# MULTIVARIATE CALCULUS
# MID SEMESESTER EXAM

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Abstract. This is a closed-book test with no cheat sheets allowed. There are six problems in total and two sections devoted to the notations and the definitions of essential concepts; the problems are numbered while the sections that define a concept are not. The last one is a bonus question. You have precisely 1 hour to finish the test.

## Notations and conventions

Throughout this article the symbol $\mathbb{F}$ will denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. By a *scalar* we shall always mean an element of the *scalar field* $\mathbb{F}$. All linear spaces (or vector spaces) in this set of notes are assumed to be over the scalar field $\mathbb{F}$.

Suppose $\mathcal{V}$ is a linear space over the scalar field $\mathbb{F}$. If $\mathbb{F} = \mathbb{R}$, $\mathcal{V}$ is called a *real linear space*; similarly if $\mathbb{F} = \mathbb{C}$, we speak of *complex linear spaces*. Any statement about linear spaces in which the scalar field is not explicitly mentioned is to be understood to apply to both of these cases.

## 1. Problem

Find and sketch the domain for the function $f(x,y) = \sqrt{(x^2-16)(y^2-25)}$. [*2 points*]

## 2. Problem

Deduce whether or not the following limit exists?

$$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$$

[*6 points*]

## 3. Problem

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Find the partial derivatives of the function (in two variables) $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \int_a^{\int_b^y g(s)ds} g(t)dt,$$

where $a, b \in \mathbb{R}$. [*05 points*]

*Date*: October 16, 2022.

## 4. Problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (xy)\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) Show that
  (i) $f_y(x,0) = x$ for all $x$, and
  (ii) $f_x(0,y) = -y$ for all $y$.

(b) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

*[10 points]*

## 5. Problem

Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

(a) Show that

$$\frac{\partial f}{\partial y}(x,y) = g_2(x,y).$$

(b) How should $f$ be defined (i.e. redefined) so that

$$\frac{\partial f}{\partial x}(x,y) = g_1(x,y).$$

(c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = x \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = y.$$

(d) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = x.$$

*[20 points]*

## Differentiation

Recall the following definition from your linear algebra course.

**Definition 5.1.** Suppose $\mathcal{V}$ and $\mathcal{W}$ are linear spaces over the scalar field $\mathbb{F}$. By a *linear transformation* from $\mathcal{V}$ to $\mathcal{W}$, we mean a mapping (or function) $T : \mathcal{V} \to \mathcal{W}$ from $\mathcal{V}$ to $\mathcal{W}$ such that

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

whenever $x, y \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$.

Let us now recall when a function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be *differentiable* at an interior point $(a,b)$ in the domain of $f$.

**Definition 5.2.** Let $E \subseteq \mathbb{R}^2$, let $(a, b)$ be an interior point of $E$, and let $f : E \to \mathbb{R}$. We say that $f$ is *differentiable* at $(a, b)$ (or, $f$ has a (total) derivative at $(a, b)$) if there exists (a unique) a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{h=(h_1,h_2)\to(0,0)} \frac{|f(a+h_1, b+h_2) - f(a,b) - L(h_1,h_2)|}{\sqrt{h_1^2 + h_2^2}} = 0.$$

When the total derivative exists, we set $f'(a, b) = L$, and the linear transformation $L$ is referred to as the derivative of $f$ at $(a, b)$.

**Remark 5.3.** It is perhaps worth recalling what would be the concrete nature of the linear transformation $L : \mathbb{R}^2\mathbb{R}$ if the function is differentiable at a point $(a, b)$.

$$L = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right),$$

so that

$$L(h_1,h_2) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)(h_1,h_2).$$

## 6. Bonus Problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Show that $f$ is differentiable at $(0, 0)$ but $f_x$ and $f_y$ are not continuous at $(0, 0)$.

[*10 points*]

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## 1. Problem

Find and sketch the domain for the function $f(x, y) = \sqrt{(x^2 - 16)(y^2 - 25)}$. [*2 points*]

## 2. Problem

Deduce whether or not the following limit exists?

$$\lim_{(x,y)\to(1,1)} \frac{xy^2 - 1}{y - 1}$$

[*6 points*]

The domain is the set of all points $(x,y)$ in $\mathbb{R}^2$ s.t. $(x^2-16)(y^2-25) \geq 0$. This is possible if either

Case I: $(x^2-16) \geq 0$ and $(y^2-25) \geq 0$

$\Rightarrow x^2 \geq 16$ and $y^2 \geq 25$

$\Rightarrow \{x \geq 4 \text{ or } x \leq -4\}$ and $\{y \geq 5 \text{ or } y \leq -5\}$.

$$\boxed{\{(x,y) \in \mathbb{R}^2 : |x| \geq 4 \text{ and } |y| \geq 5\}}$$ or if

Case II: $(x^2-16) < 0$ and $(y^2-25) < 0$

$\Rightarrow x^2 < 16$ and $y^2 < 25$

$\Rightarrow -4 < x < 4$ and $-5 < y < 5$

$$\boxed{\{(x,y) \in \mathbb{R}^2 : |x| < 4 \text{ and } |y| < 5\}}$$

$\therefore$ The domain of $f(x,y) =$

$$\underbrace{\{(x,y) \in \mathbb{R}^2 : |x| \geq 4, |y| \geq 5\}}_{(1 \text{ point})} \cup \underbrace{\{(x,y) \in \mathbb{R}^2 : |x| < 4, |y| < 5\}}_{(1 \text{ point})}$$

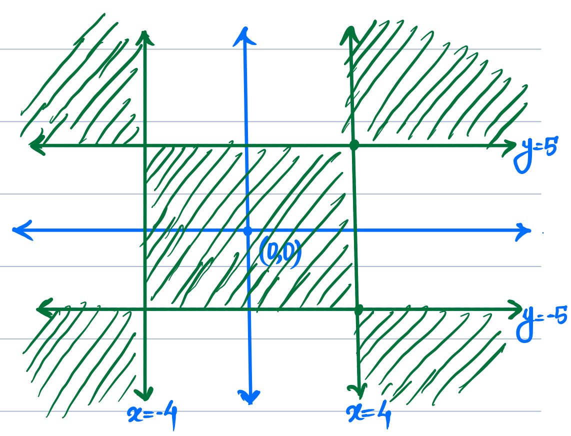
1

$\boxed{\text{Note for TAs!}}$ There are more than one ways in which the answer to this question can be written!

All of those are correct and deserve full marks.
Here are a few of those.

- $\{(x,y)\in\mathbb{R}^2 : |x|\geq 4, |y|\geq 5\} \cup \{(x,y)\in\mathbb{R}^2 : |x|<4, |y|<5\}$
- $\{(x,y)\in\mathbb{R}^2 : |x|>4, |y|>5\} \cup \{(x,y)\in\mathbb{R}^2 : |x|\leq 4, |y|\leq 5\}$
- $\{(x,y)\in\mathbb{R}^2 : x^2\geq 16, y^2\geq 25\} \cup \{(x,y)\in\mathbb{R}^2 : x^2<16, y^2<25\}$
- $\{(x,y)\in\mathbb{R}^2 : x^2>16, y^2>25\} \cup \{(x,y)\in\mathbb{R}^2 : x^2\leq 16, y^2\leq 25\}$
- The following shaded region is the domain of $f$.

However, students must express the domain algebraically, and not just graphically.



## 2. Problem

Deduce whether or not the following limit exists?

$$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$$

[6 points]

The limit does not exist. ← ① point

- Limit along the path $y=x$.

$$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{x\to 1} \frac{x^3-1}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$$

$$= 3$$

② points

- Limit along the path $x=1$

$$\lim_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1} = \lim_{y\to 1} \frac{y^2-1}{y-1}$$

$$= \lim_{y\to 1} \frac{(y-1)(y+1)}{(y-1)}$$

$$= 2$$

② points

- Since diff. paths yield diff. values and thus limit doesn't exist.

① point

## 3. Problem

Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Find the partial derivatives of the function (in two variables) $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) = \int_a^{\int_b^y g(s)ds} g(t)dt,$$

where $a, b \in \mathbb{R}$. [*05 points*]

$\frac{\partial f}{\partial x} = 0$. **Reason:** $f(x,y)$ is independent of the variable $x$.

1 point

In other words,

$\int_a^{\int_b^y g(s)ds} g(t)dt$ is a function of $y$ only.

1 point

$$\frac{\partial f}{\partial y} = g\left(\int_b^y g(s)ds\right) \cdot g(y)$$ 3 points

or, $$\frac{\partial f(c,d)}{\partial y} = g\left(\int_b^d g(s)ds\right) \cdot g(d)$$

or, $$\frac{\partial f(c,d)}{\partial y} = g\left(G(d) - G(b)\right) \cdot g(d)$$

where $G$ is the antiderivative of $g$.

## 4. Problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (xy)\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(a) Show that
  (i) $f_y(x,0) = x$ for all $x$, and
  (ii) $f_x(0,y) = -y$ for all $y$.
(b) Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

*[10 points]*

Recall that $f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

and $f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$

$\boxed{(a)}$ (i) For each $x \in \mathbb{R}$, we have

3 points

$$f_y(x,0) = \lim_{h\to 0} \frac{f(x,h)-f(x,0)}{h}$$

$$= \lim_{h\to 0} \frac{(xh)\left(\frac{x^2-h^2}{x^2+h^2}\right) - 0}{h}$$

$$= \lim_{h\to 0} x\left(\frac{x^2-h^2}{x^2+h^2}\right)$$

$$= x.$$

(ii) For each $y \in \mathbb{R}$, we get

3 points

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h}$$

$$= \lim_{h\to 0} \frac{(hy)\left(\frac{h^2-y^2}{h^2+y^2}\right) - 0}{h}$$

$$= \lim_{h\to 0} y\left(\frac{h^2-y^2}{h^2+y^2}\right) = -y.$$

$\boxed{(b)}$ Since $f_x(0,y) = -y$ for all $y \in \mathbb{R}$.

$$\Rightarrow f_{xy}(0,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,y)$$

$$= \frac{\partial}{\partial y}\left(f_x(0,y)\right)$$

$$= \frac{\partial}{\partial y}(-y)$$

$$= -1 \quad \text{for all } y \in \mathbb{R},$$

$\therefore$ In particular, $f_{xy}(0,0) = -1$

Since $f_y(x,0) = x$ for all $x \in \mathbb{R}$

$$\Rightarrow f_{yx}(x,0) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)(x,0)$$

$$= \frac{\partial}{\partial x}\left(f_y(x,0)\right)$$

$$= \frac{\partial}{\partial x}(x)$$

$$= 1 \text{ for all } x \in \mathbb{R},$$

and thus, $f_{yx}(0,0) = 1$.

Thus $\underbrace{f_{xy}(0,0) = 1}_{\leftarrow 2\text{ points}} \neq \underbrace{-1 = f_{yx}(0,0)}_{\leftarrow 2\text{ points}}$.

## 5. Problem

Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

(a) Show that

$$\frac{\partial f}{\partial y}(x,y) = g_2(x,y).$$

(b) How should $f$ be defined (i.e. redefined) so that

$$\frac{\partial f}{\partial x}(x,y) = g_1(x,y).$$

(c) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = x \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = y.$$

(d) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x,y) = y \quad \text{and} \quad \frac{\partial f}{\partial y}(x,y) = x.$$

*[20 points]*

[a] (5 points)

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\int_0^x g_1(t,0)dt + \frac{\partial}{\partial y}\int_0^y g_2(x,t)dt$$

$$= 0 + g_2(x,y)\cdot\frac{\partial}{\partial y}(y)$$

$$= g_2(x,y).$$

[b] (5 points)

Let $f(x,y) := \int_0^x g_1(t,y)dt + \int_0^y g_2(0,t)dt$

Then $\frac{\partial f}{\partial x} = g_1(x,y).$

[c] Let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

(5 points) Then clearly, $\frac{\partial f}{\partial x} = x$

and $\frac{\partial f}{\partial y} = y$

[d] On the other hand, if $f(x,y) = xy$

(5 points) Then $\frac{\partial f}{\partial x} = y$

and $\frac{\partial f}{\partial y} = x$

Remark for TAs: The solutions for parts (b), (c) and (d) can be different from what I suggested.

Be Cautious and use your wisdom!

## 6. Bonus Problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

Show that $f$ is differentiable at $(0,0)$ but $f_x$ and $f_y$ are not continuous at $(0,0)$.

[*10 points*]

We need to simply use Definition 5.2 at the point $(0,0)$.

Let $L : \mathbb{R}^2 \to \mathbb{R}$, be the linear transformation defined by the following $1\times 2$ matrix.

$$L := \left[\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right] \text{ so that,}$$

for an element $(\alpha,\beta) \in \mathbb{R}^2$, we get

matrix multiplication

$$L(\alpha,\beta) = \left[\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right]\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

<u>Now,</u> $$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^2 \sin(1/|h|) - 0}{h} = 0$$

Similarly, $\frac{\partial f}{\partial y}(0,0) = 0$

Thus, $L(\alpha,\beta) = [0,0]\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$

$\boxed{2\text{ points}}$ for guessing/obtaining the correct $L : \mathbb{R}^2 \to \mathbb{R}$

Next, we verify that $f(x,y)$ is differentiable at $(0,0)$.

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - L(h_1,h_2)|}{\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{(h_1^2+h_2^2)\sin\left(\frac{1}{\sqrt{h_1^2+h_2^2}}\right)}{\sqrt{h_1^2+h_2^2}}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \sqrt{h_1^2+h_2^2}\cdot\sin\left(1/\sqrt{h_1^2+h_2^2}\right)$$

$= 0$. Thus, $f$ is differentiable at $(0,0)$.

$\boxed{4\text{ points}}$

Next, for any point $(x,y) \neq (0,0)$, we can compute:

$$f_x(x,y) = 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - x \frac{\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}};$$

$$f_y(x,y) = 2y \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - y \frac{\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}.$$

We already know that $f_x(0,0) = f_y(0,0) = 0$.

Observe that as $(x,y) \to (0,0)$, the first terms in $f_x$ & $f_y$ both go to zero.

However, the second term in both $f_x$ and $f_y$ creates trouble, and thus the $f_x$ and $f_y$ are not continuous at $(0,0)$.

4 points