6/11/2024

# D.T. F.T.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

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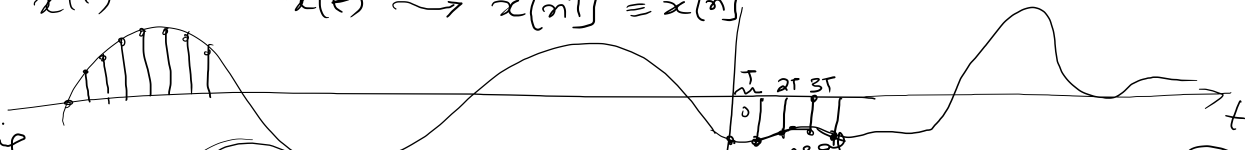
$x(t)$

$x(t) \rightsquigarrow x[nT] \equiv x[n]$




C.T. Signal

Signal is having continuum of values / amplitudes

$x(t)$ = continuum

as well as signal is also ranging over the time (independent) variable, where $t$ is also taking continuous of values

real D.T. Signal $x[n]$

C.T. Signal

$x : \mathbb{R} \to \mathbb{R}$

$x(t) \to$

$x : \mathbb{Z} \to \mathbb{R}$

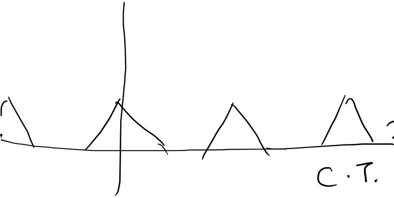
C.T. F.T.

$X : \mathbb{R} \to \mathbb{C}$

Time

$\tilde{x}(t)$ periodic

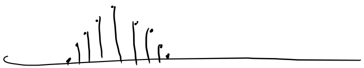
periodicity



$x(t)$

C.T. | aperiodic signal

Sampling



$x[n]$

D.T. | aperiodic

periodicity

$\tilde{x}[n]$

D.T. | periodic signal

C.T.F.S.

C.T.F.T

D.T.F.T.

D.T.F.S

Frequency

Line spectrum

Sample

$X(j\omega)$ (aperiodic

periodicity

$X(e^{j\omega})$ (periodic)

Sample

D.F. periodic spectrum

D.T.F.S.

$[0, N-1]$ frequency

DFT.

D.T.F.S.

Discrete Fourier Transform

Ex

$$\Rightarrow x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] = \boxed{\sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}}$$

periodic signal; fundamental time period $= N$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\boxed{\Rightarrow x[n] = a_0 + a_1 e^{j\omega_0 n} + a_2 e^{j2\omega_0 n} + \cdots + a_{N-1} e^{j(N-1)\omega_0 n}}$$

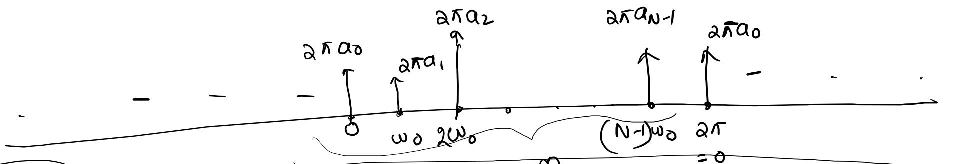
$0 - 2\pi$

$$X(e^{j\omega}) = 2\pi a_0 \delta(\omega) + 2\pi a_1 \delta(\omega - \omega_0) + \cdots + 2\pi a_{N-1} \delta(\omega - (N-1)\omega_0)$$

$$(N-1)\omega_0 = (N-1)\frac{2\pi}{N}$$

$$\downarrow$$

$$N\omega_0 = \cancel{N}\cdot\frac{2\pi}{\cancel{N}} = 2\pi \Downarrow 0$$




$a_k = \frac{1}{N}$

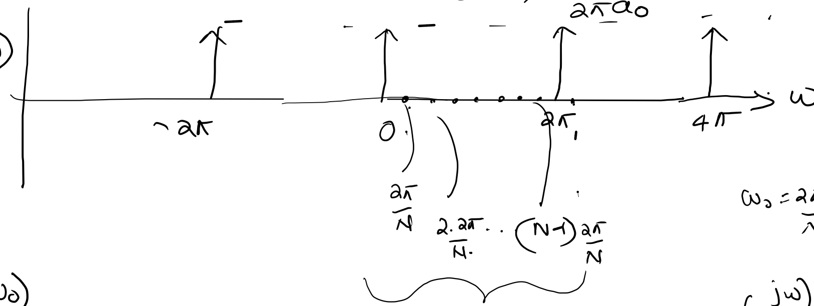
for $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$

$$\boxed{X(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)}$$

$x_0[n] = a_0$

$X_0(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_0 \, \delta(\omega - k.2\pi)$

$0 \leq \omega < 2\pi \quad ; \quad X_0(e^{j\omega}) = 2\pi a_0 \, \delta(\omega)$

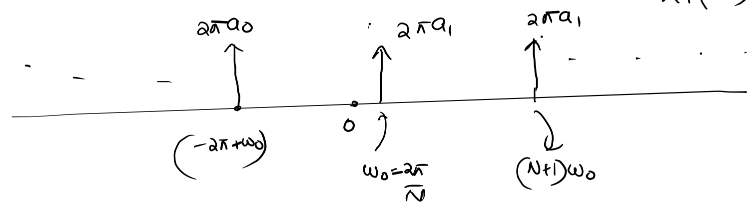



$\omega_0 = \frac{2\pi}{N}$

$x_1[n] = a_1 e^{j\omega_0 n}$

$0 \leq \omega < 2\pi \quad ; \quad X_1(e^{j\omega}) = 2\pi a_1 \, \delta(\omega - \omega_0)$

$X_1(e^{j\omega})$




$\delta(\omega) = 0 \quad \omega \neq 0$

$\int_{-\infty}^{\infty} \delta(\omega) \, d\omega = 1$

Generic D.T. periodic signal

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n}$$

$$X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \sum_{k=\langle N \rangle} a_k \delta(\omega - k\omega_0 - 2\pi l)$$

Ex

$$x[n] = \begin{cases} a^n & n \geq 0 \\ -a^{-n} & n < 0 \end{cases} \qquad 0 < a < 1$$

If $a = 1$

$$u[n] = \frac{1}{2}\left(x[n] + 1\right)$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} \left(-a^{-n}\right) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n - \sum_{n=1}^{\infty} a^n e^{j\omega n} - 1 + 1$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n - \sum_{n=0}^{\infty} \left(a e^{j\omega}\right)^n + 1$$

$$= \frac{1}{1 - a e^{-j\omega}} - \frac{1}{1 - a e^{j\omega}} + 1$$

$$= \frac{1}{1 - a e^{-j\omega}} - \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$= \frac{1 + a^2 - 2a e^{j\omega}}{1 - 2a\cos\omega + a^2}$$

If $a = 1$

$$X(e^{j\omega}) = \frac{2\left(1 \cancel{- e^{j\omega}}\right)}{\left(1 - e^{-j\omega}\right)\cancel{\left(1 - e^{j\omega}\right)}}$$

$$= \frac{2}{1 - e^{-j\omega}}$$

$$x[n] \longrightarrow \frac{2}{1-e^{-j\omega}}$$

$$u[n] = \frac{1}{2}\left\{x[n]+1\right\} \longrightarrow \frac{1}{1-e^{-j\omega}} + \pi\delta(\omega) = U(e^{j\omega})$$

$$0 \leq \omega < 2\pi$$

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## Properties

① Periodicity $\quad X(e^{j\omega}) = X\left(e^{j(\omega-2\pi)}\right)$

② Linearity

$$x_1[n] \rightleftharpoons X_1(e^{j\omega})$$

$$x_2[n] \rightleftharpoons X_2(e^{j\omega})$$

$$\alpha x_1[n] + \beta x_2[n] \rightleftharpoons \alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

③ <u>Time Shifting</u>

$$x[n] \rightleftharpoons X(e^{j\omega})$$

$$x[n-n_0] \rightleftharpoons e^{-j\omega n_0} X(e^{j\omega})$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x[n-n_0] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega(n-n_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} \underbrace{\left(X(e^{j\omega})\, e^{-j\omega n_0}\right)}\, e^{j\omega n}\, d\omega$$

(4) Shifting in the frequency domain

$$x[n] \rightleftharpoons X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \rightleftharpoons X\left(e^{j(\omega-\omega_0)}\right)$$

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$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X\left(e^{j(\omega-\omega_0)}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega-\omega_0)n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x[n]\, e^{j\omega_0 n}\right) \cdot e^{-j\omega n}$$

(5) Differencing in time domain

$$x[n] \rightleftharpoons X(e^{j\omega})$$

$$x[n] - x[n-1] \rightleftharpoons X(e^{j\omega})\left(1 - e^{-j\omega}\right)$$

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(6) Accumulation in the time domain

$$x[n] \rightleftharpoons X(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] * u[n] \rightleftharpoons X(e^{j\omega})\, U(e^{j\omega})$$

$$= \frac{X(e^{j\omega})}{1 - e^{-j\omega}} + \pi X(e^{j0})\,\delta(\omega)$$

$$0 \leq \omega < 2\pi$$

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(7) Convolution in time domain

$$x[n] \rightleftharpoons X(e^{j\omega}) \;; \quad h[n] \rightleftharpoons H(e^{j\omega})$$

$$x[n] * h[n] \rightleftharpoons X(e^{j\omega})\, H(e^{j\omega})$$

(8) Differentiation in the frequency domain

$$x[n] \rightleftharpoons X(e^{j\omega})$$

$$n\,x[n] \rightleftharpoons j\frac{d}{d\omega}X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega n}$$

$$j\frac{d}{d\omega}X(e^{j\omega}) = \sum_{n=-\infty}^{\infty}\Big(n\,x[n]\Big)e^{-j\omega n}$$

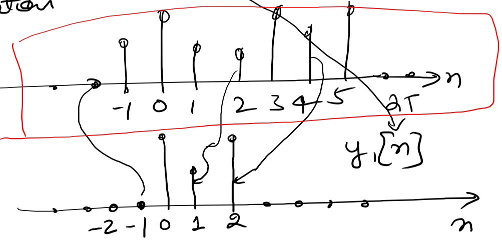
(9)

$x[n] \rightleftharpoons X(e^{j\omega})$ $\qquad x(t) \xrightarrow{T} x[n]$

downsampling

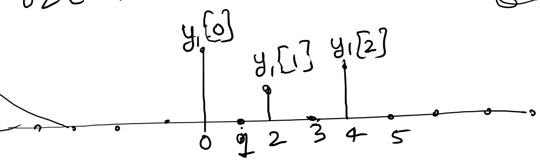
decimation

$y_1[n] = x[Ln] \Rightarrow$ pick $0^{th}$ sample, drop $(L-1)$ samples, pick $L^{th}$ sample ⋮




$$y_2[n] = \begin{cases} y_1[n/2] & ; \text{ when } n \text{ is a multiple of } 2 \\ 0 & ; \text{ otherwise} \end{cases}$$

$y_2[n]$ = inserted 1 zero between every consecutive sample of $y_1[n]$




$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Downsampling in the discrete domain

$x[n] \rightleftharpoons X(e^{j\omega})$ $\quad L = 2$

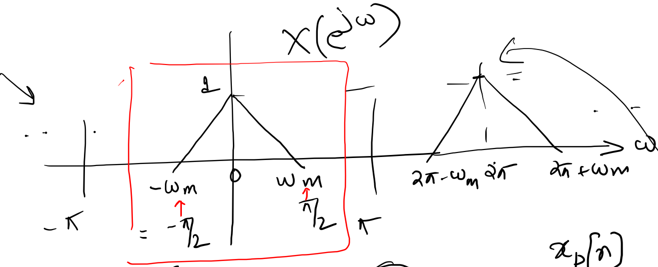
$\rightarrow y[n] = x[nL] \rightleftharpoons ?$

where $L$ is an integer.

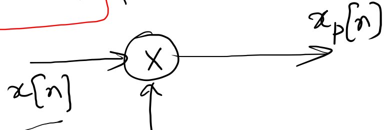
F.S.

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - Lk] = \sum_{k=\langle L\rangle} a_k e^{jk\frac{2\pi}{L}n}$$

$$\boxed{P(e^{j\omega}) = \frac{2\pi}{L} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\cdot\frac{2\pi}{L}\right)}$$



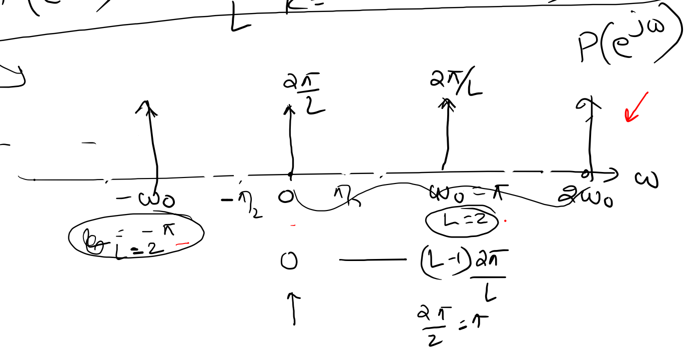

$\omega_m = \pi/2$




$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - Lk]$$

Let $L = 2$

$0 \le \omega < 2\pi$

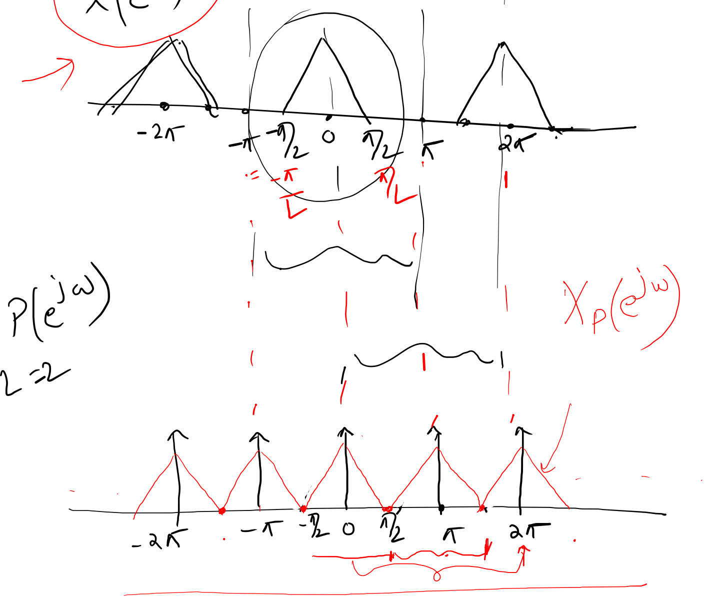


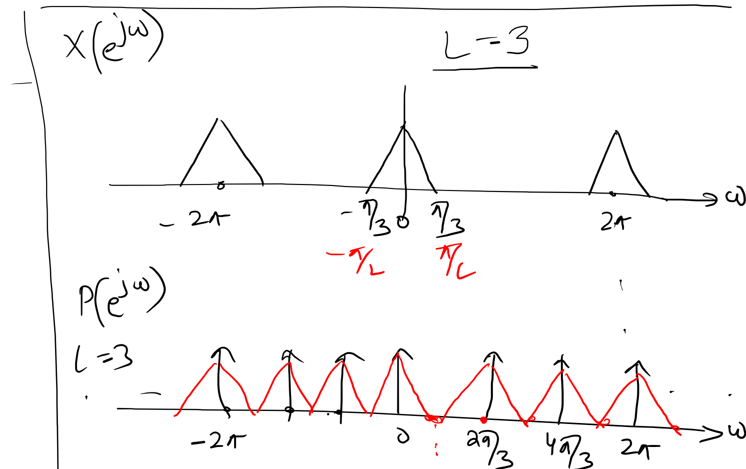
$X(e^{j\omega})$ $L=2$ $x_p[n]$ $L=2$

$2\pi$ interval

$$X_p(e^{j\omega}) = \frac{2\pi}{L}\sum_{k=0}^{L-1} X\left(e^{j\left(\omega-\frac{2\pi}{L}k\right)}\right)$$

$L=2$




$X_p(e^{j\omega})$ = red color

$0 \le \omega < 2\pi$

periodic convolution $\Rightarrow$

$y_1[n] = x[nL]$

$$\boxed{X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\, P(e^{j(\omega-\theta)})\, d\theta}$$

$$= \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{j\left(\omega - k\frac{2\pi}{L}\right)}\right)$$

convolution in the freq. domain

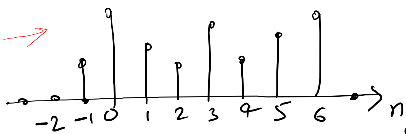
$$Y_1(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{j\left(\frac{\omega}{L} - \frac{2\pi k}{L}\right)}\right)$$

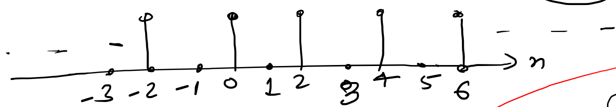
$$Y_1(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{j\frac{(\omega-2\pi k)}{L}}\right)$$

$L=2$

$x[n] \quad \times \quad p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kL]$

$L=2$

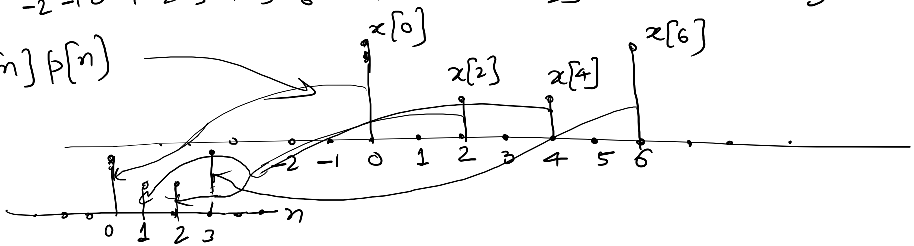




$x_p[n] = x[n]\,p[n]$

$\downarrow$

$y_1[n]$



$$x_p[n] = \begin{cases} y_1[n/L] & n \text{ is a multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{X_p(e^{j\omega/L}) = Y_1(e^{j\omega})}$$

Upsampling

$$y_2[n] = \begin{cases} x[n/L] & n = \text{multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

$y_2$ & $x$

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y_2[n]\, e^{-j\omega n}$$

$$Y_2(e^{j\omega}) = \sum_{n=\text{multiple of } L} x[n/L]\, e^{-j\omega n} + \sum_{n \neq \text{multiple of } L} 0 \cdot e^{-j\omega n}$$

$n = rL$

$r = \frac{n}{L}$

$$= \sum_{r=-\infty}^{\infty} x[r]\, e^{-j\omega r L} = \sum_{r=-\infty}^{\infty} x[r]\, e^{-j(\omega L) r}$$

$$\boxed{Y_2(e^{j\omega}) = X(e^{j\omega L})}$$

$x_p$ & $y_1$

$X_p(e^{j\omega}) = Y_1(e^{j\omega L})$

$X_p(e^{j\omega/L}) = Y_1(e^{j\omega})$

$$y_1[n] = x[Ln]$$

Let $L = 2$

$$y_1[0] = x[0]$$

$$y_1[1] = x[2]$$

$$y_1[2] = x[4]$$

$$y_1[3] = x[6]$$

$$\vdots$$

$$y_1[-1] = x[-2]$$

$$\vdots$$