CH3. Laplace Transform

$f(t)$ 의 Laplace transform

$$\boxed{F(s) \equiv \mathcal{L}[f](s) \equiv \int_0^\infty e^{-st} f(t)\, dt}$$

p86

(예제3.1)

$$f(t) = e^{at}$$

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \begin{cases} \frac{1}{s-a} & s>a \\ \infty & s<a \end{cases}$$

$$\Rightarrow \quad F(s) = \frac{1}{s-a} \qquad (s>a)$$

p87

(예제3.2)

$$g(t) = \sin t$$

$$\int e^{-st} \sin t\, dt = -\frac{1}{1+s^2} e^{-st} (\cos t + s \sin t)$$

$$\mathcal{L}[g](s) \equiv \int_0^\infty dt\, e^{-st} \sin t$$

$$= -\frac{1}{1+s^2} \left[e^{-st} (\cos t + s \sin t) \right]_{t=0}^{t=\infty}$$

$$= -\frac{1}{1+s^2} [0 - 1]$$

$$= \frac{1}{1+s^2} \qquad (s>0)$$

p89

(Ex) 사각파형

$$f(t) = \begin{cases} 1 & 0 \le t \le a \quad 2a \le t \le 3a \quad 4a \le t \le 5a \quad \cdots \\ -1 & 0 \le t \le 2a \quad 3a \le t \le 4a \quad 5a \le t \le 6a \quad \cdots \end{cases}$$

$$F(s) = \int_0^\infty dt \; e^{-st} f(t)$$

$$= \left[\int_0^a e^{-st} dt + \int_{2a}^{3a} e^{-st} dt + \int_{4a}^{5a} e^{-st} dt + \cdots \right]$$

$$\int_\alpha^\beta e^{-st} dt = \frac{1}{s} \left(e^{-\alpha s} - e^{-\beta s} \right)$$

$$- \left[\int_a^{2a} e^{-st} dt + \int_{3a}^{4a} e^{-st} dt + \int_{5a}^{6a} e^{-st} dt + \cdots \right]$$

$$= \frac{1}{s} \left[(1 - e^{-as}) + (e^{-2as} - e^{-3as}) + (e^{-4as} - e^{-5as}) + \cdots \right]$$

$$- \frac{1}{s} \left[(e^{-as} - e^{-2as}) + (e^{-3as} - e^{-4as}) + (e^{-5as} - e^{-6as}) + \cdots \right]$$

$$= \frac{1}{s} \left[1 - 2 \underbrace{\left\{ e^{-as} - e^{-2as} + e^{-3as} - e^{-4as} + \cdots \right\}}_{\frac{e^{-as}}{1 + e^{-as}}} \right]$$

$$= \frac{1}{s} \left[1 - \frac{2e^{-as}}{1 + e^{-as}} \right]$$

$$= \frac{1}{s} \frac{1 - e^{-as}}{1 + e^{-as}}$$

$$= \frac{1}{s} \frac{e^{\frac{a}{2}s} - e^{-\frac{a}{2}s}}{e^{\frac{a}{2}s} + e^{-\frac{a}{2}s}}$$

$$= \frac{1}{s} \tanh\left(\frac{a}{2} s \right)$$

//

p90

(정리 3.1)

$$\boxed{\mathcal{L}[af+bg](s) = aF(s) + bG(s)}$$

Ex) $f(t) = \alpha e^{at} + \beta e^{bt}$ what is $F(s)$?

$$F(s) = \alpha \underbrace{\mathcal{L}[e^{at}]}_{\frac{1}{s-a}} + \beta \underbrace{\mathcal{L}[e^{bt}]}_{\frac{1}{s-b}}$$ (p87 formula 5)

$$= \frac{\alpha}{s-a} + \frac{\beta}{s-b}$$

$$= \frac{(\alpha+\beta)s - (a\beta + b\alpha)}{(s-a)(s-b)}$$

Ex) $F(s) = \dfrac{s-3}{(s-1)(s-2)}$ what is $f(t)$?

$$F(s) = \frac{2}{s-1} - \frac{1}{s-2}$$

$$f(t) = 2 \overbrace{\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]}^{e^{t}} - \overbrace{\mathcal{L}^{-1}\left[\frac{1}{s-2}\right]}^{e^{2t}}$$

$$= 2e^{t} - e^{2t}$$

※

∘ Inverse Laplace Transform

If $F(s) = \mathcal{L}[f](s)$,

$$f(t) = \mathcal{L}^{-1}[F](t)$$

p93 정리 3.3

$$\mathcal{L}^{-1}[aF + bG] = a\,\mathcal{L}^{-1}[F] + b\,\mathcal{L}^{-1}[G]$$

Ex)

$$F(s) = \frac{3}{s-a} + \frac{5}{(s-b)^2}$$

$$f(t) = 3\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s-a}\right]}_{e^{at}} + 5\underbrace{\mathcal{L}^{-1}\left[\frac{1}{(s-b)^2}\right]}_{te^{bt}}$$

(p87 formula 5, 6)

$$= 3e^{at} + 5te^{bt}$$

#

§ Laplace 변환을 이용한 초기값 문제의 해

정리 3.4

$$\boxed{\mathcal{L}[f'](s) = sF(s) - f(0)}$$

Pf)

$\mathcal{L}[f'](s)$

$$\equiv \int_0^\infty dt\, \underbrace{e^{-st}}_{u}\, \underbrace{f'(t)}_{v'}$$

$u = e^{-st}$ $\quad v = f$

$u' = -se^{-st}$ $\quad v' = f'$

$$= e^{-st} f(t) \Big|_{t=0}^{t=\infty} - \int_0^\infty dt\, (-se^{-st})\, f(t)$$

$$= -f(0) + s \underbrace{\int_0^\infty dt\, e^{-st} f(t)}_{= F(s)}$$

$$= sF(s) - f(0) \quad //$$

(Ex) $\mathcal{L}[\cos t]$

Let

$f'(t) = \cos t$, $\quad f(t) = \sin t$

$$\mathcal{L}[\cos t] = s \underbrace{\mathcal{L}[\sin t]}_{\frac{1}{s^2+1}} - \overbrace{\sin 0}^{=0}$$

($p87$ 예제3.2)

$$= \frac{s}{s^2+1} \quad //$$

정리 3.5

$$\mathcal{L}[f^{(n)}] = s^n F(s) - s^{n-1} f(0) - s^{(n-2)} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$\mathcal{L}[f''] = s^2 F(s) - s f(0) - f'(0)$$

(pf) $\mathcal{L}[f'']$

$$f'' = (f')'$$

From 정리 3.4

$$\mathcal{L}[f''] = s \underbrace{\mathcal{L}[f']}_{sF(s) - f(0)} - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0) \quad ※$$

p95

예제 3.3

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{t} \qquad y(0) = 0, \quad y'(0) = 2$$

L (1)

(A) 앞 chapter 의 방법

$$y = y_g + y_p$$

(i) y_g

$$y'' + 4y' + 3y = 0$$

$$y = c_1 e^{-t} + c_2 e^{-3t}$$

$$y'' + Ay' + By = 0$$

(i) $A^2 > 4B$

$$y = c_1 e^{at} + c_2 e^{bt}$$

$$a = \frac{-A + \sqrt{A^2 - 4B}}{2} \qquad b = \frac{-A - \sqrt{A^2 - 4B}}{2}$$

(ii) y_p

$$y_p = \alpha e^{t}$$

$$y_p'' + 4y_p' + 3y_p = 8\alpha e^{t} = e^{t}$$

$$\alpha = \frac{1}{8}$$

$$\Rightarrow \quad y_p = \frac{1}{8} e^{t}$$

$$y = c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{8} e^{t}$$

$$y' = -c_1 e^{-t} - 3c_2 e^{-3t} + \frac{1}{8} e^{t}$$

$$\left.\begin{array}{l} y(0) = c_1 + c_2 + \frac{1}{8} = 0 \\ y'(0) = -c_1 - 3c_2 + \frac{1}{8} = 2 \end{array}\right\} \Rightarrow \quad c_1 = \frac{3}{4}, \quad c_2 = -\frac{7}{8}$$

$$y = \frac{3}{4} e^{-t} - \frac{7}{8} e^{-3t} + \frac{1}{8} e^{t}$$

(B) Using Laplace transform

Let

$$Y(s) = \mathcal{L}[y] \qquad (2)$$

Taking Laplace transform to Eq. (1)

$$\mathcal{L}[y''] + 4\mathcal{L}[y'] + 3\mathcal{L}[y] = \mathcal{L}[e^{t}]$$

$$\left[s^2Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=2}\right] + 4\left[sY(s) - \underbrace{y(0)}_{=0}\right] + 3Y(s) = \frac{1}{s-1}$$

$$(s^2+4s+3)Y(s) - 2 = \frac{1}{s-1}$$

$$\Rightarrow (s^2+4s+3)Y(s) = \frac{1}{s-1} + 2 = \frac{2s-1}{s-1}$$

$$\Rightarrow \quad Y(s) = \frac{2s-1}{(s-1)(s+1)(s+3)}$$

$$= \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3} \qquad (3)$$

Taking inverse Laplace transform to Eq. (3)

$$y(t) = \frac{1}{8}e^{t} + \frac{3}{4}e^{-t} - \frac{7}{8}e^{-3t}$$

*

5 이동정리와 Heaviside 함수

[1] 제 1 이동정리

p96

정리 3.6

$$\boxed{\mathcal{L}[e^{at} f(t)] = F(s-a)}$$

제1 이동정리

Pf)

$$\mathcal{L}[e^{at} f(t)]$$

$$= \int_0^\infty dt\ e^{-st}\ e^{at} f(t)$$

$$= \int_0^\infty dt\ e^{-(s-a)t}\ f(t)$$

$$\int_0^\infty dt\ e^{-st} f(t) = F(s)$$

$$= F(s-a)$$ ※

p97

(예제 3.4)

$$\mathcal{L}(t^3) = \frac{3!}{s^4} = \frac{6}{s^4}$$

(p87 formula 2)

$$\mathcal{L}(e^{7t} t^3) = \frac{6}{(s-7)^4}$$

note)

$$\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t) = e^{at} \mathcal{L}^{-1}(F(s))$$

p97

(예제 3.5)

$$F(s) = \frac{3s-1}{s^2-6s+2} \qquad \mathcal{L}^{-1}[F] = ?$$

$$F(s) = \frac{3s-1}{s^2-6s+9-7}$$

$$= \frac{3(s-3+3)-1}{(s-3)^2-7}$$

$$= \frac{3(s-3)}{(s-3)^2-7} + \frac{8}{(s-3)^2-7}$$

$$= 3\,F_1(s) + 8\,F_2(s) \qquad -①$$

where

$$\left. \begin{aligned} F_1(s) &= \frac{s-3}{(s-3)^2-7} \\ F_2(s) &= \frac{1}{(s-3)^2-7} \end{aligned} \right\} \qquad -②$$

$$\mathcal{L}^{-1}[F(s)] = 3\,\mathcal{L}^{-1}[F_1(s)] + 8\,\mathcal{L}^{-1}[F_2(s)] \qquad -③$$

$$\mathcal{L}^{-1}[F_1(s)] = e^{3t}\,\overset{\cosh\sqrt{7}t}{\underline{\mathcal{L}^{-1}\left[\frac{s}{s^2-7}\right]}} = e^{3t}\cosh\sqrt{7}t \qquad -④ \quad \text{(p88 formula 23)}$$

$$\mathcal{L}^{-1}[F_2(s)] = e^{3t}\,\mathcal{L}^{-1}\left[\frac{1}{s^2-7}\right] = e^{3t}\,\frac{1}{\sqrt{7}}\,\underset{\sinh\sqrt{7}t \ \text{(p88 formula 22)}}{\underline{\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2-7}\right)}} = \frac{1}{\sqrt{7}}\,e^{3t}\sinh\sqrt{7}t \qquad -⑤$$

④, ⑤ → ③

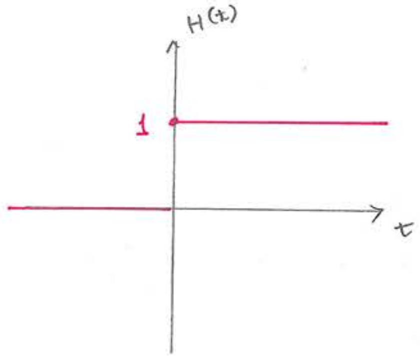
$$\mathcal{L}^{-1}[F(s)] = 3e^{3t}\cosh\sqrt{7}t + \frac{8}{\sqrt{7}}e^{3t}\sinh\sqrt{7}t$$

※

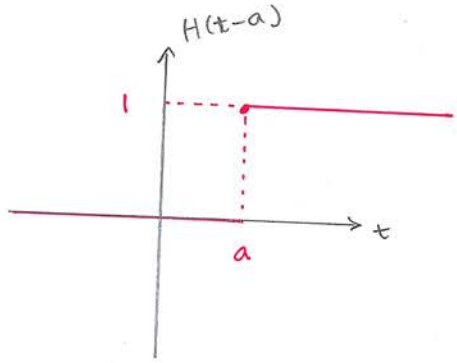
[2] Heaviside 함수 와 벽동

Heaviside 함수 (or stop 함수)

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



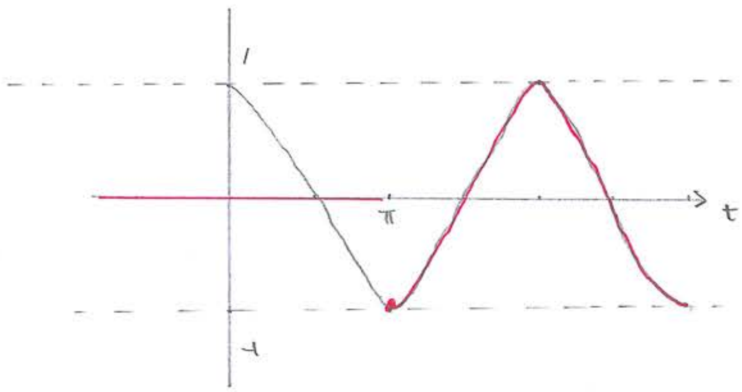
$$H(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$



$$H(t-a)\, g(t) = \begin{cases} g(t) & t \geq a \\ 0 & t < a \end{cases}$$

(Ex)

$$H(t-\pi) \cos t = \begin{cases} \cos t & t \geq \pi \\ 0 & t < \pi \end{cases}$$



맥동

$$H(t-a) - H(t-b) \qquad (a<b)$$

$$= \begin{cases} 1 & t \geq a \\ 0 & t \leq a \end{cases} - \begin{cases} 1 & t > b \\ 0 & t < b \end{cases}$$

$$= \begin{cases} 0 & t \leq a \\ 1 & a \leq t \leq b \\ 0 & t \geq b \end{cases}$$

$$* \quad [H(t-a) - H(t-b)]\, g(t) = \begin{cases} 0 & t \leq a & \text{switch off} \\ g(t) & a \leq t \leq b & \text{switch on} \\ 0 & b \leq t & \text{switch off} \end{cases}$$

[3] 제 2 이동정리

p101

정리 2.7

$$\mathcal{L}[H(t-a)f(t-a)] = e^{-as}F(s)$$

pf)

$\mathcal{L}[H(t-a)f(t-a)]$

$$\equiv \int_0^\infty dt\, e^{-st} H(t-a) f(t-a)$$

$$= \int_0^a dt\, e^{-st} \underbrace{H(t-a)}_{=0} f(t-a) + \int_a^\infty dt\, e^{-st} \underbrace{H(t-a)}_{=1} f(t-a)$$

$$= \int_a^\infty dt\, e^{-st} f(t-a)$$

$u = t-a$

$$= \int_0^\infty du\, e^{-s(u+a)} f(u)$$

$$= e^{-sa} \int_0^\infty du\, e^{-su} f(u)$$

$u \to t$

$$= e^{-sa} \underbrace{\int_0^\infty dt\, e^{-st} f(t)}_{F(s)}$$

$$= e^{-as} F(s)$$

※

note)

$$\mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)$$

p102

(예제 3.7)

$$g(t) = \begin{cases} 0 & t < 2 \\ t^2+1 & t \geq 2 \end{cases}$$

$$= H(t-2)\,(t^2+1)$$

$$= H(t-2)\left[\{(t-2)+2\}^2+1\right]$$

$$= H(t-2)\left[(t-2)^2+4(t-2)+5\right]$$

$$= H(t-2)(t-2)^2 + 4H(t-2)(t-2) + 5H(t-2)$$

$$G(s) = \mathcal{L}[g(t)]$$

$$= \mathcal{L}[H(t-2)(t-2)^2] + 4\mathcal{L}[H(t-2)(t-2)] + 5\mathcal{L}[H(t-2)]$$

$$= e^{-2s}\underbrace{\mathcal{L}[t^2]}_{\frac{2}{s^3}} + 4e^{-2s}\underbrace{\mathcal{L}[t]}_{\frac{1}{s^2}} + 5e^{-2s}\underbrace{\mathcal{L}[1]}_{\frac{1}{s}}$$

(formula 1, 2, 3 ; p87)

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{5}{s}\right]$$

p102

(예제 3.8)

$$\left.\begin{aligned} &\frac{d^2y}{dt^2} + 4\overset{y}{\cancel{\frac{dy}{dt}}} = f(t) \qquad y(0) = y'(0) = 0 \\ &f(t) = \begin{cases} 0 & t < 3 \\ t & t \geq 3 \end{cases} = H(t-3)\,t \end{aligned}\right\} \quad - ①$$

Taking Laplace transform to Eq. ①

$$\left[s^2 Y(s) - s\,\underline{\overset{=0}{y(0)}} - \underline{\overset{=0}{y'(0)}}\right] + 4Y(s) = \mathcal{L}[f(t)]$$

$$(s^2+4)Y(s) = \mathcal{L}[f(t)] \quad - ②$$

$\mathcal{L}[f(t)]$

$$= \mathcal{L}[H(t-3)\,t]$$

$$= \mathcal{L}[H(t-3)\{(t-3)+3\}]$$

$$= \mathcal{L}[H(t-3)(t-3)] + 3\mathcal{L}[H(t-3)]$$

$$= e^{-3s}\underbrace{\mathcal{L}[t]}_{\frac{1}{s^2}} + 3e^{-3s}\underbrace{\mathcal{L}[1]}_{\frac{1}{s}} \qquad \text{(제2 이동정리)}$$

$$= e^{-3s}\left[\frac{1}{s^2} + \frac{3}{s}\right]$$

$$= \frac{3s+1}{s^2}e^{-3s} \quad - ③$$

③ → ②

$$Y(s) = \frac{3s+1}{s^2(s^2+4)}e^{-3s} \quad - ④$$

$$\frac{3s+1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{3s+1}{s^2} - \frac{3s+1}{s^2+4} \right]$$

$$= \frac{3}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s^2} - \frac{3}{4}\frac{s}{s^2+4} - \frac{1}{4}\frac{1}{s^2+4} \quad - ⑤$$

⑤ → ④

$$Y(s) = \frac{3}{4}\frac{1}{s}e^{-3s} + \frac{1}{4}\frac{1}{s^2}e^{-3s} - \frac{3}{4}\frac{s}{s^2+4}e^{-3s} - \frac{1}{4}\frac{1}{s^2+4}e^{-3s} \quad - ⑥$$

$$y(t) = \mathcal{L}^{-1}[Y]$$

$$= \frac{3}{4}\mathcal{L}^{-1}\left[\frac{1}{s}e^{-3s}\right] + \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-3s}\right] - \frac{3}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}e^{-3s}\right] - \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}e^{-3s}\right]$$

$\llcorner$ ⑦

(i) $\mathcal{L}^{-1}\left[\frac{1}{s}e^{-3s}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

Since

$$\mathcal{L}^{-1}\left[e^{-as}F(s)\right] = H(t-a)f(t-a) \quad - ⑧$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-3s}\right] = H(t-3) \quad - ⑨$$

(ii) $\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-3s}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-3s}\right] = H(t-3)\,(t-3) \quad - ⑩$$

(iii) $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\,e^{-3s}\right]$

$\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\right] = \cos 2t$ (see formula 12, p88)

$\mathcal{L}^{-1}\left[\dfrac{s}{s^2+4}\,e^{-3s}\right] = H(t-3)\cos 2(t-3)$ — 11

(iv) $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\,e^{-3s}\right]$

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\right] = \dfrac{1}{2}\mathcal{L}^{-1}\left[\dfrac{2}{s^2+4}\right] = \dfrac{1}{2}\sin 2t$ (see formula 11, p88)

$\mathcal{L}^{-1}\left[\dfrac{1}{s^2+4}\,e^{-3s}\right] = H(t-3)\,\dfrac{1}{2}\sin 2(t-3)$ — 12

9, 10, 11, 12 $\Rightarrow$ 7

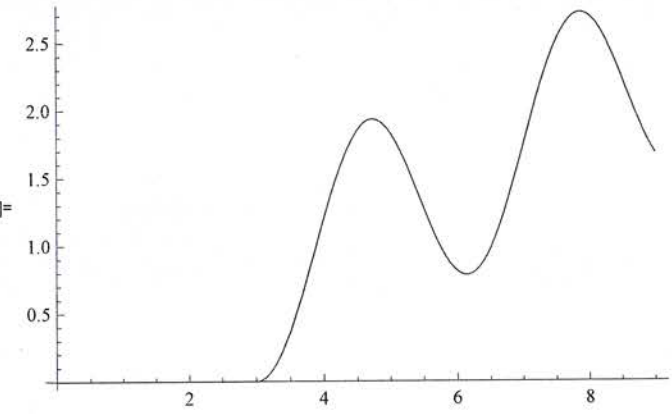
$$y(t) = \frac{3}{4}H(t-3) + \frac{1}{4}H(t-3)(t-3) - \frac{3}{4}H(t-3)\cos 2(t-3) - \frac{1}{8}H(t-3)\sin 2(t-3)$$

$$= H(t-3)\left[\frac{1}{4}t - \frac{3}{4}\cos 2(t-3) - \frac{1}{8}\sin 2(t-3)\right]$$

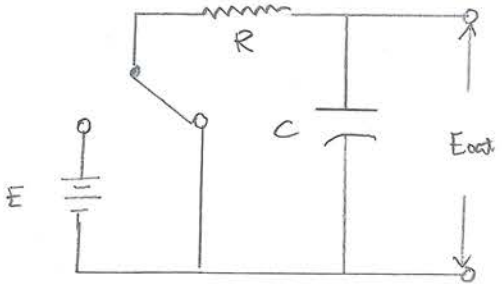
$$= \begin{cases} 0 & t < 3 \\ \dfrac{1}{8}\left[2t - 6\cos 2(t-3) - \sin 2(t-3)\right] & t \geq 3 \end{cases}$$

```
In[9]:= f[t_] := If[t < 3, 0, (1/8) (2 t - 6 Cos[2 (t - 3)] - Sin[2 (t - 3)])];
        Plot[f[t], {t, 0, 9}]
```

Out[10]=



[4] 전기회로 분석



$E = 10(V)$

$R = 250{,}000\,(\Omega)$

$C = 10^{-6}\,(F)$

$E_{out} = ?$ (출력전압)

$$E \Rightarrow E\left[H(t-2) - H(t-3)\right] \quad -① $$

$$iR + \frac{q}{C} = E\left[H(t-2) - H(t-3)\right]$$

$$i = \frac{dq}{dt}$$

$$\left.\begin{aligned} &\Rightarrow R\frac{dq}{dt} + \frac{1}{C}q = E\left[H(t-2) - H(t-3)\right] \\ &q(0) = 0 \end{aligned}\right\} \quad -②$$

Takig Laplace transform to Eq. ②

$$R\left[sQ(s) - q(0)\right] + \frac{1}{C}Q(s) = E\left[\mathcal{L}\left[H(t-2)\right] - \mathcal{L}\left[H(t-3)\right]\right] \quad -③$$

$$\left.\begin{aligned} &\mathcal{L}\left[H(t-2)\right] = e^{-2s}\mathcal{L}[1] = \frac{1}{s}e^{-2s} \\ &\mathcal{L}\left[H(t-3)\right] = e^{-3s}\mathcal{L}[1] = \frac{1}{s}e^{-3s} \end{aligned}\right\} \quad -④$$

제2 이동정리

④ → ③

$$\left(Rs + \frac{1}{C}\right)Q(s) - R\underbrace{q(0)}_{=0} = \frac{E}{s}\left(e^{-2s} - e^{-3s}\right)$$

$$Q(s) = \left(Rs + \frac{1}{C}\right)^{-1}\frac{E}{s}\left(e^{-2s} - e^{-3s}\right)$$

$$= \frac{CE}{s(RCs+1)}\left(e^{-2s} - e^{-3s}\right) \quad -⑤$$

Using

$$\frac{1}{s(RCs+1)} = \frac{1}{s} - \frac{RC}{RCs+1} \qquad -⑥$$

$$Q(s) = CE\left(\frac{1}{s} - \frac{RC}{RCs+1}\right)\left(e^{-2s} - e^{-3s}\right)$$

$$= CE\left[\frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} - \frac{RC}{RCs+1}e^{-2s} + \frac{RC}{RCs+1}e^{-3s}\right] \qquad -⑦$$

$$q(t) = CE\,\mathcal{L}^{-1}\left[\frac{1}{s}e^{-2s}\right] - CE\,\mathcal{L}^{-1}\left[\frac{1}{s}e^{-3s}\right] - RC^2E\,\mathcal{L}^{-1}\left[\frac{1}{RCs+1}e^{-2s}\right]$$

$$+ RC^2E\,\mathcal{L}^{-1}\left[\frac{1}{RCs+1}e^{-3s}\right] \qquad -⑧$$

$$\mathcal{L}^{-1}\left[e^{-as}F(s)\right] = H(t-a)\,f(t-a) \qquad -⑨$$

(i) $\mathcal{L}^{-1}\left[\frac{1}{s}e^{-2s}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-2s}\right] = H(t-2) \qquad -⑩$$

(ii) $\mathcal{L}^{-1}\left[\frac{1}{s}e^{-3s}\right] = H(t-3) \qquad -⑪$

(iii) $\mathcal{L}^{-1}\left[\frac{1}{RCs+1}e^{-2s}\right]$

$$\mathcal{L}^{-1}\left[\frac{1}{RCs+1}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{RC}\,\frac{1}{s+\frac{1}{RC}}\right]$$

$$= \frac{1}{RC}\,\mathcal{L}^{-1}\left[\frac{1}{s+\frac{1}{RC}}\right]$$

$$= \frac{1}{RC}\,e^{-\frac{t}{RC}}$$

(formula 5, p87)

$$\mathcal{L}^{-1}\left[\frac{1}{RCs+1}e^{-2s}\right] = H(t-2)\,\frac{1}{RC}\,e^{-\frac{1}{RC}(t-2)} \quad \text{— ⑫}$$

(iv) $$\mathcal{L}^{-1}\left[\frac{1}{RCs+1}e^{-3s}\right] = H(t-3)\,\frac{1}{RC}\,e^{-\frac{1}{RC}(t-3)} \quad \text{— ⑬}$$

⑩, ⑪, ⑫, ⑬ → ⑧

$$q(t) = CE\,H(t-2) - CE\,H(t-3) - RC^2E\,H(t-2)\,\frac{1}{RC}\,e^{-\frac{1}{RC}(t-2)}$$

$$+RC^2E\,H(t-3)\,\frac{1}{RC}\,e^{-\frac{1}{RC}(t-3)}$$

$$= CE\,H(t-2)\left[1-e^{-\frac{1}{RC}(t-2)}\right] - CE\,H(t-3)\left[1-e^{-\frac{1}{RC}(t-3)}\right] \quad \text{— ⑭}$$

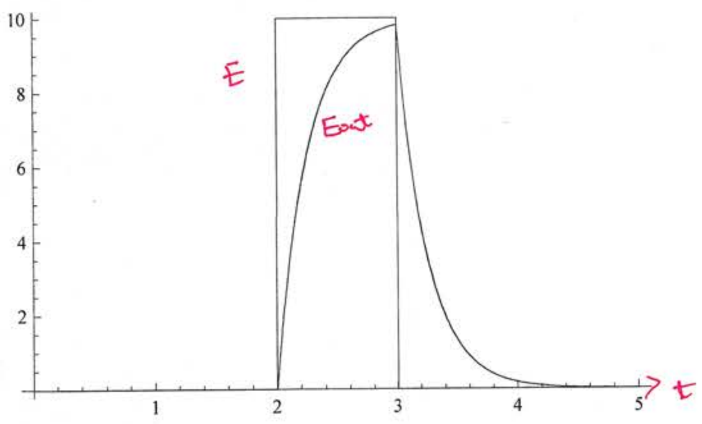
$$E_{out} = \frac{q}{C}$$

$$= E\,H(t-2)\left[1-e^{-\frac{1}{RC}(t-2)}\right] - E\,H(t-3)\left[1-e^{-\frac{1}{RC}(t-3)}\right]$$

$$= 10\,H(t-2)\left[1-e^{-4(t-2)}\right] - 10\,H(t-3)\left[1-e^{-4(t-3)}\right]$$

In[22]:=
```
Ein[t_] := If[2 < t < 3, 10, 0];
H[t_] := If[t ≥ 0, 1, 0];
Eout[t_] := 10 H[t - 2] (1 - Exp[-4 (t - 2)]) - 10 H[t - 3] (1 - Exp[-4 (t - 3)]);
Plot[{Ein[t], Eout[t]}, {t, 0, 5}, PlotStyle → {Red, Blue}]
```

Out[25]=



§ Convolution

정의 : convolution

$$(f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

f와 g의 convolution

$*$ $(f*g)(t) = (g*f)(t)$

(pf)

$$\begin{aligned}
&(g*f)(t) \\
&= \int_0^t g(t-\tau)\,f(\tau)\,d\tau \\
&= \int_t^0 (-ds)\,g(s)\,f(t-s) \\
&= \int_0^t ds\; f(t-s)\,g(s) \\
&= \int_0^t d\tau\; f(t-\tau)\,g(\tau) \\
&= (f*g)(t)
\end{aligned}$$

$s = t-\tau$

$s \to \tau$

$*$

정리 3.8

$$\mathcal{L}[f*g] = F(s)\,G(s)$$

Convolution theorem

(pf) 생략

$*$ $\mathcal{L}^{-1}[FG] = (f*g)(t) = \int_0^t d\tau\; f(t-\tau)\,g(\tau)$

p109

(예제3.10)

$$f(t) = 2t^2 + \int_0^t f(t-\tau)e^{-\tau}\,d\tau \qquad -① \qquad$$ 적분방정식

what is $f(t)$?

Taking Laplace transform to Eq.①

$$F(s) = 2\underbrace{\mathcal{L}[t^2]}_{\frac{2}{s^3}} + \mathcal{L}[\underbrace{\int_0^t f(t-\tau)e^{-\tau}d\tau}_{(f*g)(t)}] \quad \text{where } g(t)=e^{-t}$$

(formula 3, p87)

$$= \frac{4}{s^3} + F(s)\underbrace{\mathcal{L}[e^{-t}]}_{\frac{1}{s+1}} \quad (\because \text{convolution theorem})$$

(formula 5, p87)

$$= \frac{4}{s^3} + \frac{1}{s+1}F(s)$$

$$\Rightarrow F(s) = \frac{4(s+1)}{s^4} = 4\frac{1}{s^3} + 4\frac{1}{s^4} \qquad -②$$

Taking inverse Laplace transform to Eq.②

$$f(t) = 4\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s^4}\right]$$

$$= 4\cdot\frac{1}{2}\underbrace{\mathcal{L}^{-1}\left[\frac{2}{s^3}\right]}_{t^2} + 4\cdot\frac{1}{6}\underbrace{\mathcal{L}^{-1}\left[\frac{3!}{s^4}\right]}_{t^3} \quad \text{(formula 3, p87)}$$

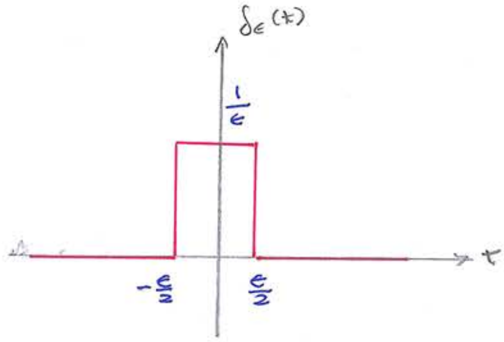
$$= 2t^2 + \frac{2}{3}t^3$$

※

p110 § 단위 충격 함수 와 Dirac-δ 함수

$$\delta_\epsilon(t) = \frac{1}{\epsilon}\left[H\left(t+\frac{\epsilon}{2}\right) - H\left(t-\frac{\epsilon}{2}\right)\right]$$

$$= \begin{cases} 0 & t < -\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & -\frac{\epsilon}{2} \le t < \frac{\epsilon}{2} \\ 0 & \frac{\epsilon}{2} \le t \end{cases}$$



$$\boxed{\text{Dirac-}\delta \text{ 함수} \quad \delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)}$$

① $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

② $\delta(-t) = \delta(t)$

③ $\mathcal{L}[\delta(t)] = 1$ ~~: δ-함수의 여과 특성 (filtering property)~~

④ $\int_0^{\infty} f(t)\,\delta(t-a)\,dt = 1$ $(a > 0)$: δ-함수의 여과 특성 (filtering property)

(pf)

③ $\mathcal{L}[\delta_\epsilon(t)]$

$$= \frac{1}{\epsilon}\Big[\underbrace{\mathcal{L}[H(t+\tfrac{\epsilon}{2})]}_{\frac{1}{s}e^{\frac{\epsilon}{2}s}} - \underbrace{\mathcal{L}[H(t-\tfrac{\epsilon}{2})]}_{\frac{1}{s}e^{-\frac{\epsilon}{2}s}} \Big]$$

(제2 이동정리)

$$= \frac{1}{\epsilon}\,\frac{1}{s}\underbrace{\left[e^{\frac{\epsilon}{2}s} - e^{-\frac{\epsilon}{2}s}\right]}_{2\sinh\frac{\epsilon}{2}s}$$

$\left(\because \sinh\theta \equiv \dfrac{e^{\theta} - e^{-\theta}}{2}\right)$

$$= \frac{2}{s}\,\frac{\sinh\frac{\epsilon}{2}s}{\epsilon}$$

$\Rightarrow \mathcal{L}[\delta(t)]$

$$= \lim_{\epsilon\to 0}\mathcal{L}[\delta_\epsilon(t)]$$

$$= \frac{2}{s}\lim_{\epsilon\to 0}\frac{\sinh\frac{\epsilon}{2}s}{\epsilon}$$

로피탈 정리

$$= \frac{2}{s}\lim_{\epsilon\to 0}\frac{\frac{s}{2}\cosh\frac{\epsilon}{2}s}{1}$$

$$= 1$$

④ Filtering property

$$\int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt \qquad \delta_\epsilon(t-a) = \begin{cases} 0 & t < a-\frac{\epsilon}{2} \\ \frac{1}{\epsilon} & a-\frac{\epsilon}{2} \le t < a+\frac{\epsilon}{2} \\ 0 & a+\frac{\epsilon}{2} \le t \end{cases}$$

$$= \int_0^{a-\frac{\epsilon}{2}} f(t)\,\underbrace{\delta_\epsilon(t-a)}_{}^{=0}\,dt$$

$$+ \int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(t)\,\underbrace{\delta_\epsilon(t-a)}_{}^{=\frac{1}{\epsilon}}\,dt + \int_{a+\frac{\epsilon}{2}}^{\infty} f(t)\,\underbrace{\delta_\epsilon(t-a)}_{}^{=0}\,dt$$

$$= \frac{1}{\epsilon}\int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(t)\,dt$$

$$\Rightarrow \int_0^\infty f(t)\,\delta(t-a)\,dt$$

$$= \lim_{\epsilon \to 0} \int_0^\infty f(t)\,\delta_\epsilon(t-a)\,dt$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\underline{\int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} f(t)\,dt}$$

$$= f(a)\int_{a-\frac{\epsilon}{2}}^{a+\frac{\epsilon}{2}} dt = \epsilon f(a)$$

$$= f(a)$$

※.

p113 (예제 3.11)

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = \delta(t-3) \qquad y(0) = y'(0) = 0 \qquad -①$$

Taking Laplace transform to Eq. ①

$$[s^2Y(s) - \underbrace{sy(0)}_{=0} - \underbrace{y'(0)}_{=0}] + 2[sY(0) - \underbrace{y(0)}_{=0}] + 2Y(s) = \mathcal{L}[\delta(t-3)]$$

$$(s^2+2s+2)\,Y(s) = \mathcal{L}[\delta(t-3)] \qquad -②$$

$$\mathcal{L}[\delta(t-3)]$$

$$\equiv \int_0^\infty dt\; e^{-st}\,\delta(t-3)$$

↓ filtering property

$$= e^{-3s} \qquad -③$$

③ → ②

$$Y(s) = \frac{e^{-3s}}{s^2+2s+2} \qquad -④$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+2s+2}\right] \qquad -⑤$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]}_{\sin t} \qquad (\because \mathcal{L}^{-1}[F(s-a)] = e^{at}f(t))\ \text{제1 이동정리}$$

sin t (formula 11, p88)

$$= e^{-t}\sin t \qquad -⑥$$

Using ⑥

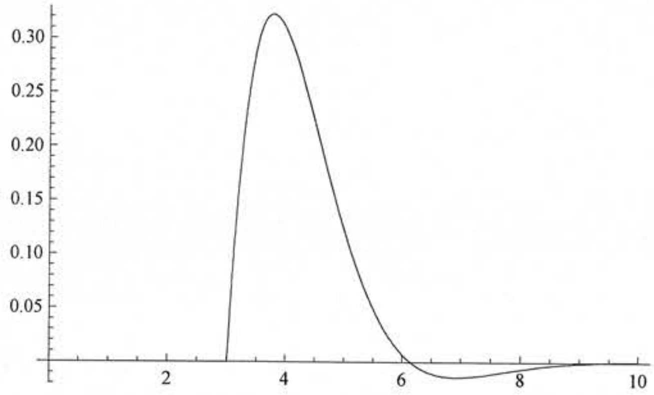
$$y(t) = \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2+2s+2}\right]$$

$$= H(t-3)\, e^{-(t-3)} \sin(t-3)$$

$\left(\because \mathcal{L}^{-1}[e^{-as}F(s)] = H(t-a)f(t-a)\right)$ 제2 이동정리

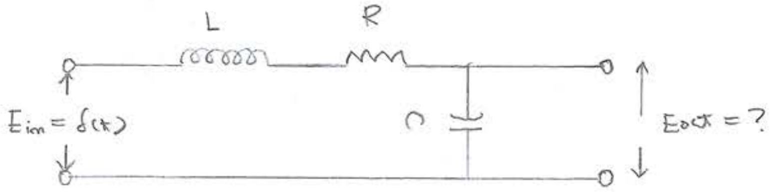
```
In[26]:= H[t_] := If[t ≥ 0, 1, 0];
         Plot[H[t - 3] Exp[- (t - 3)] Sin[t - 3], {t, 0, 10}]
```

Out[27]=



P113

(예제 3.12)



$L = 1H$

$R = 10\,\Omega$

$C = 0.01F$

$$\left.\begin{aligned} &L\frac{di}{dt} + iR + \frac{q}{C} = E_{in} = \delta(t) \\ &i = \frac{dq}{dt} \end{aligned}\right)$$

$$\left.\begin{aligned} &L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = \delta(t) \\ &q(0) = \frac{dq}{dt}(0) = 0 \end{aligned}\right) \quad - ①$$

Taking Laplace transform to Eq. ①

$$L\left[s^2Q(s) - s\underbrace{q(0)}_{=0} - \underbrace{q'(0)}_{=0}\right] + R\left[sQ(s) - \underbrace{q(0)}_{=0}\right] + \frac{1}{C}Q(s) = 1$$

$$\left(Ls^2 + Rs + \frac{1}{C}\right)Q(s) = 1$$

$$Q(s) = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

$$= \frac{1}{L}\,\frac{1}{s^2 + \frac{R}{L}s + \frac{R^2}{4L^2} + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{1}{L}\,\frac{1}{\left(s + \frac{R}{2L}\right)^2 + \left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} \quad - ②$$

$$q(t) = \mathcal{L}^{-1}[Q(s)]$$

$$= \frac{1}{L}\mathcal{L}^{-1}\left[\frac{1}{(s+\frac{R}{2L})^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})}\right]$$

$$= \frac{1}{L} e^{-\frac{R}{2L}t} \mathcal{L}^{-1}\left[\frac{1}{s^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})}\right] \quad (\because \mathcal{L}^{-1}[F(s-a)] = e^{at} f(t))$$

제1 이동정리

$$= \frac{1}{L} e^{-\frac{R}{2L}t} \frac{1}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \underbrace{\mathcal{L}^{-1}\left[\frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{s^2 + (\frac{1}{LC} - \frac{R^2}{4L^2})}\right]}_{\sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t}$$

(formula 11 : p88)

$$= \frac{1}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} e^{-\frac{R}{2L}t} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t \qquad - ②$$

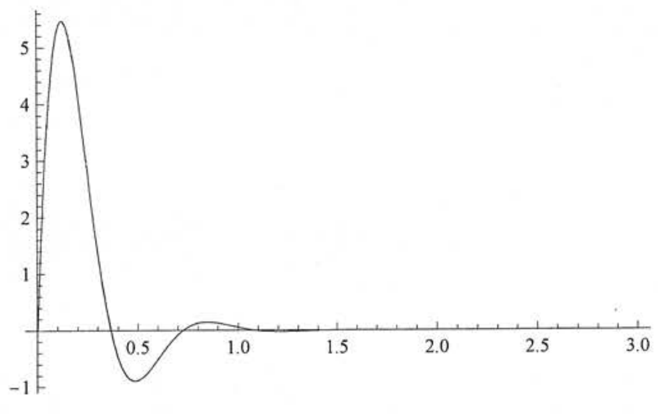
$$E_{out}(t) = \frac{1}{C} q(t)$$

$$= \frac{1}{LC\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} e^{-\frac{R}{2L}t} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t$$

$$= \frac{20}{\sqrt{3}} e^{-5t} \sin(5\sqrt{3}\, t)$$

```
In[29]:= Plot[(20 / Sqrt[3]) Exp[-5 t] Sin[5 Sqrt[3] t], {t, 0, 3}, PlotRange → All]
```

Out[29]=



(예제 3.13)

$$\left.\begin{aligned} x'' - 2x' + 3y' + 2y &= 4 \\ 2y' - x' + 3y &= 0 \end{aligned}\right\} \quad -①$$

$$x(0) = x'(0) = y(0) = 0$$

Taking Laplace transform to Eq. ①

$$[s^2X(s) - s\underbrace{x(0)}_{=0} - \underbrace{x'(0)}_{=0}] - 2[sX(s) - \underbrace{x(0)}_{=0}] + 3[sY(s) - \underbrace{y(0)}_{=0}] + 2Y(s) = 4\frac{1}{s}$$

$$(s^2 - 2s)X(s) + (3s+2)Y(s) = \frac{4}{s} \quad -②$$

$$2[sY(s) - \underbrace{y(0)}_{=0}] - [sX(s) - \underbrace{x(0)}_{=0}] + 3Y(s) = 0$$

$$-sX(s) + (2s+3)Y(s) = 0 \quad -③$$

$$\Rightarrow \quad X(s) = \frac{2(2s+3)}{s^2(s+2)(s-1)} \qquad Y(s) = \frac{2}{s(s+2)(s-1)} \quad -④$$

$$X(s) = \frac{-\frac{7}{2}s - 3}{s^2} + \frac{1}{6}\frac{1}{s+2} + \frac{10}{3}\frac{1}{s-1}$$

$$= -\frac{7}{2}\frac{1}{s} - 3\frac{1}{s^2} + \frac{1}{6}\frac{1}{s+2} + \frac{10}{3}\frac{1}{s-1} \quad -⑤$$

$$Y(s) = -\frac{1}{s} + \frac{1}{3}\frac{1}{s+2} + \frac{2}{3}\frac{1}{s-1} \quad -⑥$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$= -\frac{7}{2}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s}\right]}_{=1} - 3\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s^2}\right]}_{t} + \frac{1}{6}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}} + \frac{10}{3}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]}_{e^{t}}$$

(p89)

$$= -\frac{7}{2} - 3t + \frac{1}{6}e^{-2t} + \frac{10}{3}e^{t}$$

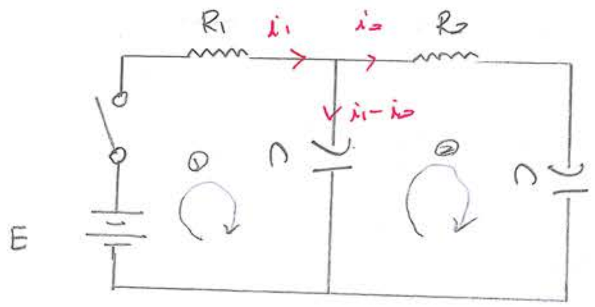
$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$= -\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s}\right]}_{=1} + \frac{1}{3}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]}_{e^{-2t}} + \frac{2}{3}\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s-1}\right]}_{e^{t}}$$

$$= -1 + \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t}$$

P118

(예제 3.15)



$E = 10(V)$

$R_1 = 40(\Omega)$

$R_2 = 60(\Omega)$

$C = \frac{1}{120}(F)$

Let

$$i_1 = \frac{dq_1}{dt} \qquad i_2 = \frac{dq_2}{dt} \qquad \text{— ①}$$

From loop ①

$$i_1 R_1 + \frac{q_1 - q_2}{C} = E \qquad \text{— ②}$$

From loop ②

$$i_2 R_2 + \frac{q_2}{C} = \frac{q_1 - q_2}{C} \qquad \text{— ③}$$

Using Eq. ① Eq. ② and ③ becomes

$$R_1 \frac{dq_1}{dt} + \frac{1}{C}(q_1 - q_2) = E \qquad \text{— ④}$$

$$R_2 \frac{dq_2}{dt} + \frac{1}{C}(-q_1 + 2q_2) = 0 \qquad \text{— ⑤}$$

$$q_1(0) = q_2(0) = 0$$

Taking Laplace transform to Eq ④ and ⑤

$$R_1\left[s Q_1(s) - \underbrace{q_1(0)}_{=0} \right] + \frac{1}{C}\left[Q_1(s) - Q_2(s) \right] = E\,\frac{1}{s}$$

$$\left(R_1 s + \frac{1}{C}\right) Q_1(s) - \frac{1}{C} Q_2(s) = \frac{E}{s} \qquad -⑥$$

$$R_2\left[s Q_2(s) - \underbrace{q_2(0)}_{=0} \right] + \frac{1}{C}\left[-Q_1(s) + 2 Q_2(s) \right] = 0$$

$$-\frac{1}{C} Q_1(s) + \left(R_2 s + \frac{2}{C}\right) Q_2(s) = 0 \qquad -⑦$$

From ⑥ and ⑦

$$Q_1(s) = \frac{E\,(C R_2 s + 2)}{s\left[C R_1 R_2 s^2 + (2R_1 + R_2) s + \frac{1}{C} \right]} = \frac{1}{4}\,\frac{s+4}{s(s+1)(s+6)} \qquad -⑧$$

$$Q_2(s) = \frac{E}{s\left[C R_1 R_2 s^2 + (2R_1 + R_2) s + \frac{1}{C} \right]} = \frac{1}{2}\,\frac{1}{s(s+1)(s+6)} \qquad -⑨$$

$$Q_1(s) = \frac{1}{4}\left[\frac{2}{3}\,\frac{1}{s} - \frac{3}{5}\,\frac{1}{s+1} - \frac{1}{15}\,\frac{1}{s+6} \right]$$

$$= \frac{1}{6}\,\frac{1}{s} - \frac{3}{20}\,\frac{1}{s+1} - \frac{1}{60}\,\frac{1}{s+6}$$

$$\Rightarrow q_1(t) = \frac{1}{6} - \frac{3}{20} e^{-t} - \frac{1}{60} e^{-6t}$$

$$\Rightarrow i_1 = \frac{dq_1}{dt} = \frac{3}{20} e^{-t} + \frac{1}{10} e^{-6t}$$

$$Q_2(s) = \frac{1}{2}\left[\frac{1}{6}\,\frac{1}{s} - \frac{1}{5}\,\frac{1}{s+1} + \frac{1}{30}\,\frac{1}{s+6}\right]$$

$$= \frac{1}{12}\,\frac{1}{s} - \frac{1}{10}\,\frac{1}{s+1} + \frac{1}{60}\,\frac{1}{s+6}$$

$$q_2(t) = \frac{1}{12} - \frac{1}{10}e^{-t} + \frac{1}{60}e^{-6t}$$

$$\Rightarrow i_2(t) = \frac{dq_2}{dt} = \frac{1}{10}e^{-t} - \frac{1}{10}e^{-6t}$$

$\divideontimes$