CTD L-1

- **Linear circuit** →

i/p → System → o/p

x → System → y

combine

$k.x$ → System → $k.y$

x_1 → System → y_1

$(kx + x_1)$ → System → $(ky + y_1)$

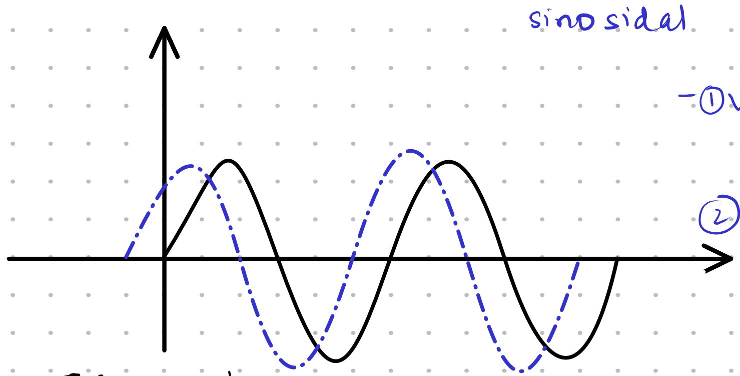
→ linear circuit consist of ① R, L, C linear element
② Independent voltage / current
③ "Liner dependent sources"

→ full response of circuit = transiant + steady state.

- due to suddun change
- natural response
- RLC.

- due to source
- forced response
- V, I.

Sinoside & Complex Numbers



sinosidal voltage source

-① $V = V_0 \sin \omega t$ or $V = V_0 \cos \omega t$ . . .

② $V = V_0 \sin(\omega t + \phi)$

②.is Leading ① or ① is lagging ②

Trigonometry

- $\sin(-\theta) = -\sin\theta$
- $\sin(\theta + 90) = -\cos\theta$
- $\cos(-\theta) = \cos\theta$
- $\cos(\theta + 90) = \sin\theta$.

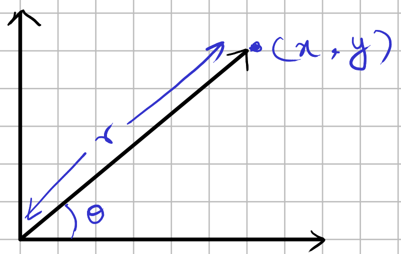
→ $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

- complex number $\rightarrow$ $x+iy$

Eulars identity $\rightarrow$ $e^{i\theta} = \cos\theta + i\sin\theta$

complex no $\rightarrow$ rectangular form $(x+iy)$

complex no $\rightarrow$ polar form $(r\angle\theta)$



conversion :

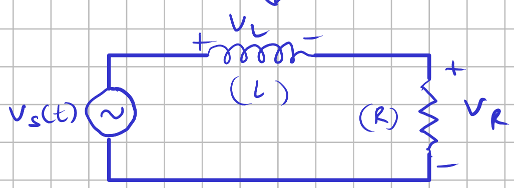
① $\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$ (from polar to rect.)

② $\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(x/y) \end{aligned}}$ (from rect. to polar)

Steady State Analysis :

$\rightarrow$ assumming at $t \rightarrow \infty$ (transient finished)

consider :



$$V_s(t) = V\cos(\omega t)$$

$$V_s(t) = iR + L\frac{di}{dt}$$

* consider $i(t) = A\cos\omega t + B\sin\omega t$

$$V\cos\omega t = AR\cos\omega t + BR\sin\omega t + L(-A\omega\sin\omega t + B\omega\cos\omega t)$$

$$= \cos\omega t(AR + \omega LB) + \sin\omega t(BR - A\omega L)$$

$t = 0$

$\hookrightarrow$ $V = AR + \omega LB \rightarrow A = \dfrac{V - \omega LB}{R}$

$t = \pi/2\pi \rightarrow 0 = BR - A\omega L \rightarrow B = \dfrac{A\omega L}{R}$

$$AR = V - \frac{A\omega^2L^2}{R} \rightarrow AR^2 = VR - A\omega^2L^2$$

$$\boxed{A = \frac{VR}{\omega^2L^2+R^2}}$$

$$\boxed{B = \frac{\omega LV}{\omega^2L^2+R^2}}$$

$$i(t) = I_0 \cos(\omega t + \phi)$$

$$= I_0 (\cos \omega t \cos\phi - \sin \omega t \sin \phi) \quad \text{— (1)}$$

and from above statement

$$i(t) = \frac{VR}{\omega^2 L^2 + R^2} \cos(\omega t) + \frac{V\omega L}{\omega^2 L^2 + R^2} \sin \omega t \quad \text{— (11)}$$

compare

$$I_0 \cos\phi = \frac{VR}{\omega^2 L^2 + R^2} \qquad I_0 \sin\phi = \frac{-V\omega L}{\omega^2 L^2 + R^2}$$

$$I_0 = \sqrt{\frac{V^2R^2 + V^2\omega^2 L^2}{(\omega^2L^2+R^2)^2}} = \boxed{\frac{V}{\sqrt{\omega^2L^2+R^2}} = I_0}$$

$$\boxed{\phi = -\tan^{-1}\left(\frac{\omega L}{R}\right)}$$

$$\boxed{i(t) = \frac{V}{\sqrt{\omega^2L^2+R^2}} \cos\left(\omega t - \tan^{-1}(\omega L/R)\right)}$$

Phasor →

$$i(t) = I_0 \cos(\omega t + \phi)$$

$$= \mathrm{Re}\{I_0 e^{j\omega t + j\phi}\}$$

remove $e^{j\omega t}$

$$= I e^{j\phi}$$

$$I_0 e^{j\phi} = \underbrace{I_0 \angle \phi}_{I_0}$$

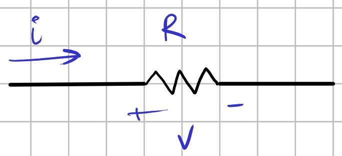
→ phasor is a complex number.

represented by capital letters.

Phasor representation in RLC circuit

$$\boxed{v = V_m e^{j(\omega t + \phi)} \qquad i = I_m e^{j(\omega t + \phi)}}$$

① Resistor →

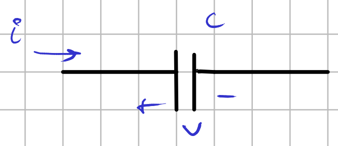


$$v = iR$$

$$V_m e^{j(\cancel{\omega t} + \phi)} = R \times I_m e^{j(\cancel{\omega t} + \phi)}$$

$$\underbrace{V_m e^{j\phi}}_{V} = R \underbrace{I_m e^{j\phi}}_{I} \Rightarrow \boxed{V = RI}$$

② capacitor ↴



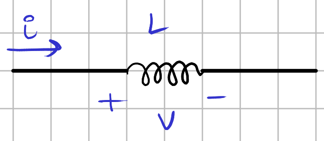
$$i = c\frac{dv}{dt}$$

$$I_m e^{j(\cancel{\omega t}+\phi)} = (cj\omega)V e^{j(\cancel{\omega t}+\phi)}$$

$$\Rightarrow \underbrace{I_m e^{j(\phi)}}_{I} = (cj\omega)\,\underbrace{V_m e^{j\phi}}_{V}$$

$$\rightarrow \boxed{V = \frac{1}{j\omega c}\cdot I}$$

③ Inductor ↴



$$v = L\frac{di}{dt} \Rightarrow V_m e^{j(\omega t+\phi)} = Lj\omega I e^{j(\cancel{\omega t}+\phi)}$$

$$= \underbrace{V_m e^{j\phi}}_{V} = j\omega L\,\underbrace{I_m e^{j\phi}}_{I}$$

$$\boxed{V = j\omega L\cdot I}$$

Impedence → (Z) $Z = \frac{V}{I}$ (Ω) → $\boxed{Z = x + iy}$

resistance

reactance

inductor or capacitor

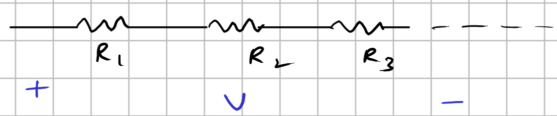
Admittance → (Y) $Y = \frac{1}{Z}$ moh or siemen (s)

Time domain	→ frequency domain
change with time	complex number

to solve circuit convert time domain to frequency domain

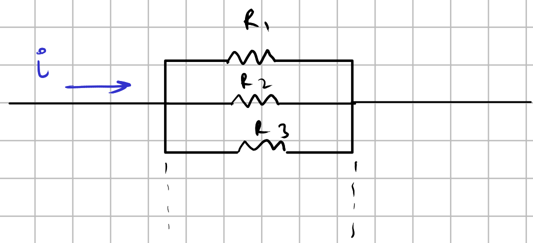
voltage domain ↴

$$V_k = V\frac{R_k}{R_1+R_2+\dots+R_n}$$

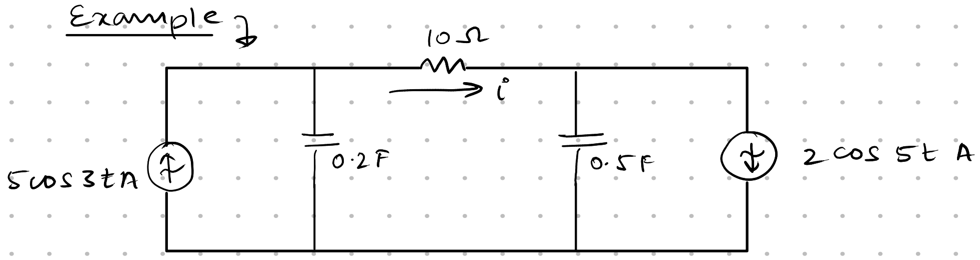


current division ↴

$$i_k = i\frac{1/R_k}{1/R_1+1/R_2+\dots}$$

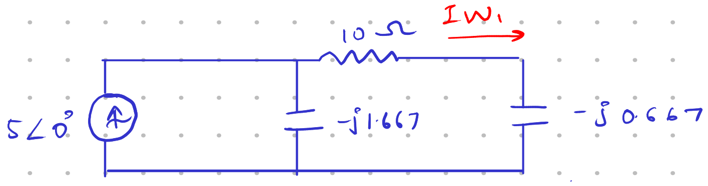


Example 7



↳ as 2 different frequncies we will use $\boxed{\text{supperposition principal}}$

① consider Left source $w_1 = 3 \text{ rad/s}$



$$I_{w_1} = 5\angle 0^\circ \left[\frac{-j(1.667)}{-j(1.667)+10-j(0.667)} \right]$$

$$= 5 \left[\frac{-j(1.667)}{10 - j(2.334)} \right]$$

$$= 5 \left[\frac{-16.67j + (3.89)}{100 - 5.447} \right]$$

$$= 5(-0.17j + 0.0411)$$

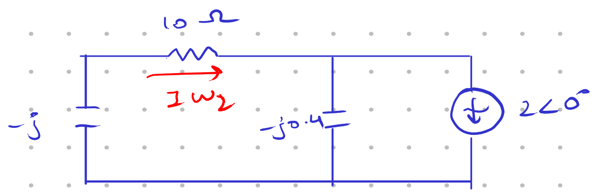
$$= 0.205 - 0.85j$$

in polar form → $r = \sqrt{0.042 + 0.72} = 0.874$

$$\theta = \tan^{-1}(\quad) = -76.4$$

$$= 0.874 \angle -76.4 \text{ A}$$

→ right part



$$I_{w_2} = +2\angle 0^\circ \left(\frac{-j0.4}{-j0.4 + 10 - j} \right)$$

$$= 79.2 \angle -82.03^\circ \text{ mA}$$

Applying superposition

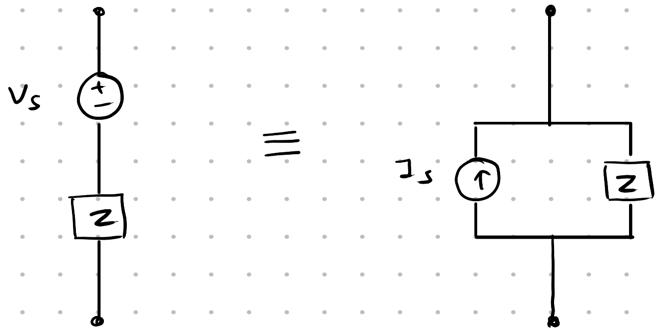
$I = \cancel{I_{w_1}} + I_{w_2} \implies i(t) = i_{w_1}(t) + i_{w_2}(t)$

$i_{w_1}(t) = 811.7 \cos(3t - 76.86) \text{ mA}$

$i_{w_2}(t) = 79.23 \cos(5t - 82.03) \text{ mA}$

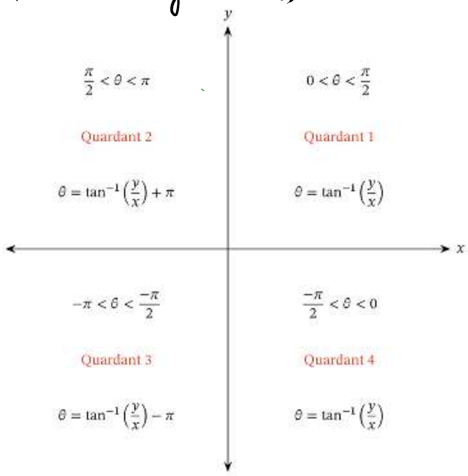
Phasors can only be when their corresponding frequency matches.

Source Transformation →

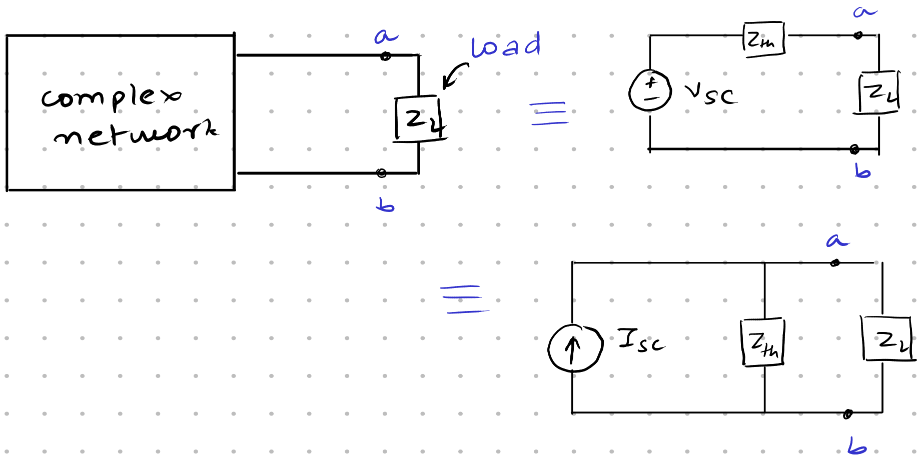


$\boxed{I_s = V_s / Z}$

Polar angles →



Thevenin & Norton's Equivalent →



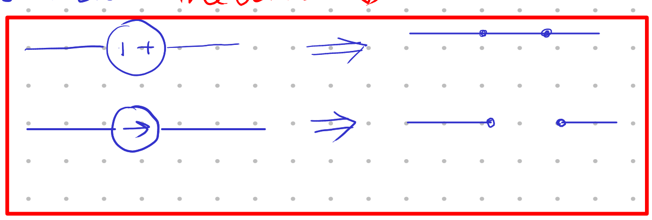
→ Thevenin →

(A) Remove load resistance in theveni but short in Norton

(B) calculate V_{sc}

(C) Remove all dependent sources. Thevenin →

(D) calculate Z_{th}.

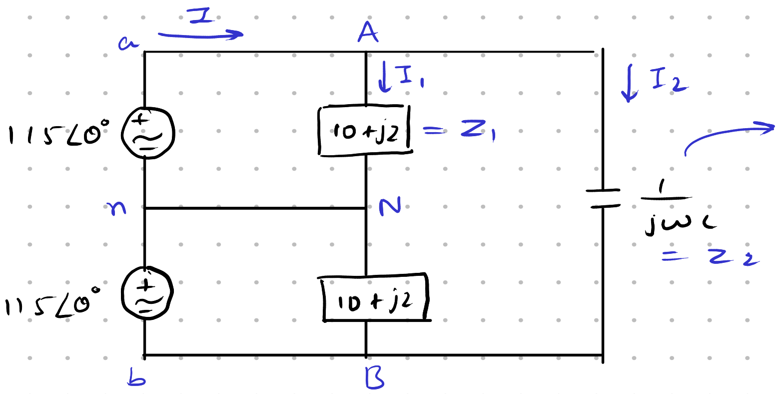


POWER ANALYSIS

$PF = \cos(\theta - \phi)$ $\hookrightarrow$ if +ve $\longrightarrow$ lagging (inducting)

-ve $\longrightarrow$ leading (capacitive)

* always write leading or lagging with P.F.



$\theta = \phi$

$I_{aA} = I_1 + I_2$

$= \frac{115\angle 0^\circ}{10+2j} + \frac{230\angle 0}{(1/j\omega c)}$

S = complex power of total load $= \frac{1}{2} V I^*$

$PF = \frac{Re\{S\}}{|S|} = 1$

$\Rightarrow R\{S\} = |S| \Rightarrow Im\{S\} = 0$

$S = \frac{1}{2} V_{aN} I_{an}^* + \frac{1}{2} V_{bn} I_{NB}^* + \frac{1}{2} V_{ab} I_{AB}^*$

$= \frac{1}{2}\left[115\angle 0^\circ \times \left(\frac{V_{an}}{Z_1}\right)^* + 115\angle 0^\circ \left(\frac{V_{bn}}{Z_1}\right)^* + 230\angle 0^\circ \left(\frac{V_{ab}}{Z_2}\right)^* \right]$

$= \frac{1}{2}\left[\frac{|V_{an}|^2}{|Z_1|^2} Z_1 + \frac{|V_{bn}|^2}{|Z_1|} Z_1 + \frac{|V_{ab}|^2}{|Z_2|^2} Z_2 \right]$

same

$= \frac{1}{2}\left[\frac{2|V_{an}|^2}{|Z_1|^2} Z_1 + 4\frac{|V_{an}|^2}{|Z_2|^2} |Z_2| \right]$

$= \frac{115^2}{104}(10+2j) + \frac{2(115)^2}{(1/\omega c)^2}\left(\frac{-j}{\omega c}\right)$

$= 115^2\left[\frac{10+2j}{104} - 2j(\omega c) \right]$

$= 115^2\left[\frac{10}{104} + 2j\left[\frac{1}{104} - \omega c\right] \right]$

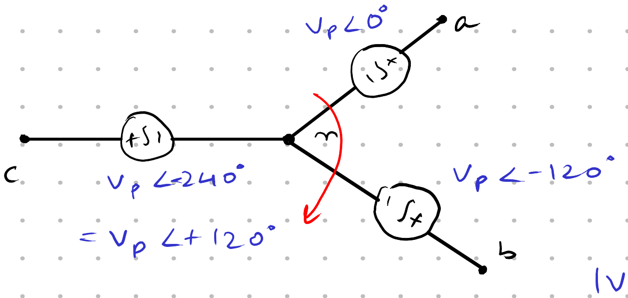
$\Rightarrow \omega c = \frac{1}{104}$ $\quad c = \frac{1}{104 \times \omega} = \frac{1}{104 \times 2\pi 60}$

$= 9.6 \times 10^{-3} \times 3.18 \times 10^{-3}$

$$= \quad 0.\underbrace{0\,305} \times 10^{-3} = 30.5 \times 10^{-6}$$

$$= 30.5 \ \mu F.$$

Three Phase Sources



$V_{an} = V_p\angle 0°$

$V_{bn} = V_p\angle -120°$

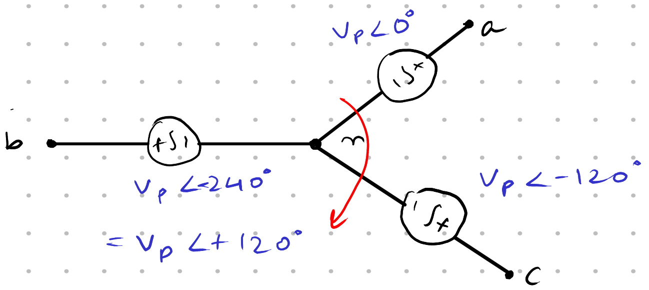
$V_{cn} = V_p\angle -240°$

abc or positive sequence.

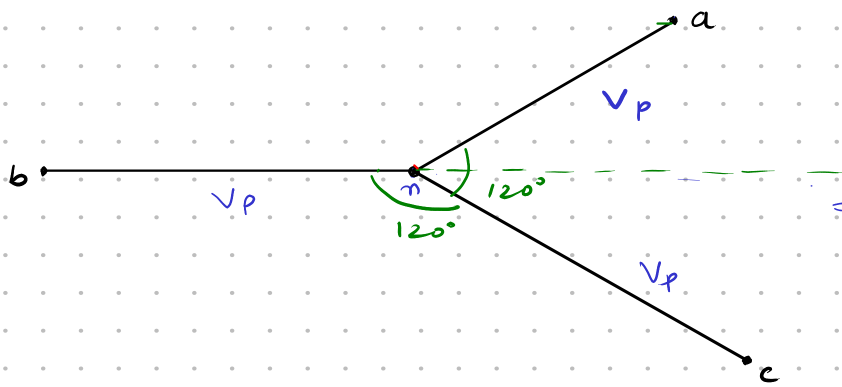
$|V_{an}| = |V_{bn}| = |V_{cn}| = V_p$

$V_{an} + V_{bn} + V_{cn} = 0$

Balanced source.



→ cba or -ve source.



V_{ab}, V_{ac}, V_{bc}

$$V_{ab} = V_{an} - V_{bn}$$

$$= \sqrt{V_p^2 + V_p^2 - 2V_p^2\cos(120°)}$$

$$= \sqrt{2V_p^2 - 2V_p^2 \times -\frac{1}{2}}$$

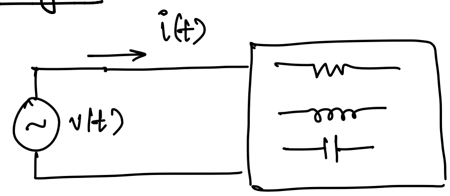
$$= \sqrt{3V_p^2} = \sqrt{3}V_p\angle 30°$$

Question

→

Module 3

AC Power Analysis



$\cos A . \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

Instantaneous Power : $p(t) = v(t)\, i(t)$

$$v(t) = V_m \cos(\omega t + \theta)$$

$$i(t) = I_m \cos(\omega t + \phi)$$

$$p(t) = V_m \cos(\omega t + \theta) . I_m \cos(\omega t + \phi)$$

$$= \underbrace{\frac{V_m I_m}{2} \cos(\theta - \phi)}_{\substack{\text{constant} \\ \text{(DC term)}}} + \underbrace{\frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi)}_{\substack{\text{twice frequency} \\ \text{(harmonic)}}}$$

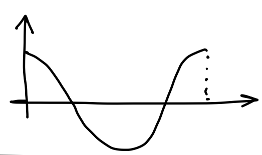
Average Power :

$$P_{avg} = \frac{1}{T}\int_0^T p(t)\, dt$$

$$= \underbrace{\frac{1}{T}\int_0^T \frac{V_m I_m}{2} \cos(\theta - \phi) . dt}_{} + \underbrace{\frac{1}{T}\int_0^T \frac{V_m I_m}{2} \cos(2\omega t + \theta + \phi) dt}_{0}$$

$$= \frac{V_m I_m}{2} \cos(\theta - \phi) + 0$$

Average of Cos



$$\boxed{P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi)}$$

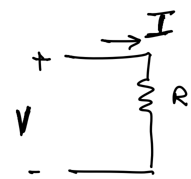
$$P_{avg} = \frac{1}{2} \mathrm{Re}\{\mathbf{V}\mathbf{I}^*\}$$

$$\mathbf{I} = I_m e^{j\phi}$$
$$\mathbf{I}^* = I_m e^{-j\phi}$$

$$\cancel{P_{avg} = \mathbf{V}\mathbf{I}^*}$$

No phasor for power !

Average power absorbed by a purely resistive element

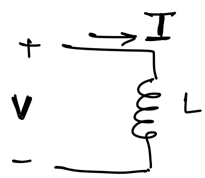


$$P_{avg} = \frac{V_m I_m}{2} \cos\underbrace{(\theta - \phi)}_{0^\circ}$$

$$\mathbf{V} = \mathbf{I}R$$

$$P_{avg} = \frac{V_m I_m}{2} = \frac{I_m^2 R}{2} = \frac{V_m^2}{2R}$$

Purely inductive element



$$P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi)$$

$$= \frac{V_m I_m}{2} \cos(90^\circ)$$

$$\boxed{P_{avg} = 0}$$

$$\mathbf{V} = j\omega L \mathbf{I}$$

$$(\theta - \phi) =$$

$$V_m e^{j\theta} = j\omega L I_m e^{j\phi}$$

$$\Rightarrow \underbrace{\frac{V_m}{I_m}}\, e^{j\theta - \phi} = j\omega L$$

$$\theta - \phi = 90^\circ$$

Purely capacitive element

$$P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi)$$

$$\theta - \phi = -90^\circ$$

$$\boxed{P_{avg} = 0}$$

$p(t)$

$$P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi)$$ ✓

$$P_{avg} = Re\left\{\frac{V I^*}{2}\right\}$$

$$v(t) = V_m \cos(\omega t + \theta)$$

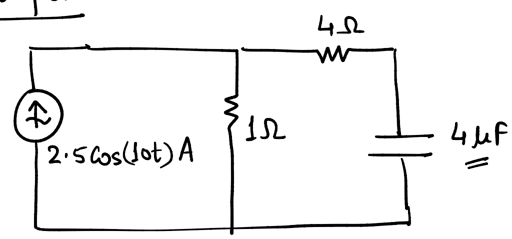
$$\downarrow$$

$$v(t) = Re\left\{V_m e^{j\omega t + j\theta}\right\}$$

$$\downarrow \; e^{j\omega t}$$

$$V = V_m \angle \theta$$

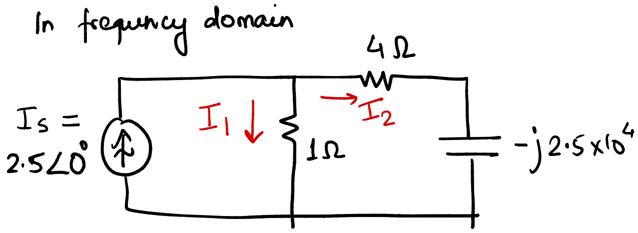
Example



① Find power delivered to each element at $t = 0, 10, 20$ ms

② Find average power delivered to each element.

In frequency domain



Power delivered to 1Ω

$$p_{1\Omega}(t) = v_{1\Omega}(t)\, i_{1\Omega}(t)$$

$$I_{1\Omega} = I_1 = I_s \frac{4 - j2.5\times10^4}{1 + 4 - j2.5\times10^4} \approx I_s \quad \Rightarrow i_{1\Omega}(t) = 2.5 \cos(10t) \text{ A}$$

$$V_{1\Omega} = (I_1)(1\Omega) \quad \Rightarrow \quad v_{1\Omega}(t) = 2.5 \cos(10t) \text{ V}$$

$$p_{1\Omega}(t) = 2.5 \cos(10t) \cdot 2.5 \cos(10t)$$

$$p_{1\Omega}(t) = 3.125 + 3.125 \cos(20t) \text{ Watts}$$

$$p_{1\Omega}(t = 0) = 6.25 \text{ W}$$

$$p_{1\Omega}(t = 10\text{ms}) = 6.1877 \text{ W}$$

$$p_{1\Omega}(t = 20\text{ms}) = 6.0033 \text{ W}$$

Average power delivered to 1Ω

$$P_{avg\,1\Omega} = 3.125 \text{ W}$$

Power delivered to 4Ω :-

$$p_{4\Omega}(t) = v_{4\Omega}(t) \cdot i_{4\Omega}(t)$$

$$I_{4\Omega} = I_2 = I_s \frac{1}{5 - j2.5\times10^{4}} = 10^{-4}\angle 90^\circ \text{ A} \rightarrow i_{4\Omega}(t)$$

$$V_{4\Omega} = 4(I_{4\Omega}) = 4\times10^{-4}\angle 90^\circ \text{ V} \rightarrow v_{4\Omega}(t)$$

$$p_{4\Omega}(t) = 2\times10^{-8} + 2\times10^{-8}\cos(20t) \text{ W}$$

$$p_{4\Omega}(t=0) = 4\times10^{-8} \text{ W}$$

$$p_{4\Omega}(t=10ms) = 3.96\times10^{-8} \text{ W}$$

$$p_{4\Omega}(t=20ms) = 3.84\times10^{-8} \text{ W}$$

$$P_{avg4\Omega} = 2\times10^{-8} \text{ W}$$

Power delivered to capacitor :

$$p_c(t) =$$

$$P_{avg.c} =$$

$$I_c = I_2 = 10^{-4}\angle 90^\circ \text{ A} \rightarrow i_c(t) \qquad \Rightarrow p(t) = i_c(t)v_c(t)$$

$$V_c = (-j2.5\times10^{4})I_2 = 2.5 \text{ V} \rightarrow v_c(t)$$

$$\cancel{\underbrace{\underline{I_c}\times\underline{V_c}}_{\text{Phasor of power}} \rightarrow p(t)}$$

$$i_c(t) = 10^{-4}\cos(10t + 90^\circ)$$

$$v_c(t) = 2.5\cos(10t)$$

$$p_c(t) = 1.25\times10^{-4}\cos(20t + 90^\circ) \text{ W}$$

$$p_c(t=0) = 0 \text{ W}$$

$$p_c(t=10ms) = 2.48\times10^{-5} \text{ W}$$

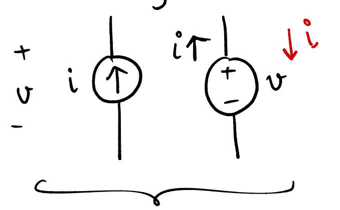
$$p_c(t=20ms) = 4.86\times10^{-5} \text{ W}$$

$$P_{avg.c} = 0$$

① $p_{1\Omega}(t) + p_{4\Omega}(t) + p_C(t) = \text{constant } 1$ ✗

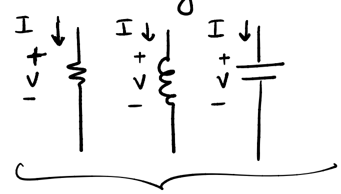
② $P_{avg,1\Omega} + P_{avg,4\Omega} + P_{avg,C} = \text{constant } 2$ ✓ $= P_{avg,\,source}$

Active sign convention

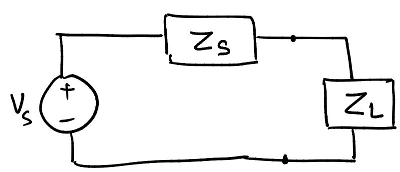


$v \cdot i = -20\,W$

Passive sign convention



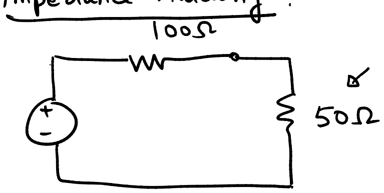
Maximum Power Transfer :



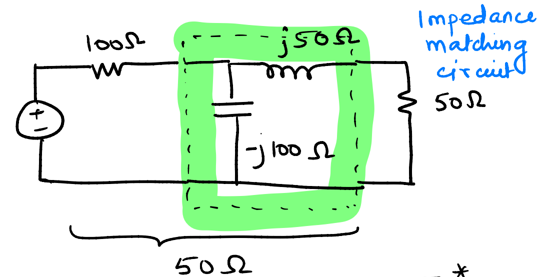
$R_S = R_L$

Max power delivered to load, when $\boxed{Z_L = Z_S^*}$ ✓

Impedance Matching :-

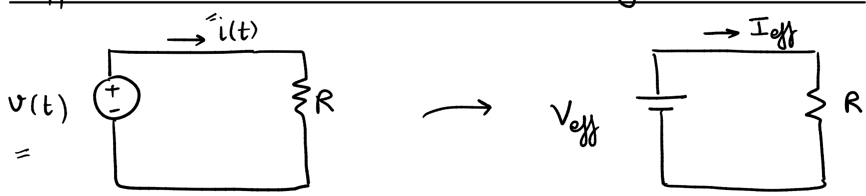


⇒



$Z_L = Z_S^*$

Effective value (RMS value) of voltage and current :—



$v(t) \leftarrow \qquad P_{avg} = P_{avg}$

$$\Rightarrow \frac{1}{T}\int_0^T i^2(t)\,R\,dt = I_{eff}^2 R = \frac{V_{eff}^2}{R}$$

$$\Rightarrow I_{eff} = \underbrace{\sqrt{\frac{1}{T}\int_0^T i^2(t)\,.\,dt}} =$$

Root mean square

Sinusoidally varying sources

$$i(t) = I_m \cos(\omega t + \phi)$$

$2\cos^2 x = \cos 2x + 1$

$$I_{eff} = \sqrt{\frac{1}{T}\int_0^T I_m^2 \cos^2(\omega t + \phi)\,dt}$$

$T = \frac{2\pi}{\omega} = \frac{1}{f}$

$$\boxed{I_{eff} = \frac{I_m}{\sqrt{2}}}$$ rms value

Suppose, $\mathbf{I} = 10\angle 90^\circ$ A

$$\Rightarrow \mathbf{I} = \frac{10}{\sqrt{2}}\angle 90^\circ \text{ A rms}$$

$$\boxed{V_{eff} = \frac{V_m}{\sqrt{2}}}$$ rms value

$$V_1 = 100\angle -60^\circ \text{ V rms}$$

$$V_1 = 100\sqrt{2}\ \angle -60^\circ \text{ V}$$

Apparent Power

$$P_{apparent} = I_{eff} V_{eff}$$

Power Factor

$$PF = \frac{\text{average power}}{\text{apparent power}} = \frac{(V_m I_m / 2)\cos(\theta - \phi)}{V_{eff} I_{eff}} = \cos(\theta - \phi)$$

$$PF = \cos\underbrace{(\theta - \phi)}_{\text{Power factor angle}}$$

Review

Instantaneous Power : $p(t) = v(t)\, i(t)$

Average Power : $P_{avg} = \frac{V_m I_m}{2} \cos(\theta - \phi) = \frac{1}{2} \mathrm{Re}\{ \mathbf{V}\mathbf{I}^* \}$

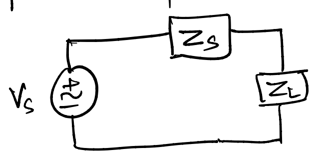
$$P_{avg,sources} = \sum P_{avg,elements}$$

P_{avg} for resistive load : $P_{avg} \neq 0$

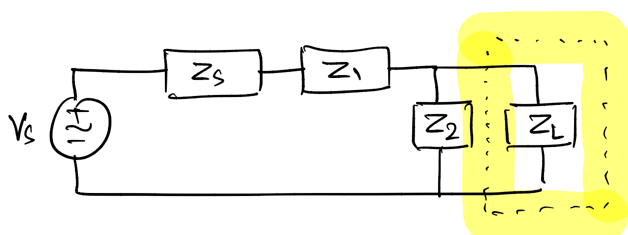
P_{avg} for reactive load : $P_{avg} = 0$

$Z = R + jX$ (resistive: R, reactive: X)

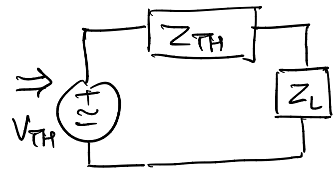
Max. power Transfer :



$$Z_L = Z_S^*$$



$\Rightarrow$



$$\boxed{Z_L = Z_{TH}^*}$$

Effective V and I values (RMS values)

$$I_{eff} = \sqrt{\frac{1}{T} \int_0^T i^2(t)\, dt}$$

$$i(t) = I_m \cos(\omega t + \phi)$$

$$I_{eff} = \frac{I_m}{\sqrt{2}} \quad , \quad V_{eff} = \frac{V_m}{\sqrt{2}}$$

$$\mathbf{I} = I_m \angle \phi = \sqrt{2}\, I_{eff} \angle \phi$$

I_{eff} A rms

V_{eff} V rms

Apparent Power

$P_{apparent} = I_{eff} \cdot V_{eff}$ VA (Volt-Ampere)

Power Factor

$$PF = \cos(\theta - \phi) = \frac{P_{average}}{P_{apparent}}$$

For purely resistive load: $PF = 1$ (max) $\theta - \phi$ 90°

For purely reactive load: $PF = 0$ (min) $\theta - \phi$ -90°

Assume PF of a load $= 0.5$

$$\cos(\theta - \phi) = 0.5$$

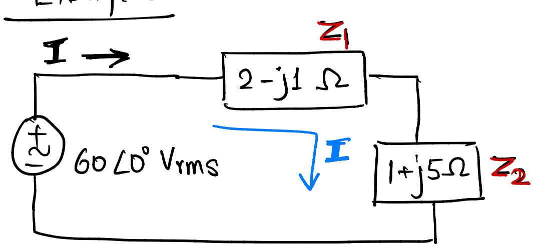
$$\Rightarrow (\theta - \phi) = \pm 60^\circ$$

$PF = 0.5$ leading $\rightarrow$ capacitive $(\theta - \phi) < 0$

0.5 lagging $\rightarrow$ inductive $(\theta - \phi) > 0$

$\boxed{\mathbf{I} \text{ w.r.t. } \mathbf{V}}$

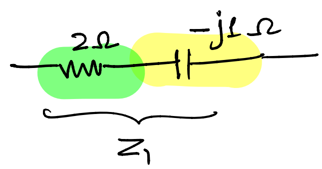
Example



Find
1. Average power delivered to each loads
2. Average power supplied by source
3. Apparent power supplied by source
4. PF of combined load.

(1) $P_{avg, z_1} = \frac{1}{2} V_m I_m \cos(\theta - \phi) = \frac{1}{2} \text{Re}\{\mathbf{V}\mathbf{I}^*\}$

$= \underline{I_{eff}^2 R}$



$\mathbf{I} =$

Writing KVL;

$$-60\angle 0^\circ \text{ Vrms} + \mathbf{I}(2 - j1 + 1 + j5) = 0$$

$$\Rightarrow \mathbf{I} = \frac{60\angle 0^\circ \text{ Vrms}}{3 + j4\ \Omega} = \underbrace{12}_{I_{eff}}\angle -53.13^\circ \text{ A rms}$$

$$P_{avg, z_1} = (12)^2\, 2 = 288 \text{ W}$$

$\mathbf{I} = 12 \angle -53^\circ$ A rms

$i(t) = \underbrace{12}_{I_{eff}} \cos(\omega t - 53^\circ)$ A rms

I_{eff} (A rms)	versus	I_m (A)

$$I_{eff} = \frac{I_m}{\sqrt{2}} \Rightarrow I_m = I_{eff}\sqrt{2}$$

$I_{eff} = 12$

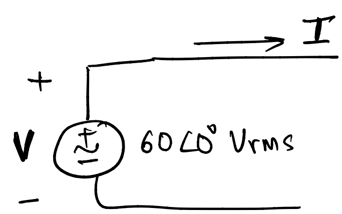
$I_m = I_{eff}\sqrt{2} = 12\sqrt{2}$

$i(t) = \underbrace{12\sqrt{2}}_{I_m} \cos(\omega t - 53^\circ)$ A

$$P_{avg} = \frac{1}{2} V_m I_m \cos(\theta - \phi) = V_{eff} I_{eff} \cos(\theta - \phi)$$

$$P_{avg, Z_2} = I_{eff}^2 R = (12)^2 1 = 144 \text{ W}$$

(2) $P_{avg,\, source} = V_{eff} I_{eff} \cos(\theta - \phi)$



$\mathbf{V}_{source} = 60 \angle 0^\circ$ V rms

$\mathbf{I}_{source} = 12 \angle -53.13^\circ$ A rms

$P_{avg, source} = (60)(12) \cos(0 - (-53.13^\circ)) = 432$ W

$P_{avg, z_1} = 288$ W , $P_{avg, z_2} = 144$ W

$P_{avg, z_1} + P_{avg, Z_2} = 432$ W

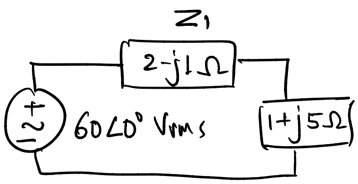
(3) $P_{apparent,\, source} = I_{eff} V_{eff} = (60)(12) = 720$ VA

(4) PF of combined loads = PF of source

$PF = \cos(\theta - \phi) = \cos(\underbrace{0^\circ + 53.13^\circ}) = 0.6$ lagging

$(\theta - \phi) > 0 \rightarrow$ lagging

$(\theta - \phi) < 0 \rightarrow$ leading



$$P_{avg,Z_1} = I_{eff}^2 R$$

$$P_{avg,Z_1} = V_{eff}\, I_{eff} \cos(\theta - \phi)$$

$$\mathbf{V}_{Z_1} = 60\angle 0^\circ \text{ Vrms} \times \frac{2-j1}{3+j4}$$

$$= \underbrace{26.83}\ \angle -79.7^\circ \ \text{Vrms}$$

$$\mathbf{I}_{Z_1} = \mathbf{I} = 12\angle -53.13^\circ \ \text{A rms}$$

$$P_{avg,Z_1} = 26.83 \times 12 \times \cos(-79.7^\circ - (-53.13^\circ)) = 288 \text{ W}$$

$$P_{average} = \frac{1}{2}\text{Re}\{\mathbf{V}\mathbf{I}^*\} \quad ; \quad \mathbf{V} = V_m\angle\theta, \quad \mathbf{I} = I_m\angle\phi$$

$$\mathbf{V}_{eff} = V_{eff}\angle\theta = \frac{V_m}{\sqrt{2}}\angle\theta$$

$$\mathbf{I}_{eff} = I_{eff}\angle\phi = \frac{I_m}{\sqrt{2}}\ \phi$$

$$P_{average} = \text{Re}\{\mathbf{V}_{eff}\mathbf{I}_{eff}^*\}$$

Complex Power

$$\mathbf{S} = \mathbf{V}_{eff}\mathbf{I}_{eff}^* \quad \text{(unit: VA)}$$

Not a phasor of power.
This is a complex number.

$$\mathbf{S} = (V_{eff}\angle\theta^\circ)(I_{eff}\angle -\phi) = V_{eff}\cdot I_{eff}\, e^{j(\theta-\phi)}$$

$$\Rightarrow \mathbf{S} = \underbrace{V_{eff} I_{eff}\cos(\theta-\phi)}_{\substack{P \\ \text{average power}}} + j\,\underbrace{V_{eff} I_{eff}\sin(\theta-\phi)}_{\substack{Q \\ \text{reactive power}}}$$

(VA) (W) (VAR)

$$\mathbf{S} = P + jQ$$

$$|\mathbf{S}| = V_{eff} I_{eff} = P_{apparent} \quad (VA)$$

$Z = R + jX$

Purely resistive load $Z = R$, $X = 0$

$$\mathbf{S} = \underbrace{P}_{V_{eff} I_{eff} \cos(\theta - \phi)} + jQ \longrightarrow V_{eff} I_{eff} \underbrace{\sin(\theta - \phi)}_{0}$$

$P \neq 0$

$Q = 0$

Purely reactive load $Z = jX$, $R = 0$

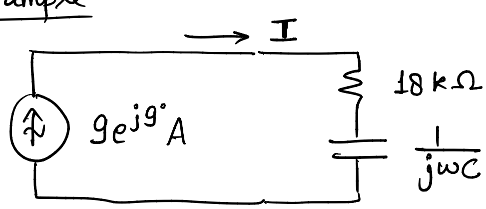
$\mathbf{S} = P + jQ$

$P = 0$

$Q \neq 0$

Q signifies energy flow rate into or out of reactive component of load.

Example



Assume $C = 1\,\mu F$, $\omega = 45$ rad/s

Find:

1. Complex power provided by source
2. Time-average power absorbed by combined load.

Find

3. Reactive power absorbed by combined load
4. Apparent power absorbed by " "
5. Power factor of " "

$S = 1.158 \times 10^6 \angle -50.99^\circ$ VA

$P_{avg} = 7.29 \times 10^5$ W

$Q = -9 \times 10^5$ VAR

$|S| = P_{apparent} = 1.158 \times 10^6$ VA

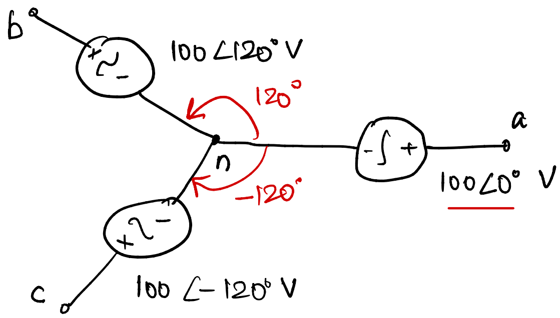
$pf = 0.629$ leading

$p(t) = 7.28 \times 10^5 + 1.158 \times 10^6 \cos(90t - 33^\circ)$ W

Module 4

Polyphase Circuits
(Chapter 12 of textbook)

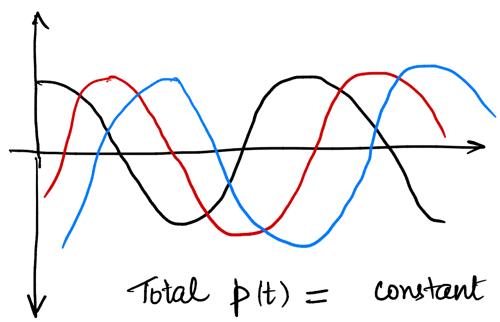
Three-phase source



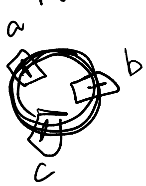
$$V_{an} = 100\angle 0^\circ \ \text{V}$$
$$V_{bn} = 100\angle 120^\circ \ \text{V}$$
$$V_{cn} = 100\angle -120^\circ \ \text{V}$$

$$\left.\begin{aligned} |V_{an}| = |V_{bn}| &= |V_{cn}| \\ \rightarrow \quad V_{an} + V_{bn} + V_{cn} &= 0 \end{aligned}\right\} \text{Balanced source}$$

Freq. Domain	Time Domain
$100\angle 0^\circ$	$100\cos(\omega t)$
$100\angle 120^\circ$	$100\cos(\omega t + 120^\circ)$
$100\angle -120^\circ$	$100\cos(\omega t - 120^\circ)$



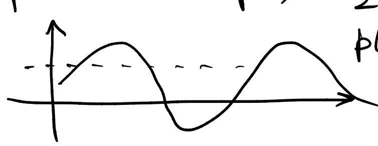
Motor



Total $p(t)$ = constant for three-phase sources

For single phase source: $p(t) = \frac{1}{2} V_m I_m \cos(\theta - \phi) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta - \phi)$

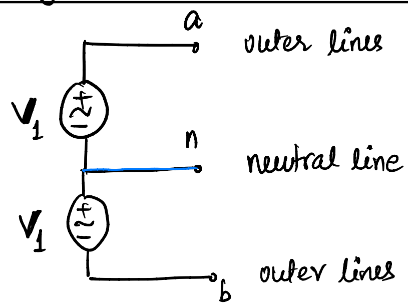
$p(t) \neq$ constant



Advantages of three-phase sources are
- constant power and constant torque to large motors.
- more economical since motors are more efficients.

Single-Phase Three-Wire Source (Two-phase source)

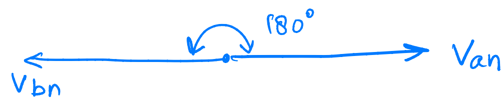
$\theta^\circ, \theta^\circ - 180^\circ$



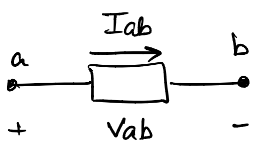
$V_1 = V_m \angle \theta^\circ$

$V_{an} = V_1$
$V_{nb} = V_1$

$$\left\{ \begin{aligned} V_{an} &= V_1 = V_m \angle \theta^\circ \\ V_{bn} &= -V_1 = V_m \angle \theta^\circ - 180^\circ \end{aligned} \right\}$$



Double script Notations:



V_{an} = voltage at point a w.r.t. voltage at point n

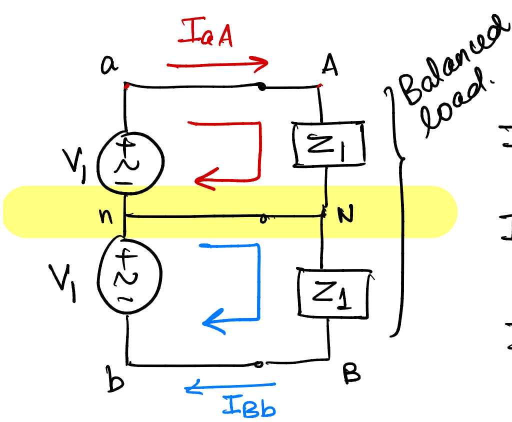
$V_{ab} = V_{an} - V_{bn} = V_{an} + V_{nb}$

$V_{bn} = -V_{nb}$

I_{ab} = current flowing from point a to point b.

Balanced Single-Phase Three-Wire Source:

$$\left. \begin{aligned} |V_{an}| &= |V_{bn}| \\ V_{an} + V_{bn} &= 0 \end{aligned} \right\} \text{Balanced source}$$



$V_{an} = V_1$, $V_{bn} = -V_1$

$$I_{aA} = \frac{V_1}{Z_1}$$

$$I_{Bb} = \frac{V_1}{Z_2}$$

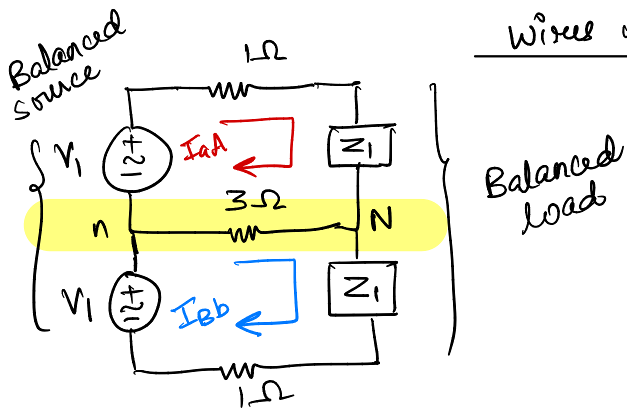
$$I_{nN} = I_{Bb} - I_{aA} = \frac{V_1}{Z_2} - \frac{V_1}{Z_1}$$

Assume $Z_1 = Z_2$ (Balanced load)

$$I_{nN} = \frac{V_1}{Z_1} - \frac{V_1}{Z_1} = 0$$

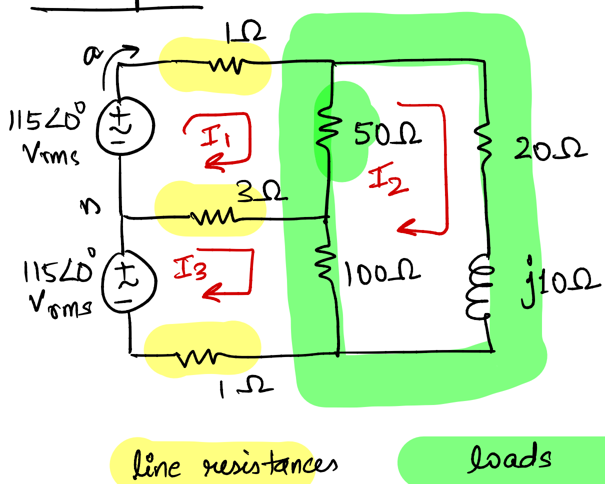
$\left.\begin{array}{l}\text{For balanced source} \\ \text{and balanced load}\end{array}\right\}$ current in neutral line $I_{nN} = 0$

Wires with non-zero resistance



$I_{nN} = 0 = I_{Bb} - I_{aA}$

Example



Find

① Average power delivered to each of three loads

② Avg power lost in outerlines and neutral line.

③ Avg power provided by source.

Using mesh analysis

$$-115\angle 0^\circ + 1(I_1) + 50(I_1 - I_2) + 3(I_1 - I_3) = 0$$

$$50(I_2 - I_1) + (20 + j10)I_2 + 100(I_2 - I_3) = 0$$

$$-115\angle 0^\circ + (I_3 - I_1)3 + (I_3 - I_2)100 + I_3(1) = 0$$

$\longrightarrow$ $I_1 = 11.24\angle -19.83^\circ$ Arms

$I_2 = 9.389\angle -24.47^\circ$ Arms

$I_3 = 10.37\angle -21.80^\circ$ Arms

$I_1 - I_2 = \cancel{11.24} - 9.389$

Avg power delivered to loads:

$$P_{avg,50\Omega} = |I_2 - I_1|^2 (50)$$
$$= 206 \text{ W}$$

$$P_{avg,100\Omega} = 117 \text{ W}$$

$$P_{avg,\underbrace{20+j10\Omega}} = I_2^2(20) = 1763 \text{ W}$$

Total power delivered to loads = 2086 W

$P_{avg} = V_{eff} I_{eff} \cos(\theta - \phi)$

$P_{avg} = I_{eff}^2 R$

$P_{avg} = \frac{1}{2} I_m^2 R$

Avg power lost in lines

$$P_{avg.\,aA} = |I_1|^2 1 = 126 \text{ W}$$

$$P_{avg.\,Bb} = (I_3)^2 1 = 108 \text{ W}$$

$$P_{avg.\,nN} = |I_3 - I_1|^2 3 = 3 \text{ W}$$

Total power lost in lines = 237W

Power supplied by sources

$V_{an} = 115\angle 0^\circ$ Vrms

$I_{na} = I_1$

$$P_{avg,an} = V_{an} I_{na} \cos(\theta - \phi)$$
$$= 115 \times 11.24 \cos(0 - (-19.83^\circ))$$
$$= 1216 \text{ W}$$

$$P_{avg,nb} = 115 \times 10.37 \cos(21.80^\circ)$$
$$= 1107 \text{ W}$$

Total power supplied = 2323 W

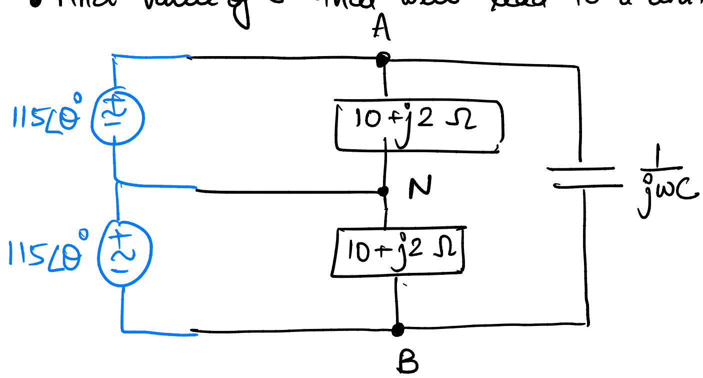
$$\boxed{P_{supplied} = P_{delivered} + P_{loss}}$$

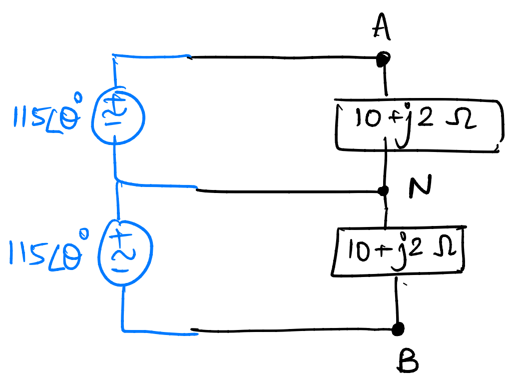
① Example: Consider $V_{12} = 9\angle 30^\circ$, $V_{32} = 3\angle 130^\circ$

Find $V_{21} = 9\angle -150^\circ$, $V_{13} = V_{12} + V_{23}$

② Consider a single-phase three-wire balanced source connected to a balanced load. $f = 50\,Hz$, $V_{an} = 115\angle 0^\circ\,V$

- Find power factor of the load if capacitor is omitted.
- Find value of C that will lead to a unity power factor of total load.

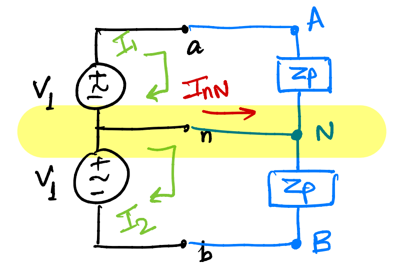




$PF = \cos(\theta - \phi)$

Review

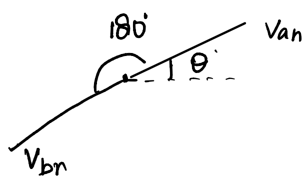
Single-phase Three-wire sources (Two-phase)



$$V_1 = V_p \angle \theta^\circ$$

$$V_{an} = V_a - V_n = V_1 = V_p \angle \theta^\circ$$

$$V_{bn} = -V_1 = -V_p \angle \theta^\circ = V_p \angle \theta^\circ - 180^\circ$$



Balanced source
$$\begin{cases} |V_{an}| = |V_{bn}| = V_p \\ V_{an} + V_{bn} = 0 \end{cases}$$

$I_{nN} = 0 \leftarrow$ when source and load are balance.

Time-domain	Freq. domain
$v_{an}(t) = V_p \cos(\omega t + \theta^\circ)$	$V_{an} = V_1 = V_p \angle \theta^\circ$
$v_{bn}(t) = V_p \cos(\omega t + \theta - 180^\circ)$	$V_{bn} = -V_1 = V_p \angle \theta^\circ - 180^\circ$

$$p(t) = v(t)\, i(t)$$

$$P_{avg} = V \cdot I \quad \times$$

$$P_{avg} = \frac{1}{2} \mathrm{Re}\{ V \cdot I^* \} \quad \checkmark$$

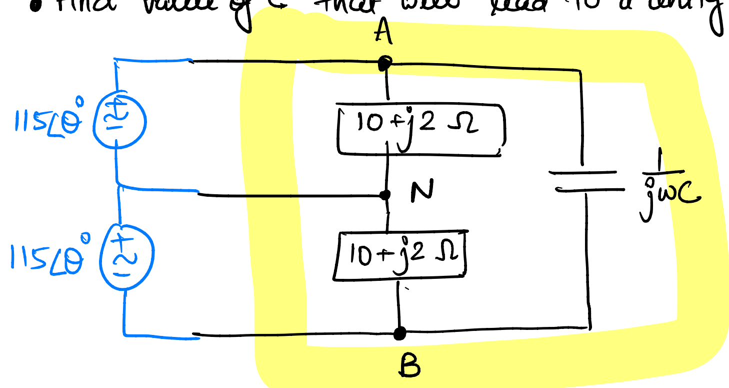
① **Example**: Consider $V_{12} = 9\angle 30^\circ$, $V_{32} = 3\angle 130^\circ$

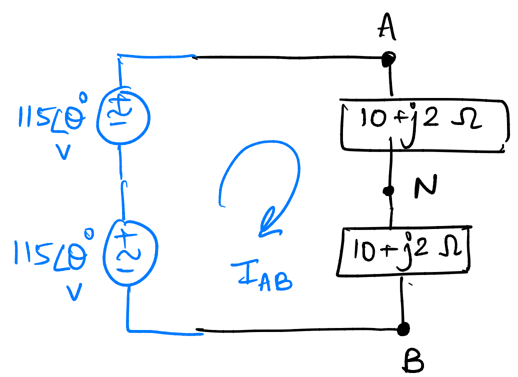
Find $V_{21} = 9\angle -150^\circ$, $V_{13} = V_{12} + V_{23}$

② Consider a single-phase three-wire balanced source connected to a balanced load. $f = 50\,Hz$, $V_{an} = 115\angle\theta^\circ\,V$

→• Find power factor of the load if capacitor is omitted.

• Find value of C that will lead to a unity power factor of total load.





$$PF = \cos(\theta - \phi)$$

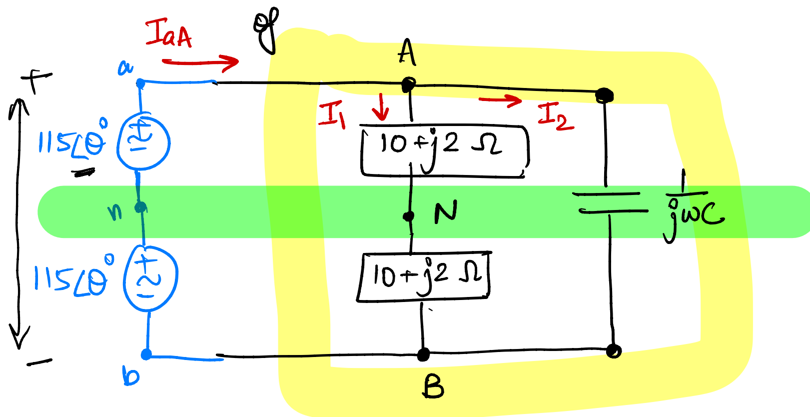
$$I_{AB} = \frac{115\angle\theta^\circ + 115\angle\theta^\circ}{2(10+j2)} = \frac{230\angle\theta^\circ}{2(10+j2)}$$

$$= 11.27\angle\theta^\circ - 11.3^\circ\ A$$

$$\phi = \theta^\circ - 11.3^\circ$$

$$PF = \cos(\underbrace{\theta - \theta + 11.3^\circ}) = 0.98 \text{ lagging}$$

$$PF = 1 \Rightarrow \cos(\theta - \phi) = 1 \Rightarrow \theta - \phi = 0 \Rightarrow \theta = \phi$$



$\theta = \phi$

$$I_{aA} = I_1 + I_2$$
$$= \frac{115\angle\theta^\circ}{10+j2} + \frac{230\angle\theta^\circ}{\left(\frac{1}{j\omega C}\right)}$$
$$= \underbrace{\quad}$$

S = Complex power of total load $= \frac{1}{2} V I^*$

$$PF = \frac{Re\{S\}}{|S|} \Rightarrow PF = 1 \Rightarrow |S| = Re\{S\} \Rightarrow \boxed{Im\{S\} = 0}$$

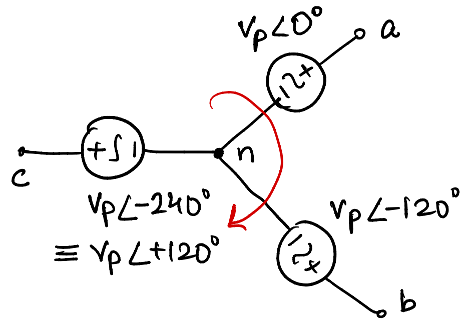
$$\boxed{Z_p = 10 + j2}$$
$$\boxed{Z_c = \frac{1}{j\omega C}}$$

$$S = \frac{1}{2} V_{an} I_{AN}^* + \frac{1}{2} V_{nb} I_{NB}^* + \frac{1}{2} V_{ab} I_{AB}^*$$
$$= \frac{1}{2} V_{an} \left(\frac{V_{an}}{Z_p}\right)^* + \frac{1}{2} V_{nb} \left(\frac{V_{nb}}{Z_p}\right)^* + \frac{1}{2} V_{ab} \left(\frac{V_{ab}}{Z_c}\right)^*$$
$$= \underline{\frac{1}{2} \frac{|V_{an}|^2}{|Z_p|^2} Z_p} + \underline{\frac{1}{2} \frac{|V_{nb}|^2}{|Z_p|^2} Z_p} + \frac{1}{2} \frac{|V_{ab}|^2}{|Z_c|^2} Z_c$$
$$= \frac{|V_{an}|^2}{|Z_p|^2} Z_p + \frac{1}{2} \frac{|V_{ab}|^2}{|Z_c|^2} Z_c$$

$$S = \frac{115^2}{104}(10 + 2j) + \frac{1}{2} \frac{(230)^2}{(1/\omega C)^2} \left(\frac{-j}{\omega C}\right)$$

$$Im(S) = 0 \Rightarrow C = \frac{1}{\omega \times 104} = \frac{1}{100\pi \times 104} = 30.6\ \mu F$$

Three-phase Sources



$V_{an} = V_p \angle 0^\circ$

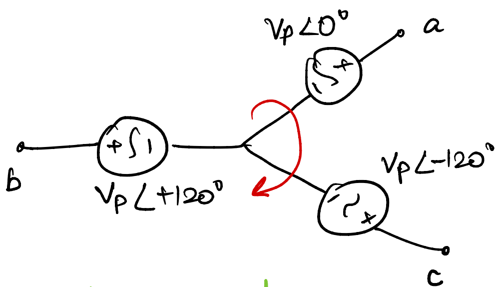
$V_{bn} = V_p \angle -120^\circ$

$V_{cn} = V_p \angle 120^\circ$

$$\begin{cases} |V_{an}| = |V_{bn}| = |V_{cn}| = V_p \\ V_{an} + V_{bn} + V_{cn} = 0 \end{cases}$$

↳ Balanced source.

→ abc or positive sequence

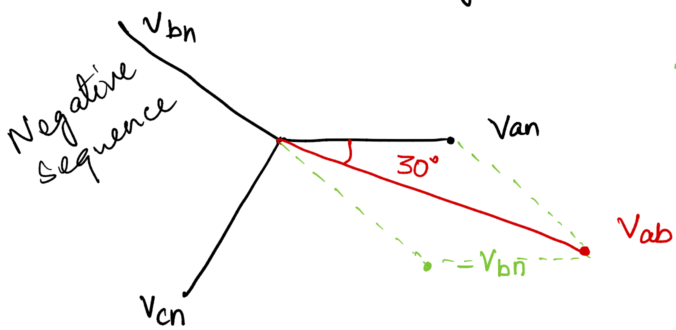


Balanced source

cba or negative sequence

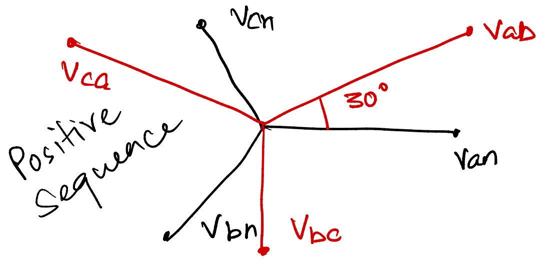
Phase voltages: V_{an}, V_{bn}, V_{cn}

Line-to-line or line voltages: V_{ab}, V_{bc}, V_{ca}



$V_{ab} = V_{an} - V_{bn} = V_{an} + (-V_{bn})$

$\boxed{V_L = \sqrt{3}\, V_p}$



$V_{ab} = V_{an} + V_{nb}$

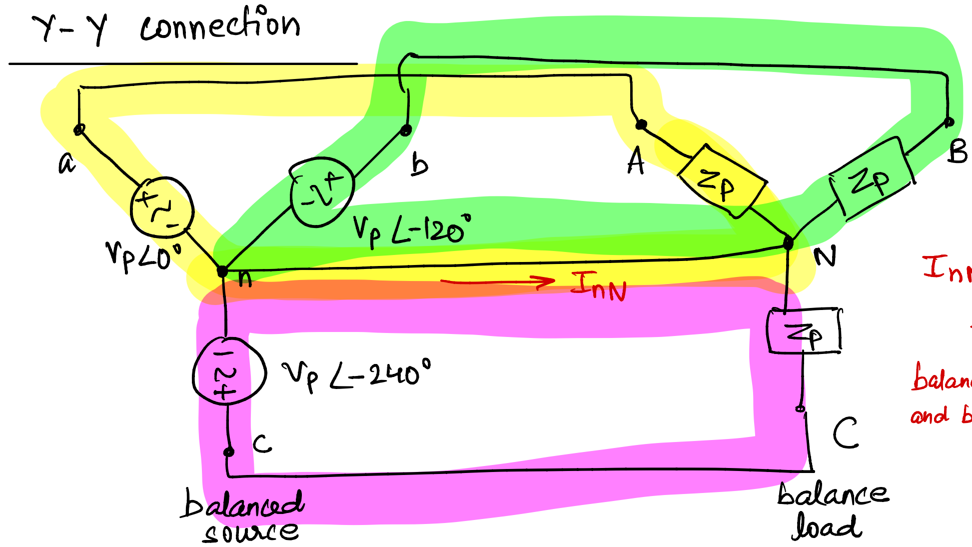
$= V_{an} - V_{bn}$

$= \sqrt{3}\, V_p \angle 30^\circ$

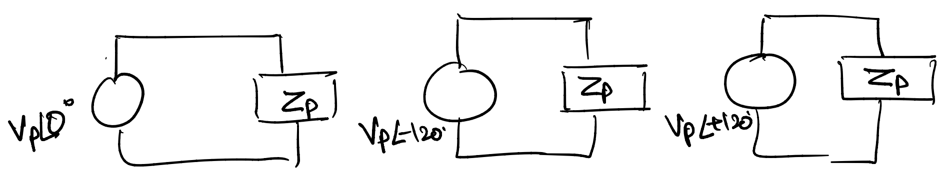
$V_{bc} = \sqrt{3}\, V_p \angle -90^\circ$

$V_{ca} = \underbrace{\sqrt{3}\, V_p}_{V_L} \angle -210^\circ$

Y-Y connection



$I_{nN} = 0$ for balanced source and balanced load



"per-phase" analyses

$I_{aA} = \frac{V_{an}}{Z_p}$

$I_{bB} = \frac{V_{bn}}{Z_p} = \frac{V_{an}\angle -120°}{Z_p} = I_{aA}\angle -120°$

$I_{cC} = \frac{V_{cn}}{Z_p} = \frac{V_{an}\angle +120°}{Z_p} = I_{aA}\angle +120°$

$I_{nN} = I_{aA} + I_{bB} + I_{cC} = 0$

$V_{an} = V_p\angle 0°$

$V_{bn} = V_p\angle -120°$

$= V_{an}\angle -120°$

$= V_{an}\,(1\angle -120°)$

Example: Consider three-phase balanced Y-Y connected system.

$V_{an} = 200\angle 0°\ V\,rms$

$Z_p = 100\angle 60°\ \Omega$

Find phase and line voltages.

Find phase and line currents.

Find total average power delivered to load.

Phase Voltages:

$V_{an} =$

$V_{bn} = 200 \angle -120^\circ$ Vrms

$V_{cn} = 200 \angle +120^\circ$ Vrms

Line voltages: $V_{ab} = \sqrt{3} V_p \angle 30^\circ = 200\sqrt{3} \angle 30^\circ$ Vrms

$V_{bc} = 200\sqrt{3} \angle -90^\circ$ Vrms

$V_{ca} = 200\sqrt{3} \angle -210^\circ$ Vrms

Line currents:

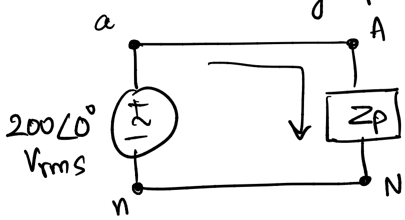
$I_{aA} = \dfrac{V_{an}}{Z_p} = \dfrac{200\angle 0^\circ}{100\angle 60^\circ} = 2 \angle -60^\circ$ Arms

$I_{bB} = 2 \angle -180^\circ$ Arms

$I_{cC} = 2 \angle -300^\circ$ Arms

Total Average power:

Average power in phase A



$P_{avg, A} = \frac{1}{2} \mathrm{Re}\{V I^*\} = \mathrm{Re}\{V_{eff} I_{eff}^*\}$

$= \mathrm{Re}\{V_{an} I_{AN}\}$

$= \mathrm{Re}\{200 \angle 0^\circ \times 2\angle -60^\circ\}$ W

$= \mathrm{Re}\{400 \angle -60^\circ\}$

$= 400 \cos(-60^\circ)$

$= 200$ W

$P_{avg, B} = 200$ W

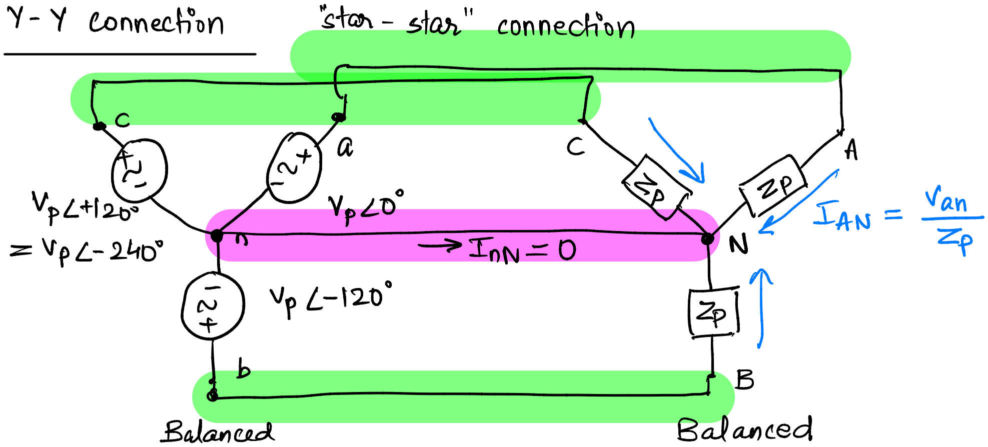
$P_{avg, C} = 200$ W

P_{avg} total $= 600$ W.

Three phase systems

Y - Y connection

"star - star" connection



Phase Voltages : $V_{AN} = V_{an}$, $V_{BN} = V_{bn}$, $V_{CN} = V_{cn}$

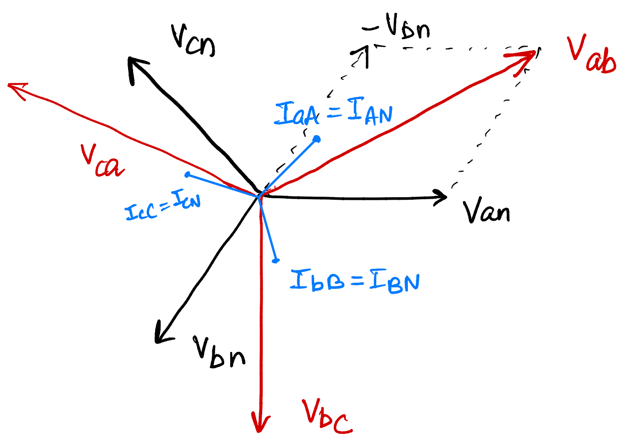
Line Voltages : V_{ab} , V_{bc} , V_{ca}

Line Currents : I_{aA} , I_{bB} , I_{cC}

Phase currents : $I_{AN} = I_{aA}$, $I_{BN} = I_{bB}$, $I_{CN} = I_{cC}$

$$\underbrace{\begin{aligned} V_{an} &= V_p\angle 0^\circ \\ V_{bn} &= V_p\angle{-120^\circ} \\ V_{cn} &= V_p\angle{-240^\circ} \end{aligned}}_{\text{phase voltages}}$$

$$\underbrace{\begin{aligned} V_{ab} &= V_{an} - V_{bn} = \sqrt{3}\,V_p\angle 30^\circ \\ V_{bc} &= V_{bn} - V_{cn} = \sqrt{3}\,V_p\angle{-90^\circ} \\ V_{ca} &= V_{cn} - V_{an} = \sqrt{3}\,V_p\angle{-210} \end{aligned}}_{\text{line voltages}}$$



V_p = magnitude of phase voltages $= |V_{AN}| = |V_{BN}| = |V_{CN}| = V_p$

V_L = magnitude of line voltages $= |V_{ab}| = |V_{bc}| = |V_{ca}| = \sqrt{3}V_p$

$$\boxed{V_L = \sqrt{3}V_p}$$

Line currents = phase currents.

$$I_{aA} = I_{AN} = \frac{V_{an}}{Z_p} = \frac{V_{AN}}{Z_p}$$

$$I_{bB} = I_{BN} = \frac{V_{BN}}{Z_p}$$

$$I_{cC} = I_{CN} = \frac{V_{CN}}{Z_p}$$

$$\boxed{\begin{aligned} Z_p &= \frac{V_{AN}}{I_{AN}} \\ &= \frac{V_p\angle\theta^\circ}{I_p\angle\phi^\circ} \\ &= \frac{V_p}{I_p}\angle\theta-\phi^\circ \end{aligned}}$$

I_p = magnitude of phase currents $= |I_{AN}| = |I_{BN}| = |I_{CN}|$

I_L = magnitude of line currents $= |I_{aA}| = |I_{bB}| = |I_{cC}|$

$$\boxed{I_p = I_L}$$

P_{avg} for any phase $= \frac{1}{2}\mathrm{Re}\{V I^*\}$

$V_{AN} = V_p\angle 0^\circ$

$I_{AN} = I_p\angle\phi^\circ$

$$\begin{aligned} P_{avg,A} &= \frac{1}{2}\mathrm{Re}\{V_{AN} I_{AN}^*\} \\ &= \frac{1}{2}\mathrm{Re}\{V_p\angle\theta^\circ (I_p\angle\phi^\circ)^*\} \\ &= \frac{1}{2} V_p I_p \cos(\theta-\phi) \\ &= \frac{1}{2}\frac{1}{\sqrt{3}} V_L I_L \cos(\theta-\phi) \end{aligned}$$

P_{avg} per phase $= P_{avg\,A\,phase} = P_{avg\,B\,phase} = P_{avg\,C\,phase} = \frac{1}{2}V_pI_p\cos(\theta-\phi')$

$$\boxed{P_{avg} \text{ per phase} = \frac{V_L I_L}{\sqrt{3}}\cos(\theta-\phi)\ \text{, if } V_L, I_L \text{ are rms values.}}$$

$$P_{avg}\ \text{total} = \frac{3}{2}V_pI_p\cos(\theta-\phi)$$

$$= 3V_pI_p\cos(\theta-\phi)\ \text{, if } V_p\ \&\ I_p \text{ are rms values.}$$

$$\boxed{P_{avg\,total} = \sqrt{3}V_LI_L\cos(\theta-\phi)\ \text{, if } V_L\ \&\ I_L \text{ are rms values.}}$$

Total instantaneous power

$$p_{total}(t) = p_A(t) + p_B(t) + p_C(t)$$

$$p_A(t) = v_{AN}(t)\, i_{AN}(t)$$

$$v_{AN}(t) = V_p \cos(\omega t + \theta)$$

$$i_{AN}(t) = I_p \cos(\omega t + \phi)$$

$$p_A(t) = V_p I_p \cos(\omega t + \phi)\cos(\omega t + \theta)$$

$$= \frac{V_p I_p}{2}\left[\cos\underbrace{(\theta - \phi)}_{\beta} + \cos\underbrace{(2\omega t + \theta + \phi)}_{\alpha}\right]$$

$$= \frac{V_p I_p}{2}\left[\cos\alpha + \cos\beta\right]$$

$$p_B(t) = v_{BN}(t)\, i_{BN}(t) = \frac{V_p I_p}{2}\left[\cos\beta + \cos(\alpha - 240^\circ)\right]$$

$$p_C(t) = \frac{V_p I_p}{2}\left[\cos\beta + \cos(\alpha + 240^\circ)\right]$$

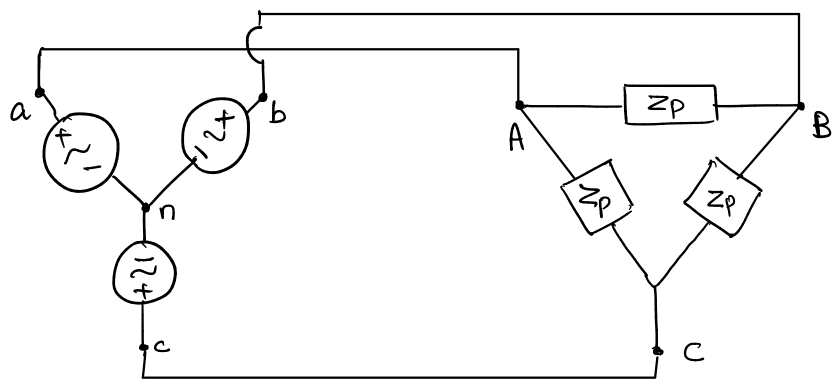
$$p_{total}(t) = \frac{V_p I_p}{2}\left[3\cos\beta + \cos\alpha + \underbrace{\cos(\alpha - 240^\circ) + \cos(\alpha + 240^\circ)}_{-\cos\alpha}\right]$$

$$p_{total}(t) = 3\frac{V_p I_p}{2}\cos\beta = \frac{3 V_p I_p}{2}\cos(\theta - \phi)$$

$$P_{avg} = \frac{1}{T}\int_0^T p(t)\,dt$$

Y– Δ Connections

Y- connected source
Δ- connected load (no neutral node)



Phase voltages: $V_{AB} = V_{ab}$, $V_{BC} = V_{bc}$, $V_{CA} = V_{ca}$

Line voltages: V_{ab} , V_{bc}, V_{ca} } ← same as phase voltages

$$\left.\begin{aligned} V_{AB} &= V_{ab} = V_{an} - V_{bn} = \sqrt{3}\, V_{an} \angle 30^\circ \\ V_{BC} &= V_{bc} = \sqrt{3}\, V_{an} \angle -90^\circ \\ V_{CA} &= V_{ca} = \sqrt{3}\, V_{an} \angle -210^\circ \end{aligned}\right\} \text{Line voltages} = \text{phase voltages}$$

V_P = magnitude of phase voltage $= |V_{AB}| = |V_{BC}| = |V_{CA}| = \sqrt{3}|V_{an}|$

V_L = magnitude of line voltage $= |V_{ab}| = |V_{bc}| = |V_{ca}| = \sqrt{3}|V_{an}|$

$$\boxed{V_P = V_L}$$

Line currents: $I_{aA} = I_{AB} - I_{CA} = \sqrt{3}\, I_{AB} \angle -30^\circ$

$I_{bB} = \sqrt{3}\, I_{BC} \angle -30^\circ$

$I_{cC} = \sqrt{3}\, I_{CA} \angle -30^\circ$

Phase currents : $I_{AB} = V_{AB} / Z_P$

$I_{BC} = V_{BC} / Z_P$

$I_{CA} = V_{CA} / Z_P$

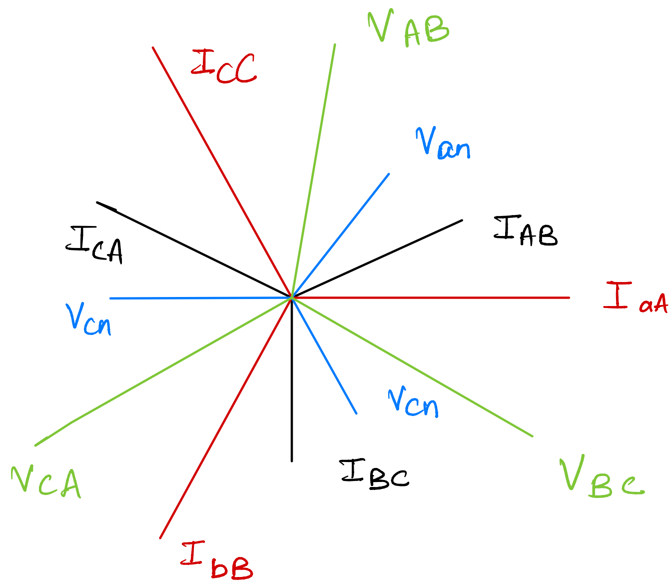
$$\boxed{Z_P = \frac{V_{AB}}{I_{AB}} = \frac{V_P \angle \theta^\circ}{I_P \angle \phi^\circ} = \frac{V_P}{I_P} \angle \theta - \phi^\circ}$$

I_P = magnitude of phase currents $= |I_{AB}| = |I_{BC}| = |I_{CA}|$

I_L = magnitude of line currents $= |I_{aA}| = |I_{bB}| = |I_{cC}|$

$$\Rightarrow I_L = \sqrt{3}\, \underbrace{|I_{AB}|}_{I_P}$$

$$\Rightarrow \boxed{I_L = \sqrt{3}\, I_P}$$



Similar to Y-Y connection:

$$P_{avg \text{ per phase}} = \frac{1}{2} Re \left\{ V_{AB} I_{AB}^{*} \right\} = \frac{1}{2} V_p I_p \cos(\theta - \phi)$$

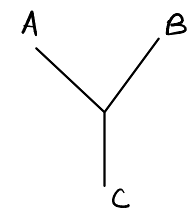
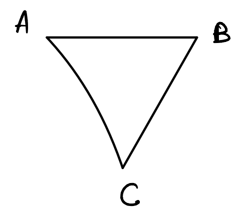
$$\boxed{P_{avg \text{ per phase}} = \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi) \text{ , if } V_L, I_L \text{ are rms values.}}$$

$$P_{avg \text{ total}} = \frac{3}{2} V_p I_p \cos(\theta - \phi)$$

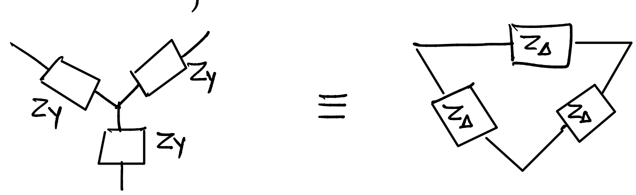
$$= 3 V_p I_p \cos(\theta - \phi) \text{ , if } V_p \text{ \& } I_p \text{ are rms values.}$$

$$\boxed{P_{avg \text{ total}} = \sqrt{3} V_L I_L \cos(\theta - \phi) \text{ , if } V_L \text{ \& } I_L \text{ are rms values.}}$$

Y-Y connection versus Y-Δ connection

		
Phase voltages Line voltages	V_{AN} , V_{BN} , V_{CN} V_{ab} , V_{bc} , V_{ca}	V_{AB} , V_{BC} , V_{CA} V_{ab} , V_{bc} , V_{ca} same
Phase currents Line currents	I_{AN}, I_{BN}, I_{CN} I_{aA}, I_{bB}, I_{cC} same	I_{AB}, I_{BC}, I_{CA} I_{aA}, I_{bB} , I_{cC}
	$V_L = \sqrt{3}\, V_p$ $I_L = I_p$	$V_L = V_p$ $I_L = \sqrt{3}\, I_p$
Z_p	$Z_p = \frac{V_p}{I_p} \angle \theta - \phi$	$Z_p = \frac{V_p}{I_p} \angle \theta - \phi$
P_{avg} per phase	$\frac{1}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$	$\frac{1}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$
P_{avg} total	$\frac{3}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$	$\frac{3}{2} V_p I_p \cos(\theta - \phi)$ $= \frac{1}{2} \frac{V_L I_L}{\sqrt{3}} \cos(\theta - \phi)$

Star to delta load transformation :-



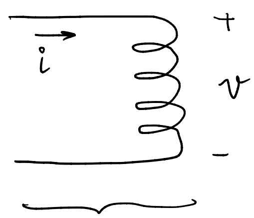
$$\boxed{Z_Y = \frac{Z_\Delta}{3}}$$

Module 5

Magnetically coupled circuits

(Chapter 13 of textbook)

" Pure linear inductors "
→ no resistance



$$v = L\frac{di}{dt}$$

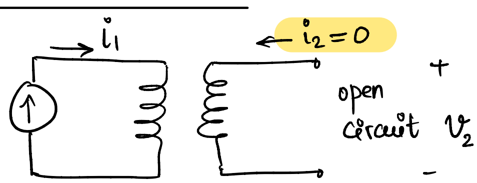
i leads to magnetic flux $\boxed{i \propto \phi}$

Magnetic flux leads to induced v $\boxed{v \propto d\phi/dt}$ $\Rightarrow$ $v \propto di/dt$

$$v = L\frac{di}{dt}$$

DC → const i → const. mag. flux → $v = 0$

Mutual Inductance :



$$v_2 \propto \frac{di_1}{dt}$$

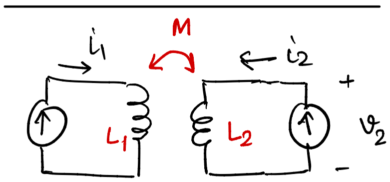
$$\Rightarrow v_2 = M\frac{di_1}{dt}$$

↳ mutual inductance

Henry

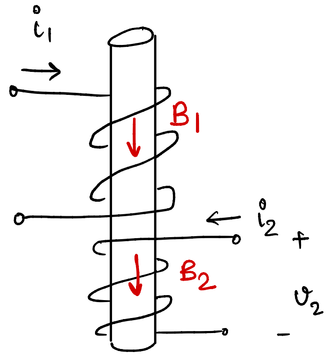
L : self inductance

Now consider : $i_2 \neq 0$



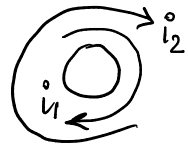
$v_2 \propto \frac{di_1}{dt}$, $v_2 \propto \frac{di_2}{dt}$

$$v_2 = \underbrace{L_2\frac{di_2}{dt}}_{\text{Self inductance}} + \underbrace{M\frac{di_1}{dt}}_{\text{Mutual inductance}}$$

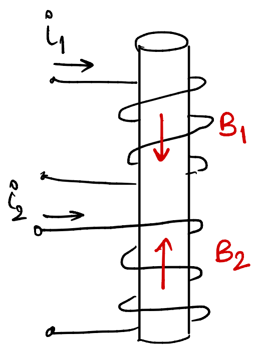


$$\boxed{v_2 = \underbrace{L_2\frac{di_2}{dt}}_{B_2} + \underbrace{M\frac{di_1}{dt}}_{B_1}}$$

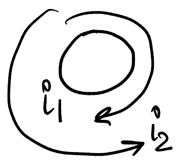
Top view



Additive magnetic flux



Top view

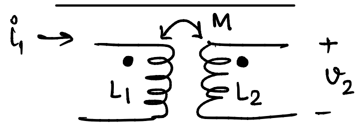


Substractive magnetic flux

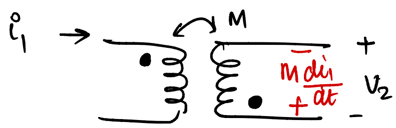
$$v_2 = \frac{L_2\, di_2}{dt} - M\frac{di_1}{dt}$$

Dot convention

Assume $i_2 = 0$

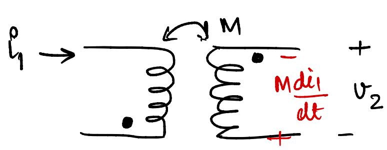


$$v_2 = M\frac{di_1}{dt}$$

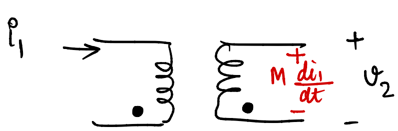


$$v_2 = -M\frac{di_1}{dt}$$

Current entering at dotted terminal
↓ leads to
+ve voltage reference at dotted terminal.



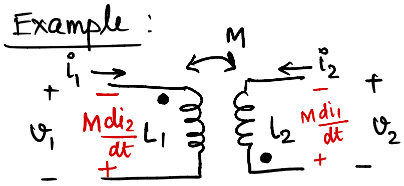
$$v_2 = -M\frac{di_1}{dt}$$



$$v_2 = M\frac{di_1}{dt}$$

Current entering at undotted terminal
↓ leads to
+ve voltage reference at undotted terminal.

Example :



$M = 2H$

(A) Find v_2 if $i_1 = -8e^{-t}$ A, $i_2 = 0$

$$v_2 = -M\frac{di_1}{dt} + \underbrace{L_2\frac{di_2}{dt}}_{0}$$

$$v_2 = -M\frac{di_1}{dt} = -2\frac{d}{dt}(-8e^{-t}) = -2\times(-8)\times(-1)\,e^{-t}$$

$$= -16\,e^{-t} \text{ V}$$

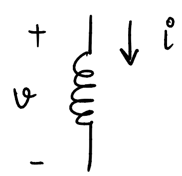
(B) Find v_1 if $i_1 = 0$ and $i_2 = 5\sin 45t$ A.

$$v_1 = \underbrace{L_1\frac{di_1}{dt}}_{0} - M\frac{di_2}{dt}$$

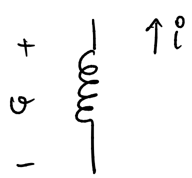
$$v_1 = -M\frac{d}{dt}(5\sin 45t) = -2\times 5\times 45\cos 45t$$

$$= -450\cos 45t \text{ V}$$

Sign Convention for self inductance :



$v = L\frac{di}{dt}$ (current entering from +ve terminal)



$v = -L\frac{di}{dt}$ (current leaving from +ve terminal)

Phasor Representation :

$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

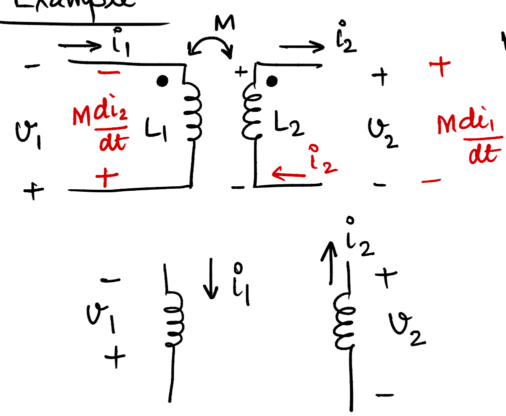
$v = L\frac{di}{dt}$ time domain

$V = j\omega L I_1$ freq. domain

for sinusoidally varying sources

$$V_1 = j\omega L_1 I_1 + j\omega M I_2$$
$$V_2 = j\omega L_2 I_2 + j\omega M I_1$$

Example

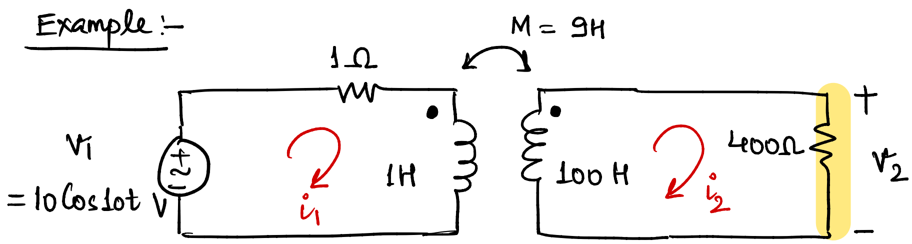


Write phasor equations for V_1 and V_2.

$$V_1 = -j\omega L_1 I_1 + j\omega M I_2$$

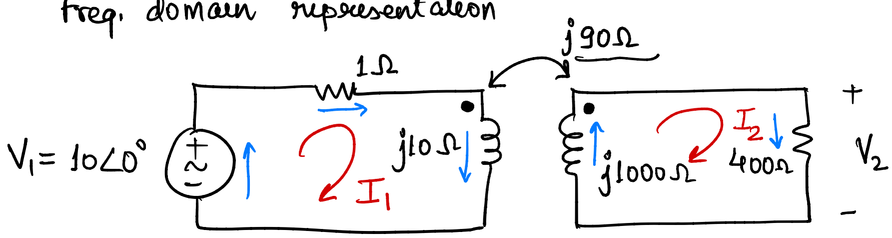
$$V_2 = -j\omega L_2 I_2 + j\omega M I_1$$

Example :-



Find V_2.

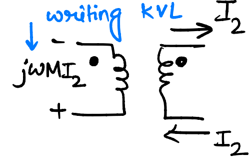
Freq. domain representation



Mesh 1 : $-10\angle 0° + I_1(1) + (j10)I_1 - (j90)I_2 = 0$

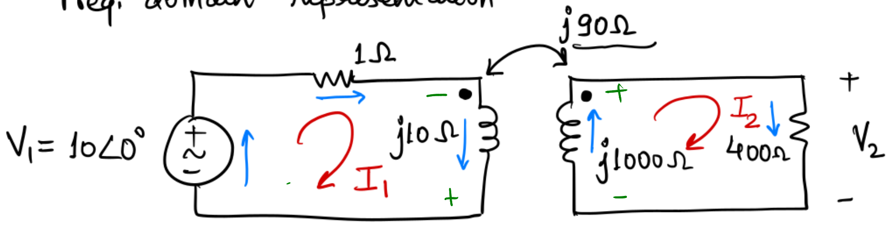
$$\Rightarrow 10\angle 0° = (1 + j10)I_1 - j90 I_2$$

direction in which writing KVL



Last term

Freq. domain representation



in direction of current there is potential drop across inductor.

→ In this mam take potential drop instead of potential gain in KVL

$$-10\angle 0° + I_1(1) + j10 I_1 - j90 I_2 = 0$$

$$+j1000 I_2 - j90 I_1 + 400 I_1 = 0$$

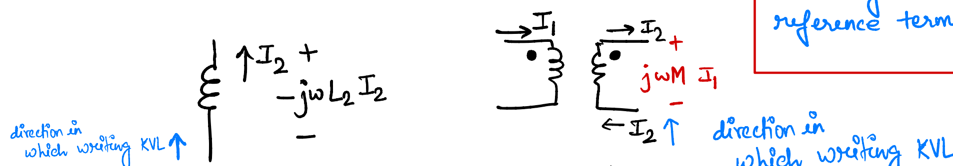
$$I_2 (400 + j1000) = j90 I_1$$

Ignore this normal convention will be followed

Mesh 2: $-(-j1000)I_2 \; -(j90)I_1 \; + (400)I_2 \; = 0$

$$\Rightarrow \; -j90\, I_1 + (400+j1000)\, I_2 = 0$$

Always remember to put a minus sign if while writing KVL terms you encounter a −ve reference terminal first.



First term

Second term

$$(1+j10)\, I_1 - j90\, I_2 = 10$$

$$(-j90)\, I_1 + (400+j1000)\, I_2 = 0$$

Cramer's rule: $\quad I_1 = \dfrac{D_1}{D_0} \quad , \quad I_2 = \dfrac{D_2}{D_0}$

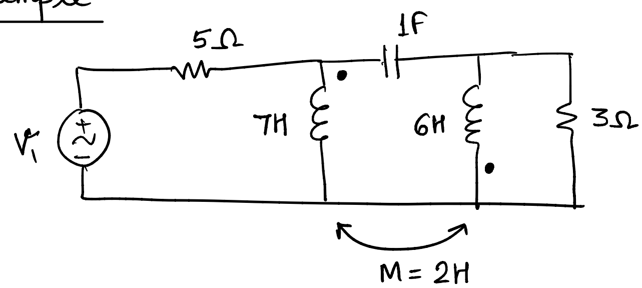
$$D_0 = \begin{vmatrix} 1+j10 & -j90 \\ -j90 & 400+j1000 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} 10 & -j90 \\ 0 & 400+j1000 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 1+j10 & 10 \\ -j90 & 0 \end{vmatrix}$$

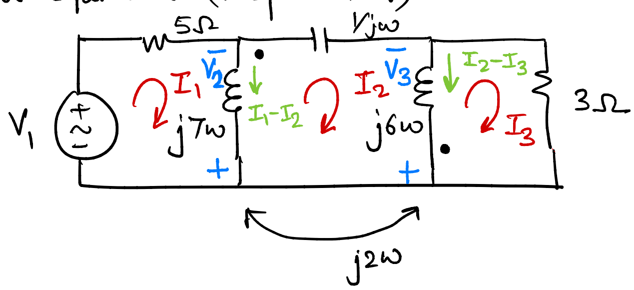
$$V_2 = 400\, I_2$$

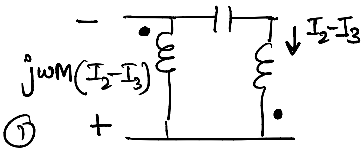
Example



Write phasor mesh equations.

Phasor equivalent (freq. domain)





Mesh 1: $-V_1 + 5I_1 - V_2 = 0$ ——— ①

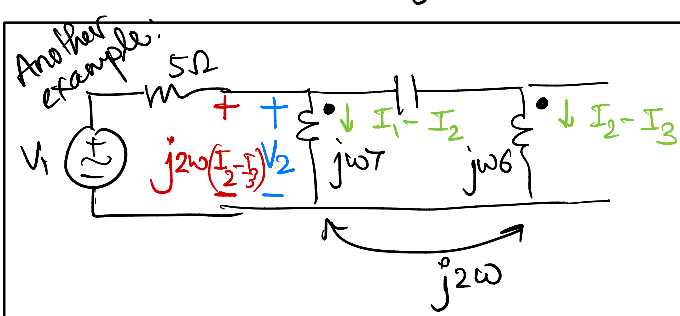
$$V_2 = -j7\omega(I_1 - I_2) + j2\omega(I_2 - I_3)$$

Mesh 3: $+V_3 + 3I_3 = 0$ ——— ②

$$V_3 = -j6\omega(I_2 - I_3) + j2\omega(I_1 - I_2)$$

Mesh 2: $+V_2 + \frac{1}{j\omega}I_2 - V_3 = 0$ ——— ③

Another example:



$$-V_1 + 5I_1 + V_2 = 0$$

$$V_2 = +j\omega 7(I_1 - I_2) + j2\omega(I_2 - I_3)$$

Mesh equations:

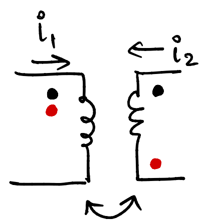
$$V_1 = (5 + j\omega 7)I_1 - j9\omega I_2 + j2\omega I_3$$

$$0 = 2j\omega I_1 - 8j\omega I_2 + (3 + j6\omega) I_3$$

$$0 = -9j\omega I_1 + \left(17j\omega + \frac{1}{j\omega}\right) I_2 - 8j\omega I_3$$

Energy stored :-

$$w(t) = \frac{1}{2} L (i(t))^2 \qquad \text{for only 1 inductor}$$



$$w(t) = \frac{1}{2} L_1 (i_1(t))^2 + \frac{1}{2} L_2 (i_2(t))^2 \pm M\, i_1(t)\, i_2(t)$$

$+M\, i_1 i_2$

$-M\, i_1 i_2$

+ve if both i_1 & i_2 are entering at dotted (or undotted)

−ve if one of i_1 & i_2 is at undotted & another is at dotted.

Coupling coefficient :-

$$M \leq \sqrt{L_1 L_2}$$

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

$$0 \leq k \leq 1$$

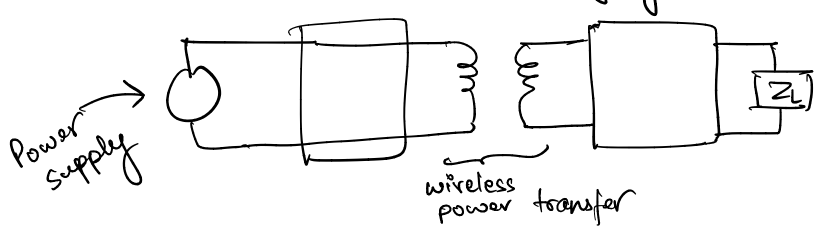
$k \to 0$ poor coupling or no coupling

$k \to 1$ tight coupling

k depends on : distance between coils, size of coils, ferrite core

wireless power transfer ← inductively coupled

electric vehicle charging



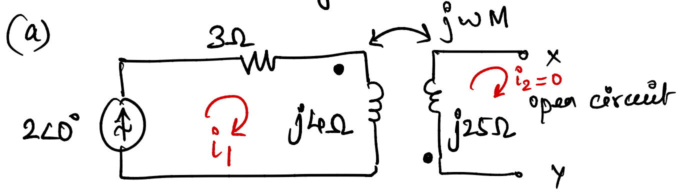
Phone battery.

Example : let $i_s = 2\cos 10t$ A. find total energy stored in network at $t=0$ if $k = 0.6$ and

(a) xy terminals are open circuited.

(b) xy terminals are short circuited.

(a)



$M = k\sqrt{L_1 L_2} = 0.6\sqrt{0.4 \times 2.5} = 0.6$ H

$j\omega M = j6\ \Omega$

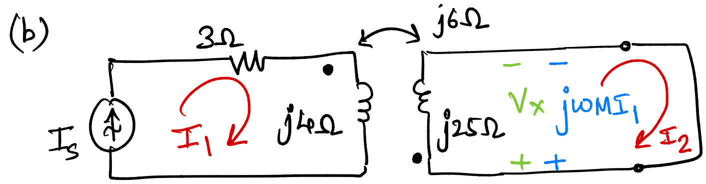
$$w(t) = \frac{1}{2} L_1 (i_1(t))^2 + \underbrace{\frac{1}{2} L_2 (i_2(t))^2}_{0} \pm \underbrace{M\, i_1(t)\, i_2(t)}_{0}$$

$$j\omega L_1 = j4 \Rightarrow L_1 = \frac{4}{\omega} = 0.4$$

$$w(t) = \frac{1}{2}(0.4)(2\cos 10t)^2 = 0.8 \cos^2 10t$$

$$\text{at } t=0 \Rightarrow w(t=0) = 0.8 \text{ J}$$

(b)



$V_x = 0$

$$w(t) = \frac{1}{2} L_1 (i_1(t))^2 + \frac{1}{2} L_2 (i_2(t))^2 + M\, i_1(t)\, i_2(t)$$

+ve sign because both I_1 and I_2 are entering from dotted terminals.

$$I_1 = I_s = 2\angle 0°$$

$$V_x = +(j25) I_2 + j\omega M I_1 = 0$$

Vx comes from two terms:
① self inductance of L_2
② mutual inductance M

$$\Rightarrow I_2 = \frac{-j6\ I_1}{j25} = -0.24 I_1$$

$$\Rightarrow I_1 = -0.48\angle 0° \text{ A} = 0.48 \cos(10t + 180°) \text{ A}$$

$$w(t=0) = \frac{1}{2}(0.4)(2^2) + \frac{1}{2}(0.48)^2(2.5) + (0.6)(2)(-0.48)$$

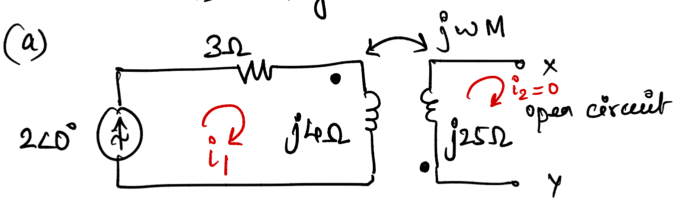
$$= 0.512 \text{ J}$$

Example: let $i_s = 2\cos 10t$ A. find total energy stored in network at $t=0$ if $k = 0.6$ and

(a) xy terminals are open circuited.

(b) xy terminals are short circuited.

(a)



$M = k\sqrt{L_1 L_2} = 0.6\sqrt{0.4\times 2.5} = 0.6$ H

$j\omega M = j6\ \Omega$

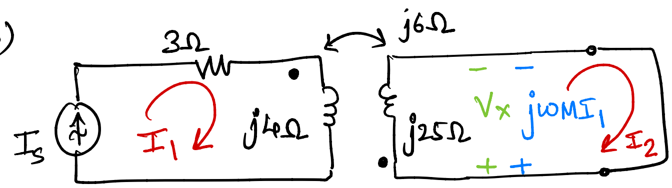
$$w(t) = \frac{1}{2}L_1(i_1(t))^2 + \underbrace{\frac{1}{2}L_2(i_2(t))^2}_{0} \pm \underbrace{M\,i_1(t)\,i_2(t)}_{0}$$

$$j\omega L_1 = j4 \quad\Rightarrow\quad L_1 = \frac{4}{\omega} = 0.4$$

$$w(t) = \frac{1}{2}(0.4)(2\cos 10t)^2 = 0.8\cos^2 10t$$

at $t=0 \quad\Rightarrow\quad w(t=0) = 0.8$ J

(b)



$V_x = 0$

$$w(t) = \frac{1}{2}L_1(i_1(t))^2 + \frac{1}{2}L_2(i_2(t))^2 + M\,i_1(t)\,i_2(t)$$

+ve sign because both I_1 and I_2 are entering from dotted terminals.

$$I_1 = I_s = 2\angle 0^\circ$$

Vx comes from two terms:
① self inductance of L_2
② mutual inductance M

$$V_x = +(j25)I_2 + j\omega M I_1 = 0$$

$$\Rightarrow\quad I_2 = \frac{-j6\ I_1}{j25} = -0.24 I_1$$

$$\Rightarrow\quad I_1 = -0.48\angle 0^\circ \text{ A} = 0.48\cos(10t + 180^\circ) \text{ A}$$

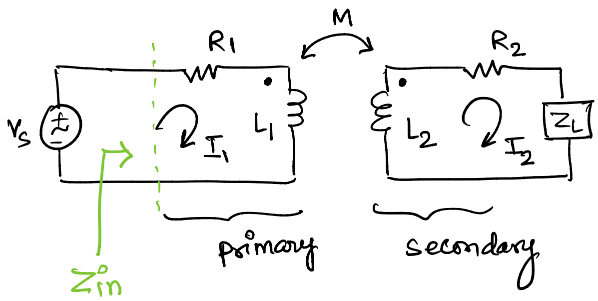
$$w(t=0) = \frac{1}{2}(0.4)(2^2) + \frac{1}{2}(0.48)^2(2.5) + (0.6)(2)(-0.48)$$

$$= 0.512 \text{ J}$$

Linear Transformer

$v \propto di/dt$ "linear"

↳ no magnetic core



Mesh equations:

$$-V_s + I_1R_1 + (j\omega L_1)I_1 - (j\omega M)I_2 = 0$$
$$(j\omega L_2)I_2 + R_2I_2 + Z_LI_2 - (j\omega M)I_1 = 0$$

$$\Rightarrow \quad V_s = \overbrace{(R_1 + j\omega L_1)}^{Z_{11}} I_1 - j\omega M I_2$$
$$0 = (-j\omega M)I_1 + \underbrace{(R_2 + j\omega L_2 + Z_L)}_{Z_{22}} I_2$$

$$\left.\begin{aligned} V_s &= Z_{11}I_1 - j\omega M I_2 \\ 0 &= -j\omega M I_1 + Z_{22}I_2 \end{aligned}\right\} \leftarrow \quad I_2/I_1 = \frac{j\omega M}{Z_{22}}$$

Input impedance : $Z_{in} = \dfrac{V_s}{I_1} = Z_{11} - j\omega M \dfrac{I_2}{I_1}$

$$Z_{in} = Z_{11} - \frac{j\omega M \times j\omega M}{Z_{22}}$$

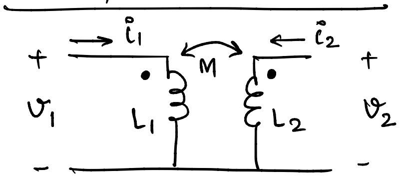
$$\boxed{Z_{in} = Z_{11} + \underbrace{\frac{\omega^2M^2}{Z_{22}}}_{\text{Reflected impedance}}}$$

$$Z_{11} = R_1 + j\omega L_1$$
$$Z_{22} = R_2 + j\omega L_2 + Z_L$$

$k = M/\sqrt{L_1L_2}$

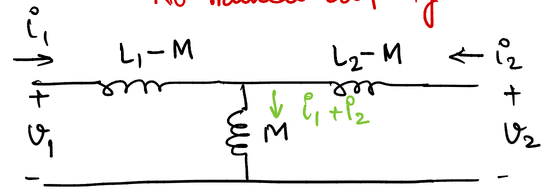
$k = 0 \qquad Z_{in} = Z_{11}$

T Equivalent Network



$\equiv$

" No mutual coupling "



$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

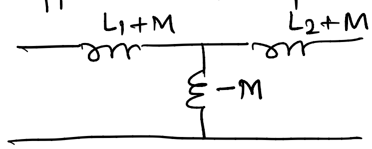
$$v_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$v_1 = (L_1 - M)\frac{di_1}{dt} + M\left(\frac{di_1}{dt} + \frac{di_2}{dt}\right)$$

$$\Rightarrow v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

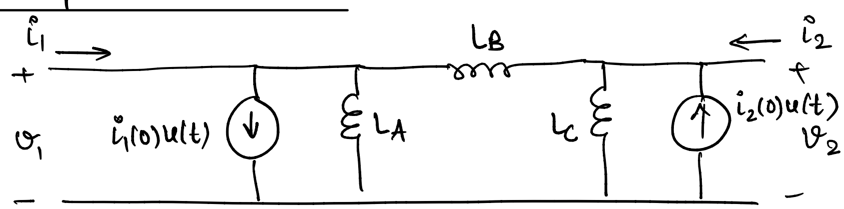
Similarly $v_2 =$

If dots are on opposite side ; replace M by − M.



Π Equivalent Network

$u(t)$



$$L_A = \frac{L_1 L_2 - M^2}{L_2 - M}$$

$$L_B = \frac{L_1 L_2 - M^2}{M}$$

$$L_C = \frac{L_1 L_2 - M^2}{L_1 - M}$$

Try at home.
Nodal analysis.

$$v = L\, di/dt \Rightarrow i = \frac{1}{L}\int_{-\infty}^{t} v\, dt$$

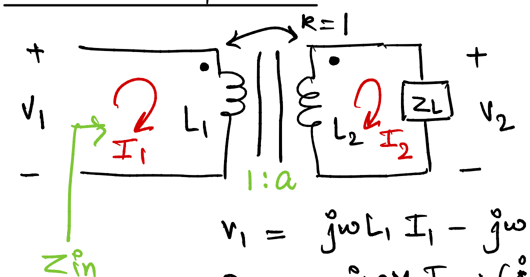
$$\Rightarrow i = \underbrace{i(0)\, u(t)} + \frac{1}{L}\int_0^t v(t)$$

please verify!

If dots are on opposite sides , Replace M by − M

Ideal Transformer

$k=1$ "iron core"
L_1 & $L_2 \gg Z_L$ } Ideal transformer



$L_2 \propto N_2^2$
$L_1 \propto N_1^2$

$$V_1 = j\omega L_1 I_1 - j\omega M I_2$$
$$0 = -j\omega M I_1 + (j\omega L_2 + Z_L) I_2$$

Turn Ratio

$$a = \frac{L_2}{L_1}$$

$$a = \frac{N_2^2}{N_1^2}$$

$$Z_{in} = \frac{V_1}{I_1} = j\omega L_1 - j\omega M\left(\frac{I_2}{I_1}\right)$$

$$= j\omega L_1 - \frac{j\omega M\,(j\omega M)}{j\omega L_2 + Z_L}$$

$$\boxed{Z_{in} = j\omega L_1 + \frac{\omega^2 M^2}{Z_L + j\omega L_2}}$$

Simplifying $\quad Z_{in} = \dfrac{Z_L}{\underbrace{(Z_L/j\omega L_1)}+a^2}$

$j\omega L_1 \gg Z_L$

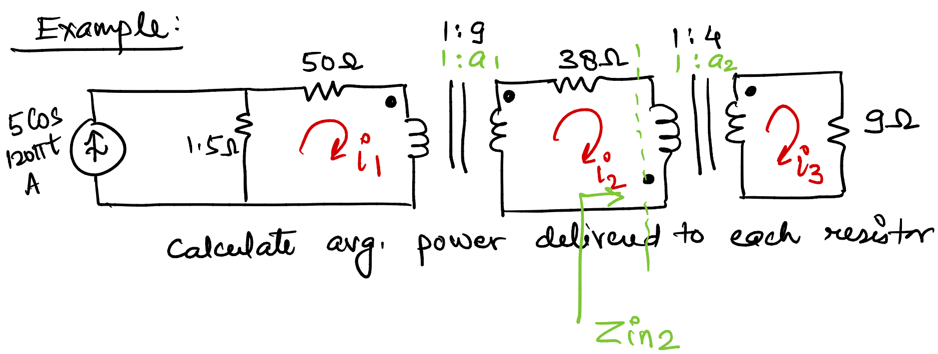
$$\boxed{Z_{in} = \frac{Z_L}{a^2}}$$

Current Ratio $\dfrac{I_2}{I_1} = \dfrac{1}{a}$

Voltage Ratio $\dfrac{V_2}{V_1} = a$

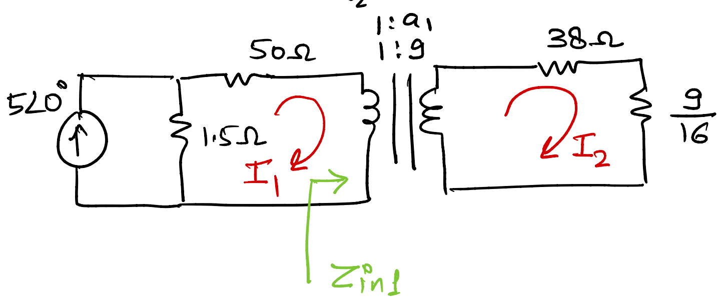
} Replace a by −a if dotes are on the opposite sides.

Example:

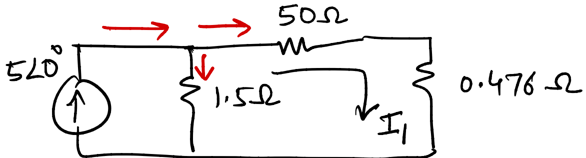


Calculate avg. power delivered to each resistor.

$$Z_{in2} = \frac{Z_L}{a_2^2} = \frac{9}{16}\ \Omega$$



$$Z_{in1} = \frac{(9/16)+38}{a_1^2} = \frac{(9/16)+38}{81} = 0.476\ \Omega$$



$$P_{avg} = \frac{1}{2}|I|^2 R$$

Current division: $I_1 = 5\angle 0^\circ \left(\frac{1.5}{1.5+50+0.476} \right)$

$$\Rightarrow I_1 = 0.144\angle 0^\circ \ \text{A}$$

$$\frac{I_2}{I_1} = \frac{1}{a_1} \quad \Rightarrow \quad I_2 = \frac{0.144\angle 0^\circ}{9} = 0.016\angle 0^\circ \ \text{A}$$

$$\frac{I_3}{I_2} = \frac{1}{-a_2} \quad \Rightarrow \quad I_3 = \frac{0.016\angle 0^\circ}{-4} = -4\angle 0^\circ \ \text{mA}$$

$$P_{avg\ 1.5\Omega} = \frac{1}{2}(5-0.144)^2\ 1.5 = 17.67\text{W}$$

$$P_{avg\ 50\Omega} = \frac{1}{2} \times 0.144^2 \times 50 = 0.518\ \text{W}$$

$$P_{avg\ 38\Omega} = \frac{1}{2} \times 0.016^2 \times 38 = 0.0048\ \text{W}$$

$$P_{avg\ 9\Omega} = \frac{1}{2} \times (-4)^2 \times 10^{-6} \times 9 = 72\ \mu\text{W}$$

Module 6

Circuit analysis in s-domain

(textbook chapter 14)

$$V_m \cos(\omega t + \theta) = \mathrm{Re}\{V_m e^{j(\omega t+\theta)}\}$$

$$\Downarrow \quad \text{Remove } \mathrm{Re}\{\ \},\ e^{j\omega t}$$

$$V_m e^{j\theta} = V_m \angle\theta$$

$$L\frac{di}{dt} \Rightarrow j\omega L I$$

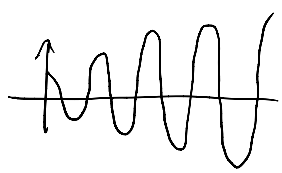
Consider $\boxed{v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)}$ —— ⊛

$\sigma = 0,\ \omega = 0$ $\quad v(t) = V_m$ DC source

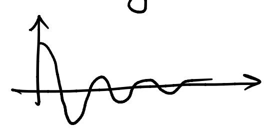
$\sigma = 0,\ \omega \neq 0$ $\quad v(t) = V_m \cos(\omega t + \theta)$ sinusoidally varying (AC)

$\sigma \neq 0,\ \omega = 0$ $\quad v(t) = V_m e^{\sigma t} \cos(\theta) = V_1 e^{\sigma t}$ "exponential signal"

$\sigma \neq 0,\ \omega \neq 0$



$\sigma > 0$



$\sigma < 0$

$$v(t) = V_m e^{\sigma t} \cos(\omega t + \theta)$$
$$= V_m e^{\sigma t} \left[\frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2} \right]$$
$$= \frac{V_m}{2} e^{j\theta} \left[e^{\overbrace{(\sigma + j\omega)}^{}t}\Big|_{s} + e^{(\sigma - j\omega)t}\Big|_{s^*} \right]$$

s : complex frequency

$s = \sigma + j\omega$

σ = Neper frequency (neper/sec)

ω = radian frequency $= 2\pi f$ (radians/sec)

Laplace Transform :-

$$f(t) \longrightarrow \boxed{\text{Laplace Transform}} \longrightarrow F(s)$$

$$f(t) \xrightarrow[\text{Transform}]{\text{Laplace}} F(s)$$

$$F(s) \xrightarrow[\text{Transform}]{\text{Inverse Laplace}} f(t)$$

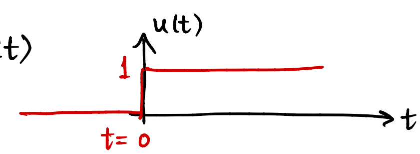
One-sided Laplace Transform

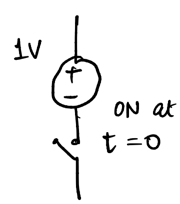
$$F(s) = \mathcal{L}(f(t)) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} e^{st} F(s)\, ds$$

$$\boxed{f(t) \Longleftrightarrow F(s)}$$

Unit step Function : $u(t)$





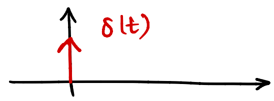
$$\mathcal{L}(u(t)) = \int_{0^-}^{\infty} e^{-st} u(t)\, dt = \int_{0}^{\infty} e^{-st}(1)\, dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_0^{\infty} = \frac{-1}{s}\left[\underbrace{e^{-s(\infty)}}_{0} - \underbrace{e^{-s(0)}}_{1}\right]$$

$$= \frac{1}{s}$$

$$\boxed{u(t) \Longleftrightarrow \frac{1}{s}}$$

Unit Impulse function : $\delta(t)$



$\delta(t) = 0$ when $t \neq 0$

$$\boxed{\delta(t) \Longleftrightarrow 1}$$

Exponential function

$$\boxed{e^{-\alpha t} u(t) \Longleftrightarrow \frac{1}{s+\alpha}}$$

Example:

$$\frac{1}{s+5} \Longleftrightarrow e^{-5t} u(t)$$

$$\boxed{t e^{-\alpha t} u(t) \Longleftrightarrow \frac{1}{(s+\alpha)^2}}$$

Laplace Transform Theorems :-

(1) Linearity Theorem :-

$$\mathcal{L}\{f_1(t) + f_2(t)\} = \mathcal{L}\{f_1(t)\} + \mathcal{L}\{f_2(t)\} \quad \text{—— (1)}$$

$$\mathcal{L}\{k f(t)\} = k \, \mathcal{L}\{f(t)\} \quad \text{—— (2)}$$

k constant.

Example: $V(s) = \frac{7}{s} - \frac{31}{s+17}$. find $v(t)$

$$v(t) = \mathcal{L}^{-1}\{V(s)\} = \mathcal{L}^{-1}\left\{\frac{7}{s} - \frac{31}{s+17}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{7}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-31}{s+17}\right\}$$

$$= \underbrace{7\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}} - 31\,\mathcal{L}^{-1}\left\{\frac{1}{s+17}\right\}$$

$$= 7u(t) - 31 e^{-17t} u(t) = \{7 - 31e^{-17t}\} u(t)$$

Zeros & Poles :-

$$V(s) = \frac{N(s)}{D(s)}$$

Zeros : values of s that lead to $N(s) = 0$

Poles : values of s that lead to $D(s) = 0$

$$V(s) = \frac{s-3}{s(s+6)}$$

zeros : $s = 3$

Poles : $s = 0, \ s = -6$

Example :- $V(s) = \dfrac{1}{(s+\alpha)(s+\beta)}$. find $v(t)$

"Method of Residues"

$$V(s) = \frac{A}{(s+\alpha)} + \frac{B}{(s+\beta)}$$

$$\Rightarrow \frac{1}{(s+\alpha)(s+\beta)} = \frac{A}{(s+\alpha)} + \frac{B}{(s+\beta)}$$

$$\Rightarrow 1 = A(s+\beta) + B(s+\alpha)$$

when $s = -\alpha$ $\Rightarrow 1 = A(-\alpha+\beta) + B(-\alpha+\alpha)$

$$\Rightarrow 1 = A(\beta-\alpha)$$

$$\Rightarrow \boxed{A = \frac{1}{\beta-\alpha}}$$

when $s = -\beta$ $\Rightarrow 1 = A(-\beta+\beta) + B(-\beta+\alpha)$

$$\Rightarrow \boxed{B = \frac{-1}{\beta-\alpha}}$$

$$V(s) = \frac{1/(\beta-\alpha)}{s+\alpha} + \frac{-1/(\beta-\alpha)}{s+\beta}$$

$$v(t) = \frac{1}{(\beta-\alpha)} e^{-\alpha t} u(t) + \frac{-1}{(\beta-\alpha)} e^{-\beta t} u(t)$$

$$= \frac{1}{(\beta-\alpha)} \left(e^{-\alpha t} - e^{-\beta t}\right) u(t)$$

Example : $I(s) = \dfrac{7s+5}{s^2+s}$. find $i(t)$

zeros : $s = -5/7$

Poles : $s = 0, -1$

$$I(s) = \frac{7s+5}{s^2+s} = \frac{7s+5}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$\Rightarrow \quad 7s+5 = A(s+1) + B(s)$$

when $s=0 \Rightarrow 5 = A(1) + 0 \Rightarrow A = 5$

when $s=-1 \Rightarrow -2 = 0 + B(-1) \Rightarrow B = 2$

$$I(s) = \frac{5}{s} + \frac{2}{s+1}$$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{5}{s} + \frac{2}{s+1}\right\} = 5u(t) + 2e^{-t}u(t)$$

Repeated Poles :

$$F(s) = \frac{2}{s^3 + 12s^2 + 36s}$$

$$= \frac{2}{s(s^2 + 12s + 36)}$$

$$= \frac{2}{s(s+6)^2}$$

$$F(s) = \frac{A}{s} + \frac{B}{(s+6)} + \frac{C}{(s+6)} \quad \Rightarrow \quad 2 = 0$$

For repeated poles :

$$V(s) = \frac{N(s)}{(s-p)^n}$$

$$V(s) = \frac{a_n}{(s-p)^n} + \frac{a_{n-1}}{(s-p)^{n-1}} + \frac{a_{n-2}}{(s-p)^{n-2}} + \cdots \frac{a_1}{(s-p)}$$

$$F(s) = \frac{2}{s(s+6)^2} = \frac{A}{s} + \frac{B}{(s+6)^2} + \frac{C}{s+6}$$

$$\Rightarrow \quad 2 = A(s+6)^2 + Bs + Cs(s+6)$$

when $s=0$ $\Rightarrow$ $2 = A(36) + B(0) + C(0)$

$\Rightarrow$ $A = 1/18$

when $s=-6$ $\Rightarrow$ $2 = A(0) + B(-6) + C(0)$

$\Rightarrow$ $B = -1/3$

Use Differentiation

$$\frac{2}{s} = \frac{A(s+6)^2}{s} + B + C(s+6)$$

$$\Rightarrow \quad 2 = A(s+6)^2 + Bs + Cs(s+6)$$

differentiate

$$\Rightarrow \quad 0 = 2A(s+6) + B + C(s+6) + C(s)$$

when $s=-6$ $\Rightarrow$ $0 = 0 + B + C(-6)$

$\Rightarrow$ $C = +B/6 = -1/18$

$$F(s) = \frac{A}{s} + \frac{B}{(s+6)^2} + \frac{C}{(s+6)} = \frac{1/18}{s} + \frac{-1/3}{(s+6)^2} + \frac{-1/18}{(s+6)}$$

$$f(t) = \frac{1}{18}u(t) - \frac{1}{3}te^{-6t}u(t) - \frac{1}{18}e^{-6t}u(t)$$

$$= \frac{1}{18}\left(1-(6t+1)e^{-6t}\right)u(t)$$

Theorems :-

(1) Time differentiation theorem :-

$$f(t) \Leftrightarrow F(s)$$

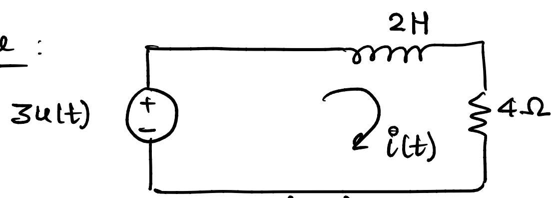
$$\boxed{\frac{d}{dt}f(t) \Longleftrightarrow sF(s) - f(0^-)}$$

② Time integration theorem :-

$$f(t) \Longleftrightarrow F(s)$$

$$\boxed{\int_{0^-}^{t} f(t)\, dt \Longleftrightarrow \frac{F(s)}{s}}$$

Example :



Find $i(t)$

$i(0^-) = 5A$

write KVL : $-3u(t) + L\frac{di}{dt} + 4\,i(t) = 0$

$$\Rightarrow -3u(t) + 2\frac{di(t)}{dt} + 4i(t) = 0$$

Take Laplace Transform : $i(t) \Longleftrightarrow I(s)$

$$\Rightarrow \frac{-3}{s} + 2\{sI(s) - \underbrace{i(0^-)}_{5}\} + 4\,I(s) = 0$$

$$\Rightarrow -\frac{3}{s} + 2sI(s) - 10 + 4I(s) = 0$$

$$\Rightarrow I(s) = \frac{\frac{3}{s} + 10}{2s+4} = \frac{3+10s}{s(2s+4)}$$

$$I(s) = \frac{3+10s}{s(2s+4)} = \frac{A}{s} + \frac{B}{(2s+4)}$$

$$\Rightarrow 3+10s = A(2s+4) + B(s)$$

when $s=0 \Rightarrow 3 = 4A + 0 \Rightarrow A = 0.75$

when $s=-2 \Rightarrow -17 = 0 - 2B \Rightarrow B = 17/2$

$$I(s) = \frac{0.75}{s} + \frac{17/2}{(2s+4)}$$

$$i(t) = 0.75\,u(t) + \frac{17}{4}e^{-2t}u(t)$$