2}\,dy$

$\Gamma(q) = 2\int_0^\infty x^{2q-1} e^{-x^2}\,dx$

$\Rightarrow \Gamma(p)\Gamma(q) = 4\int_0^\infty dx \int_0^\infty dy\ x^{2q-1} y^{2p-1} e^{-(x^2+y^2)}$ $\quad \left(\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dx\,dy = r\,dr\,d\theta \end{array} \right)$

$= 4\int_0^\infty dr \int_0^{\frac{\pi}{2}} d\theta\, r\,(r\cos\theta)^{2q-1}(r\sin\theta)^{2p-1} e^{-r^2}$

$= 4\int_0^\infty dr\ r^{2p+2q-1} e^{-r^2}\,dr\ \underbrace{\int_0^{\frac{\pi}{2}} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta}_{\frac{1}{2}B(p,q)}$

$= 2B(p,q)\int_0^\infty dr\ r^{2p+2q-1} e^{-r^2}$ $\quad \left(x = r^2,\ dr = \dfrac{dx}{2\sqrt{x}}\right)$

$$= 2B(p,q)\int_0^\infty \frac{dx}{2\sqrt{x}}\ x^{p+q-\frac{1}{2}}\ e^{-x}$$

$$= B(p,q)\underline{\int_0^\infty dx\ x^{p+q-1}\ e^{-x}}$$

$\Gamma(p+q)$

$$= B(p,q)\ \Gamma(p+q)$$

$$\Rightarrow\quad B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$ ※

Ex)

$$\int_0^\infty \frac{x^3}{(1+x)^5}\,dx$$

$$= B(4,1)$$

$$= \frac{\Gamma(4)\,\Gamma(1)}{\Gamma(5)}$$

$$= \frac{3!\ 0!}{4!}$$

$$= \frac{1}{4}$$ ※

[3] Stirling's formula

$$\boxed{\text{For large } n \quad n! \sim n^n e^{-n} \sqrt{2\pi n}}$$ Stirling's formula

pf)

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x} dx = \int_0^\infty e^{p\ln x} e^{-x} dx = \int_0^\infty e^{p\ln x - x} dx \quad \text{— ①}$$

Define

$$y = \frac{x-p}{\sqrt{p}} \quad \text{— ②}$$

② → ①

$$\Gamma(p+1) = \int_{-\sqrt{p}}^\infty e^{p\ln(\sqrt{p}y+p) - (p+\sqrt{p}y)} \sqrt{p}\, dy$$

$$= \sqrt{p}\int_{-\sqrt{p}}^\infty e^{p\ln[p(1+\frac{y}{\sqrt{p}})] - p - \sqrt{p}y} dy$$

$$= \sqrt{p}\int_{-\sqrt{p}}^\infty e^{p\ln p + p\ln(1+\frac{y}{\sqrt{p}}) - p - \sqrt{p}y} dy \quad \text{— ③}$$

Now, we assume that p is very large.

Taylor Expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \quad \text{— ④}$$

Thus,

$$\ln(1+\frac{y}{\sqrt{p}}) = \frac{y}{\sqrt{p}} - \frac{1}{2}\frac{y^2}{p} + \cdots \quad \text{— ⑤}$$

⑤ → ③

$$\Gamma(p+1) = \sqrt{p}\int_{-\sqrt{p}}^\infty e^{p\ln p + p(\frac{y}{\cancel{\sqrt{p}}} - \frac{1}{2}\frac{y^2}{p} + \cdots) - p - \cancel{\sqrt{p}}y} dy$$

$$\cong \sqrt{p}\, e^{p\ln p - p}\int_{-\sqrt{p}}^\infty e^{-\frac{y^2}{2}} dy$$

$$= \sqrt{p}\, e^{p\ln p - p} \left[\underbrace{\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy}_{\sqrt{2\pi}} - \underbrace{\int_{-\infty}^{-\sqrt{p}} e^{-\frac{y^2}{2}} dy}_{\approx 0 \ (\because p \text{ is very large})} \right]$$

$$\approx \sqrt{2\pi p}\; p^p\, e^{-p}$$

put $p = n$. Then

$$n! \simeq \sqrt{2\pi n}\; n^n e^{-n} \quad \text{for large } n$$

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