Discrete Mathematics for Computer Scientists

Solutions to Recurrences

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Discrete Mathematics for Computer Scientists
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Section 4.3

Growth Rates of Solutions to Recurrences

- Divide and Conquer Algorithms
- Recursion Trees
- Three Different Behaviors

Divide and Conquer Algorithms

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In the previous section we analyzed recurrences of the form

$$T(n) = \begin{cases} a & \text{if } n = b \\ c \cdot T(n-1) + d & \text{if } n > b \end{cases}$$

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These corresponded to the analyses of recursive algorithms in which a problem of size n is solved by recursively solving a problem(s) of size $n-1$.

We will now look at recurrences that arise from recursive algorithms in which problems of size n are solved by recursively solving problems of size n/m, for some fixed m. These recurrences will be in the form

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We will now look at recurrences that arise from recursive algorithms in which problems of size n are solved by recursively solving problems of size n/m, for some fixed m. These recurrences will be in the form

$$T(n) = \begin{cases} \text{something given} & \text{if } n \leq b \\ c \cdot T(n/m) + d & \text{if } n > b \end{cases}$$

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Someone has chosen a number x between 1 and n.
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- Is x greater than k?
- Is x equal to k?

Our strategy will be to always ask greater than questions,
at each step halving our search range,
until the range only contains one number,
when we ask a final equal to question

Binary Search Example

Binary Search Example

1 32 48 64

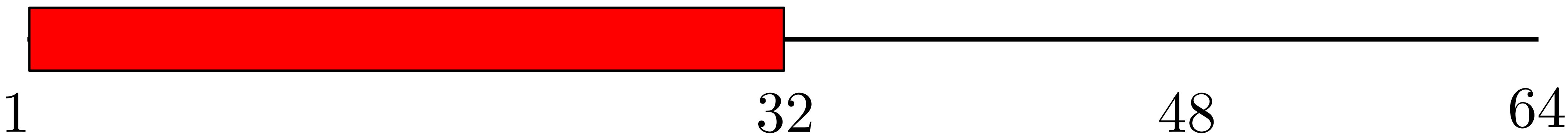
Binary Search Example

1 32 48 64

Is $x > 32$?

halfway mark

Binary Search Example

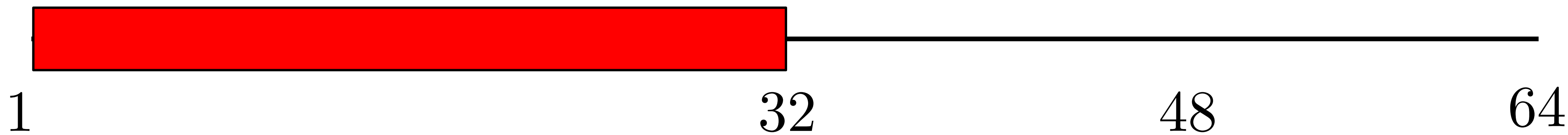


Is $x > 32$? Answer: Yes

eliminate what X wouldn't be

↳ reduce possibility

Binary Search Example

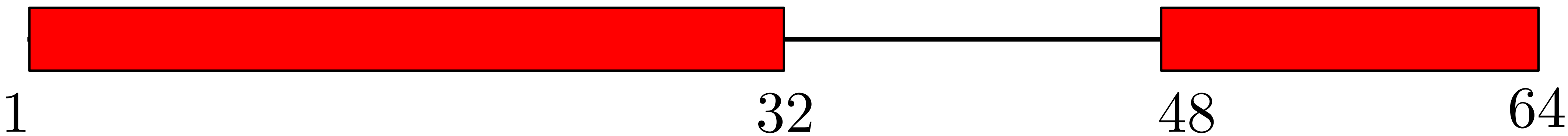


Is $x > 32$? Answer: Yes

Is $x > 48$?

take other half and split in half

Binary Search Example



Is $x > 32$? Answer: Yes

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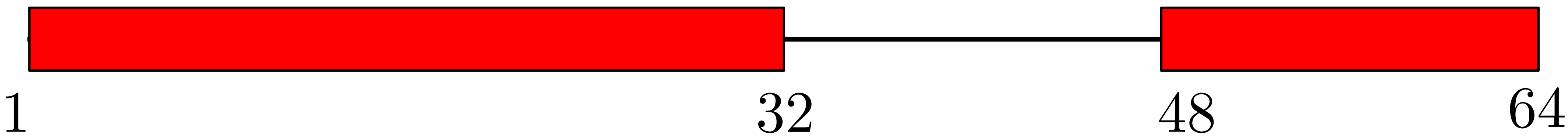
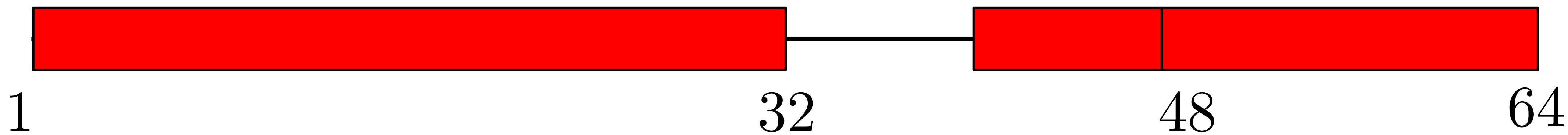


Is $x > 32$? Answer: Yes

Is $x > 48$? Answer: No

Is $x > 40$?

Binary Search Example



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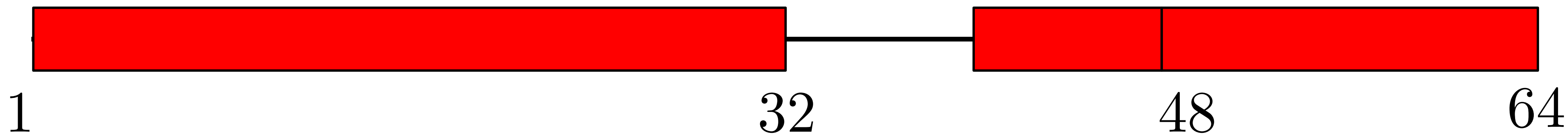

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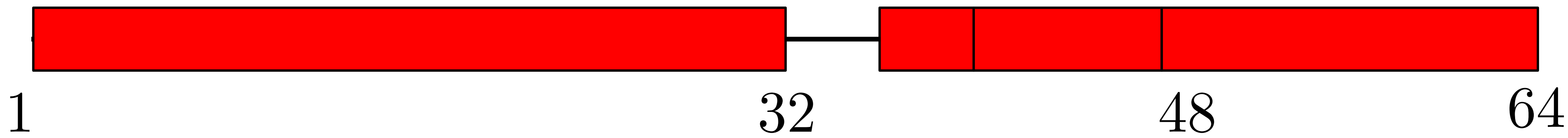

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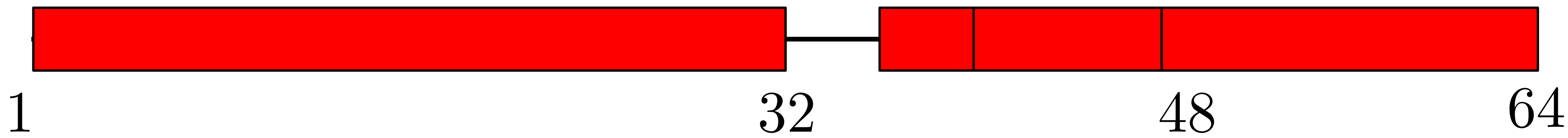

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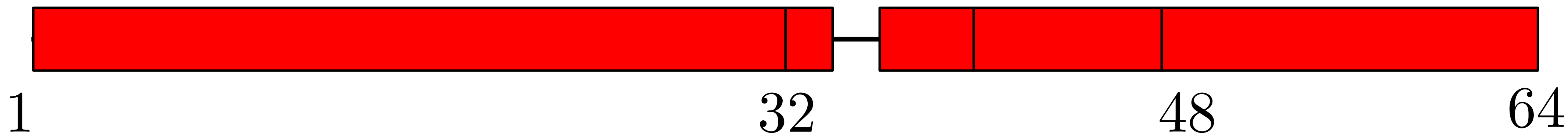

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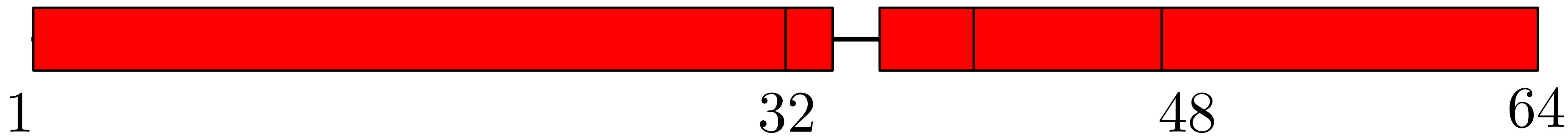

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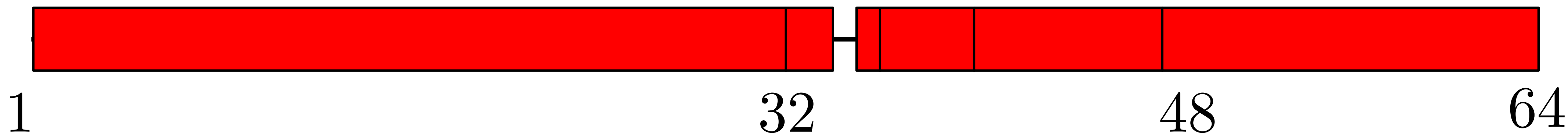

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Keep asking questions until it's in 1 range

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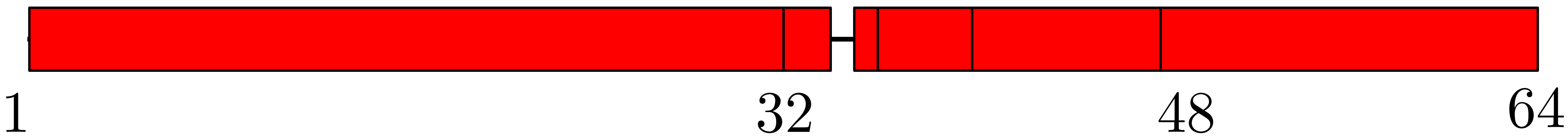

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Is $x > 35$? Answer: No
Is $x = 35$?

to make sure x is not in decimal

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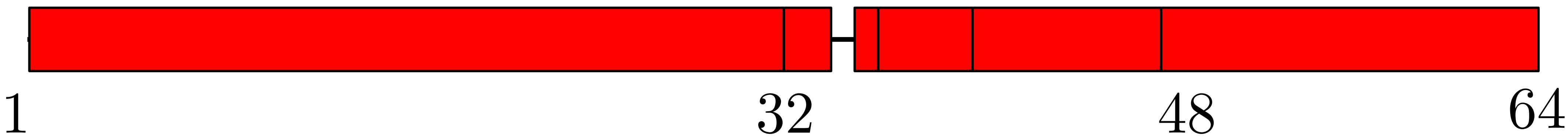

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what formula do we need to get the # of questions?

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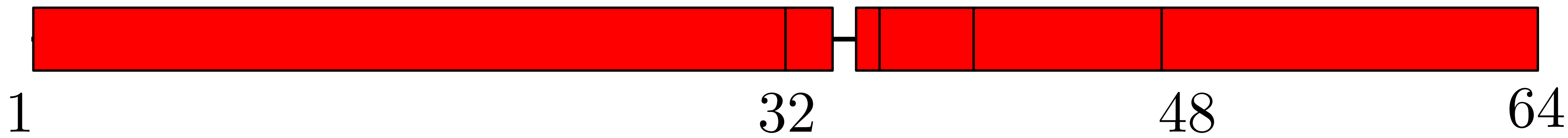

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Method: Each guess reduces the problem to one in which the range is only **half** as big.

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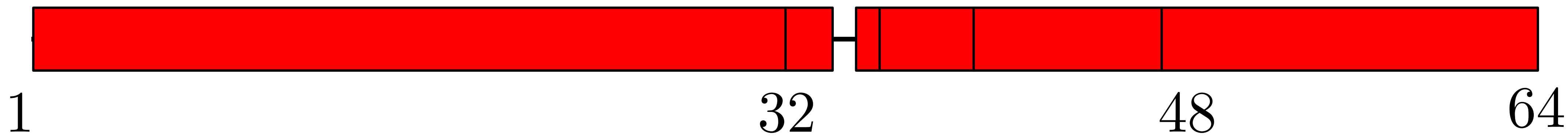

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Method: Each guess reduces the problem to one in which the range is only **half** as big.

This divides the original problem into one that is only half as big; we can now (recursively) conquer this smaller problem.

Note: Our derivation that, when n is a power of 2, $T(n)$, the number of questions in a binary search on $[1, n]$, satisfies

$$T(n) = \begin{cases} T(n/2) + 1 & \text{if n} \geq \text{2,} \\ 1 & \text{if n} = \text{1.} \end{cases}$$

was actually, implicitly, an inductive proof. This is similar to what we saw with the tower of Hanoi recurrence. We did not write out all the formal steps of the inductive proof, though.

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$$(*) \qquad T(n) = \begin{cases} T(\lceil n/2 \rceil) + C_1 & \text{if n} \geq 2, \\ C_2 & \text{if n} = 1, \end{cases}$$

In order to avoid complications we will (usually) assume that n is a power of 2 (or sometimes 3 or 4) and also often that constants such as C_1, C_2 are 1. This will let us replace a recurrence such as (*) by one such as (**).

$$(**) \qquad T(n) = \begin{cases} T(n/2) + 1 & \text{if n} \geq 2, \\ 1 & \text{if n} = 1. \end{cases}$$

In practice, the solution of (*) will be very close to the solution of (**) (this can be proven mathematically) so, as in this class, we can restrict ourselves to (**) without losing much.

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To solve some problem of size n, we
(ii) solve 3 subproblems of size $n-1$ and
(ii) do n units of additional work.

We will start off by examining the recurrence

$$(*) \qquad T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + n & \text{if } n > 1 \\ T(1) & \text{if } n = 1 \end{cases}$$

This corresponds to solving a problem of size n, by
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We will now see how to "solve" (*), first by algebraically iterating the recurrence, and then by using a recursion tree (which is a visual method for iterating the recurrence).

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$$= 4T\left(\frac{n}{4}\right) + 2n \quad = 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

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$$= 8T\left(\frac{n}{8}\right) + 3n$$

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$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n \qquad &&= 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n \\
&= 4T\left(\frac{n}{4}\right) + 2n &&= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n \\
&= 8T\left(\frac{n}{8}\right) + 3n \\
&\vdots \qquad\qquad \vdots
\end{aligned}
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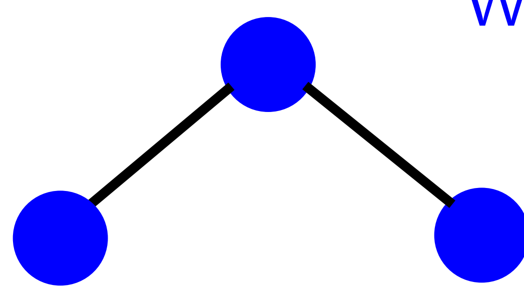
2	$n/2$		



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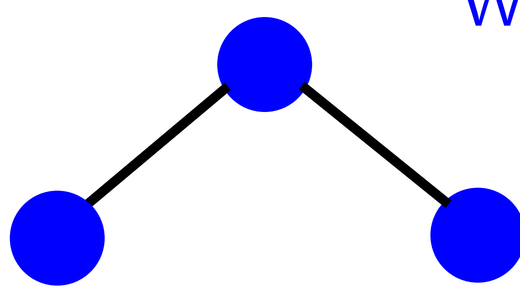

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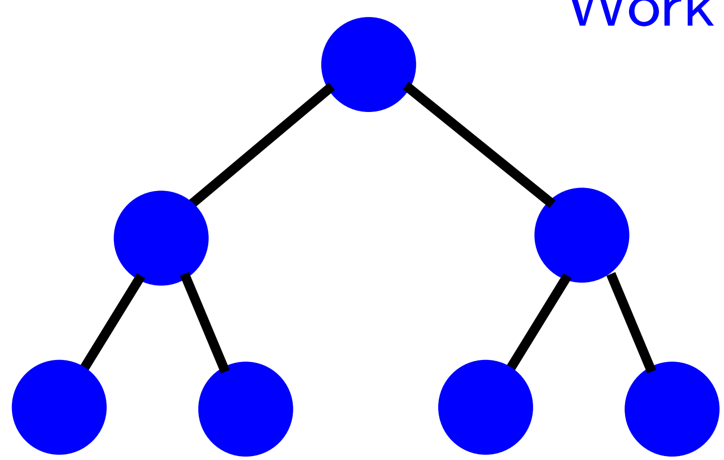

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$2^2 = 4$	$n/4$		

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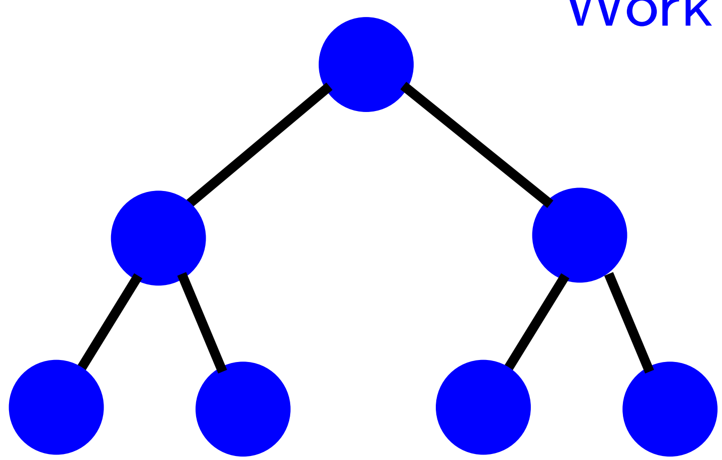

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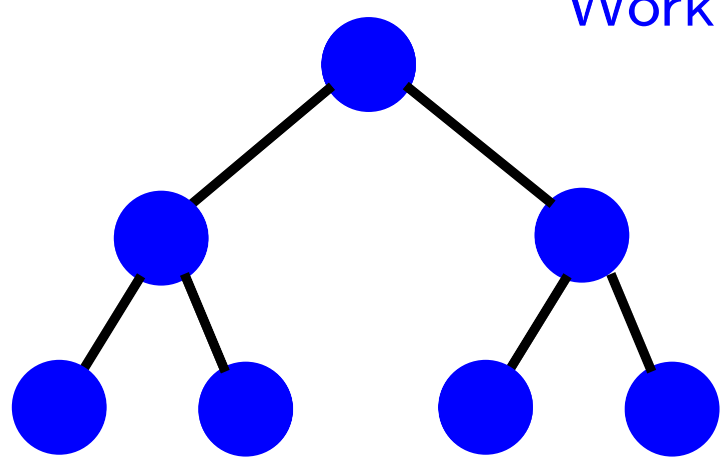
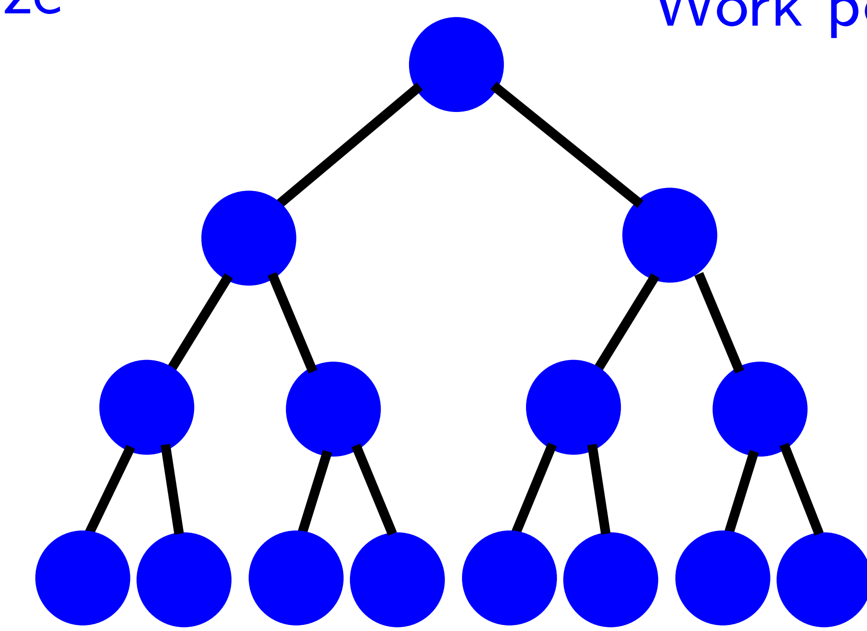
$2^2 = 4$	$n/4$	$n/4$	$4(n/4) = n$



Example: Visually "iterating" the recurrence

$$T(n) = 2T\left(\frac{n}{2}\right) + n \;= 4T\left(\frac{n}{4}\right) + 2n \;= 8T\left(\frac{n}{8}\right) + 3n.$$

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
2	$n/2$	$n/2$	$2(n/2) = n$
$2^2 = 4$	$n/4$	$n/4$	$4(n/4) = n$
$2^3 = 8$	$n/8$		



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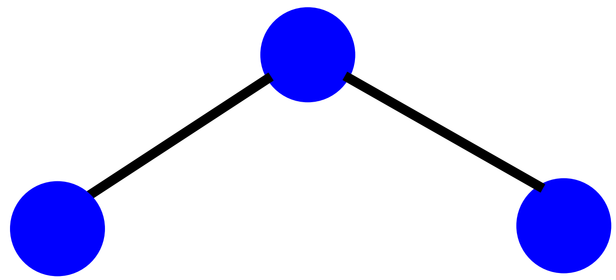
# of Probl	Probl Size		Work per Probl	Work per Level
1	n		n	n
2	$n/2$			

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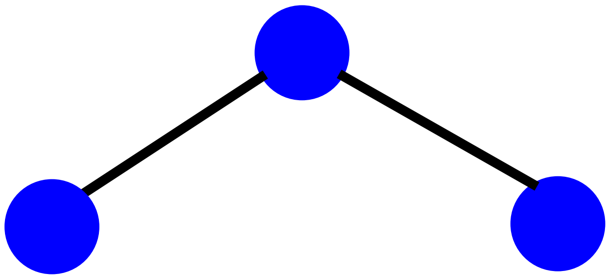

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
2	$n/2$	$n/2$	$2(n/2) = n$

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1	n	n	n
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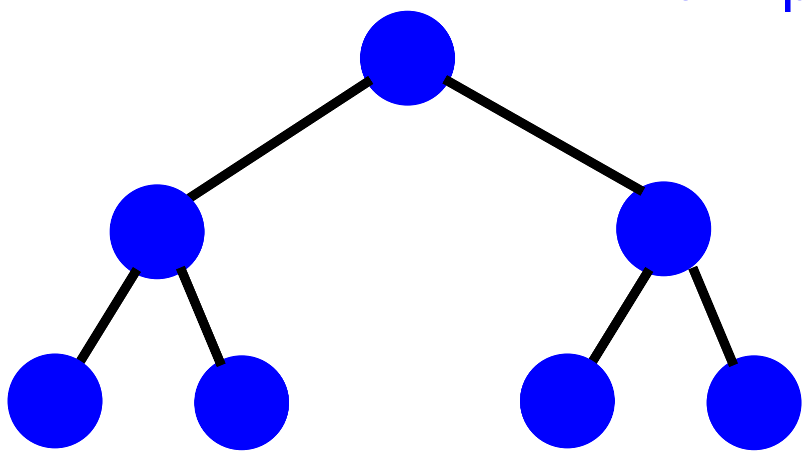
$2^2 = 4$	$n/4$		



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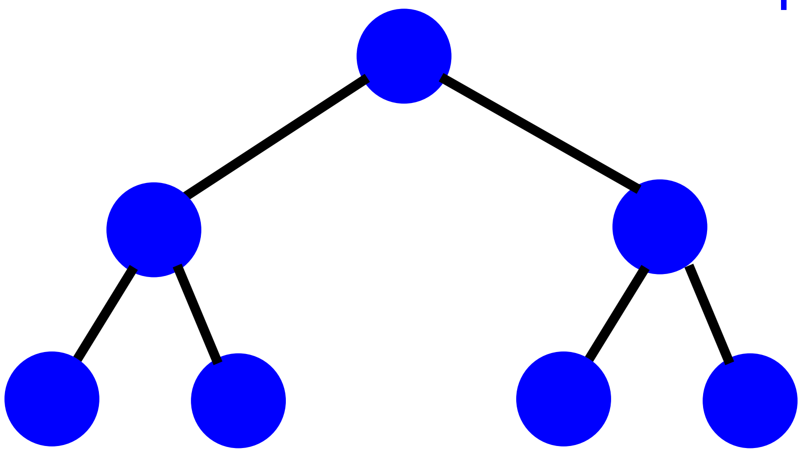

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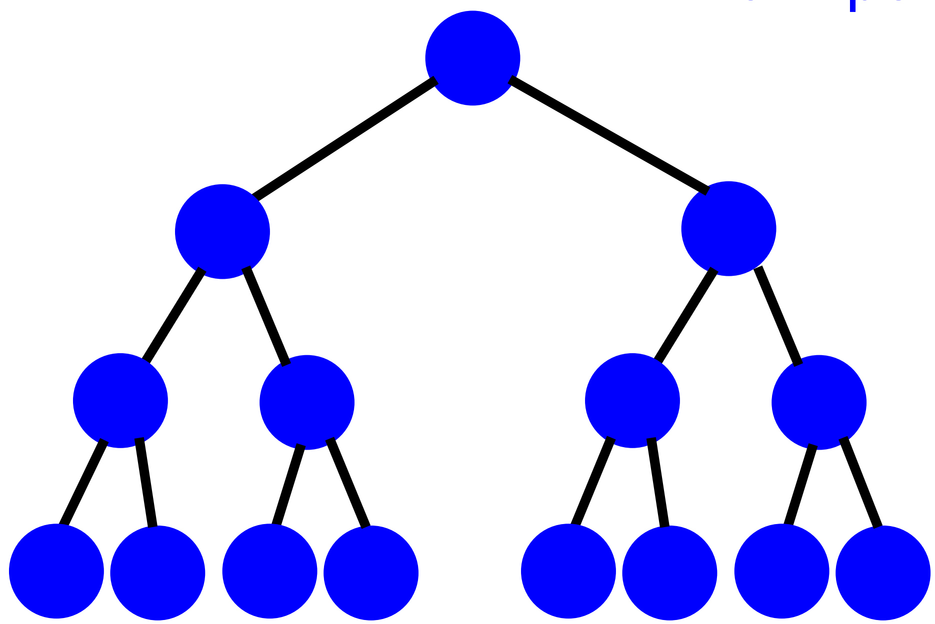

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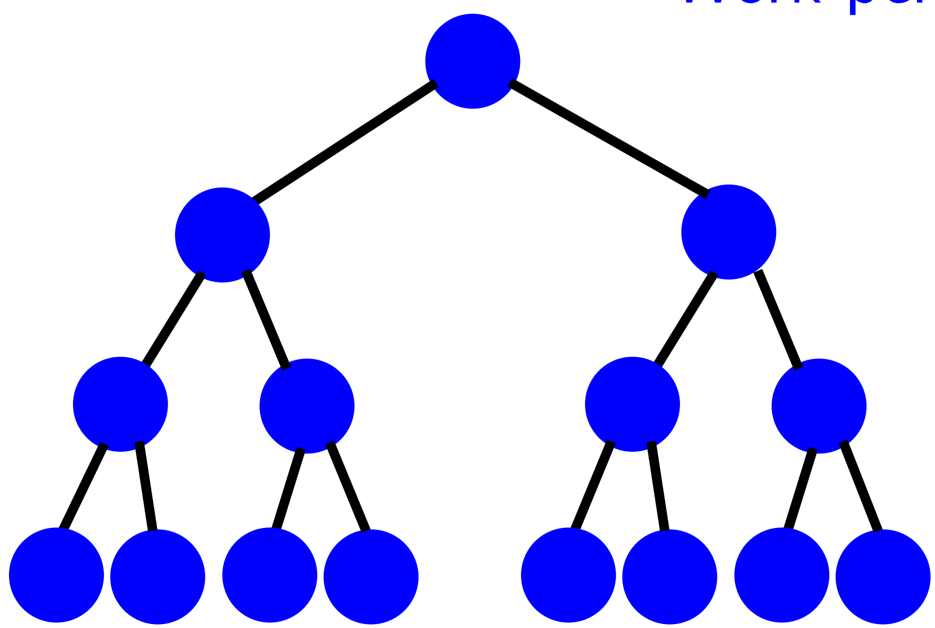

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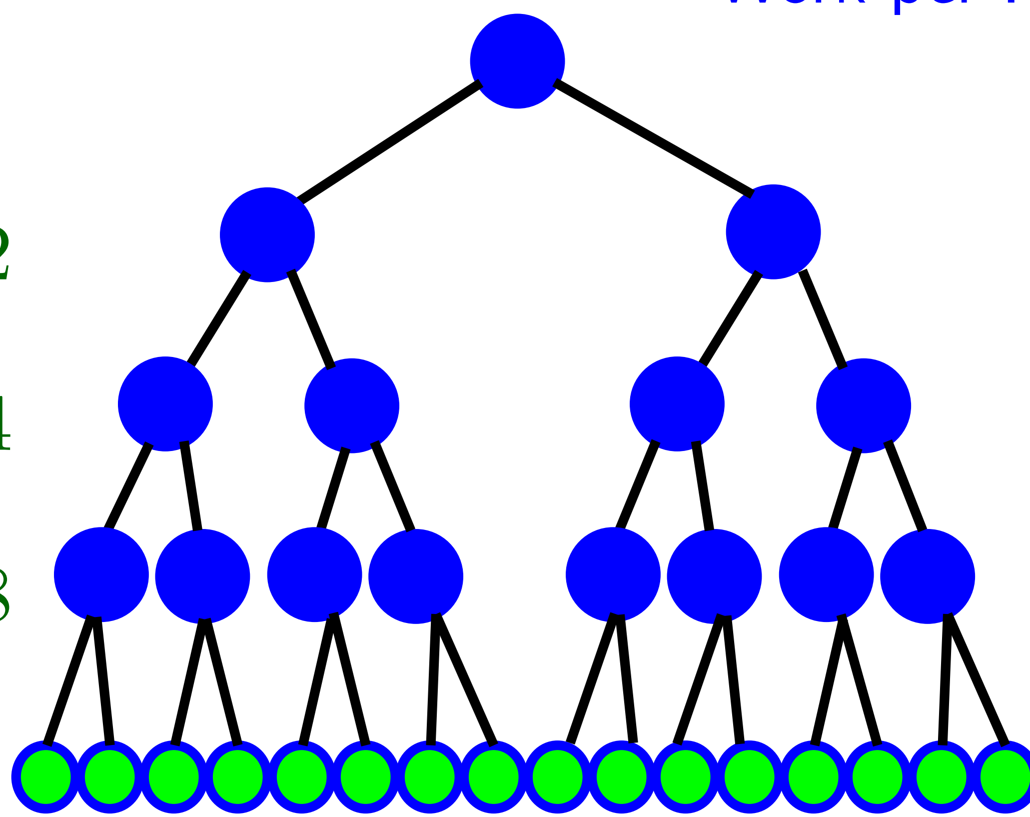

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$2^{\log n} = n$	1		

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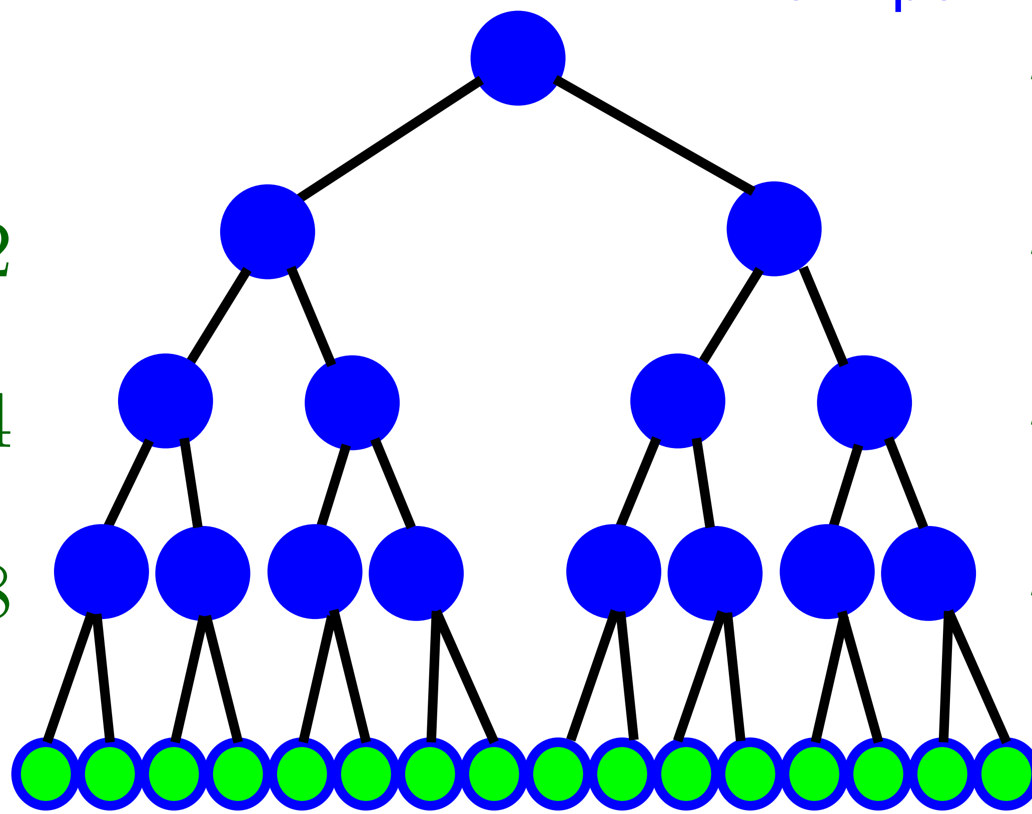

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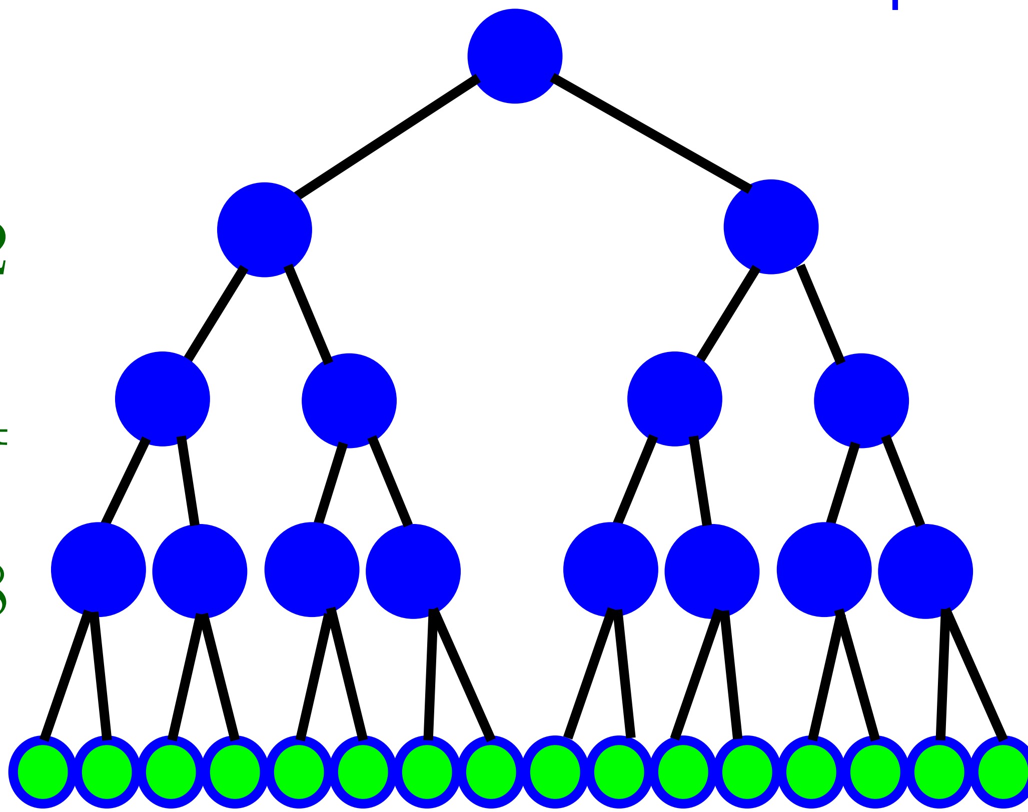

$(1 + \log_2 n)$ levels $\Rightarrow$ total work $= n \log_2 n + nT(1)$

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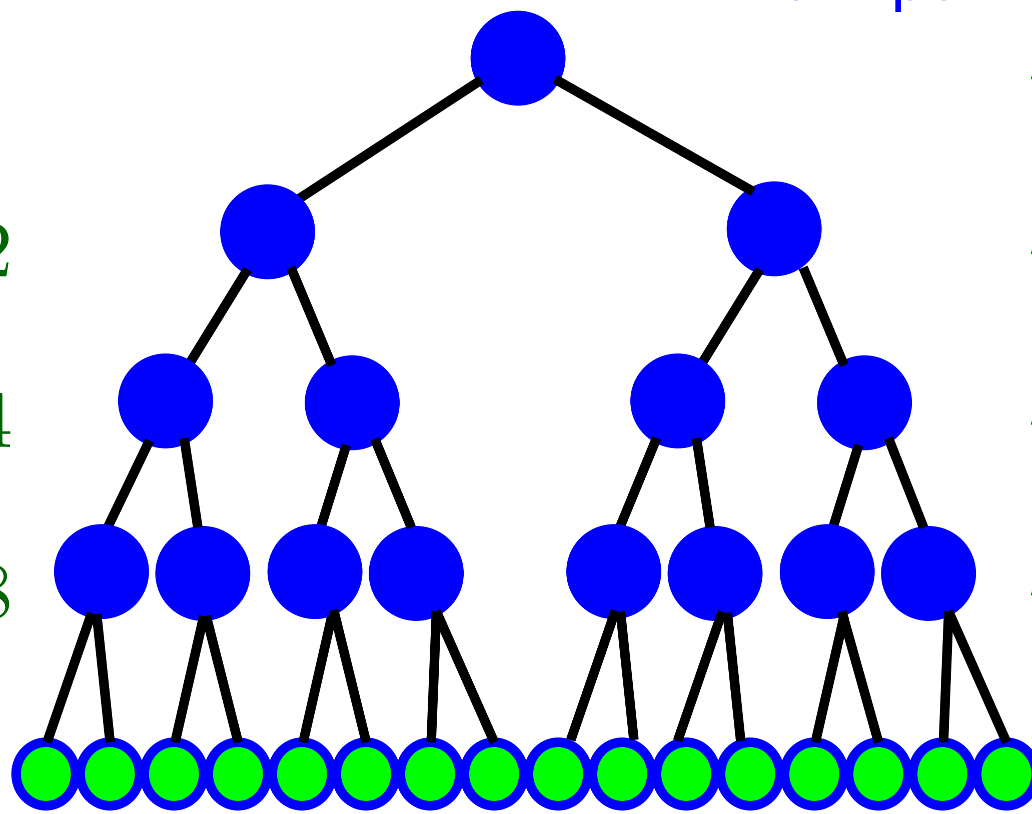

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5 levels $\Rightarrow$ total work $= 4n + 2n$

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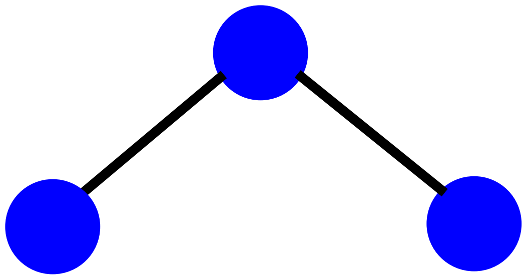
1	n		n	n
2	$n/2$			

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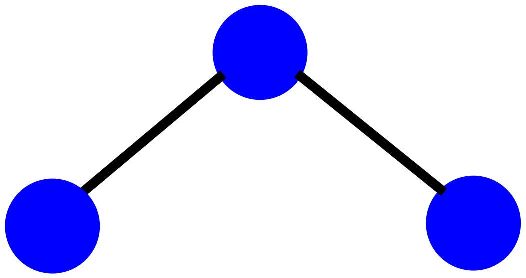
1	n		n	n
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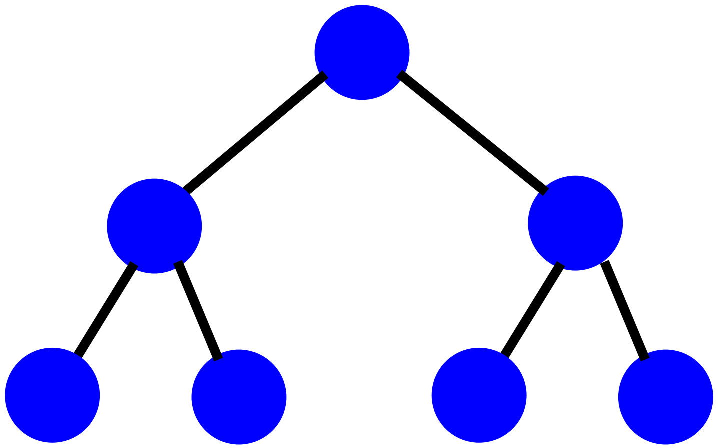
1	n		n	n
2	$n/2$		$n/2$	$2(n/2) = n$
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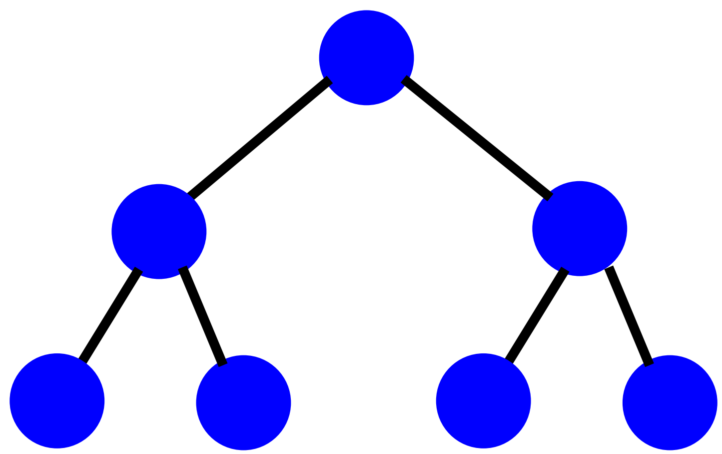

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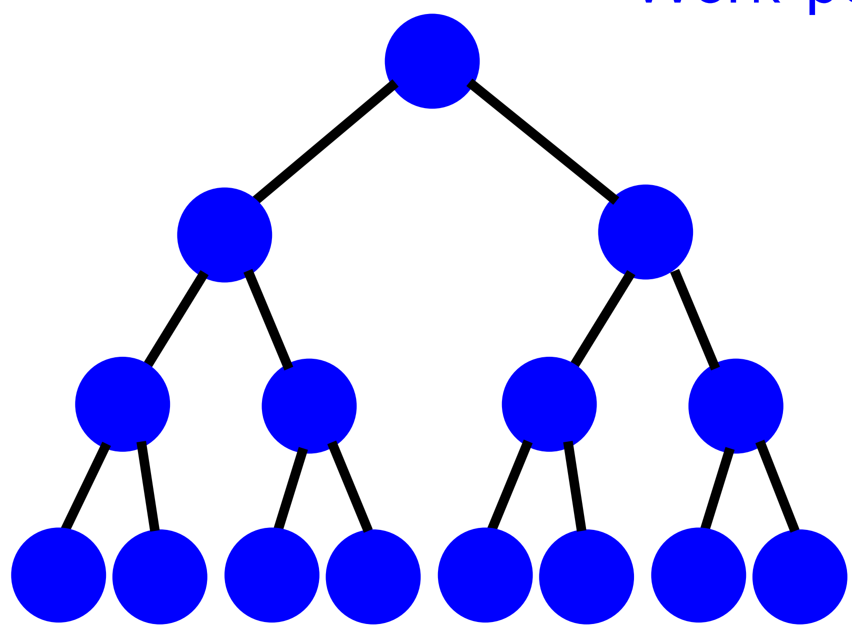

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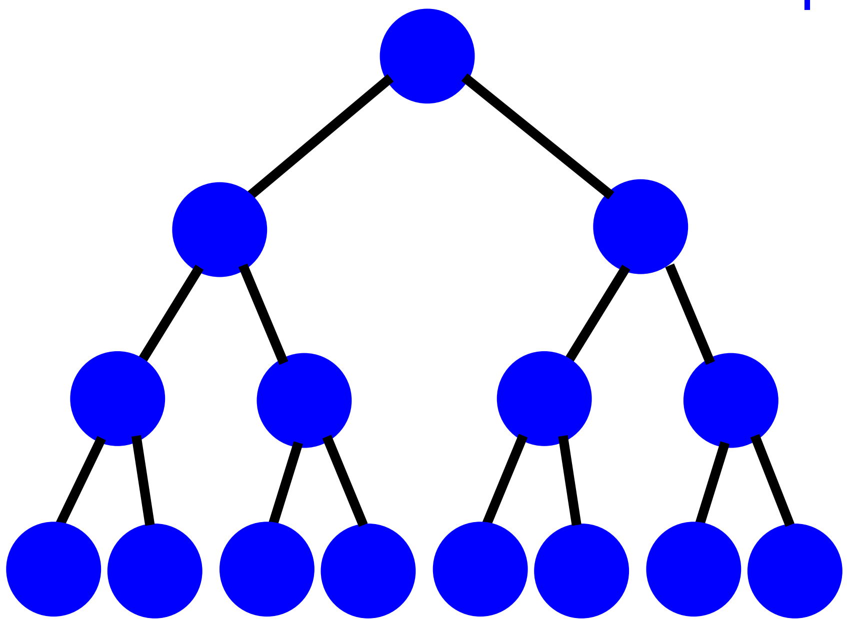

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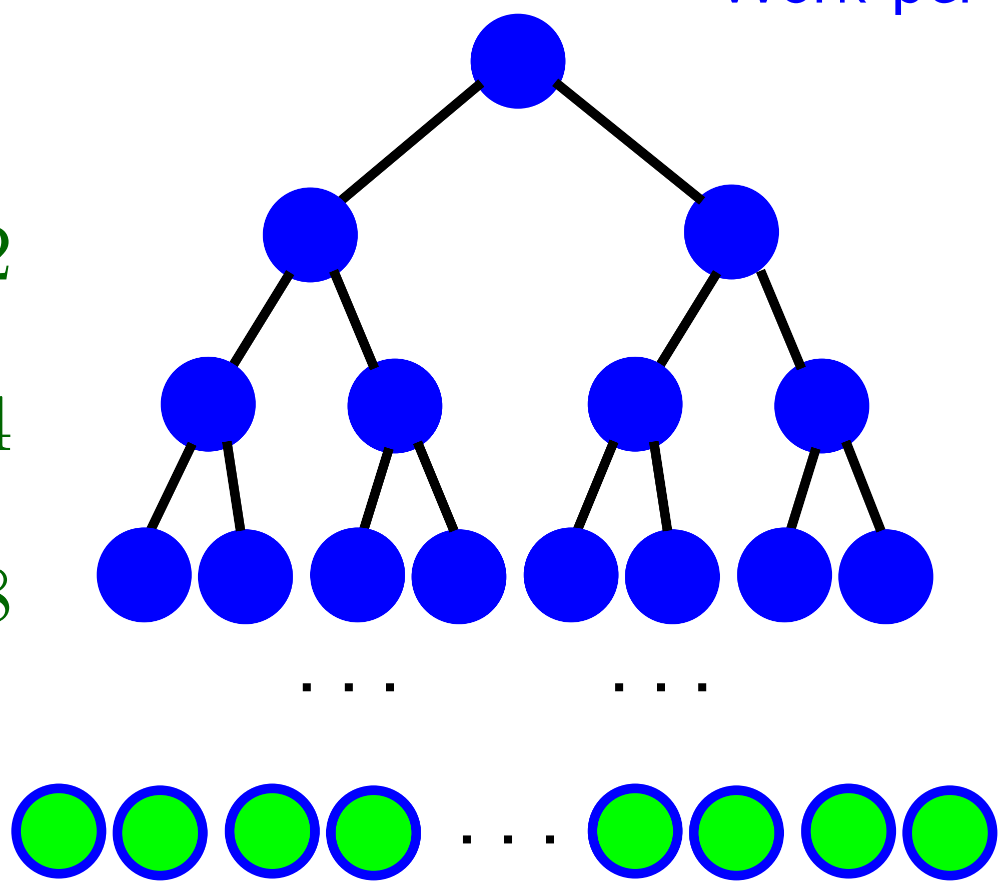

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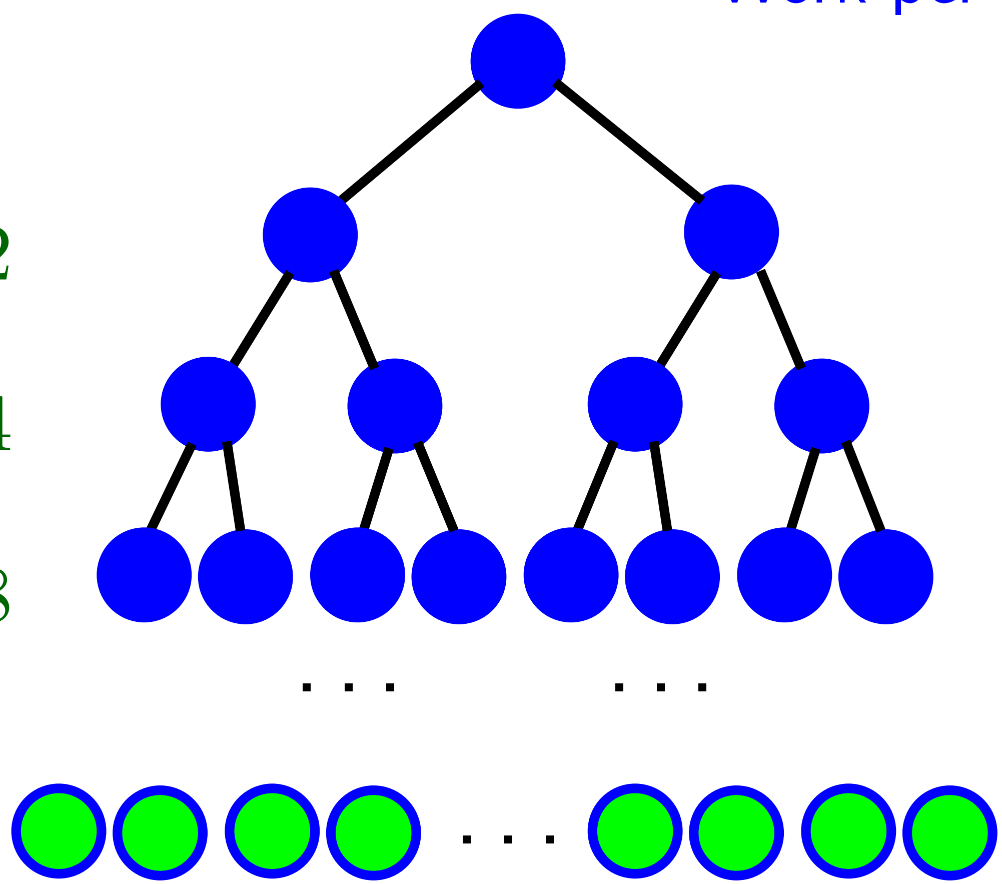

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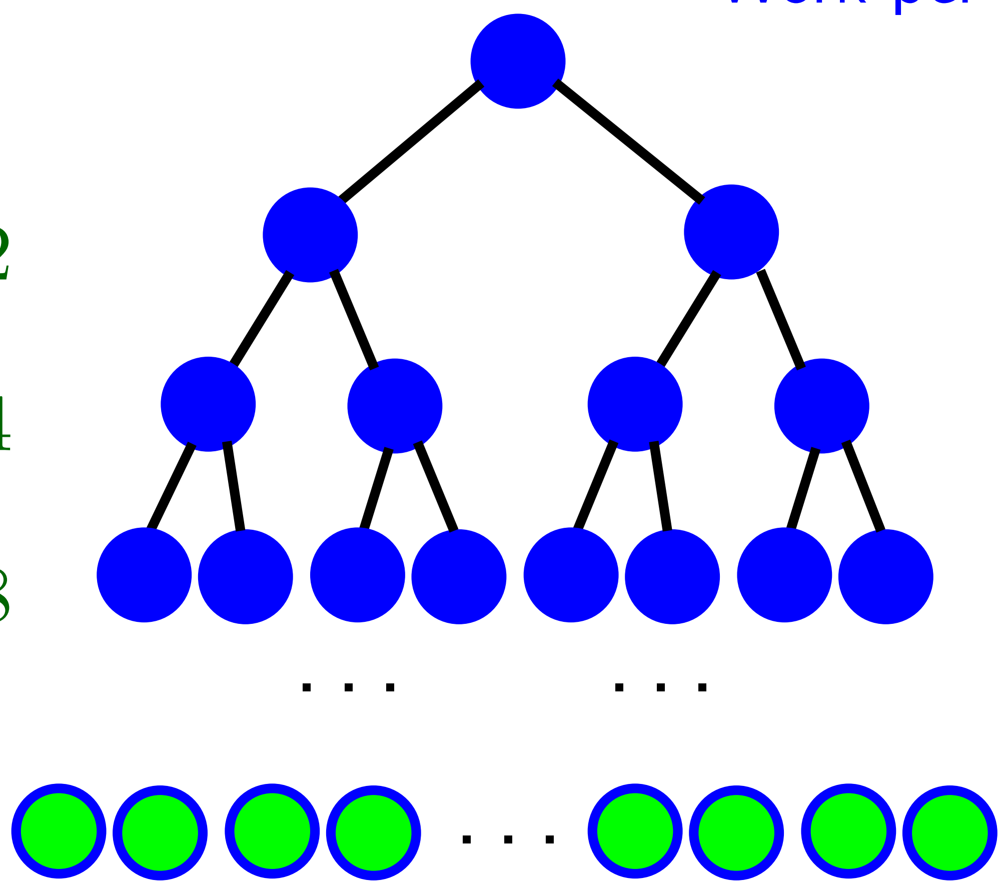

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$(1 + \log_2 n)$ levels $\Rightarrow$ total work $= n \log_2 n + 2n$

We just derived that the solution to

$$T(n) = \begin{cases} 2T(n/2) + n & \text{if } n \geq 2, \\ T(1) & \text{if } n = 1. \end{cases}$$

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We derived this in two different, but similar, ways; first by algebraically iterating the recurrence. Second by drawing the recursion tree.

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We derived this in two different, but similar, ways; first by algebraically iterating the recurrence. Second by drawing the recursion tree.

Note: Technically, we still need to use induction to prove *that our solution is correct. Practically, we never explicitly perform this step, since it is obvious how the induction would work (the* $\boxed{\ldots}$ *in the algebraic iteration and the recursion tree, are really hiding an inductive step).*

Example 2

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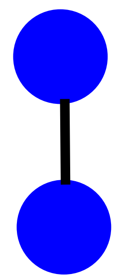
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	1	1
1	$n/2$		



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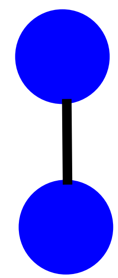

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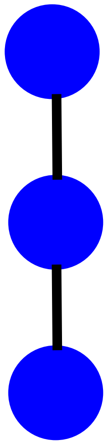
1	$n/2$	1	1
1	$n/4$		



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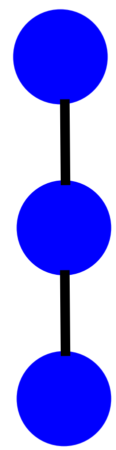
1	n		1	1
1	$n/2$		1	1
1	$n/4$		1	1

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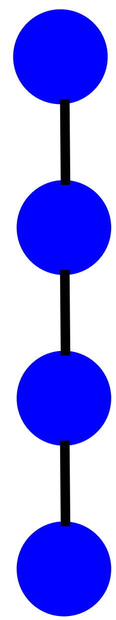

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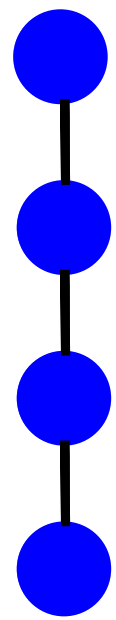

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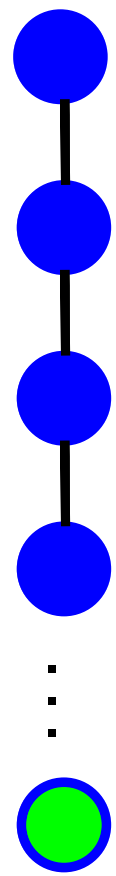

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$\vdots$	$\vdots$		
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$\vdots$	$\vdots$	$\vdots$	$\vdots$
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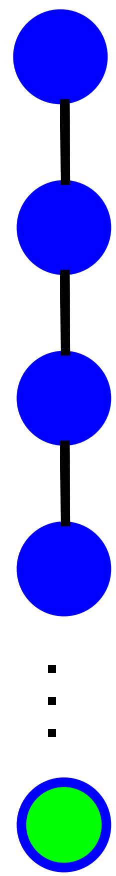

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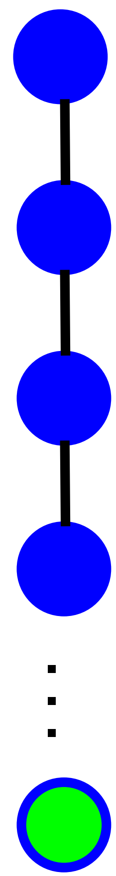
1	n	1	1
1	$n/2$	1	1
1	$n/4$	1	1
1	$n/8$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$1 = n/2^{\log_2 n}$	1	1



$(1 + \log_2 n)$ levels $\Rightarrow$ total work $= 1 + \log_2 n$

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$$T(n) = \begin{cases} T(n/2) + n & \text{if } n \geq 2, \\ 1 & \text{if } n = 1. \end{cases}$$

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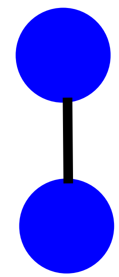
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$		



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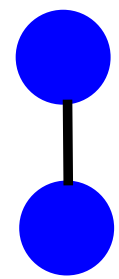

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$

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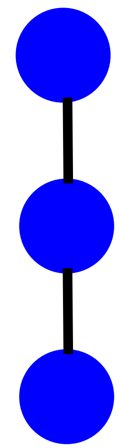
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$		



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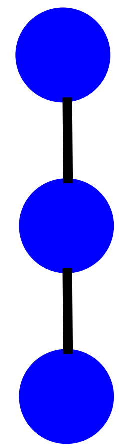

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$

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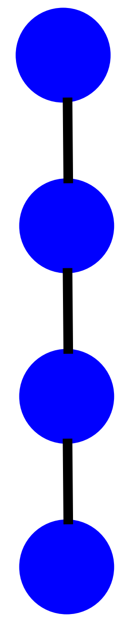

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$		

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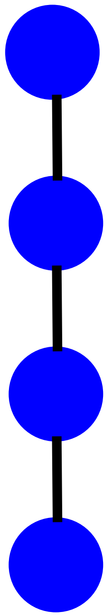
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$



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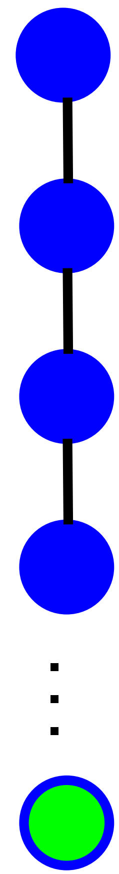

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$		
1	1		

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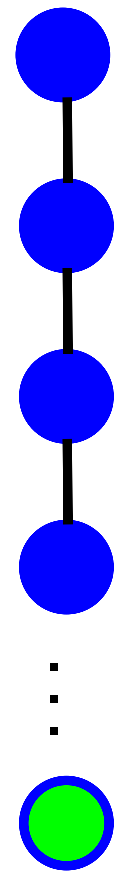

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1

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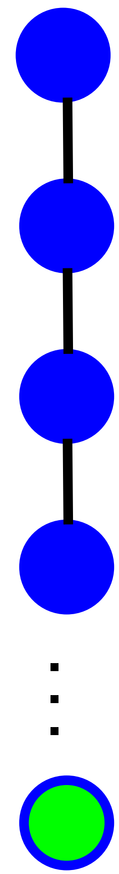

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
1	$n/2$	$n/2$	$n/2$
1	$n/4$	$n/4$	$n/4$
1	$n/8$	$n/8$	$n/8$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	1	1

$(1 + \log_2 n)$ levels. Total work $= n + \frac{n}{2} + \frac{n}{4} + \cdots + 2 + 1$

Total amount of work:

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Theorem 4.4 tells us that the value of the geometric series is $O(1)$ (in fact it is < 2) so, the total amount of work done is $O(n)$.

Example 4

$$T(n) = \begin{cases} 3T(n/3) + n & \text{if n} \geq 3, \\ 1 & \text{if n} < 3. \end{cases}$$

Example 4

assume n *is power of* 3

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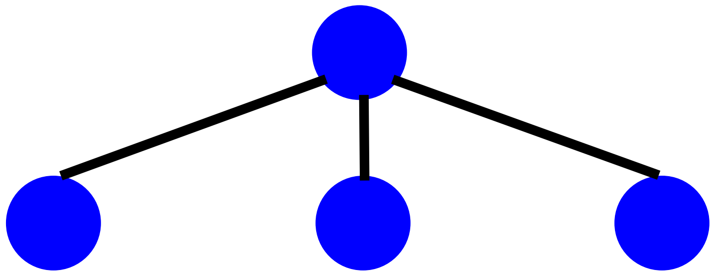
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$		



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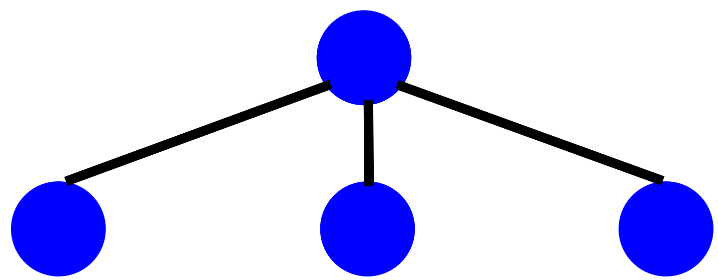
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$



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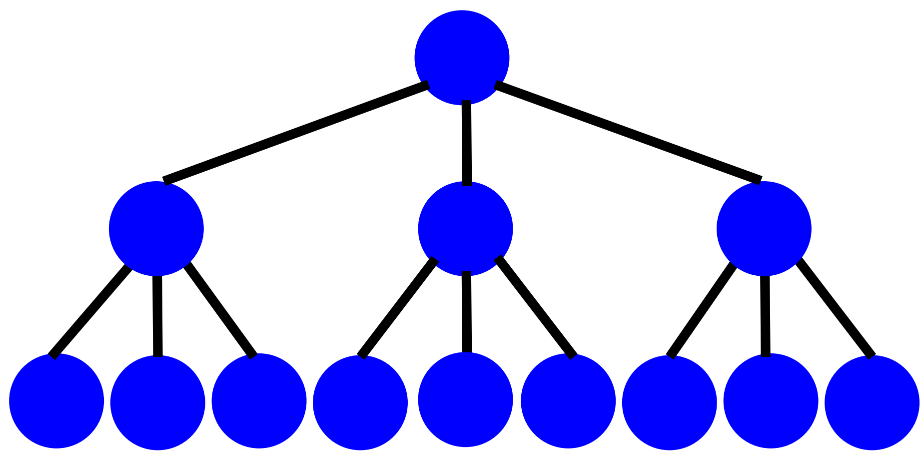

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$		

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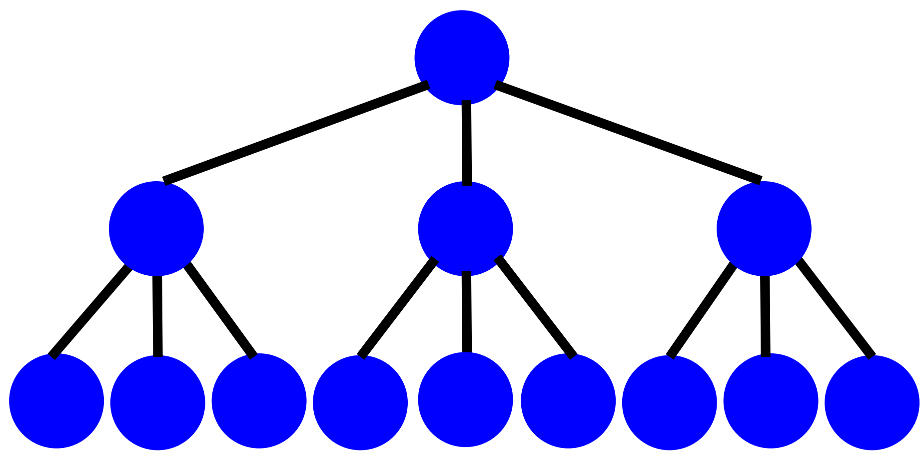

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$

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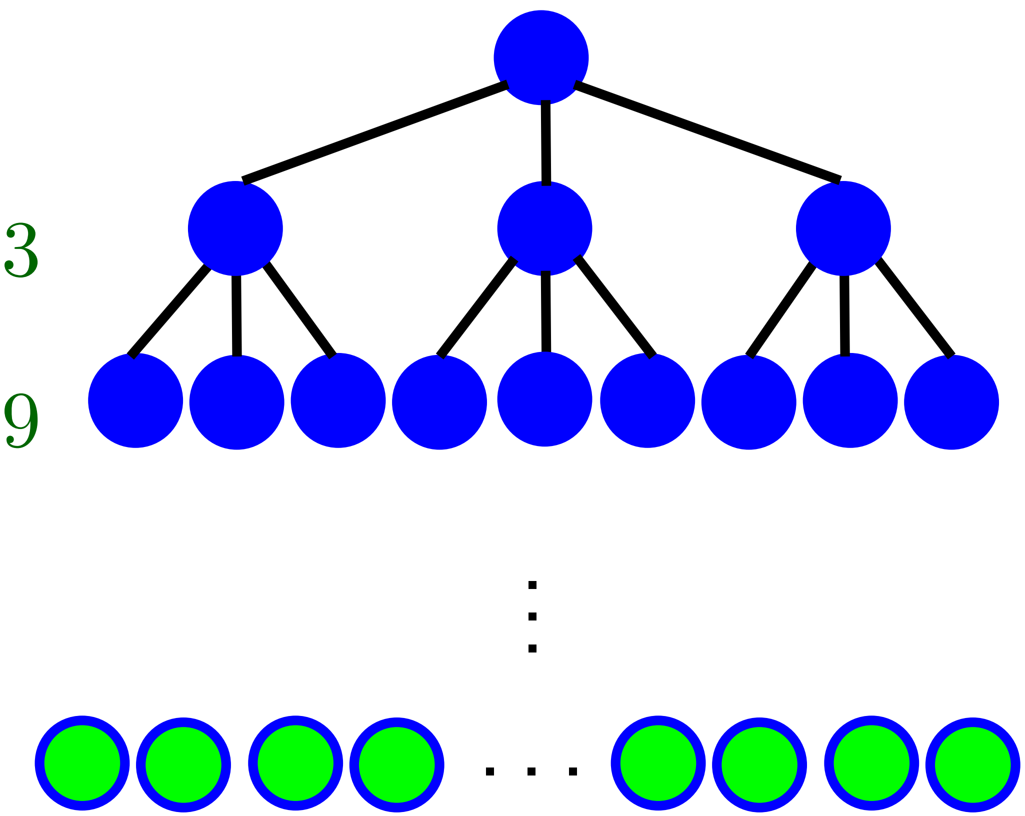

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
$\vdots$	$\vdots$		
$3^{\log_3 n} = n$	1		

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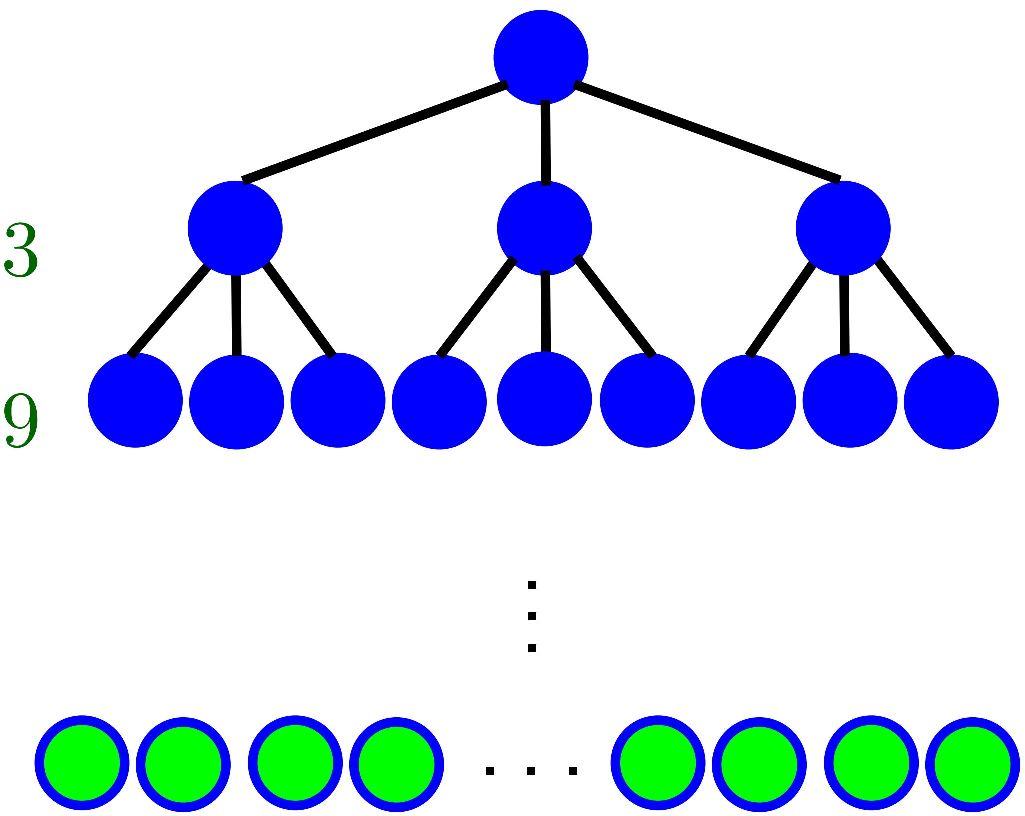

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{\log_3 n} = n$	1	1	$n(1) = n$

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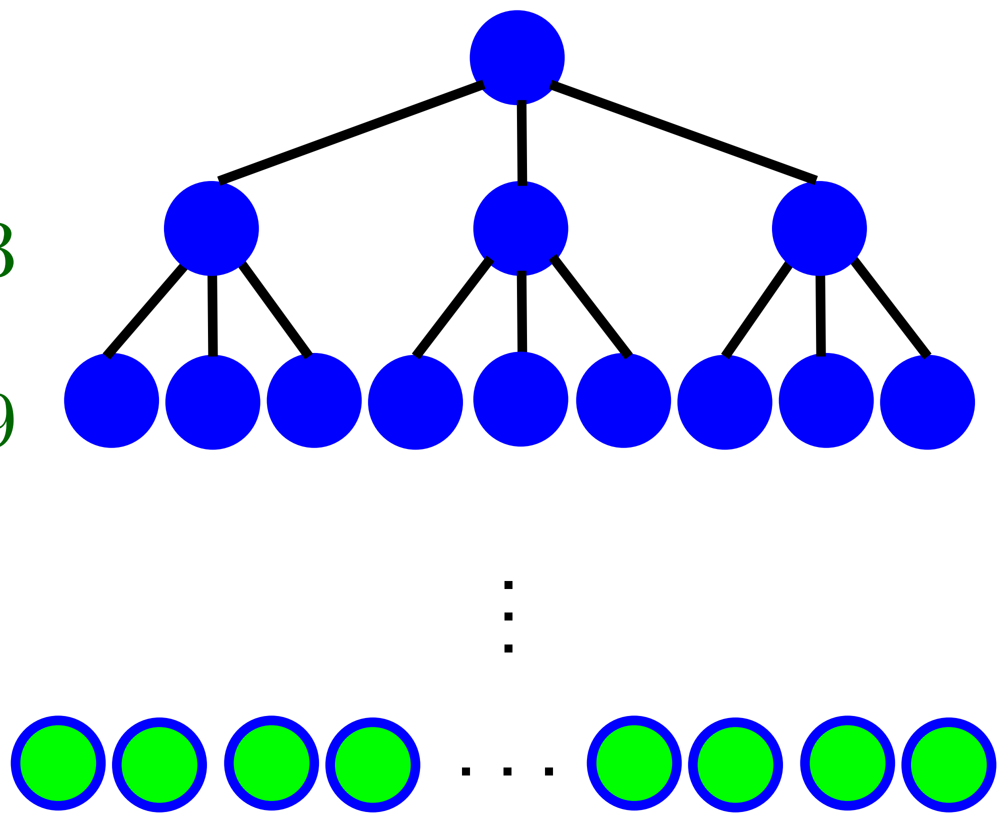
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
3	$n/3$	$n/3$	$3(n/3) = n$
$3^2 = 9$	$n/9$	$n/9$	$9(n/9) = n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$3^{\log_3 n} = n$	1	1	$n(1) = n$



$(1 + \log_3 n)$ levels $\Rightarrow$ total work $= n(1 + \log_3 n)$

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# of Probl	Probl Size	Work per Probl	Work per Level

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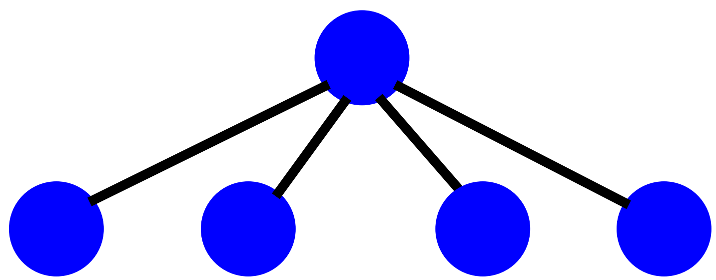
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$		



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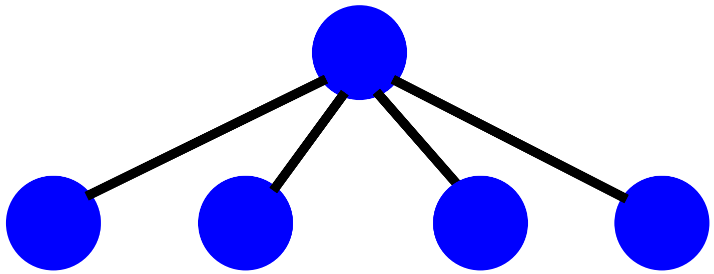
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$



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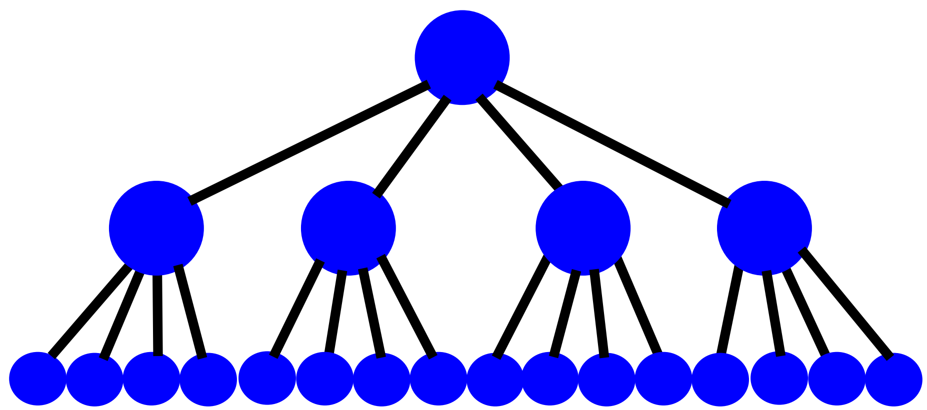

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$		

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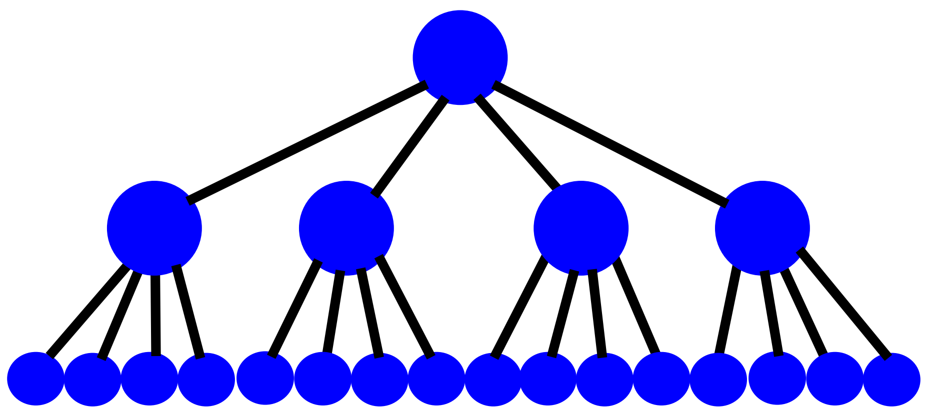

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$	$n/4$	$16(n/4) = 4n$

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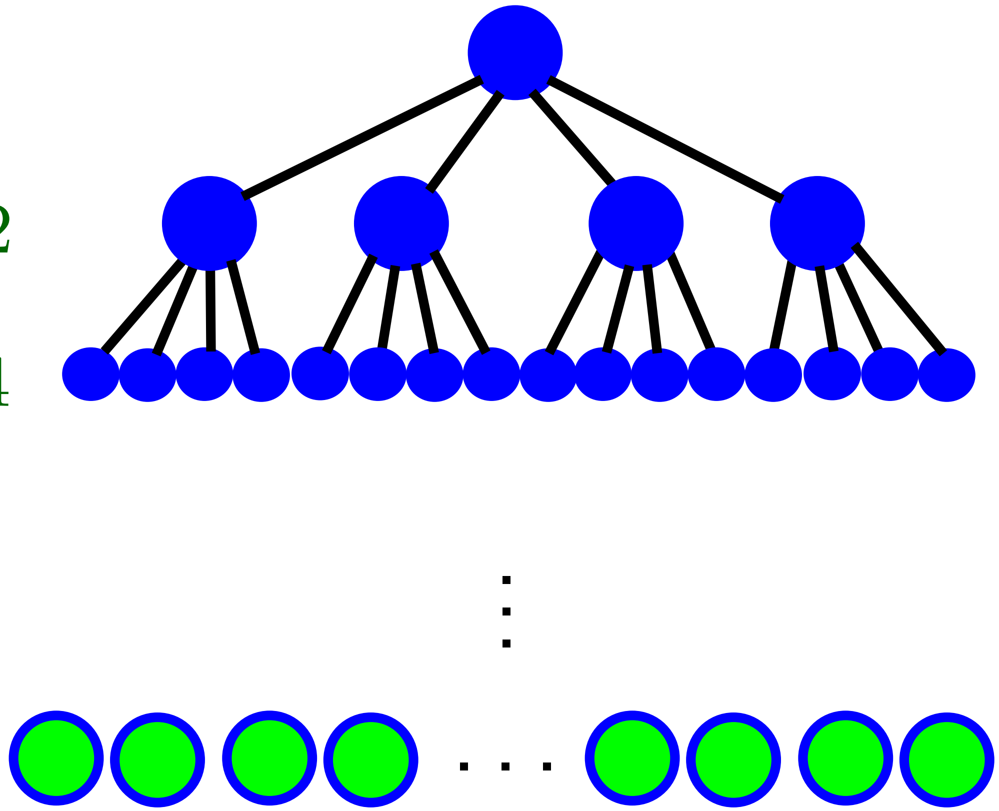
# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$	$n/4$	$16(n/4) = 4n$
$\vdots$	$\vdots$		
$4^{\log_2 n} = n^2$	1		



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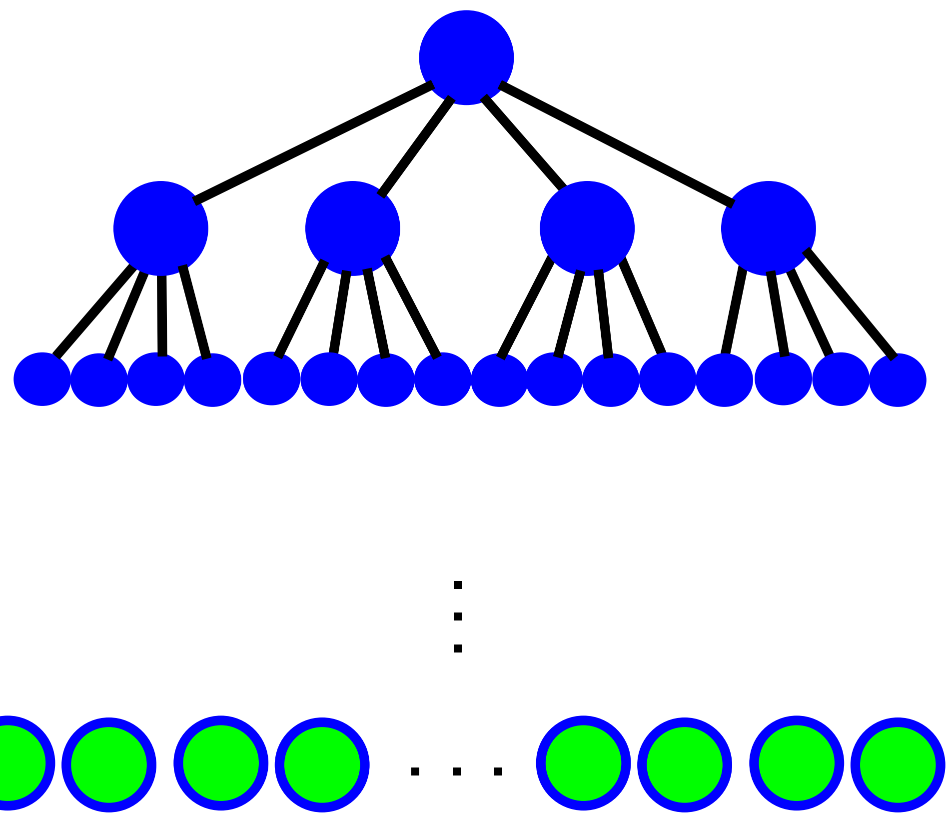

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$	$n/4$	$16(n/4) = 4n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4^{\log_2 n} = n^2$	1	1	$n^2 \cdot 1 = n^2$

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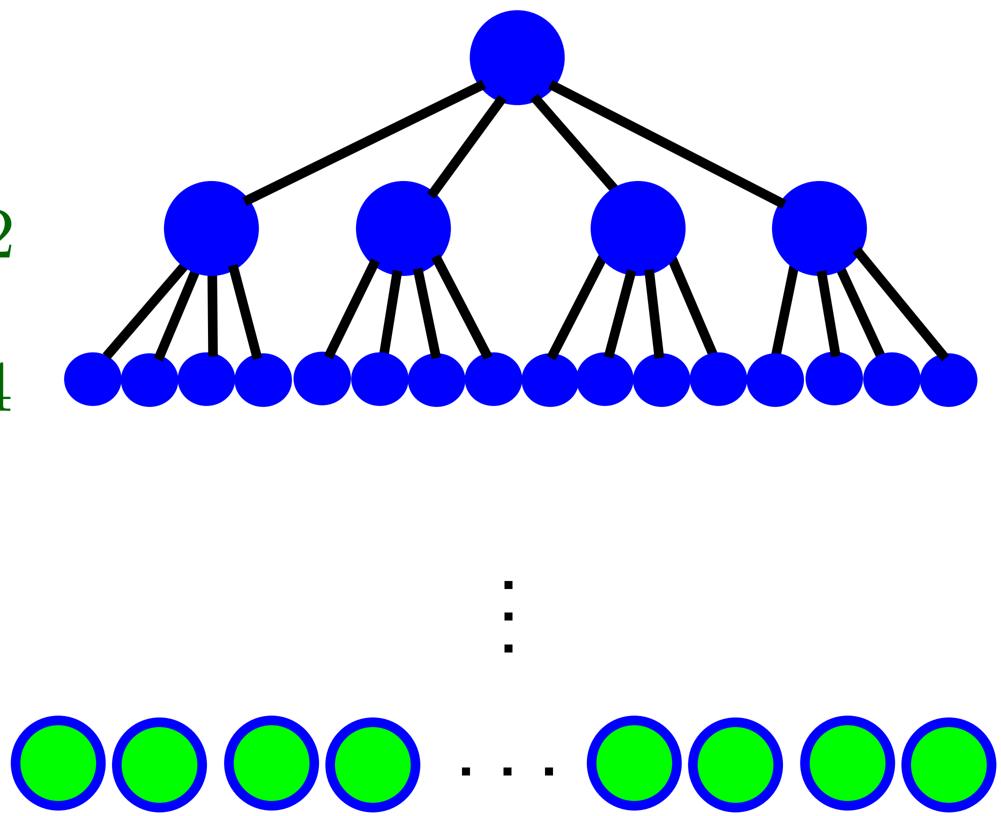

# of Probl	Probl Size	Work per Probl	Work per Level
1	n	n	n
4	$n/2$	$n/2$	$4(n/2) = 2n$
$4^2 = 16$	$n/4$	$n/4$	$16(n/4) = 4n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$4^{\log_2 n} = n^2$	1	1	$n^2 \cdot 1 = n^2$

total work $= n + 2n + 4n + \cdots + 2^{\log_2 n} n$

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$$= 2n^2 - n$$

Growth Rates of Solutions to Recurrences

- Divide and Conquer Algorithms
- Recursion Trees
- Three Different Behaviors

Three Different Behaviors

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Compare the recursion tree diagrams for the recurrences

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Three Different Behaviors

Compare the recursion tree diagrams for the recurrences

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$$T(n) = T(n/2) + n$$
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- all three trees have depth $1 + \log_2 n$
- in each case, the size of each subproblem is half the size of the parent problem
- differ in the amount of work done per level

Lemma 4.7:
Suppose that we have a recurrence of the form
$$T(n) = aT\left(\frac{n}{2}\right) + n,$$
where a is a positive integer and $T(1)$ is nonnegative.
Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

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We already proved Case 1 ($a = 1$) in Example 3.

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3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

Proof:

We already proved Case 1 ($a = 1$) in Example 3.

We already proved Case 2 in Example 1.

Lemma 4.7:
Suppose that we have a recurrence of the form
$$T(n) = aT\left(\frac{n}{2}\right) + n,$$
where a is a positive integer and $T(1)$ is nonnegative.
Then we have the following big Θ bounds on the solution:

1. If $a < 2$, then $T(n) = \Theta(n)$.
2. If $a = 2$, then $T(n) = \Theta(n \log n)$.
3. If $a > 2$, then $T(n) = \Theta(n^{\log_2 a})$.

Proof:

We already proved Case 1 $(a = 1)$ in Example 3.

We already proved Case 2 in Example 1.

We will now prove Case 3.

$T(n) = aT\left(\frac{n}{2}\right) + n$ where $a > 2$. Assume n is a power of 2.

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At Level i, there are a^i nodes,
each corresponding to a problem of size $n/2^i$.

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Summing over the $1 + \log_2 n$ levels, we get

$$a^{\log_2 n} T(1) \quad + \quad n \sum_{i=0}^{(\log_2 n)-1} \left(\frac{a}{2}\right)^i .$$

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Summing over the $1 + \log_2 n$ levels, we get

$$\underbrace{a^{\log_2 n} T(1)}_{\text{Work at bottom level}} \quad + \quad \underbrace{n \sum_{i=0}^{(\log_2 n)-1} \left(\frac{a}{2}\right)^i}_{\text{Work on non-bottom levels}} .$$

Total work is

$$a^{\log_2 n} T(1) + n \sum_{i=0}^{(\log_2 n)-1} \left(\frac{a}{2}\right)^i .$$

The sum is a geometric series.

Because $a/2 \neq 1$, Theorem 4.4 tells us that the sum will be big Θ of the largest term.

Because $a > 2$, the largest term in this case is clearly the last one, namely, $(a/2)^{(\log_2 n)-1}$.

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$$a^{\log_2 n} = \left(2^{\log_2 a}\right)^{\log_2 n} = \left(2^{\log_2 n}\right)^{\log_2 a} = n^{\log_2 a}$$

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and we are done!

As an example of Case 3 consider

$$T(n) = \begin{cases} 4T(n/2) + n & \text{if n} \geq 2, \\ 1 & \text{if n} = 1. \end{cases}$$

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$a = 4$ so the Theorem says that

$$T(n) = \Theta\left(n^{\log_2 a}\right) = \Theta\left(n^{\log_2 4}\right) = \Theta(n^2)$$

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This matches with the exact answer of $2n^2 - n$, which we already derived in Example 5.