Tutorial 10

Problem 1)

Obtain the expression for the quality factor of a parallel RLC circuit.

Problem 2)

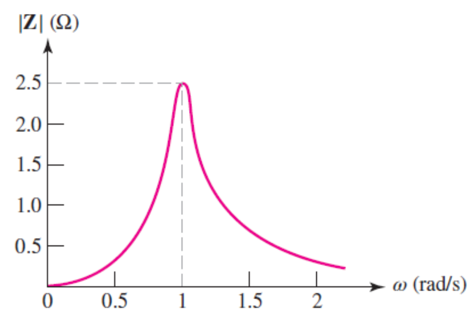
Derive the expressions of the exponential damping coefficient and natural resonant frequency in terms of quality factor.

Problem 3)

(a) Show that the half-power bandwidth is inversely proportional to the quality factor.
(b) Show that the resonant frequency is the geometric mean of half-power frequencies.

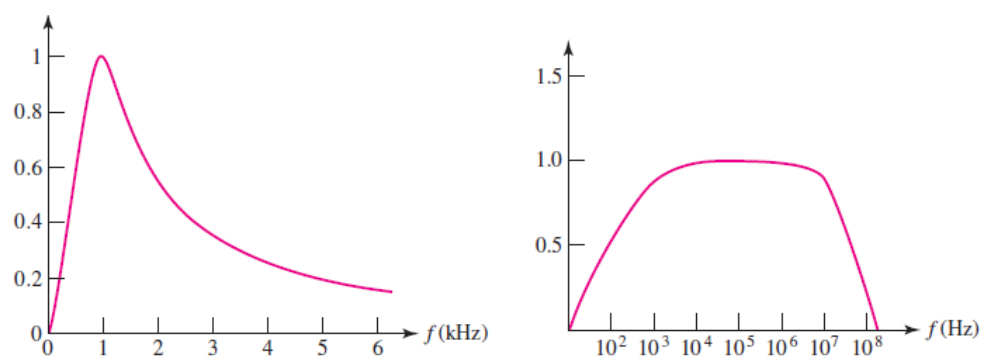
Problem 4)

A parallel *RLC* circuit is constructed such that it has the impedance magnitude characteristic plotted in Fig. 15.60. (*a*) Determine the resistor value. (*b*) Determine the capacitor value if a 1 H inductor was used. (*c*) Obtain values for the bandwidth, Q_0, and both the low half-power frequency and the high half-power frequency.



Problem 5)

Find the bandwidth of each of the response curves shown in Figures.



(*a*) (*b*)

Problem 6)

Design a low-pass filter with a break frequency of 1450 rad/s.

Problem-1

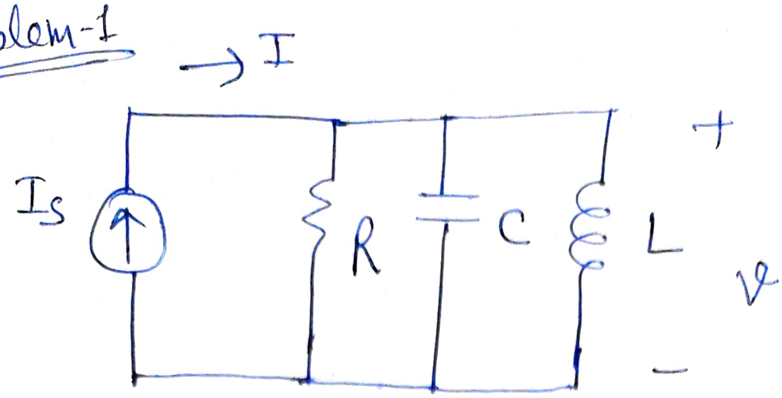


fig: Parallel RLC ckt.

Q = Quality factor:

We define Q as

$$Q = \frac{2\pi \text{ max. energy stored}}{\text{total energy Lost per period}} \quad (1)$$

$$Q = \frac{2\pi \left[w_L(t) + w_C(t) \right]_{max.}}{P_R T.}$$

Q at resonance frequency is denoted by Q_0

Now, we select the current forcing function

$$i(t) = I_m \cos w_0 t$$

voltage response at resonance,

$$v(t) = R\,i(t) = R I_m \cos w_0 t$$

The energy stored in capacitor is given by -

$$w_C(t) = \tfrac{1}{2} C v^2$$

$$= \frac{I_m^2 R^2 C}{2} \cos^2 \omega_0 t$$

The instantaneous energy stored in inductor is given by-

$$w_L(t) = \frac{1}{2} L i_L^2 = \frac{1}{2} L \left(\frac{1}{L} \int v dt \right)^2$$

$$= \frac{1}{2L} \left[\frac{R I_m}{\omega_0} \sin \omega_0 t \right]^2$$

$$w_L(t) = \frac{I_m^2 R^2 C}{2} \sin^2 \omega_0 t$$

So, The total instantaneous stored energy is therefore constant:

$$w(t) = w_L(t) + w_C(t) = \frac{I_m^2 R^2 C}{2} \quad \text{(maximum value)}$$

The energy lost in the resistor in one period is the average power absorbed by the resistor

$$\boxed{P_R = \frac{1}{2} I_m^2 R}$$

Now

$$\boxed{P_R T = \frac{1}{2 f_0} I_m^2 R}$$

Now By putting all the values in eq^n (1)

$$Q_o = \frac{2\pi \times I_m^2 R^2 C/2}{I_m^2 R/2 \; f_o}$$

$$\boxed{Q_o = 2\pi f_o RC = \omega_o RC} \quad [2\pi f_o = \omega_o]$$

Also, by putting $\omega_o = \frac{1}{\sqrt{LC}}$, we get:

$$\boxed{Q_o = R\sqrt{\frac{C}{L}} = \frac{R}{|X_C|} = \frac{R}{|X_L|}}$$

Problem-2

Differential eq^n. for parallel RLC ckt.

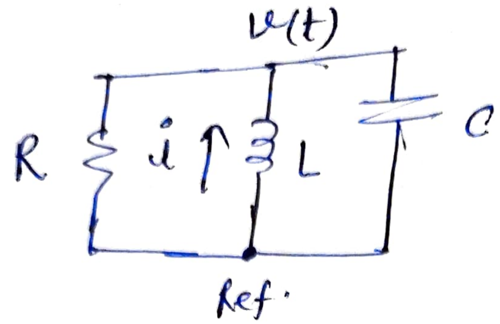


fig: Source free RLC ckt.

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)\,dt + C\frac{dv}{dt} = 0 \quad \text{--- (1)}$$

Laplace transform of eq^n (1)

$$\frac{V(s)}{R} + \frac{1}{Ls}V(s) + CsV(s) = 0$$

$$\frac{V(s)\times C}{s}\left[\frac{\cancel{s}}{RC} + \frac{1}{LC} + s^2\right] = 0$$

$$\frac{V(s)\,C}{s}\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right] = 0$$

$$V(s) \neq 0$$

So

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$S_1, S_2 = \frac{-\frac{1}{RC} \pm \sqrt{\left(\frac{1}{RC}\right)^2 - 4\times\frac{1}{LC}}}{2\times 1}$$

$$S_1, S_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - w_0^2}$$

So, $\boxed{\alpha = \frac{1}{2RC}}$ & $\boxed{w_0 = \frac{1}{\sqrt{LC}}}$

When the response is underdamped.

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$S_{1,2} = -\alpha \pm \sqrt{\alpha^2 - w_0^2}$$

$$\sqrt{\alpha^2 - w_0^2} = \sqrt{-1}\sqrt{w_0^2 - \alpha^2} = j\sqrt{w_0^2 - \alpha^2}$$

where $j = \sqrt{-1}$

Now,

$$\boxed{w_d = \sqrt{w_0^2 - \alpha^2}}$$

w_d = natural freq.

As we know -

$Q_0 = w_0 RC$

So, $RC = \frac{Q_0}{w_0}$

and $\boxed{\alpha = \frac{1}{2RC} = \frac{w_0}{2Q_0}}$ & $w_d = \sqrt{w_0^2 - \alpha^2}$

exponential damping coefficient

So,

$$w_d = \sqrt{w_0 - \left(\frac{w_0}{2Q_0}\right)^2}$$

$$w_d = \sqrt{w_0^2 \left(1 - \frac{1}{2Q_0}\right)^2}$$

$$\boxed{w_d = w_0 \sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}}$$

Natural freq.

As.

$$s^2 + \frac{1}{RC} s + \frac{1}{LC}$$

We can write this eq^n in terms of α and w_0 so eq^n is

$$s^2 + 2\alpha s + w_0^2 \quad -(A)$$

or in the form with damping factor

$$s^2 + 2\xi w_0 s + w_0^2 \quad -(B)$$

By comparing eq^n (A) & (B)

$$\boxed{\xi = \frac{\alpha}{\omega_0} = \frac{1}{2Q_0}}$$

$\hookrightarrow$ damping factor

Problem-3

(a) The admittance of the parallel RLC ckt.

$$Y = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right) \quad \text{— (1)}$$

and $Q_0 = \omega_0 RC = \frac{1}{\sqrt{LC}} RC = R\sqrt{\frac{C}{L}} \times \frac{\sqrt{L}}{\sqrt{L}} = \frac{R\sqrt{LC}}{L}$

$$\boxed{C = \frac{Q_0}{\omega_0 R}} \quad \text{— (2)}$$

Also

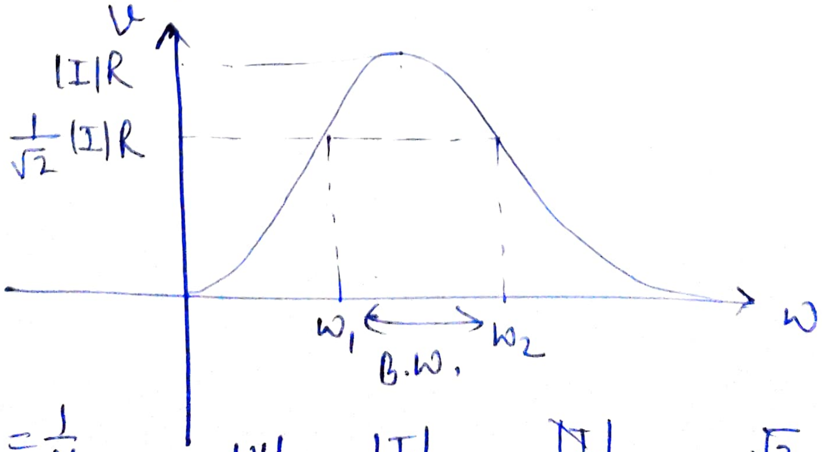
$$Q_0 = \frac{R\sqrt{LC}}{L} = \frac{R}{L \times \omega_0}$$

So, $\boxed{L = \frac{R}{\omega_0 Q_0}}$ — (3)

from eq^n (1), (2) & (3)

$$Y = \frac{1}{R} + j\left(\frac{\omega Q_0}{\omega_0 R} - \frac{\omega_0 Q_0}{\omega R}\right)$$

$$Y = \frac{1}{R} + j\frac{Q_0}{R}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \quad \text{— (4)}$$



$\frac{V}{I} = Z = \frac{1}{Y}$, $|Y| = \frac{|I|}{|V|} = \frac{|I|}{\frac{1}{\sqrt{2}}|I|R} = \frac{\sqrt{2}}{R}$

V is $\frac{1}{\sqrt{2}}$ times, Y is $\sqrt{2}$ times for half power freq.

$$|Y| = \frac{\sqrt{2}}{R}$$

from eqn (4)

$$\Rightarrow \left[\frac{1}{R} + \frac{Q_0}{R} j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] = \frac{\sqrt{2}}{R}$$

$$\Rightarrow \cancel{\frac{1}{R}}\left[1 + Q_0 j\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] = \frac{\sqrt{2}}{\cancel{R}}$$

$$\Rightarrow Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \pm 1$$

$$\Rightarrow Q_0\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = -1$$

and

$$Q_0\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = +1$$

solving we have -

$$\omega_1 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0}\right] \quad -(5)$$

$$\omega_2 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0}\right] \quad -(6)$$

$$B.W. = w_2 - w_1$$

$$= w_0 \left[\cancel{\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}} + \frac{1}{2Q_0} - \cancel{\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}} + \frac{1}{2Q_0} \right]$$

$$B.W. = w_0 \left[\frac{1}{2Q_0} + \frac{1}{2Q_0} \right]$$

$$B.W. = w_0 \left[\frac{1}{Q_0} \right]$$

$$\boxed{BW = \frac{w_0}{Q_0}}$$

So. $\boxed{B.W \propto \frac{1}{Q_0}}$

(b) We have to show

$$\sqrt{w_1 w_2} = w_0 \qquad \text{— (7)}$$

square both side.

$$w_1 w_2 = w_0$$

from eq^n (5) & (6)

$$w_0 \left[\sqrt{1+\left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0} \right] w_0 \left[\sqrt{1+\left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0}\right.$$

$$= w_0^2 \left[1 + \cancel{\left(\frac{1}{2Q_0}\right)^2} + \cancel{\frac{1}{2Q_0}\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}} - \cancel{\frac{1}{2Q_0}\sqrt{1+\left(\frac{1}{2Q_0}\right)^2}} - \cancel{\left(\frac{1}{2Q_0}\right)^2} \right]$$

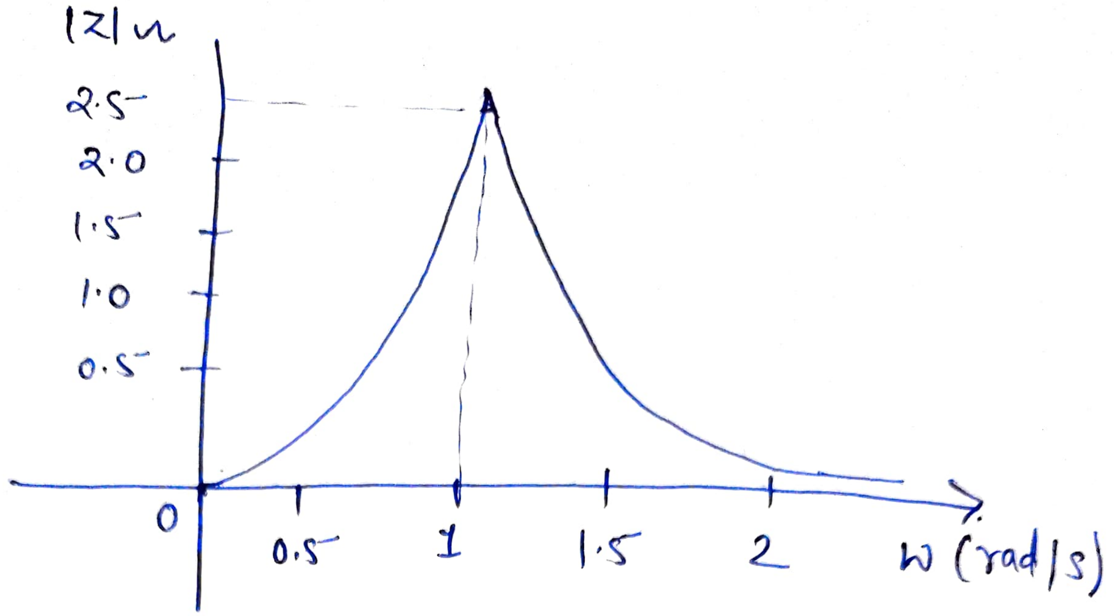
So

$$\omega_1 \omega_2 = \omega_0^2$$

or

$$\boxed{\omega_0 = \sqrt{\omega_1 \omega_2}}$$

P-4



(a) Resonance frequency $w_0 = 1$ rad/s. (from graph)

At resonance impedance = Resistance (ckt purely resistive)

So $\boxed{R = 2.5\Omega}$

(b) As $w_0 = \frac{1}{\sqrt{LC}}$

$$1 = \frac{1}{\sqrt{1 \times C}}$$

$$\sqrt{C} = \frac{1}{1}$$

$\boxed{C = 1F}$ As

(c) $Q = w_0 RC$

$= 1 \times 2.5 \times 1$

$$\boxed{Q = 2.5}$$

(d) $B.W = \frac{w_0}{Q_0} = \frac{1}{2.5} = 0.4$

$$\boxed{B.W = 0.4 \text{ rad/s.}}$$

$\Rightarrow$ $B.W = w_2 - w_1 \Rightarrow 0.4 = w_2 - w_1$ — (1)

& $w_0 = \sqrt{w_1 w_2} \Rightarrow 1 = \sqrt{w_1 w_2} \Rightarrow 1 = w_1 w_2$

$w_2 = \frac{1}{w_1}$ — (2)

from eqn (1) & (2)

$0.4 = \frac{1}{w_1} - w_1$

$0.4 = \frac{1 - w_1^2}{w_1}$

$0.4 w_1 = 1 - w_1^2$

$w_1^2 + 0.4 w_1 - 1 = 0$

$w_1 = \frac{-0.4 \pm \sqrt{(0.4)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$

$w_1 = \frac{-0.4 \pm \sqrt{0.16 + 4}}{2}$

$$\omega_1 = \frac{-0.4 \pm \sqrt{4.16}}{2}$$

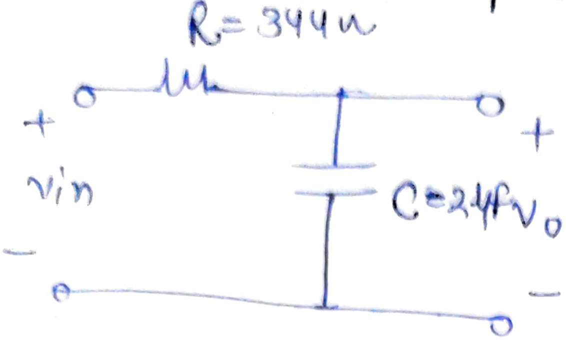
$$\omega_1 = -0.2 \pm 1.01$$

$$\boxed{\omega_1 = 0.81 \text{ rad/s.}}$$

$$\boxed{\omega_2 = \frac{1}{0.81} = 1.23 \text{ rad/s.}}$$

P-6

The ckt for simple low pass filter.



from the circuit

$$V_o(s) = \frac{1/sC}{1/sC + R} \times V_{in}(s) = \frac{1/sC}{\frac{1+sCR}{sC}} = \frac{1}{1+sCR} V_{in}(s)$$

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1}{1+sCR}$$

$\Downarrow$

$$H(s) = \frac{1}{1+sCR} \Rightarrow H(j\omega) = \frac{1}{1+j\omega CR}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega CR)^2}}$$

At break or cut-off freq.

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(\omega_c CR)^2}}$$

$$1+(\omega_c CR)^2 = 2$$

$$\omega_c CR = 1$$

$$\boxed{\omega_c = \frac{1}{RC}}$$

$$\Downarrow$$

Assume $\boxed{C = 2\times10^{-6} F}$.

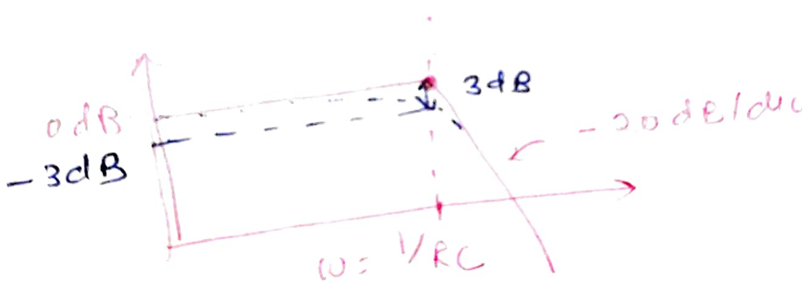
$$1450 = \frac{1}{R\times C}$$

$$1450 = \frac{1}{2\times10^{-6}\times R}$$

$$R = \frac{1}{2\times10^{-6}\times1450}$$

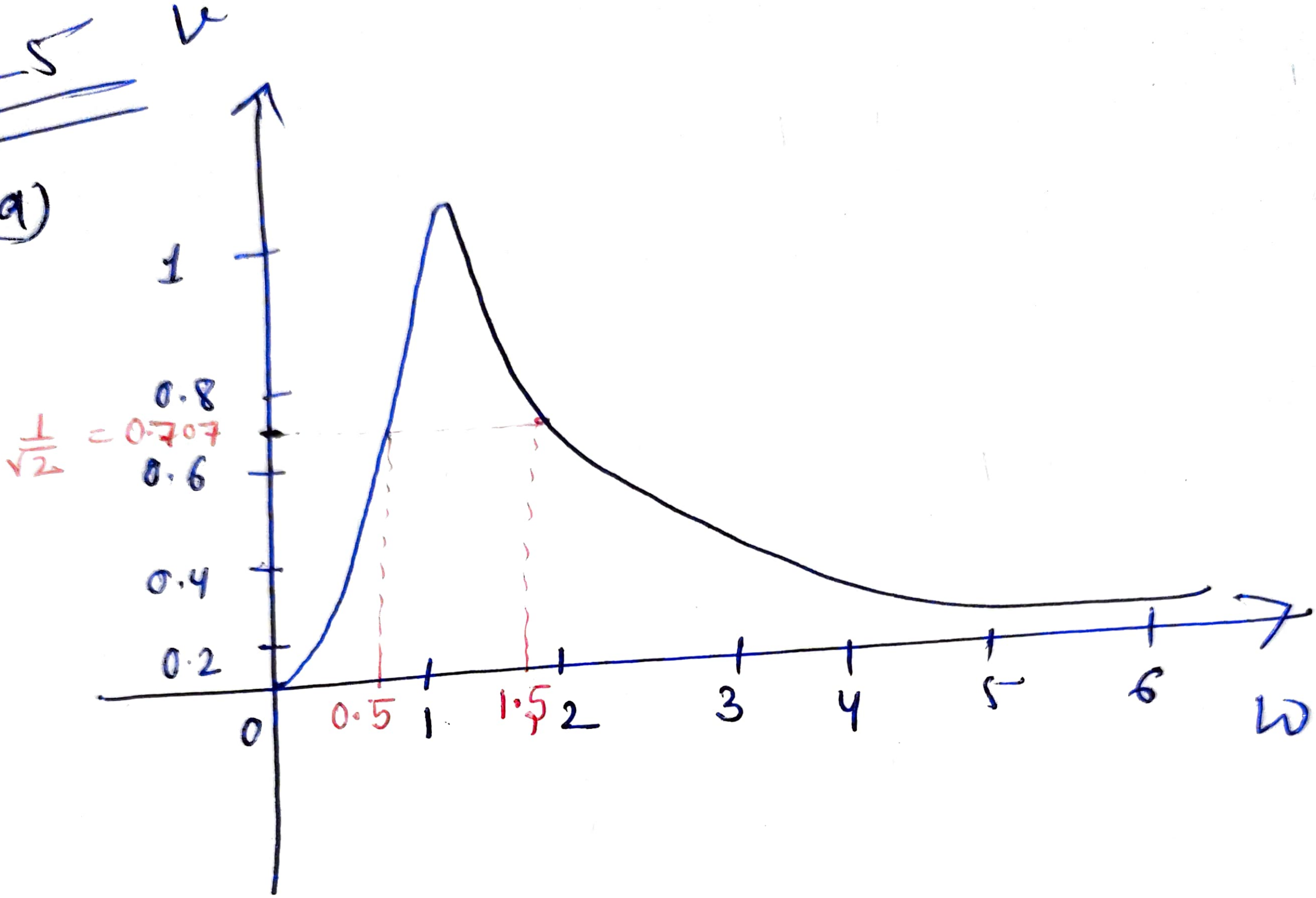
$$R = 0.000344\times10^{6}$$

$$\boxed{R = 344\,\Omega}$$



P-5

(a)

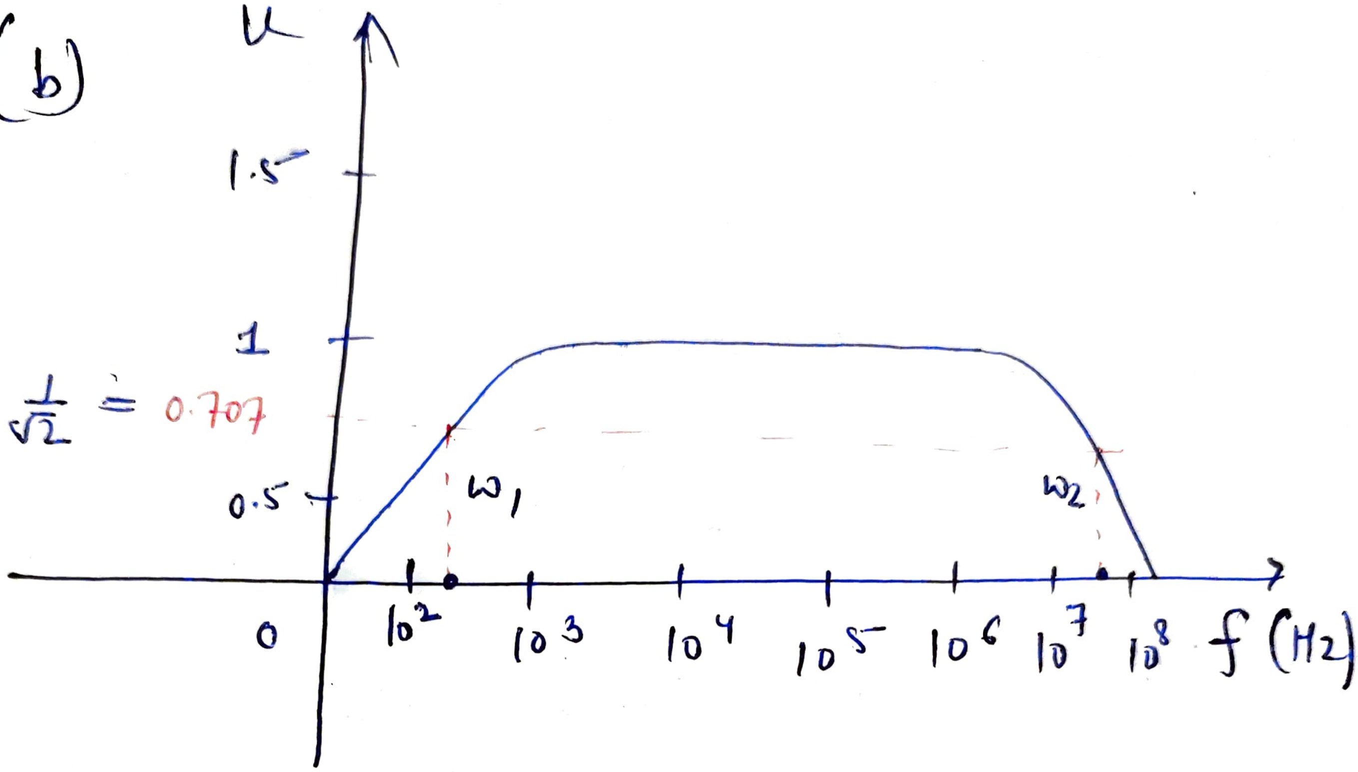


$BW \simeq W_2 - W_1$

$\simeq 1.5 - 0.5$

$\boxed{BW \simeq 1 KHz}$

(b)



$$BW \simeq 10^7 - 10^2$$

$$B.W \simeq 10^7$$

$$BW = 10 \times 10^6$$

$$\boxed{BW = 10\ MHz}$$