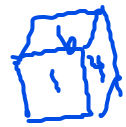
# CHAPTER 5: PROBABILITY

 roll die

## 5.1 Introduction to Probability

**Why Study Probability?**

**Probability (chance/likelihood)**

$$\text{Probability of an event occurring } = \frac{\text{Number of desired outcomes}}{\text{Total number of outcomes}}$$

$\frac{1}{6}$

**Definition**: We define the following for a given probability experiment:
(a) Outcome $x$ – a result of the experiment
(a) *Sample space* $S$ – set of all possible outcomes (or elements) of the experiment
(b) *Event* $E$ – set of desired outcomes (subset of the sample space)
(c) *Probability weight* $P(x)$ – relative likelihood of outcome $x$ occurring

to test fairness, see if it's getting closer to $\frac{1}{6}$ each time you roll

Example: (Roll of a die) Consider the model in describing the rolling of a fair die.

every outcome is equally likely

unfair

a loaded die could have heavier sides or a chip

Sample space (set of Outcomes):

$$\{1, 2, 3, 4, 5, 6\}$$

Probability weights:

$$P(1)=\frac{1}{6}, P(2)=\frac{1}{6}, P(3)=\frac{1}{6},$$
$$P(4)=\frac{1}{6}, P(5)=\frac{1}{6}, P(6)=\frac{1}{6}$$

all added to 1

Example: (Pair of dice) Determine the probability distribution when a pair of fair dice are tossed.

Solution: Suppose we denote by $(a, b)$ the outcome for the two dice where $a$ and $b$ are integers between1 and 6 representing the value of each die. The sample space thus has 36 possible outcomes, i.e. 36 two-element lists (Multiplication Principle):

$$\begin{aligned} S &= \{1,2,3,4,5,6\}\times\{1,2,3,4,5,6\} \\ &= \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ &\quad (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), \\ &\quad (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), \\ &\quad (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), \\ &\quad (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), \\ &\quad (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

essentially counting problems

how do you want to count the pairs

compute the sample space → what collection?

notice

Since each outcome is equally likely, the probability of each outcome is $P(x)=1/36$ for each $x \in S$.

$P(x) = \frac{1}{36}$

order matters → $(5,2) \neq (2,5)$

**Definition**: A *probability distribution (or measure)* $P$ on a sample space $S$ is a function that satisfies the following rules:

repetition allowed → $(3,3)=(3,3)$

unfair die: add probability of each outcome every probability can't be negative

totally possible

1. $P(E) \geq 0$ for any $E \subseteq S$
2. $P(S) = 1$ and $P(\varnothing) = 0$ (not possible)
3. $P(E \cup F) = P(E) + P(F)$ for any two disjoint events $E$ and $F$, i.e. $E \cap F = \varnothing$, where we define

$$P(E) = \sum_{x \in E} P(x)$$

unfair

event

for any event $E$.

NOTE: $P(x) = 0$ describes an outcome that never occurs and $P(x) = 1$ describes an outcome that occurs with absolute certainty.

fair experiment

**Definition**: If all outcomes are equally likely, i.e. $\boxed{P(x) = \frac{1}{|S|}}$ for every outcome $x$, then $P(x)$ is called a *uniform probability distribution*.

**Theorem**: If $P(x)$ is a uniform probability distribution, then for any event $E$, we have

$$P(E) = \frac{|E|}{|S|}$$

fair

Proof: This follows from the calculation

$$P(E) = \sum_{x \in E} P(x) = \sum_{x \in E} \frac{1}{|S|} = \frac{1}{|S|} \sum_{x \in E} 1$$

$$= \frac{|E|}{|S|} \qquad = \frac{\text{(number of desired outcomes)}}{\text{(total number of possible outcomes)}}$$

**Some Examples of Probability Computations**

Example: What is the probability of obtaining a sum of 6 on the roll of a pair of fair dice?

$|S| = 36$ outcomes

Solution: Here, our event is

$$E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

Since $P(x)$ is a uniform probability distribution, i.e. $P(x) = 1/36$ for all $x \in S$, we have

$$P(E) = \sum_{x \in E} P(x) = P((1,5)) + P((2,4)) + P((3,3)) + P((4,2)) + P((5,1))$$

$$= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$$

$$= \frac{5}{36} = \frac{|E|}{|S|}$$

Example: (Coin Tossing) What is the probability of obtaining exactly 2 heads in five tosses of a coin?

order matters 5-el tuple

repetition → H & T can be repeated

$|S| = 2^5$
$= 32$

HHTTT (Anagram)
|E| = # anagrams

permutations is 120 (5!)



Solution: Since there are two possibilities for each toss, head (H) or tail (T), it follows by the Multiplication Principle that the sample space for five tosses has $2\cdot 2\cdot 2\cdot 2\cdot 2 = 32$ outcomes:

$$S = \{\text{HHHHH}, \text{THHHH}, \text{HTHHH}, \ldots, \text{TTTTT}\}$$

Since each outcome is equally likely, we have $P(x) = 1/32$ for all $x \in S$.

To determine the size of our event *E*, i.e., those outcomes containing exactly two heads, we first consider the set *A* of all possible 5-element (ordered) permutations generated from the symbols $H_1$, $H_2$, $T_1$, $T_2$, $T_3$ without repetition where we distinguish the two H's and three T's (in math). The number of such lists equals 5! = 120 (Multiplication Principle). Now, define an equivalence relation on *A* such that two permutations are equivalent if the letter H's are rearrangements of each other and the letter T's are rearrangements of each other, e.g. $H_1H_2T_1T_2T_3 : H_2H_1T_2T_3T_1$ so that they read the same in English (without distinguishing the same letters). The size of each equivalence class is $m = 2!3! = 12$ (2! ways of rearranging the indices 1, 2 on the letters H and 3! ways of rearranging the indices 1, 2, 3 on the letters T), e.g.

HHTTT = [{H1,H2,T1,T2,T3}] =
{{H1,H2,T1,T2,T3},{H1,H2,T1,T3,T2},{H1,H2,T2,T1,T3},{H1,H2,T2,T3,T1},{H1,H2,T3,T1,T2},{H1,H2,T3,T2,T1},
{H2,H1,T1,T2,T3},{H2,H1,T1,T3,T2},{H2,H1,T2,T1,T3},{H2,H1,T2,T3,T1},{H2,H1,T3,T1,T2},{H2,H1,T3,T2,T1}}

It follows that the number of such equivalence classes, which is the same as the number of outcomes in our event *E*, equals

HTHTT → {1, 3}

$$|E| = \frac{|A|}{m} = \frac{5!}{2!3!} = \boxed{\binom{5}{2}} = 10$$

binomial coefficient

Thus, the probability of event *E* is

2 el subset pool of 5

$$P(E) = \frac{|E|}{|S|} = \frac{10}{32} = \frac{5}{16}$$

all cards same suite

Exercise: (Poker hands) What is the probability of getting a flush hand in poker (including straight and royal flushes)?

each card is different

Solution: Let *S* denote all possible 5-card poker hands. Then the size of *S* equals $C(52,5)$ (i.e. 52 choose 5). Since each hand is equally likely, we have $P(x) = 1/C(52,5)$ for all $x \in S$. Let *E* be the event of obtaining a flush hand. What is the size of *E*, or the number of possible flush hands? We employ the Multiplication Principle: there are 52 choices for the first card, 12 for the second, 11 for the third, 10 for the fourth, and 9 for the fifth card. But order doesn't matter, so we need divide by 5!, the size of the corresponding equivalence class. It follows that

assumes order matters

$$|E| = \frac{52\cdot 12\cdot 11\cdot 10\cdot 9}{5!} = 4\frac{13!}{5!8!} = 5148$$

permutation of flush

values of cards matter

5 el subsets

Thus,

$$P(E) = \frac{|E|}{|S|} = \frac{5148}{2598960} \approx 0.002\ (0.2\%)$$

permutations of poker hand

Q ♡    5 ♡

52    12

↑ # choices

Alternate Solution: View a flush hand as a 2-element list $(s, V)$, where *s* denotes the suit of the hand and *V* denotes a 5-element subset of card values of the hand.

♡ {Q, 12, 10, 5, J}

$$4 \cdot \binom{13}{5}$$

4 suites

13 cards in a suite

choose a suite
already picked 1 card out from suite

**Probability and Hashing** (See PowerPoint slides)

Example: What is the probability of randomly hashing a set of 3 elements into a hash table with 20 locations so that there is no collision, i.e. each element is hashed to a different slot?

Solution: Let $(a_1, a_2, a_3)$ denote the three elements represented as a 3-tuple. Then hashing these 3 elements into a hash table can be represented by another 3-tuple, which denotes the memory locations of the 3 elements (chosen from a pool of 20 with repetition allowed). For example, the 3-tuple (4,16,7) indicates that the first element $a_1$ is hashed to slot 4, $a_2$ is hashed to slot 16, and $a_3$ is hashed to slot 7. Let $S$ denote the set of all possible 3-tuples of locations and $E \subset S$ the subset of permutations (i.e. event of no collision where the elements are all hashed to different slots). By the Product Principle, the size of each set is

permutation

Pochhammer notation

$$|S| = 20 \cdot 20 \cdot 20 = 20^3$$

$$|E| = 20 \cdot 19 \cdot 18 = 20^{\underline{3}}$$

Thus, assuming that elements are hashed randomly, in which case the probability distribution is uniform, the probability of hashing all three elements to different slots and avoid collision is given by

$$P(E) = \frac{|E|}{|S|} = \frac{20^{\underline{3}}}{20^3} = 0.855$$

opposite of given w/ same sample space

1 is total possibilities

complement

NOTE:

1. Observe that the probability of collision can be thought of as a complementary event $S - E$:

$\bar{E} = S - E$

$$P(\text{Collision}) = P(S - E) = P(S) - P(E) = 1 - \frac{20^{\underline{3}}}{20^3} = 0.145$$

$\frac{|S|}{|S|}$

2. If we increase the number of elements to 6, then the probability of no collision drops to less than half:

$$P(E) = \frac{|E|}{|S|} = \frac{20^{\underline{6}}}{20^6} = 0.43605$$

less than half
no random hashing

**Complementary Probabilities**

**Theorem**: If $E$ and $F$ are two complementary events, i.e. $E = S - F$ (or equivalently $F = S - E$ ), then

$$P(E) = 1 - P(F)$$

$P(E) + P(F) = 1$

or equivalently,

$$\boxed{P(F) = 1 - P(E)}$$

NOTE: $\bar{E} = S - E$ is called the complement of $E$.

Example: (Birthday Problem) Four people are chosen at random. What is the probability that two or more people have the same birthday?

$(d_1, d_2, d_3, d_4)$ — repetition allowed
— can assume either way why order matters or not

Solution: Consider a sample space ($S$, $P$) where $S$ is the set of all outcomes represented as 4-element tuples (or multisets) of birthdays of the four people chosen. Then $|S| = 365^4$ by the Multiplication Principle. Let $E$ be the event that two or more people have the same birthday. We consider the complement $\bar{E}$ where everyone has a different birthday. Then $|\bar{E}| = 365 \cdot 364 \cdot 363 \cdot 362 = 365_4$. It follows that the probability of 4 people have different birthdays is

less than 1

$$\frac{365}{365} \cdot \frac{364}{365} \cdots$$

$$P(\bar{E}) = \frac{|\bar{E}|}{|S|} = \frac{365_4}{365^4}$$

easier to count complement

Thus,

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{365_4}{365^4} \approx 0.0164 \ \ (1.64\%)$$

NOTE: If we increase the number of people chosen to 23, then

replace 4 w/ 23

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{365_{23}}{365^{23}} \approx 0.5073 \ \ (50.73\%)$$

so it is more likely than not that two people will have the same birthday. This is surprising since there are 365 different birthdays, which is relatively large compared to 23.

## 5.2 Unions and Intersections

### The Probability of a Union of Events

Example: What is the probability on the roll of a fair die that an even number or a prime number appears?

intersection (double counts)

Solution: Our event can be considered ~~the~~ union of two other events:

$$E = \{2,4,6\} \text{ (even number)}$$

$$F = \{2,3,5\} \text{ (prime number)}$$

write out what each event is

merge

$$E \cup F = \{2,3,4,5,6\}$$

Then $P(E \cup F) = 5/6$. Observe that

$$P(E \cup F) \neq P(E) + P(F) = 3/6 + 3/6 = 6/6 = 1$$

since the right hand side double counts the outcomes that are in both events $E$ and $F$. The correct formula is

$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

$$= 3/6 + 3/6 - 1/6 = 5/6$$

intersect

overlap (intersection)

where $E \cap F = \{2\}$.

**Proposition**:

a) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

b) If $E \cap F = \varnothing$, then $P(E \cup F) = P(E) + P(F)$

empty

NOTE: The proof relies on the same counting argument as that for the formula

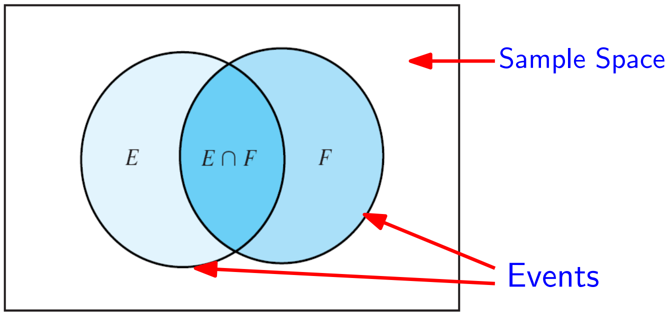
$$|E \cup F| = |E| + |F| - |E \cap F|$$

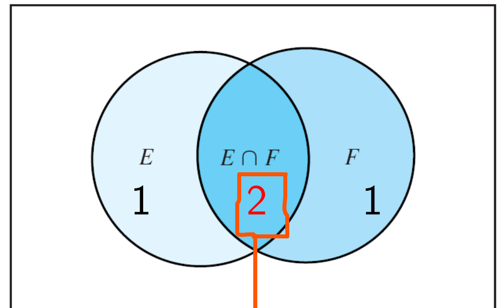
inclusion exclusion

Dividing both sides by $|S|$ yields the part a) of the proposition. We generalize this concept to the union of an arbitrary number of sets.

**Principle of Inclusion and Exclusion for Counting**

I. Two Sets:

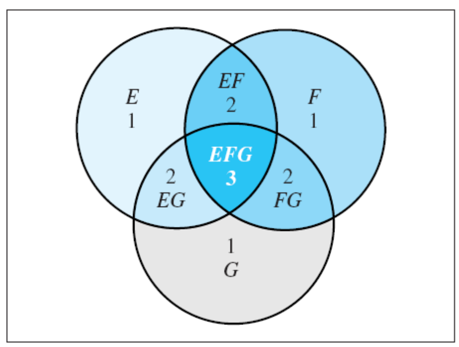



$$|E \cup F| = |E| + |F| - |E \cap F|$$

\# of sets each element belongs to

II. Three Sets:




repetition not allowed
order doesn't matter
↑
subset

$$|E \cup F \cup G| = |E| + |F| + |G| - |E \cap F| - |E \cap G| - |F \cap G| + |E \cap F \cap G|$$

single sets exclude include

III. Four Sets

$$\begin{aligned}|E_1 \cup E_2 \cup E_3 \cup E_4| &= |E_1| + |E_2| + |E_3| + |E_4| \quad \binom{4}{1}\\ &-|E_1 \cap E_2| - |E_1 \cap E_3| - |E_1 \cap E_4| - |E_2 \cap E_3| - |E_2 \cap E_4| - |E_3 \cap E_4| \leftarrow \binom{4}{2}\\ &+|E_1 \cap E_2 \cap E_3| + |E_1 \cap E_2 \cap E_4| + |E_1 \cap E_3 \cap E_4| + |E_2 \cap E_3 \cap E_4| \leftarrow \binom{4}{3}\\ &-|E_1 \cap E_2 \cap E_3 \cap E_4| \leftarrow \binom{4}{4}\end{aligned}$$

**Theorem**: (Principle of Inclusion-Exclusion) Given $n$ sets $E_1, E_2, \ldots, E_n$, we have

$$\left|\bigcup_{k=1}^{n} E_k\right| = \sum_{k=1}^{n} (-1)^{k+1} \left( \underbrace{\sum_{1 \le i_1 < \ldots < i_k \le n} \left| E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k} \right|}_{\text{all ways to intersect } k \text{ sets}} \right)$$

k = index counter (# subsets to intersect)

alternation of add & subtract

all ways to intersect k sets

**Application: Derangements**

every element out of its natural position

**Definition**: Let $S = \{1,2,3,\ldots,n\}$.

(a) A *permutation* $P$ of $S$ is a rearrangement of $S$, i.e. $P$ is an $n$-element repetition-free list chosen from the $n$ elements of $S$.

(b) A *derangement* $D$ of $S$ is a permutation of $S$ where every element of $D$ is out of its natural position, i.e. the element $j$ cannot occupy the $j^{\text{th}}$-position.

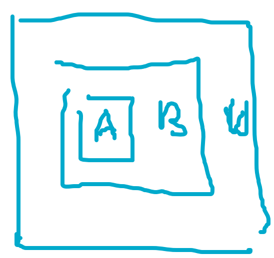
Example: The permutation $(2,4,1,3)$ is a derangement whereas the permutation $(3,2,1,4)$ is not (elements 2 and 4 are in their natural $2^{\text{nd}}$ and $4^{\text{th}}$ positions).

**Theorem**: The number of derangements of $S = \{1,2,3,\ldots,n\}$ equals

$$\boxed{n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}}$$

Proof: Denote by $U$ to be all permutations of $S$ and denote by $A$ the set of derangements of $S$, i.e. set of 'good lists' (permutations). Then define $B$ to be the set of bad lists, i.e. those lists where at least one element is in its natural position. It follows that $A = U - B$. We now count $B$ using the Inclusion-Exclusion Principle: define subsets

(natural position)




$$B_1 = \{\text{lists in } B \text{ with element 1 in position 1}\}$$
$$B_2 = \{\text{lists in } B \text{ with element 2 in position 2}\}$$
$$\ldots$$
$$B_n = \{\text{lists in } B \text{ with element } n \text{ in position } n\}$$

Since $B = B_1 \cup B_2 \cup \ldots \cup B_n$, it follows that

$$|B| = |B_1 \cup B_2 \cup \ldots \cup B_n|$$

$$= |B_1| + |B_2| + \ldots + |B_n| \qquad \binom{n}{1} \text{ such terms}$$

each individual set

$$-|B_1 \cap B_2| - |B_1 \cap B_3| - \ldots - |B_{n-1} \cap B_n| \qquad \binom{n}{2} \text{ such terms}$$

$$+|B_1 \cap B_2 \cap B_3| + |B_1 \cap B_2 \cap B_3| + \ldots |B_1 \cap B_2 \cap B_3| \qquad \binom{n}{3} \text{ such terms}$$

$$\ldots$$

$$+(-1)^{n+1}|B_1 \cap B_2 \cap \ldots \cap B_n| \qquad \binom{n}{n} \text{ such terms}$$

inclusion exclusion

It remains to determine all the cardinalities on the right hand side. By the Multiplication Principle we have

( 1 , _ , _ , _ , _ )
natural position

$$|B_1| = 1\cdot(n-1)\cdot(n-2)\cdots 1 = (n-1)!$$
$$|B_2| = (n-1)\cdot 1\cdot(n-2)\cdots 1 = (n-1)!$$
$$\ldots$$
$$|B_n| = (n-1)\cdot(n-2)\cdots 1\cdot 1 = (n-1)!$$

( _ , 2 , _ , _ , _ )
np

Similarly, since

$$B_{a_1} \cap B_{a_2} = \{\text{lists in } B \text{ with elements } a_1 \text{ and } a_2 \text{ in positions } a_1 \text{ and } a_2\text{, respectively}\}$$

for all 2-element subsets $\{a_1, a_2\}$ chosen from $n$ elements, we have

$$|B_{a_1} \cap B_{a_2}| = (n-2)!$$

In general, we have

$$|B_{a_1} \cap B_{a_2} \cap \ldots \cap B_{a_k}| = (n-k)!$$

for all $k$-element subsets $\{a_1, a_2, \ldots, a_k\}$ chosen from $n$ elements. It follows that

$$|B| = n\cdot(n-1)! - \binom{n}{2}\cdot(n-2)! + \binom{n}{3}\cdot(n-3)! + \ldots + \binom{n}{n}\cdot(n-n)!$$
$$= \sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\cdot(n-k)!$$

Thus, the number of derangements equals

$$|A| = |U| - |B|$$
$$= n! - \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\cdot(n-k)!$$
$$= \sum_{k=0}^{n}(-1)^{k+2}\binom{n}{k}\cdot(n-k)!$$
$$= \sum_{k=0}^{n}(-1)^{k}\frac{n!}{k!(n-k)!}\cdot(n-k)!$$
$$= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

translates to every cap out of his / her own natural position

permutation

label students by #

Example: (Graduation Cap Problem) At a high school commencement, all ten graduating seniors toss their caps into the air. After retrieving their caps, they discover that no student is holding his or her own cap. In how many different ways could this have happened?

Solution: We view such a distribution of 10 caps as a derangement of 10 elements. For example, suppose the position of the derangement indicates the student and the value of an element indicates the cap. Then (2,4,6,8,10,1,3,5,7,9) means student 1 received cap 2, student 2 received cap 4, student 3 received cap 6, etc. Let $A$ denote the set of all such derangements. Then the size of $A$ is given by

interpretation

order matters
repetition not allowed

$$10!\sum_{k=0}^{10}\frac{(-1)^k}{k!}=10!\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\pm\ldots+\frac{1}{10!}\right)=\frac{10!}{0!}-\frac{10!}{1!}+\frac{10!}{2!}-\frac{10!}{3!}\pm\ldots+\frac{10!}{10!}=1334961$$

not probability

**Principle of Inclusion and Exclusion for Probability** (see power point slides)

**Theorem**: (Principle of Inclusion-Exclusion for Probability) Given $n$ events $E_1, E_2, \ldots, E_n$, then

$$P\left(\bigcup_{k=1}^{n} E_k\right)=\boxed{\sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1<\ldots<i_k\le n} P(E_{i_1}\cap E_{i_2}\cap\ldots\cap E_{i_k})\right)}$$

Example: (Graduation Cap Problem Revisited) At a high school commencement, all ten graduating seniors toss their caps into the air. After retrieving their caps, they discover that no student is holding his or her own cap. What is the probability of this happening?

Solution: Let $S$ denote the set of permutations representing all ways that students can retrieve their caps and $E\subset S$ the subset of derangements where no student retrieves his or her cap. Then by our previous calculation, the probability of $E$ occurring is given by

$$P(E)=\frac{|E|}{|S|}=\frac{10!\sum_{k=0}^{10}\frac{(-1)^k}{k!}}{10!}=\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\pm\ldots+\frac{1}{10!}\right)=\frac{16481}{44800}\approx 0.36788$$

→ all permutations

NOTE: Observe that if the number of students $n\to\infty$, then the probability approaches (using calculus)

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k}{k!}=\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\pm\ldots=\frac{1}{e}\approx 0.36787944$$

same

### 5.3 Conditional Probability and Independence

sample space

**Conditional Probability**

changes

Example: On the roll of a pair of fair dice, what is the probability that the numbers on the dice sum to a prime number given that the first die shows a 3?

→ condition restriction

Solution: Our event can be considered as the intersection of two other events $E$ (prime outcomes) and $F$ (first die is a 3):

$$E=\{(1,1),(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(4,1),(4,3),(5,2),(5,6),(6,1),(6,5)\}$$

← possibilities of prime #

sample space, restrict to condition

$$F=\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$$

→ condition

$$E\cap F=\{(3,2),(3,4)\}$$

However, the fact we have assumed the event $F$ to have already occurred means that our sample space is now restricted to $B$ (instead of the sample space $S$ consisting of 36 outcomes). Thus, the probability of $E$ given $F$ is

NOT

fair experiment

$$\boxed{P(E\mid F)=\frac{|E\cap F|}{|F|}}=2/6=1/3$$

Observe that the same answer can be gotten as a ratio of corresponding probabilities:

helpful in definition

ratio of 2 other probabilities

$$\boxed{P(E\mid F)=\frac{P(E\cap F)}{P(F)}}=\frac{\frac{|E\cap F|}{|S|}}{\frac{|F|}{|S|}}=\frac{2/36}{6/36}=1/3$$

unfair experiment

NOTE: Observe that $P(E\mid F)<P(E)=15/36$.

**Definition**: (Conditional Probability) Let $(S, P)$ be a sample space. Suppose $E$ and $F$ are two given events and $F$ satisfies $P(F)\neq 0$. The *conditional probability* $P(E\mid F)$, i.e., the probability of $E$ given $F$, is defined to be

$$\boxed{P(E\mid F)=\frac{P(E\cap F)}{P(F)}}$$

$$P(E)=\frac{|E|}{|S|}=\frac{2^4}{2^5}$$

Example: A coin is flipped five times.
a) What is the probability that the first flip is TAILS given that exactly three of the five flips are HEADS?

define & denote events on a sample space

Solution: Let $S$ be all possible outcomes from five flips of a coin and denote by

5-el tuple

$$E=\{\text{first flip is TAILS}\}$$
$$F=\{\text{three out of five flips are HEADS}\}$$

Then

$$E\cap F=\{\text{first flip is TAILS and three out of (last) four flips are HEADS}\}$$

Since $|S|=2^5$, $|F|=\binom{5}{3}=10$, and $|E\cap F|=\binom{4}{3}=4$, we have

HTHHT

SUBSET $\to \{1,3,4\}$

3 el subset, 5 elements

$$P(E\mid F)=\frac{P(E\cap F)}{P(F)}=\frac{4/2^5}{10/2^5}=\frac{2}{5}$$

4 elements, 1 remove

NOTE: Observe that here $P(E\mid F)\neq P(E)=1/2$.

b) What is the probability that the first flip is TAILS given that the last flip comes up HEADS?

Solution: Define similarly the events

$$E=\{\text{first flip is TAILS}\}$$
$$F=\{\text{last flip is HEADS}\}$$

Then

$$E\cap F=\{\text{first flip is TAILS and last flip is HEADS}\}$$

Since again $|S|=2^5$, but $|F|=2^4$ and $|E\cap F|=2^3$, we have

T, _ _ _, H

$1\cdot 2\cdot 2\cdot 2\cdot 1=2^3$

union = adding (or) (binary addition)



commutative property

$$P(E\,|\,F)=\frac{P(E\cap F)}{P(F)}=\frac{2^3/2^5}{2^4/2^5}=\frac{1}{2}$$

NOTE: Observe that here $P(E\,|\,F)=P(E)=1/2$ so that $P(E)=\dfrac{P(E\cap F)}{P(F)}$, or equivalently,

$$\boxed{P(E\cap F)=P(E)P(F)}$$

This is because event $E$ is independent of event $F$ and motivates a product principle that we shall elucidate on later.

intersection (and)

product (binary multiply)

**Bayes' Theorem**

**Theorem**: (Bayes) Let $E$ and $F$ be two given events. Then

$$\boxed{P(E|F)=\frac{P(F\,|\,E)P(E)}{P(F)}}$$

Proof: We equate $P(E\,|\,F)=\dfrac{P(E\cap F)}{P(F)}$ and $P(F\,|\,E)=\dfrac{P(F\cap E)}{P(E)}$ by observing that $P(E\cap F)=P(F\cap E)$.

positive result GIVEN they used steroids

Example: A manufacturer claims that its drug test will detect steroid use, i.e. show a positive result for an athlete who uses performance-enhancing steroids, 95% of the time. But the manufacturer also admits that steroid-free athletes test positive 15% of the time (false positive rate). Suppose an athlete tested positive for steroids using this drug test. What is the probability that this athlete actually uses steroids given that it known 10% of all athletes use steroids?

unknown

green, red dot over lap

Solution: Let

$E$ = the event that an athlete uses steroids
$E'$ = the event that an athlete does not use steroids
$F$ = the event that a drug test gives a positive result on any athlete


Then

$$P(E)=0.1$$
$$P(E')=1-P(E)=0.9$$
$$P(F)=P(F\cap E)+P(F\cap E')=P(F\,|\,E)P(E)+P(F\,|\,E')P(E')$$
$$=(0.95)(0.1)+(0.15)(0.9)=0.23$$

F = {Red} ∪ {Green} $=(E\cap F)\cup(E'\cap F)$

It follows from Bayes Theorem that

$$P(E|F)=\frac{P(F\,|\,E)P(E)}{P(F)}=\frac{(0.95)(0.1)}{0.23}=0.413$$

Thus, the probability that an athlete who tests positive actually uses steroids is 0.413.

**Independence**

**Definition**: We say that $E$ and $F$ are *independent* events if $P(E|F)=P(E)$.

**Theorem**: (Product Principal for Independent Probabilities) If $E$ and $F$ are independent events, then

Test for independence

$$\boxed{P(E \cap F) = P(E)P(F)}$$

Example: A bag contains 10 balls consisting of 5 black ones and 5 white ones. Suppose two balls are drawn from the bag. What is the probability of picking a black ball on the first draw and also a black one on the second draw, assuming

a) Replacement: The first ball drawn is put back in the bag
b) No replacement: The first ball is NOT put back in the bag

like the problem resets

Solution:

a) Let $S$ denote the set of all outcomes for drawing 2 balls from the bag. Then $|S| = 10 \cdot 10 = 100$.

Define

$$E = \{\text{first ball black}\}$$
$$F = \{\text{second ball black}\}$$

independent

10 balls each time 40 open bag (2-el tuple)

Since

blk ch ball choices

$$|E| = 5 \cdot 10 = 50$$
$$|F| = 10 \cdot 5 = 50$$
$$|E \cap F| = 5 \cdot 5 = 25$$

we have

$$P(E \cap F) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(E)P(F)$$

Thus, $E$ and $F$ are independent events.

b) Again, let $S$ denote the set of all outcomes for drawing 2 balls from the bag. Then $|S| = 10 \cdot 9 = 90$. Define

$$E = \{\text{first ball black}\}$$
$$F = \{\text{second ball black}\}$$

Since

$$|E| = 5 \cdot 9 = 45$$
$$|F| = 5 \cdot 5 + 5 \cdot 4 = 45 \text{ (WB+BB)}$$
$$|E \cap F| = 5 \cdot 4 = 20$$

2 cases (b/c # of balls change)

we have

$$P(E \cap F) = \frac{20}{90} = \frac{2}{9} \neq \frac{1}{4} = P(E)P(F)$$

Thus, in this case $E$ and $F$ are NOT independent events.

**Independent Trials Processes**

**Definition**: A probability experiment that occurs in stages is called an *independent trials process* if the probability at any stage is independent of any other stage.

one trial → another

Example: (Repeated independent trials) Suppose a die is rolled four times. What is the probability that all rolls were 3's?

(given)

$|S| = 6^4$

$P(E) = \frac{1}{6^4}$

E

$|E| = 1^4$

more rigorous

Solution: Since each roll (stage) is independent of all other rolls, we can apply the notion of independence repeatedly to obtain

AND

$P(G) = P(E \cap F)$
$= P(E) \cdot P(F)$

G

$$
\begin{aligned}
P(3333) &= P(\overbrace{(3 \text{ on roll } 1)}^{E} \cap \overbrace{(3 \text{ on rolls } 2,3,4)}^{F}) \\
&= P(3 \text{ on roll } 1) \cdot P(3 \text{ on rolls } 2,3,4) \quad (E, F \text{ independent events}) \\
&= P(3 \text{ on roll } 1) \cdot P((3 \text{ on roll } 2) \cap (3 \text{ on rolls } 3,\, 4)) \\
&= P(3 \text{ on roll } 1) \cdot P(3 \text{ on roll } 2) \cdot P(3 \text{ on rolls } 3,\, 4) \\
&\ldots \\
&= P(3 \text{ on roll } 1) \cdot P(3 \text{ on roll } 2) \cdot P(3 \text{ on roll } 3) \cdot P(3 \text{ on roll } 4) \\
&= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \\
&= \frac{1}{1296} \approx 0.00077 \ (0.077\%)
\end{aligned}
$$

split into 4 trials (each roll)

**Tree Diagrams**

different weights

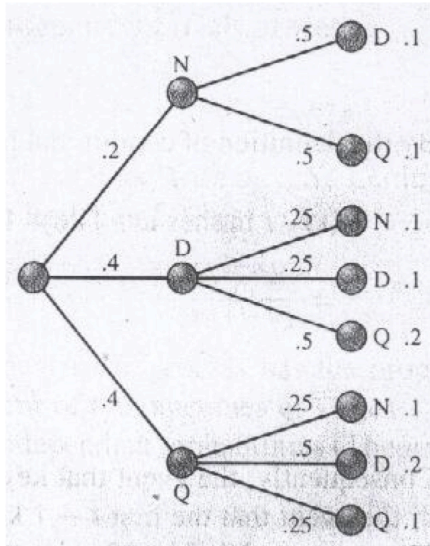
If a probability experiment proceeds in stages, then it is possible to represent it by a tree diagram where each stage corresponds to a branching.

Example:

a) NO REPLACEMENT: A cup contains 5 coins: one nickel, two identical dimes, and two identical quarters. Two coins are drawn from the cup one at a time without replacement. Sketch a tree diagram to compute to probability of each outcome assuming that the drawing of coins is a fair process.
b) REPLACEMENT: Repeat part a) but assume replacement so that the first coin is put back in the cup before drawing the second coin. What is the difference in the two tree diagrams? How can you recognize a tree diagram for an independent trials process?

Solution:

a) The corresponding tree diagram is sketched below where the values along each branch (from left to right) denote the probability of drawing a coin and the values on the far right denote the probability of each outcome (drawing two coins).




$P(N D)$

$\downarrow$

$P(N) \cdot P(D|N)$

$\frac{1}{5} \cdot \frac{2}{4} = \frac{2}{20}$

$0.2 \cdot 0.5 = 0.1$

b) Left as an exercise.

Door 1 Door 2 Door 3

goat

**The Monty Hall Problem**

http://en.wikipedia.org/wiki/Monty_hall_problem

$\frac{1}{3} \leftarrow$ originally picking probability

$\rightarrow$ simple solutions $\rightarrow$ switch