Generalized uncertainty principle and *d*-dimensional quantum mechanics

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Nonrelativistic quantum mechanics with a generalized uncertainty principle (GUP) is examined in d-dimensional free particle and harmonic oscillator systems. The Feynman propagators for these systems are exactly derived within the first order of the GUP parameter.

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I. INTRODUCTION

The existence of a minimal length seems to be a model-independent feature of quantum gravity [1–3]. It appears as various expressions in loop quantum gravity [4,5], string theory [6,7], path-integral quantum gravity [8–10], and black hole physics [11]. From the aspect of quantum mechanics, the existence of a minimal length results in the modification of the Heisenberg uncertainty principle (HUP) [12,13] $\Delta P \Delta Q \geq \frac{\hbar}{2}$, because ΔQ should be larger than the minimal length. Various modifications of HUP, called the generalized uncertainty principle (GUP), were suggested in Refs. [14,15].

The purpose of this paper is to examine the d-dimensional nonrelativistic quantum mechanics when the GUP has the following specific form:

$$\Delta P_i \Delta Q_i \geq \frac{\hbar}{2}[1 + \alpha(\Delta \mathbf{P}^2 + \langle \mathbf{P} \rangle^2) + 2\alpha(\Delta P_i^2 + \langle P_i \rangle^2)] \quad (i = 1, 2, \ldots, d) \tag{1.1}$$

where α is a GUP parameter, which has the dimension $(\text{momentum})^{-2}$. Using $\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|$, Eq. (1.1) induces the modification of the commutation relation as

$$[Q_i, P_j] = i\hbar(\delta_{ij} + \alpha \delta_{ij} \mathbf{P}^2 + 2\alpha P_i P_j), \qquad [Q_i, Q_j] = [P_i, P_j] = 0. \tag{1.2}$$

The existence of the minimal length is easily shown at $d = 1$. In this case, Eq. (1.1) is expressed as

$$\Delta P \Delta Q \geq \frac{\hbar}{2}(1 + 3\alpha \Delta P^2) \tag{1.3}$$

if $\langle P \rangle = 0$. Then, the equality of Eq. (1.3) yields

$$\Delta Q^2 \geq \Delta Q_{\text{min}}^2 = 3\alpha\hbar^2. \tag{1.4}$$

In Fig. 1, the allowed region and minimal length of Eq. (1.3) are plotted when $\hbar = \alpha = 1$.

If α is small, Eq. (1.2) can be solved as

$$P_i = p_i(1 + \alpha \mathbf{p}^2) + \mathcal{O}(\alpha^2), \qquad Q_i = q_i, \tag{1.5}$$

where p_i and q_i obey the usual HUP. Using Eq. (1.5) and Feynman's path-integral technique [16,17], the Feynman propagator (or kernel) has been exactly derived up to $\mathcal{O}(\alpha)$ for the $d = 1$ free particle case [18,19]. Also, the propagator for a simple harmonic oscillator (SHO) was derived recently in Ref. [20].

In this paper, we will derive the Feynman propagators for d-dimensional free particle and SHO systems within the first order of α in the GUP-corrected quantum mechanics. In Sec. II, we derive the Feynman propagators for $d = 2$ free particle and SHO systems up to $\mathcal{O}(\alpha)$. In Sec. III, we extend the results to d-dimensional systems. In Sec. IV, a brief conclusion is given. In the Appendix, we summarize

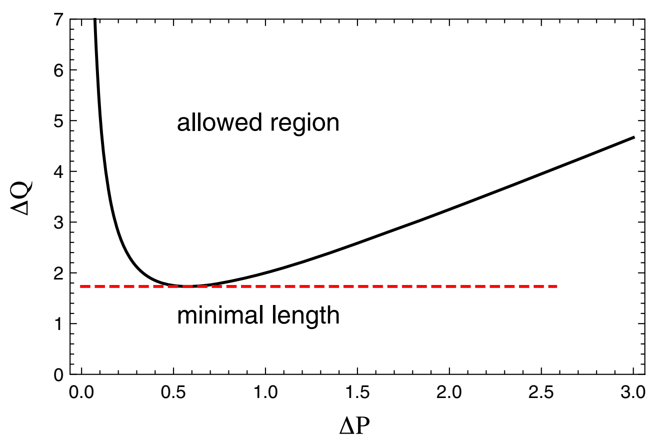

FIG. 1. The minimal length and allowed region of one-dimensional GUP [Eq. (1.3)] when $\hbar = \alpha = 1$.

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several d-dimensional Gaussian integral formulas, which are frequently used in the main text.

II. TWO-DIMENSIONAL FREE PARTICLE AND SHO SYSTEMS

A. Free particle case

The Hamiltonian for a two-dimensional free particle system is

$$H_F = \frac{1}{2m}(P_1^2 + P_2^2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{\alpha}{m}(p_1^2 + p_2^2)^2 + \mathcal{O}(\alpha^2). \tag{2.1}$$

Then, the corresponding Schrödinger equation can be written as

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2}\right) + \frac{\alpha\hbar^4}{m}\left(\frac{\partial^4}{\partial q_1^4} + \frac{\partial^4}{\partial q_2^4} + 2\frac{\partial^2}{\partial q_1^2}\frac{\partial^2}{\partial q_2^2}\right)\right] \times \phi(\mathbf{q}) = \mathcal{E}\phi(\mathbf{q}). \tag{2.2}$$

The eigenfunction and corresponding eigenvalue of the Schrödinger equation are

$$\phi(\boldsymbol{q}) = \frac{1}{2\pi}e^{i\boldsymbol{k}\cdot\boldsymbol{q}}, \qquad \mathcal{E} = \frac{\hbar^2}{2m}|\boldsymbol{k}|^2 + \frac{\alpha\hbar^4}{m}|\boldsymbol{k}|^4. \tag{2.3}$$

Using Eqs. (2.3), ((A1), and (A2), one can straightforwardly derive the Feynman propagator $K_F[\boldsymbol{q}_f, t_f : \boldsymbol{q}_0, t_0]$ as follows:

$$\begin{aligned}
K_F[\boldsymbol{q}_f, t_f : \boldsymbol{q}_0, t_0] &= \int_{-\infty}^{\infty} d\mathrm{k}\phi(\boldsymbol{q}_f)\phi^*(\boldsymbol{q}_0)e^{-\frac{i}{\hbar}\mathcal{E}T} \\
&= \frac{m}{2\pi i\hbar T}\left[1 + \frac{8i\alpha\hbar m}{T} - \frac{8\alpha m^2|\mathbf{q_f} - \mathbf{q_0}|^2}{T^2}\right] \\
&\quad \times \exp\left[\frac{im}{2\hbar T}|\mathbf{q_f} - \mathbf{q_0}|^2\left\{1 - 2\alpha m^2\left(\frac{|\mathbf{q_f} - \mathbf{q_0}|}{T}\right)^2\right\}\right] \\
&\quad + \mathcal{O}(\alpha^2),
\end{aligned} \tag{2.4}$$

where $T = t_f - t_0$.

B. SHO case

For a two-dimensional SHO, the Hamiltonian can be written as

$$H_{SHO} = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{\alpha}{m}(p_1^2 + p_2^2)^2 + \frac{1}{2}m\omega^2(q_1^2 + q_2^2) + \mathcal{O}(\alpha^2). \tag{2.5}$$

Then, the corresponding action functional is

$$S[q_1, q_2] = \int_0^T dt L(q_1, \dot{q}_1 : q_2, \dot{q}_2), \tag{2.6}$$

where

$$L(q_1, \dot{q}_1 : q_2, \dot{q}_2) = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \alpha m^3(\dot{q}_1^2 + \dot{q}_2^2)^2 - \frac{1}{2}m\omega^2(q_1^2 + q_2^2) + \mathcal{O}(\alpha^2). \tag{2.7}$$

Thus, the classical solutions obey

$$\begin{aligned}
&\ddot{q}_1 + \omega^2 q_1 - 4\alpha m^2[(3\dot{q}_1^2 + \dot{q}_2^2)\ddot{q}_1 + 2\dot{q}_1\dot{q}_2\ddot{q}_2] + \mathcal{O}(\alpha^2) = 0, \\
&\ddot{q}_2 + \omega^2 q_2 - 4\alpha m^2[(\dot{q}_1^2 + 3\dot{q}_2^2)\ddot{q}_2 + 2\dot{q}_1\dot{q}_2\ddot{q}_1] + \mathcal{O}(\alpha^2) = 0.
\end{aligned} \tag{2.8}$$

Up to $\mathcal{O}(\alpha)$, Eq. (2.8) can be solved as follows:

$$\begin{aligned}
q_1 &= A_1\cos\omega t + B_1\sin\omega t + \alpha\Big[F_1\cos\omega t + G_1\sin\omega t \\
&\quad + \frac{m^2\omega^2}{8}[-4C_3(\omega t)\cos\omega t + 4C_1(\omega t)\sin\omega t - C_2\cos 3\omega t - C_4\sin 3\omega t]\Big], \\
q_2 &= A_2\cos\omega t + B_2\sin\omega t + \alpha\Big[F_2\cos\omega t + G_2\sin\omega t \\
&\quad + \frac{m^2\omega^2}{8}[-4\tilde{C}_3(\omega t)\cos\omega t + 4\tilde{C}_1(\omega t)\sin\omega t - \tilde{C}_2\cos 3\omega t - \tilde{C}_4\sin 3\omega t]\Big],
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned} C_1 &= -3A_1(A_1^2 + A_2^2 + B_1^2) - 2A_2B_1B_2 - A_1B_2^2, \\ C_2 &= 3[A_1^3 - 2A_2B_1B_2 + A_1(A_2^2 - 3B_1^2 - B_2^2)], \\ C_3 &= -3A_1^2B_1 - 2A_1A_2B_2 - A_2^2B_1 - 3B_1(B_1^2 + B_2^2), \\ C_4 &= 3[3A_1^2B_1 + 2A_1A_2B_2 - B_1(-A_2^2 + B_1^2 + B_2^2)], \end{aligned} \tag{2.10}$$

and $\tilde{C}_j = C_j(1 \leftrightarrow 2)$ for $j = 1, 2, 3, 4$. Now, we impose the boundary conditions

$$\begin{aligned} q_x(t=0) &= q_{0,1}, \qquad q_y(t=0) = q_{0,2}, \\ q_x(t=T) &= q_{f,1}, \qquad q_y(t=T) = q_{f,2}. \end{aligned} \tag{2.11}$$

Then, it is straightforward to show

$$\begin{aligned} A_1 &= q_{0,1}, \qquad A_2 = q_{0,2} \qquad F_1 = \frac{m^2\omega^2}{8}C_2, \qquad F_2 = \frac{m^2\omega^2}{8}\tilde{C}_2, \\ B_1 &= \frac{1}{\sin\omega T}(q_{f,1} - q_{0,1}\cos\omega T), \qquad B_2 = \frac{1}{\sin\omega T}(q_{f,2} - q_{0,2}\cos\omega T), \\ G_1 &= \frac{m^2\omega^2}{8\sin\omega T}[(4\omega T C_3 - C_2)\cos\omega T - 4\omega T C_1 \sin\omega T + C_2\cos 3\omega T + C_4 \sin 3\omega T], \\ G_2 &= \frac{m^2\omega^2}{8\sin\omega T}[(4\omega T \tilde{C}_3 - \tilde{C}_2)\cos\omega T - 4\omega T \tilde{C}_1 \sin\omega T + \tilde{C}_2\cos 3\omega T + \tilde{C}_4 \sin 3\omega T]. \end{aligned} \tag{2.12}$$

Inserting the classical solutions into the action, one can derive the classical action in a form

$$S_{cl} = S_0 + \alpha S_1 + \mathcal{O}(\alpha^2), \tag{2.13}$$

where

$$\begin{aligned} S_0 &= \frac{m\omega}{2\sin\omega T}[(|\boldsymbol{q}_0|^2 + |\boldsymbol{q}_f|^2)\cos\omega T - 2\boldsymbol{q}_0\cdot\boldsymbol{q}_f], \\ S_1 &= -\frac{m^3\omega^3}{32\sin^4\omega T}[(12\omega T + 8\sin 2\omega T + \sin 4\omega T)(|\boldsymbol{q}_0|^4 + |\boldsymbol{q}_f|^4) \\ &\quad - 4(12\omega T\cos\omega T + 11\sin\omega T + 3\sin 3\omega T)(\boldsymbol{q}_0\cdot\boldsymbol{q}_f)(|\boldsymbol{q}_0|^2 + |\boldsymbol{q}_f|^2) \\ &\quad + 4(4\omega T + 2\omega T\cos 2\omega T + 5\sin 2\omega T)[2(\boldsymbol{q}_0\cdot\boldsymbol{q}_f)^2 + |\boldsymbol{q}_0|^2|\boldsymbol{q}_f|^2]]. \end{aligned} \tag{2.14}$$

When $\omega \to 0$, S_0 and S_1 reduce to

$$\lim_{\omega\to 0} S_0 = \frac{m}{2T}|\boldsymbol{q}_f - \boldsymbol{q}_0|^2, \qquad \lim_{\omega\to 0} S_1 = -\frac{m^3}{T^3}|\boldsymbol{q}_f - \boldsymbol{q}_0|^4, \tag{2.15}$$

which is consistent with the exponential factor of Eq. (2.4).

Now, let us derive the Feynman propagator (or kernel) $K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T]$ of this system. It is well known that the propagator is related to the Schrödinger equation as follows:

$$K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T] = \sum_{n_1,n_2} \psi_{n_1,n_2}(\boldsymbol{q}_f)\psi^*_{n_1,n_2}(\boldsymbol{q}_0)e^{-(i/\hbar)E_{n_1,n_2}T}, \tag{2.16}$$

where $\psi_{n_1,n_2}(\boldsymbol{q})$ and E_{n_1,n_2} are the eigenfunction and corresponding eigenvalue of the Schrödinger equation derived from the Hamiltonian (2.5). If we treat the term $\frac{\alpha}{m}(p_1^2 + p_2^2)^2$ as a small perturbation, it is possible to derive the eigenvalue

$$\begin{aligned} E_{n_1,n_2} &= (n_1 + n_2 + 1) + \frac{\alpha m\hbar^2\omega^2}{2}[3(n_1 + n_2)^2 \\ &\quad + 5(n_1 + n_2) - 2n_1n_2 + 4] + \mathcal{O}(\alpha^2), \\ &\quad (n_1, n_2 = 0, 1, 2, \ldots). \end{aligned} \tag{2.17}$$

Of course, it is also possible to derive the eigenfunction $\psi(\boldsymbol{q})$ within the first order of α by applying the perturbation method. However, the derivation of the Feynman propagator in this way is extremely cumbersome for the following reasons: First, the unperturbed Hamiltonian generates the degenerate eigenspectrum. Thus, we should use the perturbation method with degeneracy, which is more complicated. Second, even though we derive all eigenfunctions up to the first order of α, the right-hand side of Eq. (2.16) has too many terms, and each term involves the double summation with respect to n_1 and n_2. Thus, derivation of the Feynman propagator by making use of Eq. (2.16) is highly complicated and tedious. Thus, we will choose the alternative method in the following.

In Ref. [20], the Feynman propagator for one-dimensional SHO is derived by making use of Eq. (2.16). The final result is expressed as

$$K[q_f, q_0 : T] = \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega T}}[1 + \alpha f(q_f, q_0 : T)]e^{\frac{i}{\hbar}(S_0 + \alpha S_1)} + \mathcal{O}(\alpha^2), \tag{2.18}$$

where

$$\begin{aligned} f(q_f, q_0 : T) = {} & \frac{3i\hbar m\omega}{8\sin^2\omega T}(2\omega T + 5\sin\omega T\cos\omega T + \omega T\cos 2\omega T) \\ & - \frac{3m^2\omega^2}{8\sin^3\omega T}[2\omega T\{3\cos\omega T(q_0^2 + q_f^2) - 2(2 + \cos 2\omega T)q_0 q_f\} \\ & + 10\sin\omega T(q_0^2 + q_f^2 - 2q_0 q_f\cos\omega T) - 6\sin^3\omega T(q_0^2 + q_f^2)]. \end{aligned} \tag{2.19}$$

Of course, $S_0 + \alpha S_1$ is a classical action. We assume that the elevation of dimension does not change the Feynman propagator drastically. Thus, we take an ansatz as follows:

$$K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T] = \frac{m\omega}{2\pi i\hbar \sin\omega T}[1 + \alpha f(\mathbf{q_f}, \mathbf{q_0} : T)]e^{\frac{i}{\hbar}(S_0 + \alpha S_1)} + \mathcal{O}(\alpha^2), \tag{2.20}$$

where

$$\begin{aligned} f(\mathbf{q_f}, \mathbf{q_0} : T) = {} & \beta_1\frac{i\hbar m\omega}{\sin^2\omega T}(2\omega T + 5\sin\omega T\cos\omega T + \omega T\cos 2\omega T) \\ & + \beta_2\frac{m^2\omega^2}{\sin^3\omega T}[2\omega T\{3\cos\omega T(|\boldsymbol{q}_0|^2 + |\boldsymbol{q}_f|^2) - 2(2 + \cos 2\omega T)\boldsymbol{q}_0\cdot\mathbf{q_f}\} \\ & + 10\sin\omega T(|\boldsymbol{q}_0|^2 + |\boldsymbol{q}_f|^2 - 2(\boldsymbol{q}_0\cdot\boldsymbol{q}_f)\cos\omega T) \\ & - 6\beta_3\sin^3\omega T(|\boldsymbol{q}_0|^2 + |\boldsymbol{q}_f|^2)]. \end{aligned} \tag{2.21}$$

Of course, S_0 and S_1 are given in Eq. (2.14). If Eq. (2.21) is a correct ansatz, the constants β_j $(j = 1, 2, 3)$ should be determined by making use of the integral equation

$$\int d\boldsymbol{q}K[\boldsymbol{q}_f, \boldsymbol{q} : T_2]K[\boldsymbol{q}, \boldsymbol{q}_0 : T_1] = K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T], \tag{2.22}$$

where $T_2 = T - T_1$. Using Eq. (2.20), one can show that the left-hand side of Eq. (2.22) reduces to

$$\begin{aligned} \int d\boldsymbol{q}K[\boldsymbol{q}_f, \boldsymbol{q} : T_2]K[\boldsymbol{q}, \boldsymbol{q}_0 : T_1] = {} & \left(\frac{m\omega}{2\pi i\hbar}\right)^2\frac{1}{\sin\omega T_1\sin\omega T_2}\exp\left[\frac{im\omega}{2\hbar}(|\boldsymbol{q}_f|^2\cot\omega T_2 + |\boldsymbol{q}_0|^2\cot\omega T_1)\right] \\ & \times\int d\boldsymbol{q}e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}}\left[1 + \alpha\{f(\boldsymbol{q}, \boldsymbol{q}_0 : T_1) + f(\boldsymbol{q}_f, \boldsymbol{q} : T_2)\}\right. \\ & \left. + \frac{i\alpha}{\hbar}\{S_1(\boldsymbol{q}, \boldsymbol{q}_0 : T_1) + S_1(\boldsymbol{q}_f, \boldsymbol{q} : T_2)\} + \mathcal{O}(\alpha^2)\right], \end{aligned} \tag{2.23}$$

where

$$a = -\frac{im\omega}{2\hbar}\frac{\sin\omega T}{\sin\omega T_1 \sin\omega T_2}, \qquad \boldsymbol{b} = -\frac{im\omega}{2\hbar}\frac{\boldsymbol{q}_f \sin\omega T_1 + \boldsymbol{q}_0 \sin\omega T_2}{\sin\omega T_1 \sin\omega T_2}. \tag{2.24}$$

Using the integral formula (A2) at $d = 2$, one can show straightforwardly

$$\int d\boldsymbol{q}\{S_1(\boldsymbol{q},\boldsymbol{q}_0 : T_1) + S_1(\boldsymbol{q}_f,\boldsymbol{q} : T_2)\}e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} = \frac{\pi}{a}e^{|\boldsymbol{b}|^2/a}[S_1(\boldsymbol{q}_f,\boldsymbol{q}_0 : T) + \Delta S], \tag{2.25}$$

where

$$\begin{aligned}\Delta S = &-\frac{m^3\omega^3}{32\sin^4\omega T_1}\Bigg[(12\omega T_1 + 8\sin 2\omega T_1 + \sin 4\omega T_1)\frac{2}{a^2}\left(1 + \frac{2|\boldsymbol{b}|^2}{a}\right)\\ &- 8(12\omega T_1\cos\omega T_1 + 11\sin\omega T_1 + 3\sin 3\omega T_1)\frac{\boldsymbol{q}_0\cdot\boldsymbol{b}}{a^2}\\ &+ 8(4\omega T_1 + 2\omega T_1\cos 2\omega T_1 + 5\sin 2\omega T_1)\frac{|\boldsymbol{q}_0|^2}{a}\Bigg]\\ &+ (T_1 \to T_2, \boldsymbol{q}_0 \to \boldsymbol{q}_f).\end{aligned} \tag{2.26}$$

By making use of Eqs. (2.25) and (A1) at $d = 2$, one can show that our ansatz (2.20) really satisfies the integral equation (2.22) if

$$\beta_1 = 1, \qquad \beta_2 = -\frac{1}{2}, \qquad \beta_3 = 1. \tag{2.27}$$

Inserting Eq. (2.27) into Eq. (2.21), one can derive the Feynman propagator for a $d = 2$ SHO system explicitly. In the $\omega \to 0$ limit, it is easy to show that

$$\lim_{\omega\to 0} f(\mathbf{q_f},\mathbf{q_0} : T) = \frac{8i\hbar m}{T} - \frac{8m^2}{T^2}|\boldsymbol{q}_f - \boldsymbol{q}_0|^2, \tag{2.28}$$

which exactly coincides with the first order of α in the prefactor of the free-particle Feynman propagator given in Eq. (2.4).

In order to confirm that the Feynman propagator we derived is correct, we examine the other properties of the Feynman propagator. First, it should be a solution of the time-dependent Schrödinger equation. In fact, one can show straightforwardly that

$$\left[i\hbar\frac{\partial}{\partial T} - H_2\right]K[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = \mathcal{O}(\alpha^2), \tag{2.29}$$

where

$$\begin{aligned}H_2 = &-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial q_{f,1}^2} + \frac{\partial^2}{\partial q_{f,2}^2}\right) + \frac{\alpha\hbar^4}{m}\left(\frac{\partial^2}{\partial q_{f,1}^2} + \frac{\partial^2}{\partial q_{f,2}^2}\right)^2\\ &+ \frac{1}{2}m\omega^2|\boldsymbol{q}_f|^2.\end{aligned} \tag{2.30}$$

Thus, the Feynman propagator obeys the time-dependent Schrödinger equation within the first order of α. If we impose $K[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = 0$ for $T < 0$, Eq. (2.29) is modified in the form

$$\left[i\hbar\frac{\partial}{\partial T} - H_2\right]K[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = \xi\delta(\boldsymbol{q}_f - \boldsymbol{q}_0)\delta(T), \tag{2.31}$$

where ξ is some constant. If we integrate Eq. (2.31) from 0^- to 0^+ in T, we get

$$\lim_{T\to 0}K[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = \lim_{T\to 0}K_F[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = \frac{\xi}{i\hbar}\delta(\boldsymbol{q}_f - \boldsymbol{q}_0), \tag{2.32}$$

where $K_F[\boldsymbol{q}_f,\boldsymbol{q}_0 : T]$ is a Feynman propagator for free particles given in Eq. (2.4). If we integrate Eq. (2.32) with respect to $\boldsymbol{q}_f$ and use Eqs. (A1) and (A2) with $d = 2$, one can show that $\xi = i\hbar + \mathcal{O}(\alpha^2)$. Thus, the Feynman propagator satisfies the initial condition up to the first order of α in the form

$$\lim_{T\to 0}K[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = [1 + \mathcal{O}(\alpha^2)]\delta(\boldsymbol{q}_f - \boldsymbol{q}_0). \tag{2.33}$$

III. *d*-DIMENSIONAL FREE PARTICLE AND SHO SYSTEMS

A. Free particle case

From the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + \frac{\alpha\hbar^4}{m}\nabla^4\right]\phi(\boldsymbol{q}) = \mathcal{E}\phi(\boldsymbol{q}) \tag{3.1}$$

where ∇^2 is a Laplacian, it is easy to derive

$$\phi(\boldsymbol{q}) = \frac{1}{(2\pi)^{d/2}}e^{i\boldsymbol{k}\cdot\boldsymbol{q}}, \qquad \mathcal{E} = \frac{\hbar^2}{2m}|\boldsymbol{k}|^2 + \frac{\alpha\hbar^4}{m}|\boldsymbol{k}|^4. \tag{3.2}$$

Then, the Feynman propagator defined as

$$K_F[\boldsymbol{q}_f,\boldsymbol{q}_0 : T] = \int d\boldsymbol{k}\phi(\boldsymbol{q}_f)\phi^*(\boldsymbol{q}_0)e^{-\frac{i}{\hbar}\mathcal{E}T} \tag{3.3}$$

can be computed easily using the integral formulas presented in the Appendix. The final expression is

$$K_F[\boldsymbol{q}_f, \boldsymbol{q}_0 : T] = \left(\frac{m}{2\pi i\hbar T}\right)^{d/2}\left[1 + \frac{d(d+2)i\alpha\hbar m}{T} - \frac{2(d+2)\alpha m^2}{T^2}|\boldsymbol{q}_f - \boldsymbol{q}_0|^2\right]e^{\frac{i}{\hbar}S_{cl}} + \mathcal{O}(\alpha^2), \tag{3.4}$$

where S_{cl} is a classical action given by

$$S_{cl} = \frac{m}{2T}|\boldsymbol{q}_f - \boldsymbol{q}_0|^2\left(1 - \frac{2\alpha m^2}{T^2}|\boldsymbol{q}_f - \boldsymbol{q}_0|^2\right). \tag{3.5}$$

B. SHO case

We note that the classical action for this case is also $S_{cl} = S_0 + \alpha S_1 + \mathcal{O}(\alpha^2)$, where S_0 and S_1 have the same expressions as Eq. (2.14). This only difference is the fact that $\boldsymbol{q}_0$ and $\boldsymbol{q}_f$ are d-dimensional vectors. Now, we take an ansatz as follows:

$$K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T] = \left(\frac{m\omega}{2\pi i\hbar \sin\omega T}\right)^{d/2}[1 + \alpha f(\mathbf{q_f}, \mathbf{q_0} : T)]e^{\frac{i}{\hbar}(S_0 + \alpha S_1)} + \mathcal{O}(\alpha^2), \tag{3.6}$$

where we assume that the prefactor $f(\mathbf{q_f}, \mathbf{q_0} : T)$ is the same form as Eq. (2.21). Then, one can show

$$\int d\boldsymbol{q}\{S_1(\boldsymbol{q}, \boldsymbol{q}_0 : T_1) + S_1(\boldsymbol{q}_f, \boldsymbol{q} : T_2)\}e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} = \left(\frac{\pi}{a}\right)^{d/2}e^{|\boldsymbol{b}|^2/a}[S_1(\boldsymbol{q}_f, \boldsymbol{q}_0 : T) + \Delta S], \tag{3.7}$$

where

$$\begin{aligned}\Delta S = &-\frac{(d+2)m^3\omega^3}{32\sin^4\omega T_1}\left[(12\omega T_1 + 8\sin 2\omega T_1 + \sin 4\omega T_1)\frac{1}{a^2}\left(\frac{d}{4} + \frac{|\boldsymbol{b}|^2}{a}\right)\right.\\ &- 2(12\omega T_1\cos\omega T_1 + 11\sin\omega T_1 + 3\sin 3\omega T_1)\frac{\boldsymbol{q}_0\cdot\boldsymbol{b}}{a^2}\\ &\left. + 2(4\omega T_1 + 2\omega T_1\cos 2\omega T_1 + 5\sin 2\omega T_1)\frac{|\boldsymbol{q}_0|^2}{a}\right]\\ &+ (T_1 \to T_2, \boldsymbol{q}_0 \to \boldsymbol{q}_f).\end{aligned} \tag{3.8}$$

Using Eq. (3.8), it is straightforward to show that the Feynman propagator (3.6) obeys the integral equation (2.22) only if

$$\beta_1 = \frac{d(d+2)}{8}, \qquad \beta_2 = -\frac{d+2}{8}, \qquad \beta_3 = 1. \tag{3.9}$$

Thus, inserting Eq. (3.9) into Eq. (2.21), one can derive the Feynman propagator for the d-dimensional SHO system up to the first order of α. It is easy to show that the $\omega \to 0$ limit in this case reduces to Eq. (3.4). As we discussed in the case of $d = 2$, it is straightforward to show that the Feynman propagator satisfies the time-dependent Schrödinger equation and initial condition up to the first order of α.

IV. CONCLUSION

We derive the Feynman propagators for d-dimensional free particle and SHO systems explicitly within the first order of α in the GUP-corrected nonrelativistic quantum mechanics. From the Feynman propagator, one can derive the energy-dependent Green's function $\hat{G}[\mathbf{q_f}, \mathbf{q_0} : \epsilon]$ by taking a Laplace transform to the Euclidean propagator $G[\boldsymbol{q}_f, \boldsymbol{q}_0 : \tau] \equiv K[\boldsymbol{q}_f, \boldsymbol{q}_0 : T = -i\tau]$. This is a useful quantity, because the eigenvalue and eigenfunction of the Schrödinger equation appear as the minus pole and its residue of $\hat{G}[\mathbf{q_f}, \mathbf{q_0} : \epsilon]$. Also, one can derive the thermal state from the Euclidean propagator by letting $\tau \to 1/(k_B T)$ [16], where k_B and T are the Boltzmann constant and external temperature. From Eq. (2.4), the energy-dependent Green's function for the two-dimensional free particle case can be derived in a form

$$\hat{G}_F[\mathbf{q_f}, \mathbf{q_0} : \epsilon] = \frac{m}{\pi\hbar}[(1 + 8\alpha\hbar m\epsilon)K_0(z) - 2\alpha\hbar m\epsilon z K_1(z)], \tag{4.1}$$

where $z = \sqrt{2m\epsilon/\hbar}|\boldsymbol{q}_f - \boldsymbol{q}_0|$, and $K_\nu(z)$ is a modified Bessel function. For the case of two-dimensional SHO, the energy-dependent Green's function can be derived if $\alpha = 0$ [21]. This is expressed as a winding number representation in a form

$$\hat{G}[\mathbf{q_f}, \mathbf{q_0} : \epsilon] = \frac{1}{2\pi\omega|\boldsymbol{q}_f||\boldsymbol{q}_0|} W_{-\frac{\epsilon}{2\omega}, \frac{m}{2}}\left(\frac{m\omega}{\hbar} q_+^2\right) \times M_{-\frac{\epsilon}{2\omega}, \frac{m}{2}}\left(\frac{m\omega}{\hbar} q_-^2\right) \times \sum_{j=-\infty}^{\infty} \frac{\Gamma((1+j+\epsilon/\omega)/2)}{\Gamma(1+j)} e^{ij\theta}, \tag{4.2}$$

where $\theta = \cos^{-1}(\boldsymbol{q}_f \cdot \boldsymbol{q}_0/(|\boldsymbol{q}_f||\boldsymbol{q}_0|))$, $q_+ = \max(|\boldsymbol{q}_f|, |\boldsymbol{q}_0|)$, $q_- = \min(|\boldsymbol{q}_f|, |\boldsymbol{q}_0|)$, and $W_{\kappa,\mu}(z)$ and $M_{\kappa,\mu}(z)$ are the Whittaker's functions. However, it seems to be highly nontrivial to derive the energy-dependent Green's function within $\mathcal{O}(\alpha)$ due to the nontrivial expression of the prefactor $f(\mathbf{q_f}, \mathbf{q_0} : T)$.

It seems to be of interest to derive the Feynman propagators for the nonrelativistic and spin-1/2 Aharonov-Bohm systems. As shown in Refs. [22,23] the spin-1/2 system contains a two-dimensional δ-function potential. The effect of a two-dimensional δ function was extensively discussed in the usual quantum mechanics [23–25] by applying the self-adjoint extension [26,27] and renormalization scheme. Furthermore, the effect of a one-dimensional δ function potential was discussed recently in the GUP-corrected quantum mechanics [28]. By applying the method of Ref. [28], we hope to discuss how GUP affects the Zeeman interaction of the spin with the magnetic field in the future.

APPENDIX: *d*-DIMENSIONAL GAUSSIAN INTEGRAL FORMULAS

In this appendix, we summarize the several d-dimensional Gaussian integral formulas, which are used in the main text frequently. If we assume that $\boldsymbol{x}$ and $\boldsymbol{q}$ are d-dimensional vectors, one can derive

$$\begin{aligned} \int d\boldsymbol{q} e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a}, \\ \int d\boldsymbol{q} |\boldsymbol{q}|^2 e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a} \frac{1}{a}\left(\frac{d}{2} + \frac{|\boldsymbol{b}|^2}{a}\right), \\ \int d\boldsymbol{q} (\boldsymbol{x}\cdot\boldsymbol{q}) e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a} \frac{\boldsymbol{x}\cdot\boldsymbol{b}}{a}. \end{aligned} \tag{A1}$$

Also, one can show straightforwardly

$$\begin{aligned} \int d\boldsymbol{q} (\boldsymbol{x}\cdot\boldsymbol{q})^2 e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a} \left[\frac{|\boldsymbol{x}|^2}{2a} + \frac{(\boldsymbol{x}\cdot\boldsymbol{b})^2}{a^2}\right], \\ \int d\boldsymbol{q} (\boldsymbol{x}\cdot\boldsymbol{q}) |\boldsymbol{q}|^2 e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a} \left(\frac{d+2}{2} + \frac{|\boldsymbol{b}|^2}{a}\right) \frac{(\boldsymbol{x}\cdot\boldsymbol{b})}{a^2}, \\ \int d\boldsymbol{q} |\boldsymbol{q}|^4 e^{-a|\boldsymbol{q}|^2 + 2\boldsymbol{b}\cdot\boldsymbol{q}} &= \left(\frac{\pi}{a}\right)^{d/2} e^{|\boldsymbol{b}|^2/a} \frac{1}{a^2}\left(\frac{d(d+2)}{4} + \frac{(d+2)|\boldsymbol{b}|^2}{a} + \frac{|\boldsymbol{b}|^4}{a^2}\right). \end{aligned} \tag{A2}$$

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