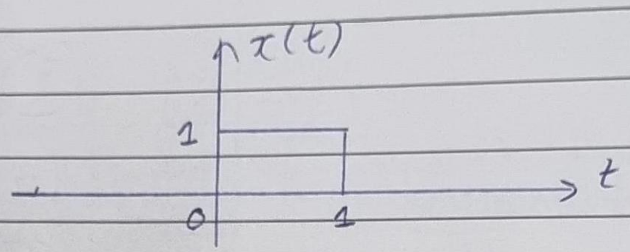
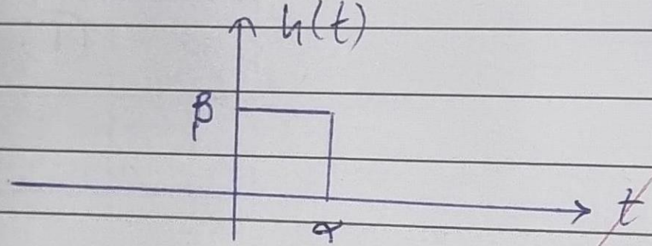
Ans)

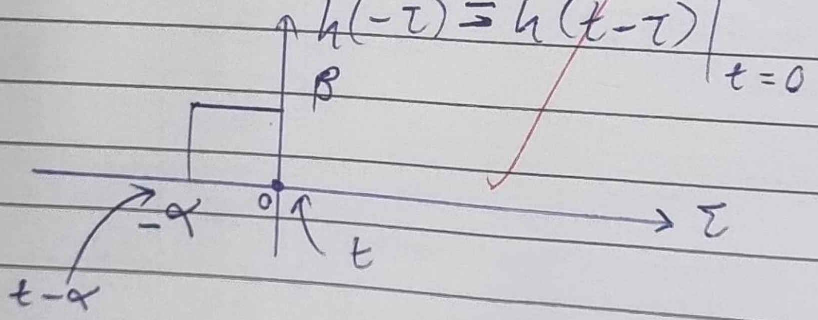
$$x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \beta\, x(t/\alpha)$$

(a) $y(t) = x(t) * h(t)$




$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Case-1 When $t<0$

$\Rightarrow$ No overlap b/w $x(\tau)$ & $h(t-\tau)$

$\Rightarrow$ $y(t)=0$

Case-2 $\quad 0 \le t < \alpha$

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^t 1 \cdot \beta \, d\tau$$

$$= \beta [\tau]_0^t = \beta t$$

+1.5

Case-3 $\quad \alpha \le t < 1$

$$y(t) = \int_{t-\alpha}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-\alpha}^{t} (1)(\beta)\, d\tau$$

$$= \beta [\tau]_{t-\alpha}^{t} = \alpha\beta$$

+1.5

Case-4 $\quad 1 \le t < \alpha+1$

$$y(t) = \int_{t-\alpha}^{1} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-\alpha}^{1} (1)(\beta) \cdot d\tau$$

$$= \beta[\tau]_{t-\alpha}^{1} = \beta(1-t+\alpha) = -\beta t + \alpha\beta + \beta$$

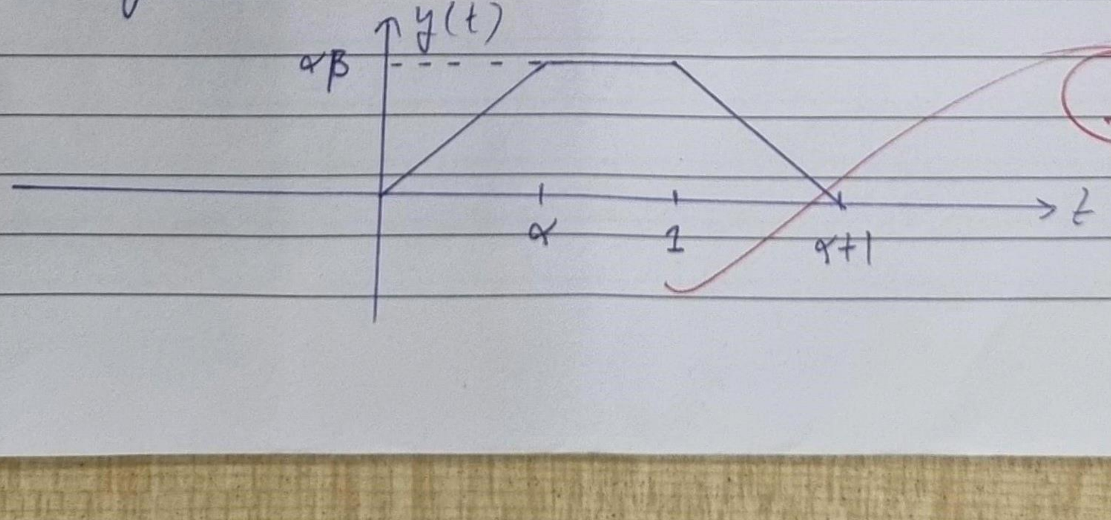
+1.5

Case-5 $\quad t \ge \alpha+1$

$\Rightarrow$ No overlap b/w $x(\tau)$ & $h(t-\tau)$

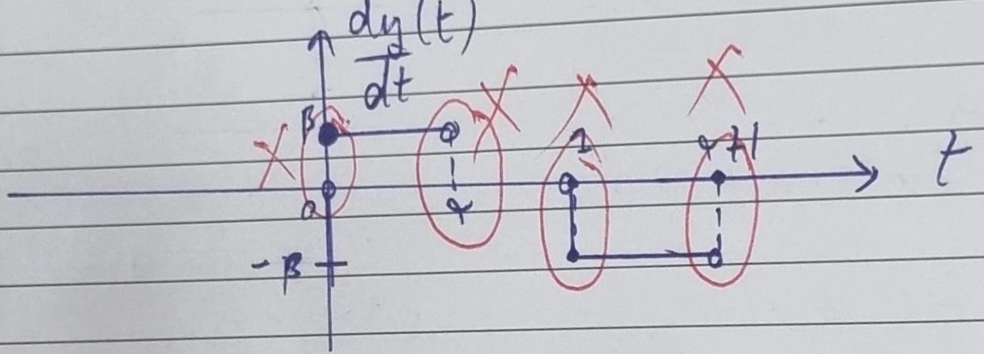
$\Rightarrow$ $y(t) = 0$

+0.25




+1

(b) $$\frac{dy(t)}{dt} = \begin{cases} 0 & t<0 \\ \beta & 0\le t<\alpha \\ 0 & \alpha \le t < 1 \\ -\beta & 1\le t<\alpha+1 \\ 0 & t\geq \alpha+1\end{cases}$$




For 3 points of discontinuity,

$\alpha = 1$ because then the no. of discontinuous points becomes 3.

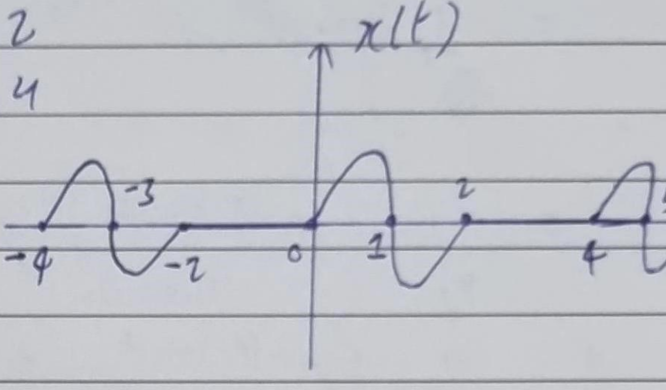
For $\beta$, there is no such restriction. ×3

$\Rightarrow \beta > 0$

$$\boxed{\begin{array}{c}\alpha = 1 \\ \beta > 0\end{array}}$$

Ans3

$$x(t) = \begin{cases} \sin \pi t & 0 \le t \le 2 \\ 0 & 2 < t \le 4 \end{cases}$$




$T = 4$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{2}t}$$

$$a_0 = \frac{1}{T}\int_T x(t)\,dt$$

$$= \frac{1}{4}\int_0^4 x(t)\,dt = \frac{1}{4}\left[\int_0^2 x(t)\,dt + \cancelto{0}{\int_2^4 x(t)\,dt}\right]$$

$$= \frac{1}{4}\int_0^2 \sin\pi t\,dt$$

$$= \frac{1}{4}\left[\frac{-\cos\pi t}{\pi}\right]_0^2$$

$$= \frac{1}{4\pi}\left((-\cos 2\pi) - (-\cos 0)\right) = 0$$ 1

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt \qquad k \neq 0$$

$$= \frac{1}{4}\int_0^4 x(t)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{4}\left[\int_0^2 x(t)e^{-jk\omega_0 t}\,dt + \cancelto{0}{\int_2^4 x(t)e^{-jk\omega_0 t}\,dt}\right]$$

$$= \frac{1}{4}\int_0^2 \sin\pi t\, e^{-jk\omega_0 t}$$

$$= \frac{1}{4}\int_0^2 \left(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right)e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{8j}\int_0^2 e^{j(\pi-kw_0)t} - e^{-j(\pi+kw_0)t}\, dt$$

$$= \frac{1}{8j}\left[\frac{e^{j(\pi-kw_0)t}}{j(\pi-kw_0)} + \frac{e^{-j(\pi+kw_0)t}}{+j(\pi+kw_0)}\right]_0^2$$

$$= \frac{1}{8j}\left[\frac{e^{j(\pi-kw_0)t}}{j(\pi-kw_0)} + \frac{e^{-j(\pi+kw_0)t}}{j(\pi+kw_0)}\right]_0^2$$

$$= \frac{1}{8j}\left[\frac{e^{j2(\pi-kw_0)}}{j(\pi-kw_0)} + \frac{e^{-j2(\pi+kw_0)}}{j(\pi+kw_0)}\right] - \left[\frac{1}{j(\pi-kw_0)} + \frac{1}{j(\pi+kw_0)}\right]$$

$$= \frac{1}{8j}\left[\frac{e^{-j2kw_0}}{j(\pi-kw_0)} + \frac{e^{-j2kw_0}}{j(\pi+kw_0)}\right] - \left[\frac{1}{j(\pi-kw_0)} + \frac{1}{j(\pi+kw_0)}\right]$$

$$= \frac{1}{8j}(e^{-j2kw_0} - 1)\left[\frac{1}{j(\pi-kw_0)} + \frac{1}{j(\pi+kw_0)}\right]$$

$$= \frac{1}{8\cancel{j}}(e^{-j2kw_0}-1)\left(\frac{2\pi}{j^2(\pi^2-k^2w_0^2)}\right)$$

$$= \frac{-1}{8}(e^{-j\pi k}-1)\frac{2\pi}{\left(\pi^2 - k^2\frac{\pi^2}{4}\right)}$$

$$= \frac{-1}{\cancel{8}}(e^{-j\pi k}-1)\frac{\cancel{8}}{(4-k^2)\pi}$$

$$= \frac{1-e^{-j\pi k}}{(4-k^2)\pi}$$

6

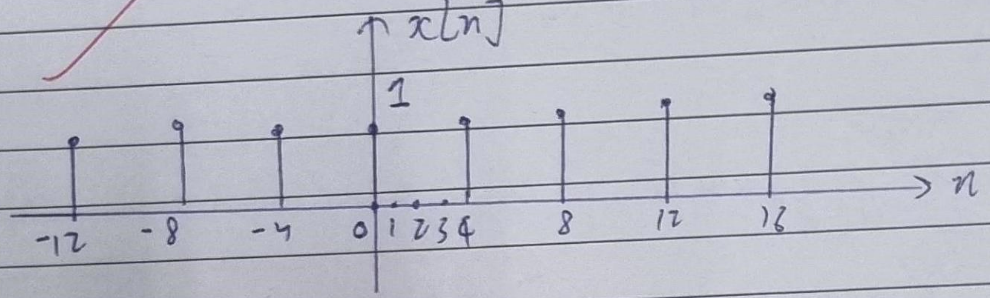
$$a_k = \begin{cases} 0 & k=0 \\ \dfrac{1-e^{-j\pi k}}{(4-k^2)\pi} & k \neq 0 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi/2 t}$$

Ans 4

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$$

$$= \dots + \delta[n+8] + \delta[n+4] + \delta[n] + \delta[n-4] + \delta[n-8] + \dots$$




$$N = 4$$

$$\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x[n] = \sum_{k=\langle 4\rangle} a_k e^{jk\frac{\pi}{2}n}$$

$$a_0 = \frac{1}{N} \sum_{k=\langle N\rangle} x[k]$$

$$= \frac{1}{4} \sum_{k=0}^{3} x[k] = \frac{1}{4}(1+0+0+0) = \frac{1}{4}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk\frac{\pi}{2}n} \qquad k \neq 0$$

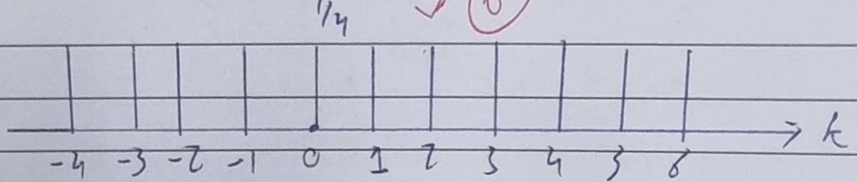
$$= \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n}$$

$$= \frac{1}{4}\left(1 \cdot e^{-jk\frac{\pi}{2}\cdot 0} + 0 + 0 + 0\right) = \frac{1}{4}$$

1.5

$$a_k = \frac{1}{4} \quad \forall\, k \in \mathbb{Z}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\pi/2\, n} \qquad (N=4)$$

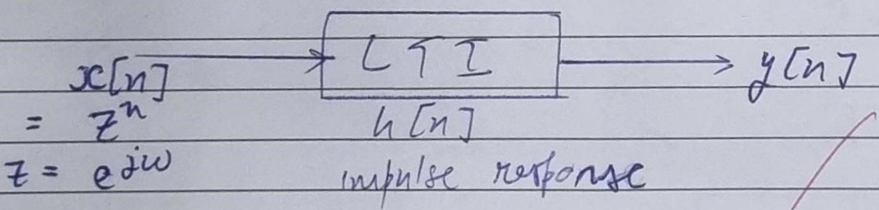



Line spectrum of $x[n]$

(b) $H(e^{j\omega})$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$y[n] = \sum_{k=\langle N\rangle} a_k H(e^{jk\omega_0}) n$$




$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$= z^n \boxed{\sum_{k=-\infty}^{\infty} h[k]\, z^{-k}} \rightarrow H(z)$$

$$= z^n H(z)$$

$$= e^{j\omega n} H(e^{j\omega}) \qquad [z = e^{j\omega}]$$

$\Rightarrow$ If $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$

then $y[n] = \sum_{k=\langle N \rangle} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$ [Property of LTI System]

$y[n] = \sum_{k=\langle 4 \rangle} a_k H(e^{jk\pi/2}) e^{jk\pi/2 n}$

for $x[n] = \sum_{k=\langle 4 \rangle} a_k e^{jk\pi/2 n}$

Also, $a_k = \frac{1}{4} \quad \forall k \in \mathbb{Z}$

$\Rightarrow y[n] = \frac{1}{4} H(e^{j0}) \cancel{e^{j0}}^{1} + \frac{1}{4} H(e^{j\pi/2}) e^{j\pi/2 n}$
$+ \frac{1}{4} H(e^{j\pi}) e^{j\pi n} + \frac{1}{4} H(e^{j3\pi/2}) e^{j3\pi/2 n}$ — ①

②

$y[n] = \cos\left(\frac{5\pi n}{2} + \frac{\pi}{4}\right)$

$= \frac{e^{j\left(\frac{5\pi n}{2} + \pi/4\right)} + e^{-j\left(\frac{5\pi n}{2} + \pi/4\right)}}{2}$

$= \frac{e^{j\pi/4}}{2}\left(e^{j\frac{5\pi n}{2}}\right) + \frac{e^{-j\pi/4}}{2}\left(e^{-j\frac{5\pi}{2}n}\right)$

$e^{j\frac{5\pi}{2}n} = \left(e^{j5\pi/2}\right)^n = \left(e^{j(2\pi + \pi/2)}\right)^n = e^{j\pi/2 n}$

$e^{-j5\pi/2 n} = \left(e^{-j5\pi/2}\right)^n = \left(e^{j(-2\pi - \pi/2)}\right)^n = e^{-j\pi/2 n}$

$= \left(e^{j(2\pi - 3\pi/2)}\right)^n = e^{j3\pi/2 n}$

$y[n] = \frac{e^{j\pi/4}}{2}\left(e^{j\pi/2 n}\right) + \frac{e^{-j\pi/4}}{2}\left(e^{j3\pi/2 n}\right)$ — ②

②

From eqⁿ ① & ② we get

$$\frac{1}{4}H(e^{j0}) + \frac{1}{4}H(e^{j\pi/2})e^{j\pi/2 n} + \frac{1}{4}H(e^{j\pi})e^{j\pi n} + \frac{1}{4}H(e^{j3\pi/2})e^{j\frac{3\pi}{2}n}$$

$$= \frac{e^{j\pi/4}}{2}(e^{j\pi/2 n}) + \frac{e^{-j\pi/4}}{2}(e^{j3\pi/2 n})$$

$$\Rightarrow \boxed{\begin{aligned} H(e^{j0}) &= 0 \\ H(e^{j\pi}) &= 0 \end{aligned}}$$

+ 0.5

$$\frac{1}{4}H(e^{j\pi/2}) = \frac{e^{j\pi/4}}{2}$$

$$\boxed{H(e^{j\pi/2}) = 2e^{j\pi/4}}$$

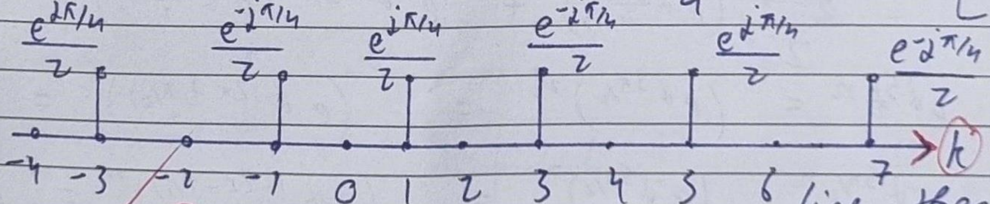
0.5 + 1

$$\frac{1}{4}H(e^{j3\pi/2}) = \frac{e^{-j\pi/4}}{2}$$

$$\boxed{H(e^{j3\pi/2}) = 2e^{-j\pi/4}}$$

(c) $y[n] = \sum_{k=\langle 4\rangle} a_k H(e^{jk\pi/2}) e^{jk\pi/2 n}$

F.S. Coeff $\rightarrow a_k H(e^{jk\pi/2}) = \frac{1}{4}H(e^{jk\pi/2})$ $\left[a_k = \frac{1}{4}\right]$




line spectrum of $y[n]$

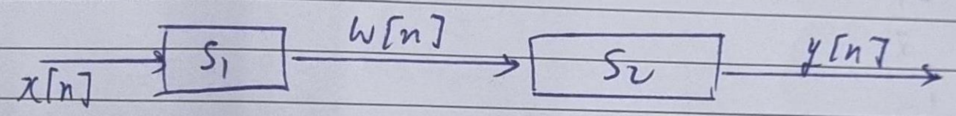
2

(d) Comparing line spectrum of $x[n]$ & $y[n]$ we can conclude that LTI system acts as a filter removing the frequencies of 0 and $2\omega_0$ (i.e. $\pi$) and making the amplitude of other 2 frequencies ($\omega_0$ & $3\omega_0$ i.e. $\frac{\pi}{2}$ & $\frac{3\pi}{2}$) as $\frac{1}{4}H(e^{j\pi/2})$ & $\frac{1}{4}H(e^{j3\pi/2})$ in the line spectrum of $y[n]$ where $H(e^{j\pi/2}) = 2e^{j\pi/4}$

& $H(e^{j3\pi/2}) = 2e^{-j\pi/4}$

1

Ans 2




$$S_1: \quad w[n] = \frac{1}{2}w[n-1] + x[n]$$

$$S_2: \quad y[n] = \alpha y[n-1] + \beta w[n]$$

$$y[n] = \alpha y[n-1] + \beta\left(\frac{1}{2}w[n-1] + x[n]\right) \quad - ①$$

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n] \quad - ②$$

From $S_2$: $\quad y[n-1] = \alpha y[n-2] + \beta w[n-1]$

$$w[n-1] = \frac{y[n-1] - \alpha y[n-2]}{\beta}$$

Putting this in eqn ①

$$y[n] = \alpha y[n-1] + \frac{\cancel{\beta}}{2}\left(\frac{y[n-1] - \alpha y[n-2]}{\cancel{\beta}}\right) + \beta x[n] \quad - ③$$

Comparing ③ & ②, we get

$-\frac{1}{8} y[n-2] + \frac{3}{4} y[n-1] + x[n] = -\frac{\alpha}{2} y[n-2] + \left(\frac{\alpha+1}{2}\right) y[n-1]$
$+ \beta x[n]$

Comparing coeff.

$\beta = 1$
$-\frac{\alpha}{2} = -\frac{1}{8}$
$\alpha = \frac{1}{4}$

X

(b) $w[n] = \frac{1}{2} w[n-1] + x[n]$

For S2

For impulse response $x[n] = \delta[n]$

$h[n] = \frac{1}{2} h[n-1] + \delta[n]$

For $n = 0$
$h[0] = 1$ $\left[h[-1] = 0\right]$ because system S1 is causal

$n = 1 \rightarrow h[1] = \frac{1}{2}$

$n = 2 \Rightarrow h[2] = \frac{1}{4}$

$\vdots$

$n \quad h[n] = \frac{1}{2^n}$

IIIT-D
SUPPLEMENTARY SHEET

नाम Name: Adit Goel

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दिनांक Date: 04-10-2024 (DD MM YYYY)

पाठ्यक्रम Course Title: Signals & Systems

पृष्ठ संख्या Sheet No.: 2

$\Rightarrow h[n] = \frac{1}{2^n} u[n]$ for S1 +1.5

For S2

$y[n] = \frac{1}{4} y[n-1] + w[n]$

For impulse response $w[n] = \delta[n]$

Use different notation

$h[n] = \frac{1}{4} h[n-1] + \delta[n]$

For $n=0 \rightarrow h[0] = 1$ $\quad [h[-1]=0] \leftarrow$ because S2 is causal

$n=1 \rightarrow h[1] = 1/4$

$n=2 \rightarrow h[2] = 1/4^2$

$\vdots$

$n \rightarrow h[n] = 1/4^n$

+1.5

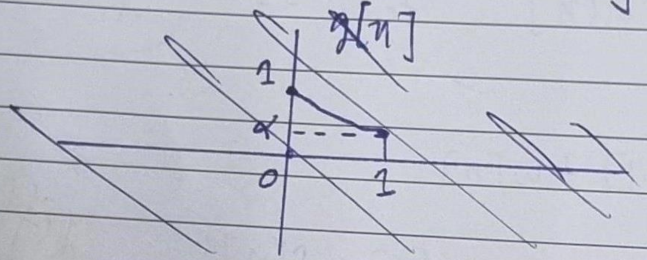
$\Rightarrow h[n] = \frac{1}{4^n} u[n]$ for S2

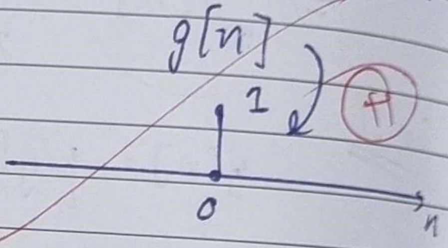
$h[n] = h_1[n] * h_2[n]$

$\rightarrow$ Impulse response of Cascaded system.

Ans (a) $x[n] = \alpha^n u[n]$

$$g[n] = x[n] - \alpha x[n-1]$$
$$= \alpha^n u[n] - \alpha \cdot \alpha^{n-1} u[n-1]$$
$$= \alpha^n \left[u[n] - u[n-1]\right] = \alpha^n \delta[n] = \alpha^0 \delta[n]$$




(b) $x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} \cancel{\alpha^n} \, x[k]\, h[n-k]$$
$$= \sum_{k=-\infty}^{\infty} \alpha^k u[k]\, h[n-k]$$
$$= \sum_{k=0}^{\infty} \alpha^k h[n-k]$$

? $\because u[k] = 0$ for $k < 0$