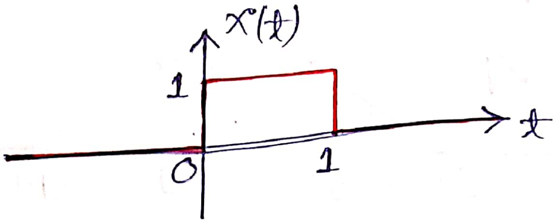
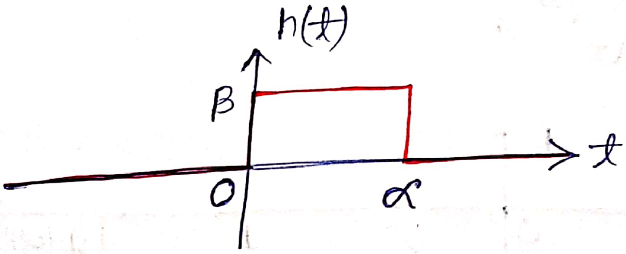
Sol:- (a) (1)

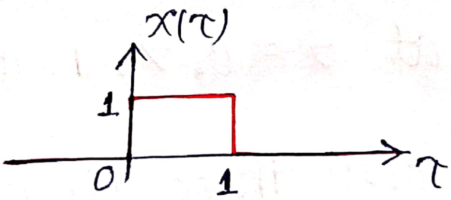
Given that $x(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$

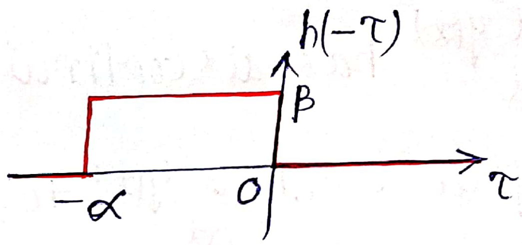


$h(t) = \beta \cdot x(t/\alpha), \quad 0 < \alpha \leq 1$



Now, $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$





Case(I): $t \leq 0$

$y(t) = 0$ → (0.25 Marks)

Case(II): $0 \leq t \leq \alpha$

$$y(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau = \int_0^t (1) \cdot \beta\, d\tau = (\beta \cdot t)$$

→ (1.5 Marks)

Case(III): $\alpha \leq t \leq 1$

$$y(t) = \int_{(t-\alpha)}^{t} x(\tau) \cdot h(t-\tau)\, d\tau = \int_{(t-\alpha)}^{t} (1)(\beta)\, d\tau = \beta[t - t + \alpha] = (\alpha\beta)$$

→ (1.5 Marks)

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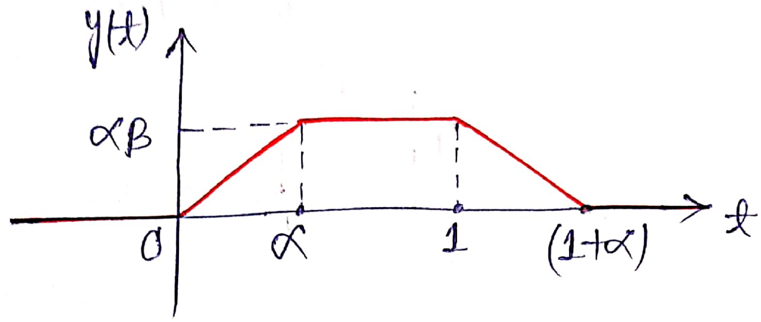
Case(IV): $1 \leq t \leq (1+\alpha)$

$$y(t) = \int_{(t-\alpha)}^{1} x(\tau)\cdot h(t-\tau)\,d\tau = \int_{(t-\alpha)}^{1} (1)(\beta)\,d\tau = \beta\left[1-t+\alpha\right]$$

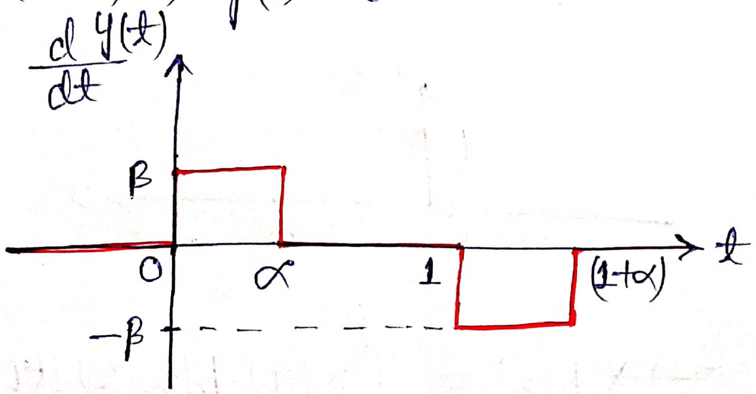
→ (1.5 Marks)



→ (1 Mark)

Case(V): $t \geq (1+\alpha)$, $y(t) = 0$ → (0.25 Mark)

Sol:- (b)(1)



$\frac{d}{dt}y(t)$ has discontinuities at $t = 0, \alpha, 1,$ and $(1+\alpha)$

If we want $\frac{d}{dt}y(t)$ to have only three discontinuities, then we need to ensure that $\alpha = 1$, where β can take any real value.

→ (3 Marks)

Sol:- (a)(2)

Given that - $w[n] = \frac{1}{2} w[n-1] + x[n]$ —— (1)

$$y[n] = \alpha y[n-1] + \beta w[n] \quad \text{—— (2)}$$

$$y[n] = -\frac{1}{8} y[n-2] + \frac{3}{4} y[n-1] + x[n] \quad \text{—— (3)}$$

$$x[n] \longrightarrow \boxed{S_1} \xrightarrow{w[n]} \boxed{S_2} \longrightarrow y[n]$$

S_1, S_2 : Causal LTI system

$\because$ $y[n] = \alpha y[n-1] + \beta w[n]$

From this we may write,

$$w[n] = \frac{1}{\beta} y[n] - \frac{\alpha}{\beta} y[n-1] \quad \text{—— (4)}$$

and

$$w[n-1] = \frac{1}{\beta} y[n-1] - \frac{\alpha}{\beta} y[n-2] \quad \text{—— (5)}$$

Multiplying both side of eqn(5) with $(\frac{1}{2})$, we get-

$$\frac{1}{2} w[n-1] = \frac{1}{2\beta} y[n-1] - \frac{\alpha}{2\beta} y[n-2] \quad \text{—— (6)}$$

Subtracting eqn(4) & eqn(6), we get-

$$w[n] - \frac{1}{2} w[n-1] = \frac{1}{\beta} y[n] - \frac{\alpha}{\beta} y[n-1] - \frac{1}{2\beta} y[n-1] + \frac{\alpha}{2\beta} y[n-2] \quad \text{—— (7)}$$

By eqn(1) & eqn(7), we get-

$$x[n] = \frac{1}{\beta} y[n] - \frac{\alpha}{\beta} y[n-1] - \frac{1}{2\beta} y[n-1] + \frac{\alpha}{2\beta} y[n-2]$$

$$y[n] = \left(\alpha + \frac{1}{2}\right) y[n-1] - \left(\frac{\alpha}{2}\right) y[n-2] + \beta x[n] \quad \text{—— (8)}$$

Comparing eq^n(3) and eq^n(8), we get —

$$\left(\alpha+\frac{1}{2}\right)=\left(\frac{3}{4}\right) \quad \text{and} \quad \beta=1$$

$$\alpha=\left(\frac{1}{4}\right) \quad \text{and} \quad \beta=1 \qquad \rightarrow (2\times2 \text{ Marks})$$

Sol ÷ (b)(2)

Let impulse response of system S_1 & S_2 are $h_1[n]$ & $h_2[n]$ respectively.

By eq^n(1), $w[n]=\frac{1}{2}w[n-1]+x[n]$

$$h_1[n]-\frac{1}{2}h_1[n-1]=\delta[n]$$

$$\therefore h_1[n]=\left(\frac{1}{2}\right)^n u[n] \quad \text{—— (9)}$$

$\rightarrow$ (1.5 Marks)

$$x[n]=\delta[n] \longrightarrow \overset{h_1[n]}{\boxed{S_1}} \longrightarrow w[n]$$

$$w[n]=x[n]*h_1[n]$$
$$=\delta[n]*h_1[n]$$
$$=h_1[n]$$

By eq^n(2), $y[n]=\alpha y[n-1]+\beta w[n]$

$$y[n]=\frac{1}{4}y[n-1]+w[n]$$

$$h_2[n]-\frac{1}{4}h_2[n-1]=\delta[n]$$

$$\therefore h_2[n]=\left(\frac{1}{4}\right)^n u[n] \quad \text{—— (10)}$$

$\rightarrow$ (1.5 Marks)

$$w[n]=\delta[n] \longrightarrow \overset{h_2[n]}{\boxed{S_2}} \longrightarrow y[n]$$

$$y[n]=w[n]*h_2[n]$$
$$=\delta[n]*h_2[n]$$
$$=h_2[n]$$

$\therefore$ Impulse response of cascaded systems,

$$h[n]=h_1[n]*h_2[n]$$

$$=\sum_{K=-\infty}^{\infty} h_1[K]\cdot h_2[n-K]$$

$$\therefore h[n] = \sum_{K=0}^{\infty} \left(\frac{1}{2}\right)^K \left(\frac{1}{4}\right)^{n-K} u[n-K]$$

$$= \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K \left(\frac{1}{4}\right)^{n-K} = \left(\frac{1}{4}\right)^n \sum_{K=0}^{n} \left(\frac{1}{2}\right)^K \cdot \left(\frac{1}{4}\right)^{-K}$$

$$= \left(\frac{1}{4}\right)^n \sum_{K=0}^{n} \left(\frac{1}{2}\right)^{-K} = \left(\frac{1}{4}\right)^n \cdot \frac{1\left[2^{n+1} - 1\right]}{[2-1]}$$

$$= \left(\frac{1}{2}\right)^{2n} \cdot \left[2^{n+1} - 1\right]$$

$$= \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n]$$

→ (3 Marks)

SOL(3): Given signal,

$$x(t) = \begin{cases} \sin(\pi t), & 0 \leq t \leq 2 \\ 0, & 2 < t \leq 4 \end{cases}$$

FTP, $T_0 = 4$

$$\omega_0 = \frac{2\pi}{T_0} = \left(\frac{2\pi}{4}\right) = \left(\frac{\pi}{2}\right)$$

$$x(t) \xrightleftharpoons{FS} C_K$$

we know that –

$$C_K = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jK\omega_0 t}\cdot dt$$

Case(I): $C_0 = \frac{1}{4}\int_4 x(t)\,dt = \frac{1}{4}\left[-\frac{\cos(\pi t)}{\pi}\right]_0^2$

$$= \frac{1}{4\pi}\left[-1+1\right] = 0 \qquad \rightarrow \text{(1 Mark)}$$

Case(II): $C_K = \frac{1}{4}\int_0^2 \sin(\pi t)\cdot e^{-jK\omega_0 t}\cdot dt$

$$= \frac{1}{4}\int_0^2 \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right) e^{-jK\frac{\pi}{2}t}\, dt$$

$$= \left(\frac{1}{4\times 2j}\right)\left\{\int_0^2 e^{j\pi\left(1-\frac{K}{2}\right)t}\cdot dt - \int_0^2 e^{-j\pi\left(1+\frac{K}{2}\right)t}\cdot dt\right\}$$

$$= \left(\frac{1}{8j}\right)\left\{\frac{e^{j\pi\left(1-\frac{K}{2}\right)t}}{j\pi\left(1-\frac{K}{2}\right)} - \frac{e^{-j\pi\left(1+\frac{K}{2}\right)t}}{-j\pi\left(1+\frac{K}{2}\right)}\right\}_0^2$$

$$= \left(\frac{1}{8j}\right)\left\{\frac{e^{j\pi\left(1-\frac{K}{2}\right)t}}{j\pi\left(1-\frac{K}{2}\right)} + \frac{e^{-j\pi\left(1+\frac{K}{2}\right)t}}{j\pi\left(1+\frac{K}{2}\right)}\right\}_0^2$$

$$= \frac{1}{(8j)}\left\{ \frac{e^{j2\pi\left(1-\frac{K}{2}\right)} - e^{0}}{j\pi\left(1-\frac{K}{2}\right)} + \frac{e^{-j2\pi\left(1+\frac{K}{2}\right)} - e^{0}}{j\pi\left(1+\frac{K}{2}\right)} \right\}$$

$$= \left(\frac{1}{8j}\right)\left\{ \frac{e^{j\pi(2-K)} - e^{j2\pi}}{j\pi\left(1-\frac{K}{2}\right)} + \frac{e^{-j\pi(2+K)} - e^{-j2\pi}}{j\pi\left(1+\frac{K}{2}\right)} \right\}$$

$$= \left(\frac{1}{8j}\right)\left\{ \frac{e^{-j\pi K} - 1}{j\pi\left(1-\frac{K}{2}\right)} + \frac{e^{-j\pi K} - 1}{j\pi\left(1+\frac{K}{2}\right)} \right\}$$

$$= \left\{ e^{-j\pi K} - 1 \right\} \times \left(\frac{1}{8j}\right) \times \left\{ \frac{j\pi\left(1+\frac{K}{2}\right) + j\pi\left(1-\frac{K}{2}\right)}{(j\pi)^2\left(1-\frac{K^2}{4}\right)} \right\}$$

$$= \frac{(-1)^K - 1}{\pi(K^2-4)}$$

$\rightarrow$ (5 Mark)

SOL: (4)(a)

Given that $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$

The signal $x[n]$ is periodic with period $N=4$.

$$\therefore a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] \cdot e^{-jk\omega_0 n}$$

$$= \frac{1}{4} \sum_{n=0}^{3} x[n] \cdot e^{-jk\left(\frac{2\pi}{4}\right)n}$$

$$= \frac{1}{4}, \quad \text{for all '}k\text{'} \qquad \text{—— (1)}$$

→ (1.5 Marks)

SOL (4)(b):

$$\begin{array}{c} x[n] \\ \updownarrow^{FS} \\ a_k \end{array} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ \text{System}\end{array}}_{H(e^{j\omega_0})} \longrightarrow y[n] \overset{FS}{\rightleftharpoons} b_k$$

$$b_k = a_k \cdot H(e^{jk\omega_0}) \qquad \omega_0 = \left(\frac{2\pi}{4}\right) = \left(\frac{\pi}{2}\right)$$

Therefore,

$$y[n] = \sum_{k=\langle N \rangle} b_k \cdot e^{jk\omega_0 n}$$

$$= \sum_{k=0}^{3} a_k \cdot H(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

$$= a_0 \cdot H(e^{j0}) \cdot e^{j0} + a_1 \cdot H(e^{j\pi/2}) \cdot e^{j(\pi/2)n} + a_2 \cdot H(e^{j3\pi/2}) \cdot e^{j(3\pi/2)n}$$
$$+ a_3 \cdot H(e^{j\pi}) \cdot e^{j\pi n}$$

$$= \frac{1}{4} H(e^{j0}) \cdot e^{j0} + \frac{1}{4} H(e^{j\pi/2}) \cdot e^{j(\pi/2)n} + \frac{1}{4} H(e^{j3\pi/2}) \cdot e^{nj(3\pi/2)}$$
$$+ \frac{1}{4} \cdot H(e^{j\pi}) \cdot e^{j\pi n} \qquad \text{—— (2)}$$

→ (2 Marks)

Given that $y[n] = \cos\left(\frac{5\pi}{2}n + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)$

$$= \frac{1}{2} e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} + \frac{1}{2} e^{-j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$$

$$= \frac{1}{2} \cdot e^{j\pi/4} \cdot e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{4}} \cdot e^{j3\frac{\pi}{2}n} \quad \text{———} (3)$$

→ (2 Marks)

Comparing eq^n(2) & eq^n(3), we obtain—

$$\left.\begin{aligned} &\text{For, } K=0, \quad H(e^{j0}) = 0 \\ &K=1, \quad \frac{1}{4} H(e^{j\pi/2}) = \frac{1}{2} e^{j\pi/4} \\ &\qquad H(e^{j\pi/2}) = 2 \cdot e^{j\pi/4} \\ &K=2, \quad H(e^{j3\pi/2}) = 0 \\ &K=3, \quad \frac{1}{4} H(e^{j\pi}) = \frac{1}{2} e^{-j\pi/4} \\ &\qquad H(e^{j\pi}) = 2 e^{-j\pi/4} \end{aligned}\right\} \rightarrow (4 \times 0.5 \text{ Marks})$$

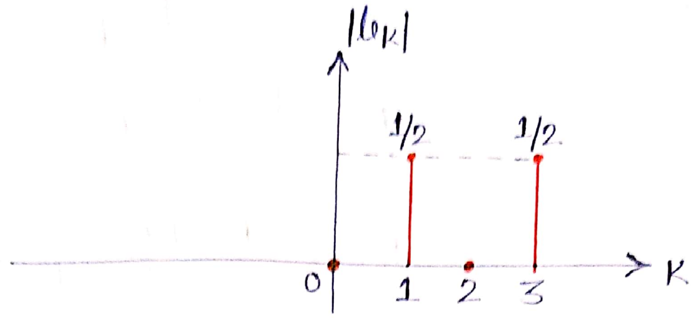
SOL (4)(c) : $y[n] \xrightleftharpoons{FS} b_K$ Over one time period—

$b_1 = \frac{1}{2} e^{j\pi/4}$ $\qquad$ $b_3 = \frac{1}{2} e^{-j\pi/4}$

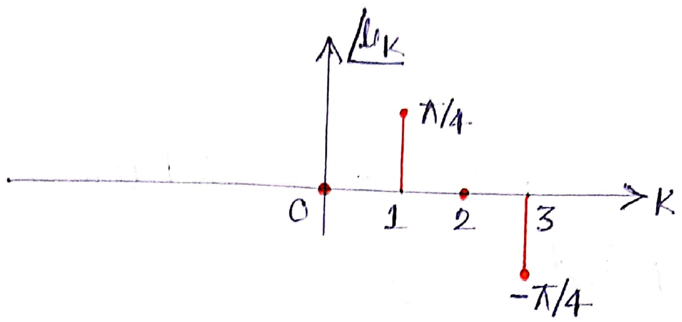
$|b_1| = \left(\frac{1}{2}\right)$ $\qquad$ $|b_3| = \left(\frac{1}{2}\right)$

$\angle b_1 = (\pi/4)$ $\qquad$ $\angle b_3 = -(\pi/4)$

$b_0 = b_2 = 0$



→ (1 Mark)



→ (1 Mark)

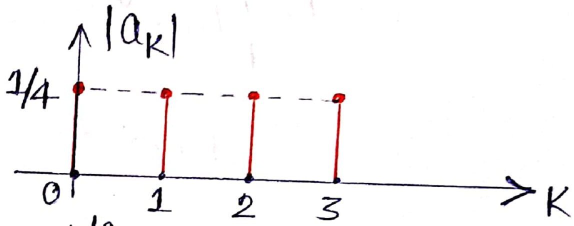
SOL (4)(a)÷ $x[n] \xleftrightarrow{FS} a_K$

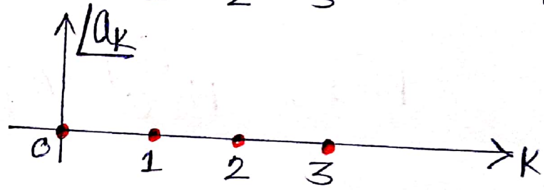
over one time period–

$$a_0 = a_1 = a_2 = a_3 = \left(\frac{1}{4}\right)$$

$$\therefore\ |a_0| = |a_1| = |a_2| = |a_3| = (1/4)$$

$$\&\ \ \angle a_0 = \angle a_1 = \angle a_2 = \angle a_3 = 0$$





→ (0.5 Mark)

SOL (4)(d)÷ By comparing the line spectrum of $x[n]$ & $y[n]$,

(I) The given LTI system block the two freq component $= 0, 2\omega_0 = 0, \pi$ (over one time period)

(II) Magnitude & Phase alter but frequency remains same.

(III) Time period remains same.

→ (1 Mark)

SOL:- (a)(5)

Given that-

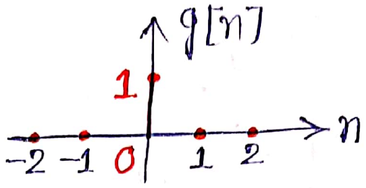
$$x[n] = \alpha^n \cdot u[n] \qquad \text{——— (1)}$$

$$g[n] = x[n] - \alpha \cdot x[n-1] \qquad \text{——— (2)}$$

$$= \alpha^n u[n] - \alpha \cdot \alpha^{n-1} \cdot u[n-1]$$

$$= \alpha^n u[n] - \alpha^n u[n-1]$$

$$= \alpha^n \delta[n] = \delta[n] \qquad \text{——— (3)}$$



$\longrightarrow$ (1 Mark)

SOL:- (b)(5) By eq^n(2) & eq^n(3), we obtain-

$$g[n] = x[n] - \alpha x[n-1] = \delta[n]$$

$$= x[n] * \{\delta[n] - \alpha \cdot \delta[n-1]\} = \delta[n] \qquad \text{——— (4)}$$

Using eq^n(4), we may write-

$$x[n] * \{\delta[n+1] - \alpha \cdot \delta[n]\} = \delta[n+1]$$

$$x[n] * \{\delta[n+2] - \alpha \cdot \delta[n+1]\} = \delta[n+2]$$

$$x[n] * \{\delta[n-1] - \alpha \cdot \delta[n-2]\} = \delta[n-1]$$

Given that-

$$x[n] * h[n] = \left(\tfrac{1}{2}\right)^n \{u[n+2] - u[n-2]\}$$

$$= \left(\tfrac{1}{2}\right)^n \{\delta[n+2] + \delta[n+1] + \delta[n] + \delta[n-1]\}$$

$$= \left(\tfrac{1}{2}\right)^n \delta[n+2] + \left(\tfrac{1}{2}\right)^n \delta[n+1] + \left(\tfrac{1}{2}\right)^n \cdot \delta[n] + \left(\tfrac{1}{2}\right)^n u[n-1]$$

$$= 4\,\delta[n+2] + 2\,\delta[n+1] + \delta[n] + \tfrac{1}{2}\,\delta[n-1]$$

$$\therefore x[n] * h[n] = 4\, x[n] * \{\delta[n+2] - \alpha\delta[n+1]\}$$
$$+ 2\, x[n] * \{\delta[n+1] - \alpha \cdot \delta[n]\}$$
$$+ x[n] * \{\delta[n] - \alpha\, \delta[n-1]\}$$
$$+ \frac{1}{2} x[n] * \{\delta[n-1] - \alpha\, \delta[n-2]\}$$

$$= x[n] * \{4\delta[n+2] - 4\alpha\delta[n+1] + 2\delta[n+1] - 2\alpha\,\delta[n]$$
$$+ \delta[n] - \alpha\,\delta[n-1] + \frac{1}{2}\delta[n-1] - \frac{\alpha}{2}\delta[n-2]\}$$

$$= x[n] * \{4\delta[n+2] + (2-4\alpha)\delta[n+1] + (1-2\alpha)\delta[n]$$
$$+ \left(\frac{1}{2} - \alpha\right)\delta[n-1] - \frac{\alpha}{2}\delta[n-2]\}$$

Therefore,

$$h[n] = \{4\,\delta[n+2] + (2-4\alpha)\,\delta[n+1] + (1-2\alpha)\,\delta[n]$$
$$+ \left(\frac{1}{2} - \alpha\right)\delta[n-1] - \frac{\alpha}{2}\,\delta[n-2]\}$$

$\rightarrow$ (5 Marks)