Discrete Mathematics for Computer Scientists

Big O Notation

Alg 1 ? Alg 2

$f(n) \leq g(n)$

n = size of list

Assume n is large

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A Quick and Dirty Introduction to Big O Notation

Which function is "bigger"?

$\frac{1}{10}n^2$ or $100n + 10000$

Answer depends upon value of n.

In Computer Science we are usually interested in what happens when our problem input size gets large. Asymptotic = n is large

$100n + 10{,}000 \le \frac{1}{10} n^2 , \; n \ge x_0$

$\underbrace{100n + 10{,}000}_{\text{line}} \le \underbrace{\tfrac{1}{10} n^2}_{\text{quadratic}}$

x_0 is different

type of function

$\underbrace{100n + 10000}_{f} = O(\underbrace{\tfrac{1}{10} n^2}_{g})$

$f = O\, g$

still holds

$100n + 10{,}000 \le [\underbrace{\tfrac{1}{50}}_{\text{irrelevant}} n^2$

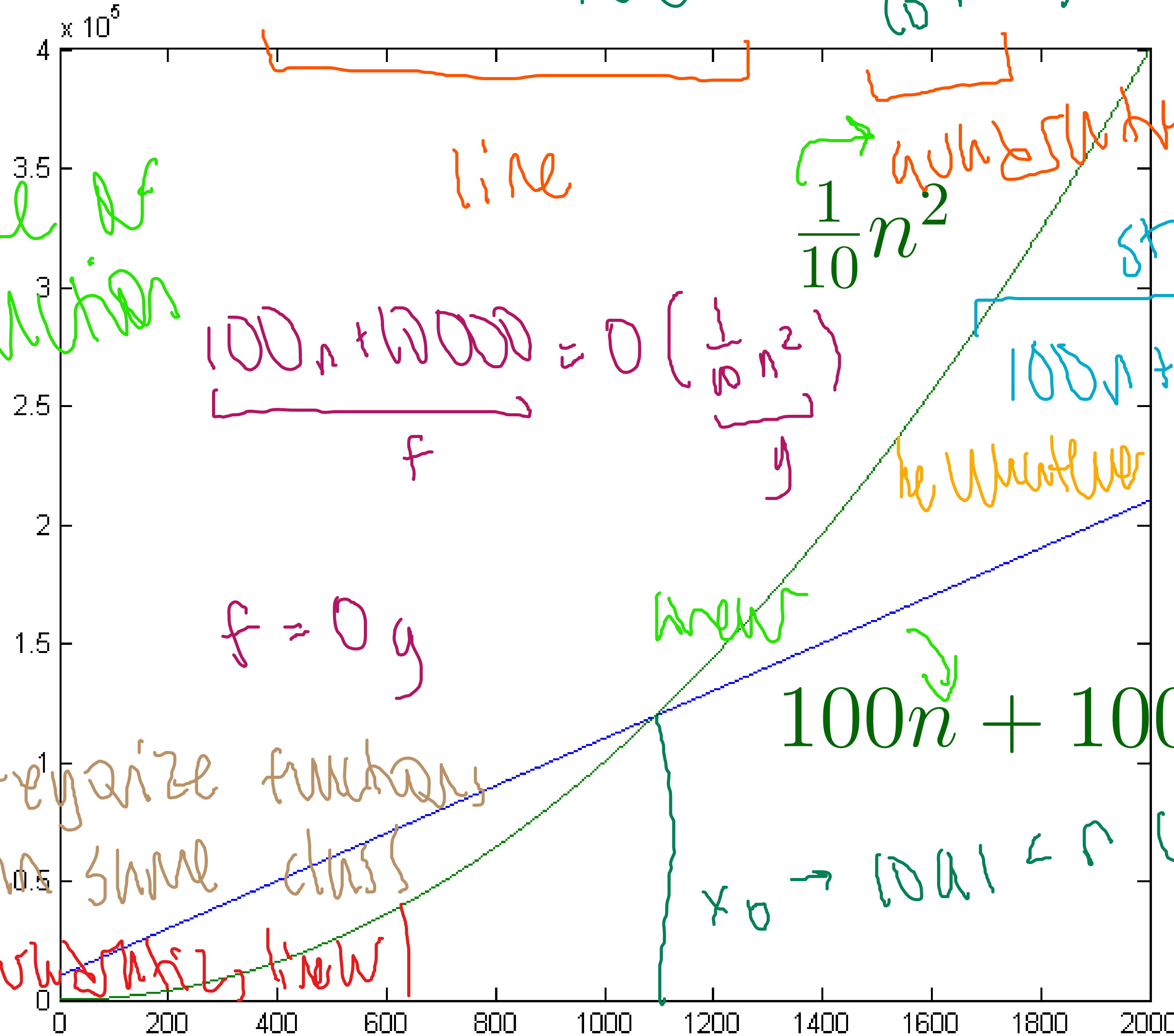
be whatever

linear

$x_0 \to 1001 < n$ (large)

categorize functions if in same class

(quadratic, linear)



Notice that when n is "large enough" $\frac{1}{10}n^2$ gets much bigger than $100n + 10000$ and stays larger.

Function $f(n) = O(g(n))$:

(read: $f(n)$ is O of $g(n)$)

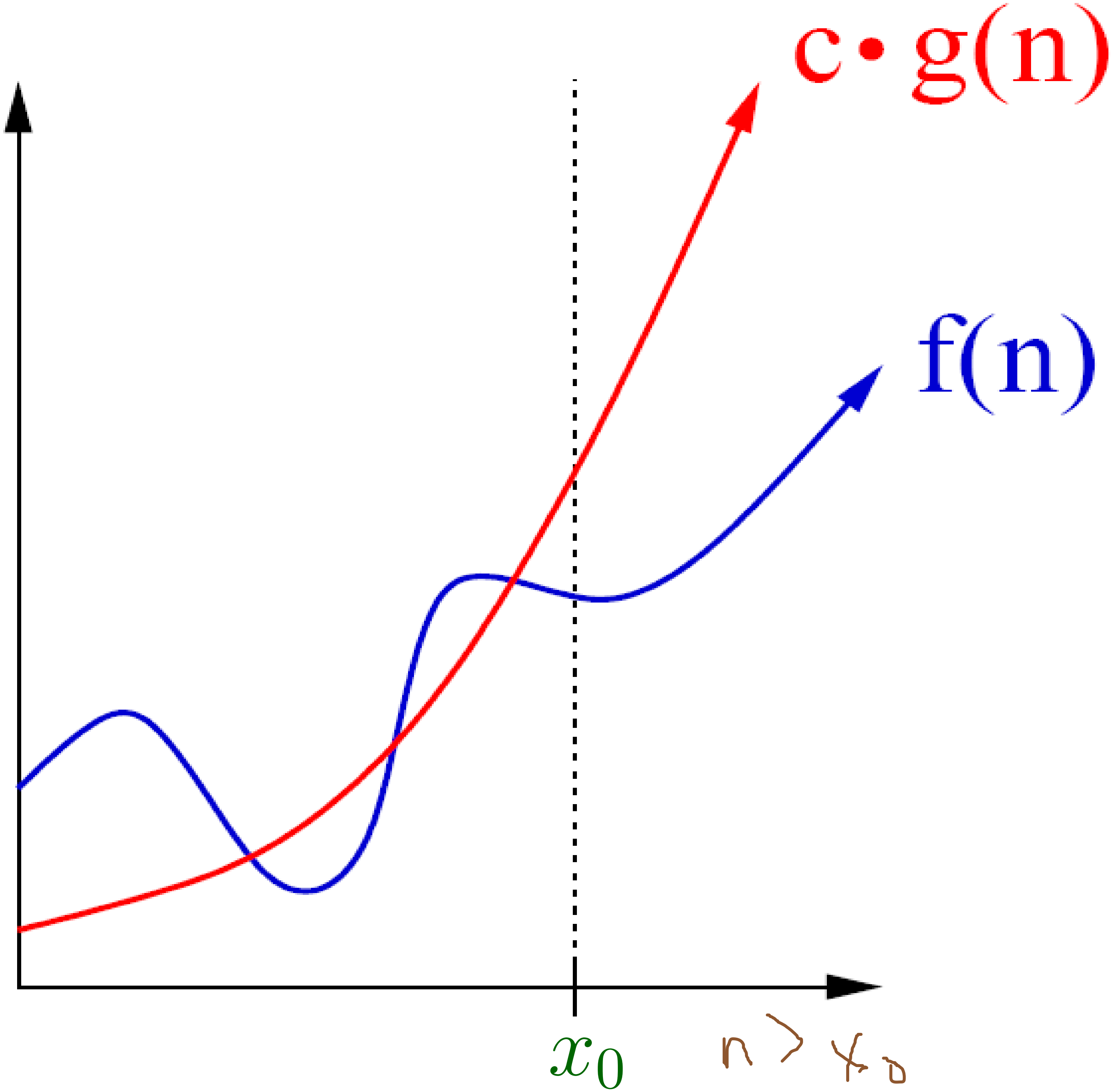
If (i) There is some positive $x_0 \in R$

(ii) There is some positive $c \in R$

such that

$\forall x \geq x_0 \qquad f(x) \leq cg(x)$.

starting pt


c • g(n)
f(n)
x_0
n > x₀

Let $x_0 = 1091$.

Can verify, $\forall n > x_0,\ 100n + 10000 \leq \frac{1}{10}n^2$.

Thus $\quad 100n + 10000 = O(\frac{1}{10}n^2)$.

$c = 1$

Note that the opposite is **not** true!

Why? *(Proof by contradiction)*

$$\underbrace{\frac{1}{10}n^2}_{g} \leq c \cdot \underbrace{(100n + 10000)}_{f} \quad \forall n \geq x_0$$

Can you find x_0, c to make this inequality true? NO

$g \neq O(f)$

More Examples:

negligible

dominant

$$4n^2$$

$$8n^2 + \boxed{2n} - \boxed{3}$$

$$n^2/5 + \sqrt{n} - 10\log n$$

$$n(n-3)$$

are all $O(n^2)$.

Two functions $f(n), g(n)$ have the same order of growth if
$f(n) = O(g(n))$ and $g(n) = O(f(n))$.

In this case we say

$$f(n) = \Theta(g(n))$$

which is the same as

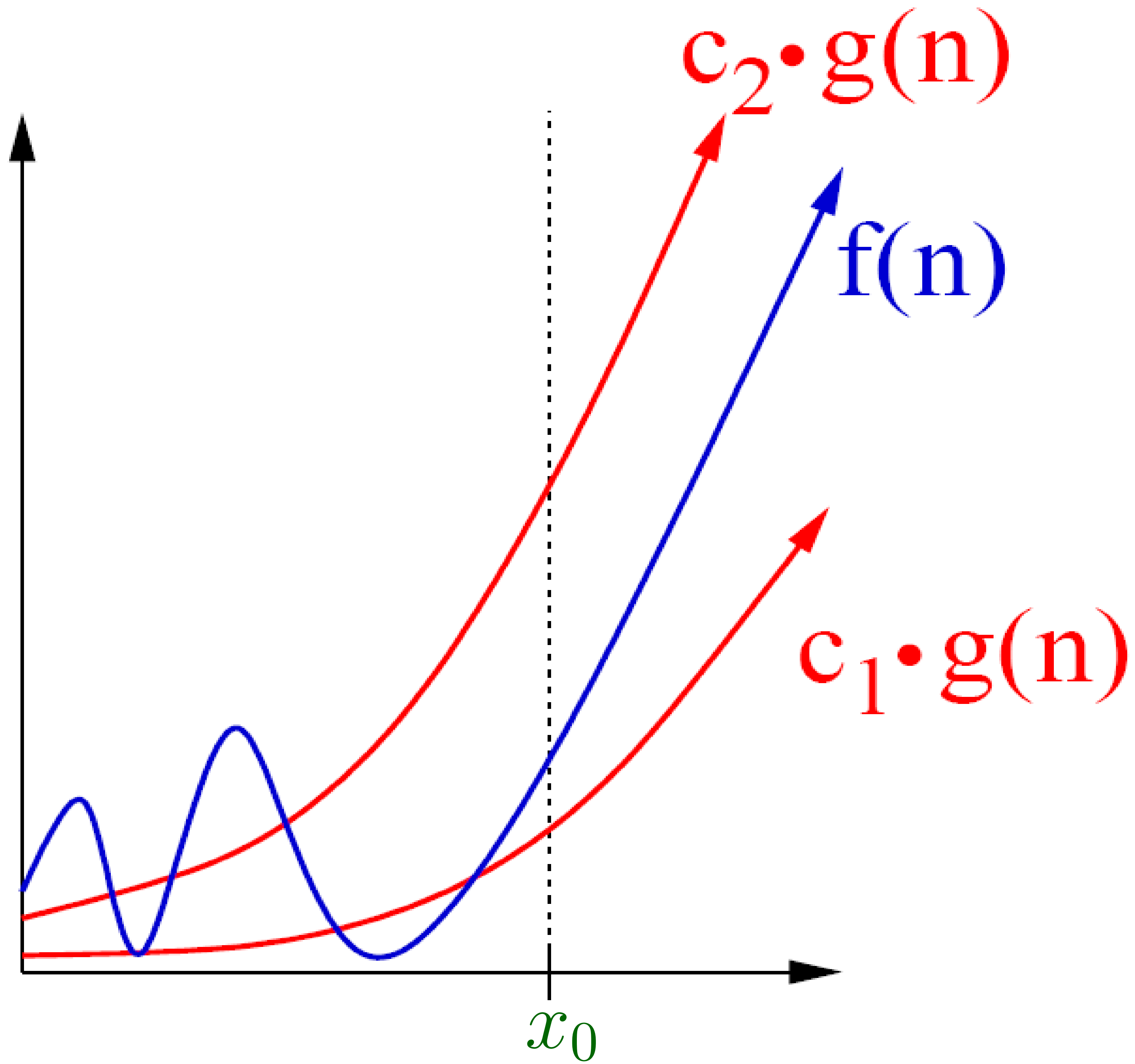
$$g(n) = \Theta(f(n))$$

$f(n) = n^2$

$g(n) = 3n^2 + 100$

$f \leq g \Rightarrow f = O(g)$

$g \leq f \Rightarrow g = O(f)$


$c_2 \cdot g(n)$
$f(n)$
$c_1 \cdot g(n)$
x_0

Examples ($f(n) = \Theta(g(n))$):

- $3n^2 + 4n = \Theta(n)$? No
- $3n^2 + 4n = \Theta(n^2)$? Yes
- $3n^2 + 4n = \Theta(n^3)$? No, but $O(n^3)$
- $n/5 + 10n\log n = \Theta(n^2)$? No, but $O(n^2)$
- $n^2/5 + 10n\log n = \Theta(n\log n)$? No
- $n^2/5 + 10n\log n = \Theta(n^2)$? Yes