opposite is continuous

1. Count how many ways to do something

seperate integers, sometimes fractions

Discrete Structures
Lecture 1: Counting

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2. Proofs

convince someone of answer

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3. Applications

probability, etc.

Abstract

This lecture covers combinatorial techniques to solve counting problems.

Contents

1 Notation

bold

1.1 Set Notation

- $\mathbb{N} = \{0, 1, 2, \ldots\}$ - Natural numbers (non-negative integers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ - Integers ($\mathbb{Z}^+$ - positive integers)
- $\mathbb{Q}$ - Rational numbers (fractions)
- $\mathbb{R} = (-\infty, \infty)$ - Real numbers ($\mathbb{R}^+$ - positive real numbers)
- $|S|$ - Cardinality (or size) of a set S (i.e., the number of elements in S) tell how many elements are in it
- $A \cup B$ - union of sets A and B (combine two sets) merge
- $A \cap B$ - intersection of sets A and B (set of elements in common)

1.2 Sigma Notation

Definition 1.1. Sigma notation is a type of shorthand notation used to represent a long sum in the symbolic form (called sigma sum)

$$\sum_{k=m}^{n} f(k) = f(m) + f(m+1) + f(m+2) + \ldots + f(n) \tag{1}$$

integers

where

- k - integer index (relative position of each term) that increments by one
- m - starting value of index

Σ

1. sigma
2. find formula for pattern
3. insert formula
4. insert range

$f(k) = ?$
$f(1) = 1$
$f(2) = 3$

- n - ending value of index
- $f(k)$ - formula that generates the value of each term in terms of k

Example 1.1. Examples of sigma notation.

not always precise, reader must recognize pattern

$f(k)=1$

(a) Manipulating sigma sums:

$$\sum_{k=1}^{50}\underbrace{(2k-1)}_{\text{formula}} = 1+3+5+\ldots+99$$
$$= (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3)-1+\ldots+(2\cdot 50-1)$$
$$= 2(1+2+3+\ldots+50)-(1+1+1+\ldots+1)$$

– group both the 1's & 2's

$$= 2\sum_{k=1}^{50}k - \sum_{k=1}^{50}1$$

more than 1 answer for sigma sum

simplify (split)

what formula you plug to get 1 (always)

(b) Changing initial value of index:

→ need to know how many terms in between

$$\boxed{152}+157+162+\ldots+352 = \sum_{k=0}^{40}(152+5k)$$
$$= \sum_{k=30}^{70}(5k+2)$$

ranges differ

$$= 5\sum_{k=30}^{70}k - \sum_{k=30}^{70}2$$
$$= 5\left(\sum_{k=1}^{70}k - \sum_{k=1}^{2}9k\right) - \sum_{k=1}^{41}2$$

→ find pattern

$30+31+\ldots 70$

$152+5\times 0$
$2+5\times 30$

(c) Sigma sums are NOT multiplicative:

$$1\cdot 2+2\cdot 3+\ldots+99\cdot 100 = \sum_{k=1}^{99}k(k+1) \neq \left(\sum_{k=1}^{100}k\right)\cdot\left(\sum_{k=1}^{99}(k+1)\right)$$

1.3 Pi Notation

Definition 1.2. Pi notation (similar to sigma notation) is used to represent a long product in the symbolic form (called pi product)

$$\prod_{k=m}^{n}f(k) = f(m)\cdot f(m+1)\cdot f(m+2)\cdots f(n) \tag{2}$$

Example 1.2. Examples of pi notation and factorials.

factorials

4 - 10 = 7

4 5 6 7 8 9 10

(a) $\prod_{k=1}^{10} k = 1 \cdot 2 \cdot 3 \cdots 10 = 10!$

10 falling 7

not factorial

(b) $\prod_{k=4}^{10} k = 4 \cdot 5 \cdot 6 \cdots 10 = (4)^7 = (10)_7 = \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdots 10}{1 \cdot 2 \cdot 3} = \dfrac{10!}{3!}$

4 rising 7

NOTE: $(4)^7$ and $(10)_7$ are different forms of Pochhammer notation

(c) $\prod_{k=1}^{10} (2k) = 2 \cdot 4 \cdot 6 \cdots 10 = 2^{10}(1 \cdot 2 \cdot 3 \cdots 10) = 2^{10} \cdot 10!$

even #s

(d) $\prod_{k=1}^{10} (2k+1) = 3 \cdot 5 \cdot 7 \cdots 21 = \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 20 \cdot 21}{2 \cdot 4 \cdot 6 \cdots 20} = \dfrac{21!}{2^{10} \cdot 10!}$

NOTE: We define $0! = 1$ (why?)

2 Basic Counting, a.k.a. Combinatorics (Section 1.1)

2.1 For-Loops

cocalc

□ indention shows it is in the for-loop

An important application of counting arises in computer programming where one needs to determine how many times for-loops are executed in order to estimate run-time.

Example 2.1. How many times is Line 3 executed in the following pseudo-code consisting of a two nested for-loops?

cmd line

range for i

(1) for $i = 1$ to m
(2) □ for $j = 1$ to n
(3) print (i, j)

NOTE: The indices i and j are called counters.

Solution. We construct a two-dimensional grid to list the set of values (i, j) that are printed (in order along rows):

tally row

how many pairs in each row

(i,j)	$j=1$	$j=2$	$j=3$	...	$j=n$
$i=1$	$(1,1)$	$(1,2)$	$(1,3)$	...	$(1,n)$
$i=2$	$(2,1)$	$(2,2)$	$(2,3)$	...	$(2,n)$
...	...	...	...	...	...
$i=m$	$(m,1)$	$(m,2)$	$(m,3)$	...	(m,n)

S-1, S-2, ..., S-m

Since the dimension of this grid is $m \times n$ (m rows by n columns), the total number of pairs (i, j) can be calculated by summing up the number of pairs, namely n, in each row:

$$\overbrace{n + n + \ldots + n}^{m \text{ times}} = \boxed{m \cdot n}$$

product principle

count students
group them
find size of each

This is the same as the number of times Line 3 is executed. Here, we are implicitly using a sum principle in terms of sets consisting of pairs in each row:

call all the pairs

$$\begin{aligned}
S_1 &= \{(1,1),(1,2),...,(1,n)\} = \{(1,j) : j = 1,2,...,n\}\\
S_2 &= \{(2,1),(2,2),...,(2,n)\} = \{(2,j) : j = 1,2,...,n\}\\
&\dots\\
S_m &= \{(m,1),(m,2),...,(m,n)\} = \{(m,j) : j = 1,2,...,n\}
\end{aligned}$$

don't expect same size

This is formalized in the following theorem.

Theorem 2.1 (Sum Principle)**.** *Let S be the union of a family of m mutually disjoint finite sets, i.e., $S = S_1 \cup S_2 \cup \ldots \cup S_m$. Then the cardinality of S is given by the sum*

no pairs in common

$$|S| = \left|\bigcup_{i=1}^{m} S_i\right| = |S_1| + |S_2| + \ldots + |S_m| \tag{3}$$

divide & conquer → divide into rows & add accordingly

NOTE: Useful mnemonic: Cardinality of union equals sum of cardinalities.

As a corollary, if $S_i = n$ for all $i = 1,2,...,m$, i.e., all sets are the same size as in the example above, then this illustrates the following product principle.

Corollary 2.2 (Product Principle)**.**

$$|S| = \left|\bigcup_{i=1}^{m} S_i\right| = |S_1| + |S_2| + \ldots + |S_m| = \overbrace{n + n + \ldots + n}^{m\ \textit{times}} = m \cdot n \tag{4}$$

Example 2.2. How many times is Line 3 executed in the following pseudo-code where the counter j for the inner for-loop depends on the counter i for the outer for-loop?

(1) for $i = 1$ to $n-1$ → range
(2) for $j = i+1$ to n → dependent on i
(3) print (i,j)

$i=1$ $j=i+1=2$, $j=3$ ... $j=n$

Solution. Again, we construct a two-dimensional grid to list the set of values (i,j) that are printed (in order along rows):

pairs

$i=2$ $j=i+1=3$

m rows

n columns

blank not in range

(i,j)	$j=1$	$j=2$	$j=3$	...	$j=n$
$i=1$		$(1,2)$	$(1,3)$	...	$(1,n)$
$i=2$			$(2,3)$	...	$(2,n)$
...	...	...	...	...	...
$n-1$ $\cancel{i=m}$					$\cancel{(m,n)}$ $(n-1,n)$

$n-1$
$n-2$
2
1

If we define the sets $S_i = \{(i,j) : j = i+1, i+2, ..., n\}$ (denoting the pairs in the i-th row of the table above) and apply the Sum Principle (observe that the Product Principle does not apply since the sets S_i are of different sizes), then the total number of such pairs (i,j) is given by a sum consisting of consecutive integers:

$$|S| = \left|\bigcup_{i=1}^{m} S_i\right| = |S_1| + |S_2| + \ldots + |S_m| = \underline{(n-1)} + \underline{(n-2)} + \ldots + \underline{1} \tag{5}$$

$n\left(\frac{n-1}{2}\right)$ → consecutive integers

handshake

$n=10$ $|S| = 9+8+7+\ldots+1$
$= 1+2+3+\ldots+9$

quickest: add opposite ends
$1+9$, $2+8$ ... 10 (5)

2.2 Summing Consecutive Integers

We present an efficient formula for summing consecutive integers, which is very useful since these sums appear frequently in counting problems.

last term $(n-1)$

Theorem 2.3 (Hand-Shake Formula)**.** consecutive integers

$$1+2+3+\ldots+(n-1)=\sum_{k=1}^{n-1} k=\frac{n(n-1)}{2} \tag{6}$$

Proof. We consider two different arrangements of the given sum:

take OG sum (s)
reverse sum
add equations

$$\begin{aligned} s &= 1+2+\ldots+(n-1) \\ s &= (n-1)+(n-2)+\ldots+1 \end{aligned}$$

Summing these two equations yields

always get the same sum (n)

$$2s=\overbrace{n+n+\ldots+n}^{(n-1)\text{ times}}=n(n-1)$$

Hence, we conclude $s=n(n-1)/2$ as desired. ∎

NOTE: This formula can also be proven geometrically by observing that the set $S = S_1 \cup \ldots \cup S_n$ in the previous example is half of the full $n \times n$ grid (with the main diagonal deleted).

more about how you got it

Exercise 2.1 (Hand-Shake Problem)**.** Suppose 100 people attend a party. How many hand-shakes occurred if everyone shakes hands with each other?

$100 \cdot \frac{99}{2}$

$99+98\ldots$

2.3 Does Order Matter?

Definition 2.1. Let x and y denote two elements.

(a) An *ordered pair* (x, y) (also called a two-element *tuple*) is a collection of two elements (we allow $x = y$) where the order in which the elements are listed is distinguished, i.e., order matters. If the elements x and y are assumed to be distinct, then (x, y) is also called a two-element *permutation*.

(b) A two-element *set* $\{x, y\}$ is a collection of two distinct elements ($x \neq y$) where order does not matter.

Example 2.3.

(a) The ordered pair (2, 5) (or permutation) is considered different from the ordered pair (5, 2). Also, (2, 2) is considered an ordered pair.

(b) The set $\{2, 5\}$ is the same as $\{5, 2\}$. Also, $\{2, 2\}$ is considered a one-element set (NOT two).

Remark. Ordered pairs and two-element subsets are different types of collections; one cannot compare them in general, although they may be related to each other in certain contexts as the following example demonstrates.

Example 2.4. How many two-element subsets can be constructed from the set of n elements (pool of possible choices)?

Solution. Denote the elements by the numbers in the set $S = \{1, 2, \ldots, n\}$. Let's first construct all possible ordered pairs (i, j) where i and j are chosen from S:

(i,j)	$j=1$	$j=2$	$j=3$	$\ldots$	$j=n$
$i=1$	$\cancel{(1,1)}$	$(1,2)$	$(1,3)$	$\ldots$	$(1,n)$
$i=2$	$(2,1)$	$\cancel{(2,2)}$	$(2,3)$	$\ldots$	$(2,n)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$i=n$	$(n,1)$	$(n,2)$	$(n,3)$	$\ldots$	$\cancel{(n,n)}$

If we delete ordered pairs where $i = j$ (marked in the grid above), then observe that there is a 2-to-1 correspondence between permutations ($i \neq j$) and two-element subsets, namely each subset $\{x, y\}$ corresponds to two permutations, (x, y) and (y, x):

$$\begin{gathered}\{1,2\} \leftrightarrow (1,2),(2,1) \\ \{1,3\} \leftrightarrow (1,3),(3,1) \\ \ldots \\ \{n-1,n\} \leftrightarrow (n-1,n),(n,n-1)\end{gathered}$$

Since the number of permutations in the grid is $n^2 - n$ (subtract those n ordered pairs where $i = j$ from the total), it follows that the number of subsets is half this amount, i.e., $n(n-1)/2$.

3 Counting Different Types of Collections (Section 1.2)

Definition 3.1. When a collection of objects is formed and each element is chosen from a different pool, then the collection is generically called a LIST. In this case, the order of the elements must be specified before the list can be written down (any ordering can be used since they all describe the same collection).

Definition 3.2. When objects are chosen from the same pool, there are four different types of collections (TUPLE, PERMUTATION, SUBSET, MULTI-SET) depending on whether order matters and repetition is allowed:

handout

SAME POOL FOR EACH CHOICE	Order Matters	Order Does Not Matter
Repetition Allowed	TUPLE ("List")	MULTI-SET
Repetition Not Allowed	PERMUTATION	SUBSET ("Combination")

Example 3.1. How many different 3-character passwords can be created assuming the first character is any upper-case letter (A-Z), the second character is any digit (0-9), and the third character is any lower-case letter (a-z).

tuple list

Solution. We view each password as a 3-element list and tabulate them in a 3-dimensional box grid (consisting of 26 layers stacked on top of each other):

26

all varying except the letter

Layer A

10

$(A, 0, a)$	$(A, 0, b)$	...	$(A, 0, z)$
$(A, 1, a)$	$(A, 1, b)$	...	$(A, 1, z)$
...	...	...	...
$(A, 9, a)$	$(A, 9, b)$	...	$(A, 9, z)$

10 26

Layer B

$(B, 0, a)$	$(B, 0, b)$	...	$(B, 0, z)$
$(B, 1, a)$	$(B, 1, b)$	...	$(B, 1, z)$
...	...	...	...
$(B, 9, a)$	$(B, 9, b)$	...	$(B, 9, z)$

0 1 2 3 4 5

6 7 8 9 10

10 26

Layers are A-Z

26 letters

...

Layer Z

$(Z, 0, a)$	$(Z, 0, b)$	...	$(Z, 0, z)$
$(Z, 1, a)$	$(Z, 1, b)$	...	$(Z, 1, z)$
...	...	...	...
$(Z, 9, a)$	$(Z, 9, b)$	...	$(Z, 9, z)$

10 26

$26(10 \cdot 26)$

Since each layer has dimensions 10×26 and there are 26 layers, the number of such passwords equals the volume of the box grid, $26 \cdot 10 \cdot 26 = 6760$. Observe that the answer involves multiplying the number of choices in the pool for each character. This is an application of the following Product Principle.

Theorem 3.1 (Product Principle (Version 2)). *Let S be a collection consisting of m-element lists (or tuples or permutations) where order matters. Suppose*

1. *There are $\boxed{i_1}$ choices for choosing the first element*
2. *There are $\boxed{i_2}$ choices for choosing the second element (given any fixed choice for the first element)*
3. *There are i_j choices for choosing the j-th element (given any fixed choices for the first $j - 1$ elements.*

Then the total number of lists in S is given by

mish if it's a collection

$$|S| = i_1 \cdot i_2 \cdots 1_m = \prod_{j=1}^{m} i_j \tag{7}$$

f $S = \{1, 2, 3\}$

$T = \{1, 2, 3, 4\}$

$f(1) = 3$ $\longleftrightarrow$ $(3, 1, 4)$

$f(2) = 1$ order?

$f(3) = 4$ rep?

S = domain
T = range

3.1 Functions as Lists

rule where you input and it gives an output

Definition 3.3. A *function* $\boxed{f : S \to T}$ is a relationship between the elements of a set S (called the *domain*) and the elements of a set T (called the *range*) such that each element $s \in S$ corresponds to exactly one element $t \in T$. We shall write $f(s) = t$ or as an ordered-pair $(x, y) \in f$ where $y = f(x)$ to express this relationship. The set of elements $\text{im}(f) = f(S) = \{f(s) : s \in S\}$ is called the *image* of S.

integers

Lemma 3.2. *If $S = \{1, 2, \ldots, k\}$ denotes a set of k-elements and $f : S \to T$ is a function, then the image of S, i.e.,*

$$f(S) = \{f(1), f(2), \ldots, f(k)\}$$

is a k-element tuple consisting of elements in T.

3 is size of domain

Example 3.2. How possible distinct functions can be defined where each maps the set $S = \{1, 2, 3\}$ to the set $T = \{1, 2, 3, 4\}$?

4 is the size of range

Solution. We equate each function $f : S \to T$ with its image, which we view as a 3-element tuple:

4 4 4 write out output

range domain

$$f \leftrightarrow f(S) = (f(1), f(2), f(3))$$

Since there are 4 choices for choosing each of the elements $f(1)$, $f(2)$, and $f(3)$, it follows from the Product Principle that the number of such tuples (same as the number distinct functions f) equals $4 \cdot 4 \cdot 4 = \boxed{4^3} = 64$.

how many choices of output

Theorem 3.3. *Let $S = \{1, 2, \ldots, m\}$ and $T = \{1, 2, \ldots, n\}$. Then the number of distinct functions $f : S \to T$ equals n^m.*

Definition 3.4 (One-to-one)**.** A function f is called one-to-one (or an injection) if it satisfies the property: $(x, z) \in f$ and $(y, z) \in f$ implies $x = y$, or equivalently, $f(x) = f(y) \to x = y$.

Remark. The contrapositive of this definition (to be defined later) is: $x \neq y \to f(x) \neq f(y)$.

Example 3.3. How many distinct one-to-one functions can be defined where each maps the set $S = \{1, 2, 3\}$ to the set $T = \{1, 2, 3, 4\}$?

how many choices do you have

Solution. Since the values for $f(1)$, $f(2)$, and $f(3)$ must all be distinct (f is a one-to-one function), the number of choices for choosing each element equals 4, 3, and 2, respectively. Thus, the number of one-to-one functions (viewed as 3-element permutations) equals $4 \cdot 3 \cdot 2 = 24$ by the Product Principle.

$\frac{f(1)}{4 \text{ choices}}$ $\frac{f(2)}{3}$ $\frac{f(3)}{2}$

Definition 3.5 (Onto)**.** Let $f : S \to T$. We say that f is onto T (or a surjection) provided that for every element $t \in T$ there is at least one element $s \in S$ so that $f(s) = t$, i.e., $\text{im}(f) = T$ (image equals range). Thus, every t is the image of some s.

Example 3.4.

(a) Find a function $f : S \to T$ that is onto but NOT one-to-one. HINT: Which set has more elements then, S or T?

(b) Find a function f mapping S to T (or vice versa) that is one-to-one but NOT onto.

(c) How many distinct onto functions can be defined mapping the set $S = \{1, 2, 3\}$ to itself?

(d) Challenge: How many distinct onto functions can be defined mapping the set $T = \{1, 2, 3, 4\}$ to the set $S = \{1, 2, 3\}$? HINT: Use Principle of Inclusion-Exclusion (to be discussed later)

Definition 3.6 (Bijection Principle)**.** Two sets S and T (possible infinite) are said to have the same cardinality (or size) if there exists a one-to-one correspondence (or bijection) between the elements of S and the elements of T, i.e., there exists a function $f : S \to T$ that is both one-to-one and onto.

Remark. A permutation can be viewed as the image of a bijection from a finite set to itself.

3.2 Counting Permutations

Definition 3.7. Let S be a set consisting of n elements. A k-element permutation of S a permutation P whose elements, chosen from S, are all distinct (thus $k \leqslant n$). If $k = n$, then P is also said to be a permutation of S (arrangement of its elements).

3 slots

Example 3.5. Let $S = \{1, 2, 3, 4\}$ (pool) and X denote the collection of all 3-element permutations of S, which we enumerate (numeric order):

$$X = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}$$

$(\underline{4}, \underline{3}, \underline{2})$ how many choices

The following theorem, similar to the Product Principle, gives a formula to efficiently count the number of permutations in X, namely $4 \cdot 3 \cdot 2 = 24$.

Theorem 3.4. *The number of k-element permutations of an n-element set is given by the k-th falling factorial power of n:*

to count

$$\{1, 2, 3, \ldots n\} \quad (n)_k = n(n-1)(n-2)\cdots(n-k+1) = \boxed{\frac{n!}{(n-k)!}} \tag{8}$$

1 2 3 k

n n-1 n-2 ...

Remark. Your textbook uses the alternate notation $n^{\underline{k}}$ instead of $(n)_k$.

permutation

3.3 Counting Subsets

order doesn't matter, repetition not allowed

Definition 3.8. We write $\binom{n}{k}$ or $C(n, k)$, called a *binomial coefficient* and pronounced "n choose k", to denote the number of k-element subsets that can be made by choosing elements from a set S containing n elements.

combination

Example 3.6. The binomial coefficient $\binom{4}{3}$ denotes the number of 3-element subsets that can be made by choosing elements from a set S containing 4 elements, say $S = \{1, 2, 3, 4\}$. In particular, there are 4 such subsets: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$. Thus, $\binom{4}{3} = 4$.

The following table enumerates all k-element subsets of S for $k = 0, 1, \ldots, 4$.

empty curly brackets

$n=4$

$\binom{4}{5} = 0$ can't pick more than pool here

empty set

repetition not allowed

example

k	k-element subsets of $S = \{1,2,3,4\}$	$\binom{4}{k}$ = Number of k-element subsets
0	$\varnothing$ = {}	$\binom{4}{0} = 1$
1	$\{1\}, \{2,\}, \{3\}, \{4\}$	$\binom{4}{1} = 4$
2	$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$	$\binom{4}{2} = 6$
~~1~~ 3	$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$	$\binom{4}{3} = 4$
~~1~~ 4	$\{1,2,3,4\}$	$\binom{4}{4} = 1$

3.3.1 Formula for Binomial Coefficients

To obtain an efficient formula for computing the value of a binomial coefficient, we carefully study the connection between k-element permutations (no repetition of elements) and k-element subsets through an example.

$n=5$

Example 3.7. Let $S = \boxed{\{1,2,3,4,5\}}$ be a set with 5 elements. The following table shows a six-to-one correspondence between 3-element permutations and 3-element subsets of S, namely each subset $\{a_1, a_2, a_3\}$ is represented by a collection (equivalence class) of six permutations where each contains the elements a_1, a_2, a_3. grouping

how many classes

how many permutations $3 \cdot 2 \cdot 1 = 6$

3-element subset	3-element permutations (equivalence class)
$\{1,2,3\}$	123, 132, 213, 231, 312, 321 6
$\{1,2,4\}$	124, 142, 214, 241, 412, 421
$\{1,2,5\}$	125, 152, 215, 251, 512, 521
...	...
$\{3,4,5\}$	345, 354, 435, 453, 534, 543

$1 \leftrightarrow 3!$

#subsets = #classes

Thus, counting the number of subsets in the table above is the same as counting the number of equivalence classes of permutations. To count the latter, observe that each equivalence class has exactly six permutations because of the Product Principle: given a 3-element subset $\{a_1, a_2, a_3\}$ as a pool, there are $3 \cdot 2 \cdot 1 = 3! = 6$ ways to make a 3-element permutation. Since the total number of 3-element permutations equals $5 \cdot 4 \cdot 3 = 60$, it follows that

1-1

$$\text{Number of equivalence classes} = \frac{\text{Number of 3-element permutations}}{\text{Size of each equivalence class}}$$

5 from domain 5

$$= \frac{5 \cdot 4 \cdot 3}{3!} = \frac{60}{6} = 10$$

Thus,

$$\text{Number of 3-element subsets} = \text{Number of equivalence classes} = 10$$

Remark. The quotient formula used above to calculate the number of equivalence classes is an application of the Quotient Principle that will be formally discussed later.

More generally, let $S = \{1, 2, \ldots, n\}$ be a set containing n elements. To count the number of k-element subsets of S, i.e., those made by choosing elements from S, we first count all k-element permutations of S:

$$\text{Number of } k\text{-element permutations} = (n)_k = n(n-1)(n-2)\cdot(n-k+1) \quad \frac{n!}{(n-k)!}$$

relationship $1 \leftrightarrow k!$

Next, observe that each k-element subset $\{a_1, \ldots, a_k\}$ corresponds to an equivalence class of k-element permutations (each containing the elements $a_1, a_2, \ldots, a_k$) whose size equals

$$\text{Size of each equivalence class} = k(k-1)(k-2)\cdots 1 = k!$$

because of the Product Principle. It follows that

$$\begin{aligned}
\text{Number of } k\text{-element subsets} &= \text{Number of equivalence classes} \\
&= \frac{\text{Number of } k\text{-element permutations}}{\text{Size of each equivalence class}} \\
&= \frac{(n)_k}{k!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \\
&= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)(n-k-1)\cdot 1}{(n-k)(n-k-1)\cdot 1} \\
&\boxed{= \frac{n!}{k!(n-k)!}}
\end{aligned}$$

counting subsets

Thus, we have just derived an efficient formula for computing binomial coefficients, which we formally state as a theorem.

Theorem 3.5. *Let n and k be non-negative integers such that $0 \leqslant k \leqslant n$. Then the number of k-element subsets that can be made from an n-element set equals*

n choose k

$$\binom{n}{k} = C(n,k) = \frac{n!}{k!(n-k)!} \tag{9}$$

Example 3.8. Given a class of 10 students, how many ways can a team of 6 students be formed?

Solution. We view each team as a 6-element subset since normal usage of the word "team" is taken to mean an unordered collection of distinct students. It follows that the number of such teams is the same as the number of 6-element subsets chosen from 10 elements, which we calculate by applying the theorem above:

what kind of collection?
- order?
- repetition

handout 1

team: order doesn't matter

team: can't pick one person twice, so repetition not allowed

$$\binom{10}{6} = \frac{10!}{6!4!} = 210$$

4 Binomial Coefficients (Section 1.3)

Recall that the quantity $\binom{n}{k}$ (number of k-element subsets from a pool of n elements) is called a binomial coefficient. In this section we explain why the term "binomial" is used.

$$\frac{\cancel{10}^{5} \cdot \cancel{9}^{3} \cdot \cancel{8}^{2} \cdot 7 \cdot \cancel{6!}}{\cancel{6!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}_{1} \cdot 1} = 210$$

simplification

$\binom{4}{2} = 6$

4.1 Figurate Triangle

The following table, called the Figurate Triangle, gives values of binomial coefficients $\binom{n}{k}$ for small values of n (pool size) and k (subset size).

k

answers

$\binom{5}{2} = 10$

n

$\binom{n}{k}$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	2	1			
$n=3$	1	3	3	1		
$n=4$	1	4	6	4	1	
$n=5$	1	5	10	10	~~4~~ 5	1

each row starts w/ 1,

each column begins w/ 1

right triangle

4.2 Pascal's Triangle

A symmetric version of the Figurate Triangle is known as Pascal's Triangle.

$\binom{5}{2} \rightarrow$ where in table

$6 = \binom{?}{?} = \binom{4}{2}$

n	$\binom{n}{0} \cdots \binom{n}{k} \cdots \binom{n}{n}$
$n=0$	1
$n=1$	1 1
$n=2$	1 2 1
$n=3$	1 3 3 1
$n=4$	1 4 6 4 1
$n=5$	1 5 10 10 5 1

k=0 k=2

10 equal to 2 #s above it (4+6)

↳ Pascal's relationship

We observe the following interesting patterns in Pascal's Triangle:

1. The first and last entry in each row equals 1, e.g., $\binom{5}{0} = \binom{5}{5} = 1$.
2. The second entry in each row equals the row position, e.g., $\binom{5}{1} = 5$.
3. The entries in each row are symmetric, e.g., $\binom{5}{2} = \binom{5}{3} = 10$.
4. Each entry is equal to the sum of the two entries in the row above it (Pascal relationship), e.g.,
$$\binom{5}{2} = 10 = 4 + 6 = \binom{4}{1} + \binom{4}{2}$$
5. Each entry is equal to the sum of the entries along an adjacent diagonal up to that entry, e.g.,
$$\binom{4}{2} = 6 = 1 + 2 + 3 = \binom{1}{1} + \binom{2}{1} + \binom{3}{1}$$

$\frac{5!}{2!3!} = \frac{5!}{3!2!}$

2 el. = 3 el. why? every two set you have stuff you choose some you don't

↳ compliments

$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

These patterns are formally stated in the next theorem.

Theorem 4.1. *The following properties hold for Pascal's Triangle:*

1. $\binom{n}{0} = \binom{n}{n} = 1$
2. $\binom{n}{1} = n$
3. $\binom{n}{k} = \binom{n}{n-k}$
4. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ *(Pascal's relationship)*
5. $\binom{n}{k} = \binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n-1}{k-1}$

know

Proof of Pascal's relationship (PR). We give a combinatorial proof where we pose a counting problem and solve it in two different ways, each leading to one side of PR. Consider the problem of counting the number of k-element subsets chosen from a pool of n elements, say $S = \{1, 2, \ldots, n\}$.

Solution 1: One solution is simply $\binom{n}{k}$ (by definition), which gives the left-hand side of PR.

Solution 2: On the other hand, another way to count these subsets is to divide them into two types: those that contain the largest element n and those that do not. For example, we can divide 2-element subsets of $S = \{1, 2, 3, 4, 5\}$ as follows:

Type I (contains $n = 5$): $\{1, \mathbf{5}\}$, $\{2, \mathbf{5}\}$, $\{3, \mathbf{5}\}$, $\{4, \mathbf{5}\}$

Type II (does not contain $n = 5$): $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$

Observe that subsets of Type I (with $n = 5$ removed) correspond to 1-element subsets of $T = \{1, 2, 3, 4\}$ and those of Type II correspond to 2-element subsets of T. It follows that the number of 2-element subsets equals

$$\binom{4}{1} + \binom{4}{2}$$

In general, the number of k-element subsets of Type I (with element n removed) correspond to $(k-1)$-element subsets of $T = \{1, 2, \ldots, n-1\}$ and those of Type II correspond to $(k-1)$-element subsets of T. It follows that the number of k-element subsets equals

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

This gives the right-hand side of PR.

But solving a problem in two different ways should yield the same answer. Thus, the left-hand and right-hand sides of PR must equal each other, namely

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal's relationship

■

4.3 The Binomial Theorem

why 3?

Consider the following power expansions of the binomial $x + y$:

except 0^0

if $x, y = 1$ when 2^n = all mat

$$
\begin{aligned}
(x+y)^{\boxed{0}} &= 1 && = 1\\
(x+y)^{\boxed{1}} &= x + y && = 1x + 1y\\
(x+y)^2 &= x^2 + 2xy + y^2 && = 1x^2 + 2xy + 1y^2\\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 && = 1x^3 + 3x^2y + 3xy^2 + 1y^3\\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 && = 1x^4 + 4x^3y + \boxed{6}x^2y^2 + 4xy^3 + 1y4
\end{aligned}
$$

coefficients we parts in Pascal's Triangle

4 choose 2

The appearance of Pascal's Triangle (coefficients in red) is not coincidental.

look @ power of y

Theorem 4.2 (Binomial Theorem). in the case of 4 $\sum_{k=0}^{4} \binom{4}{k} x^{4-k}$

total of n factors

$$
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{k} x^{n-k} y^k + \ldots + \binom{n}{n} y^n \quad (10)
$$

Proof. We first illustrate by example how to derive the coefficient of xy^2 in the expansion $(x+y)^3$ given by the terms in red below: FOIL

k factors of 4

$$
\begin{aligned}
(x+y)^3 = (x+y)(x+y)(x+y) &= (x+y)(xx + xy + yx + yy)\\
&= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy
\end{aligned}
$$

3 red terms

The connection is to represent each term in red by a 2-element subset describing the positions of its two y factors as shown in the following table: pool size Position

terms = # subsets $\rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Term	Positions of the y factors
xyy	$\{2, 3\}$
yxy	$\{1, 3\}$
yyx	$\{1, 2\}$

2 3

1 _ 3 position

Thus, the coefficient of xy^2 equals the number of 2-element subsets chosen from a pool of 3 (positions):

$$
\binom{3}{2} xy^2 = 3xy^2
$$

why 3? because it's $\binom{3}{2}$

In general, to compute the coefficient of $x^{n-k}y^k$, i.e., the number of terms with k factors of y, we record the positions of the y factors in each term, which correspond precisely to a k-element subset of $\{1, 2, \ldots, n\}$. Thus, the coefficient of $x^{n-k}y^k$ equals the number of these subsets, which we express as a binomial coefficient:

$(x+y)^8 = \ldots + \ldots + x^3y^5$

$$
\binom{n}{k} x^{n-k} y^k
$$

$n-k=3$ $k=5$ $n=8$

$\binom{8}{5}$

∎

if it was just x+y

$(x+2y)^8 = (x+z)^8 = \ldots + \binom{8}{5} x^3 z^5$

z-sub

$\binom{8}{5} x^3 (2y)^5$

Example 4.1. What is the coefficient of x^3y^5 in the expansion of $(x+\boxed{2}y)^8$?

Solution. We apply the binomial theorem with $2y$ in place of y:

if it were

$$(x+2y)^8 = \binom{8}{0}x^8 + \binom{8}{1}x^7(2y) + \ldots + \binom{8}{5}x^3(2y)^5 + \ldots + \binom{8}{8}y^8$$

Multiplying out the term in red yields $(-1)^5$

neg, and a neg to others ↓ neg coeff

$$\binom{8}{5}x^3(2y)^5 = \frac{8!}{5!3!}2^5x^3y^5 = 1792x^3y^5$$

Thus, the desired coefficient of x^3y^5 equals $\boxed{1792.}$

Setting $x = y = 1$ in the Binomial Theorem yields the following result.

Corollary 4.3.

$$2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} \tag{11}$$

$\binom{n}{k}$

Remark. A combinatorial proof of the corollary can be argued as follows: count all possible subsets that can be made from a set A containing n elements in two different ways. One solution is to view each subset as a n-element tuple consisting of a collection of 1's (indicating inclusion of elements from A) and 0's (indicating exclusion of elements from A). The other solution is to count separately all subsets of size k, where $k = 0, 1, \ldots, n$.

$S\{1, 2, \ldots 10\}$

A B

4.4 Labeling Objects and Trinomial Coefficients

AND

Example 4.2. subset

(a) How many ways are there to place 10 distinct objects into two bins, called bin A and bin B, so that bin A contains 6 objects and bin B contains 4 objects? (Or equivalently, how many ways are there to place 10 labels on 10 distinct objects, where 6 are labeled as A and 4 are labeled as B).

2 el list subsets

(b) How many ways are there to place 10 distinct objects into three bins, called bin A, bin B, and bin C, so that bin A contains 5 objects, bin B contains 3 objects, and bin C contains 2 objects?

order of subset matters

Solution.

(a) We view such an arrangement as a two-element (nested) list (S_1, S_2) where S_1 is a subset consisting of 6 objects that are placed in bin A and S_2 is a subset consisting of 4 objects placed in bin B. Now, the number of possibilities for S_1 equals $\binom{10}{6}$; moreover, for each choice of S_1 there are $\binom{4}{4}$ possibilities for S_2. Thus, the total number of such lists is given by the Product Principle: AND

$$\binom{10}{6}\binom{4}{4} = [10!/(6!4!)] \cdot 1 = 210$$

product principle

choices

R

X cross Y

give some relation between (8)

(b) Again, we view such an arrangement as a three-element (nested) list (S_1, S_2, S_3) consisting of a 5-element subset S_1, a 3-element subset S_2, and a 2-element subset S_3, respectively. The corresponding number of possibilities for each subset equals $\binom{10}{5}$, $\binom{5}{3}$, and $\binom{2}{2}$, respectively. Thus, the number of such lists equals an expression that we shall define as a trinomial coefficient:

(Adam, 18) (Bethy, 14) relation

$$\binom{10}{5,3,2} = \binom{10}{5}\binom{5}{3}\binom{2}{2} = \frac{10!}{5!5!}\frac{5!}{3!2!}\cdot 1 = \frac{10!}{5!3!2!} = 2520$$

can describe as pair of formula

Definition 4.1. We define a *trinomial coefficient* by the formula

X = students in class

Y = {0, 1, ... 100}

10 choose 5, 3, 2

$$\binom{n}{k_1, k_2, k_3} = \frac{n!}{k_1!k_2!k_3!} \tag{12}$$

where $n = k_1 + k_2 + k_3$.

Remark. Binomial and trinomial coefficients can be generalized to multinomial coefficients which involve placing objects into an arbitrary number of bins.

$\binom{5}{3} = \frac{5!}{3!2!}$ # 3 el perm

5 Relations (Section 1.4)

In this section we describe the notion of an equivalence relation as a technique to divide a set into equivalence classes to help solve counting problems.

Example 5.1. We saw earlier how a 3-element subset of $S = \{1, 2, 3, 4, 5\}$ can be represented by an equivalence class of 3-element permutations, for example

3! = 1 + 2 + 3

permutations

size of group

$$(1)\ \overset{\text{subset}}{\{1,2,3\}} \leftrightarrow \overset{\text{equivalence class}}{\{123, 132, 213, 231, 312, 321\}}\ (6)$$

{123, 312}

Observe that all the permutations above have a common relationship: each contains the same three elements 1, 2, 3. This defines a relation, namely two permutations x and y are related, and write (x, y), $x \; R \; y$ or $x \sim y$, if they contain exactly the same set of elements, for example, $123 \sim 312$. However, this relation is more special that it appears (called an equivalence relation), which we discuss further later on.

$\frac{60}{6} = 10$

(,) R = ~ = tilda (relations)

Definition 5.1. Let X and Y be two sets.

(a) A *relation* R from X to Y is a set of ordered pairs (x, y) where $x \in X$ and $y \in Y$ to indicate that x and y are related, which we also express by writing $(x, y) \in R$ (or $x \; R \; y$ or $x \sim y$). In other words, R is a subset of the Cartesian product $X \times Y = \{(x, y) : x \in X, y \in Y\}$ (the set of all ordered pairs).

"not multiply X cross Y"

indicate something in common

(b) A relation R on X is a relation from X to X, i.e. ordered pairs (x_1, x_2) with $x_1, x_2 \in X$.

Example 5.2. Let $X = \{1, 2, 3, 4\}$ and $Y = \{4, 5, 6\}$. Then

"relation on X": X = Y

X cross X

cartesian

1 is in X
3 not in Y

1st el in X,
2nd el in X?
yes

(X=4)

if true for every pair, then yes

(a) $R = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,4)\} \subset X \times X$ is a relation on X.

1st el in Y
2nd el in X

(b) $S = \{(1,3),(2,6),(4,4),(3,4),(4,2)\} \subset X \times Y$ is NOT a relation from X to Y.

(c) $T = \{(5,2),(4,1),(4,3),(6,3)\} \subset Y \times X$ is a relation from Y to X. ✓

(d) $U = \{(1,2),(2,3),(3,4),(4,5)\} \subset X \times X$ is NOT a relation on X since $(4,5) \notin X \times X$.

5 not in X

Example 5.3 (Set equivalence relation)**.** Let $\boxed{X}$ denote the set of all 3-element permutations of $S = \{1,2,3,4,5\}$. As introduced earlier, we define a relation R on X to consist of order pairs (x,y) where x and y are permutations that contain exactly the same elements, i.e., form the same subset. For example, $(123, 132) \in R$ since 123 is related to 132. We will refer to this relation as the set-equivalence relation.

123

pair them to a common relationship

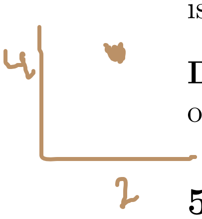
f(0)=0

F(1)=1

domain

f(2)= 2^2=4

(2,4)



5.1 Functions as Relations

Example 5.4. Let $f(x) = x^2$ be a function defined on the integers $\mathbb{Z} = \{\ldots,2,1,0,1,2,\ldots\}$ (domain). The set of ordered pairs (infinitely many)

$$R = \{(x, f(x)) | x \in \mathbb{N}\} = \{\ldots,(-2,4),(-1,1),(0,0),(1,1),(2,4),\ldots\}$$

relation

is in fact a relation from $\mathbb{Z}$ to $\mathbb{N}$ (natural numbers).

Definition 5.2. A *relation of a function* $f(x)$ with domain D is defined to be the set of ordered pairs $R = \{(x, f(x)) | x \in D\}$, also called the *graph* of f.

→ all pairs that come out of the function

5.2 Equivalence Relation

To count the size of a set X, it useful to partition it into a union of mutually disjoint subsets. To guarantee that this can be done, we seek a relation R on X that satisfies the following three properties: reflexive, symmetric, and transitive. → only if ALL 3 are true

Definition 5.3. A relation R on a set X is said to be an *equivalence relation* on X if it satisfies the following properties:

make sure situation makes sense

every object related to itself

1. Reflexive: For all $x \in X$, we require that $(x,x) \in R$.

2. Symmetric: For all $x,y \in X$ ($x = y$ is allowed), we require that if $(x,y) \in R$, then $(y,x) \in R$. reverse pair

   NOTE: If we define the *inverse* relation, $R^{-1} = \{(y,x) : (x,y) \in R\}$ where all pairs in R are reversed, then this symmetric relation can be expressed by requiring $R = R^{-1}$.

3. Transitive: For all $x,y,z \in X$, we require that if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$. only when you have 3 values

check all possibilities

Example 5.5. Determine whether or not the following relations are equivalence relations:

a) ref? $(x,x) \in R$ for all $\in X$] 4,4 missing
does it have (1,1), (2,2), etc.
symm? missing (3,1) reverse pair
trans? (1,3), (3,2) → (1,2) ← missing

b) yes
c) fails
d) yes

definition: as long as you have same 3 elements (order doesn't matter)

(1,4) & (4,1) missing = irrelevant to symm

(a) The relation $R = \{(1,1), (1,3), (2,2), (2,3), (3,2), (3,3)\}$ defined on the set $X = \{1,2,3,4\}$.

(b) The $\boxed{\text{set-equivalence relation}}$ R defined on the set X consisting of all 3-element permutations of $\{1,2,3,4\}$. satisfies all 3

not ref

make sample values for x & y

(c) The relation $R = \{(x,y) : x, y \in \boxed{\mathbb{Z}}, |x - y| = 1\}$ (called the neighbor relation).

prove w/ counter example

if x,y 1 apart, reversed

(d) Let $X = \{A : A \subset \mathbb{Z}, |A| < \infty\}$ (all finite subsets of $\mathbb{Z}$) where we define a relation R on X as follows: $(A, B) \in R$ if $|A| = |B|$ (called the same-size equivalence relation).

size of A = size of B

$= |-1(-x+y)| = |-1(y-x)|$

$= |-1| \cdot |y-x| = 1$

ok proof

5.3 Other Properties of Relations

order doesn't matter when have equality of size

Example 5.6 (Anti-symmetry)**.** Consider the *smaller-than-or-equal-to* ($\leqslant$) relation R defined on $\mathbb{R}$ (all real numbers) by

inequality [equality is symm]

$$R = \{(x,y) : x, y \in \mathbb{R}, x \leqslant y\}$$

Observe that R is reflexive and transitive, but not symmetric, and thus NOT an equivalence relation. For example, we have $(1,2) \in R$ since $1 \leqslant 2$, but $(2,1) \in R$ since $2 \nleqslant 1$. The only way in which both $(x,y) \in R$ and $(y,x) \in R$ are true is if $x = y$. Any relation that has this property is said to be *anti-symmetric*.

other properties: NOT in book for independent reading

5.4 Equivalence Classes

Given an equivalence relation, we can group all elements that are related to each other into a set called an equivalence class.

if you have equivalence relation, groups don't overlap

Definition 5.4. Let R be an equivalence relation on a set S. The *equivalence class* of $x \in R$, denoted by S_x (or $[x]$), is the subset of elements of S that are related (or equivalent) to x, i.e.,

$$S_x = \{z \in S : (x,z) \in R\}$$

Example 5.7. Let $S = \{1,2,3,4\}$ and define the following equivalence relation R on S:

$$R = \{(1,1), (2,2), (2,3), (3,2), (3,3), (4,4)\}$$

partition

Here are the equivalence classes of R:

2,2 related 2,3 related

S

1 2 3 4

(a) $S_1 = \{1\}$

(b) $S_2 = \{2,3\} = S_3$ equivalent class

(c) $S_4 = \{4\}$

Example 5.8. Let $X = \{1,2,3,4\}$ and $S = 2^X$ denote the set of all subsets of X (called the *power set* of X). Define R to be the same-size-as equivalence relation on S, i.e.,

$$R = \{(A,B) : A, B \in S, |A| = |B|\}$$

Here are the equivalence classes of R:

$S = S_1 \cup S_2 \cup S_3$

partition

2^x (power set) ←

{ 1 set el, 2 el set, 3 el set, 4, empty set

$\emptyset$

$S = S_{\{1\}} \cup S_{\{1,2\}} \ldots \cup S_{\varnothing}$

partition of S

all el belong to one class for equiv relation

(a) $S_{\{1\}} = \{\{1\},\{2\},\{3\},\{4\}\}$ (1-element subsets)

(b) $S_{\{1,2\}} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ (2-element subsets)

(c) $S_{\{1,2,3\}} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$ (3-element subsets)

(d) $S_{\{1,2,3,4\}} = \{\{1,2,3,4\}\}$ (1-element subsets) 4

(e) $S_{\varnothing} = \{\varnothing\}$ (empty set)

S_x, S_y don't overlap

S_x & S_y empty

Lemma 5.1 (Properties of Equivalence Classes)**.** *Let R be an equivalence relation on a set S. The following properties hold:*

1. $x \in S_x$
2. $y \in S_x$ *if and only if* $(x, y) \in R$ — x & y related
3. $(x, y) \in R$ *if and only if* $S_x = S_y$ — $S_x = S_y$
4. *If* $S_x \cap S_y$ *is nonempty, then* $S_x = S_y$.

S_x

S

if $S_x \cap S_y$ is nonempty, they're the same

E.h

$x \neq y$ (symm)

Proof of Property 4. Suppose $S_x \cap S_y$ is nonempty. Let $z \in S_x \cap S_y$. Then $z \in S_x \Rightarrow (x, z) \in R$ and $z \in S_y \Rightarrow (y, z) \in R$ by property 2. By transitivity of the equivalence relation R, we have $(x, y) \in R$. Thus, by property 3, we conclude that $S_x = S_y$. ■

The following theorem justifies what we said earlier: grouping elements into equivalence classes guarantees that each element belongs to exactly one class, i.e., equivalence classes are disjoint (no elements in common).

Theorem 5.2 (Equivalence Classes)**.** *Let R be an equivalence relation on a set S. The equivalence classes of R are nonempty, mutually (pair-wise) disjoint subsets of S whose union is all of S, i.e.*

1. $S_x \cap S_y \neq \varnothing \Rightarrow S_x = S_y$ — classes don't overlap
2. $\bigcup_{x \in S} S_x = S$

correspond

$R \rightarrow P$

equiv rel $\leftarrow$ partition of S

5.5 Partitions

The previous theorem shows that an equivalence relation yields a partition of a set S into mutually disjoint subsets, which we define formally.

Definition 5.5. Let S be a set. A *partition* of S is a collection P consisting of nonempty, mutually (pair-wise) disjoint subsets of S whose union is all of S, i.e.

1. $A \cap B = \varnothing$ for any two distinct subsets $A, B \in P$
2. $\bigcup_{A \in P} A = S$

In fact, the converse of the previous theorem is also true.

remember properties

Theorem 5.3. *Let P be a partition S. Then P defines an equivalence relation R on S as follows:*

$$(x, y) \in R \text{ if and only if } x, y \in A \text{ form some } A \in P.$$

Moreover, each disjoint subset A of P forms an equivalence class of R.

5.6 Partial and Total Orders

Recall the inequality relation $\leqslant$ (smaller-than-or-equal-to) defined on the real numbers $\mathbb{R}$ discussed earlier. Such a relation allows us to compare and order any two real numbers. We also saw that the relation $\leqslant$ is reflexive, anti-symmetric, and transitive. This leads us to generalize the notion of smaller-than-or-equal-to relation defined on arbitrary sets (called a partial order):

Definition 5.6. Let R be a relation defined on a set S.

(a) If R is reflexive, anti-symmetric, and transitive, then R said to be a *partial order*. In that case, S is said to be a *partially ordered set* (or *poset* for short).

(b) If R is a partial order with the additional property that any two elements of S can always be compared, i.e., either $(a, b) \in R$ or $(b, a) \in R$ for any $a, b \in S$ (this means that at least one pair (a, b) or (b, a) must belong to R), then R is said to be a *total order*. In that case, S is said to be a *totally ordered set*.

(c) If R is a total order and S has the property that every nonempty subset of S has a smallest element, then S is said to be a *well-ordered set*.

Remark. Not every subset has a smallest element. For example, if $S = \mathbb{Z}$, then the subset of even integers, $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$ has no smallest element.

Example 5.9.

(a) The relation $\leqslant$ defined on $\mathbb{R}$ is a partial order (and thus $\mathbb{R}$ is said to be a partially ordered set), but NOT a total order.

(b) The relation $\leqslant$ defined on $\mathbb{N}$ is a total order and moreover, $\mathbb{N}$ is a well-ordered set since every subset of $\mathbb{N}$ has a smallest element (this property of $\mathbb{N}$ is assumed to be true because of the Well-Ordering Principle).

(c) Let S denote the power set of $\mathbb{Z}$ and define a relation $\subseteq$ (subset of) on S by the following requirement: two subsets $A, B \in S$ are related if $A \subseteq B$. Then $\subseteq$ is a partial order. Observe that $\subseteq$ is NOT a total order; for example, the subsets $A = \{1, 2\}$ and $B = \{2, 3\}$ cannot be compared since $A \nsubseteq B$ and $B \nsubseteq A$.

6 Equivalence Relations in Counting (Section 1.5)

Symmetry Principle: If a formula has symmetry (i.e., interchanging two variables doesn't change the result), then a proof that explains this symmetry is likely to give us additional insight into the formula and help in solving counting problems.

Example 6.1. The formula for binomial coefficients,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

is symmetric since we can interchange k with $n-k$ and not change the result. This follows from the fact that any k-element subset A of an n-element set S corresponds to its complement, $S - A$ (elements of S not in A), which is an $(n-k)$-element subset.

required

We now formally present the Quotient Principle mentioned earlier.

Theorem 6.1 (Quotient Principle)**.** *Let R be an equivalence relation defined on a finite set S containing p elements. If all the equivalence classes of R have the same size r and there are q such equivalence classes, then*

$$q = \frac{p}{r} \tag{13}$$

$\dfrac{\text{total}}{\text{size of each class}}$

Remark. In general, equivalence classes are not equal in size. The theorem above applies only to relations whose equivalence classes are all the same size.

Quotient Principle, Version 2: If we can partition a set of size p into q blocks of size r, then

$$q = \frac{p}{r}$$

6.1 Counting Equivalence Classes

Example 6.2. Use an equivalence relation and the Quotient Principle to determine the number of 5-element subsets chosen from $S = \{1, 2, \ldots, 7\}$.

Solution. We first count the number of 5-element permutations of S, whose collection we denote by the set X. By the product principle, the number of elements of X equals

$$(7)_5 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 7!2!$$

since there are 7 choices for the first element of a permutation, 6 choices for the second, etc. To count the corresponding number of 5-element subsets, we define an equivalence relation R on S as follows: $(x, y) \in R$ (x and y are related) if and only if the permutations x and y generate the same subset (i.e., contain exactly the same elements). For example, we have $(13567, 53617) \in R$. We leave it for the reader to check that this relation is reflexive, symmetric, and transitive. It follows that we can represent each 5-element subset as the equivalence class of permutations that generate it:

$$\{1,3,5,6,7\} \leftrightarrow \{13567, 13576, 13657, 13675, 13756, 13765, \ldots, 76531\}$$

Thus, the number of 5-element subsets is the same as the number of equivalence classes of R. To determine this number, we'll first need to compute the size of each equivalence class, i.e., the number of 5-element permutations that generate a given subset, say $\{a_1, a_2, a_3, a_4, a_5\}$. Again, we apply the product principle (5 choices for the first element, 4 choices for the second, etc.) to obtain

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

By the quotient principle, the number of equivalence classes equals

$$q = \frac{p}{r} = \frac{7!/2!}{5!} = \frac{7!}{5!2!} = 21$$

which also gives the number of 5-element subsets.

Example 6.3. Four people sit down at a round table to play cards. How many different playing seating charts are possible?

cwt want neighbors?

Solution. Suppose we label the four people as $\boxed{A, B, C, D.}$ Then any seating of the four people can be described by a 4-element permutation of the set $S = \{A, B, C, D\}$. The number of such permutations equals 4! (Product Principle). However, this answer is not correct if we assume that in a seating chart it does not matter which chair a person sits on but only their relative position around the table. To account for this, we define an equivalence relation R on the set of permutations as follows: $x \sim y$ if and only if the permutations x and y yield the same seating chart where each person has the same person to their right in both permutations. Thus, each distinct seating chart corresponds to an equivalence class of permutations. How large is each equivalence class? Observe that if we perform a right (circular) shift of the elements of any permutation (i.e., have each person move the same number of seats around the table), then it yields the same seating chart, e.g. $ABCD \sim DABC$. Since there are 4 possible shifts, the size of each equivalence class equals $r = 4$. Hence, the number of equivalence classes equals (by the Quotient Principle)

seating chart = equiv class

$$q = \frac{p}{r} = \frac{4!}{4} = 3! = 6$$

counted 4er permutations

which also gives the number of distinct seating charts.

equiv

Example 6.4. In how many different ways can the letters in each of the following words be rearranged to form new words (called anagrams) which may or may not make sense?

diff words

(a) ROWAN 5 el permutation product principle

(b) HELLO

$HEL_1L_2O = 5!$

Define equiv

$x = HEL_1L_2O \sim HEL_2L_1O$

(c) PROFESSORS

Solution.

(a) If we view each anagram as a permutation, then the number of anagrams equals the number of 5-element permutations, which equals (by the Product Principle)

$$\text{(5 choices for first letter)(4 choices for second)...(1 choice for fifth)} = 5! = 120$$

(b) If we distinguish the two letter L's, say refer to them as L_1 and L_2, then the number of permutations of HEL_1L_2O equals $5! = 120$ as in part (a). Next, we define an equivalence relation R as follows: $x \sim y$ if and only if the permutations x and y produce the same anagram in English (by removing the subscripts). For example, $HL_2EL_1O \sim HL_2EL_1O) \in R$ since HL_2EL_1O and HL_2EL_1O read the same as English words. Now, each equivalence class of R consists of two permutations since we can swap L_1 and L_2 (or really their indices) without changing the word (in English), and thus by the Product Principle:

PROOF

5!/2!

$$\text{(2 choices for the first index of L)(1 choice for the second index of L)} = 2!$$

Thus, the size of each equivalence class equals 2! and the number of equivalence classes equals (by the quotient principle).

Perm → subsets
subsets → multisets

(c) We leave this as an exercise.

6.2 Multisets

We now discuss how to count the fourth and last type of collection: multisets. Here is a classic problem where mutilsets arise (we'll revisit this problem and solve it later).

Bookshelf Problem: How many ways are there to arrange 7 identical books onto 5 different shelves of a bookcase?

Let's first review the definition of multisets.

angled brackets means it's a multiset

Definition 6.1. A *multiset* $\langle a_1, a_2, \ldots, a_n \rangle$ is an unordered collection of elements with repetition allowed. The number of times that an element appears in a multiset is called its *multiplicity*. Thus, the *cardinality* (or size) of a multiset is thus the sum of all its multiplicities.

⇐ multiplicity

Example 6.5. The multiset $\langle 4, 1, 2, 1, 2, 5, 1, 2, 5, 2 \rangle$ has 10 elements where the multiplicities of the elements $1, 2, 4, 5$ are $3, 4, 1, 2$, respectively.

We next introduce notation to count multisets similar to binomial coefficients for counting subsets.

Pascal's relationship

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

⟨1, 1⟩ ≠ ⟨1⟩

pool size   size of multiset

Definition 6.2. We denote the number of k-element multisets that can be formed by choosing elements from an n-element set by a *repeating binomial coefficient* and write $\left(\!\binom{n}{k}\!\right)$ (pronounced "n choose k repeated").

$\left(\!\binom{n}{k}\!\right)$

Example 6.6. Let us compute $\left(\!\binom{3}{k}\!\right)$ for k=0,1,2,3,4.

$S = \{1, 2, 3\}$

Solution. Here is a table of values for $\left(\!\binom{3}{k}\!\right)$.

order doesn't matter, allows repetition

k	k-element multisets	$\left(\!\binom{3}{k}\!\right)$
0	⟨⟩ (empty multiset)	1
1	⟨1⟩, ⟨2⟩, ⟨3⟩ ⟨1,1⟩ ⟨2,2⟩ ⟨3,3⟩	3
2	⟨1, 1⟩, ⟨1, 2⟩, ⟨1, 3⟩, ~~⟨1, 4⟩, ⟨1, 5⟩, ⟨1, 6⟩~~	6
3	⟨1, 1, 1⟩, ⟨1, 1, 2⟩, ⟨1, 1, 3⟩, ⟨1, 2, 2⟩, ⟨1, 2, 3⟩, ⟨1, 3, 3⟩, ⟨2, 2, 2⟩, ⟨2, 2, 3⟩, ⟨2, 3, 3⟩, ⟨3, 3, 3⟩	10
4	⟨1, 1, 1, 1⟩, ⟨1, 1, 1, 2⟩, ⟨1, 1, 2, 2⟩, ⟨1, 2, 2, 2⟩, ⟨2, 2, 2, 2⟩, ⟨1, 1, 1, 3⟩, ⟨1, 1, 2, 3⟩, ⟨1, 1, 3, 3⟩, ⟨1, 2, 2, 3⟩, ⟨1, 2, 3, 3⟩, ⟨1, 3, 3, 3⟩, ⟨2, 2, 2, 3⟩, ⟨2, 2, 3, 3⟩, ⟨2, 3, 3, 3⟩, ⟨3, 3, 3, 3⟩	15

$\left(\!\binom{3}{2}\!\right)$

$= \left(\!\binom{3}{3}\!\right)$

$= \left(\!\binom{3}{4}\!\right)$

6.3 Pascal-like Relationship for Multisets

The following table of values for repeating binomial coeficients, analogous to Pascal's Triangle, reveals a recursive pattern similar to Pascal's relationship.

$56 = 35 + 21$

$\left(\!\binom{6}{3}\!\right) = \left(\!\binom{5}{3}\!\right) + \left(\!\binom{6}{2}\!\right)$

figurate rectangle

$\boxed{\left(\!\binom{n}{k}\!\right)}$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$n = 0$	1	0	0	0	0	0	0
$n = 1$	1	1	1	1	1	1	1
$n = 2$	1	2	3	4	5	6	7
$n = 3$	1	3	6	10	15	21	28
$n = 4$	1	4	10	20	35	56	84
$n = 5$	1	5	15	35	70	126	210
$n = 6$	1	6	21	56	126	252	462

The following theorem describes this pattern mathematically.

Theorem 6.2. *Let n and k be positive integers. Then*

$$\boxed{\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n-1}{k}\!\right) + \left(\!\binom{n}{k-1}\!\right)} \tag{14}$$

6.4 Formula for Multisets (Stars and Bars Encoding)

We introduce a 'stars-and-bars' encoding of multisets as follows: let M be a k-element multiset chosen from a set containing n elements. Assuming that the elements of M are arranged in increasing order, we make a correspondence with a sequence of stars and bars, denoted by M^*, where we use a star "$*$" to represent each element and a vertical bar "|" to separate the elements into n compartments corresponding to the n elements.

* = book

| = divider

Example 6.7. Here is the encoding for the 7-element multiset

$$M = \langle 1, 1, 1, 2, 3, 3, 5 \rangle \quad \leftrightarrow \quad M^* = * * *| * | * *||*$$

which separates the 7 elements into 5 compartments. Thus, M^* can be viewed as an 11-element list consisting of 7 stars and 4 bars. To determine the number of such encodings, we make another correspondence between M^* and a subset P denoting the positions of the stars in M^*. Continuing with the previous example, we have

can't have 2 of same #

$$M^* = * * *| * | * *||* \quad \leftrightarrow \quad P = \{1, 2, 3, 5, 7, 8, 11\}$$

→ # of elements

where P is a 7-subset chosen from 11 elements. Thus, we have a one-to-one correspondence between multisets M and subsets P. Since the number of 7-elements subsets chosen from 11 elements is given by the binomial coefficient $\binom{11}{7}$, it follows that the number of 7-element multisets chosen from 5 elements is the same, i.e.,

$$\left(\!\!\binom{5}{7}\!\!\right) = \binom{11}{7} = \frac{11!}{7!4!} = 330$$

Example 6.8. The following table gives the correspondence between 3-element multisets chosen from 3 elements with 3-element subsets chosen from 5 elements.

$\left(\!\!\binom{3}{3}\!\!\right)$

Multiset	Stars-and-bars	Subset
$\langle 1,1,1 \rangle$	$***\|\|$	$\{1,2,3\}$
$\langle 1,1,2 \rangle$	$**\|*\|$	$\{1,2,4\}$
$\langle 1,1,3 \rangle$	$**\|\|*$	$\{1,2,5\}$
$\langle 1,2,2 \rangle$	$*\|**\|$	$\{1,3,4\}$
$\langle 1,2,3 \rangle$	$*\|*\|*$	$\{1,3,5\}$
$\langle 1,3,3 \rangle$	$*\|\|**$	$\{1,4,5\}$
$\langle 2,2,2 \rangle$	$\|***\|$	$\{2,3,4\}$
$\langle 2,2,3 \rangle$	$\|**\|*$	$\{2,3,5\}$
$\langle 2,3,3 \rangle$	$\|*\|**$	$\{2,4,5\}$
$\langle 3,3,3 \rangle$	$\|\|***$	$\{3,4,5\}$

$\binom{5}{3} = \binom{3+(3-1)}{3}$

The same argument can be used as in the previous example to obtain

$$\left(\!\!\binom{3}{3}\!\!\right) = \binom{3+3-1}{3} = \binom{5}{3} = 10$$

The correspondence above can be applied in general to k-element multisets chosen from a set containing n elements to derive the following formula in terms of binomial coefficients.

Theorem 6.3. *Let $n, k \in \mathbb{N}$. Then*

subset

pool

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \boxed{\frac{(n+k-1)!}{k!(n-1)!}} \tag{15}$$

size

multiset

Example 6.9 (Bookshelf Problem visited)**.** How many ways are there to arrange 7 identical books onto 5 different bookshelves?

Solution. If we think of each bookshelf as a compartment, then any arrangement of the 7 books can be represented by a stars-and-bars encoding with 7 stars and 4 bars, e.g. $M^* = * * *|*|**||*$ (3 books in the 1st shelf, 1 book in the 2nd shelf, 2 books in the 3rd shelf, 0 books in the 4th shelf, and 1 book in the 5th shelf). This corresponds to the 7-element multiset $M = \langle 1, 1, 1, 2, 3, 3, 5 \rangle$ chosen from 5 elements, or alternatively, a 7-element subset $P = \{1, 2, 3, 5, 7, 8, 11\}$ chosen from 11 elements (7+5-1), indicating the positions of the stars in M. Thus, the number of arrangements of 7 books onto 5 bookshelves is the same as the number of 7-element subsets chosen from 11 elements:

$$\left(\binom{5}{7}\right) = \binom{5+7-1}{7} = \binom{11}{7} = 330$$

References

[1] Discrete Mathematics for Computer Scientists (1st Edition), Cliff L. Stein, Robert Drysdale, and Kenneth Bogart, Addison-Wesley, 2011, ISBN-13: 978-0132122719.