END SEM EXAM SOLUTION

SOL(1): Given that – $x[n] \rightleftharpoons X(z)$

where, $X(z) = \log(1 + az^{-1}), \quad |z| > a$

$$\because \quad X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} (-n) \cdot x[n] \cdot z^{-n-1}$$

$$\therefore \quad -n \cdot x[n] \rightleftharpoons z \frac{d}{dz} X(z)$$

$$n \cdot x[n] \rightleftharpoons -z \frac{d}{dz} X(z) \quad \rightarrow (1 \text{ POINT})$$

By using the property of differentiation in freq.,

$$n \cdot x[n] \rightleftharpoons -z \cdot \frac{d\,X(z)}{dz}$$

$$n \cdot x[n] \rightleftharpoons -z \cdot \frac{1}{(1+az^{-1})} \cdot (-az^{-2})$$

$$n \cdot x[n] \rightleftharpoons \frac{az^{-1}}{1+az^{-1}}$$

$$n \cdot x[n] \rightleftharpoons 1 - \frac{1}{1+a \cdot z^{-1}}$$

$$\therefore n \cdot x[n] = \delta[n] - (-a)^n \cdot u[n]$$

$$\therefore x[n] = \frac{1}{n}\left\{\delta[n] - (-a)^n \cdot u[n]\right\}$$

→ (4 POINTS

SOL(2): Given that —

Causal LTI system with impulse response $h(t)$,

where $h(t)$ satisfies the differential eqn —

$$\frac{d}{dt}h(t) + 2\,h(t) = (e^{-4t})\,u(t) + 6 \cdot u(t)$$

Taking Laplace Transform of both side —

$$S\,H(S) + 2\,H(S) = \frac{1}{(S+4)} + \frac{6}{S}$$

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$$(S+2)\,H(S) = \frac{1}{(S+4)} + \frac{b}{S}$$

$$H(S) = \frac{S + b(S+4)}{S(S+2)(S+4)} \rightarrow \text{(1 POINT)}$$

$$\frac{Y(S)}{X(S)} = \frac{S + b(S+4)}{S(S+2)(S+4)}$$

$$(S^3 + 6S^2 + 8S)\,Y(S) = (b+1)S\,X(S) + 4b\,X(S)$$

Taking inverse Laplace transform of both side–

$$\frac{d^3}{dt^3}y(t) + 6\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t)$$

$$= (b+1)\frac{d}{dt}x(t) + 4b\,x(t) \qquad \text{—— (1)}$$

Given that– input to the system, $x(t) = e^{2t}, \ \forall t$

Output to the system, $y(t) = \frac{1}{6}e^{2t}, \ \forall t$

Put the above value in eqn (1), we get–

$$\frac{d^3}{dt^3}\left(\frac{1}{6}e^{2t}\right) + 6\cdot\frac{d^2}{dt^2}\left(\frac{1}{6}e^{2t}\right) + 8\cdot\frac{d}{dt}\left(\frac{1}{6}e^{2t}\right)$$

$$= (b+1)\frac{d}{dt}\left(e^{2t}\right) + 4b\cdot e^{2t}$$

$$\frac{8}{6}e^{2t} + 6\times\frac{4}{6}e^{2t} + 8\times\frac{2}{6}e^{2t} = (b+1)\cdot 2e^{2t} + 4b\cdot e^{2t}$$

$$\frac{8}{6} + 4 + \frac{16}{6} = 2(b+1) + 4b$$

$$\therefore\ b = 1$$

$\rightarrow$ (5 POINTS)

$\therefore$ System Function,

$$H(S) = \frac{S + (S+4)}{S(S+2)(S+4)} = \frac{2S+4}{S(S+2)(S+4)}$$

$\rightarrow$ (2 POINTS)

SOL(3): Given a causal discrete time LTI system, whose input $x[n]$ & output $y[n]$ are related by-

$$y[n] - \frac{1}{4} \cdot y[n-1] = x[n] \quad \text{--- (1)}$$

Let, $x[n] \xrightleftharpoons{DTFS} a_k$

$y[n] \xrightleftharpoons{DTFS} b_k$

Consider an input, $x[n] = e^{j\omega n}$

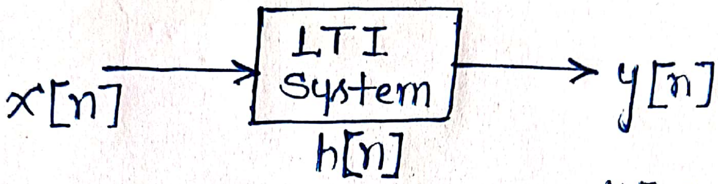
$\therefore$ Response to this input, $y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$

Substituting $x[n]$ & $y[n]$ in given difference eqn,

$$H(e^{j\omega}) \cdot e^{j\omega n} - \frac{1}{4} \cdot e^{-j\omega} \cdot e^{j\omega n} \cdot H(e^{j\omega}) = e^{j\omega n}$$

Therefore, $$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{4} \cdot e^{-j\omega}} \quad \text{--- (2)}$$

where $H(e^{j\omega})$ = Transfer function of LTI system



$$y[n] = x[n] * h[n]$$

If $x[n]$ is periodic with FTP 'N' then –

$$x[n] = \sum_{k=\langle N\rangle} a_k \cdot e^{jk\cdot\frac{2\pi}{N}\cdot n}$$

$$\therefore y[n] = \sum_{r=-\infty}^{\infty} h[r]\cdot x[n-r] = \sum_{r=-\infty}^{\infty} h[r]\cdot \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}(n-r)}$$

$$= \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \cdot \left\{ \sum_{r=-\infty}^{\infty} h[r]\cdot e^{-jk\frac{2\pi}{N}r} \right\}$$

$$= \sum_{k=\langle N\rangle} a_k \cdot e^{jk\frac{2\pi}{N}n} \cdot H\left(e^{j\frac{2\pi}{N}k}\right)$$

$$\therefore y[n] = \sum_{k=\langle N\rangle} a_k \cdot H\left(e^{j2\pi k/N}\right)\cdot e^{jk(2\pi/N)n}$$

$\rightarrow$ (2 POINTS)

For causal LTI system, we know that —

$$\boxed{y[n] = \sum_{k=\langle N \rangle} a_k \cdot H(e^{j2\pi k/N}) \cdot e^{jk(2\pi/N)n}}$$

$$b_k = H(e^{j2\pi k/N}) \cdot a_k \quad \text{——— (3)}$$

Given input, $x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\sin\left(\frac{\pi}{3}n\right)$

$$\omega_1 = \frac{\pi}{4} \qquad \omega_2 = \frac{\pi}{3}$$

$$N_1 = \frac{2\pi}{\omega_1} = 8 \qquad N_2 = \frac{2\pi}{\omega_2} = 6$$

$$\therefore N = LCM(N_1, N_2) = 24$$

$$\therefore \omega_0 = \frac{2\pi}{N} = \left(\frac{\pi}{12}\right)$$

$$x[n] = \frac{e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}}{2} + 2\cdot\frac{e^{j\frac{\pi}{3}n} - e^{-j\frac{\pi}{3}n}}{2j}$$

$$= \frac{1}{2}\left\{e^{j\cdot 3\cdot\frac{\pi}{12}n} + e^{-j\cdot 3\cdot\frac{\pi}{12}\cdot n}\right\}$$

$$+ \frac{2}{2j}\left\{e^{j\cdot 4\cdot\frac{\pi}{12}\cdot n} - e^{-j\cdot 4\cdot\frac{\pi}{12}\cdot n}\right\}$$

$$= \left\{\frac{1}{2}e^{j\cdot 3\cdot\frac{\pi}{12}n} + \frac{1}{2}e^{-j\cdot 3\cdot\frac{\pi}{12}\cdot n} + \frac{1}{j}e^{j\cdot 4\cdot\frac{\pi}{12}\cdot n} - \frac{1}{j}e^{-j\cdot 4\cdot\frac{\pi}{12}\cdot n}\right\}$$

$$= \sum_{K=3,-3,4,-4} a_K\cdot e^{jK\omega_0 n}$$

$\therefore$ Non zero FS coefficient of $x[n]$ are—

$$a_3 = a_{-3} = \left(\frac{1}{2}\right)$$

$$a_4 = \left(\frac{1}{j}\right)$$

$$a_{-4} = \left(\frac{-1}{j}\right)$$

$\rightarrow$ (4×0.~~5~~ POINT)

By eq^n(3)—

$$y[n] = \sum_{K=3,-3,4,-4} a_K\cdot H(e^{j2\pi K/N})\cdot e^{jK(2\pi/N)\cdot n}$$

$$= \sum_{K=3,-3,4,-4} b_K\cdot e^{jK(2\pi/N)n}$$

$\therefore$ Non zero FS coefficient of $y[n]$ are—

$$b_3 = a_{+3}\cancel{\times} H(e^{j3\cdot 2\pi/N}) = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}e^{-j\cdot 3\cdot\frac{\pi}{12}}}$$

$$= \frac{1}{2}\cdot\frac{4}{4-e^{-j\pi/4}} = \frac{2}{4-e^{-j\pi/4}}$$

$\rightarrow$ (0.25 POINT)

$$b_{-3} = a_{-3} \times H(e^{-j3\cdot 2\pi/N})$$

$$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4} \cdot e^{-j(-3)\cdot \frac{\pi}{12}}} = \frac{1}{2} \cdot \frac{4}{4 - e^{j\pi/4}}$$

$$= \frac{2}{4 - e^{j\pi/4}} \quad \rightarrow \text{(0.25 POINT)}$$

$$b_4 = a_4 \times H(e^{+j4\cdot 2\pi/N}) = \left(\frac{1}{j}\right) \cdot \frac{1}{1 - \frac{1}{4} \cdot e^{-j\times 4 \times \frac{\pi}{12}}}$$

$$= \frac{-4j}{4 - e^{-j\pi/3}} \quad \rightarrow \text{(0.25 POINT)}$$

$$b_{-4} = a_{-4} \times H(e^{-j4\cdot 2\pi/N}) = \left(\frac{-1}{j}\right) \cdot \frac{1}{1 - \frac{1}{4} \cdot e^{-j\times(-4)\times \pi/12}}$$

$$= \frac{4j}{4 - e^{j\pi/3}} \quad \rightarrow \text{(0.25 POINT)}$$

SOL (4): (a) Given causal LTI system is described by the difference eqⁿ,

$$y[n] - a\cdot y[n-1] = b\cdot x[n] + x[n-1]$$

Taking DTFT of both side, we get—

$$Y(e^{j\omega}) - a\cdot e^{-j\omega} \cdot Y(e^{j\omega}) = b \cdot X(e^{j\omega}) + e^{-j\omega} \cdot X(e^{j\omega})$$

$$\therefore H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(b + e^{-j\omega})}{(1 - a e^{-j\omega})}$$

$$\because |H(e^{j\omega})| = 1, \quad \forall \omega \quad \text{(Given)}$$

$$|b + e^{-j\omega}| = |1 - a \cdot e^{-j\omega}|$$

$$|b + \cos\omega - j\sin\omega| = |1 - a\cos\omega + ja\sin\omega|$$

$$(b+\cos\omega)^2 + (-\sin\omega)^2 = (1 - a\cos\omega)^2 + (a\sin\omega)^2$$

$$b^2 + \cos^2\omega + 2b\cos\omega + \sin^2\omega$$
$$= 1 + a^2\cos^2\omega - 2a\cos\omega + a^2\sin^2\omega$$

$$1 + b^2 + 2b\cos\omega = 1 + a^2 - 2a\cos\omega$$

This is possible only if $b = (-a)$ → (2 POINTS)

(b) Given that — $a = (-1/2)$ & input $x[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

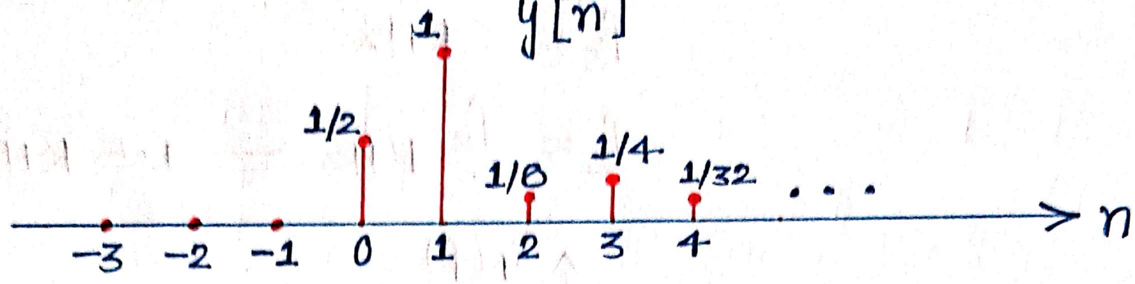
$$\therefore H(e^{j\omega}) = \frac{\frac{1}{2} + e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

also,
$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} \cdot e^{-j\omega}}$$

therefore,
$$Y(e^{j\omega}) = H(e^{j\omega}) \cdot X(e^{j\omega})$$
$$= \left(\frac{\frac{1}{2} + e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}\right)\left(\frac{1}{1 - \frac{1}{2}e^{-j\omega}}\right)$$
$$= \frac{5/4}{1 - \frac{1}{2} \cdot e^{-j\omega}} - \frac{3/4}{1 + \frac{1}{2} \cdot e^{-j\omega}}$$

Taking the inverse Fourier transform, we get —

$$y[n] = \frac{5}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{3}{4}\left(-\frac{1}{2}\right)^n \cdot u[n]$$ →(3 POINTS)

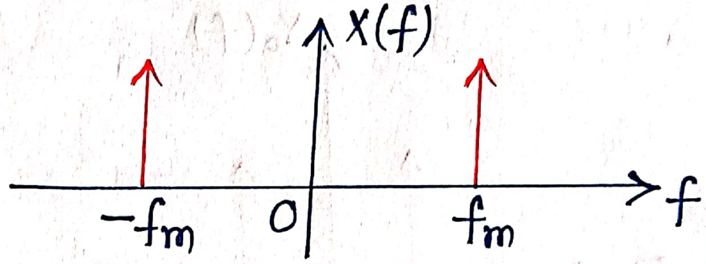


→ (1 POINT)

SOL(5): Given signal, $x(t) = 10 \cdot \cos(2\pi f_m t) \rightleftharpoons X(f)$

$$\therefore\ X(\omega) = \frac{10}{2}\left[\delta(\omega + \omega_m) + \delta(\omega - \omega_m)\right] \cdot 2\pi$$

$$X(\omega) = 5\left[\delta(\omega + \omega_m) + \delta(\omega - \omega_m)\right] \cdot 2\pi$$



$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) \quad , \text{ where } p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{K=-\infty}^{\infty} a_K \cdot e^{jK\omega_s t}$$

Periodic with period (T_s), $\omega_s = (2\pi/T_s) = (2\pi f_s)$

$$1 \rightleftharpoons 2\pi\,\delta(\omega)$$

$$e^{jK\omega_s t} \rightleftharpoons 2\pi\,\delta(\omega - K\omega_s)$$

$$\therefore a_K = \frac{1}{T_s}\int_{T_s} \delta(t) \cdot e^{-jK\omega_s t} \cdot dt = \frac{1}{T_s}$$

$$\therefore P(j\omega) = \sum_{K} \frac{1}{T_s} \cdot 2\pi\,\delta(\omega - K\omega_s) = \frac{2\pi}{T_s} \sum_{K=-\infty}^{\infty} \delta(\omega - K\omega_s)$$

$$\therefore X_P(j\omega) = \frac{1}{2\pi}\left\{ X(j\omega) * P(j\omega) \right\}$$

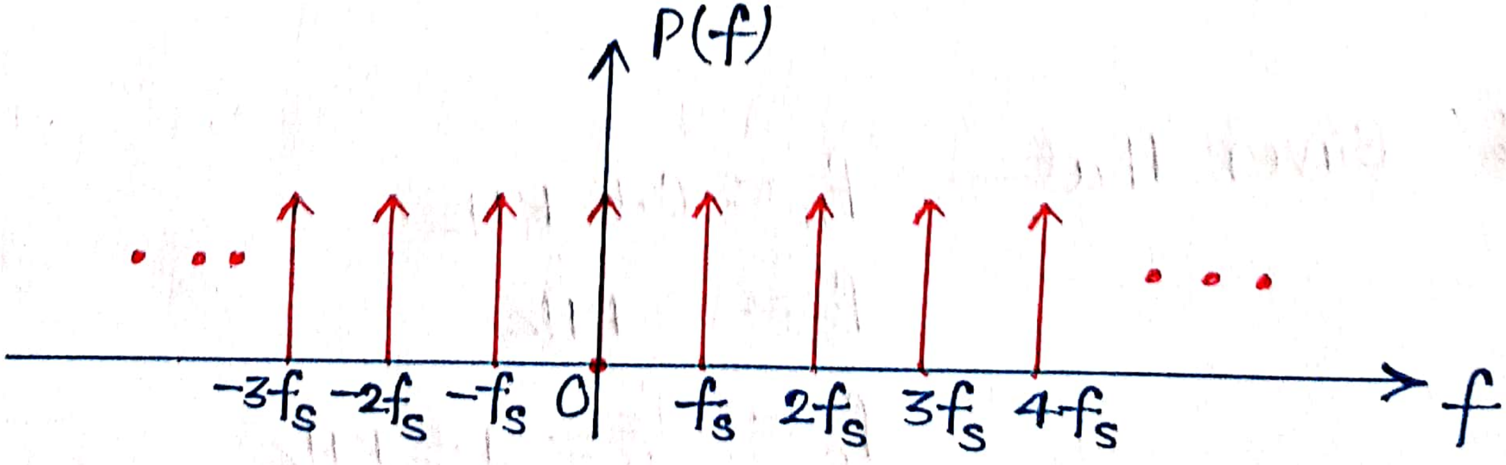
$$= \frac{1}{2\pi}\left\{ X(j\omega) * \left[\frac{2\pi}{T_s} \sum_{K=-\infty}^{\infty} \delta(\omega - K\omega_s) \right] \right\}$$

$$= \frac{1}{T_s}\, X[j(\omega - K\omega_s)]$$

$\therefore P(\omega) =$ Fourier transform of impulse train $[p(t)]$

$$= \frac{2\pi}{T_s} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

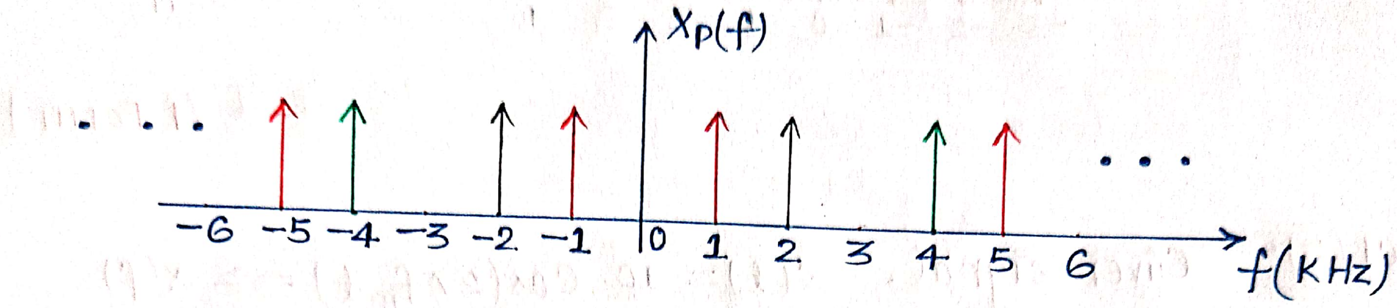
where, $f_s =$ sampling frequency



Part (a): Given that — $f_m = 2\ KHz$

$f_s = 3\ KHz$

$f_c = \frac{f_s}{2}\ KHz = 1 \cdot 5\ KHz$

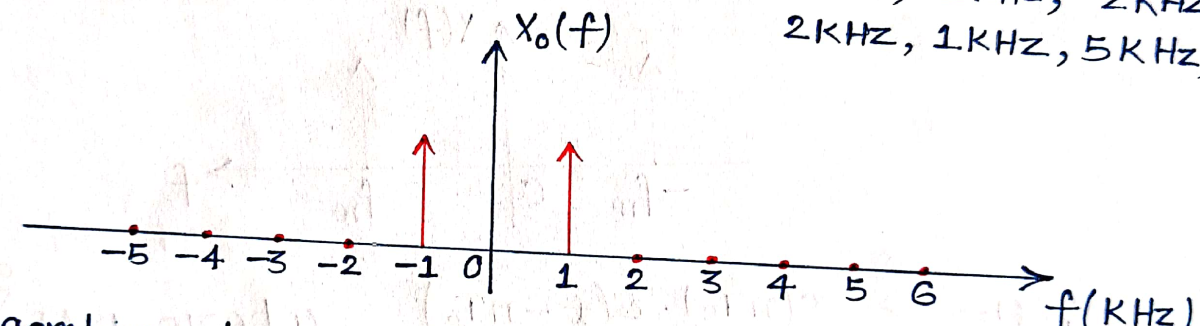


→ (1 POINT)

where, $X_p(f) = X(f) * P(f)$

Freq. components present in $X_p(f) = nf_s \pm f_m$

$\cdots -f_s \pm f_m, \pm f_m, f_s \pm f_m, \cdots \Rightarrow \cdots -5\ KHz, -1\ KHz, -2\ KHz,$
$2\ KHz, 1\ KHz, 5\ KHz, 4\ KHz, \cdots$



Freq components present in $X_o(f) = \pm 1\ KHz$ → (1 POINT)

Here, $X_o(f) = A \cdot [\delta(f+1) + \delta(f-1)]$

$\therefore\ x_o(t) = A \cdot \cos[2\pi \cdot (1) \cdot t] = A \cos(2\pi t)$

Frequency of reconstructed signal $= 1\ KHz$

→ (1 POINT)

Part (b): Given that — $f_m = 0 \cdot 5\ KHz$

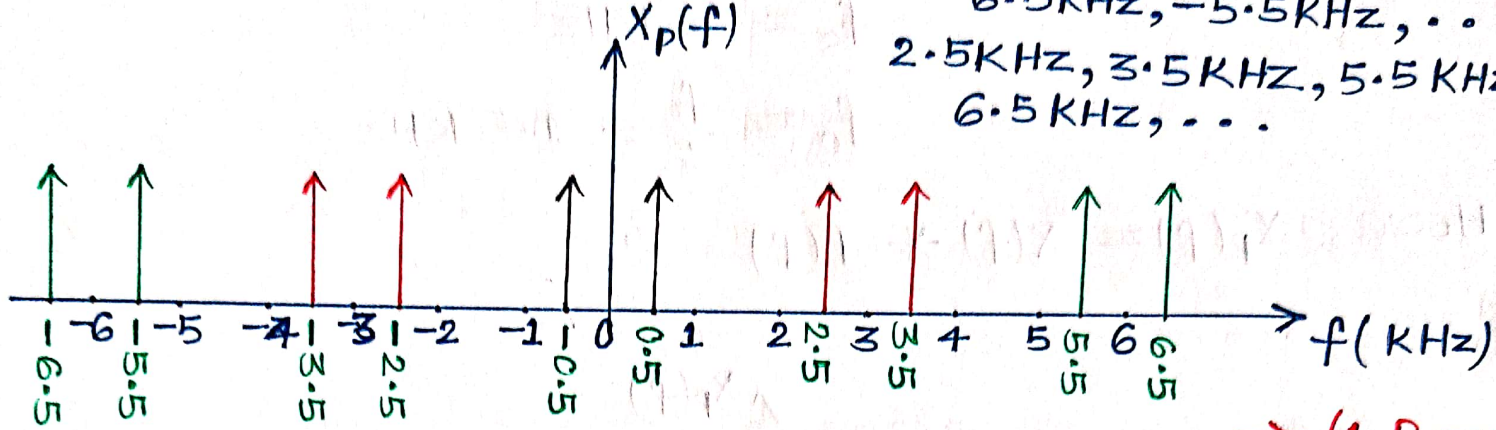
$f_s = 3\ KHz$

$f_c = \frac{f_s}{2} = 1 \cdot 5\ KHz$

$\because X_p(f) = X(f) * P(f)$

Freq. components present in $X_p(f)$
$= nf_s \pm f_m$

$\cdots -6.5\,KHz, -5.5\,KHz, \ldots,$
$2.5\,KHz, 3.5\,KHz, 5.5\,KHz,$
$6.5\,KHz, \ldots$



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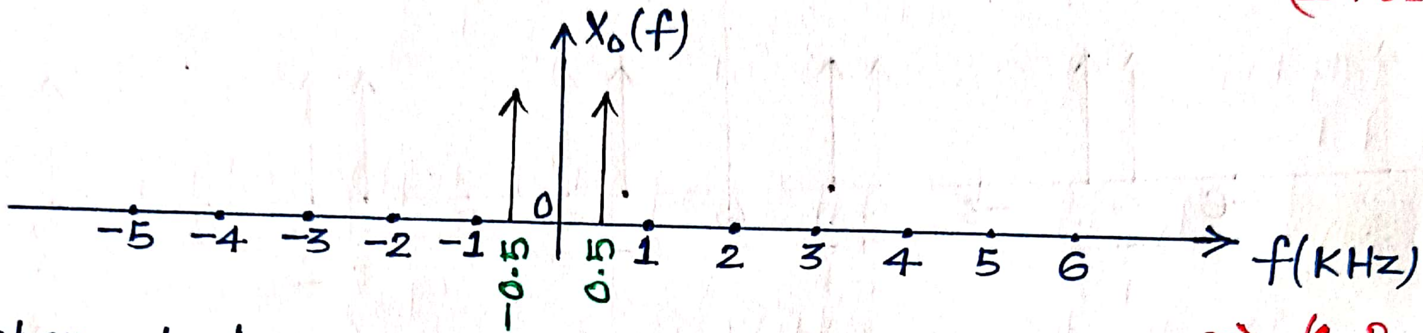


→ (1 POINT)

Freq components present in $X_0(f) = \pm 0.5\,KHz$

Here $X_0(f) = B\cdot[\delta(f+0.5) + \delta(f-0.5)]$

$\therefore x_0(t) = B\cdot \cos[2\pi(0.5)t] = B\cdot \cos(\pi t)$

Frequency of reconstructed signal $= 0.5\,KHz$

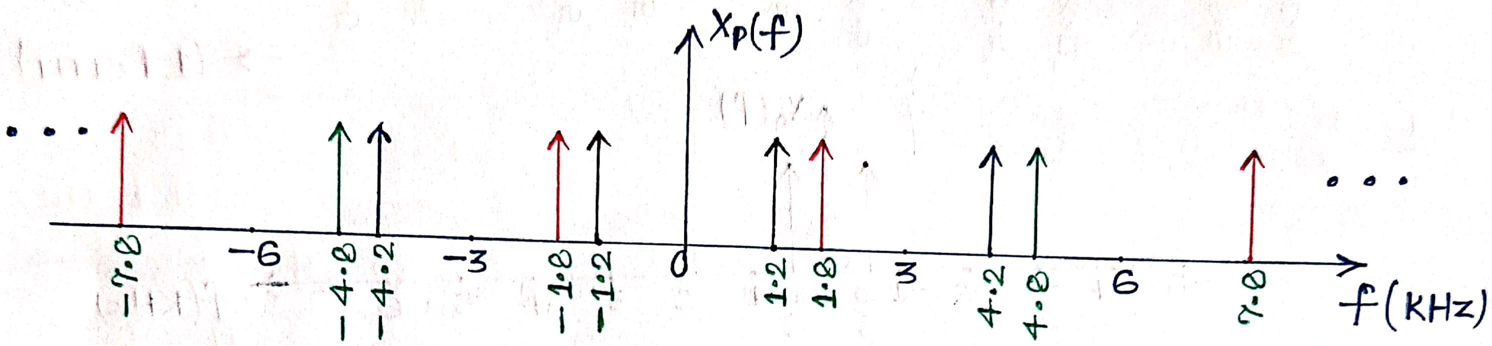
→ (1 POINT)

Part (C): Given that — $f_m = 4.8$ KHz

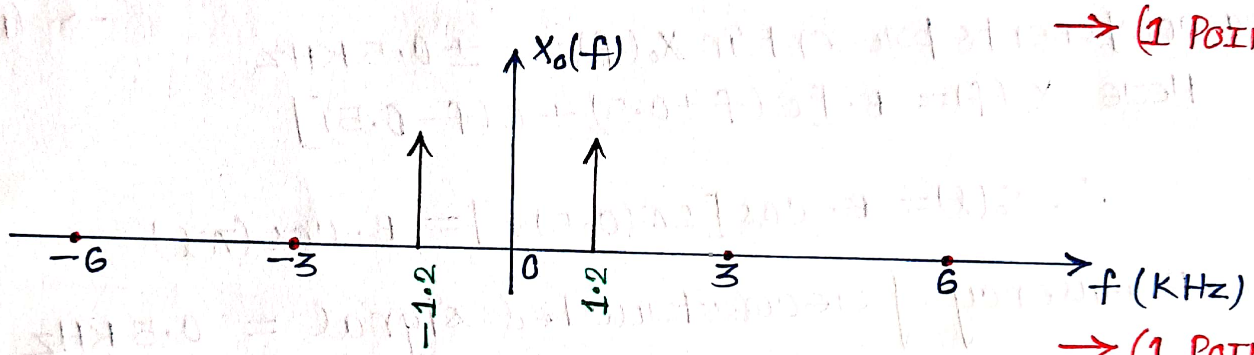
$f_s = 3$ KHz

$f_c = \frac{f_s}{2} = 1.5$ KHz

Here, $X_p(f) = X(f) * P(f)$



→ (1 POINT)



→ (1 POINT)

∴ Freq components present in $X_p(f) = nf_s \pm f_m$

$\dots -3f_s \pm f_m, -2f_s \pm f_m, -f_s \pm f_m, \pm f_m, f_s \pm f_m, 2f_s \pm f_m,$
$3f_s \pm f_m, \dots$

$\Rightarrow$ ... −13.8 KHz, −4.2 KHz, −10.8 KHz, −1.2 KHz, −7.8 KHz, 1.8 KHz, ±4.8 KHz, −1.8 KHz, 7.8 KHz, 1.2 KHz, 10.8 KHz, 4.2 KHz, 13.8 KHz, ...

∴ Freq components present in $X_o(f) = \pm 1.2$ KHz

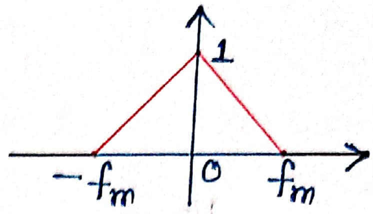
Here, $X_o(f) = c \cdot [\delta(f+1.2) + \delta(f-1.2)]$

$x_o(t) = c \cdot \cos[2\pi(1.2)t] = c \cdot \cos(2.4\pi t)$

∴ Freq. of reconstructed signal = 1.2 KHz

→ (1 POINT)

Q. (6): Given message signal $= x(t) \rightleftharpoons X(f)$



where, $f_m = 8\,KHz$

Given carrier signal $= \cos(\omega_c t) = \cos(2\pi f_c t)$

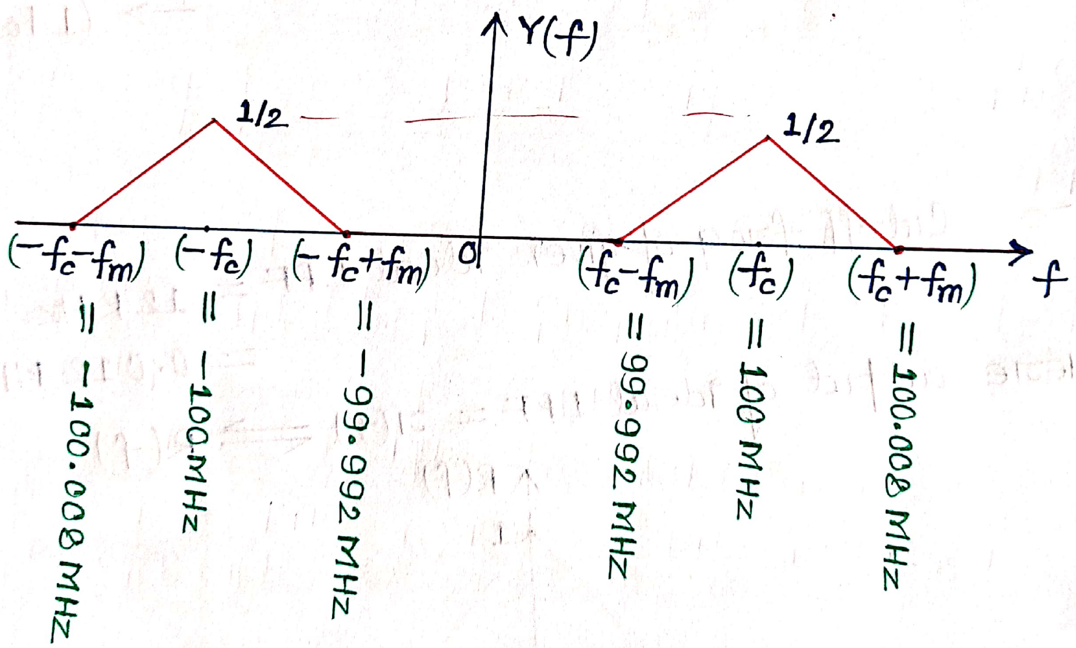
where, $f_c = 100\,MHz$

Part (a) —

$$y(t) = x(t) \times \cos(2\pi f_c t)$$

$$\therefore Y(\omega) = \left[X(\omega) * \left\{ \frac{\delta(\omega+\omega_c) + \delta(\omega-\omega_c)}{2} \right\} 2\pi \right] \frac{1}{2\pi}$$

$$= \frac{1}{2} \left\{ X(\omega+\omega_c) + X(\omega-\omega_c) \right\} \cancel{2\pi}$$

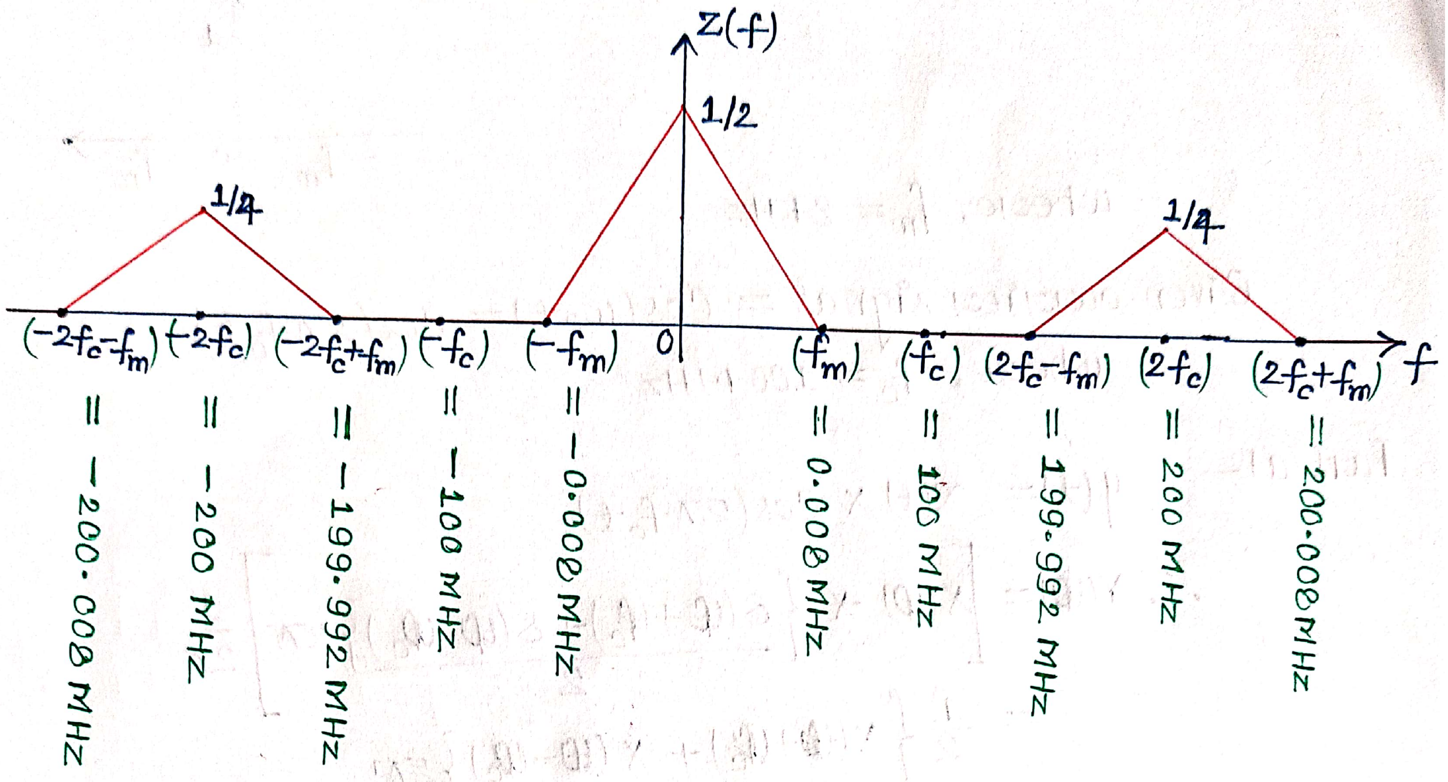


→ (1 POINT)

Part (b) — $z(t) = y(t) \times \cos(2\pi f_c t)$

$$\therefore Z(\omega) = \left[Y(\omega) * \frac{1}{2} \left\{ \delta(\omega+\omega_c) + \delta(\omega-\omega_c) \right\} 2\pi \right] \frac{1}{2\pi}$$

$$= \frac{1}{2} \left\{ Y(\omega+\omega_c) + Y(\omega-\omega_c) \right\} \cancel{2\pi}$$

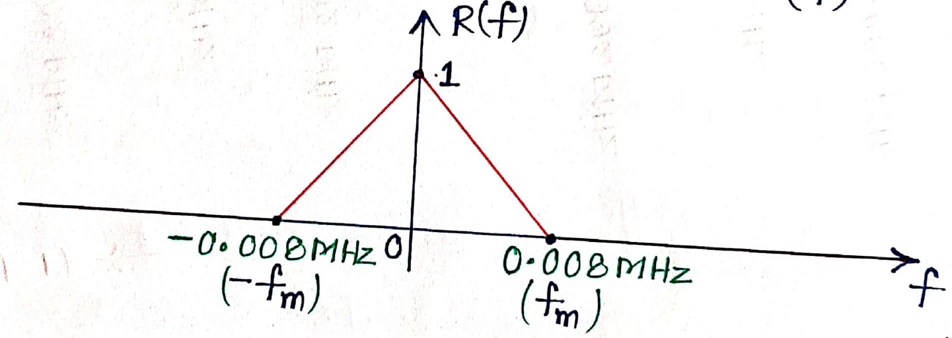


→ (1 POINT)

Part (c) — Cutoff freq of given ideal LPF $= 12\ KHz$

$= 0.012\ MHz$

Here output of ideal LPF $= r(t) \rightleftharpoons R(f)$



→ (1 POINT)

Part (d) — By the final output, we can say that $X(f)$ & $R(f)$ have identical frequency spectrum. ~~both are same signal~~. Hence, message signal $x(t)$ transmitted over ideal noiseless channel, can be recovered ~~received~~ perfectly ~~same~~ at the receiver ~~output~~ o/p.

→ (1 POINT)