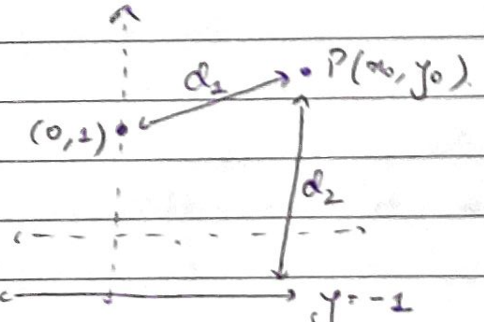
Q2



$$d_1 = d_2$$

$$\Rightarrow \sqrt{x_0^2 + (y_0-1)^2} = y_0 + 1$$

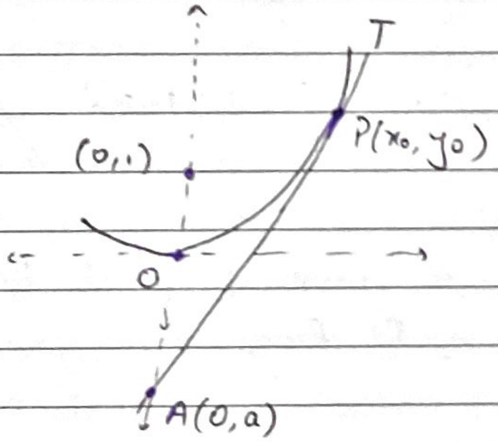
$$\Rightarrow x_0^2 + (y_0-1)^2 = (y_0+1)^2$$

$$\Rightarrow x_0^2 = 4y \quad //$$

(a) Hence, $f(x) = \frac{x^2}{4}$ [3 Marks]

Domain: $\mathbb{R}$, Range: $\mathbb{R}_{\geq 0}$ [2 Marks]

(b)



$$\left.\frac{df}{dx}\right|_{x=x_0} = \left.\frac{x}{2}\right|_{x=x_0} = \frac{x_0}{2}.$$

$$\therefore T: x_0 x - 2y = x_0^2 - 2y_0^2$$

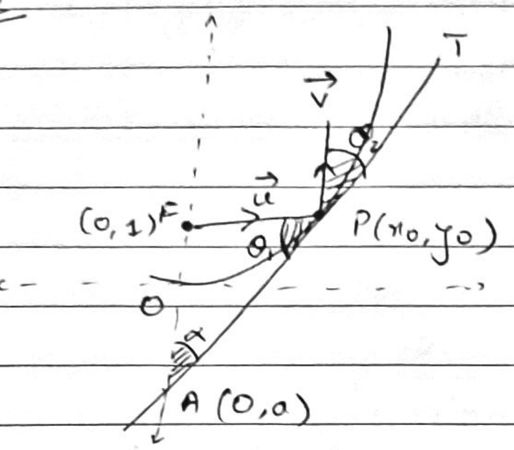
$$= \frac{x_0^2}{2} \quad //$$

[5 Marks]

Putting $x = 0$:

$$-2y = \frac{x_0^2}{2} \Rightarrow y = a = -\frac{x_0^2}{4} \quad //$$

Now



$\vec{u} = x_0\hat{i} + (y_0 - 1)\hat{j}$

$\vec{v} = 0\hat{i} + 1\hat{j}$

To prove: $\theta_1 = \theta_2$

Since $\vec{v}$ is $\parallel^{l}$ to the y-axis

$\alpha = \theta_2$.

Hence, it is enough to show that

$\theta_1 = \alpha$. [5 Marks]

$\Leftrightarrow |FA| = |FP|$

$$|FA| = 1 + |a| = 1 + \frac{x_0^2}{4}$$

$$|FP| = \|\vec{u}\| = \sqrt{x_0^2 + \left(\frac{x_0^2}{4} - 1\right)^2}$$

$$= \sqrt{\frac{x_0^4}{16} + 1 + \frac{x_0^2}{2}}$$

$$= \sqrt{\left(\frac{x_0^2}{4} + 1\right)^2} = 1 + \frac{x_0^2}{4} \quad //$$

[5 Marks]

→ Another approach can be using the inner product b/w $\vec{u}$ & the line, as well as $\vec{v}$ & the line.

Q3

Range of g : $(-\infty, -3] \cup (1, \infty)$.

Hence, the discontinuity of f at -1 does not impact the composition.

[1 Mark]

• For discontinuity of f at 3:

$g(x) = 3$

↳ $x^2+1 = 3$, $x \geq 0$

$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

$\Rightarrow x = \sqrt{2}$ $(\because x > 0)$ [2 Marks]

↳ $x-3 = 3$ $x \leq 0$

$\Rightarrow x = 6$ (Rejected since $x \leq 0$) [2 Marks]

→ Hence $f \circ g(x)$ is discontinuous at $x = \sqrt{2}$. [2 Marks]

→ At $x=0$: $\lim_{x\to 0^-} f(g(x)) = \lim_{x \to 0^-} f(x-3) = f(-3)$

$\lim_{x\to 0^+} f(g(x)) = \lim_{x\to 0^+} f(x^2+1) = f(1)$

Hence, $f \circ g(x)$ is discontinuous at $x=0$ if $f(1) \neq f(-3)$.

[3 Marks]

Q4

(a) Can easily verify that $f_x = g_x = 4x^3$

$f_y = -g_y = 4y^3$

All are 0 at $(0,0)$. [1 Mark]

(b) • $x^4 \geq 0^4 \quad \forall x \in \mathbb{R}$

$y^4 \geq 0^4 \quad \forall y \in \mathbb{R}$.

$\therefore \quad x^4 + y^4 \geq 0^4 \quad \forall (x,y)$

Hence f has a minima at $(0,0)$ [2 Marks]

• if $x > 0$ & $y = 0$: $x^4 - y^4 > 0$

$x = 0$ & $y > 0$: $x^4 - y^4 < 0$

Hence g has a saddle point at $(0,0)$ [2 Marks]

$\therefore$ (b) (i) No [1 Mark]

(ii) No [1 Mark]

(iii) No [1 Mark]

(c) Compute $f_{xx} f_{yy} - f_{xy}^2$ for f.

$g_{xx} g_{yy} - g_{xy}^2$ for g.

Both come out to be zero. Hence, this tells us that second derivative test is inconclusive when the discriminant is zero.

$\because$ f has a minima & g has a saddle pt.

[2 Marks]