

Phasors Vs S-domain

zero initial conditions & sinusoidally varying source "steady-state"

$$\begin{aligned} \text{time-domain} \\ V(t) &= V_0 \cos(\omega t + \theta) \\ &= \operatorname{Re} \left\{ V_0 e^{j\omega t + j\theta} \right\} \end{aligned}$$

freq. domain

$$V = V_0 e^{j\theta}$$

Real frequency: ω
 $j\omega L$

$$\begin{aligned} V(t) &= V_0 e^{-\sigma t} \cos(\omega t + \theta) \\ &= \operatorname{Re} \left\{ V_0 e^{-\sigma t} e^{j\theta} e^{j\omega t} \right\} \\ &\quad e^{(\sigma + j\omega)t} \end{aligned}$$

$$V(s) = V_0 e^{j\theta}$$

$$s = \sigma + j\omega$$

"complex frequency"

R

L

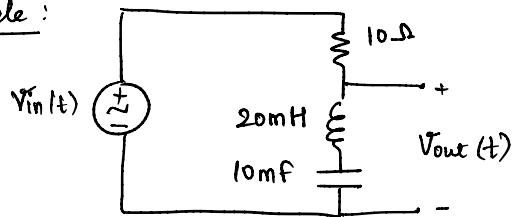
C

R

$$L(\sigma + j\omega) = sL$$

$$\frac{1}{(j\omega + \sigma)C} = \frac{1}{sC}$$

Example:

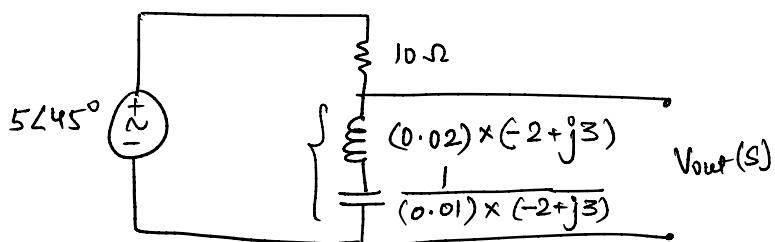


Consider $V_{in}(t) = 5e^{-2t} \cos(3t + 45^\circ) V$
 find s , $V_{out}(t)$.

$$s = \sigma + j\omega = -2 + j3$$

freq. domain representation:

$$V_{in}(s) = 5 e^{j45^\circ} = 5 \angle 45^\circ$$



$$V_{out}(s) = 5 \angle 45^\circ \times \frac{\frac{0.02}{(0.01)} \times (-2 + j3) + \frac{1}{(0.01)(-2 + j3)}}{10 + \frac{0.02}{(0.01)}(-2 + j3) + \frac{1}{(0.01)(-2 + j3)}} = 5.86 \angle -24.4^\circ$$

$$V_{out}(t) = \operatorname{Re} \{ 5.86 e^{j(24.4^\circ)} e^{(5+j\omega)t} \}$$

$$= 5.86 e^{-2t} \cos(3t - 24.4^\circ) \text{ V}$$

Note: for zero initial condition, we can use phasors
for $e^{st} \cos(\omega t + \theta)$.

$$\begin{aligned} R &\rightarrow R \\ L &\rightarrow sL \\ C &\rightarrow \frac{1}{sC} \end{aligned}$$

Ques: Can we use phasors for $\delta(t)$, $u(t)$, $\sin(\omega t) u(t)$?

NO

s is variable

Laplace transform.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \Rightarrow f(s) =$$

Refer to table 14.1, 14.2

$\delta(t)$	1
$u(t)$	$1/s$
$t u(t)$	$1/s^2$
$e^{-at} u(t)$	$1/(s+a)$
$t e^{-at} u(t)$	$1/(s+a)^2$
$\sin(\omega t) u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t) u(t)$	$\frac{s}{s^2 + \omega^2}$
$e^{-at} \sin(\omega t) u(t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos(\omega t) u(t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$

$f(t) \Leftrightarrow F(s)$	
$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$k f(t)$	$k F(s)$
$d f/dt$	$s F(s) - f(0)$
$\int_0^t f(t) dt$	$F(s)/s$
$f_1(t) * f_2(t)$	$F_1(s) F_2(s)$
$f(t-a) u(t-a)$	$e^{-as} F(s)$
$f(t) e^{-at}$	$F(s+a)$

Initial and final value theorems:

$$\begin{aligned} f(0^+) &= \lim_{s \rightarrow \infty} s F(s) \\ f(\infty) &= \lim_{s \rightarrow 0} s F(s) \end{aligned}$$

{ all poles of $sF(s)$ lie in LHP

$$\begin{aligned}
 & \mathcal{L} \left\{ \sin(\omega t) u(t) \right\} \\
 &= \mathcal{L} \left\{ \frac{1}{2j} \left\{ e^{j\omega t} - e^{-j\omega t} \right\} u(t) \right\} \\
 &= \frac{1}{2j} \left[\underbrace{\mathcal{L} \left\{ e^{j\omega t} u(t) \right\}}_{=} - \underbrace{\mathcal{L} \left\{ e^{-j\omega t} u(t) \right\}}_{=} \right] \\
 &= \frac{1}{2j} \left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right] \\
 &= \left(\frac{1}{2j} \right) \frac{s + j\omega - s - j\omega}{(s - j\omega)(s + j\omega)} = \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}
 \end{aligned}$$

Examples:

① Find $f(0^+)$ and $f(\infty)$ for $f(s) = \frac{4e^{-2s}(s+50)}{s}$

$$\begin{aligned}
 f(0^+) &= \lim_{s \rightarrow \infty} s f(s) = \lim_{s \rightarrow \infty} \frac{4e^{-2s}(s+50)}{s} \underset{x \neq}{\cancel{\infty}} \\
 &= \lim_{s \rightarrow \infty} \frac{4(s+50)}{e^{2s}}
 \end{aligned}$$

$\frac{\infty}{\infty}$ = indeterminate

$$\frac{1}{\infty} = 0$$

L'Hopital's Rule: $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$

$$f(0^+) = \lim_{s \rightarrow \infty} \frac{4(s+50)}{e^{2s}} = \lim_{s \rightarrow \infty} \frac{4}{2e^{2s}} = \frac{1}{\infty} = 0$$

Final value: $f(\infty) = \lim_{s \rightarrow 0} s f(s) = \lim_{s \rightarrow 0} 4e^{-2s}(s+50) = 200$

② Find $f(0^+)$ & $f(\infty)$ for $f(s) = \frac{5s^2 + 10}{2s(s^2 + 3s + 5)}$

$$f(0^+) = \lim_{s \rightarrow \infty} \frac{s(5s^2 + 10)}{2s(s^2 + 3s + 5)} = \lim_{s \rightarrow \infty} \frac{5 + 10/s^2}{2(1 + 3/s + 5/s^2)} = \frac{5}{2}$$

$$f(\infty) = \lim_{s \rightarrow 0} \frac{5s^2 + 10}{2(s^2 + 3s + 5)} = \frac{10}{2(5)} = 1$$

Examples find ILT of $f(s) = \frac{s^2 + 6}{s^2 + 7}$

$$\begin{array}{r} 1 \\ s^2 + 7 \int s^2 + 6 \\ \underline{-s^2 - 7} \\ 0 - 1 \end{array}$$

$$F(s) = \frac{s^2 + 6}{s^2 + 7} = 1 - \frac{1}{s^2 + 7}$$

$$f(s) = 1 - \frac{1}{s^2 + (\sqrt{7})^2} = 1 - \underbrace{\frac{\sqrt{7}}{\sqrt{7}(s^2 + (\sqrt{7})^2)}}_{\omega} \sin(\sqrt{7}t) u(t)$$

$$f(t) = \delta(t) - \frac{1}{\sqrt{7}} \sin(\sqrt{7}t) u(t)$$

Example: find ILT of $\frac{s+2}{s^2 + 2s + 4} = \frac{(s+1) + 1}{(s+1)^2 + (\sqrt{3})^2}$

find roots: $s^2 + 2s + 4 = 0$

$$s = \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{12}}{2} = -1 \pm j\sqrt{3}$$

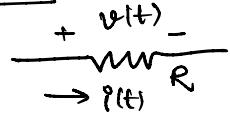
$$f(s) = \frac{s+2}{(s+1-j\sqrt{3})(s+1+j\sqrt{3})} = \frac{s+2}{(s+1)^2 - (j\sqrt{3})^2} = \frac{s+2}{(s+1)^2 + (\sqrt{3})^2}$$

$$f(s) = \underbrace{\frac{s+1}{(s+1)^2 + (\sqrt{3})^2}}_{\text{conjugate}} + \underbrace{\frac{1}{(s+1)^2 + (\sqrt{3})^2} \times \frac{\sqrt{3}}{\sqrt{3}}}_{\text{conjugate}}$$

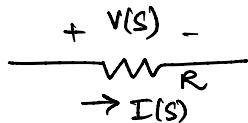
$$f(t) = e^{-t} \cos(\sqrt{3}t) u(t) + \frac{1}{\sqrt{3}} e^{-t} \sin(\sqrt{3}t) u(t)$$

Impedances in s-domain:

Resistor:



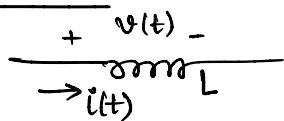
$$v(t) = i(t) R$$



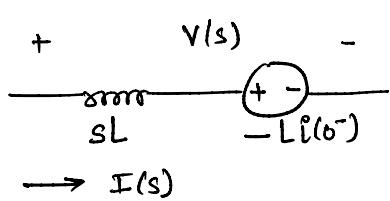
$$V(s) = I(s) R$$

$$\frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

Inductor:



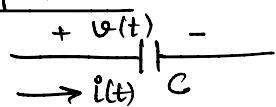
$$v(t) = L \underbrace{\frac{di(t)}{dt}}_{\text{Take LT:}}$$



$$\Rightarrow V(s) = L(sI(s) - i(0^-))$$

$$V(s) = sL I(s) - \underbrace{L i(0^-)}$$

Capacitor:



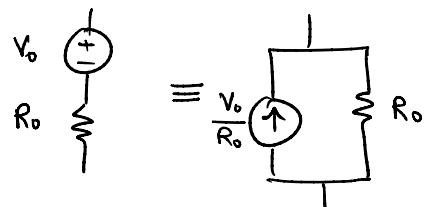
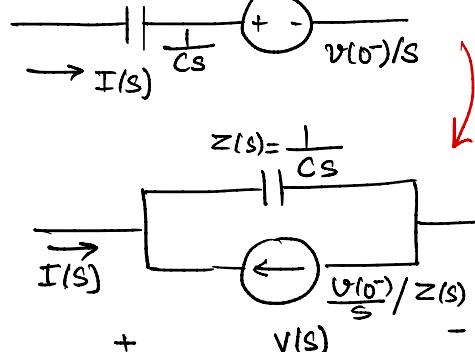
$$i(t) = C \frac{dv(t)}{dt}$$

Take LT:

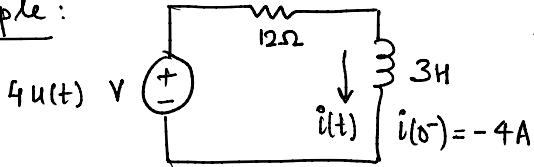
$$\Rightarrow I(s) = C [sV(s) - v(0^-)]$$

$$\Rightarrow I(s) = Cs V(s) - Cv(0^-)$$

$$\Rightarrow V(s) = \frac{1}{Cs} I(s) + \frac{v(0^-)}{s}$$

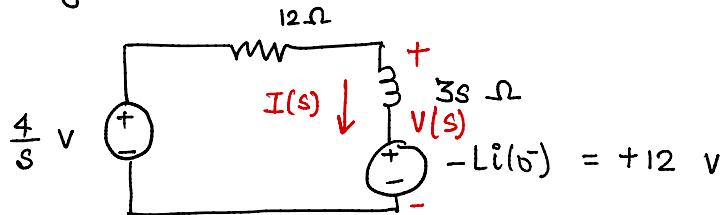


Example :



find $i(t)$. find $v(t)$ across inductor.

frequency domain representation:



$$4u(t) \Leftrightarrow \frac{4}{s}$$

$$\text{write KVL eqn: } -\frac{4}{s} + 12I(s) + (3s)I(s) + 12 = 0$$

$$\Rightarrow I(s) = \frac{\frac{4}{s} - 12}{12 + 3s} = \frac{4 - 12s}{(12 + 3s)s}$$

$$I(s) = \frac{A}{s} + \frac{B}{12+3s}$$

$$\Rightarrow 4 - 12s = A(12 + 3s) + B(s)$$

$$\text{at } s=0 \Rightarrow 4 = A(12+0) + 0 \Rightarrow A = \frac{4}{12}$$

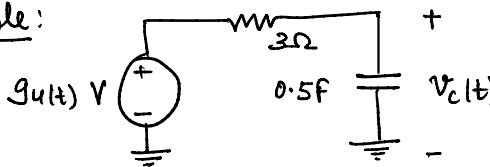
$$\text{at } s=-4 \Rightarrow 4 - 12(-4) = A(0) + B(-4) \Rightarrow B = -13$$

$$I(s) = \frac{\frac{4}{12}}{s} + \frac{-13}{12+3s} = \frac{1}{3}\left(\frac{1}{s}\right) - \left(\frac{13}{3}\right)\left(\frac{1}{4+s}\right)$$

$$i(t) = \frac{1}{3}u(t) - \frac{13}{3}e^{-4t}u(t) \quad A$$

$$\boxed{\begin{aligned} u(t) &\Leftrightarrow \frac{1}{s} \\ e^{-at}u(t) &\Leftrightarrow \frac{1}{s+a} \end{aligned}}$$

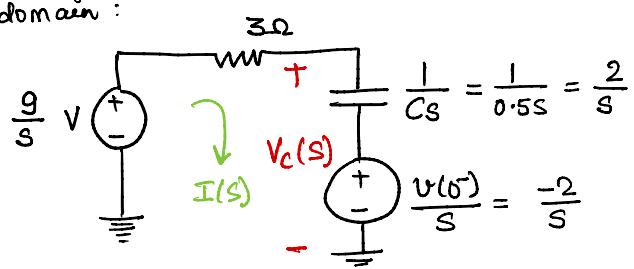
Example :



Given: $v_c(0^-) = -2V$

find $v_c(t)$.

freq. domain:



$$V_c(s) = I(s) \times \frac{2}{s} - \frac{2}{s}$$

$$\cancel{KVL} \quad -\frac{g}{s} + 3I(s) + \frac{2}{s}(I(s)) - \frac{2}{s} = 0$$

$$\Rightarrow I(s) = \frac{11}{3s+2} = \frac{11/3}{s+2/3}$$

$$i(t) = \frac{11}{3} e^{-\frac{2}{3}t} u(t)$$

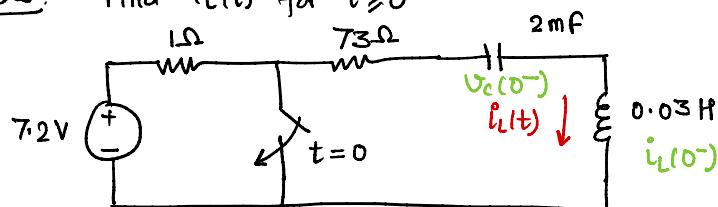
$$V_c(s) = \frac{2}{s} I(s) - \frac{2}{s} = \frac{18 - 6s}{s(3s+2)} = \frac{A}{s} + \frac{B}{3s+2}$$

$$A = 9, \quad B = -33$$

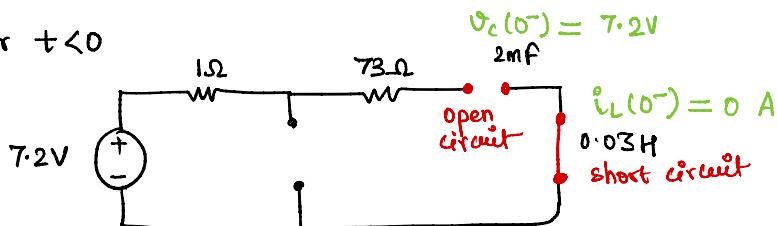
$$V_c(s) = \frac{9}{s} - \frac{33}{3s+2} = \frac{9}{s} - \frac{11}{s+2/3}$$

$$v_c(t) = 9u(t) - 11e^{-\frac{2}{3}t} u(t) \text{ V}$$

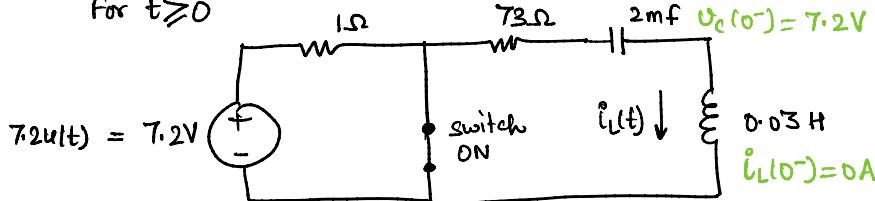
Example: find $i_L(t)$ for $t \geq 0$



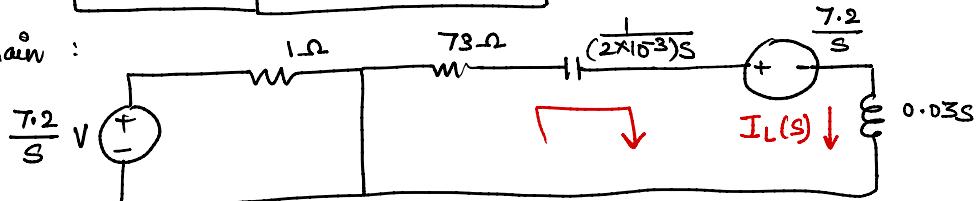
for $t < 0$



for $t \geq 0$



freq. domain :



$$\begin{aligned}
 \text{KVL: } & 73 I(s) + \frac{1}{(2 \times 10^3)s} I(s) + \frac{7.2}{s} + 0.03s I(s) = 0 \\
 \Rightarrow I(s) &= \frac{-7.2/s}{73 + \frac{1}{(2 \times 10^3)s} + 0.03s} = \frac{-7.2}{0.03s^2 + 73s + 500} \\
 \Rightarrow I(s) &= \frac{-7.2/0.03}{s^2 + \frac{73}{0.03}s + \frac{500}{0.03}} = \frac{A}{s+a} + \frac{B}{s+b}
 \end{aligned}$$

find a & b

$$\begin{aligned}
 s^2 + \frac{73}{0.03}s + \frac{500}{0.03} &= 0 \quad \leftarrow \text{Roots} \\
 \Rightarrow s^2 + 2433s + 16666 &= 0 \\
 \Rightarrow s &= \frac{-2433 \pm \sqrt{2433^2 - 4(1)(16666)}}{2(1)} \approx -7, -2426 \\
 s^2 + 2433s + 16666 &= (s+7)(s+2426)
 \end{aligned}$$

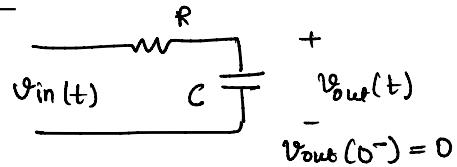
$$\begin{aligned}
 I(s) &= \frac{-7.2/0.03}{(s+2426)(s+7)} = \frac{A}{(s+2426)} + \frac{B}{s+7} \\
 \Rightarrow \frac{-7.2}{0.03} &= -240 = A(s+7) + B(s+2426) \\
 \text{at } s=-7 &\Rightarrow B = -240/2419 = -0.099 \\
 \text{at } s=-2426 &\Rightarrow A = +240/2419 = 0.099
 \end{aligned}$$

$$I(s) = \frac{0.099}{s+2426} + \frac{-0.099}{s+7}$$

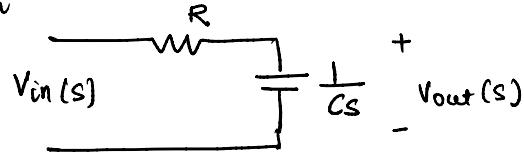
$$\hat{i}(t) = 0.099 \left[e^{-2426t} u(t) - e^{-7t} u(t) \right] A.$$

Transfer function :

Let's consider:



Freq. domain



$$V_{out}(s) = \frac{1/Cs}{R + 1/Cs} V_{in}(s)$$

$$\Rightarrow \underbrace{\frac{V_{out}(s)}{V_{in}(s)}}_{\text{transfer function. } H(s)} = \frac{1}{sRC + 1}$$

transfer function. $H(s)$

$$V_{out}(t) = \mathcal{L}^{-1} \{ V_{out}(s) \} = \mathcal{L}^{-1} \{ H(s) \cdot V_{in}(s) \}$$

defines the system.

Stability :

stable system : bounded output for bounded input.

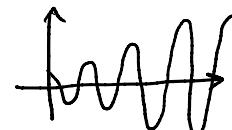
unstable system : unbounded output for bounded input.

Marginally stable : either oscillating or bounded offset.

$$e^{\sigma t} \cos(\omega_0 t) u(t)$$

① $\sigma > 0$ let's say $\sigma = 2$
unstable

$$e^{2t} \cos(\omega_0 t) u(t)$$



② $\sigma < 0$ let's say $\sigma = -3$
stable

$$e^{-3t} \cos(\omega_0 t) u(t)$$



③ $\sigma = 0$ $\cos(\omega_0 t) u(t)$
marginally stable.



Stability Criterion

$$\text{System transfer fn } H(s) = k \frac{(s-z_1)(s-z_2)(s-z_3)\dots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\dots(s-p_n)}$$

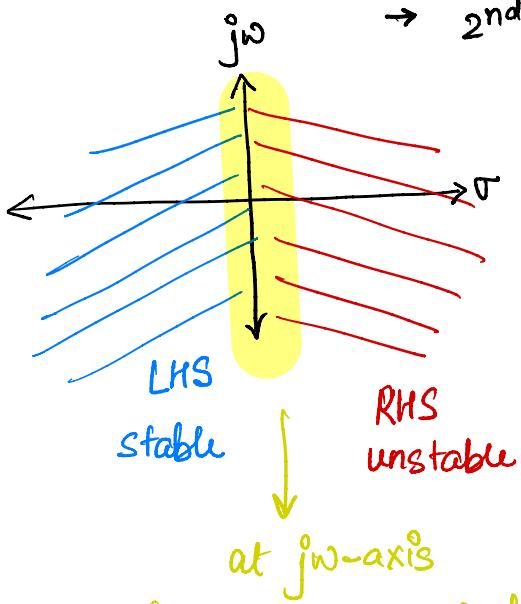
- System is stable if
All poles of $H(s)$ lie on LHS of complex plane
- Marginally stable if
1st order pole at $s=0$ OR 1st order complex conjugate pole pair at $j\omega\text{-axis}$
- Unstable system if
All other cases
 - Any poles on RHS OR
 - 2nd or higher order pole at $s=0$ OR
 - 2nd or higher order conjugate pole pair at $j\omega\text{-axis}$

Stability Criterion

$$\text{System transfer fn } H(s) = k \frac{(s-z_1)(s-z_2)(s-z_3)\dots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\dots(s-p_n)}$$

- System is stable if
All poles of $SH(s)$ lie on LHS of complex plane
- Marginally stable if
1st order pole at $s=0$ OR 1st order complex conjugate pole pair at $j\omega$ -axis
- Unstable system if
All other cases
 - Any poles on RHS OR
 - 2nd or higher order pole at $s=0$ OR
 - 2nd or higher order conjugate pole pair at $j\omega$ -axis

$$SH(s) = \underbrace{\frac{1}{s}}_{\text{1st order}} \quad \underbrace{\frac{1}{s^2}}_{\text{2nd order}} \quad \underbrace{\frac{1}{s^3}}_{\text{3rd order}} \quad \underbrace{\frac{1}{s^n}}_{n \geq 2}$$



(could be marginally stable or unstable)

$f(t)$

①

$$e^{-2t} u(t)$$

$s \cdot f(s)$

$$s \cdot \frac{1}{s+2}$$

②

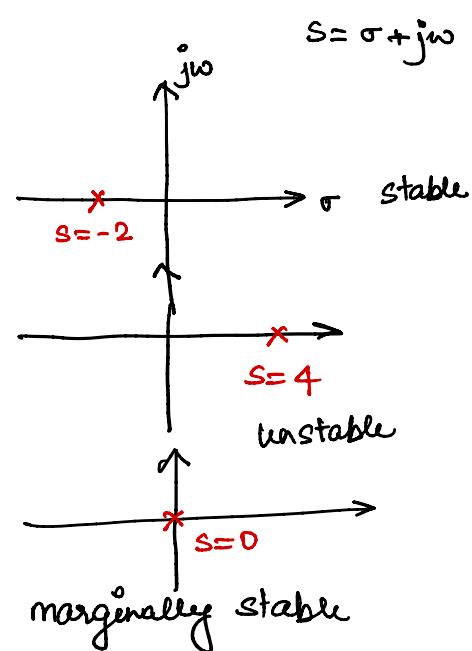
$$e^{4t} u(t)$$

$$s \cdot \frac{1}{s-4}$$

③

$$u(t)$$

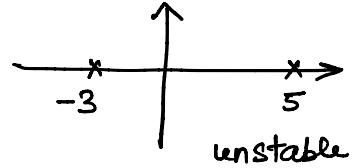
$$s \cdot \frac{1}{s}$$



④

$$e^{-3t} u(t) + e^{+5t} u(t)$$

$$S \cdot \left\{ \frac{1}{s+3} + \frac{1}{s-5} \right\}$$



⑤

$$S \cdot \frac{8}{(s+1)(s+2)(s+3)}$$

stable

⑥

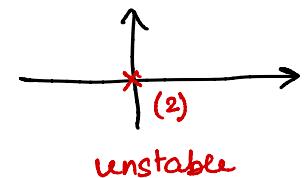
$$S \cdot \frac{14}{(s+1-\sqrt{2})(s-1+\sqrt{2})}$$

unstable

⑦

$t u(t)$

$$S \cdot \frac{1}{s^2}$$



⑧

$$e^{-3t} \sin(\omega_0 t) u(t)$$

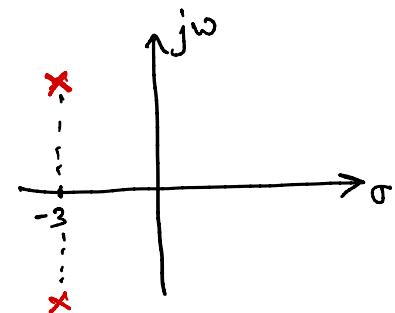
$$S \cdot \frac{\omega_0}{(s+3)^2 + \omega_0^2}$$

$$(s+3)^2 + \omega_0^2 = 0$$

$$\Rightarrow (s+3)^2 = -\omega_0^2$$

$$\Rightarrow s+3 = \pm \sqrt{-1} \omega_0$$

$$\Rightarrow s = -3 \pm j\omega_0$$

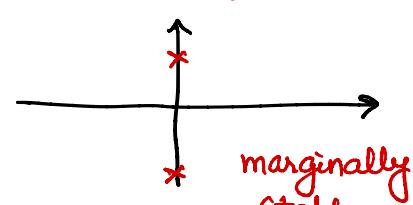


⑨

$$\sin(\omega_0 t) u(t)$$

$$S \cdot \frac{\omega_0}{s^2 + \omega_0^2}$$

$$s = \pm j\omega_0$$

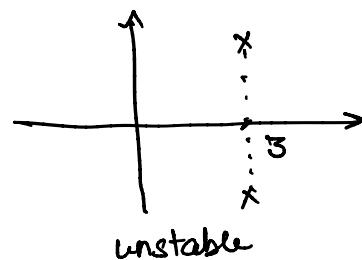


⑩

$$e^{3t} \sin(\omega_0 t) u(t)$$

$$S \cdot \frac{\omega_0}{(s-3)^2 + \omega_0^2}$$

$$\text{poles: } s = 3 \pm j\omega_0$$



⑪

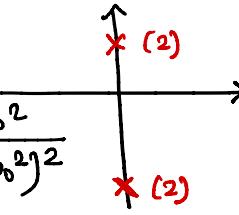
$$t \cos(\omega_0 t) u(t)$$

$$f(t) \Leftrightarrow f(s)$$

$$tf(t) \Leftrightarrow -\frac{d}{ds}(f(s))$$

$$\mathcal{L} \left\{ t \cos(\omega_0 t) u(t) \right\} = -\frac{d}{ds} \left\{ \frac{s}{s^2 + \omega_0^2} \right\} = -\frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$$

Please check! ↗



Final Value Theorem :

$$f(\infty) = f(t \rightarrow \infty) = \lim_{s \rightarrow 0} s F(s), \quad \text{when all poles are on LHS.}$$

Initial value theorem:

$$f(0) = f(t=0) = \lim_{s \rightarrow \infty} s F(s).$$

$$\rightarrow f(t) = \sin(\omega_0 t) u(t)$$

$$\text{final value theorem : } \lim_{s \rightarrow 0} s F(s) = \lim_{s \rightarrow 0} s \frac{\omega_0}{s^2 + \omega_0^2} = \frac{0}{\omega_0^2} = 0$$

$f(\infty) = \text{Indeterminate.}$

\times

\uparrow

Chapter 15 (9th edition of textbook)

Frequency Response

$$\text{Capacitor Impedance } Z_C = \frac{1}{sC}$$

$$s = \sigma + j\omega$$

Transfer function :

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$$V_{out}(s) = \underbrace{V_{in}(s)}_{\text{Input}} \cdot \underbrace{H(s)}_{\text{System}}$$

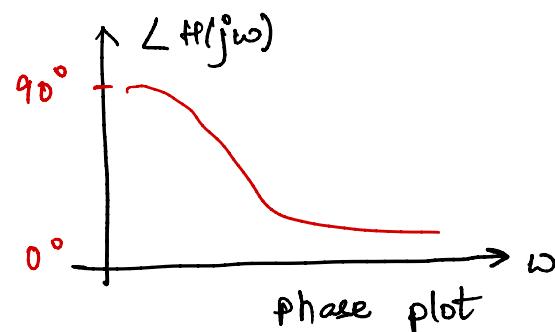
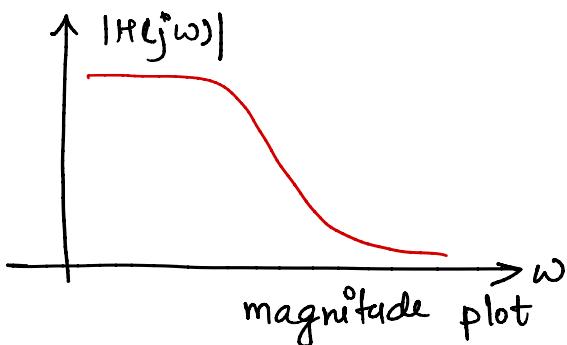
$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \text{voltage gain } V/V$$

$$H(j\omega) = \frac{I_{out}(j\omega)}{I_{in}(j\omega)} = \text{current gain } A/A$$

$$H(j\omega) = \frac{V_{out}(j\omega)}{I_{in}(j\omega)} = \text{transfer impedance } \frac{V}{A} \text{ or } -\Omega$$

$$H(j\omega) = \frac{I_{out}(j\omega)}{V_{in}(j\omega)} = \text{transfer admittance } \frac{A}{V} \text{ or } \Omega^{-1}$$

$$H(j\omega) = |H(j\omega)| \angle H(j\omega)$$



Decibels (dB) scale

$$H_{dB} = 20 \log_{10} |H(j\omega)|$$

$$\log_{10} 1 = 0$$

$$\log_{10} 10 = 1$$

$$\log_{10} 2 = 0.3$$

$$\log_{10}(0.1) = -1$$

$$\log_{10}(10^{-k}) = -k$$

$\log_{10}(0)$ \Rightarrow undefined.

Find H_{dB} at $\omega = 146$ rad/s if $H(s) = 20(s+100)$.

$$\begin{aligned} H_{dB} &= 20 \log_{10} |20(j\omega + 100)| = 20 \log \left\{ 20 \times \sqrt{100^2 + \omega^2} \right\} \\ &= 20 \log_{10} \left\{ 20 \times \sqrt{100^2 + 146^2} \right\} \\ &= 70.9 \text{ dB} \end{aligned}$$

$$H(s) = K \frac{(s-z_1)(s-z_2) \cdots (s-z_m)}{(s-p_1)(s-p_2) \cdots (s-p_n)}$$

$$\begin{aligned} H_{dB} &= 20 \log|K| + 20 \log|(s-z_1)| + 20 \log|(s-z_2)| + \dots \\ &\quad - 20 \log|(s-p_1)| - 20 \log|(s-p_2)| + \dots \end{aligned}$$

$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$20 \log_{10} |H(j\omega)| = 20 \log_{10} |H_1(j\omega)| + 20 \log_{10} |H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

$$\bar{A} = g_1 e^{j\theta_1}$$

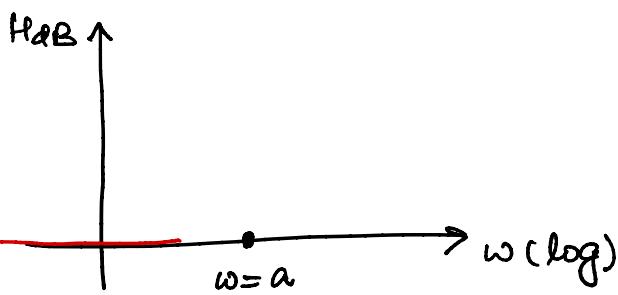
$$\bar{B} = g_2 e^{j\theta_2}$$

$$AB = (g_1 g_2) e^{j(\theta_1 + \theta_2)}$$

$$(1) \text{ Consider: } H(s) = 1 + \frac{s}{\alpha} \quad \text{Simple Zero}$$

$$H(j\omega) = 1 + \frac{j\omega}{\alpha}$$

$$H_{dB} = 20 \log \left| 1 + \frac{j\omega}{\alpha} \right| = 20 \log \sqrt{1 + \left(\frac{\omega}{\alpha} \right)^2}$$



lets consider $\omega \ll a$

$$HdB = 20 \log(\sqrt{1}) = 0 \text{ dB}$$

$\omega \gg a$

$$HdB = 20 \log \sqrt{\left(\frac{\omega}{a}\right)^2} = 20 \log \left(\frac{\omega}{a}\right) \text{ dB}$$

$$\omega = \underline{10a}$$

$$HdB = 20 \text{ dB}$$

$$\omega = \underline{100a}$$

$$HdB = \underline{40} \text{ dB}$$

$$\omega = \underline{1000a}$$

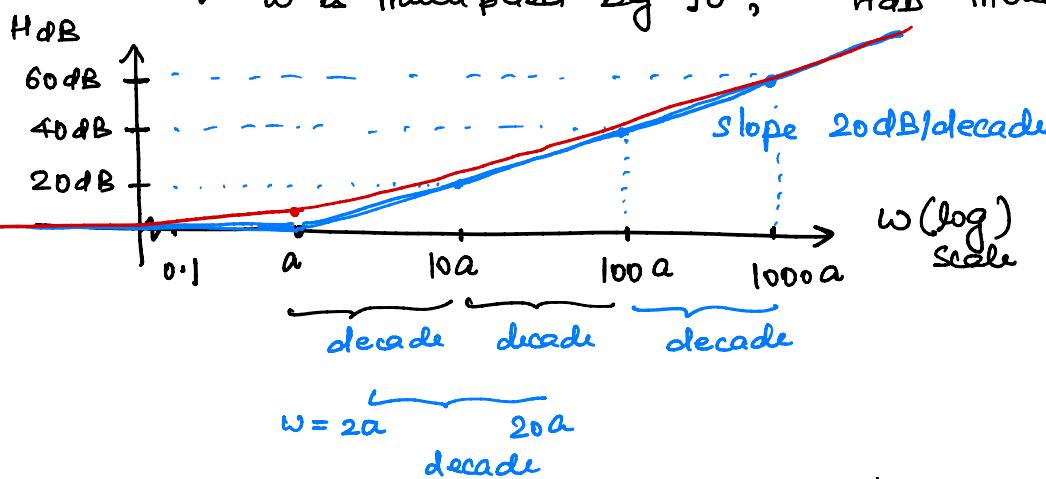
$$HdB = 20 \log \left(\frac{1000a}{a}\right) = \underline{60} \text{ dB}$$

$$\omega = \underline{10^4a}$$

$$HdB = \underline{80} \text{ dB}$$

$$y = mx + c$$

(Linear scale) ω is multiplied by 10, HdB increases by 20 dB



$$\text{slope} = \frac{\Delta y}{\Delta x}$$

semilog

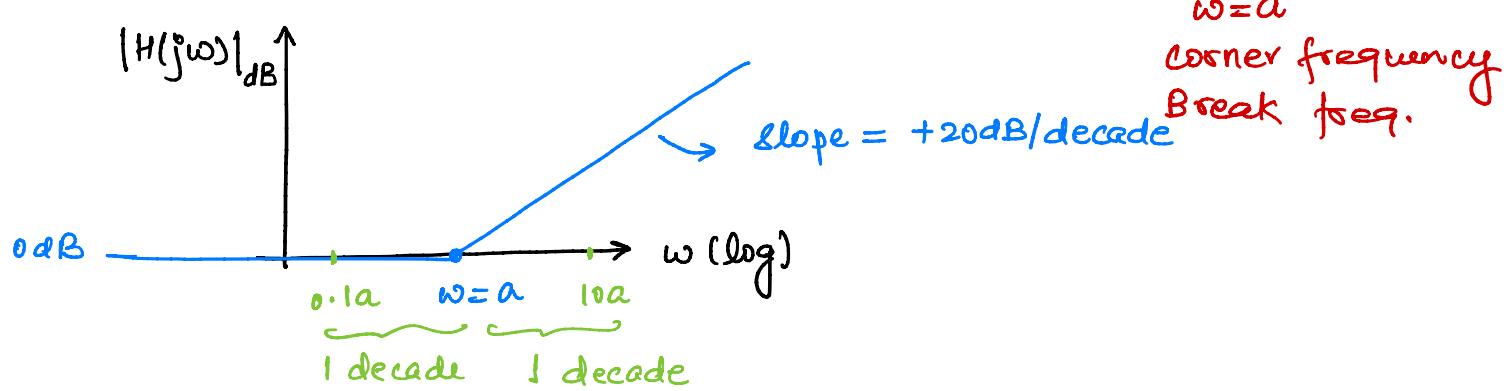
Bode plot

straight line plot

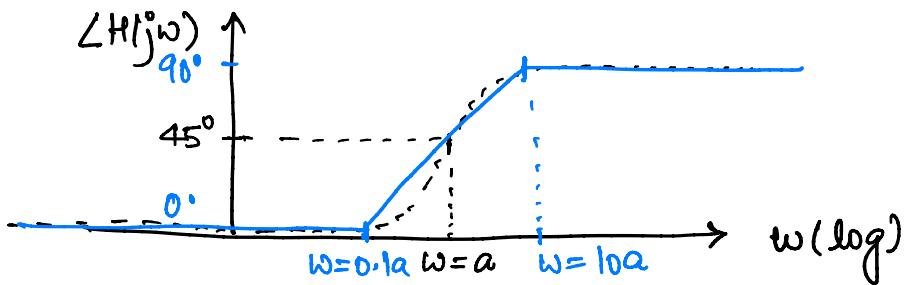
$$\text{at } \omega = a, \quad H(j\omega) = 1 + \frac{j\omega}{a} = 1 + j$$

$$HdB = 20 \log_{10} \sqrt{2} = 20 \times \frac{1}{2} \times 0.3 = 3 \text{ dB}$$

$$\text{Simple zero : } H(s) = 1 + s/a \Rightarrow H(j\omega) = \left[1 + \frac{j\omega}{a} \right]$$



$$\text{Phase plot : } \angle H(j\omega) = \angle \left\{ 1 + \frac{\omega}{a} j \right\} = \tan^{-1} \left(\frac{\omega}{a} \right)$$



ω	$\tan^{-1}(\omega/a)$
0	0°
0.001a	~0°
a	45°
1000a	~90°
∞	90°

$$\text{Simple pole : } H(s) = \frac{1}{1 + s/a}$$

$$H(j\omega) = \frac{1}{1 + j\omega/a}$$

$$|H(j\omega)|_{dB} = 20 \log_{10} \left(\left| \frac{1}{1 + j\omega/a} \right| \right) = 20 \log_{10} \left\{ \frac{1}{\sqrt{1 + (\omega/a)^2}} \right\}$$

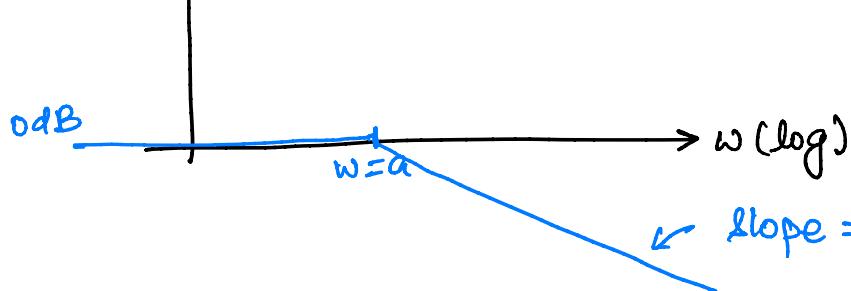
lets consider $\omega \ll a$

$$|H(j\omega)|_{dB} = 0$$

$$|H(j\omega)|_{dB}$$

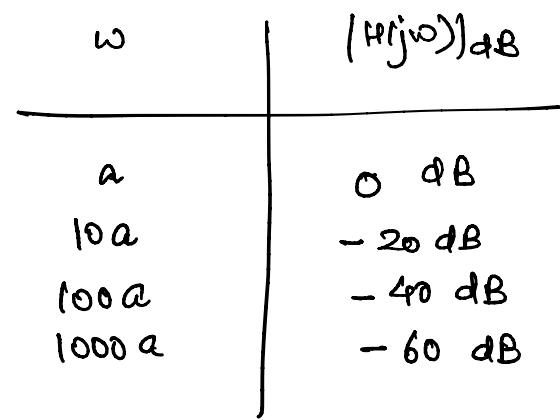
lets consider $\omega \gg a$

$$|H(j\omega)|_{dB} = -20 \log_{10} \left(\frac{\omega}{a} \right)$$



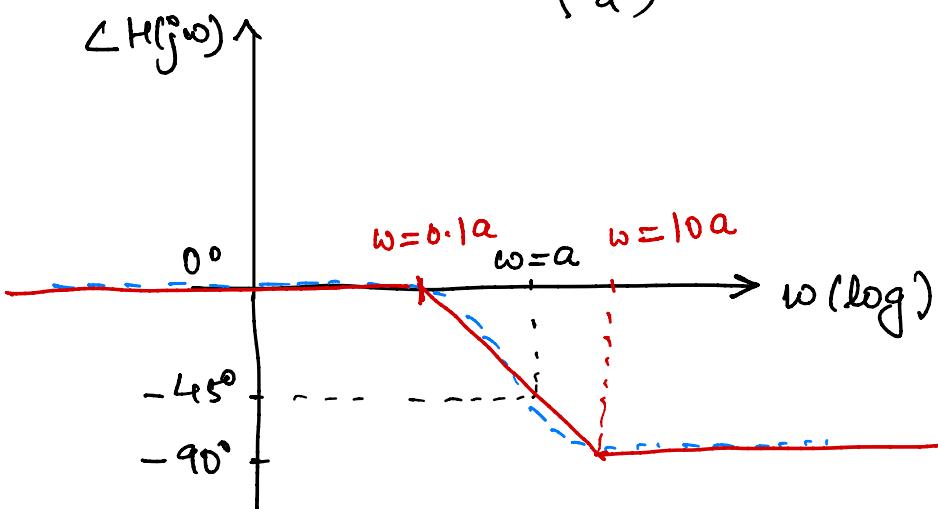
$$\log \frac{1}{x} = -\log x$$

$$\text{slope} = -20 \text{ dB/decade}$$



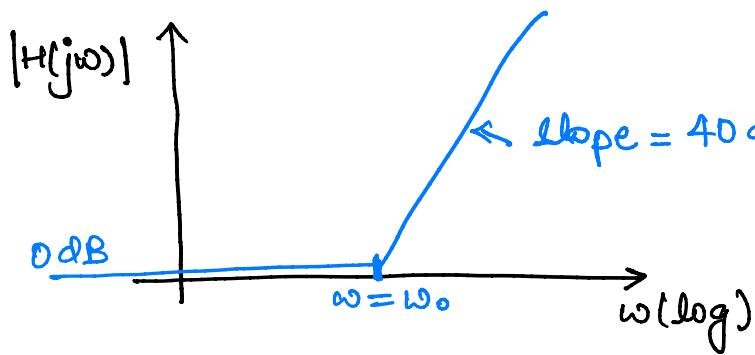
$$\angle(H(j\omega)) = \angle \left\{ \frac{1}{1+j\omega/a} \right\} = \angle \left\{ \frac{(1-j\omega/a)}{(1+j\omega/a)((-j\omega/a))} \right\}$$

$$= -\tan^{-1}\left(\frac{\omega}{a}\right)$$



ω	$-\tan^{-1}(\omega/a)$
0	0°
0.0001	~0°
a	-45°
$10000a$	~-90°
∞	-90°

Complex conjugate pair zero: $H(s) = 1 + 2\zeta\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2$

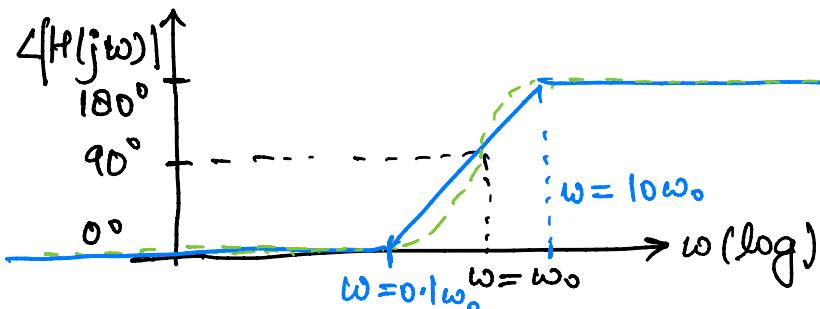


when $\omega \ll \omega_0$

$$|H(j\omega)|_{dB} = 20\log(1) = 0 \text{ dB}$$

when $\omega \gg \omega_0$

$$\begin{aligned} |H(j\omega)|_{dB} &= 20\log\left(\frac{j\omega}{j\omega_0}\right)^2 \\ &= 40\log\left(\frac{\omega}{\omega_0}\right) \text{ dB} \end{aligned}$$



Example : Draw Bode plots of $H(s) = \frac{s+1}{s(s+2)^2}$.

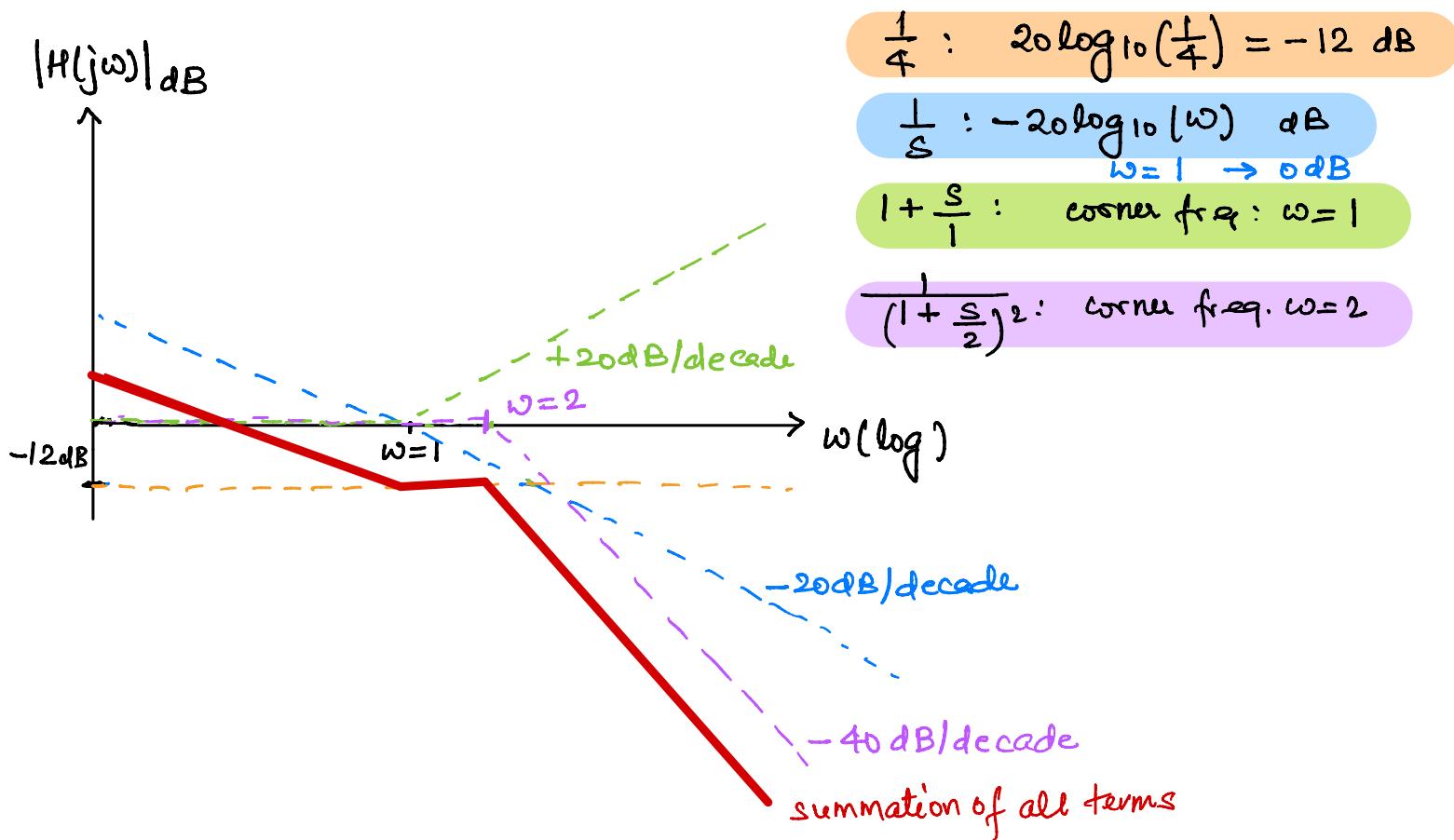
$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$\begin{aligned} H_{dB} &= 20 \log_{10} |H(j\omega)| = 20 \log_{10} |H_1(j\omega) H_2(j\omega)| \\ &= 20 \log_{10} |H_1(j\omega)| + 20 \log_{10} |H_2(j\omega)| \end{aligned}$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

$$\begin{aligned} &(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \end{aligned}$$

$$H(s) = \frac{s+1}{s(s+2)^2} = \frac{(1+s/1)}{s \left(1+\frac{s}{2}\right)^2 \cdot 4} = \frac{1}{4} \cdot \frac{1}{s} \cdot \left(1+\frac{s}{1}\right) \frac{1}{\left(1+\frac{s}{2}\right)^2}$$



$\angle H(j\omega)$

$\frac{1}{4}$: 0° always

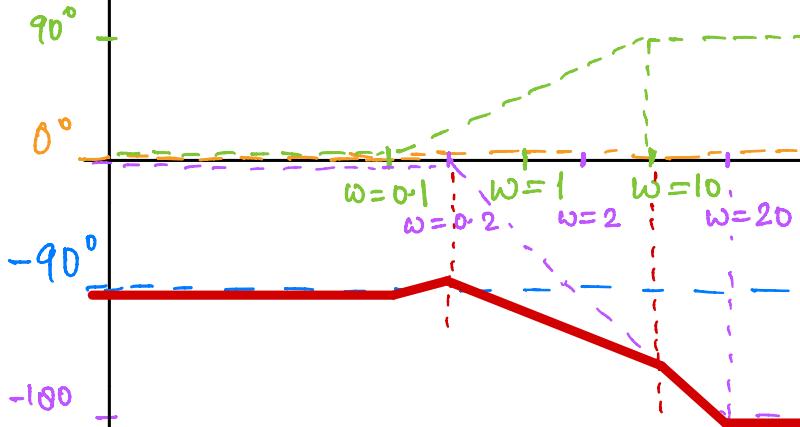
$\frac{1}{S}$: -90° always

$1 + \frac{S}{1}$: simple zero
corner freq: $\omega = 1$

$\omega(\log)$

$\frac{1}{(1 + \frac{S}{2})^2}$: 2nd order pole at
corner freq. $\omega = 2$

summation of all terms



Example : Draw Bode plots of $H(s) = \frac{s+1}{s(s+2)^2}$.

Zeros: $s = -1$

Poles: $s=0$

$s=-2$ }
 $s=-2$ }
2nd order pole

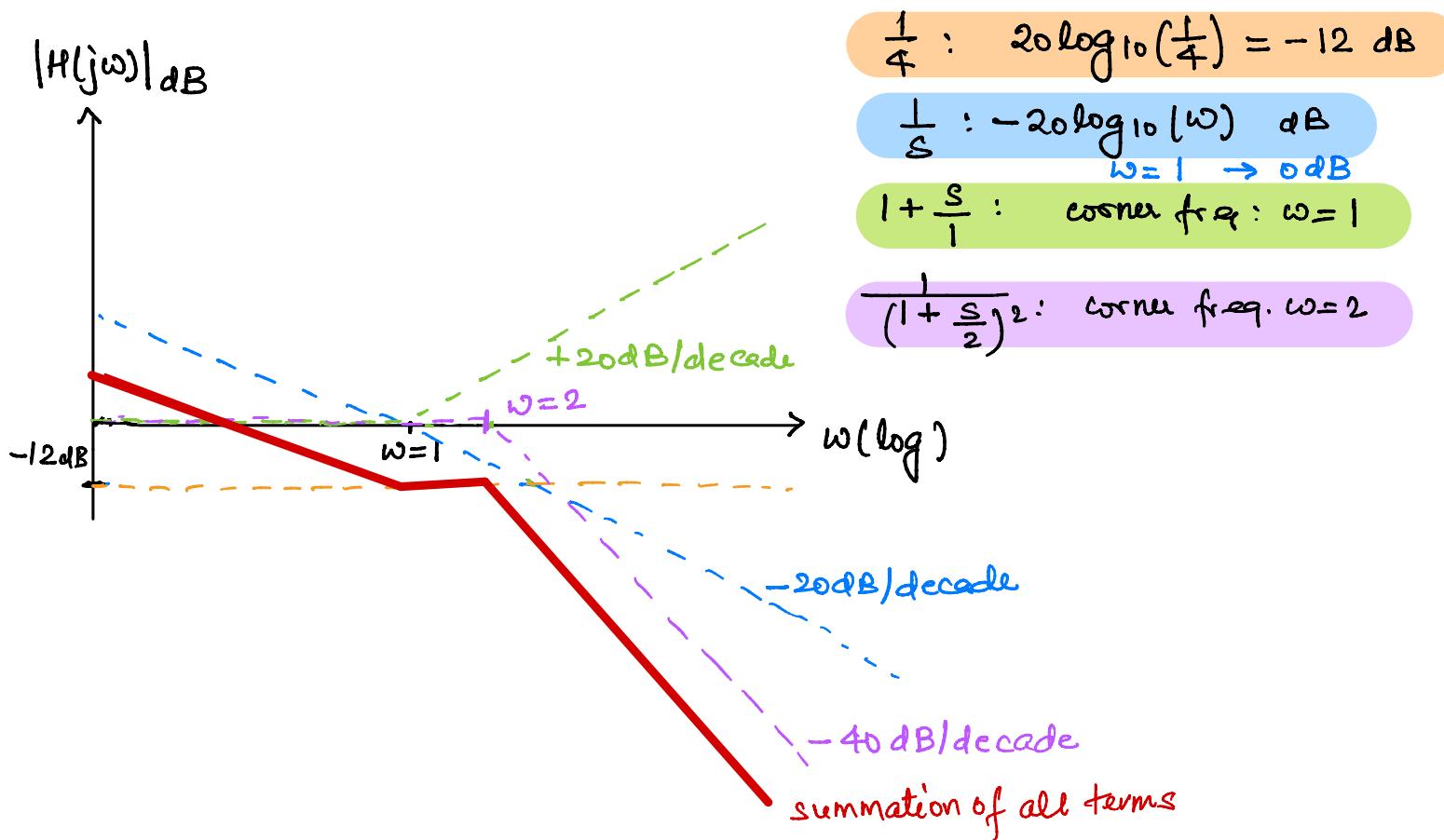
$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$\begin{aligned} H_{dB} &= 20 \log_{10} |H(j\omega)| = 20 \log_{10} |H_1(j\omega) H_2(j\omega)| \\ &= 20 \log_{10} |H_1(j\omega)| + 20 \log_{10} |H_2(j\omega)| \end{aligned}$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

$$\begin{aligned} &(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \end{aligned}$$

$$H(s) = \frac{s+1}{s(s+2)^2} = \frac{(1+s/1)}{s \left(1+\frac{s}{2}\right)^2 \cdot 4} = \frac{1}{4} \cdot \frac{1}{s} \cdot \left(1+\frac{s}{1}\right) \frac{1}{\left(1+\frac{s}{2}\right)^2}$$



$\angle H(j\omega)$

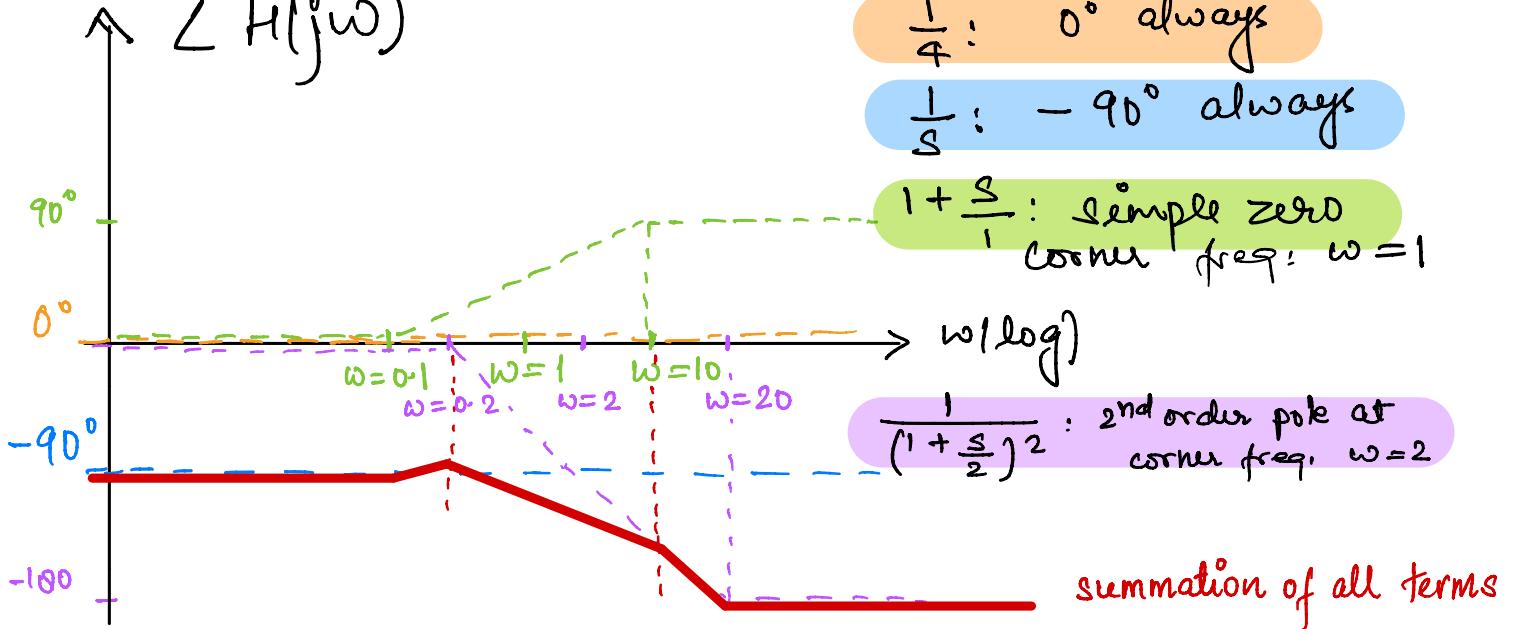
$\frac{1}{4}$: 0° always

$\frac{1}{s}$: -90° always

$1 + \frac{s}{1}$: simple zero
corner freq: $\omega = 1$

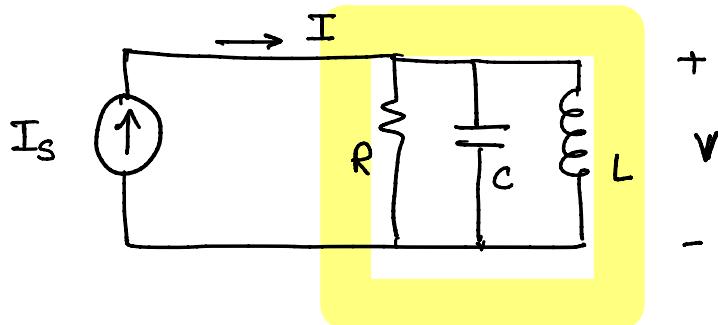
$\frac{1}{(1 + \frac{s}{2})^2}$: 2nd order pole at
corner freq. $\omega = 2$

summation of all terms



Resonance :

A network is in resonance when V and I at network input terminals are in phase.



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Assume : $I_s = I_0 \cos(\omega t + \phi)$

$$\begin{aligned} Y(j\omega) &= \frac{1}{Z(j\omega)} = \frac{1}{R} + \frac{1}{(1/j\omega C)} + \frac{1}{j\omega L} \\ &= \frac{1}{R} + j \left\{ \omega C - \frac{1}{\omega L} \right\} \end{aligned}$$

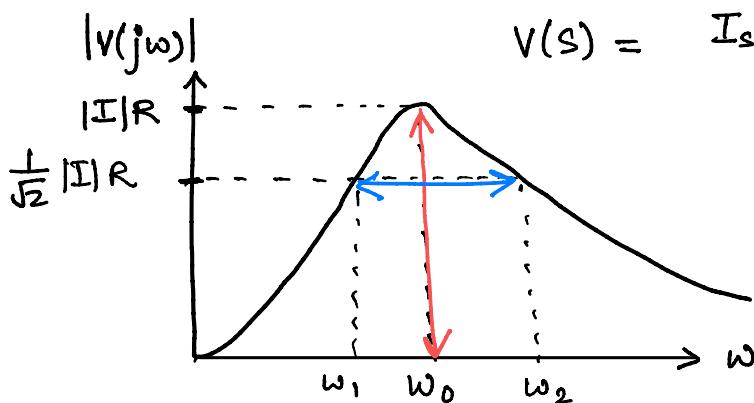
for resonance: V and I are in phase.

$$\begin{aligned} \omega C - \frac{1}{\omega L} &= 0 \\ \Rightarrow \omega &= \frac{1}{\sqrt{LC}} = \omega_0 \quad \text{"Resonant freq."} \end{aligned}$$

Assume $I_s = I_0 e^{\sigma t} \cos(\omega t + \phi)$

$$\begin{aligned} Y(s) &= \frac{1}{Z(s)} = \frac{1}{R} + \frac{1}{(1/cs)} + \frac{1}{sL} \\ &= \frac{s^2 + s(V_{RC}) + V_{LC}}{(1/c)s} \end{aligned}$$

$$\left. \begin{array}{l} V_{out} = V \\ I_{in} = I_s \end{array} \right\} \text{Transfer function} = \frac{V(s)}{I_s(s)} = Z(s) = \frac{1}{Y(s)}$$



$$V(s) = I_s(s) \cdot Z(s)$$

ω_1, ω_2 : half power freq.

height depends on R

width depends on L and C as well.

At resonance:

$$I_S = I_R$$

$$I_C = -I_L \Rightarrow I_C + I_L = 0$$

Quality Factor:

$$Q = 2\pi \frac{\text{max. energy stored}}{\text{total energy lost per period}}$$
$$= 2\pi \frac{[W_L(t) + W_C(t)]_{\text{max}}}{P_R T}$$

P_R = average power lost in resistor $= \frac{1}{2} |I|^2 R$ watts

T = time period (sec)

$$W_L(t) = \frac{1}{2} L i_L^2(t) \text{ Joules}$$

$$W_C(t) = \frac{1}{2} C V_C^2(t) \text{ Joules}$$

$$Q = \omega_0 R C = R \sqrt{\frac{C}{L}}$$

$$\alpha = \text{damping factor} = \frac{1}{2RC} = \frac{\omega_0}{2Q_0}$$

$$\omega_d = \text{damped freq.} = \sqrt{\omega_0^2 - \alpha^2} = \omega_0 \sqrt{1 - (1/2Q_0)^2}$$

Half Power Bandwidth (BW)

$$BW = \omega_2 - \omega_1 = \omega_0 / Q_0$$

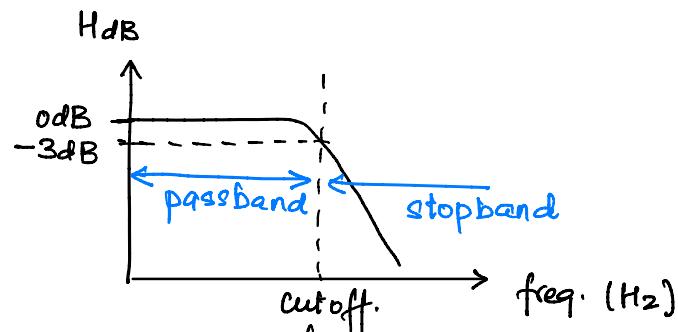
$$\text{where } \omega_0 = \sqrt{\omega_1 \omega_2}$$

$$BW \propto \frac{1}{Q_0}$$

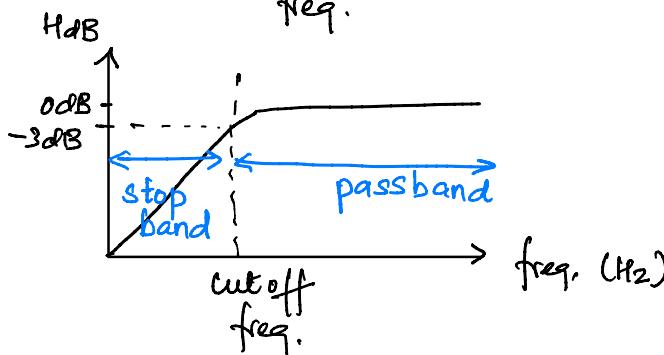
$Q_0 > 5$: High-Q circuit

Filter Design:

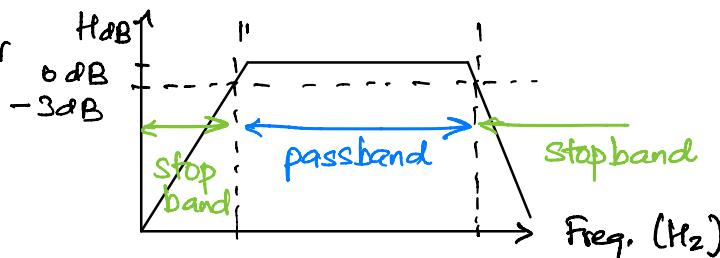
① Low pass filter



② High pass filter

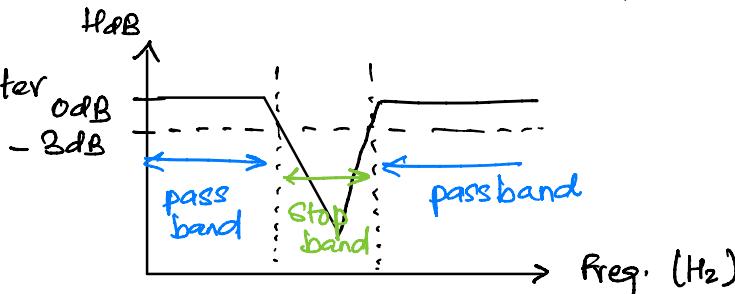


③ Band Pass filter



$$\frac{V_{out}(s)}{V_{in}(s)} = H(s)$$

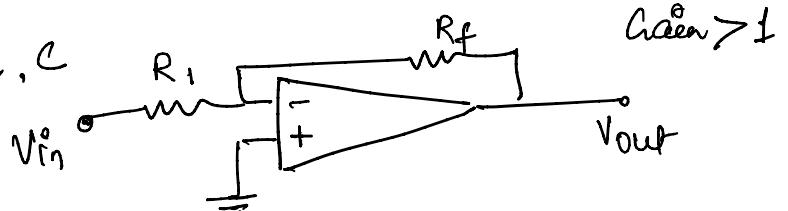
④ Band Stop filter



Passive filter : $gain \leq 1 \quad \text{i.e. } H_{dB} \leq 0 \text{ dB}$

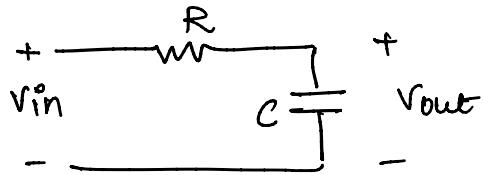
using R, L, C

Active filter : opamp and R, L, C

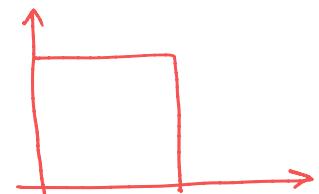
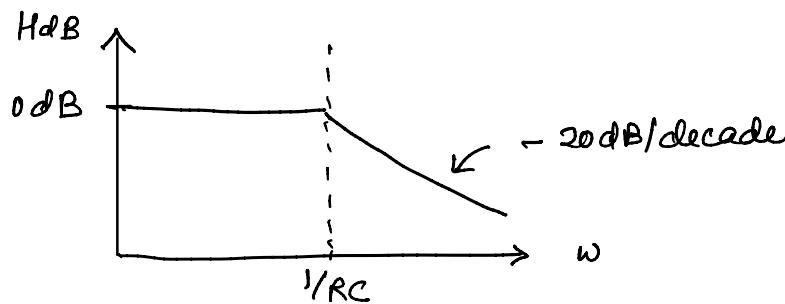


Filter can have $gain > 1 \quad \text{i.e. } H_{dB} > 0 \text{ dB}$

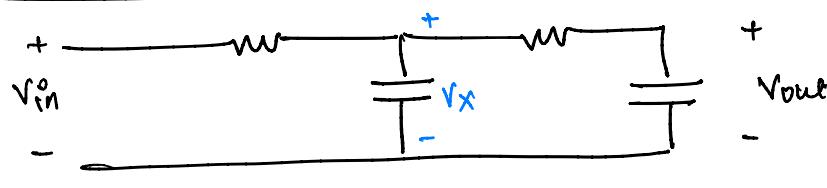
1st order LPF using R, L, C,



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1/Cs}{1/Cs + R} = \frac{1}{1 + s/(RC)}$$

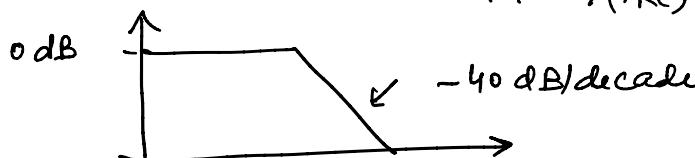


2nd order LPF :-

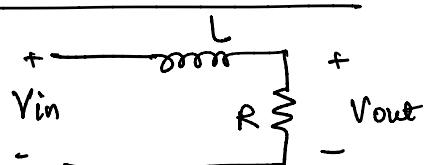


$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{V_{out}(s)}{V_x} \cdot \frac{V_x}{V_{in}}$$

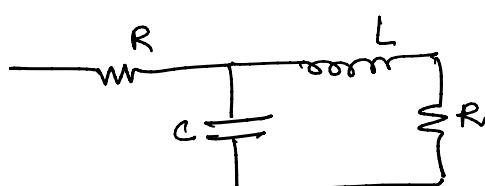
$$\Rightarrow H(s) = \frac{1}{1 + s/(RC)} \cdot \frac{1}{1 + s/(RC)}$$



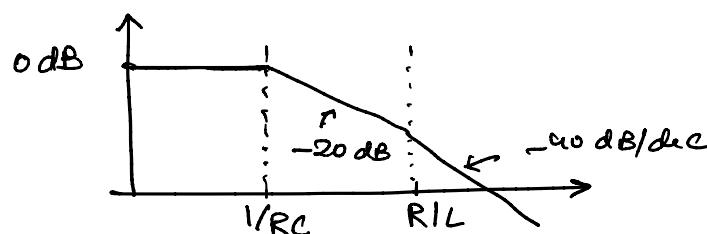
Another 1st order LPF :-



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + Ls} = \frac{1}{1 + s/(RL)}$$

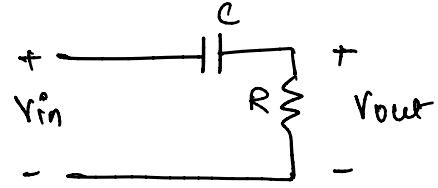


$$H(s) = \frac{1}{1 + s/(RC)} \cdot \frac{1}{1 + s/(RL)}$$



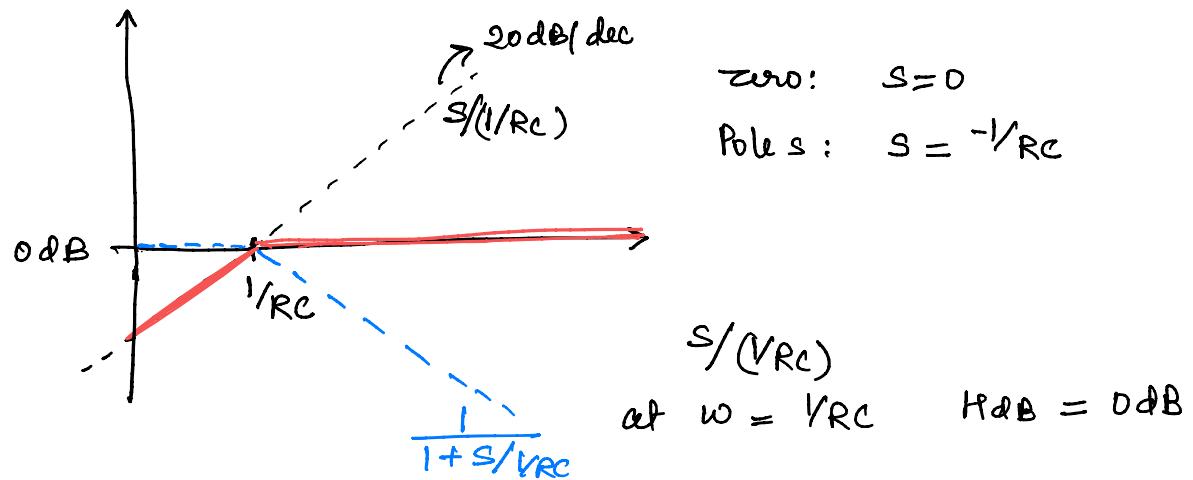
lets assume
 $1/RC < R/L$

High Pass Filter :-

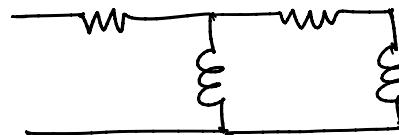
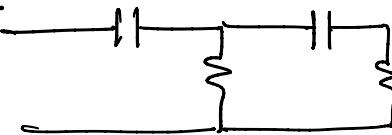


$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + 1/CS} = \frac{RCs}{RCs + 1}$$

$$= \frac{s/(1/RC)}{1 + s/(1/RC)}$$



2nd order:



General transfer functions for nth order filters:

$$LPF: H(s) = \frac{K a_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

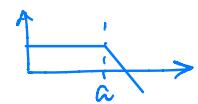
$$HPF: H(s) = \frac{K s^n}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

$$BPF: H(s) = \frac{K s^{n/2}}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}, n \text{ is even}$$

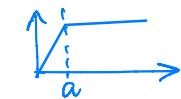
$$BSF: H(s) = \frac{K (s^2 + \omega_0^2)^{n/2}}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}, n \text{ is even}$$

Simplest form

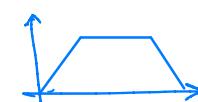
$$\frac{1}{1 + s/a}$$



$$\frac{s/a}{1 + s/a}$$



$$\frac{s}{s^2 + as + b}$$



$$\frac{s^2 + \omega_0^2}{s^2 + as + b}$$

