Monte Carlo Project 2023-2024

Asian Option

Writer :
Sixtine Sphabmixay

Abstract : In this academic project realized for the course of Monte Carlo for finance at Paris-Dauphine University (Master MASEF), we discussed and applied Monte Carlo and finite difference methods in pricing asian options. Different approaches have been proposed in the academic literature for pricing these options. We will first study the three schemes reported in the article of Lapeyre and Temam. The first one is based on Riemann integral, the second on trapezoidal method and the last one is directly derived from the Black Scholes model. These schemes give an estimation of the average price of the underlying. We will use them to get the price of an Asian option with a Monte Carlo method. We will then derive a finite difference method to solve the PDE presented in the article of Rogers and Shi in order to get an other estimation of the price of an Asian option.

Contents

1 Introduction

Options are versatile financial instruments used for various purposes, including speculation, hedging, and risk management. Traders and investors use options to take advantage of price movements, protect against potential losses, and create more complex trading strategies. The options market plays a crucial role in financial markets, providing liquidity and flexibility for market participants. Now, options are financial derivatives that give the holder the right, but not the obligation, to buy or sell an underlying asset at a predetermined price (called the strike) within a specified period (until expiration). Analytic formulas are not always available to price options when derivatives are complicated. Common methods used to price such derivatives include Monte Carlo simulation or numerical methods, such as finite difference used to solve PDE.

Asian options are a type of financial derivative whose payoff is determined by the average price of the underlying asset over a specific time period, as opposed to the asset's price at the option's expiration. The use of the average price over a period makes Asian options less sensitive to short-term fluctuations in the price of the underlying asset compared to European or American options, which depend only on the asset's price at maturity. This characteristic can make Asian options attractive in certain market conditions.
They come in various forms, such as fixed strike Asian options (where the strike price is predetermined) and floating strike Asian options (where the strike price is determined at the end of the option's life based on the average asset price). The valuation of Asian options is more complex than that of standard options, often requiring advanced mathematical models such as Monte Carlo simulations or numerical methods for accurate pricing.

1.1 General framework

The Black and Scholes model is widely used in finance to describe the price of an asset at time t. In that framework we consider a financial market composed of a risky asset (of price S_t at time t) and a risk-free one (of price S_t^0 at time t). We represent the randomness of the market by the probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_t, \mathbb{P})$ where $(\mathcal{F}_t)_t$ is the filtration of the information available and $\mathbb{P}$ is the real world probability measure.

The price S_t is given by the following stochastic differential equation :

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where μ and σ are two positive constants representing the drift and the volatility of the stock price, and $(B_t)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. From the Girsanov theorem, we know that there exists a risk neutral probability $\mathbb{Q}$ under which the process $W_t = B_t + \frac{\mu - r}{\sigma} t$ is a Brownian motion. We can then rewrite the dynamics of S_t in this risk-neutral world as

$$dS_t = S_t(rdt + \sigma dW_t)$$

Applying Itô's formula, we get that the solution of this SDE is

$$S_t = S_{t_0} \exp\left(\sigma W_t + (r - \frac{\sigma^2}{2})t\right)$$

where S_{t_0} is the price of the risky asset at the beginning of the modeling.

The riskless asset satisfies the ordinary differential equation

$$dS_t^0 = rS_t^0 dt$$

As said before, the payoff of Asian options is determined by the average price of the underlying asset over a specific time period. Thus the price of an Asian option with maturity T can be written

$$V(t, S, A) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}(f(S_t, A_S(t_0, t)))$$

where

$$A_S(t_0, t) = \frac{1}{t - t_0} \int_{t_0}^{t} S_u du$$

We chose $t_0 = 0$ for the rest of this project. f can take various expression depending of the type of the option :

- For a call with fixed strike K : $f(s, a) = (a - K)_+$
- For a put with fixed strike K : $f(s, a) = (K - a)_+$
- For a call with floating strike : $f(s, a) = (s - a)_+$
- For a put with floating strike : $f(s, a) = (a - s)_+$

2 Pricing by Monte Carlo

The main difficulty for pricing Asian option is to estimate $Y_T = \int_0^T S_t dt$. We will study three schemes proposed in the article of Lapeyre and Temam which estimate this integral.
We will divide the time interval $[0,T]$ into N steps of size $h = \frac{T}{N}$ and define the times $t_k = \frac{kT}{N} = kh$. We will consider in this part the case where $f(s,a) = (a-K)_+$.

2.1 Riemann scheme

The first scheme proposed by Lapeyre and Temam is based on a widely used method to estimate integrals which is the Riemann sum. Riemann sums are approximations used to calculate the definite integral of a function over a closed interval. The Riemann sum is based on dividing the interval into subintervals and approximating the area under the curve of the function within each subinterval.
Now, since we can simulate S_t at any given time t, Y_T can be approached by Riemann sums :

$$\begin{aligned} Y_T^{r,N} &= \sum_{k=0}^{N-1} S_{t_k}(t_{k+1} - t_k) \\ &= h \sum_{k=0}^{N-1} S_{t_k} \end{aligned}$$

as $\forall k \in \{0, ..., n-1\}$, $t_{k+1} - t_k = h$

Using this scheme, the approximate value of an Asian call option with fixed strike is given by

$$\frac{e^{-rT}}{M} \sum_{j=1}^{M} (\frac{h}{T} \sum_{k=0}^{N-1} S_{t_k} - K)_+$$

where M is the number of Monte Carlo simulation.

The time complexity of this algorithm is $\mathcal{O}(\frac{1}{NM})$ as this is true for every kind of Monte Carlo methods. This time complexity involves the step error and the Monte Carlo error of order $\frac{\sigma}{\sqrt{M}}$.

2.2 Trapezoidal scheme

This scheme is equivalent to the trapezoidal method, as it will be proved later in this part. As in $\mathcal{L}^2$ the closest random variable to $(\frac{1}{T}\int_0^T S_u du - K)_+$ when the $(S_{t_k}, k = 0, \ldots, N)$ are known is $\mathbb{E}((\frac{1}{T}\int_0^T S_u du - K)_+ | \mathcal{B}_h)$ where $\mathcal{B}_h$ is the σ-field generated by the $(S_{t_k}, k = 0, \ldots, N)$.

Since $W_u|W_{t_k}=x, W_{t_{k+1}}=y$ follows a gaussian distribution of mean $\frac{t_{k+1}-u}{h}x+\frac{u-t_k}{h}y$ and variance $\frac{(t_{k+1}-u)(u-t_k)}{h}$, for $t_k \leq u \leq t_{k+1}$, we can compute

$$(\mathbb{E}(\frac{1}{T}\int_0^T S_u du|\mathcal{B}_h)-K)_+ = (\frac{1}{T}\int_0^T \mathbb{E}(S_u|\mathcal{B}_h)du-K)_+$$

as a function of $(W_{t_k}, k=0,\ldots,N)$.

Indeed, using the fact that $\mathcal{L}(W_u|W_{t_k}=x, W_{t_{k+1}}=y)=\mathcal{N}(\frac{t_{k+1}-u}{h}x+\frac{u-t_k}{h}y, \frac{(t_{k+1}-u)(u-t_k)}{h})$, the definition of $(S_t)_{t\in[0,T]}$ given in the introduction and the result that gives us that if X is a gaussian random variable of mean μ and variance σ^2, we have $\mathbb{E}(e^{tX})=e^{\mu t+\frac{\sigma^2 t^2}{2}}$ we get :

$$\begin{aligned}
\mathbb{E}(\frac{1}{T}\int_0^T S_u du|\mathcal{B}_h) &= \frac{1}{T}\int_0^T \mathbb{E}(S_u|\mathcal{B}_h)du \\
&= \frac{1}{T}\int_0^T \mathbb{E}(e^{(r-\frac{\sigma^2}{2})u+\sigma B_u}|B_{t_k}=W_{t_k}, B_{t_{k+1}}=W_{t_{k+1}})du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{(r-\frac{\sigma^2}{2})u}e^{\sigma(\frac{t_{k+1}-u}{h})W_{t_k}+\sigma(\frac{u-t_k}{h})W_{t_{k+1}}+\frac{\sigma^2}{2}\frac{(t_{k+1}-u)(u-t_k)}{h}}du \\
&= \frac{1}{T}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} e^{\sigma(\frac{u-t_k}{h})(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2}{2}(\frac{u-t_k}{h})^2+ru}e^{\sigma W_{t_k}-\frac{\sigma^2}{2}t_k}du
\end{aligned}$$

By applying Taylor expansion to the exponential function and considering the change of variable $y=u-t_k$, we get :

$$e^{\sigma\frac{y}{h}(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2 y^2}{2}+ry} \sim 1+\frac{\sigma y}{h}(W_{t_{k+1}}-W_{t_k})-\frac{(\sigma y)^2}{2}+ry$$

therefore we have

$$\begin{aligned}
\mathbb{E}(\frac{1}{T}\int_0^T S_u du|\mathcal{B}_h) &= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\int_0^h e^{\sigma\frac{y}{h}(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2 y^2}{2}+ry}dy \\
&= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\int_0^h (1+\frac{\sigma y}{h}(W_{t_{k+1}}-W_{t_k})-\frac{(\sigma y)^2}{2}+ry+\mathcal{O}(h))dy \\
&= \frac{1}{T}\sum_{k=0}^{N-1} hS_{t_k}(1+\frac{\sigma}{2}(W_{t_{k+1}}-W_{t_k})+\frac{rh}{2})
\end{aligned}$$

and we get the second scheme

$$Y_T^{e,N}=\frac{h}{T}\sum_{k=0}^{N-1} S_{t_k}(1+\frac{rh}{2}+\sigma\frac{W_{t_{k+1}}-W_{t_k}}{2})$$

Now as mention before, this scheme is equivalent to the trapezoidal method. Indeed,

$$\frac{1}{T}\sum_{k=0}^{N-1} h\frac{S_{t_k}+S_{t_{k+1}}}{2} = \frac{1}{T}\sum_{k=0}^{N-1}\frac{hS_{t_k}}{2}(e^{\sigma(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2 h}{2}+rh}+1)$$

since $S_{t_{k+1}} = S_{t_k}e^{\sigma(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2 h}{2}+rh}$ and as $W_{t_{k+1}} - W_{t_k} \approx \sqrt{h}Z$ where $Z \sim \mathcal{N}(0,1)$ we can again use the Taylor expansion $\exp x = 1 + x + \frac{x^2}{2} + ...$ for h small to get

$$e^{\sigma(W_{t_{k+1}}-W_{t_k})-\frac{\sigma^2 h}{2}+rh} \sim 1 + \sigma(W_{t_{k+1}} - W_{t_k}) + rh$$

and therefore the trapezoidal method can be rewritten as

$$\frac{1}{T}\sum_{k=0}^{N-1}\frac{S_{t_k}+S_{t_{k+1}}}{2} = \frac{1}{T}\sum_{k=0}^{N-1}\frac{hS_{t_k}}{2}(2+\sigma(W_{t_{k+1}}-W_{t_k})+rh+\mathcal{O}(h))$$

and we recognize the previous scheme.

2.3 Black-Scholes scheme

For this last scheme we use the the fact that a brownian motion is a gaussian process. Therefore, using $S_u = S_{t_k}e^{\sigma(W_u-W_{t_k})+(r-\frac{\sigma^2}{2})(u-t_k)}$ for all $k \in \{0,\dots,n-1\}$, $t_k \le u \le t_{k+1}$ we can rewrite Y_T as

$$\begin{aligned} Y_T &= \frac{1}{T}\int_0^T S_u du \\ &= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\int_{t_k}^{t_{k+1}} e^{\sigma(W_u-W_{t_k})+(r-\frac{\sigma^2}{2})(u-t_k)}du \end{aligned}$$

Now we can apply a Taylor expansion to get

$$e^{\sigma(W_u-W_{t_k})+(r-\frac{\sigma^2}{2})(u-t_k)} \sim 1 + r(u-t_k) + \sigma(W_u - W_{t_k})$$

Using the change of variable $y = u - t_k$ we can rewrite Y_T as

$$\begin{aligned} Y_T &= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}\int_{t_k}^{t_{k+1}} 1 + r(u-t_k) + \sigma(W_u-W_{t_k})du \\ &= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}(\int_0^h (1+rh)dy + \sigma\int_{t_k}^{t_{k+1}}(W_u-W_{t_k})du) \\ &= \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}(h + \frac{rh^2}{2} + \sigma\int_{t_k}^{t_{k+1}}(W_u-W_{t_k})du) \end{aligned}$$

We get our last scheme :

$$Y_T^{p,N} = \frac{1}{T}\sum_{k=0}^{N-1} S_{t_k}(h + \frac{rh^2}{2} + \sigma\int_{t_k}^{t_{k+1}}(W_u - W_{t_k})du)$$

To simulate this scheme in practice we need to simulate at each step $W_{t_{k+1}}|W_{t_k}$ and $(\int_{t_k}^{t_{k+1}} W_u du|W_{t_k}, W_{t_{k+1}})$.

For the first one : By the properties of the Brownian motion, $W_{t_{k+1}} - W_{t_k}$is independent from W_{t_k} and $W_{t_{k+1}} - W_{t_k} \sim \mathcal{N}(0,h)$, so

$$\begin{aligned} W_{t_{k+1}}|W_{t_k} = x &\approx W_{t_k} + (W_{t_{k+1}} - W_{t_k}) \\ &\approx x + W_{t_{k+1}} - W_{t_k} \\ &\sim \mathcal{N}(x,h) \end{aligned}$$

so we get the following iteration $\forall k \in \{0,...,N-1\}$:

$$\begin{cases} W_{t_{k+1}} = W_{t_k} + Z_{k+1} \\ W_{t_0} = 0 \end{cases}$$

where $Z_{k+1} \sim \mathcal{N}(0,h)$ for each $k \in \{0,...,N-1\}$

For the second one, we use again the fact that $\mathcal{L}(W_u|W_{t_k} = x, W_{t_{k+1}} = y) = \mathcal{N}(\frac{t_{k+1}-u}{h}x + \frac{u-t_k}{h}y, \frac{(t_{k+1}-u)(u-t_k)}{h})$, so $(\int_{t_k}^{t_{k+1}} W_u dt|W_{t_k}, W_{t_{k+1}})$ follows a gaussian distribution of mean

$$\begin{aligned} \mathbb{E}(\int_{t_k}^{t_{k+1}} W_u du|W_{t_k} = x, W_{t_{k+1}} = y) &= \int_{t_k}^{t_{k+1}} \mathbb{E}(W_u|W_{t_k} = x, W_{t_{k+1}} = y)du \\ &= \int_{t_k}^{t_{k+1}}(\frac{t_{k+1}-u}{h}x + \frac{u-t_k}{h}y)du \\ &= \frac{h}{2}x + \frac{h}{2}y \end{aligned}$$

and variance $\frac{h^3}{12}$. Indeed,

$$\mathbb{E}((\int_{t_k}^{t_{k+1}} W_u du)^2|W_{t_k} = x, W_{t_{k+1}} = y) = 2\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \mathbb{E}(W_u W_t|W_{t_k} = x, W_{t_{k+1}} = y)dtdu$$

$= 2\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \frac{(t_{k+1}-u)(t-t_k)}{h} + (\frac{(t_{k+1}-t)x}{h} + \frac{(t-t_k)y}{h})(\frac{(t_{k+1}-u)x}{h} + \frac{(u-t_k)y}{h})dtdu$

Now,

$$2\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \frac{(t_{k+1}-u)(t-t_k)}{h}dtdu = \frac{h^3}{12}$$

and after calculations,

$$2\int_{t_k}^{t_{k+1}}\int_{t_k}^{u}(\frac{(t_{k+1}-t)x}{h}+\frac{(t-t_k)y}{h})(\frac{(t_{k+1}-u)x}{h}+\frac{(u-t_k)y}{h})dtdu=(\frac{hx}{2}+\frac{hy}{2})^2$$

So

$$\begin{aligned}
&\mathbb{V}(\int_{t_k}^{t_{k+1}}W_u du|W_{t_k}=x,W_{t_{k+1}}=y)\\
&\qquad=\mathbb{E}((\int_{t_k}^{t_{k+1}}W_u du)^2|W_{t_k}=x,W_{t_{k+1}}=y)-\mathbb{E}(\int_{t_k}^{t_{k+1}}W_u du|W_{t_k}=x,W_{t_{k+1}}=y)^2\\
&\qquad=\frac{h^3}{12}
\end{aligned}$$

Therefore, $(\int_{t_k}^{t_{k+1}}W_u dt|W_{t_k}=x,W_{t_{k+1}}=y)\sim\mathcal{N}(\frac{h}{2}x+\frac{h}{2}y,\frac{h^3}{12})$

2.4 Numerical results

Let's present some results that we got while applying these three schemes. For this we took $r=0.1$, $T=1$, $N=50$, $K=S_0=100$,$M=10^5$ and simulated the price of an Asian call option with fixed strike.

σ	Pricing Riemann scheme	C.I	Error	Variance
0.05	4.63	$[4.62, 4.65]$	0.0083	6.96
0.2	6.92	$[6.86, 6.97]$	0.026	70.7
0.3	8.93	$[8.86, 9.01]$	0.039	153.23

σ	Pricing Trapez scheme	C.I	Error	Variance
0.05	4.73	$[4.71, 4.74]$	0.0084	7.02
0.2	6.98	$[6.93, 7.03]$	0.027	71.01
0.3	8.96	$[8.85, 9.0]$	0.039	151.67

σ	Pricing BS scheme	C.I	Error	Variance
0.05	4.74	$[4.72, 4.75]$	0.0085	7.25
0.2	7.07	$[7.01, 7.12]$	0.027	73.79
0.3	9.04	$[8.96, 9.12]$	0.04	157.16

2.5 Convergence in L^p

In this section we will compare the rate of convergence of the three schemes established before. Let's start with a useful property that we will use to prove the results :

Lemma : Let $Z_t=Z_0+\int_0^t A_s dW_s+\int_0^t B_s ds$ where B_s is a vector in $\mathbb{R}^n$, A_s

a matrix in $\mathbb{R}^{n\times d}$, W_t a d dimensional Brownian motion, A and B are adapted, $\int |A_s|ds < \infty$ and $\mathbb{E}(\int B_s^2 ds) < \infty$, $p \geq 2$ then there exists a constant $C > 0$ and Z_t satisfies

$$\mathbb{E}|Z_t|^p \leq \mathbb{E}|Z_0|^p + C\int_0^t \mathbb{E}(|Z_s|^p + |A_s|^p + |B_s|^p)ds$$

The proof of this result can be found in the course of Bruno Bouchard "Monte Carlo for finance".

Proposition : With the above notations, there exists three non decreasing maps $K_1(T)$, $K_2(T)$, $K_3(T)$ such that,

$$(\mathbb{E}(\sup_{0\leq t\leq T} |Y_t^{r,N} - Y_t|^{2q}))^{\frac{1}{2q}} \leq \frac{K_1(T)}{N}$$

$$(\mathbb{E}(\sup_{0\leq t\leq T} |Y_t^{e,N} - Y_t|^{2q}))^{\frac{1}{2q}} \leq \frac{K_2(T)}{N}$$

$$(\mathbb{E}(\sup_{0\leq t\leq T} |Y_t^{p,N} - Y_t|^{2q}))^{\frac{1}{2q}} \leq \frac{K_3(T)}{N^{\frac{3}{2}}}$$

Proof. For the case of the Riemann scheme :
Using the definition of $Y_t = \int_0^T S_u du$ and $Y_t^{r,N} = Y_{t_k} + S_{t_k}(t - t_k)$ we get for all $t \in [t_k, t_{k+1}]$

$$Y_t - Y_t^{r,N} = Y_{t_k} - Y_{t_k}^{r,N} + \int_{t_k}^t (S_u - S_{t_k})du$$

As for all t, S_t is solution of the SDE $dS_t = rS_t dt + \sigma S_t dW_t$,

$$S_u = S_{t_k} + \int_{t_k}^u rS_s ds + \int_{t_k}^u \sigma S_s dW_s$$

we get,

$$Y_t - Y_t^{r,N} = Y_{t_k} - Y_{t_k}^{r,N} + \int_{t_k}^t (\int_{t_k}^u rS_s ds + \int_{t_k}^u \sigma S_s dW_s)du$$

And by using Fubini theorem,

$$Y_t - Y_t^{r,N} = Y_{t_k} - Y_{t_k}^{r,N} + \int_{t_k}^t (t-s)rS_s ds + \int_{t_k}^t \sigma(t-s)S_s dW_s$$

Now, let's apply the above lemma with $B_s = (t-s)rS_s$, $A_s = \sigma(t-s)S_s$, $Z_t = Y_t - Y_t^{r,N}$. We get

$$\begin{aligned}
\mathbb{E}|Y_t - Y_t^{r,N}|^{2q} &\leq \mathbb{E}|Y_{t_k} - Y_{t_k}^{r,N}|^{2q} + \tilde{C}\int_{t_k}^t (\mathbb{E}(|Y_s - Y_s^{r,N}|^{2q} + (t-s)^{2q})ds \\
&\leq \mathbb{E}|Y_{t_k} - Y_{t_k}^{r,N}|^{2q} + \tilde{C}h^{2q+1} + \tilde{C}\int_{t_k}^t (\mathbb{E}(|Y_s - Y_s^{r,N}|^{2q})ds
\end{aligned}$$

where $\tilde{C}$ is a constant. By the Gronwall's lemma we have

$$\mathbb{E}|Y_t - Y_t^{r,N}|^{2q} \leq (\mathbb{E}|Y_{t_k} - Y_{t_k}^{r,N}|^{2q} + \tilde{C}h^{2q+1})e^{\tilde{C}h}$$

As this is true for all $t_k \leq t \leq t_{k+1}$, we have for all $k \in \{0, ..., N-1\}$,

$$\mathbb{E}|Y_{t_{k+1}} - Y_{t_{k+1}}^{r,N}|^{2q} \leq (\mathbb{E}|Y_{t_k} - Y_{t_k}^{r,N}|^{2q} + \tilde{C}h^{2q+1})e^{\tilde{C}h}$$

By recursion we get that

$$\mathbb{E}|Y_{t_k} - Y_{t_k}^{r,N}|^{2q} \leq \hat{C}h^{2q}$$

where $\hat{C}$ is a constant. So by the Burkholder-Davis-Gundy inequality, we get the result.
We proceed in a similar way to prove the other two cases.

□

2.6 Variance reduction

In order to reduce the standard error and therefore get more precise results, one could increase the number of Monte Carlo simulation. However, this strategy can be costly as the standard error is inversely proportional to the square root of the sample n. A well known strategy to reduce the variance which avoid increasing the number of Monte Carlo simulation is the control variate technique.

This method consists of using an other random variable X such that its expectation is known, and a constant $b \in \mathbb{R}$ such that $\mathbb{V}(Y_T + b(X - \mathbb{E}(X))) \ll \mathbb{V}(Y_T)$.

As mentioned in the article of Lapeyre, we can approximate $\frac{1}{T}\int_0^T S_u du$ by $Z_T := \exp\left(\frac{1}{T}\int_0^T \log(S_u) du\right)$. This corresponds to the geometric average of Asian options.
As $Z_T = S_0 e^{\frac{T}{2}(r - \frac{\sigma^2}{2})} e^{\frac{\sigma}{T}\int_0^T (T-u) dW_u}$, let's consider $\tilde{Z} = \frac{\sigma}{T}\int_0^T (T-u) dW_u$. We got the $\tilde{Z}$ from $\frac{\sigma}{T}\int_0^T W_u du$ as $d(T-u)W_u = (T-u)dW_u - W_u du$. This quantity has obviously a normal law of mean 0 and variance

$$\begin{aligned}
\mathbb{V}(\tilde{Z}) &= \frac{\sigma^2}{T^2}\mathbb{E}((\int_0^T W_u du)^2) \\
&= \frac{\sigma^2}{T^2}\mathbb{E}(\int_0^T (T-u)^2 du) \\
&= \frac{\sigma^2 T}{3}
\end{aligned}$$

Therefore we can compute the following quantity explicitly in the case where we want to price a call option with fixed strike :

$$\mathbb{E}(e^{-rT}(Z_T - K)_+)$$

First of all, we have for $\tilde{\sigma}^2 = \frac{\sigma^2 T}{3}$

$$\mathbb{E}(e^{-rT}(Z_T - K)_+) = \frac{e^{-rT}}{\sqrt{2\pi\tilde{\sigma}^2}} \int_l^\infty (S_0 e^{\frac{T}{2}(r-\frac{\sigma^2}{2})} e^x - K) e^{-\frac{x^2}{2\tilde{\sigma}^2}} dx$$

where l is such that $Z_T \geq K$. After calculations, we get

$$\frac{\sigma}{T} \int_0^T (T-u) dW_u \geq \log \frac{K}{S_0} - \frac{T}{2}(r - \frac{\sigma^2}{2})$$

and as $\frac{\sigma}{T}\int_0^T (T-u)dW_u \approx x$ in the previous equation we get that $Z_T \geq K$ if $x \geq \log \frac{K}{S_0} - \frac{T}{2}(r - \frac{\sigma^2}{2})$. So $l = \log \frac{K}{S_0} - \frac{T}{2}(r - \frac{\sigma^2}{2})$. Now using a change of variable $y = \frac{x}{\tilde{\sigma}}$ and $u = \frac{x - \tilde{\sigma}^2}{\tilde{\sigma}}$, we get

$$\begin{aligned}
\mathbb{E}(e^{-rT}(Z_T - K)_+) &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\frac{l-\tilde{\sigma}^2}{\tilde{\sigma}}}^\infty S_0 e^{\frac{T}{2}(r-\frac{\sigma^2}{2}) + \frac{\sigma^2 T}{6}} e^{\frac{-u^2}{2}} du - K \frac{e^{-rT}}{\sqrt{2\pi}} \int_{\frac{l}{\tilde{\sigma}}}^\infty e^{-\frac{y^2}{2}} dy \\
&= e^{-rT}(S_0 \exp(C_1)\Phi(C_2 + \tilde{\sigma}) - K\Phi(C_2))
\end{aligned}$$

where Φ is the cumulative distribution function of a $\mathcal{N}(0,1)$ and

$$C_1 = \frac{T}{2}(r - \frac{\sigma^2}{2}) + \frac{\sigma^2 T}{6}$$

$$C_2 = \frac{\frac{T}{2}(r - \frac{\sigma^2}{2}) - \log \frac{K}{S_0}}{\tilde{\sigma}}$$

Therefore we can chose the random variable

$$\hat{Z} = e^{-rT}(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_u du} - K)_+$$

as our control variable for a call option with fixed strike.
As we need to approximate $\int_0^T W_u du$, we can use the three schemes presented before. Hence we consider

$$Z_T^{r,N} = e^{-rT}(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \frac{\sigma}{T}\sum_{k=0}^{N-1} h W_{t_k}} - K)_+$$

$$Z_T^{e,N} = e^{-rT}(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \frac{\sigma}{T}\sum_{k=0}^{N-1} \frac{h}{2}(W_{t_k} + W_{t_{k+1}})} - K)_+$$

$$Z_T^{p,N} = e^{-rT}(S_0 e^{(r-\frac{\sigma^2}{2})\frac{T}{2} + \frac{\sigma}{T}\sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} W_u du} - K)_+$$

As there is a strong correlation between the payoffs of the arithmetic and geometric Asian option we can pick $b = -1$.

2.7 Numerical results

Here are some numerical results using the same variables as previously : $r = 0.1$, $T = 1$, $N = 50$, $M = 10^5$, $S_0 = K = 100$. We notice that the variance has been drastically reduced compared to the standard Monte Carlo method.

σ	Pricing Riemann scheme	C.I	Error	Variance	MC Variance
0.05	4.63	$[4.63, 4.63]$	0.00015	0.002	6.96
0.2	6.99	$[6.987, 6.991]$	0.144	0.15	70.7
0.3	9.01	$[9.01, 9.02]$	0.0027	0.68	153.23

σ	Pricing Trapez scheme	C.I	Error	Variance	MC Variance
0.05	4.72	$[4.7, 4.74]$	0.0084	0.0028	7.05
0.2	6.98	$[6.98, 6.99]$	0.0011	0.12	71.38
0.3	8.96	$[8.96, 8.97]$	0.0023	0.55	153.57

σ	Pricing BS scheme	C.I	Error	Variance	MC Variance
0.05	4.72	$[4.72, 4.72]$	0.00015	0.0024	7.23
0.2	7.04	$[7.04, 7.04]$	0.0013	0.16	73.79
0.3	9.06	$[9.05, 9.06]$	0.0027	0.732	157.16

3 Pricing by PDE

Now we will present a second method to price an asian option, using a PDE established in the article of Rogers and Shi that we will solve using finite difference method. One of the advantages of the PDE approach is that it is generally faster than Monte Carlo methods. However they can be more difficult to implement.

3.1 A PDE for the price of an Asian option

Using the notation of the article we consider,

$$\phi(t, x) := \mathbb{E}((\int_t^T S_u \mu(du) - x)_+ | S_t = 1)$$

where μ is a measure with density ρ_t. $\rho_t = \frac{1}{T}$ in the case of a fixed call strike and $\rho_t = \frac{1}{T} - \delta(T - t)$ for a floating strike. Now, using the fact that

$$M_t := \mathbb{E}((\int_0^T S_u \mu(du) - K)_+ | \mathcal{F}_t)$$

is a martingale and can be rewritten as $M_t = S_t \phi(t, \xi_t)$ where $\xi_t = \frac{K - \frac{1}{t}\int_0^t S_u du}{S_t}$, we get, using Itô's formula (if we assume that ϕ has enough smoothness) that

$$\frac{\partial \phi}{\partial t} + r\phi + \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 \phi}{\partial \xi^2} - (r\xi + \rho_t)\frac{\partial \phi}{\partial \xi} = 0$$

on $[0, T]$.
If we consider $f(t,x) := e^{-r(T-t)}\phi(t,x)$, we find that f solves

$$\frac{\partial f}{\partial t} + \mathcal{G}f = 0$$

with $\mathcal{G} = \frac{\sigma^2 x^2}{2}\frac{\partial^2}{\partial x^2} - (rx + \rho_t)\frac{\partial}{\partial x}$ and

$$f(T,x) = \max(0, -x)$$

In the case of the fixed strike Asian option, and

$$f(T,x) = \max(0, -(1+x))$$

In the case of the floating strike Asian option.

Therefore, the price of an Asian call option with maturity T, fixed strike price K and initial price S_0

$$e^{-rT}\mathbb{E}(\int_0^T (S_u - K)\frac{du}{T})_+ = S_0 f(0, \frac{K}{S_0})$$

And the price of the Asian call option with maturity T and floating strike is

$$e^{-rT}\mathbb{E}(\int_0^T S_u \frac{du}{T} - S_T)_+ = S_0 f(0,0)$$

As we are in the framework of the Black-Scholes model, $\forall t \in [0,T]$, S_t follows a log-normal distribution. Let's consider now ϕ the solution of the PDE established before with the fixed strike boundary condition, and ψ, the solution of the same PDE but with the floating strike boundary condition. So using Fubini and considering $x \leq 0$, we get that

$$\begin{aligned}
\phi(t,x) &= \int_t^T \mathbb{E}(S_u | S_t = 1)\frac{du}{T} - x \\
&= \int_t^T e^{r(T-u)}\frac{du}{T} - x \\
&= \frac{e^{r(T-t)} - 1}{rT} - x
\end{aligned}$$

and

$$\psi(t,x) = \mathbb{E}(\int_t^T S_u \frac{du}{T} - S_t - x) = \frac{e^{r(T-t)} - 1}{rT} - e^{r(T-t)} - x$$

for a very large negative x.

These equations will be useful to implement the finite difference method.

3.2 Finite Difference method

Let's consider the mesh

$$\{(t_n, x_i) := (nh, \alpha + i\delta); 0 \leq n \leq m, 0 \leq i \leq l+1\}$$

where $h = \frac{T}{m}$ and $\delta = \frac{\beta-\alpha}{l+1}$. We approximate $(f(t,x))_{(t,x)\in[0,T]\times[\alpha,\beta]}$ by $(f_i^n)_{0\leq i\leq l+1, 0\leq n\leq m}$ such that $f_i^n \approx f(t_n, x_i)$.
The finite difference method with a Crank Nicolson scheme consists of approximating $\frac{\partial f}{\partial x}(t_n, x_i)$ for all $n \in \{0, ..., m-1\}$, $i = 1, ..., l$, by

$$\begin{aligned}\frac{\partial f}{\partial x}(t_n, x_i) &\approx \frac{f(t_n, x_i+\delta) - f(t_n, x_i-\delta) + f(t_{n+1}, x_i+\delta) - f(t_{n+1}, x_i-\delta)}{4\delta} \\ &\approx \frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{4\delta}\end{aligned}$$

and $\frac{\partial^2 f}{\partial x^2}(t_n, x_i)$ by

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(t_n, x_i) &\approx \frac{f(t_n, x_i+\delta) + f(t_n, x_i-\delta) - 2f(t_n, x_i) + f(t_{n+1}, x_i+\delta) + f(t_{n+1}, x_i-\delta) - 2f(t_{n+1}, x_i)}{2\delta^2} \\ &\approx \frac{f_{i+1}^n + f_{i-1}^n - 2f_i^n + f_{i+1}^{n+1} + f_{i-1}^{n+1} - 2f_i^{n+1}}{2\delta^2}\end{aligned}$$

Considering $\sigma(x) := \sigma x$ and $b(x) := -(rx + \rho_t)$, we get that

$$\mathcal{G}f(t_n, .) \approx \frac{1}{2}Af^n + \frac{1}{2}Af^{n+1} + \frac{1}{2}(g^n + g^{n+1})$$

where A is a $\mathbb{R}^{l\times l}$ tridiagonal matrix with $a_{i,i-1} = \frac{\sigma^2(x_i)}{2\delta^2} - \frac{b(x_i)}{2\delta}$, $a_{i,i} = -\frac{\sigma^2(x_i)}{\delta^2}$, $a_{i,i+1} = \frac{\sigma^2(x_i)}{2\delta^2} + \frac{b(x_i)}{2\delta}$ and for all $n \in \{0, ..., m\}$, g^n is in $\mathbb{R}^l$ and is such that $g_i^n = 0$ for all $i \in \{2, ..., l\}$ and $g_1^n = (\frac{\sigma^2(x_1)}{2\delta^2} - \frac{b(x_1)}{2\delta})\lambda(t_n, \alpha)$ where $\lambda(.,.)$ depends in our case of the choice of strike.

We also have that for all $0 \leq n \leq m-1$, f_0^n depends on the initial boundary condition and the type of asian option considered. In particular, $f_0^n = e^{-r(T-t_n)}\phi(t_n, \alpha)$ in the case of a fixed strike and $f_0^n = e^{-r(T-t_n)}\psi(t_n, \alpha)$ for a floating strike.
Moreover, for all $0 \leq n \leq m-1$, $f_{l+1}^n = 0$ and $f_i^m = f(T, x_i)$ for all $0 \leq i \leq l+1$ depends on the type of strike chosen.

Now we will use the implicit scheme to get an approximation of $\frac{\partial f}{\partial t}(t_n, x_i)$. It consists in using

$$\frac{\partial f}{\partial t}(t_n, x_i) \approx \frac{f(t_n + h, x_i) - f(t_n, x_i)}{h} \approx \frac{f_i^{n+1} - f_i^n}{h}$$

In this case we solve recursively for $n = m-1, ..., 0$

$$(I - \frac{1}{2}hA)f^n = (I + \frac{1}{2}hA)f^{n+1} + h\frac{1}{2}(g^n + g^{n+1})$$

Therefore, we need to solve at each for all $n = m-1, ..., 0$:

$$Xf^n = y$$

where $X = (I - \frac{1}{2}hA)$ and $y = (I + \frac{1}{2}hA)f^{n+1} + h\frac{1}{2}(g^n + g^{n+1})$.

3.3 Numerical results

After applying these methods in Python with the same parameters as previously, $r = 0.1$, $T = 1$, $S_0 = K = 100$ and $m = 10^3$, $l = 10^3$, $\alpha = 0$, $\beta = 2\frac{K}{S_0}$ for an Asian call with fixed strike we get the following results :

σ	Price PDE	Price Riemann scheme	Price Trapez	Price BS
0.05	4.71	4.63	4.72	4.72
0.2	7.04	6.99	6.98	7.04
0.3	9.05	9.01	8.96	9.06

We noticed that in order to get a good result for when σ is low, we need a very high number of time step and x's step. Indeed, the number of step needed to get good results seems to increase as the volatility gets lower. We illustrated this by taking $r = 0.1$, $T = 1$, $S_0 = K = 100$, $\alpha = 0$ and $\beta = 2\frac{K}{S_0}$ and obtained the following results :

σ	Price PDE, $l, m = 50$	Price PDE, $l, m = 100$	Price PDE, $l, m = 200$	Price PDE, $l, m = 300$
0.05	4.79	4.24	4.29	4.48
0.2	6.93	6.96	7.02	7.03
0.3	8.97	9.03	9.05	9.05

We only need 100 time step to get good results when $\sigma = 0.3$, 200 when $\sigma = 0.2$ and even more when $\sigma = 0.05$.

4 Aspects to explore

In this report, we presented three schemes to approximate $\frac{1}{T}\int_0^T S_u du$ and used a standard Monte Carlo approach to price Asian Options. This classic method seems to work well, however we found that the variance can be quite large. The control variate method allowed us to reduce significantly the variance, without increasing the number of simulations. We then studied the PDE established by Rogers and Shi to price Asian options, and proposed a finite different method using an implicit scheme to solve it.

We propose now some aspects to explore to price asian options :

- Use the antithetical variates technique for the three Monte Carlo schemes presented in the report. Indeed, while simulating the increments of a brownian motion to produce the path of the underlying we need to simulate a sequence of $Z_1, \dots, Z_n$ gaussian variables of mean 0 and variance 1. Using the symmetry of the gaussian, we have that $-Z_1, \dots, -Z_n$ is also a sequence of $\mathcal{N}(0,1)$ variables. One could use this fact to implement an antithetical variates method to reduce the variance of the standard Monte Carlo scheme.

- Use Cyton, which allows to write C-like code within Python which can be compiled to machine code. This can result in significant performance improvements compared to pure Python, especially for computationalnally intensive tasks.

- Use the other PDE's, such as the one established by Vecer or Dubois-Lelièvre

- Study of other numerical methods to solve PDE such as the finite element method or spectral methods which is particularly effective for problems with smooth solutions.

- Study of transform methods such as the Laplace and Fourier transform to solve linear PDE's.

References

[1] B. Lapeyre and E. Temam (2001) , Competitive Monte Carlo methods for the pricing of Asian options. *The Journal of Computational Finance*

[2] B. Bouchard (2007), Méthodes de Monte Carlo en finance

[3] L. C. G. Rogers and Z. Shi (1995), The value of an Asian option., *J. Appl. Probab*

[4] J. Claisse (2020), Monte Carlo and finite differences methods with applications in finance