A Simple Model of Superconductors: Insights from Free Fermion and Boson Gases

Mi-Ra Hwang[1], Eylee Jung[1], MuSeong Kim[2] and DaeKil Park[1,3*]

[1]*Department of Electronic Engineering,*

Kyungnam University, Changwon, 631-701, Korea

[2]*Pharos iBio Co., Ltd. Head Office: #1408, 38,*

Heungan-daero 427beon-gil, Dongan-gu, Anyang, 14059, Korea

[3]*Department of Physics, Kyungnam University, Changwon, 631-701, Korea*

Abstract

Superconductors at temperatures below the critical temperature T_c can be modeled as a mixture of Fermi and Bose gases, where the Fermi gas consists of conduction electrons and the Bose gas comprises Cooper pairs. This simple model enables the computation of the temperature dependence of $2r(T)/N$, where N is the total number of conduction electrons and $r(T)$ is the number of Cooper pairs at temperature T. Analyzing $2r(T)/N$ across various superconductors may provide significant insights into the mechanisms behind high-temperature superconductivity, especially regarding coherence in Cooper pairs.

---

[*] corresponding author, dkpark@kyungnam.ac.kr

A Fermi gas is an idealized model for an ensemble of many non-interacting fermions. This model is useful to explore the electronic and thermal properties of the certain systems with many fermions such as the behavior of charge carries in a metal [1], nucleons in an atomic nucleus, neutrons in a neutron star, and electrons in a white dwarf. Since spin-1/2 fermions obey the Fermi-Dirac statistics, the Fermi energy, the highest energy when the external temperature T is zero, can be easily computed as follows:

$$E_F = \begin{cases} \frac{\pi \hbar^2}{m} \frac{N}{S} & \text{for} \quad d = 2 \\ \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3} & \text{for} \quad d = 3, \end{cases} \tag{1}$$

where d is a dimension of the system and m is a mass of the individual spin-1/2 particle. In Eq. (1) N, S, and V are the total number of the fermions, surface area, and volume of the system respectively. Thus, if we know the concentration $n = N/S$ or N/V, we can compute E_F by applying Eq. (1). Typical values of E_F are $2 \sim 10$ eV for metals, 0.3 MeV for white dwarfs, and 30 MeV for nucleus.

If we do not know the total number of the spin-1/2 particles, it is possible to estimate N by measuring E_F experimentally as follows:

$$N = \begin{cases} \frac{mS}{\pi \hbar^2} E_F & \text{for} \quad d = 2 \\ \frac{V}{3\pi^2 \hbar^3} \left(2mE_F \right)^{3/2} & \text{for} \quad d = 3. \end{cases} \tag{2}$$

For the case of metal one can measure E_F by various methods such as the anaylsis of thermionic emission and diffusion[2], angle-resolved photoemission spectroscopy (ARPES) [3], and ultraviolet photoelectron spectroscopy (UPS) [4]. Therefore, using these experimental techniques and Eq. (2) one can estimate N.

In Ref. [5] it was suggested that in the superconductor two electrons near the Fermi surface form a bound state by the attractive force mediated by the phonon. This pair is called "Cooper pair". The formation of the Cooper pair implies the energy gap $\Delta(T)$ in the energy spectrum of superconductor. The energy gap is roughly dissolution energy, which is necessary to break the pair into the two free conduction electrons. The gap $\Delta(T)$ exhibits a decreasing behavior with increasing the external temperature and eventually, goes to zero at the critical temperature T_c. The microscopic theory of superconductivity was proposed in Ref. [6, 7], which is so called the BCS theory. The BCS theory explains many phenomena in the superconductor. One of them is that when the spin state of each Cooper pair is singlet,

the ratio between $\Delta_0 = \Delta(0)$ and T_c is given as a form:

$$\frac{\Delta_0}{k_B T_c} = \frac{2\pi}{e^\gamma} \approx 3.53 \tag{3}$$

where $\gamma = 0.5772\ldots$ is a Euler constant.

Table I: T_c and Δ_0 for various superconductors

	Material	T_c, K	$\Delta_0/k_B T_c$
Low-T_c	Hf	0.13	3.9
	Cd	0.52	3.2
	Zn	0.85	3.2
	Al	1.2	3.4
	In	3.4	3.6
	Hg	4.2	4.6
	Pb	7.2	4.3
	Nb	9.3	3.8
	$Ba_{0.6}K_{0.4}BiO_3$	18.5	3.7
	K_3C_{60}	19	3.6
High-T_c	$YBa_2Cu_3O_{7-\delta}$	87	4.0
	$Bi_2Sr_2Ca_2Cu_3O_{10}$	108	5.7
	$Tl_2Ba_2CaCu_2O_8$	112	4.5
	$Tl_2Ba_2Ca_2Cu_3O_{10}$	105	3.1
	$HgBa_2Ca_2Cu_3O_8$	131	4.3

The critical temperature T_c and $\Delta_0/k_B T_c$ is summarized for various superconductors in Table I [8]. It is shown that Eq. (3) roughly coincides with the experimental data for low-T_c superconductors. However, $\Delta_0/k_B T_c$ for high-T_c superconductors seems to be larger than 3.53. Therefore, we may need a new theory of superconductivity to explain the high-T_c superconductivity. Table I shows that Δ_0 for the high-T_c superconductor is larger than that for the low-T_c superconductor. This fact indicates that the attractive force between the two electrons in the Cooper pair becomes stronger. Thus, the phonon-mediated scenario for the Cooper pair should be modified. Many proposals were suggested in the context of the condensed matter theory[9]. Another scenario was suggested in the context of gravity theory by making use of AdS_{d+1}/CFT_d correspondence [10]. Few years ago AdS/CFT was used

to explore quark confinement[11], quark-monopole interaction[12], and Gross-Ooguri phase transition[13]. More recently, it was also applied to the Hall effect[14], Nernst effects[15–17], and strange metal[18].

Application of the AdS/CFT to the superconductivity was initiated in Ref.[19] as a name of holographic superconductivity. The ratio $\Delta_0/k_B T_c$ in the holographic scenario was investigated in Ref.[19–21]. Using simple black hole solution in AdS space, the ratio is given by

$$\frac{\Delta_0}{k_B T_c} \approx 8.2. \tag{4}$$

Eq. (4) seems to be too large than that of the high-T_c superconductors given in Table I. This fact may indicate that the holographic superconductivity is different from the conventional superconductivity arising in the condensed matter theory. Regardless of the fact I guess that holographic superconductivity may give useful insights in the future for developing a theory of the high-T_c superconductor.

In this short note, we adopt a simplified picture where conduction electrons and Cooper pairs coexist in superconductors, viewed as ideal Fermi and Bose gases, respectively. Below T_c, some conduction electrons are converted into the Cooper pairs through an attractive interaction, typically mediated by phonons in conventional superconductors. These pairs act as bosons, allowing them to condense into a coherent state, a hallmark of the superconducting phase.

This simplified model captures essential aspects of superconductivity: the transformation of fermions into bosonic pairs and their condensation into a coherent state. Although the full microscopic treatment includes complex interactions, this approach offers an accessible framework for understanding the onset of superconductivity, especially when analyzing the temperature dependence of the Cooper pair population.

As T decreases from T_c, number of Fermi (or boson) particles decreases (or increases). In this way the roles of the conduction electrons and Cooper pairs can be distinguished as follow. With decreasing T, the Fermi energy E_F of the Fermi gas become lowed, and hence the number of Cooper pairs increases, all of which condensate in the ground state of the Boson gas. In summary, the Fermi gas supplies the Cooper pairs, and all Cooper pairs play the role of condensation. We think this is the simplest picture of the superconductivity.

The advantage of this simple picture is that it allows us to compute the ratio between number of Cooper pairs and total conduction electron as follows. As far as we know, the

temperature dependence of the ratio has not been experimentally measured across various superconductors.

Let us consider a material $\mathcal{A}$, which is a superconductor at $T \leq T_c$. Let us assume that the total number of the conduction electrons is N. This can be easily estimated by measuring the mole of $\mathcal{A}$ and number of conduction electron(s) per atom. For fixed T with $T \leq T_c$ let us assume that the number of Cooper pairs is r. Then, it is evident that the number of the conduction electrons is $N - 2r$. Applying (2) it is easy to derive

$$\frac{2r}{N} = \begin{cases} 1 - \frac{mS}{\pi \hbar^2 N} E_F(T) & \text{for} \quad d = 2 \\ 1 - \frac{V}{3\pi^2 \hbar^3 N} \left(2m E_F(T)\right)^{3/2} & \text{for} \quad d = 3. \end{cases} \tag{5}$$

Since $E_F(T)$ is a Fermi energy at the temperature T, it can be measured experimentally. In this way one can measure $2r/N$ at the temperature T. If we repeat the experiment with changing the external temperature T, we can estimate the T-dependence of $2r/N$ for the material $\mathcal{A}$. If we repeat the same experiment with changing the material $\mathcal{A}$, $\mathcal{B}$, $\cdots$, we can estimate the T-dependence of $2r/N$ for many materials

Now, let us discuss how $2r/N$ exhibits with respect to T. We can conjecture easily that $2r/N$ decreases with increasing T, because coherence among Cooper pairs diminishes near T_c. However, we guess that $2r/N$ does not go to zero at $T = T_c$, because the superconducting properties need the collective motion of the Cooper pairs. I guess the temperature dependence of $2r/N$ would be either Fig. 1(a) or Fig. 1(b). In Fig. 1(a) $2r/N$ decreases from $z_0(< 1)$ to $z_1(> 0)$ as T increases from zero to T_c and there exists $T_*(> T_c)$, where $2r/N$ completely vanishes. In Fig. 1(b) the T-dependence of $2r/N$ exhibits a similar behavior with Fig. 1(a), but there is no T_*. This means that $2r/N$ approaches to zero as T approaches to ∞. It is worthwhile noting that the results of Fig. 1 are consistent with Ref. [22], where it was shown experimentally that the incoherent Cooper pairs exist even when $T > T_c$.

I guess the pattern of this figure is different in the low- and high-temperature superconductors. I think this difference may give important insights on the theory of the high-T_c superconductor.

At this stage many questions arise as follows. (1) In the Fig. 1(a)-type materials is $g = T_c/T_*$ universal or material-specific? (2)If g is material-specific, what properties of the superconductor determine g? (3) If some superconductors are Fig. 1(a)-type and others are
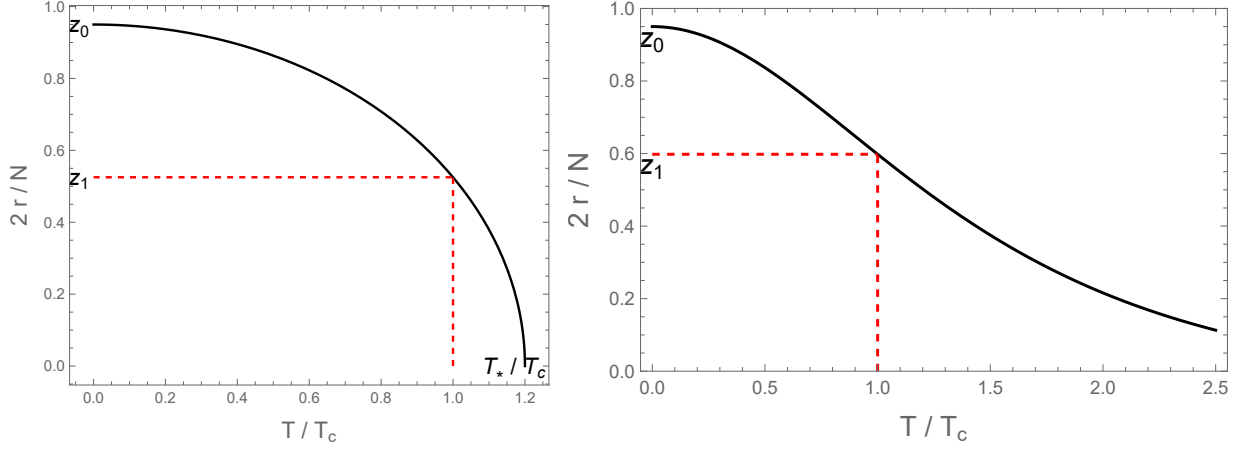
FIG. 1: (Color online) The possible temperature dependence of $2r/N$. In Fig. 1(a) $2r/N$ decreases from $z_0(< 1)$ to $z_1(> 0)$ as T increases from zero to T_c and there exists $T_*(> T_c)$, where $2r/N$ completely vanishes. In Fig. 1(b) the T dependence of $2r/N$ approaches to zero as T approaches to ∞.

Fig. 1(b)-type, what properties of the superconductors make this difference? Understanding these aspects could lead to deeper insights into the pairing mechanisms in superconductors.

---

[1] N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, (Holt Rinehart and Winston, 1976, New York).

[2] A. Tsukernik, M. Slutzky, A. Palevski, S. Luryi, and H. Shtrikman, *A method for Fermi energy measurements*, Appl. Phys. Lett. **73** (1998) 79.

[3] H. Zhang, T. Pincelli, C. Jozwiak, T. Kondo, R. Ernstorfer, T. Sato, and S. Zhou, *Angle-resolved photoemission spectroscopy*, Nat. Rev. Methods Primers, **2** (2022) 54 [arXiv:2207.06942 (cond-mat)].

[4] S. Hüfner, *Photoelectron Spectroscopy: Principles and Applications*, (Springer, Berlin, 2003).

[5] L. N. Cooper, *Bound Electron Pairs in a Degenerate Fermi Gas*, Phys. Rev. **104** (1956) 1189.

[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Microscopic Theory of Superconductivity*, Phys. Rev. **106** (1957) 162.

[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108** (1957) 1175.

[8] Table 6. 1 in C. P. Jr. Poole, H. A. Farach, and R. J. Creswick, Superconductivity (Academic Press, San Diego, 1995).

[9] E. W. Carlson, V. J. Emery, S. A. Kivelson, and D. Orgad, *Concepts in High Temperature Superconductivity*, cond-mat/0206217 and references therein.

[10] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[11] S. -J. Rey, S. Theisen, and J. -T. Yee, *Wilson-Polyakov Loop at Finite Temperature in Large N Gauge Theory and Anti-de Sitter Supergravity*, Nucl. Phys. **B 527** (1998) 171 [hep-th/9803135].

[12] D. K. Park, *Hidden functional relation in Large-N Quark-Monopole system at finite temperature*, Nucl. Phys. **B 618** (2001) 157 [hep-th/0105039].

[13] H. Kim, D. K. Park, S. Tamaryan, H. J. W. Müller-Kirsten, Gross-Ooguri Phase Transition at Zero and Finite Temperature: Two Circular Wilson Loop Case, JHEP **0103** (2001) 003 [hep-th/0101235].

[14] S. A. Hartnoll and P. Kovtun, "Hall conductivity from dyonic black holes," Phys. Rev. D **76**, 066001 (2007) [arXiv:0704.1160 [hep-th]].

[15] S. A. Hartnoll, P. K. Kovtun, M. Muller and S. Sachdev, "Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes," Phys. Rev. B **76**, 144502 (2007) [arXiv:0706.3215 [cond-mat.str-el]].

[16] S. A. Hartnoll and C. P. Herzog, "Ohm's Law at strong coupling: S duality and the cyclotron resonance," Phys. Rev. D **76**, 106012 (2007) [arXiv:0706.3228 [hep-th]].

[17] S. A. Hartnoll and C. P. Herzog, "Impure AdS/CFT," arXiv:0801.1693 [hep-th].

[18] S. Sachdev, Strange metals and black holes: insights from the Sachdev-Ye-Kitaev model, Oxford Research Encyclopedia of Physics, December 2023 [arXiv:2305.01001 (cond-mat)].

[19] S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, *Building a Holographic Superconductor*, Phys. Rev. Lett. **101** (2008) 031601 [arXiv:0803.3295 (hep-th)].

[20] G. T. Horowitz and M. M. Roberts, *Holographic superconductors with various condensates*, Phys. Rev. **D 78** (2008) 126008 [arXiv:0810.1077 (hep-th)].

[21] R. Gregory, S. Kanno, and J. Soda, *Holographic Superconductors with Higher Curvature Corrections*, JHEP **0910** (2009) 010 [arXiv:0907.3203 (hep-th)].

[22] K. J. Xu, J. He, S. D. Chen, Y. He, S. N. Abadi, C. R. Rotundu, Y. S. Lee, D. H. Lu, Q. Guo,

O. Tjernberg, T. P. Devereaux, D. H. Lee, M. Hashimoto, and Z. X. Shenl, *Anomalous normal state gap in an electron-doped cuprate*, Science **385** (2024) 796 [arXiv:2309.10208 (cond-mat)].