

C.T.

Constant coefficient differential eqn. representation of causal LTI System

$$\sum_{k=0}^N b_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M a_k \frac{d^k x(t)}{dt^k}$$

Condition of initial rest
is satisfied

$N \equiv$ Order of the system

Let $N=2 \times M=3$

$$b_2 \frac{d^2 y(t)}{dt^2} + b_1 \frac{d y(t)}{dt} + b_0 y(t) = a_0 x(t) + a_1 \frac{d x(t)}{dt} + a_2 \frac{d^2 x(t)}{dt^2} + a_3 \frac{d^3 x(t)}{dt^3}$$

if $x(t)=0$ for $t < t_0$

$\frac{d^k y(t)}{dt^k} = 0$ for $t < t_0$

$\forall k=0, 1, \dots, N$

Block Diagram representation

D.T.

$a_0 x[n] =$ scalar multiplier

$x[n-1] =$ delay element

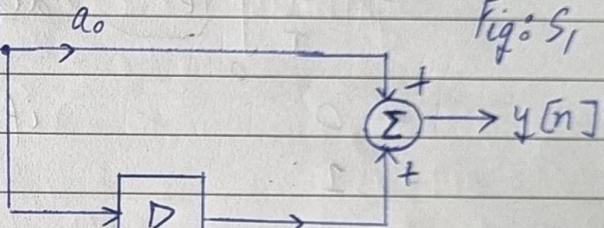
$$S_1: y[n] = a_0 x[n] + a_1 x[n-1]$$

$x[n]$

Fig: S₁

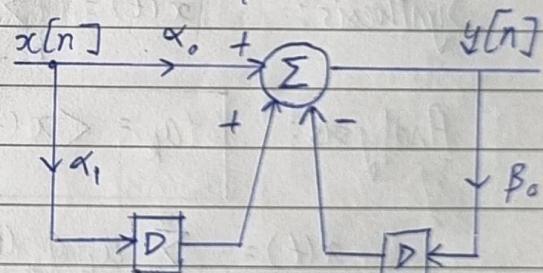
$$S_2: b_1 y[n-1] + b_0 y[n] = a_0 x[n] + a_1 x[n-1]$$

$$y[n] = \frac{a_0}{b_0} x[n] + \frac{a_1}{b_0} x[n-1] - \frac{b_1}{b_0} y[n-1]$$



$$\text{where } = a_0 x[n] + a_1 x[n-1] - b_0 y[n-1]$$

$$a_0 = a_0/b_0$$



constants

Fourier Series

C.T.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \quad \forall t \text{ for Periodic signal } x(t)$$

Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] s[n-k]$$

$$= \sum_{k=-\infty}^{\infty} a_k s[n-k]$$

Vector

basis fns.

$$\phi_k(t) = e^{jk\omega_0 t}$$

$$e^{jk\omega_0 t} = \cos k\omega_0 t + j \sin k\omega_0 t$$

$\omega_0 = \frac{2\pi}{T}$
 $T = \text{fundamental time period of } x(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau) s(t-\tau) d\tau$$

$$\phi_1(t) = e^{j\omega_0 t} \quad \phi_2(t) = e^{2j\omega_0 t}$$

$$\phi_{-1}(t) = e^{-j\omega_0 t} \quad \phi_{-2}(t) = e^{-2j\omega_0 t}$$

$$\phi_0(t) = e^{j0} = \cos 0 + j \sin 0$$

$$= 1$$

$$\langle \phi_k(t), \phi_l(t) \rangle_T \stackrel{\Delta}{=} \frac{1}{T} \int_T \phi_k(t) \phi_l^*(t) dt \equiv \text{Defining inner product}$$

$$= \frac{1}{T} \int_T e^{jk\omega_0 t} e^{-jl\omega_0 t} dt = 1 \text{ if } m=0$$

$$= \frac{1}{T} \int_T e^{j(m-k-l)\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T (e^{jm\omega_0 t}) dt = \frac{e^{jm\omega_0 T}}{jm\omega_0 T} \Big|_{m \neq 0} = \frac{e^{jm2\pi} - e^{j0}}{jm\omega_0 T} = 0$$

$$= 0 \quad \text{if } m \neq 0 \quad \text{if } k \neq l$$

$$= 1 \quad \text{if } k=l$$

Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$ $\{\phi_k(t)\}_{k \in \mathbb{Z}}$ \equiv Span of vector space

Analysis: $a_k = \langle x(t), \phi_k(t) \rangle_T$ T form an orthonormal basis (O.N.B.)

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

$$\langle x(t), \phi_l(t) \rangle_T = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k(t), \phi_l(t) \right\rangle_T = \langle \dots + a_{-10} \phi_{-10}(t) + \dots + a_0 \phi_0(t) + \dots + a_9 \phi_9(t) + \dots, \phi_l(t) \rangle_T$$

$$= \sum_{k=-\infty}^{\infty} (a_k \langle \phi_k(t), \phi_l(t) \rangle_T)$$

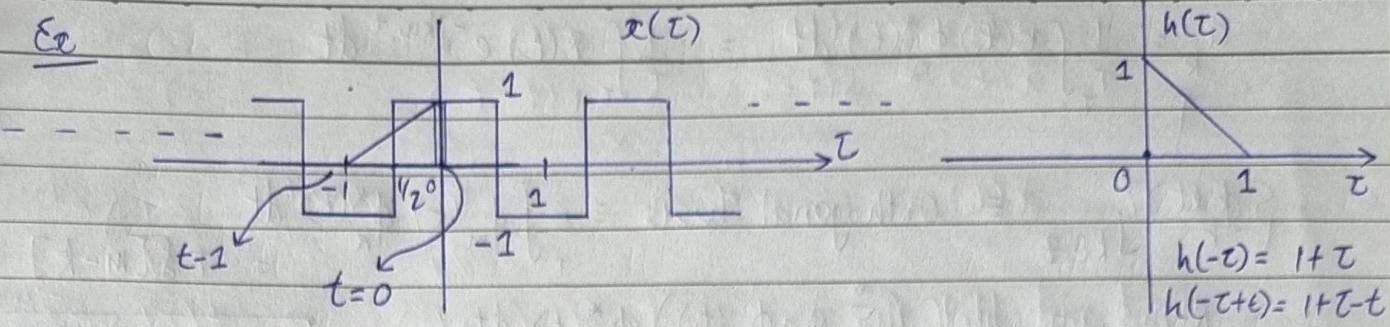
$$= \sum_{k=-\infty}^{\infty} a_k S[k-l]$$

Neelgagan $\langle x(t), \phi_l(t) \rangle_T = a_l$

$$= \dots \langle a_{-10} \phi_{-10}(t), \phi_l(t) \rangle_T + \dots + \langle a_0 \phi_0(t), \phi_l(t) \rangle_T + \dots + \langle a_9 \phi_9(t), \phi_l(t) \rangle_T$$

$$=$$

Analy
equa

Ex(Convolve $x(t) \times h(t)$)

$$\begin{aligned} y(0) &= \int_{-1}^{0} x(\tau) h(t-\tau) d\tau + \int_{-1/2}^{0} x(\tau) (h(t-\tau)) d\tau \\ &= -\int_{-1}^{-1/2} (-1)(1+\tau) d\tau + \int_{-1/2}^0 (1)(1+\tau) d\tau \\ &= -\left(\frac{\tau + \tau^2}{2}\right) \Big|_{-1}^{-1/2} + \left(\frac{\tau + \tau^2}{2}\right) \Big|_{-1/2}^0 \end{aligned}$$

Case 1: $0 \leq t \leq 1/2$

$$y(t) = -\int_{-1}^{-1/2} (1+\tau-t) d\tau + \int_{-1/2}^t (1+\tau-t) d\tau$$

C.T. Fourier Seriesperiodic signalsSynthesis

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t} \end{aligned}$$

$$\begin{aligned} \omega_0 &= \frac{2\pi}{T} = 2\pi f_0 \\ \downarrow & \\ \text{rad/s} & \\ \equiv & \text{per sec} \\ \text{or Hz} & \end{aligned}$$

 $e^{j k \omega_0 t} = \phi_k(t) = \text{pure imaginary exponential (in power)}$
 $T \equiv \text{fundamental time period}$

$$x(t) = 3 + 2e^{j\omega_0 t} + e^{j2\omega_0 t}$$

$$= 3 + 2(\cos \omega_0 t + j \sin \omega_0 t) + \cos 2\omega_0 t + j \sin 2\omega_0 t$$

$$= \sum_{k=0}^{\infty} a_k e^{j k \omega_0 t}$$

$a_0 = 3$

all other

$a_1 = 2$

 a_k 's for $k \neq 0, 1, 2$

$a_2 = 1$

are zero

Analysis equation

$$a_k = \langle x(t), \phi_k(t) \rangle_T = \frac{1}{T} \int_T x(t) \phi_k^*(t) dt$$

$$a_k = \frac{1}{T} \int_T x(t) \phi_k^*(t) dt = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$$

$\{\phi_k(t)\}_{k \in \mathbb{Z}}$ \equiv orthogonal basis $\equiv \int_T \phi_m(t) \phi_k^*(t) dt = T \delta[m-k]$

$$R.H.S = \frac{1}{T} \int_T \left(\sum_{m=-\infty}^{\infty} a_m \phi_m(t) \right) \phi_k^*(t) dt$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \left[a_m \left(\int_T \phi_m(t) \phi_k^*(t) dt \right) \right]$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \left[a_m \underbrace{\int_T e^{j(m-k)w_0 t} dt}_{\circlearrowleft} \right]$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} a_m \cdot T \delta[m-k] = a_k = L.H.S.$$

$\phi_1(t), \phi_{-1}(t)$
 \downarrow
I harmonic
 $\phi_2(t), \phi_{-2}(t)$
 \downarrow
2nd harmonics &
So on

Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$

Analysis: $a_k = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt$

e^{st} act as eigenfunctions of an LTI system

where $H(s) \equiv$ corresponding eigenvalue

$$x(t) \rightarrow \boxed{LTI} \rightarrow y(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Let us assume $x(t) = e^{st}$

where $s = j\omega_0$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau$$

$$= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

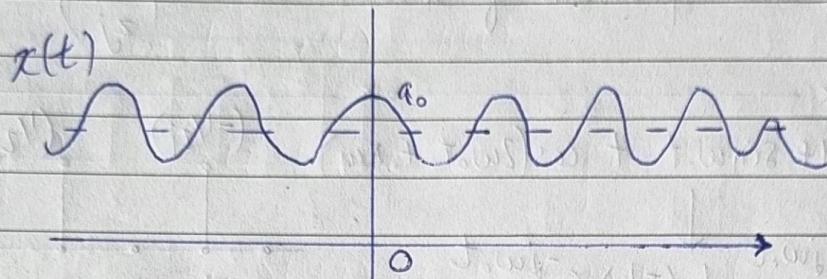
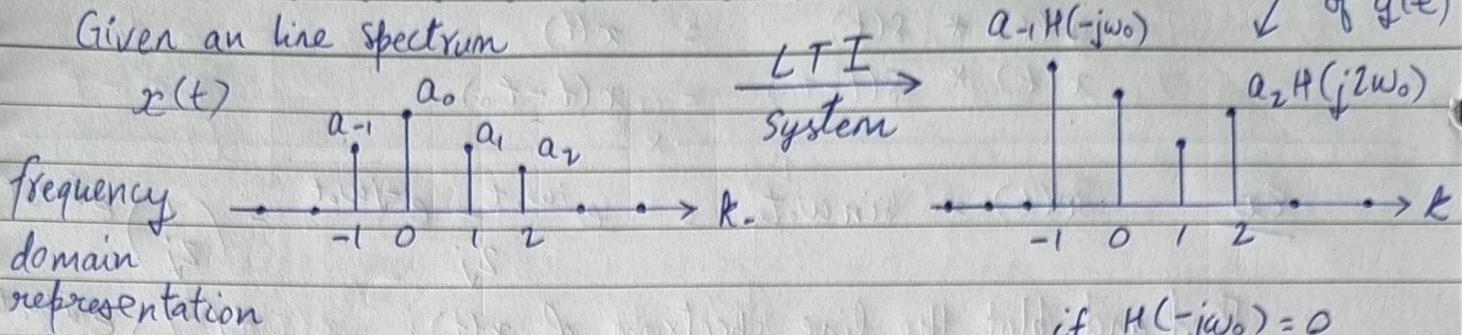
$H(s)$

$$\frac{Av}{v} = \frac{1}{H(s)}$$

No.
Date

line spectrum
of $g(t)$

Given an line spectrum



if $H(-j\omega_0) = 0$

$$a_{-1} H(-j\omega_0) = 0$$

\downarrow this freq. will
disappear in 0/b

$$\text{Given } x(t) = a_1 e^{-i\omega t} + a_0 + a_1 e^{i\omega t} + a_2 e^{i2\omega t}$$

Ex

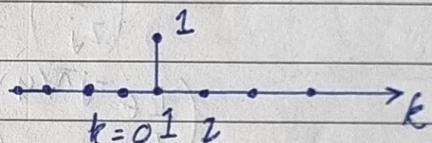
$$S: y(t) = x(t-1)$$

$$\text{let } x(t) = e^{j2t}$$

$$\omega_0 = 2\pi/T$$

line spectrum
of $Y(t)$

$$\begin{aligned} y(t) &= e^{j2(t-2)} \\ &= e^{j2t} \cdot e^{-j4} \\ &= e^{j2t} H(j2) \end{aligned}$$



$$H(jz) = e^{-j\phi}$$

$$S = j2$$

ER

What is $h(t)$?

$$h(t) = \phi(t-z)$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt = \int_{-\infty}^{\infty} s(t-2) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} f(z-t) \cdot e^{-j2\pi t} dt$$

$$= e^{-izt} \int_{-\infty}^{\infty} f(z-t) dt$$

$$= e^{-\lambda t}$$

$$x(t) * \delta(t) = x(t)$$

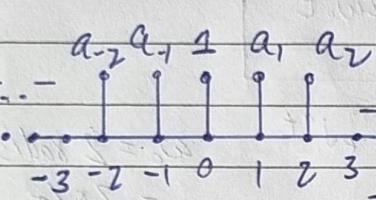
$$x(t) * \delta(t-t_0) = x(t-t_0)$$

Ex $x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \left(\frac{1}{z_j}\right) e^{j\omega_0 t} + \left(\frac{-1}{z_j}\right) e^{-j\omega_0 t}$

Find and plot the line spectrum of $x(t)$

Ex $x(t) = 1 + \sin \omega_0 t + \cos(2\omega_0 t + \pi/4)$

$$= 1 + \frac{e^{j\omega_0 t}}{z_j} + \left(\frac{-1}{z_j}\right) e^{-j\omega_0 t}$$



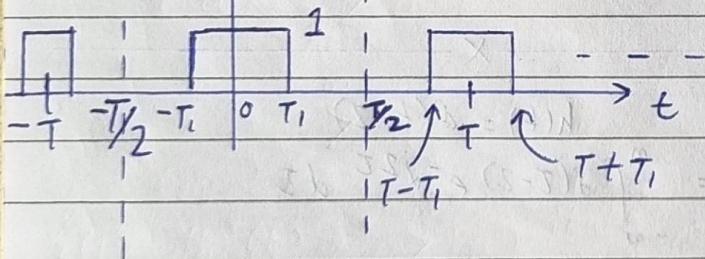
$$= 1 + \frac{1}{2} [e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}]$$

$$= 1 + \left(\frac{1}{z_j}\right) \phi_1(t) + \left(\frac{-1}{z_j}\right) \phi_{-1}(t) + \left(\frac{1}{2} e^{j\pi/4}\right) \phi_2(t) +$$

$$\left(\frac{1}{2} e^{-j\pi/4}\right) \phi_{-2}(t)$$

$$= a_0 + a_1 \phi_1(t) + a_{-1} \phi_{-1}(t) + a_2 \phi_2(t) + a_{-2} \phi_{-2}(t)$$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T_1/2 \end{cases} \quad \text{periodic}$$



① if $T \rightarrow \infty$, signal will become aperiodic

② there should be a dc bias term, so $a_0 \neq 0$

③

fundamental freq. . .

Solve it, plot the line spectrum

C.T. F.S① Linearity

$$\begin{aligned} x(t) &\xrightarrow[\text{F.S.}]{\text{F.S.}} a_k \\ y(t) &\xrightarrow[\text{F.S.}]{\text{F.S.}} b_k \end{aligned}$$

$x(t)$ & $y(t)$ are periodic
with fundamental time
period = T .

$$z(t) = [\alpha x(t) + \beta y(t)] \xrightarrow{\quad} \alpha a_k + \beta b_k$$

& $z(t) \equiv$ same fundamental time period T .

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{j k \omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$$

$$c_k = \frac{1}{T} \int_T z(t) e^{-j k \omega_0 t} dt$$

$$= \frac{1}{T} \int_T (\alpha x(t) + \beta y(t)) e^{-j k \omega_0 t} dt = \underbrace{\alpha \cdot \frac{1}{T} \int_T x(t) e^{-j k \omega_0 t} dt}_{a_k}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j k \omega_0 t}$$

$$+ \underbrace{\beta \cdot \frac{1}{T} \int_T y(t) e^{-j k \omega_0 t} dt}_{b_k}$$

$$= \alpha a_k + \beta b_k \rightarrow \text{Proved.}$$

② Time Shifting

$$x(t) \xrightarrow[\text{T}]{\text{F.S.}} a_k$$

$$\omega_0 = \frac{2\pi}{T}$$

Find F.S. coeff. of $x(t-t_0)$, where $t_0 \equiv$ constant:

$$\begin{aligned} z(t) &= x(t-t_0) \xrightarrow{T} z = c_k = a_k e^{-j k \omega_0 t_0} \\ c_k &= \frac{1}{T} \int_{-T/2}^{T/2} z(t) e^{-j k \omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t-t_0) e^{-j k \omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2-t_0}^{T/2-t_0} x(\tau) e^{-j k \omega_0 (t_0+\tau)} d\tau \end{aligned}$$

let $t-t_0 = \tau$

$dt = d\tau$

$$c_k = e^{-j k \omega_0 t_0} \left(\frac{1}{T} \int_T x(\tau) e^{-j k \omega_0 \tau} d\tau \right) = a_k e^{j k \omega_0 t_0}$$

③ Time-reversal

$$x(t) \xrightarrow[T]{F.S.} a_k$$

$$z(t) = x(-t) \xrightarrow[T]{F.S.} c_k = a_{-k}$$

$$\begin{aligned} T &= -t \\ dt &= -dt \end{aligned}$$

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(-t) e^{-jkw_0 t} dt = \frac{-1}{T} \int_{-T/2}^{T/2} x(t) e^{jkw_0 t} dt$$

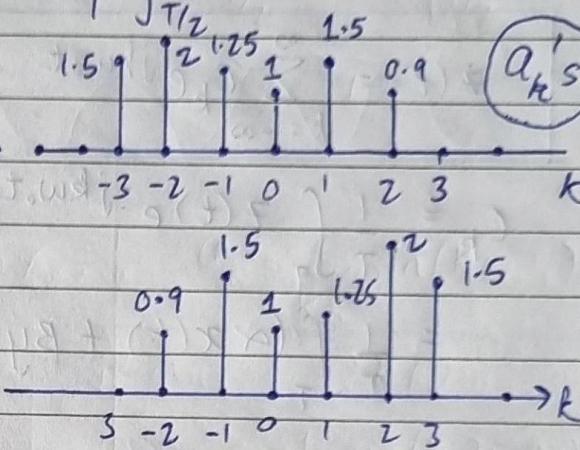
$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{jkw_0 t} dt$$

$$c_{-k} = \frac{1}{T} \int_T x(t) e^{-jkw_0 t} dt = a_k$$

$$c_k = a_{-k}$$

$$c_0 = a_0$$

$$c_1 = a_{-1}$$



④

$$x(t) \xrightarrow[T]{F.S.} a_k$$

Time period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$z(t) = x^*(t) \longrightarrow a_{-k}^*$$

Case-1

(let $x(t)$ is real)

$$x(t) = x^*(t)$$

$$\downarrow \qquad \downarrow$$

$$a_k = a_{-k}^*$$

$$a_k^* = a_{-k}$$

$$x(t) = x(t)$$

Case-2 (let $x(t)$ is real & even.)

Conclusion: a_k' 's are also real & even.

$$a_k^* = a_{-k}$$

$$a_k = a_{-k}$$

$$\leftarrow a_k^* = a_k$$

Case-3 $x(t)$ is real & odd



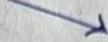
$$x(t) = x^*(t)$$



$$a_k^* = a_{-k}$$

$$a_k^* = -a_k$$

Coeff. are purely imaginary.



$$x(-t) = -x(t)$$



$$-a_k = +a_{-k}$$

(5) Time Scaling

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{j k(\omega_0 t)}$$

$$x(t) \xrightarrow[T]{F.S.} a_k \quad \alpha > 0 \quad \alpha \neq -1$$

$$z(t) = x(\alpha t) \xrightarrow[T/\alpha]{F.S.} c_k = ?$$

$$\begin{aligned} c_k &= \frac{1}{T/\alpha} \int_{-T/2\alpha}^{T/2\alpha} x(\alpha t) e^{-jk(\alpha \omega_0 t)} dt \quad \text{let } \alpha t = \tau \\ &= \frac{1}{T/\alpha} \int_{-T/2}^{T/2} x(\tau) e^{-jk\omega_0 \tau} \frac{d\tau}{\alpha} = a_k \end{aligned}$$

(6) Multiplication

$$x(t) \xrightarrow[T]{F.S.} a_k$$

$$x(t) = \sum_{l=-\infty}^{\infty} a_l e^{j l \omega_0 t}$$

$$y(t) \xrightarrow[T]{F.S.} b_k$$

$$y(t) = \sum_{m=-\infty}^{\infty} b_m e^{j m \omega_0 t}$$

$$z(t) = x(t) y(t) \xrightarrow[F.S.]{T} c_k = ?$$

$$c_k = \frac{1}{T} \int_T z(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_T (x(t) y(t)) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T} \int_T \sum_l \sum_m a_l b_m e^{j(l+m)\omega_0 t} e^{-jk\omega_0 t} dt$$

$$= \sum_l \sum_m a_l b_m \left(\frac{1}{T} \int_T e^{-j(k-l-m)\omega_0 t} dt \right) = 1 \quad k-l-m=0 \\ \quad l=k-m$$

$$= \boxed{\sum_m a_{k-m} b_m = \sum_l a_l b_{k-l}}$$

multiplication in the time domain \longleftrightarrow convolution in the frequency domain

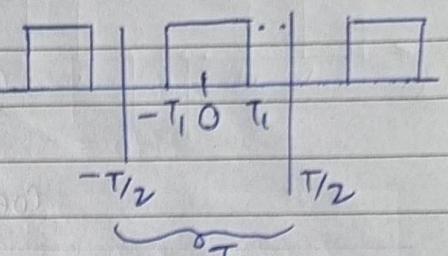
Fix T_1

\times

\times

$$\textcircled{1} \quad T = 4T_1$$

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$



$$\textcircled{2} \quad T = 8T_1$$

$$a_0 = \frac{1}{T} \int_T x(t) dt$$

$$= \frac{1}{T} \int_{-T_1}^{T_1} dt = \boxed{\frac{2T_1}{T} = a_0 = \frac{1}{2}}$$

$$a_k = \frac{1}{T} \int_{-T_1}^{T_1} e^{-jkw_0 t} dt \iff k \neq 0$$

$$a_k = \frac{1}{k\pi} \sin(Kw_0 T_1) \quad k \neq 0$$

$$= \frac{1}{k\pi} \sin\left(\frac{k2\pi}{4T_1}\right)$$

$$= \frac{1}{k\pi} \sin\left(\frac{k\pi}{2}\right)$$

⑦ Parseval's relation

$$x(t) \xrightarrow[T]{F.S} a_k$$

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_k |a_k|^2$$

D.T. F.S

Given a discrete-time periodic signal $x[n]$

$$\text{Synthesis} \equiv x[n] = \sum_{k=-N}^N a_k e^{jkw_0 n} \quad N = \text{Fundamental time period}$$

$$= \sum_{k=-N}^N a_k \phi_k[n]$$

$$0, w_0, 2w_0, \dots, (N-1)w_0, Nw_0$$

$$0, \frac{2\pi}{N}, \frac{2 \cdot 2\pi}{N}, \dots, (N-1) \cdot \frac{2\pi}{N}, 2\pi$$

$$\phi_k[n] = e^{jkw_0 n}$$

$$\phi_{k+N}[n] = \phi_k[n] = e^{jkw_0 n}$$

$$\begin{aligned}
 \langle x[n], \phi_e[n] \rangle_N &= \frac{1}{N} \sum_{n \in \{N\}} x[n] e^{-j\omega_0 n} \\
 &= \frac{1}{N} \sum_{n \in \{N\}} \left(\sum_{k \in \{N\}} a_k e^{jk\omega_0 n} \right) e^{-j\omega_0 n} \\
 &= \frac{1}{N} \sum_{n \in \{N\}} \sum_{k \in \{N\}} a_k e^{j(k-l)\omega_0 n} \\
 &= \sum_{k \in \{N\}} a_k \underbrace{\left(\frac{1}{N} \sum_{n \in \{N\}} e^{j(k-l)\omega_0 n} \right)}_0 = 1 \quad \text{when } k=l
 \end{aligned}$$

$$= \boxed{a_l = \frac{1}{N} \sum_{n \in \{N\}} x[n] \phi_e^*[n]}$$

$$= \frac{1}{N} \sum_{n \in \{N\}} x[n] e^{-jk\omega_0 n}$$

Ex $x[n] = \sin \omega_0 n$ $\omega_0 = \frac{2\pi}{N} \rightarrow$ periodic with period N .

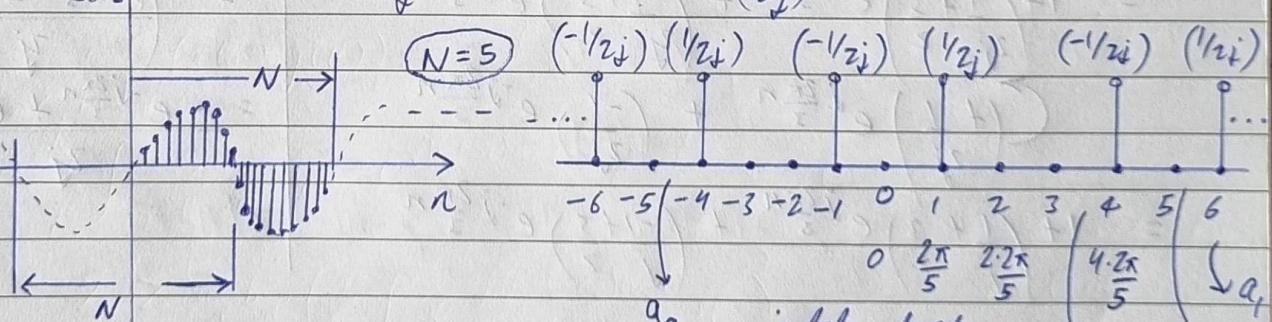
$$x[n] = \frac{1}{2j} [e^{j\omega_0 n} - e^{-j\omega_0 n}]$$

$$= \left(\frac{1}{2j}\right) e^{j\omega_0 n} + \left(\frac{-1}{2j}\right) e^{-j\omega_0 n}$$

line spectrum

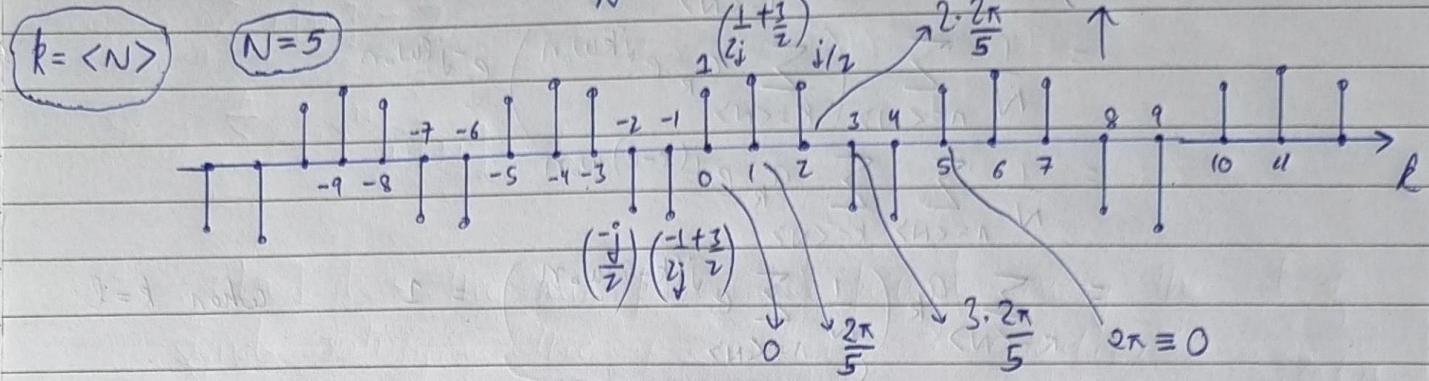
a_1, a_{-1} are non-zero, rest all are zero in $k \in \{N\}$.

$$a_1 = \frac{1}{2j}; \quad a_{-1} = \left(\frac{-1}{2j}\right)$$



independent
variable
 \equiv time

$$\text{Ex} \quad x[n] = 1 + \sin \frac{2\pi n}{N} + 3 \cos \frac{2\pi n}{N} + \cos \left(\frac{4\pi n + \pi}{2} \right)$$



$N=7$ 2 zero amplitude pts. in one N

$N=11$ 6 zero-amplitude pts. in one period in the frequency domain

$$-\frac{2\pi}{3}, 0, \frac{2\pi}{3}, \frac{2 \cdot 2\pi}{3}, 2\pi \equiv 0$$

$$\begin{aligned} e^{j2 \cdot \frac{2\pi n}{3}} &= e^{j(2\pi - \frac{2\pi}{3})n} \quad N=3 \\ &= (e^{j2\pi n}) \cdot e^{-j\frac{2\pi}{3}n} \\ &= (e^{j2\pi n}) \cdot e^{-j\frac{2\pi}{3}n} \\ &= e^{-j\frac{2\pi}{3}n} \end{aligned}$$

$$x[n] = 1 + \sin \frac{2\pi n}{3} + 3 \cos \frac{2\pi n}{3} + \cos \left(\frac{4\pi n + \pi}{2} \right)$$

$$x[n] = 1 + \frac{1}{2} \left(e^{j\frac{2\pi n}{3}} - e^{-j\frac{2\pi n}{3}} \right) + \frac{3}{2} \left(e^{j\frac{2\pi n}{3}} + e^{-j\frac{2\pi n}{3}} \right) + \cancel{\cos \frac{4\pi n}{3}} \times 0$$

$$- \sin \frac{4\pi n}{3}$$

$$\begin{aligned} &= 1 + \left(\frac{1+3}{2} \right) e^{j\frac{2\pi n}{3}} + \left(\frac{-1+3}{2} \right) e^{-j\frac{2\pi n}{3}} \\ &\quad \left(+ \frac{1}{2} \right) e^{j\frac{2\pi n}{3}} + \left(-\frac{1}{2} \right) e^{-j\frac{2\pi n}{3}} \\ &= 1 + \left(\frac{1+3}{2} \right) e^{j\frac{2\pi n}{3}} + \left(\frac{-1+3}{2} \right) e^{-j\frac{2\pi n}{3}} \end{aligned}$$

$$- \sin \frac{4\pi n}{3}$$

$$\begin{aligned} &= - \left(e^{j\frac{4\pi n}{3}} - e^{-j\frac{4\pi n}{3}} \right) \\ &= - \frac{e^{-j\frac{2\pi n}{3}} + e^{j\frac{2\pi n}{3}}}{2j} \end{aligned}$$

Ex $x[n] \equiv$ periodic with period N

Show that the F.S. coefficients of the periodic signal

$$g(t) = \sum_{k=-\infty}^{\infty} x[k] S(t - kT)$$

are periodic with period N .

Ex

① $x(t)$ is a real signal

② $x(t)$ is periodic with $T = 4$.

③ $a_k = 0$ for $|k| > 1$

④ A signal with F.S. coeff. $b_k = e^{-j\frac{2\pi k}{4}} a_k$ is odd

⑤ $\frac{1}{4} \int_0^4 |x(t)|^2 dt = \frac{1}{2}$ ≡ average power in one time period

$$= |a_{-1}|^2 + |a_0|^2 + |a_1|^2 = \frac{1}{2}$$

$$x(t) \xrightarrow[T]{F.S.} a_k$$

$x(t)$ is real

$$z(t) = x(-t+1) \xrightarrow[T]{F.S.} b_k$$

$$z(t) = x(-t+1)$$

real & odd

$$b_0 = 0$$

$$|b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$$

$$b_1 = \frac{j}{2}, b_{-1} = -\frac{j}{2}$$

$$\sqrt{|b_1|^2 + |b_{-1}|^2} = \frac{1}{2}$$

add & purely imaginary

$$b_1 = ja$$

$$b_{-1} = -ja$$

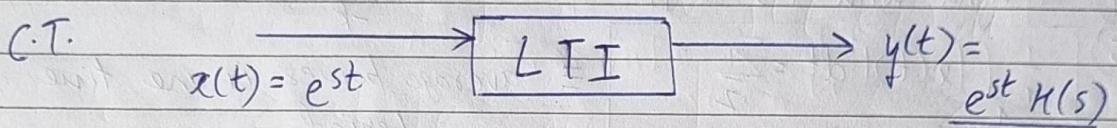
$$a^2 + a^2 = \frac{1}{2}$$

$$a^2 = \frac{1}{4}$$

$$a = \pm \frac{1}{2}$$

- ① $x[n]$ is periodic with period $N=6$
- ② $\sum_{n=0}^5 x[n] = 2$
- ③ $\sum_{n=2}^5 (-1)^n x[n] = 1$
- ④ $x[n]$ has the min. power per period among the set of signals satisfying the preceding 3 conditions

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N} kn}$$

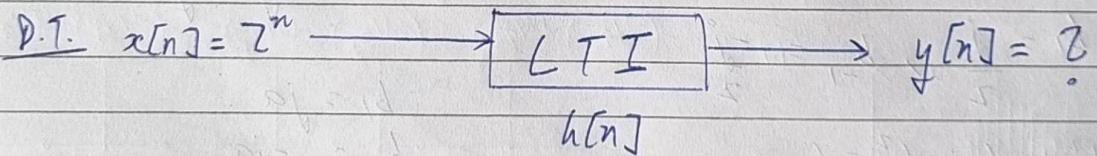


C.T. F.S. $s_k = jkw_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{s_k t} \xrightarrow{} y(t) = \sum_{k=-\infty}^{\infty} a_k H(s_k) e^{s_k t}$$

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$



$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k] 2^{n-k}$$

$$= z^n \left| \sum_{k=-\infty}^{\infty} h[k] z^{-k} \right|$$

$$y[n] = z^n H(z) \uparrow$$

$$\text{let } z = e^{j\omega_0}$$

$$x[n] = e^{j\omega_0 n} \rightarrow e^{j\omega_0 n} H(e^{j\omega_0})$$

$$\text{D.T.F.S. } x[n] = \sum_{k=-N}^{\infty} a_k e^{j k \omega_0 n} \xrightarrow[\text{system } h[n]]{LT} y[n] = \sum_{k=-N}^{\infty} a_k H(e^{j k \omega_0}) e^{j k \omega_0 n}$$

$$x[n] = \sum_{k=-5}^{\infty} a_k e^{j k \frac{2\pi}{5} n}$$

$$a_0 = 1$$

$$= 1 + \cos \omega_0 n + \frac{3}{2} \cos 2\omega_0 n$$

$$N=5$$

$$a_1 = a_{-1} = \frac{1}{2}$$

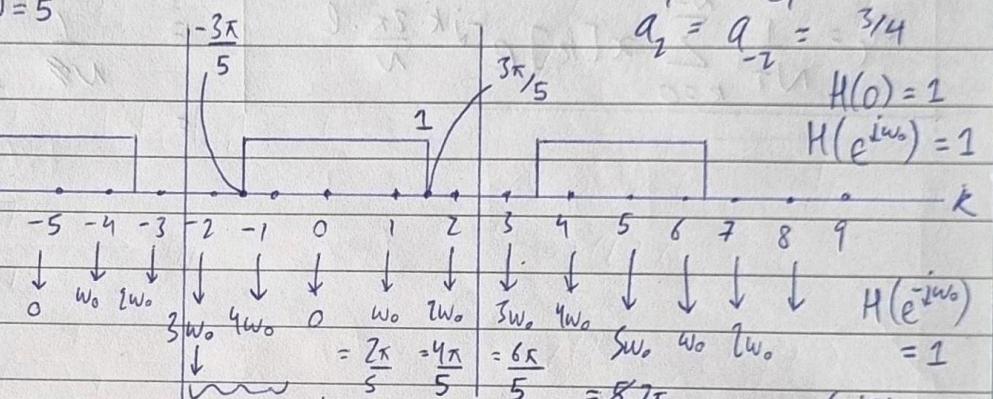
$$a_2 = a_{-2} = \frac{3}{4}$$

$$H(0) = 1$$

$$H(e^{j\omega_0}) = 1$$

$$3\omega_0 = \frac{6\pi}{5} = 2\pi - \frac{4\pi}{5}$$

$$= -\frac{4\pi}{5}$$



$$H(e^{j2\omega_0}) = 0$$

$$= 0$$

$$H(e^{-j2\omega_0}) = 0$$

Ex $x[n]$ is a periodic signal, N . Show that F.S. coeff. of a periodic signal $g(t) = \sum_{k=-\infty}^{\infty} x[k] s(t-kT)$

($N=3$)

are also periodic with period N .

$$g(t) = x[0] s(t) + x[1] s(t-T) + x[2] s(t-2T) + \dots + x[-1] s(t+T)$$

$g(t)$ is periodic with NT .

$$g(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \frac{2\pi}{NT} t}$$

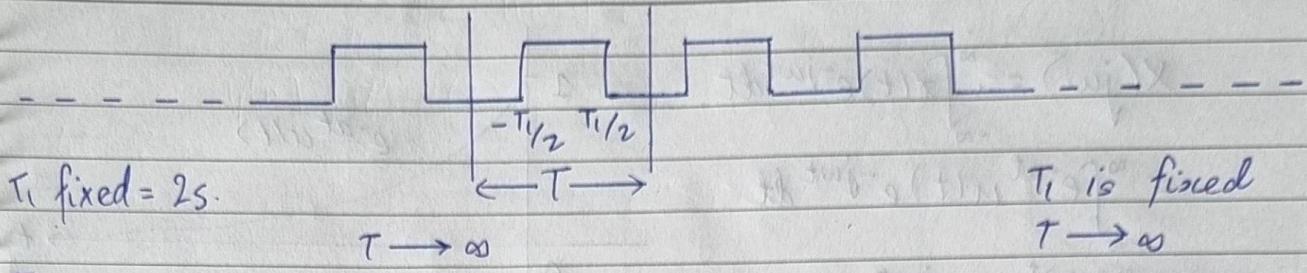
$$a_0 = \frac{1}{NT} \int_{NT} g(t) dt = \frac{x[0] + x[1] + \dots + x[N-1]}{NT}$$

$$a_l = \frac{1}{NT} \int_{NT} g(t) e^{-j l \frac{2\pi}{NT} t} dt$$

$$\begin{aligned} a_l &= \frac{1}{NT} \int_{NT} \sum_{k=-\infty}^{\infty} x[k] s(t-kT) e^{-j l \frac{2\pi}{NT} t} dt \\ &= \frac{1}{NT} \int_0^{NT} \sum_{k=0}^{N-1} x[k] e^{-j l \frac{2\pi}{NT} \cdot kT} s(t-kT) dt \\ &= \frac{1}{NT} \sum_{k=0}^{N-1} x[k] \int_0^{NT} e^{-j lk \cdot \frac{2\pi}{N}} s(t-kT) dt \end{aligned}$$

$$a_l = \frac{1}{NT} \sum_{k=0}^{N-1} x[k] e^{-j k \cdot \frac{2\pi}{N} \cdot l} = \frac{1}{N} \sum_{n=0}^{N-1} \left(\frac{x[n]}{T} \right) e^{-j l \cdot \frac{2\pi n}{N}}$$

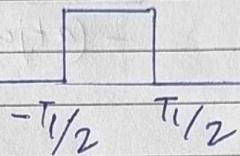
Fourier Transform



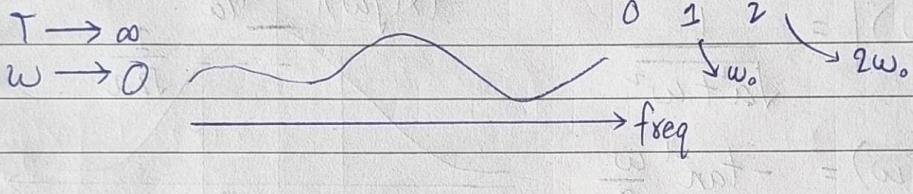
$$T = 4T_1 \Rightarrow \omega_0 = \frac{2\pi}{8}$$

$$T = 8T_1 \Rightarrow \omega_0 = \frac{2\pi}{16}$$

F.T.



as $T \rightarrow$ increasing
 $\omega_0 \rightarrow$ decreasing



C.T. F.T. Synthesis $\equiv x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

F.S.
 $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Inverse F.T.

We will use
 analysis $\equiv x(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

$$\omega = 2\pi f$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Ex $x(t) = e^{-at} u(t)$ $a > 0$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

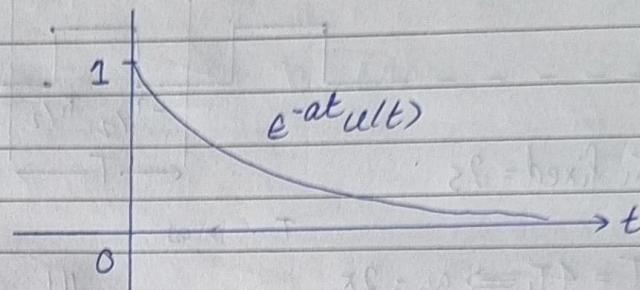
$$= \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \int_0^{\infty} e^{-(a+j\omega)t} dt = \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} = \frac{1}{a+j\omega}$$

~~$t=0$~~

$$\begin{aligned} & -at - j\omega t \\ & e^{-at} e^{-j\omega t} \\ & = 1 \cdot e^{-j0} \\ & = 1 \end{aligned}$$

$e^{-at} u(t)$



~~$t=\infty$~~

$$e^{-a \cdot \infty} \cdot e^{-j\omega \cdot \infty}$$

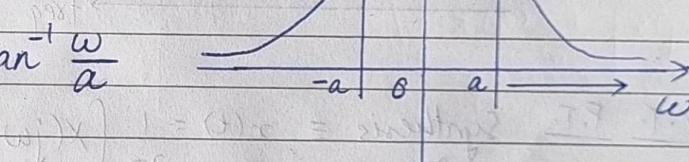
$$0 \times ? = 0$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$X(j\omega) = -\tan^{-1} \frac{\omega}{a}$$

a is increasing

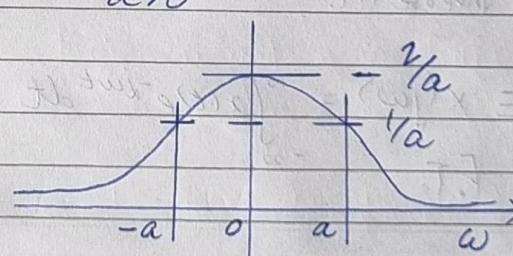
$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



Ex-2

$$x(t) = e^{-alt} \quad a > 0$$

$$X(j\omega) = \frac{2a}{a^2 + \omega^2}$$



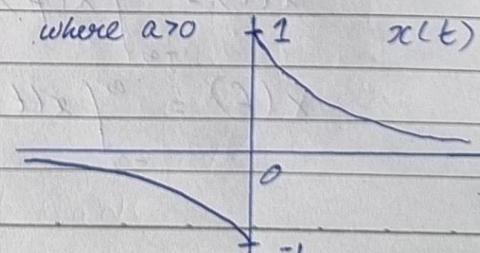
$$10 \log A^2$$

Max
ampli A
 $A^2/2$

Ex-3

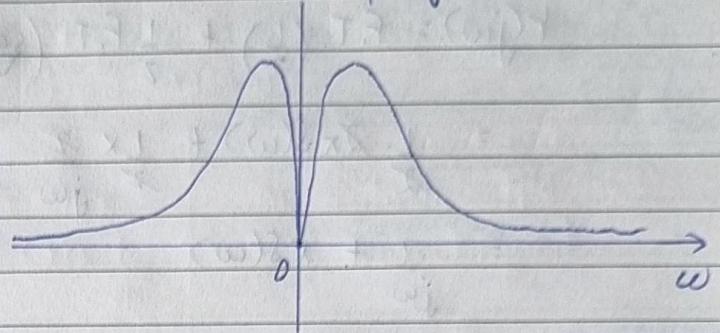
$$x(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t = 0 \\ -e^{-at} & t < 0 \end{cases}$$

where $a > 0$



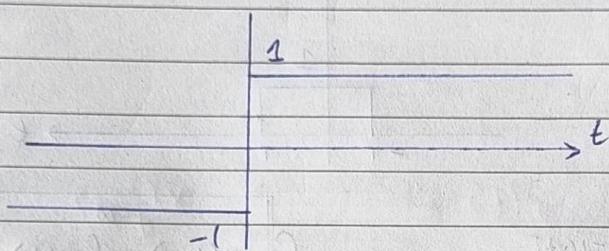
$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt - \int_{-\infty}^0 e^{at} e^{-j\omega t} dt$$

$$= \frac{-2j\omega}{a^2 + \omega^2}$$

 $|X(j\omega)|$ Case: let $a \rightarrow 0$

$$x(t) = \text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\psi(t) = \frac{1}{2} (1 + \text{sgn}(t)) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$



$$X(j\omega) = \frac{2}{j\omega}$$

Ex: 4

$$x(t) = \delta(t)$$

$$x(t) \xrightarrow{\frac{F.O.T.}{I.O.F.O.T.}} X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt$$

$$= 1$$

$$\delta(t) \xrightarrow{\quad} 1$$

$$1 \xrightarrow{\quad} 2\pi \delta(\omega)$$

Ex: 5

$$X(j\omega) = 2\pi \delta(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega) e^{j\omega t} d\omega$$

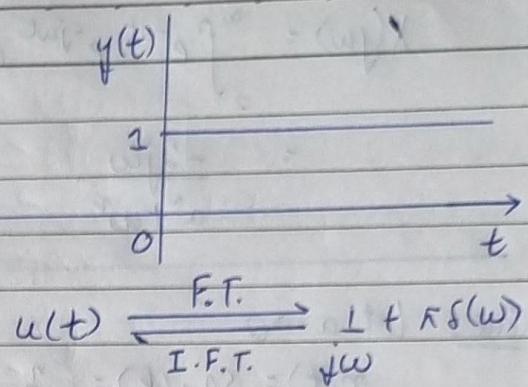
$$= 1$$

$$\text{Ex-6} \quad y(t) = \frac{1}{2} (1 + \text{sgn}(t)) = u(t)$$

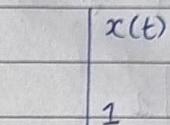
$$Y(j\omega) = \text{F.T.}\left(\frac{1}{2}\right) + \frac{1}{2} \text{F.T.}(\text{sgn}(t))$$

$$= \frac{1}{2} \cdot \frac{1}{j\omega} \pi S(\omega) + \frac{1}{j\omega} \times \frac{\pi}{j\omega}$$

$$= \frac{1}{j\omega} + \pi S(\omega)$$



$$x(t) = \begin{cases} 1 & -T \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

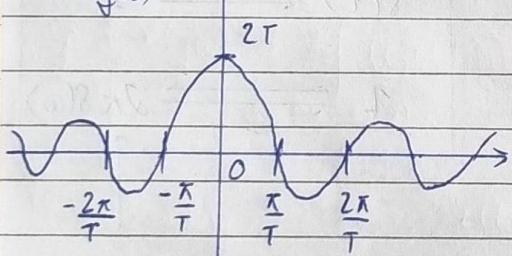


$$X(j\omega) = \int_{-T}^T e^{-j\omega t} dt$$

$$= \frac{e^{-j\omega T}}{-j\omega} \Big|_{-T}^T = \frac{e^{j\omega T} - e^{-j\omega T}}{j\omega}$$

$$= \frac{2}{\omega} \left(\frac{e^{j\omega T} - e^{-j\omega T}}{2j} \right) = \left(\frac{2}{\omega} \sin \omega T \right)$$

$$X(j\omega)$$

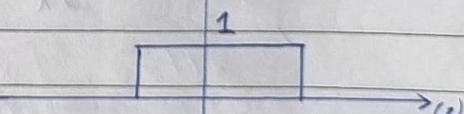
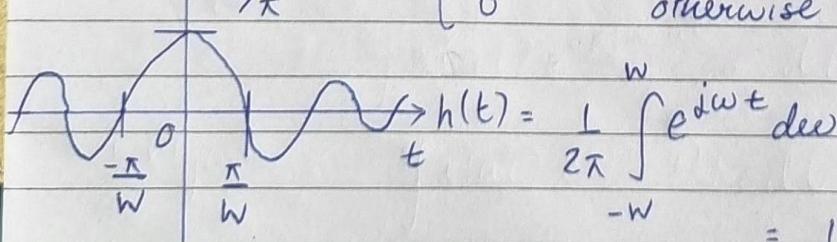


$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$H(j\omega) = \begin{cases} 1 & -W \leq \omega \leq W \\ 0 & \text{otherwise} \end{cases}$$

$$H(j\omega)$$



$$h(t) = \frac{1}{2\pi} \int_{-W}^W e^{j\omega t} d\omega = \frac{e^{j\omega t} - e^{-j\omega t}}{j2\pi t}$$

$$= \frac{1}{\pi t} \sin \omega t$$

Neelgagan

$\text{rect}\left(\frac{t}{2T}\right)$	$\xrightarrow{\text{F.T.}}$	$\frac{2 \sin \omega T}{\omega}$	time domain to freq
$\xleftarrow{\text{I.F.T.}}$			$t \rightarrow -\omega$
$\frac{1}{2\pi} \left\{ 2 \cdot \frac{\sin \omega t}{t} \right\}$	$\xrightarrow{\text{F.T.}}$	$\text{rect}\left(\frac{\omega}{2\omega}\right)$	$\times \text{ multiply by } 2\pi$
$\xleftarrow{\text{I.F.T.}}$			freq. domain to time dom
			$\omega \rightarrow -t$
			$\times \text{ divide by } 2\pi$

$$\text{rect}\left(\frac{t}{2T}\right) = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2 \sin \omega T}{\omega} \right) e^{j\omega t} d\omega \xrightarrow{\quad} x(j\omega)$$

Change $t \rightarrow -\omega$ by multiply by 2π

$$x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

↓
Change ω to $-t$

$$x(-t) = \int_{-\infty}^{\infty} x(-\omega) e^{j\omega t} d\omega$$

divide by 2π

$$x(-t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(-\omega) e^{j\omega t} d\omega$$

Ex

$$g(t) = \frac{2}{1+t^2} \quad \text{Find } G(j\omega).$$

$$G(j\omega) = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2} \right) e^{-j\omega t} dt$$

Ex

$$x(t) = e^{-|t|}$$

Prove it

$$X(j\omega) = \frac{2}{1+\omega^2}$$

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{j\omega t} d\omega$$

Charge t with $-\omega$

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{-j\omega t} d\omega$$

Properties — ① Linearity

$$\begin{array}{ccc} x(t) & \xleftrightarrow{} & X(j\omega) \\ g(t) & \xleftrightarrow{} & G(j\omega) \\ ax(t) + bg(t) & \xleftrightarrow{} & aX(j\omega) + bG(j\omega) \end{array}$$

② Differentiation in time domain

$$\begin{array}{ccc} x(t) & \xleftrightarrow{} & X(j\omega) \\ \frac{d}{dt}x(t) & \xleftrightarrow{} & j\omega X(j\omega) \end{array}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (X(j\omega) j\omega) e^{j\omega t} d\omega$$

③ Differentiation in the freq. domain

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\begin{array}{ccc} x(t) & \xleftrightarrow{} & X(j\omega) \\ tx(t) & \xleftrightarrow{} & j \frac{d}{dw} X(j\omega) \end{array}$$

$$\frac{d}{dw} X(j\omega) = \int_{-\infty}^{\infty} x(t) (-jt) e^{-j\omega t} dt$$

$$\int \frac{d}{dw} X(j\omega) = \int_{-\infty}^{\infty} (tx(t)) e^{-j\omega t} dt$$

④ Scaling

$$\begin{array}{ccc} x(t) & \xleftrightarrow{} & X(j\omega) \\ x(at) & \xleftrightarrow{} & \frac{1}{|a|} X(j\omega) \end{array}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(at) e^{-j\omega at} a \cdot dt$$

$$\frac{1}{a} X(j\omega) = \int_{-\infty}^{\infty} x(at) e^{-j(\omega/a)t} dt$$

$$a > 0$$

$$t = at$$

$$dt = adt$$

$$\omega' = \omega a$$

$$\omega = \frac{\omega}{a}$$

$$\frac{1}{a} X(j\omega) = \int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt \Rightarrow$$

⑤ Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\begin{aligned} \int_{-\infty}^{\infty} x(t)x^*(t) dt &= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t} d\omega \right] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left(\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \quad \text{proved} \end{aligned}$$

$|X(j\omega)|^2 \equiv \text{Energy spectrum}$

⑥ Convolution in time domain \Rightarrow Multiplication in the freq. domain

$$\int_{-\infty}^{\infty} x(t) h(t-\tau) dt \longleftrightarrow X(j\omega) H(j\omega)$$

$$\begin{aligned} x(t) &\longleftrightarrow X(j\omega) \\ h(t) &\longleftrightarrow H(j\omega) \end{aligned}$$

⑦ Multiplication in time domain \Rightarrow Convolution in the freq. domain

$$r(t) g(t) \longleftrightarrow \frac{1}{2\pi} (R(j\omega))^* G(j\omega)$$

Convolution

Ex. impulse response $h(t) = e^{-at} u(t) \quad a > 0$

$$\begin{aligned} &\int \frac{d}{d\omega} \left(\frac{1}{at+j\omega} \right) \\ &= \frac{1}{(at+j\omega)^2} \end{aligned}$$

i/p $x(t) = e^{-bt} u(t) \quad b > 0$

$y(t) = x(t) * h(t)$

Case I: $a = b$

$$X(j\omega) = \frac{1}{b+j\omega}$$

$$H(j\omega) = \frac{1}{a+j\omega}$$

$$\therefore a=b \Rightarrow X(j\omega) H(j\omega) = \left(\frac{1}{a+j\omega}\right)^2 = P(j\omega) = \frac{d}{d\omega} \left(\frac{1}{a+j\omega}\right)$$

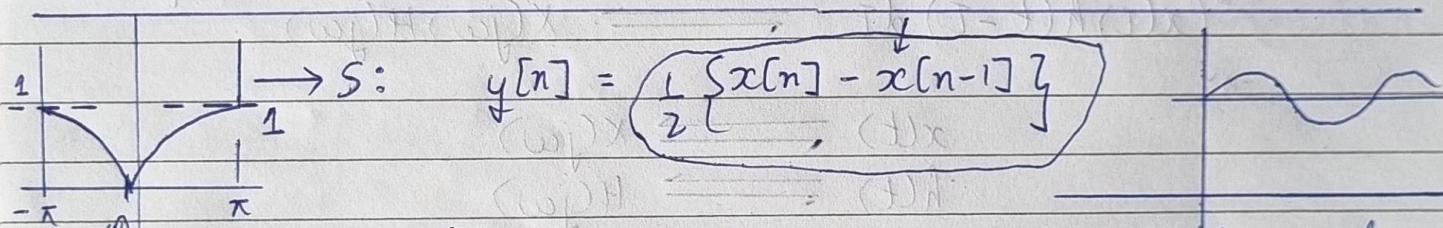
$$y(t) = te^{-at} u(t)$$

Case II $b \neq a$

$$Y(j\omega) = \left(\frac{1}{a+j\omega}\right) \left(\frac{1}{b+j\omega}\right)$$

$$Y(j\omega) = \frac{1}{b-a} \left[\frac{1}{(a+j\omega)} - \frac{1}{(b+j\omega)} \right]$$

$$y(t) = \frac{1}{b-a} e^{-at} u(t) - \left(\frac{1}{b-a}\right) e^{-bt} u(t)$$



Find the frequency response of the corresponding system -

$$\text{Let } x[n] = e^{j\omega n}$$

$$\Rightarrow e^{j\omega n} H(e^{j\omega})$$

$$y[n] = \frac{1}{2} e^{j\omega n} - \frac{1}{2} e^{j\omega(n-1)}$$

$$= e^{j\omega n} \left[\frac{1}{2} - \frac{1}{2} e^{-j\omega} \right]$$

$$x[n] \rightarrow \frac{1}{2} \{x[n] - x[n-1]\} = e^{j\omega n} \left[\frac{1}{2} - \frac{1}{2} e^{-j\omega} \right]$$

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{2}$$

$$H(e^{j\omega}) \cdot e^{j\omega n} = j e^{-j\omega/2} \left[e^{j\omega/2} - e^{-j\omega/2} \right]$$

z^n	$\rightarrow H(z) z^n$
$e^{j\omega n}$	$\rightarrow H(e^{j\omega}) e^{j\omega n}$

$$= j e^{-j\omega/2} \sin \frac{\omega}{2}$$

No.

Date

Ex $x(t) = \cos \omega t$

Find the : $\frac{e^{j\omega t} + e^{-j\omega t}}{2}$

F.T. of $x(t)$

$\left(\frac{1}{at+j\omega} \right)$