Next differentiate both sides of this equation repeatedly with respect to α (see Chapter 4, Section 12):

$$\int_0^\infty -xe^{-\alpha x}\,dx = -\frac{1}{\alpha^2} \quad \text{or} \quad \int_0^\infty xe^{-\alpha x}\,dx = \frac{1}{\alpha^2},$$

$$\int_0^\infty x^2e^{-\alpha x}\,dx = \frac{2}{\alpha^3},$$

$$\int_0^\infty x^3e^{-\alpha x}\,dx = \frac{3!}{\alpha^4}.$$

or in general

$$\int_0^\infty x^ne^{-\alpha x}\,dx = \frac{n!}{\alpha^{n+1}}. \tag{2.2}$$

Putting $\alpha = 1$, we get

$$\int_0^\infty x^ne^{-x}\,dx = n!, \qquad n = 1, 2, 3, \cdots. \tag{2.3}$$

Thus we have a definite integral whose value is $n!$ for positive integral n. We can use (2.3) to give a meaning to $0!$. Putting $n = 0$ in (2.3), we get

$$0! = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1. \tag{2.4}$$

(This agrees with our previous definition of $0!$ in Chapter 1.)

PROBLEMS, SECTION 2

In Chapter 4, Section 12, do Problems 14 to 17.

3. DEFINITION OF THE GAMMA FUNCTION; RECURSION RELATION

회귀, 귀납, 반복

So far n has been a nonnegative integer; it is natural to *define* the factorial function for nonintegral n by the definite integral (2.3). There is no real objection to the notation $n!$ for nonintegral n (and we shall occasionally use it), but it is customary to reserve the factorial notation for integral n and to call the corresponding function for nonintegral n the gamma (Γ) function. It is also rather common practice to replace n by the letter p when we do not necessarily mean an integer. Following these conventions, we define, for *any* $p > 0$

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\,dx, \qquad p > 0. \tag{3.1}$$

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For $0 < p < 1$, this is an improper integral because x^{p-1} becomes infinite at the lower limit. However, it is a convergent integral for $p > 0$ (Problem 1). For $p \le 0$, the integral diverges and so cannot be used to define $\Gamma(p)$; we shall see later how to define $\Gamma(p)$ when $p \le 0$. Then from (3.1) and (2.3) we have

$$(3.2)\qquad \begin{aligned} \Gamma(n) &= \int_0^\infty x^{n-1}e^{-x}\,dx = (n-1)!, \\ \Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,dx = n!. \end{aligned}$$

Thus

$$\Gamma(1) = 0! = 1, \qquad \Gamma(2) = 1! = 1, \qquad \Gamma(3) = 2! = 2, \qquad \Gamma(4) = 3! = 6, \quad \cdots,$$

with the usual meaning of factorial for integral n. The fact that $\Gamma(n) = (n-1)!$ and not $n!$ is unfortunate, but has to be learned in order to read the existing books and tables. Replacing p by $p + 1$ in (3.1), we can write

$$(3.3)\qquad \Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = p!, \qquad p > -1.$$

Some authors use the factorial notation $p! = \Gamma(p+1)$ even though p is not an integer; this avoids the nuisance of the $p + 1$.

Let us integrate (3.3) by parts, calling $x^p = u$, $e^{-x}\,dx = dv$; then we get

$$du = px^{p-1}\,dx, \qquad v = -e^{-x},$$

$$\begin{aligned} \Gamma(p+1) &= -x^p e^{-x}\Big|_0^\infty - \int_0^\infty (-e^{-x})px^{p-1}\,dx \\ &= p\int_0^\infty x^{p-1}e^{-x}\,dx = p\Gamma(p). \end{aligned}$$

This equation

$$(3.4)\qquad \boxed{\Gamma(p+1) = p\Gamma(p)}$$

is called the *recursion relation* for the Γ function. Given Γ of any number $p > 0$, we can use it to find $\Gamma(p+1)$. The Γ function is usually tabulated for p between 1 and 2. → 표를 참고. Using the recursion relation (3.4) we can then find $\Gamma(p)$ for p between 2 and 3; for example, $\Gamma(2.5) = 1.5\Gamma(1.5)$. Similarly, we have $\Gamma(3.5) = 2.5\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5)$, and so on. To find $\Gamma(p)$ for p between 0 and 1 from the tabulated values between 1 and 2, we write the recursion equation as $\Gamma(p) = (1/p)\Gamma(p+1)$. Then, for example, $\Gamma(0.5) = (1/0.5)\Gamma(1.5)$.

↳ $0<p<1$

PROBLEMS, SECTION 3

1. The integral in (3.1) is improper because of the infinite upper limit and it is also improper for $0 < p < 1$ because x^{p-1} becomes infinite at the lower limit. However, the integral is convergent for any $p > 0$. Prove this.

Evaluate the following Γ functions using tables and the recursion relation (3.4).

2. $\Gamma(1.7)$ 3. $\Gamma(2.7)$ 4. $\Gamma(5.7)$ 5. $\Gamma(0.7)$ 6. $\Gamma(3.3)$ 7. $\Gamma(0.3)$

0.90863 $=(5.0)(4.0)(3.0)(2.0)(1.0)\Gamma(1.0)$

Express the following integrals as Γ functions and evaluate them using a table of Γ functions.

8. $\int_0^\infty x^{2/3}e^{-x}\,dx = \Gamma(1+\frac{2}{3}) = \Gamma(\frac{5}{3})$

9. $\int_0^\infty \sqrt{x}\,e^{-x}\,dx$

10. $\int_0^\infty x^{-1/2}e^{-x}\,dx = \Gamma(1-\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$

11. $\int_0^\infty x^2e^{-x^2}\,dx$ *Hint:* Put $x^2 = u$.

12. $\int_0^\infty xe^{-x^3}\,dx$

13. $\int_0^1 x^2\left(\ln\frac{1}{x}\right)^3 dx$ *Hint:* Put $x = e^{-u}$.

14. $\int_0^1 \sqrt[3]{\ln x}\,dx$

15. $\int_0^\infty x^{-1/3}e^{-8x}\,dx$

16. A particle starting from rest at $x = 1$ moves along the x axis toward the origin. Its potential energy is $V = \frac{1}{2}m \ln x$. Write the Lagrange equation and integrate it to find the time required for the particle to reach the origin. *Caution:* $dx/dt < 0$. *Answer:* $\Gamma(\frac{1}{2})$.

17. Express as a Γ function

$$\int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{p-1} dx.$$

See Problem 13.

4. THE GAMMA FUNCTION OF NEGATIVE NUMBERS

For $p \le 0$, $\Gamma(p)$ has not so far been defined. We shall now define it by the recursion relation (3.4) solved for $\Gamma(p)$.

$$\Gamma(p) = \frac{1}{p}\,\Gamma(p+1) \tag{4.1}$$

defines $\Gamma(p)$ for $p < 0$. For example,

$$\Gamma(-0.5) = \frac{1}{-0.5}\,\Gamma(0.5), \qquad \Gamma(-1.5) = \frac{1}{-1.5}\,\frac{1}{-0.5}\,\Gamma(0.5),$$

$= \frac{-1}{-1.5}\,\Gamma(-0.5)$

and so on. Since $\Gamma(1) = 1$, we see that

$$\Gamma(p) = \frac{\Gamma(p+1)}{p} \to \infty \text{ as } p \to 0.$$

From this and successive use of (4.1), it follows that $\Gamma(p)$ becomes infinite not only at zero but also at all the negative integers. A sketch of the graph of the Γ function is left to the problems. For positive p, $\Gamma(p)$ is a continuous function passing through the

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points $p = n$, $\Gamma(p) = (n-1)!$. For negative p, as we have seen, $\Gamma(p)$ is discontinuous at the negative integers. In the intervals between the integers it is alternately positive and negative: negative from 0 to -1, positive from -1 to -2, and so on, as you can see from computations like those for $\Gamma(-0.5)$ and $\Gamma(-1.5)$ above.

PROBLEMS, SECTION 4

Evaluate the following Γ functions using (4.1) and tables.

1. $\Gamma(0.6)$
2. $\Gamma(-0.4)$
3. $\Gamma(-1.4)$
4. $\Gamma(-1.6)$
5. $\Gamma(-2.3)$
6. $\Gamma(-3.7)$
7. Using a table of Γ functions, sketch the Γ function between 1 and 2; then compute a few points and sketch it from -4 to $+4$.

5. SOME IMPORTANT FORMULAS INVOLVING GAMMA FUNCTIONS

First we evaluate $\Gamma(\frac{1}{2})$. By definition

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}}\, e^{-t}\, dt. \tag{5.1}$$

(Note that it does not matter what letter we use for the dummy variable of integration in a definite integral.) Put $t = y^2$ in (5.1); then $dt = 2y\, dy$, and (5.1) becomes

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{y}\, e^{-y^2} 2y\, dy = 2\int_0^\infty e^{-y^2}\, dy$$

or, if we like,

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2}\, dx. \tag{5.2}$$

Let us multiply these two integrals for $\Gamma(\frac{1}{2})$ together and write the result as a double integral:

$$[\Gamma(\tfrac{1}{2})]^2 = 4\int_0^\infty \int_0^\infty e^{-\overset{r^2}{(x^2+y^2)}}\, \overset{r\,dr\,d\theta}{dx\, dy}.$$

This is an integral over the first quadrant; it can be more easily evaluated in polar coordinates:

$$[\Gamma(\tfrac{1}{2})]^2 = 4\int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\theta = 4 \cdot \frac{\pi}{2} \cdot \frac{e^{-r^2}}{-2}\Bigg|_0^\infty = \pi.$$

Therefore

$$\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.} \tag{5.3}$$

We state without proof another important formula involving Γ functions (see Chapter 14, Section 7, Example 5):

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}. \tag{5.4}$$

$= \int_0^\infty \frac{r^{p-1}}{1+r}\,dr$

Notice that (5.4) also gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ if we put $p = \frac{1}{2}$. $=$

PROBLEMS, SECTION 5

1. Prove that, for positive integral n:

$$\Gamma(n+\tfrac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}\sqrt{\pi} = \frac{(2n)!}{4^n n!}\sqrt{\pi}.$$

2. Use (5.4) to show that
   (a) $\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2}+n) = (-1)^n\pi$ if $n =$ a positive integer;
   (b) $(z!)(-z)! = \pi z/\sin \pi z$, where z is not necessarily an integer; see comment after equation (3.3).

3. Prove that

$$\frac{d}{dp}\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}\ln x\,dx,$$

$$\frac{d^n}{dp^n}\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}(\ln x)^n\,dx.$$

6. BETA FUNCTIONS

The *beta function* is also defined by a definite integral:
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$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \qquad p > 0,\ q > 0. \tag{6.1}$$

There are a number of simple transformations of (6.1) which are useful to know [see (6.3), (6.4), (6.5)]. It is easy to show that (Problem 1)

$$B(p, q) = B(q, p). \tag{6.2}$$

The range of integration in (6.1) can be changed by putting $x = y/a$; then $x = 1$ corresponds to $y = a$, and (6.1) becomes

$$B(p, q) = \int_0^a \left(\frac{y}{a}\right)^{p-1}\left(1-\frac{y}{a}\right)^{q-1}\frac{dy}{a} = \frac{1}{a^{p+q-1}}\int_0^a y^{p-1}(a-y)^{q-1}\,dy. \tag{6.3}$$

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To obtain the trigonometric form of the beta function, let $x = \sin^2\theta$; then

$$dx = 2\sin\theta\cos\theta\,d\theta, \qquad (1-x) = 1-\sin^2\theta = \cos^2\theta,$$

$$x = 1 \text{ corresponds to } \theta = \pi/2.$$

With these substitutions, (6.1) becomes

$$B(p, q) = \int_0^{\pi/2} (\sin^2\theta)^{p-1}(\cos^2\theta)^{q-1} 2\sin\theta\cos\theta\,d\theta \qquad \text{or}$$

$$(6.4) \qquad B(p, q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta.$$

Finally, let $x = y/(1+y)$; then we get (Problem 2)

$$(6.5) \qquad B(p, q) = \int_0^{\infty} \frac{y^{p-1}\,dy}{(1+y)^{p+q}}.$$

PROBLEMS, SECTION 6

1. Prove that $B(p, q) = B(q, p)$ *Hint:* Put $x = 1 - y$.
2. Prove equation (6.5).
3. Show that, for integral n, m,

$$B(n, m) = \frac{1}{mC(n+m-1,\, n-1)} = \frac{1}{nC(n+m-1,\, m-1)},$$

where the C's are binomial coefficients [see Chapter 16, equation (4.5)].

[illegible]

7. THE RELATION BETWEEN THE BETA AND GAMMA FUNCTIONS

look for

You will not find tables of B functions as you did tables of Γ functions. The reason is that B functions are easily expressed in terms of Γ functions. We shall show that

$$(7.1) \qquad B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Whenever you want to evaluate a B function, you use (7.1) first and then look up the Γ functions in the tables.

To prove (7.1), we start with

$$\Gamma(p) = \int_0^\infty t^{p-1}e^{-t}\,dt$$

and put $t = y^2$. Then we have

$$\Gamma(p) = 2\int_0^\infty y^{2p-1}e^{-y^2}\,dy. \tag{7.2}$$

Similarly (the dummy integration variable can be any letter),

$$\Gamma(q) = 2\int_0^\infty x^{2q-1}e^{-x^2}\,dx.$$

Next we multiply these two equations together and change to polar coordinates:

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^\infty\int_0^\infty x^{2q-1}y^{2p-1}e^{-(x^2+y^2)}\,dx\,dy\\ &= 4\int_0^\infty\int_0^{\pi/2}(r\cos\theta)^{2q-1}(r\sin\theta)^{2p-1}e^{-r^2}r\,dr\,d\theta \qquad (7.3)\\ &= 4\int_0^\infty r^{2p+2q-1}e^{-r^2}\,dr\int_0^{\pi/2}(\cos\theta)^{2q-1}(\sin\theta)^{2p-1}\,d\theta.\end{aligned}$$

The r integral in (7.3) is $\frac{1}{2}\Gamma(p+q)$ by (7.2). The θ integral in (7.3) is $\frac{1}{2}B(p, q)$ by (6.4). Then $\Gamma(p)\Gamma(q) = 4\cdot\frac{1}{2}\Gamma(p+q)\cdot\frac{1}{2}B(p, q)$ and (7.1) follows.

Example. Find

$$I = \int_0^\infty \frac{x^3\,dx}{(1+x)^5}.$$

This is (6.5) with $(p+q) = 5$, $p - 1 = 3$ or $p = 4$, $q = 1$. Then $I = B(4, 1)$. By (7.1), this is

$$\frac{\Gamma(4)\Gamma(1)}{\Gamma(5)} = \frac{3!}{4!} = \frac{1}{4}.$$

PROBLEMS, SECTION 7

Express the following integrals as B functions, hence in terms of Γ functions, and evaluate using a table of Γ functions.

1. $\displaystyle\int_0^1 \frac{x^4\,dx}{\sqrt{1-x^2}}$
2. $\displaystyle\int_0^{\pi/2}\sqrt{\sin^3 x\cos x}\,dx$
3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^3}}$
4. $\displaystyle\int_0^1 x^2(1-x^2)^{3/2}\,dx$
5. $\displaystyle\int_0^\infty \frac{y^2\,dy}{(1+y)^6}$
6. $\displaystyle\int_0^\infty \frac{y\,dy}{(1+y^3)^2}$
7. $\displaystyle\int_0^{\pi/2}\frac{d\theta}{\sqrt{\sin\theta}}$
8. $\displaystyle\int_0^2 \frac{x^2\,dx}{\sqrt{2-x}}$

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9. Prove $B(n, n) = B(n, \frac{1}{2})/2^{2n-1}$. *Hint:* In (6.4), use the identity $2 \sin \theta \cos \theta = \sin 2\theta$ and put $2\theta = \phi$. Use this result and (5.3) to derive the ***duplication formula*** for Γ functions:

$$\Gamma(2n) = \frac{1}{\sqrt{\pi}} 2^{2n-1}\Gamma(n)\Gamma(n + \tfrac{1}{2}).$$

Check this formula for the case $n = \frac{1}{4}$ by using (5.4).

Sketch the graph of $x^3 + y^3 = 8$. Write the integrals for the following quantities and evaluate them as B functions:

10. The first quadrant area bounded by the curve.
11. The centroid of this area.
12. The volume generated when the area is revolved about the y axis.
13. The moment of inertia of this volume about its axis.

8. THE SIMPLE PENDULUM

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A simple pendulum means a mass m suspended by a string (or weightless rod) of length l so that it can swing in a plane, as shown in Figure 8.1. The kinetic energy of m is then

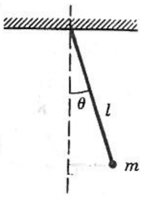

FIGURE 8.1

$$T = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m(l\dot{\theta})^2. \tag{8.1}$$

vertical

If the potential energy is zero when the string is horizontal, then at angle θ it is

$$V = -mgl \cos \theta.$$

Then the Lagrangian is (see Chapter 9, Section 5)

$$L = T - V = \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta,$$

and the Lagrange equation of motion is

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl \sin \theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or

$$\ddot{\theta} = -\frac{g}{l} \sin \theta. \tag{8.2}$$

Suppose the pendulum executes such small vibrations that $\sin \theta$ can be approximated by θ. Then (8.2) becomes the usual equation for the simple harmonic motion of a pendulum executing small vibrations, namely

$$\ddot{\theta} = -\frac{g}{l}\theta. \tag{8.3}$$

: [illegible] ; $\sin\theta \sim \theta$

$$f(x) = f(a) + f'(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^n}{n!} + \cdots$$

Evaluate the following, using either power series, a table of error functions, or asymptotic series, whichever is appropriate.

3. $\int_0^2 e^{-x^2}\,dx$
4. $\int_{0.001}^{0.002} e^{-x^2}\,dx$
5. $1 - \text{erf}\,(5) = \text{erfc}\,(5)$
6. $\frac{2}{\sqrt{\pi}}\int_{1.5}^{\infty} e^{-x^2}\,dx$
7. $\frac{2}{\sqrt{\pi}}\int_{5}^{10} e^{-x^2}\,dx$
8. $1 - \text{erf}\,(100) = \text{erfc}\,(100)$
9. $\text{erf}\,(0.7)$
10. $\int_1^{1.5} e^{-x^2/2}\,dx$
11. $\sqrt{\frac{2}{\pi}}\int_1^{\infty} e^{-x^2/2}\,dx$
12. $\int_{-\infty}^{0.003} e^{-x^2}\,dx$
13. By repeated integration by parts, find several terms of the asymptotic series for

$$\int_x^{\infty} t^{n-1}e^{-t}\,dt.$$

[Note that if $x = 0$, this integral is $\Gamma(n)$; for $x \neq 0$, it is called an *incomplete* Γ function, $\Gamma(n, x)$.]

14. Express the error function as an incomplete Γ function (Problem 13) and show that the asymptotic expansion (10.4) agrees with your result in Problem 13.
15. The integral $\int_x^{\infty} \frac{e^{-t}}{t}\,dt$ is called an *exponential integral.*
    (a) Find the asymptotic series for the exponential integral.
    (b) Express the exponential integral as an incomplete Γ function (Problem 13).
    (c) Express $\int_0^x \frac{dt}{\ln\,(1/t)}$ in terms of the exponential integral.

11. STIRLING'S FORMULA

Formulas involving $n!$ or $\Gamma(p)$ are not very convenient to simplify algebraically or to differentiate. There is an approximate formula for the factorial or Γ function known as Stirling's formula which can be used to simplify formulas involving factorials. It is

$$(11.1)\quad n! \sim n^n e^{-n}\sqrt{2\pi n} \quad\text{or}\quad \Gamma(p+1) \sim p^p e^{-p}\sqrt{2\pi p}. \qquad \text{Stirling's formula}$$

The sign $\sim$ (read "is asymptotic to") means that the ratio of the two sides

$$\frac{n!}{n^n e^{-n}\sqrt{2\pi n}}$$

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tends to 1 as $n \to \infty$. Thus we get better approximations to $n!$ as n becomes large. Actually the absolute error (difference between the Stirling approximation and the correct value) *increases*, but the relative error (ratio of the error to the value of $n!$) tends to zero as n increases. To get some idea of how this formula arises, we outline what could, with a little more detail, be a derivation of it. (For more detail, see, for example, Buck, p. 300.) Start with

$$\Gamma(p+1) = p! = \int_0^\infty x^p e^{-x}\,dx = \int_0^\infty e^{p\ln x - x}\,dx. \tag{11.2}$$

Substitute a new variable y such that

$$x = p + y\sqrt{p}.$$

Then

$$dx = \sqrt{p}\,dy,$$
$$x = 0 \text{ corresponds to } y = -\sqrt{p},$$

and (11.2) becomes

$$p! = \int_{-\sqrt{p}}^\infty e^{p\ln(p+y\sqrt{p}) - p - y\sqrt{p}}\sqrt{p}\,dy. \tag{11.3}$$

For large p, the logarithm can be expanded in the following power series:

$$\ln(p + y\sqrt{p}) = \ln p + \ln\left(1 + \frac{y}{\sqrt{p}}\right) = \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \cdots. \tag{11.4}$$

Substituting (11.4) into (11.3), we get

$$\begin{aligned} p! &\sim \int_{-\sqrt{p}}^\infty e^{p\ln p + y\sqrt{p} - (y^2/2) - p - y\sqrt{p}}\sqrt{p}\,dy \\ &= e^{p\ln p - p}\sqrt{p}\int_{-\sqrt{p}}^\infty e^{-y^2/2}\,dy \\ &= p^p e^{-p}\sqrt{p}\left[\int_{-\infty}^\infty e^{-y^2/2}\,dy - \int_{-\infty}^{-\sqrt{p}} e^{-y^2/2}\,dy\right]. \end{aligned}$$

The first integral is easily shown to be $\sqrt{2\pi}$ (Problem 9.4). The second integral tends to zero as $p \to \infty$, and we have

$$p! \sim p^p e^{-p}\sqrt{2\pi p}$$

which is (11.1). With more work, it is possible to find an asymptotic expansion for $\Gamma(p+1)$:

$$\Gamma(p+1) = p! = p^p e^{-p}\sqrt{2\pi p}\left(1 + \frac{1}{12p} + \frac{1}{288p^2} + \cdots\right). \tag{11.5}$$

This is another example of an asymptotic series which is divergent as an infinite series; however, the first term alone (Stirling's formula) is a good approximation for large p, and the second term can be used to estimate the relative error (Problem 1).

PROBLEMS, SECTION 11

1. Use the term $1/12p$ in (11.5) to show that the error in Stirling's formula (11.1) is $< 10\%$ for $p > 1$; $< 1\%$ for $p > 10$; $< 0.1\%$ for $p > 100$.

2. Compare the exact values of $n!$ and the Stirling's formula approximation for $n = 2, 5, 10, 50, 100$. Find the percentage error.

$$(50! = 3.04141 \times 10^{64}, \qquad 100! = 9.33262 \times 10^{157}.)$$

3. In statistical mechanics, we frequently use $\ln N! = N \ln N - N$, where N is of the order of Avogadro's number. Write out $\ln N!$ using Stirling's formula, compute the approximate value of each term for $N = 10^{26}$, and so justify the above approximation.

4. Use Stirling's formula to evaluate $\displaystyle\lim_{n\to\infty} \frac{(2n)!\sqrt{n}}{2^{2n}(n!)^2}$.

5. Use Stirling's formula to evaluate $\displaystyle\lim_{n\to\infty} \frac{\Gamma(n+\frac{3}{2})}{\sqrt{n}\,\Gamma(n+1)}$.

6. Use equations (3.4) and (11.1) to show that $\Gamma(p) \sim p^p e^{-p}\sqrt{2\pi/p}$.

7. Use Problem 6 to obtain $\displaystyle\frac{d}{dp} \ln \Gamma(p) \sim \ln p - \frac{1}{2p}$.

8. Sketch a graph of $y = \ln x$. Show that $\ln n!$ is between the values of the integrals $\int_2^{n+1} \ln x\, dx$ and $\int_1^n \ln x\, dx$. (*Hint:* $\ln n! = \ln 1 + \ln 2 + \ln 3 + \cdots$ is the sum of the areas of rectangles of width 1 and height up to the $\ln x$ curve at $x = 1, 2, 3, \cdots$.) By considering the values of the two integrals for very large n as in Problem 3, show that $\ln n! = n \ln n - n$ approximately for large n.

9. The following expression occurs in statistical mechanics:

$$P = \frac{n!}{(np+u)!\,(nq-u)!}\, p^{np+u} q^{nq-u}.$$

Use Stirling's formula to show that

$$\frac{1}{P} \sim x^{npx} y^{nqy} \sqrt{2\pi npqxy},$$

where $x = 1 + \dfrac{u}{np}$, $y = 1 - \dfrac{u}{nq}$, and $p + q = 1$. *Hint:* Show that

$$(np)^{np+u}(nq)^{nq-u} = n^n p^{np+u} q^{nq-u}$$

and divide numerator and denominator of P by this expression.

12. ELLIPTIC INTEGRALS AND FUNCTIONS

These are another group of integrals and related functions which have been extensively studied and tabulated. Since they may arise in applied problems, it is worth while to know enough about them to be able to use the readily available tables, to recognize when you probably have an elliptic integral, and to be able to look up and use known