Phasors Vs s-domain

Zero initial conditions & sinusoidally varying source "steady-state"

time-domain | **Freq. domain**

$v(t) = V_0 \cos(\omega t + \theta)$

$= \underline{Re}\{ V_0 \underbrace{e^{j\omega t + j\theta}} \}$

$V = V_0 e^{j\theta}$

Real frequency: ω

$j\omega L$

$v(t) = V_0 e^{\sigma t} \cos(\omega t + \theta)$

$= \underbrace{Re}\{ V_0 \underbrace{e^{\sigma t}} e^{j\theta} \underbrace{e^{j\omega t}} \}$

$e^{(\sigma + j\omega)t}$

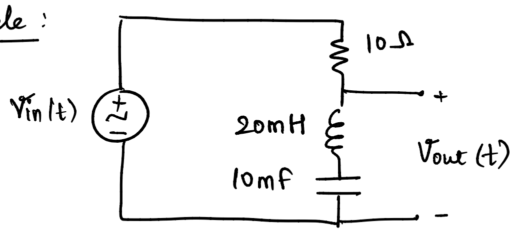
$V(s) = V_0 e^{j\theta}$

$s = \sigma + j\omega$

" complex frequency "

R	R
L	$L(\sigma + j\omega) = sL$
C	$\frac{1}{(j\omega + \sigma)C} = \frac{1}{sC}$

Example:



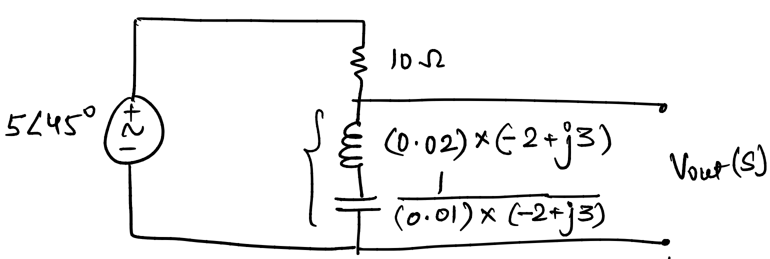
Consider $V_{in}(t) = 5e^{-2t}\cos(3t + 45^\circ)\,V$

Find s, $V_{out}(t)$.

$s = \sigma + j\omega = -2 + j3$

Freq. domain representation:

$V_{in}(s) = 5e^{j45^\circ} = 5\angle 45^\circ$



$$V_{out}(s) = 5\angle 45^\circ \times \frac{0.02 \times (-2+j3) + \frac{1}{(0.01)(-2+j3)}}{10 + (0.02)(-2+j3) + \frac{1}{(0.01)(-2+j3)}} = 5.86\angle -24.4^\circ$$

$$v_{out}(t) = \mathrm{Re}\{5.86\, e^{j(-24.4^\circ)} \cdot e^{(\sigma + j\omega)t}\}$$
$$= 5.86\, e^{-2t} \cos(3t - 24.4^\circ) \quad V$$

Note: For zero initial condition, we can use phasors for $e^{\sigma t}\cos(\omega t + \theta)$.

$R \to R$

$L \to sL$

$C \to \frac{1}{sC}$

Ques: Can we use phasors for $\delta(t)$, $u(t)$, $\sin(\omega t)\,u(t)$?

NO

s is variable

Laplace transform.

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} \cdot f(t)\, dt \implies F(s) =$$

Refer to table 14.1, 14.2

$\delta(t)$	1
$u(t)$	$1/s$
$t\,u(t)$	$1/s^2$
$e^{-at}\,u(t)$	$1/(s+a)$
$te^{-at}\,u(t)$	$1/(s+a)^2$
$\sin(\omega t)\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)\,u(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)\,u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

$f(t) \Longleftrightarrow F(s)$

$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$kf(t)$	$kF(s)$
df/dt	$sF(s) - f(0^-)$
$\int_{0^-}^{t} f(t)\,dt$	$F(s)/s$
$f_1(t) * f_2(t)$	$F_1(s)\,F_2(s)$
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$
$f(t)\,e^{-at}$	$F(s+a)$

Initial and final value theorems:

$$f(0^+) = \lim_{s\to\infty} sF(s)$$
$$f(\infty) = \lim_{s\to 0} sF(s) \quad \left\{ \text{all poles of } sF(s) \text{ lie in LHP} \right.$$

$$\mathcal{L}\{\sin(\omega t)\,u(t)\}$$

$$\sin(\omega t) = \frac{1}{2j}\underbrace{(e^{j\omega t} - e^{-j\omega t})}$$

$$= \mathcal{L}\left\{\frac{1}{2j}\{e^{j\omega t} - e^{-j\omega t}\}\,u(t)\right\}$$

$$= \frac{1}{2j}\left[\underbrace{\mathcal{L}\{e^{j\omega t}.u(t)\}} - \mathcal{L}\{e^{-j\omega t}.u(t)\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right]$$

$$= \left(\frac{1}{2j}\right)\frac{s+j\omega - s+j\omega}{(s-j\omega)(s+j\omega)} = \frac{1}{2j}\,\frac{2j\omega}{s^2+\omega^2} = \frac{\omega}{s^2+\omega^2}$$

Examples:

① Find $f(0^+)$ and $f(\infty)$ for $F(s) = \dfrac{4e^{-2s}(s+50)}{s}$

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{4e^{-2s}(s+50)}{\cancel{s}}\times\cancel{s}$$

$$= \lim_{s\to\infty}\frac{4(s+50)}{e^{2s}}$$

$$\frac{\infty}{\infty} = \text{indeterminent}$$

$$\frac{1}{\infty} = 0$$

L'Hospital's Rule: $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\dfrac{f'(x)}{g'(x)}$

$$f(0^+) = \lim_{s\to\infty}\frac{4(s+50)}{e^{2s}} = \lim_{s\to\infty}\frac{4}{2e^{2s}} = \frac{1}{\infty} = 0$$

Final value: $f(\infty) = \lim\limits_{s\to 0} sF(s) = \lim\limits_{s\to 0} 4e^{-2s}(s+50) = 200$

② Find $f(0^+)$ & $f(\infty)$ for $F(s) = \dfrac{5s^2+10}{2s(s^2+3s+5)}$

$$f(0^+) = \lim_{s\to\infty}\frac{\cancel{s}(5s^2+10)}{2\cancel{s}(s^2+3s+5)} = \lim_{s\to\infty}\frac{5+10/s^2}{2(1+3/s+5/s^2)} = \frac{5}{2}$$

$$f(\infty) = \lim_{s \to 0} \frac{5s^2+10}{2(s^2+2s+5)} = \frac{10}{2(5)} = 1$$

Examples find ILT of $F(s) = \dfrac{s^2+6}{s^2+7}$

$$\begin{array}{r|l} & \;1 \\ s^2+7 & s^2+6 \\ & \underline{-s^2 \mp 7} \\ & 0 \;\; -1 \end{array}$$

$$F(s) = \frac{s^2+6}{s^2+7} = 1 - \frac{1}{s^2+7}$$

$$F(s) = \underbrace{1}_{} - \frac{1}{s^2+(\sqrt{7})^2} = 1 - \underbrace{\frac{\sqrt{7}}{\sqrt{7}\,(s^2+(\sqrt{7})^2)}}_{}$$

$$\frac{\omega}{s^2+\omega^2} \qquad \sin(\sqrt{7}t)\,u(t)$$

$$f(t) = \delta(t) - \frac{1}{\sqrt{7}} \sin(\sqrt{7}t)\,u(t)$$

Example: find ILT of $\dfrac{s+2}{s^2+2s+4} = \dfrac{(s+1)+1}{(s+1)^2+(\sqrt{3})^2}$

find roots: $s^2+2s+4=0$

$$s = \frac{-2 \pm \sqrt{2^2-4(1)(4)}}{2} \qquad ax^2+bx+c=0, \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2 \pm j\sqrt{12}}{2} = -1 \pm j\sqrt{3}$$

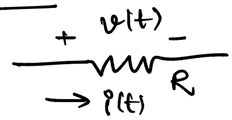
$$F(s) = \frac{s+2}{(s+1-j\sqrt{3})(s+1+j\sqrt{3})} = \frac{s+2}{(s+1)^2-(j\sqrt{3})^2} = \frac{s+2}{(s+1)^2+(\sqrt{3})^2}$$

$$F(s) = \underbrace{\frac{s+1}{(s+1)^2+(\sqrt{3})^2}}_{} + \underbrace{\frac{1}{(s+1)^2+(\sqrt{3})^2}}_{} \times \frac{\sqrt{3}}{\sqrt{3}}$$

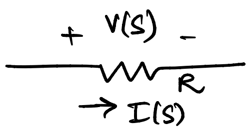
$$f(t) = e^{-t}\cos(\sqrt{3}t)\,u(t) + \frac{1}{\sqrt{3}} e^{-t}\sin(\sqrt{3}t)\,u(t)$$

Impedances in s-domain :

Resistor :



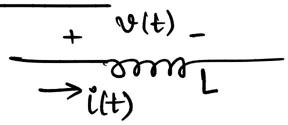
$$v(t) = i(t)\,R$$



$$V(s) = I(s)\,R$$

$$\frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

Inductor :

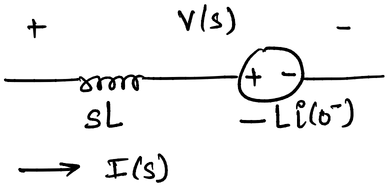


$$v(t) = L\underbrace{\frac{di(t)}{dt}}$$

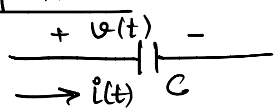
Take LT:

$$\Rightarrow \quad V(s) = L\left(s\,I(s) - i(0^-)\right)$$

$$V(s) = sL\,I(s) - \underbrace{L\,i(0^-)}$$



Capacitor :



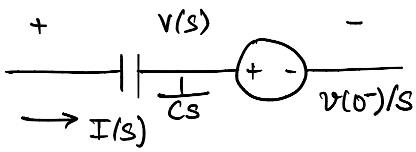
$$i(t) = C\,\frac{dv(t)}{dt}$$

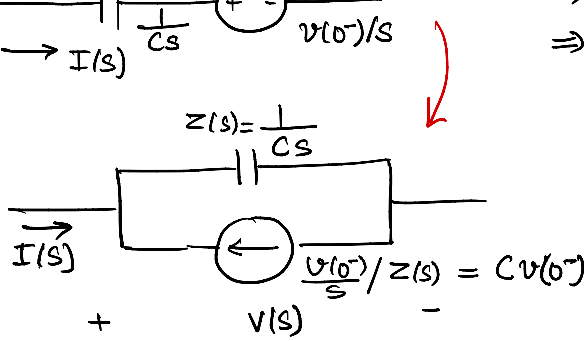
Take LT:

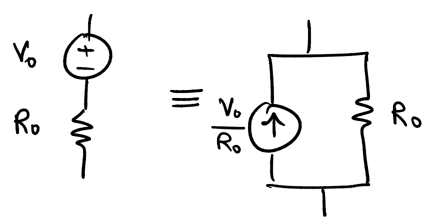
$$\Rightarrow I(s) = C\left[sV(s) - v(0^-)\right]$$

$$\Rightarrow I(s) = Cs\,V(s) - Cv(0^-)$$

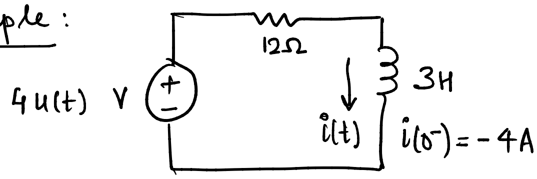
$$\Rightarrow V(s) = \frac{1}{Cs}\,I(s) + \frac{v(0^-)}{s}$$







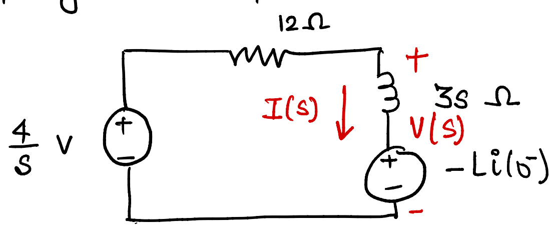
Example:



find $i(t)$. find $v(t)$ accross inductor.

frequency domain representation:

$$4u(t) \Longleftrightarrow \frac{4}{s}$$



write KVL eqn:
$$-\frac{4}{s} + 12\,I(s) + (3s)\,I(s) + 12 = 0$$

$$\Rightarrow \quad I(s) = \frac{\frac{4}{s} - 12}{12 + 3s} = \frac{4 - 12s}{(12+3s)s}$$

$$I(s) = \frac{A}{s} + \frac{B}{12+3s}$$

$$\Rightarrow \quad 4 - 12s = A(12+3s) + B(s)$$

at $s=0 \Rightarrow 4 = A(12+0) + 0 \Rightarrow A = 1/3$

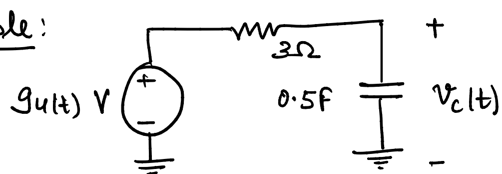
at $s=-4 \Rightarrow 4 - 12(-4) = A(0) + B(-4) \Rightarrow B = -13$

$$I(s) = \frac{1/3}{s} + \frac{-13}{12+3s} = \frac{1}{3}\left(\frac{1}{s}\right) - \left(\frac{13}{3}\right)\left(\frac{1}{4+s}\right)$$

$$i(t) = \frac{1}{3}u(t) - \frac{13}{3}e^{-4t}u(t) \quad A$$

$$\boxed{\begin{array}{c} u(t) \Longleftrightarrow \frac{1}{s} \\ e^{-at}u(t) \Longleftrightarrow \frac{1}{s+a} \end{array}}$$

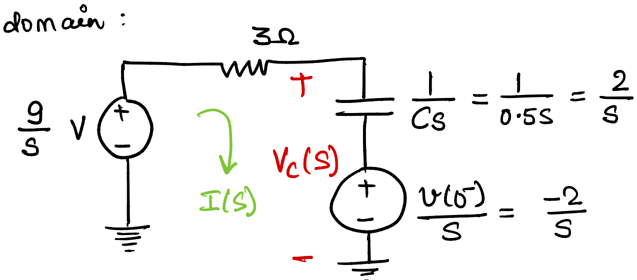
Example:



Given: $v_c(0^-) = -2V$

Find $v_c(t)$.

Freq. domain:



$$V_c(s) = I(s) \times \frac{2}{s} - \frac{2}{s}$$

KVL $\quad -\frac{9}{s} + 3I(s) + \frac{2}{s}(I(s)) - \frac{2}{s} = 0$

$$\Rightarrow \quad I(s) = \frac{11}{3s+2} = \frac{11/3}{s+2/3}$$

$$i(t) = \frac{11}{3} e^{-(2/3)t} u(t)$$

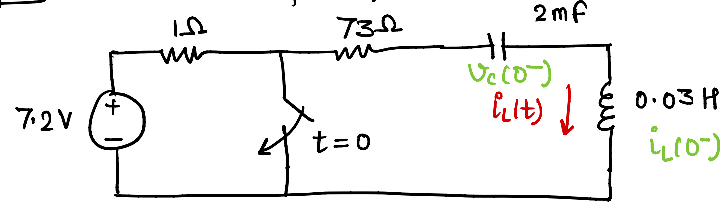
$$V_c(s) = \frac{2}{s} I(s) - \frac{2}{s} = \frac{18-6s}{s(3s+2)} = \frac{A}{s} + \frac{B}{3s+2}$$

$$A = 9 \quad , \quad B = -33$$

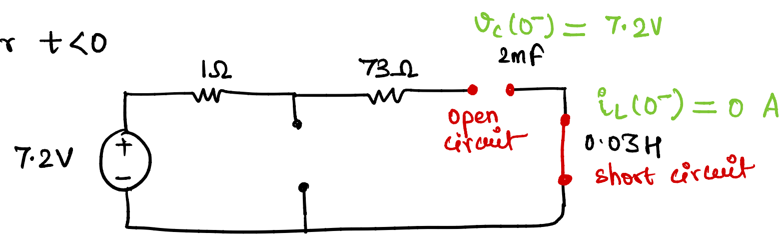
$$V_c(s) = \frac{9}{s} - \frac{33}{3s+2} = \frac{9}{s} - \frac{11}{s+2/3}$$

$$v_c(t) = 9u(t) - 11 e^{-\frac{2}{3}t} u(t) \quad V$$

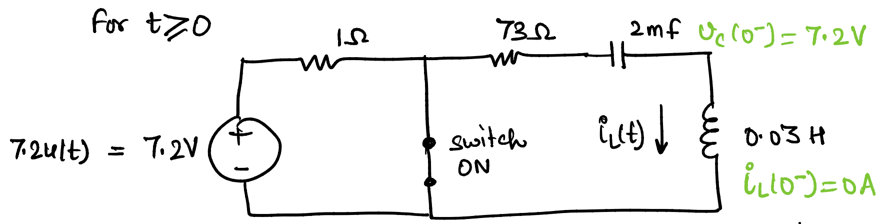
Example: Find $i_L(t)$ for $t \geq 0$



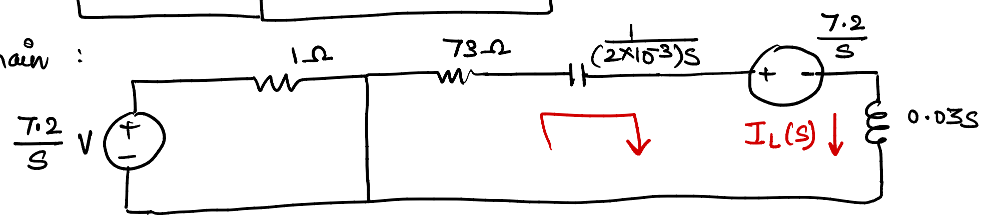
for $t<0$



for $t \geq 0$



Freq. domain :



KVL:
$$73\, I(s) + \frac{1}{(2\times 10^{-3})s} I(s) + \frac{7.2}{s} + 0.03s\, I(s) = 0$$

$$\Rightarrow I(s) = \frac{-7.2/s}{73 + \frac{1}{(2\times10^{-3})s} + 0.03s} = \frac{-7.2}{0.03s^2 + 73s + 500}$$

$$\Rightarrow I(s) = \frac{-7.2/0.03}{s^2 + \frac{73}{0.03}s + \frac{500}{0.03}} = \frac{A}{s+a} + \frac{B}{s+b}$$

find a & b

$$s^2 + \frac{73}{0.03}s + \frac{500}{0.03} = 0 \quad \Leftarrow \text{Roots}$$

$$\Rightarrow s^2 + 2433s + 16666 = 0$$

$$\Rightarrow s = \frac{-2433 \pm \sqrt{2433^2 - 4(1)(16666)}}{2(1)} \simeq -7, -2426$$

$$s^2 + 2433s + 16666 = (s+2426)(s+7)$$

$$I(s) = \frac{-7.2/0.03}{(s+2426)(s+7)} = \frac{A}{(s+2426)} + \frac{B}{s+7}$$

$$\Rightarrow \frac{-7.2}{0.03} = -240 = A(s+7) + B(s+2426)$$

at $s = -7 \Rightarrow B = -240/2419 = -0.099$

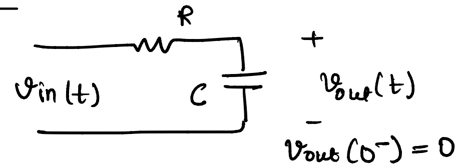
at $s = -2426 \Rightarrow A = +240/2419 = 0.099$

$$I(s) = \frac{0.099}{s+2426} + \frac{-0.099}{s+7}$$

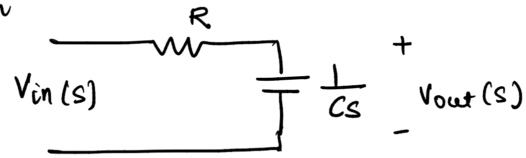
$$i(t) = 0.099\left[e^{-2426t}u(t) - e^{-7t}u(t)\right] \text{ A.}$$

Transfer function :

Lets consider :



Freq. domain



$$V_{out}(s) = \frac{1/Cs}{R + 1/Cs} V_{in}(s)$$

$$\Rightarrow \underbrace{\frac{V_{out}(s)}{V_{in}(s)}}_{\text{transfer function. } H(s)} = \frac{1}{sRC+1}$$

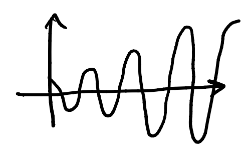
$$v_{out}(t) = \mathcal{L}^{-1}\{V_{out}(s)\} = \mathcal{L}^{-1}\{\underbrace{H(s)}_{\text{defines the system.}} \cdot V_{in}(s)\}$$

Stability :

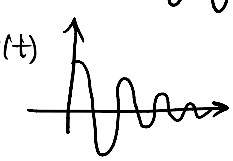
stable system : bounded output for bounded input.
Unstable system : unbounded output for bounded input.
Marginally stable : either oscillating or bounded offset.

$e^{\sigma t} \cos(\omega_0 t)\, u(t)$

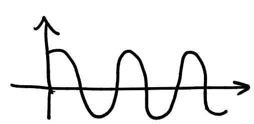
① $\sigma > 0$ lets say $\sigma = 2$ $e^{2t}\cos(\omega_0 t)\, u(t)$
unstable



② $\sigma < 0$ lets say $\sigma = -3$ $e^{-3t}\cos(\omega_0 t)\, u(t)$
stable



③ $\sigma = 0$ $\cos(\omega_0 t)\, u(t)$
marginally stable.



Stability Criterion

System transfer fn $H(s) = k\frac{(s-z_1)(s-z_2)(s-z_3)\cdots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\cdots(s-p_n)}$

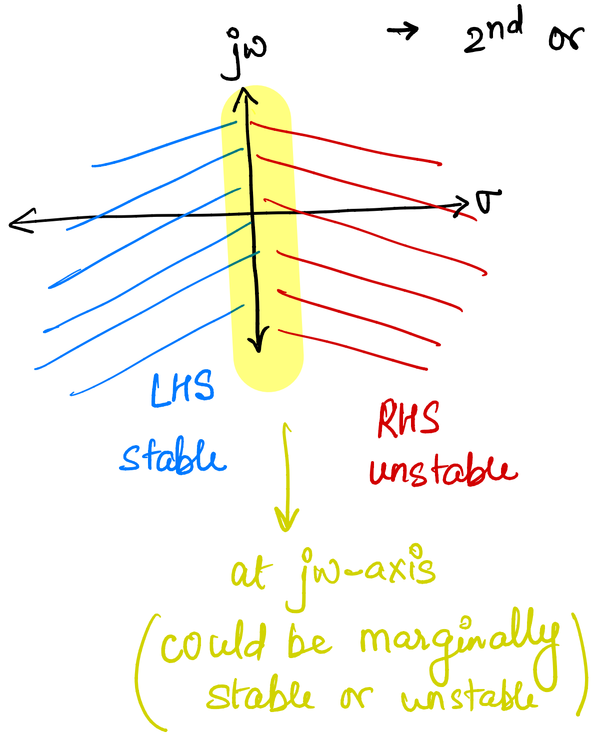
- System is stable if
  All poles of $H(s)$ lie on LHS of complex plane
- Marginally stable if
  1st order pole at $s=0$ OR 1st order complex conjugate pole pair at $j\omega$-axis
- Unstable system if
  All other cases
  - → Any poles on RHS OR
  - → 2nd or higher order pole at $s=0$ OR
  - → 2nd or higher order conjugate pole pair at $j\omega$-axis

Stability Criterion

$$\text{system transfer fn } H(S) = k\frac{(S-Z_1)(S-Z_2)(S-Z_3)\cdots(S-Z_m)}{(S-P_1)(S-P_2)(S-P_3)\cdots(S-P_n)}$$

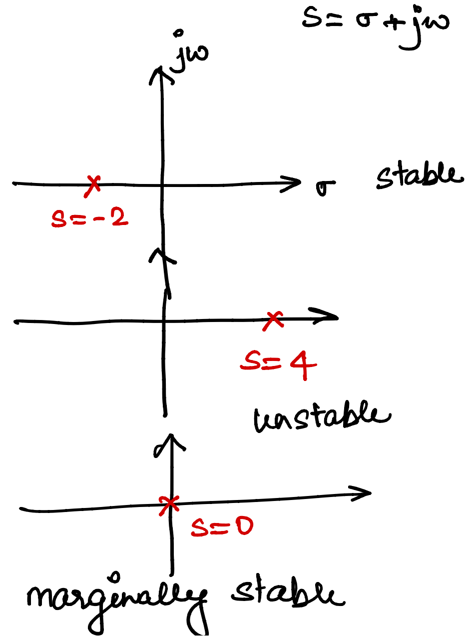
- System is stable if
  All poles of $SH(S)$ lie on LHS of complex plane
- Marginally stable if
  1st order pole at $S=0$ OR 1st order complex conjugate pole pair at $j\omega$-axis
- Unstable system if
  All other cases
  → Any poles on RHS OR
  → 2nd or higher order pole at $S=0$ OR
  → 2nd or higher order conjugate pole pair at $j\omega$-axis

$SH(S) = \underbrace{\frac{1}{S}}\quad \underbrace{\frac{1}{S^2}}_{2^{nd}\text{ order}}\quad \frac{1}{S^3}\quad \frac{1}{S^n}$ $\quad n \geq 2$

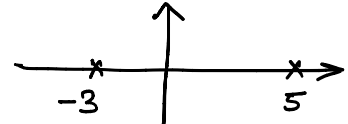


	$f(t)$	$S.F(S)$
①	$e^{-2t}u(t)$	$S.\frac{1}{S+2}$
②	$e^{4t}u(t)$	$S.\frac{1}{S-4}$
③	$u(t)$	$S.\frac{1}{S}$

$S = \sigma + j\omega$



④ $e^{-3t}u(t) + e^{+5t}u(t)$ $\quad$ $S.\left\{\frac{1}{S+3} + \frac{1}{S-5}\right\}$

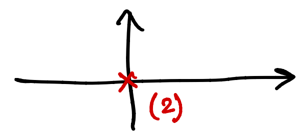


unstable

⑤ $S.\ \frac{8}{(S+1)(S+2)(S+3)}$ $\quad$ stable

⑥ $S.\ \frac{14}{(S+1-\sqrt{2})(S-1+\sqrt{2})}$ $\quad$ unstable

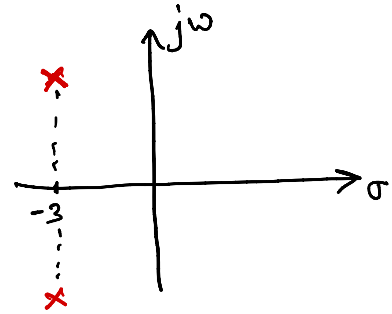
⑦ $\underline{tu(t)}$ $\quad$ $S.\ \frac{1}{S^2}$



unstable

⑧ $e^{-3t}\sin(\omega_o t)u(t)$ $\quad$ $S.\ \frac{\omega_o}{(S+3)^2 + \omega_o^2}$

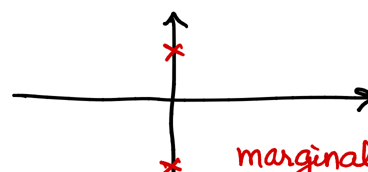
$$(S+3)^2 + \omega_o^2 = 0$$
$$\Rightarrow (S+3)^2 = -\omega_o^2$$
$$\Rightarrow S+3 = \pm\sqrt{-1}\,\omega_o$$
$$\Rightarrow S = -3 \pm j\omega_o$$



stable

⑨ $\sin(\omega_o t)u(t)$ $\quad$ $S.\ \frac{\omega_o}{S^2+\omega_o^2}$

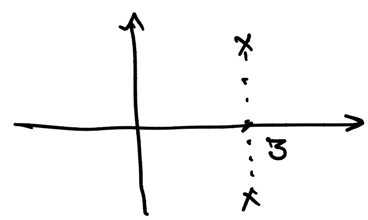
$S = \pm j\omega_o$



marginally stable

⑩ $e^{3t}\sin(\omega_o t)u(t)$ $\quad$ $S.\ \frac{\omega_o}{(S-3)^2 + \omega_o^2}$

poles: $S = 3 \pm j\omega_o$

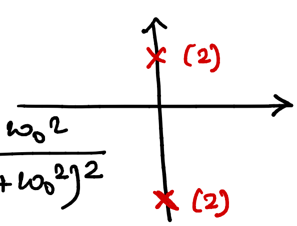


unstable

⑪ $t\cos(\omega_o t)u(t)$

$$f(t) \Leftrightarrow F(S)$$
$$tf(t) \Leftrightarrow -\frac{d}{dS}(F(S))$$

$$\mathcal{L}\{t\cos(\omega_o t)u(t)\} = -\frac{d}{dS}\left\{\frac{S}{S^2+\omega_o^2}\right\} = \frac{S^2-\omega_o^2}{(S^2+\omega_o^2)^2}$$



unstable

Please click!

Final Value Theorem :

$$f(\infty) = f(t \to \infty) = \lim_{s \to 0} s F(s),$$ when all poles are on LHS.

Initial value theorem:

$$f(0) = f(t=0) = \lim_{s \to \infty} s F(s).$$

$\longrightarrow$ $f(t) = \sin(\omega_0 t)\, u(t)$

final value theorem : $\lim_{s \to 0} sF(s) = \lim_{s \to 0} s \dfrac{\omega_0}{s^2+\omega_0^2} = \dfrac{0}{\omega_0^2} = 0$

$f(\infty)$ = indeterminent.

Chapter 15 (9th edition of textbook)

Frequency Response

Capacitor Impedance $Z_c = \dfrac{1}{sC}$

$s = \sigma + j\omega$

Transfer function :

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$$V_{out}(s) = \underbrace{V_{in}(s)}_{\text{input}} \cdot \underbrace{H(s)}_{\text{system}}$$

$s = j\omega$

$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \text{voltage gain} \quad V/V$$

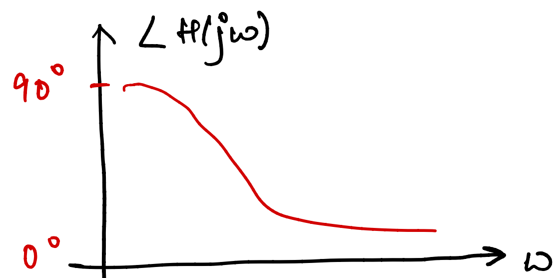
$$H(j\omega) = \frac{I_{out}(j\omega)}{I_{in}(j\omega)} = \text{current gain} \quad A/A$$

$$H(j\omega) = \frac{V_{out}(j\omega)}{I_{in}(j\omega)} = \text{transfer impedance} \quad \frac{V}{A} \text{ or } \Omega$$

$$H(j\omega) = \frac{I_{out}(j\omega)}{V_{in}(j\omega)} = \text{transfer admittance} \quad \frac{A}{V} \text{ or } \Omega^{-1}$$

$$H(j\omega) = |H(j\omega)| \angle H(j\omega)$$

magnitude plot



phase plot

Decibels (dB) scale

$$H_{dB} = 20\log_{10}|H(j\omega)|$$

$$\log_{10} 1 = 0$$
$$\log_{10} 10 = 1$$
$$\log_{10} 2 = 0.3$$
$$\log_{10}(0.1) = -1$$
$$\log_{10}(10^{-k}) = -k$$
$$\log_{10}(0) \Rightarrow \text{undefined.}$$

Find H_{dB} at $\omega = 146$ rad/s if $H(s) = 20(s+100)$.

$$H_{dB} = 20\log_{10}|20(j\omega+100)| = 20\log\left\{20\times\sqrt{100^2+\omega^2}\right\}$$

$$= 20\log_{10}\left\{20\times\sqrt{100^2+146^2}\right\}$$

$$= 70.9 \text{ dB}$$

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$H_{dB} = 20\log|K| + 20\log|(s-z_1)| + 20\log|(s-z_2)| + \cdots$$
$$- 20\log|(s-p_1)| - 20\log|(s-p_2)| + \cdots$$

$$H(j\omega) = H_1(j\omega).H_2(j\omega)$$
$$20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)|$$
$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

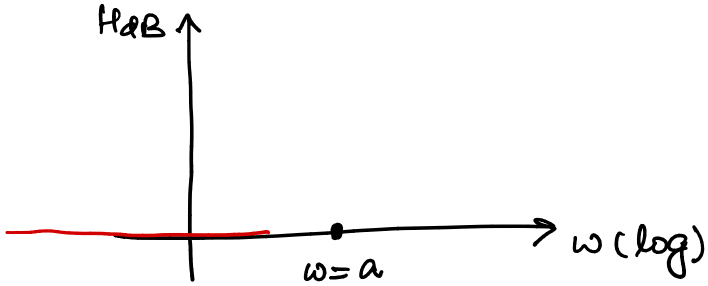
$$\bar{A} = r_1 e^{j\theta_1}$$
$$\bar{B} = r_2 e^{j\theta_2}$$
$$AB = (r_1 r_2) e^{j(\theta_1+\theta_2)}$$

---

(1) Consider : $\boxed{H(s) = 1 + \frac{s}{a}}$. Simple Zero

$$H(j\omega) = 1 + \frac{j\omega}{a}$$

$$H_{dB} = 20\log\left|1+\frac{j\omega}{a}\right| = 20\log\sqrt{1+\left(\frac{\omega}{a}\right)^2}$$



lets consider $\underline{\omega \ll a}$

$$H_{dB} = 20 \log(\sqrt{1}) = 0 \text{ dB}$$

$\underline{\omega \gg a}$

$$H_{dB} = 20 \log \sqrt{\left(\frac{\omega}{a}\right)^2} = 20 \log\left(\frac{\omega}{a}\right) \text{ dB}$$

$\omega = \underline{10a}$ $\qquad$ $H_{dB} = 20$ dB

$\omega = \underline{100a}$ $\qquad$ $H_{dB} = \underline{40}$ dB

$\omega = 1000a$ $\qquad$ $H_{dB} = 20 \log\left(\frac{1000a}{a}\right) = 60$ dB

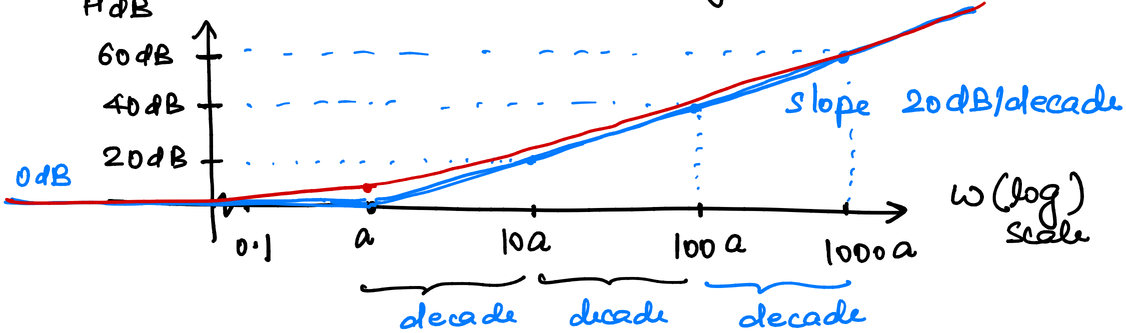
$\omega = 10^4 a$ $\qquad$ $H_{dB} = 80$ dB

$y = mx + c$

(linear scale)

ω is multiplied by 10, H_{dB} increases by 20 dB

HdB



$\text{slope} = \frac{\Delta y}{\Delta x}$

semilog

$\boxed{\text{Bode Plot}}$

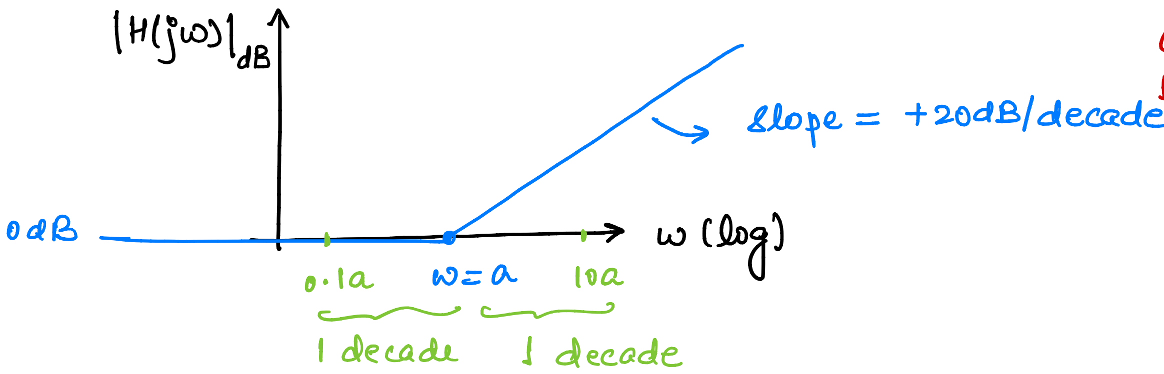
straight line plot

$\omega = \underbrace{2a \qquad 20a}_{\text{decade}}$

at $\omega = a$, $\qquad H(j\omega) = 1 + \frac{j\omega}{a} = 1 + j$

$$H_{dB} = 20 \log_{10} \sqrt{2} = 20 \times \frac{1}{2} \times 0.3 = 3\text{dB}$$

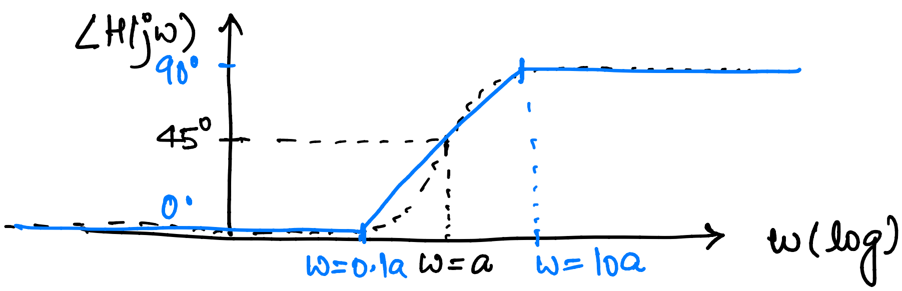
Simple zero : $H(s) = 1 + s/a \Rightarrow H(j\omega) = \left[1 + \frac{j\omega}{a}\right]$



ω=a
corner frequency
Break freq.

Phase Plot :

$$\angle H(j\omega) = \angle\left\{1 + \frac{\omega}{a}j\right\} = \tan^{-1}\left(\frac{\omega}{a}\right)$$



ω	$\tan^{-1}(\omega/a)$
0	0°
0.001a	~0°
a	45°
1000 a	~90°
∞	90°

Simple Pole : $H(s) = \dfrac{1}{1 + s/a}$

$$H(j\omega) = \frac{1}{1 + j\omega/a}$$

$$|H(j\omega)|_{dB} = 20\log_{10}\left(\left|\frac{1}{1 + j\omega/a}\right|\right) = 20\log_{10}\left\{\frac{1}{\sqrt{1 + (\omega/a)^2}}\right\}$$

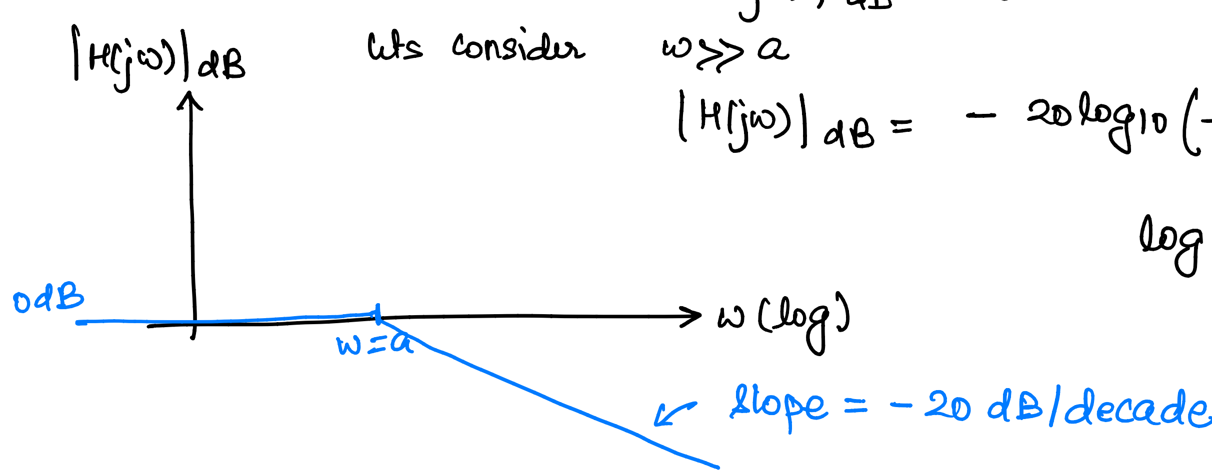
Lets consider $\omega \ll a$

$$|H(j\omega)|_{dB} = 0$$

Lets consider $\omega \gg a$

$$|H(j\omega)|_{dB} = -20\log_{10}\left(\frac{\omega}{a}\right)$$

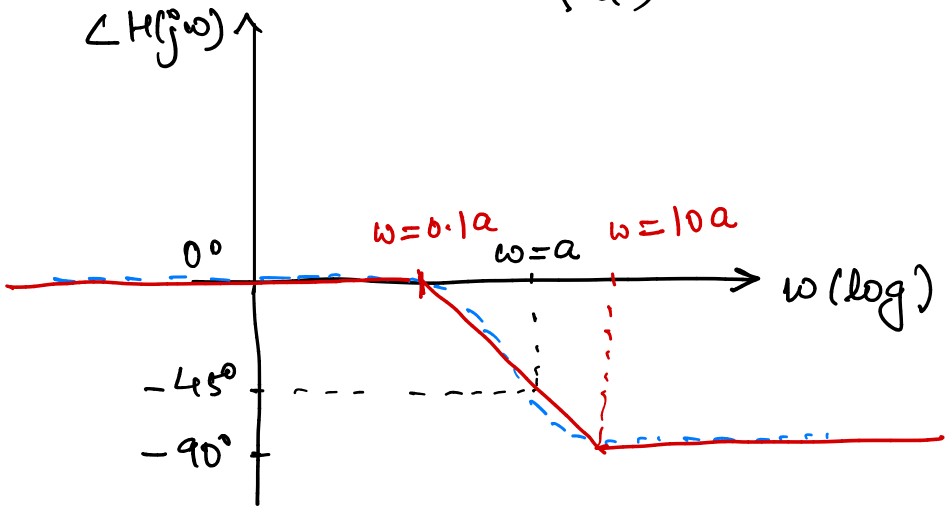
$$\log\frac{1}{x} = -\log x$$



ω	$\lvert H(j\omega)\rvert_{dB}$
a	0 dB
$10a$	−20 dB
$100a$	−40 dB
$1000a$	−60 dB

$$\angle(H(j\omega)) = \angle\left\{\frac{1}{1+j\omega/a}\right\} = \angle\left\{\frac{(1-j\omega/a)}{(1+j\omega/a)(1-j\omega/a)}\right\}$$

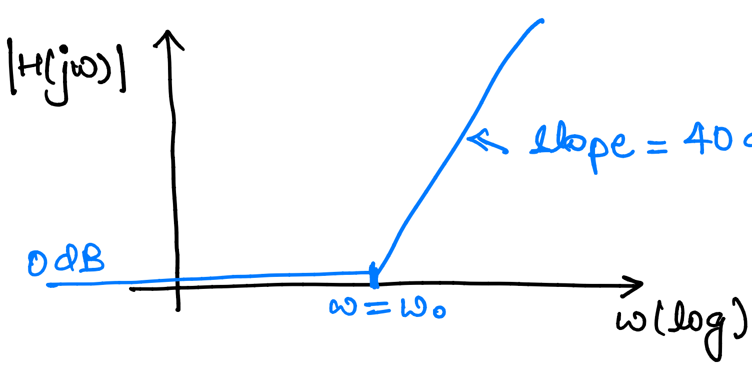
$$= -\tan^{-1}\left(\frac{\omega}{a}\right)$$



ω	$-\tan^{-1}(\omega/a)$
0	0°
0.0001	~0°
a	−45°
$10000a$	~−90°
∞	−90°

Complex conjugate pair Zero:

$$H(s) = 1 + 2\zeta\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2$$



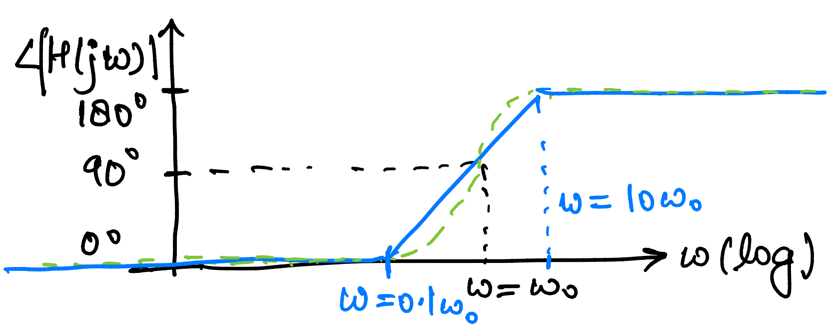
when $\omega \ll \omega_0$

$$|H(j\omega)|_{dB} = 20\log(1) = 0 \text{ dB}$$

when $\omega \gg \omega_0$

$$|H(j\omega)|_{dB} = 20\log\left(\left|\frac{j\omega}{\omega_0}\right|^2\right)$$

$$= 40\log\left(\frac{\omega}{\omega_0}\right) \text{ dB}$$



Example: Draw Bode plots of $H(s) = \frac{s+1}{s(s+2)^2}$.

$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$H_{dB} = 20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega)H_2(j\omega)|$$

$$= 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

$$\boxed{(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}}$$

$$H(s) = \frac{s+1}{s(s+2)^2} = \frac{(1+s/1)}{s\left(1+\frac{s}{2}\right)^2 \cdot 4} = \frac{1}{4}\cdot\frac{1}{s}\cdot\left(1+\frac{s}{1}\right)\frac{1}{\left(1+\frac{s}{2}\right)^2}$$

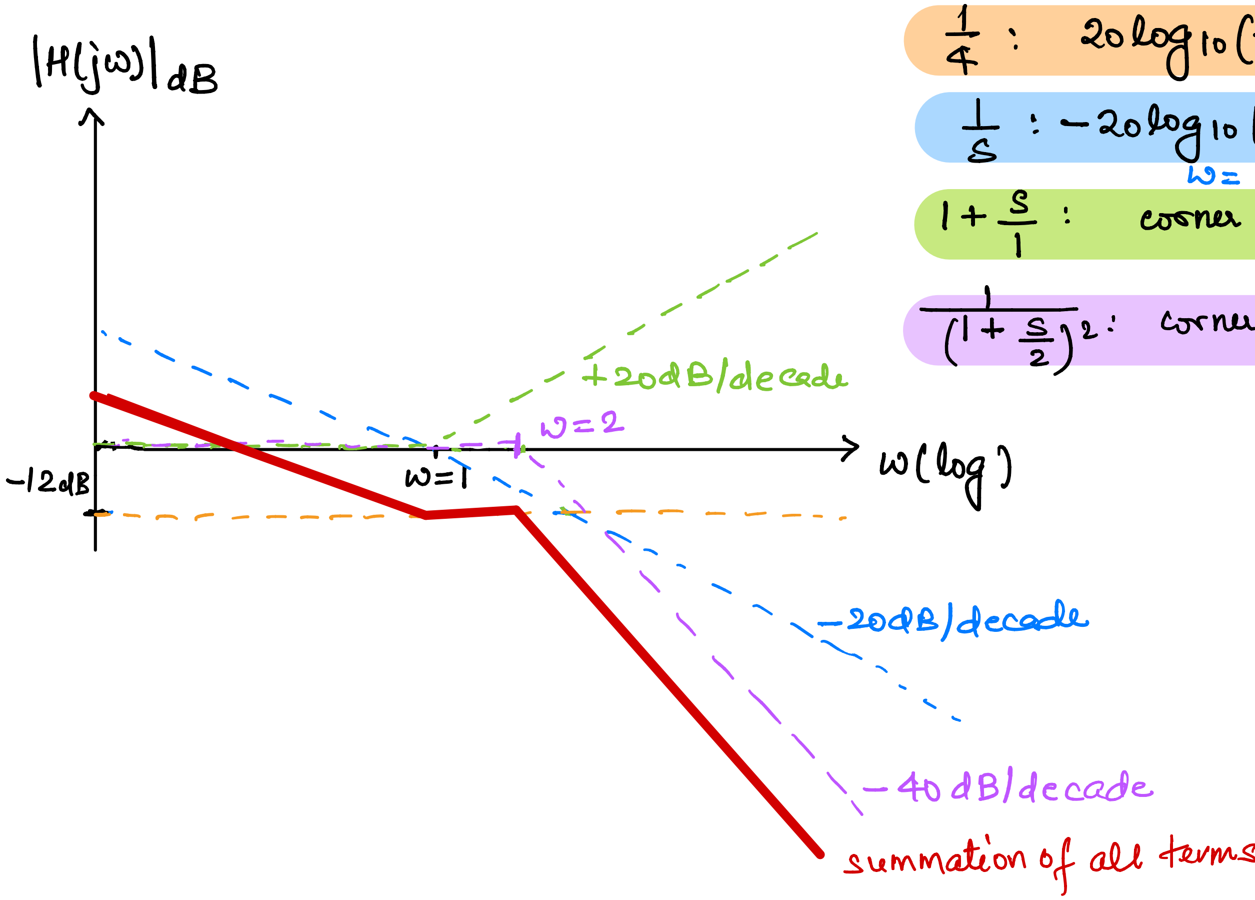
$\frac{1}{4}$: $20\log_{10}\left(\frac{1}{4}\right) = -12$ dB

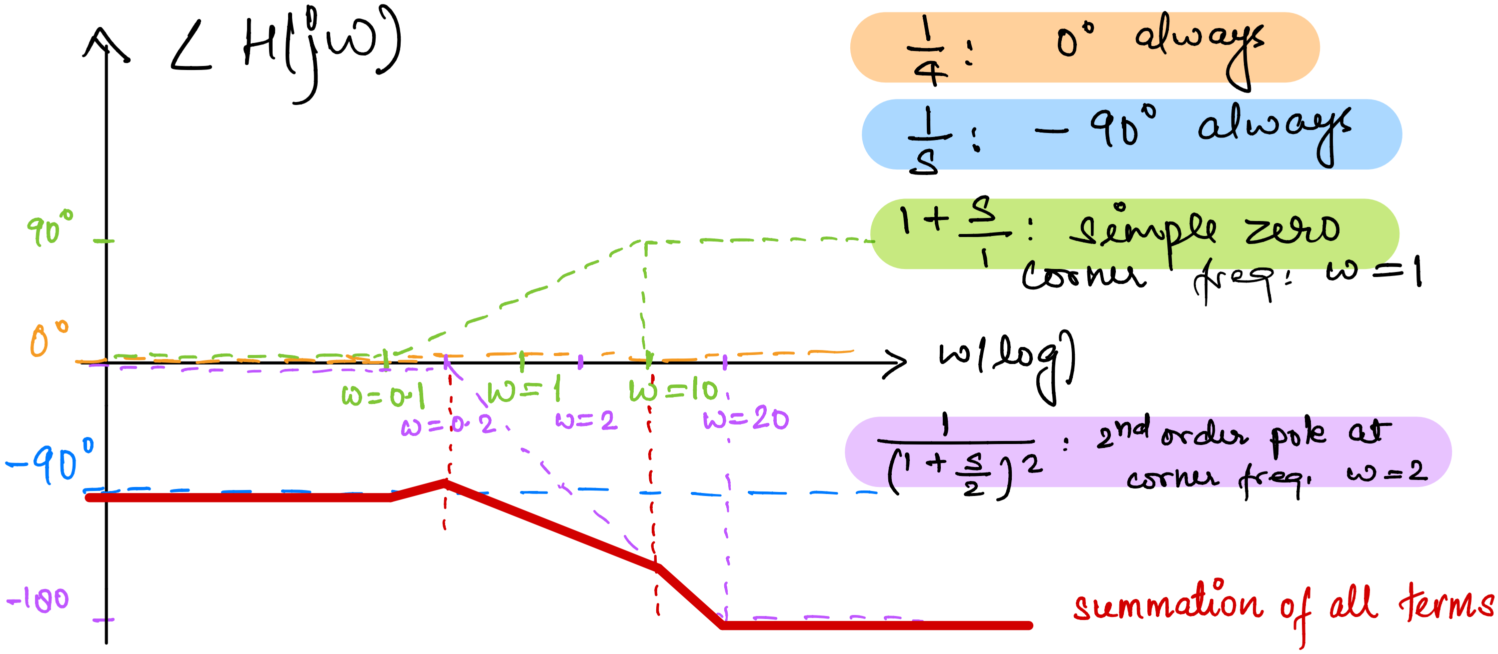
$\frac{1}{s}$: $-20\log_{10}(\omega)$ dB

$\omega = 1 \rightarrow 0$ dB

$1+\frac{s}{1}$: corner freq: $\omega = 1$

$\frac{1}{\left(1+\frac{s}{2}\right)^2}$: corner freq. $\omega = 2$





∠ H(jω)
1/4 : 0° always
1/s : − 90° always
1 + s/1 : simple zero
corner freq: ω = 1
1/(1 + s/2)² : 2nd order pole at
corner freq. ω = 2
90°
0°
−90°
−100
ω (log)
ω = 0.1
ω = 1
ω = 10
ω = 0.2
ω = 2
ω = 20
summation of all terms

Zeros: $s=-1$

Poles: $s=0$

$\left.\begin{array}{l} s=-2 \\ s=-2 \end{array}\right\}$ 2nd order pole

Example: Draw Bode plots of $H(s) = \dfrac{s+1}{s(s+2)^2}$

$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

$$H_{dB} = 20\log_{10}|H(j\omega)| = 20\log_{10}|H_1(j\omega) H_2(j\omega)|$$

$$= 20\log_{10}|H_1(j\omega)| + 20\log_{10}|H_2(j\omega)|$$

$$\angle H(j\omega) = \angle H_1(j\omega) + \angle H_2(j\omega)$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

$$H(s) = \frac{s+1}{s(s+2)^2} = \frac{(1+s/1)}{s\left(1+\frac{s}{2}\right)^2 \cdot 4} = \frac{1}{4} \cdot \frac{1}{s} \cdot \left(1+\frac{s}{1}\right) \frac{1}{\left(1+\frac{s}{2}\right)^2}$$

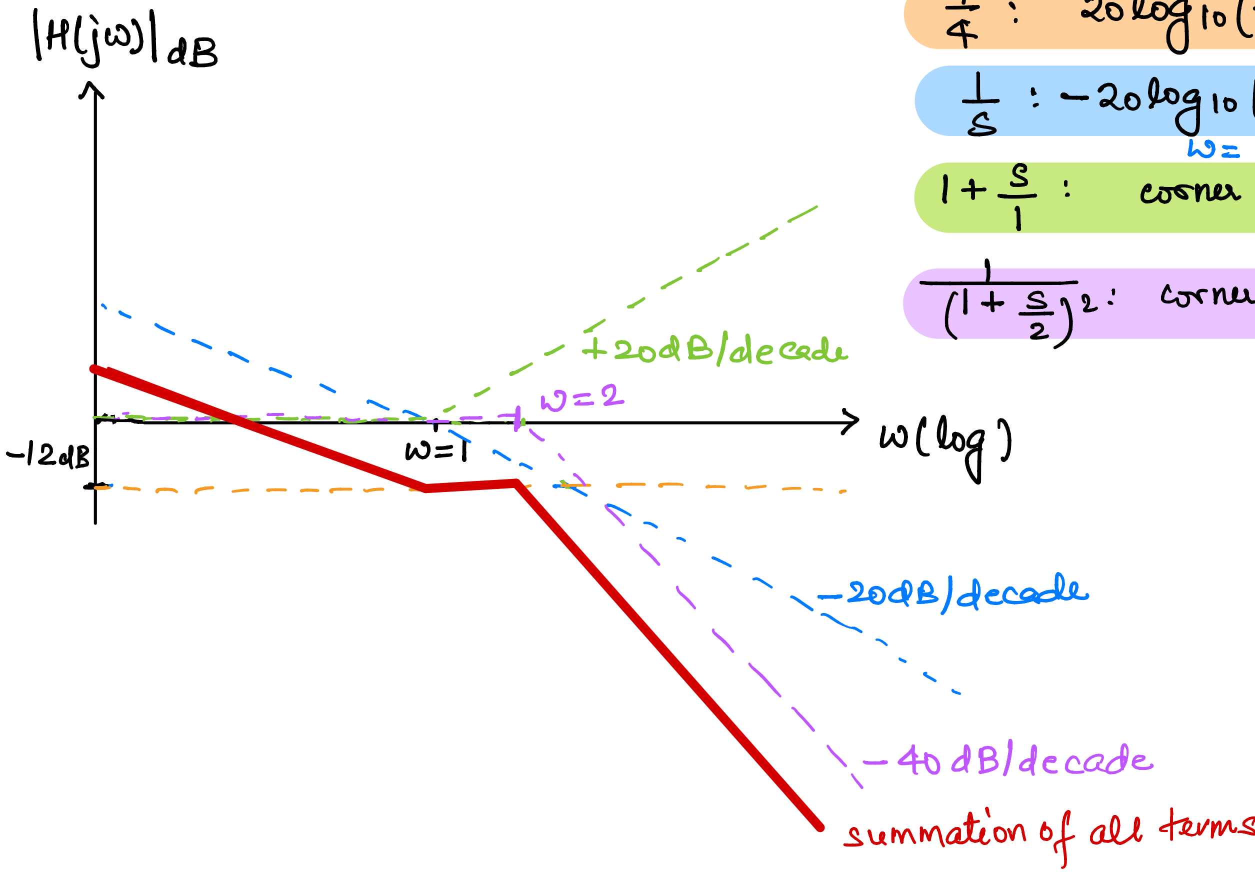
$\frac{1}{4}$: $20\log_{10}\left(\frac{1}{4}\right) = -12$ dB

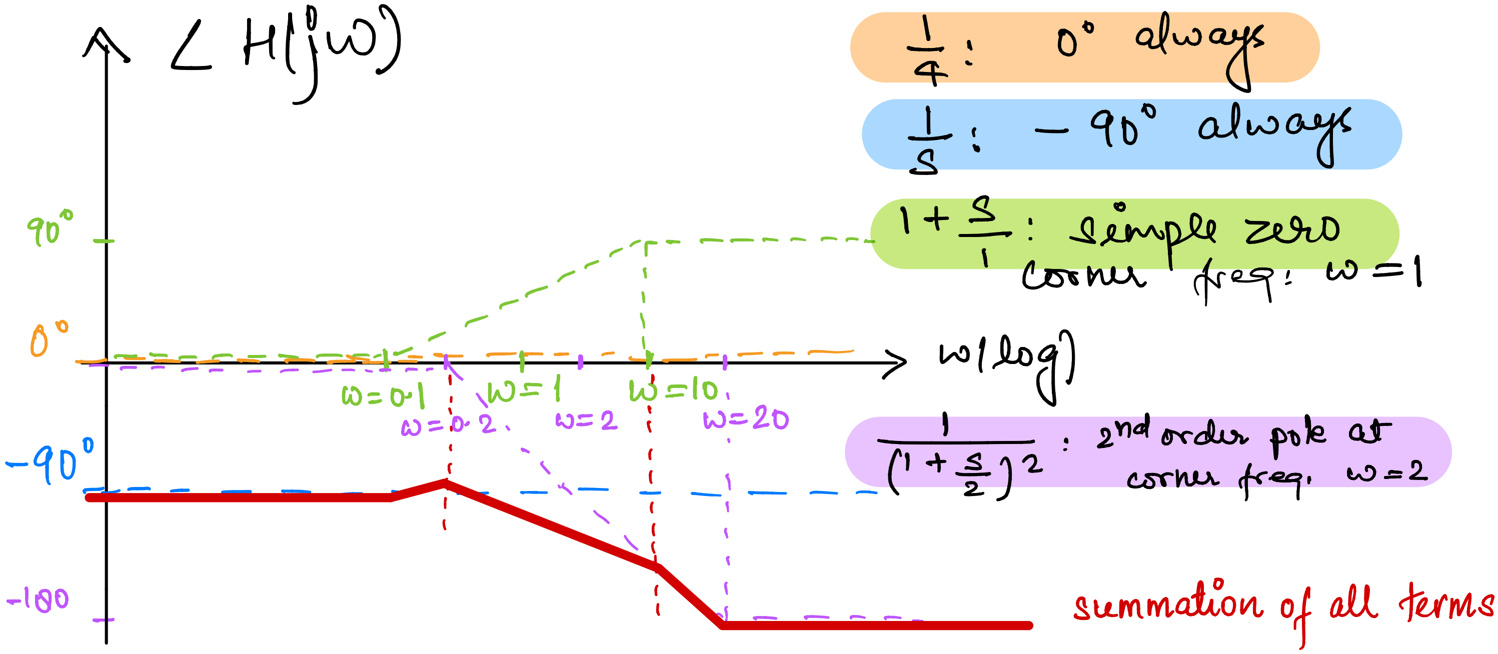
$\frac{1}{s}$: $-20\log_{10}(\omega)$ dB

$\omega = 1 \rightarrow 0$ dB

$1+\frac{s}{1}$: corner freq: $\omega = 1$

$\frac{1}{\left(1+\frac{s}{2}\right)^2}$: corner freq. $\omega = 2$

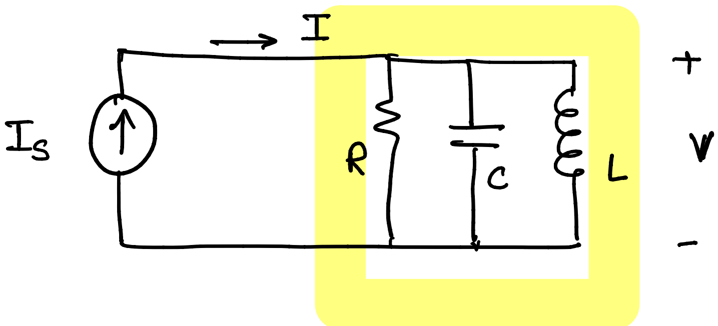




∠ H(jω)
90°
0°
-90°
-100
ω(log)
ω=0.1
ω=1
ω=10
ω=0.2
ω=2
ω=20
1/4 : 0° always
1/s : −90° always
1+s/1 : simple zero
corner freq: ω=1
1/(1+s/2)^2 : 2nd order pole at
corner freq. ω=2
summation of all terms

Resonance :

A network is in resonance when V and I at network input terminals are in phase.



$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Assume : $I_s = I_0 \cos(\omega t + \phi)$

$$Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1}{R} + \frac{1}{(1/j\omega C)} + \frac{1}{j\omega L}$$

$$= \frac{1}{R} + j\left\{\omega C - \frac{1}{\omega L}\right\}$$

For resonance: V and I are in phase.

$$\omega C - \frac{1}{\omega L} = 0$$

$$\Rightarrow \quad \omega = \frac{1}{\sqrt{LC}} = \omega_0 \quad \text{"Resonant freq."}$$

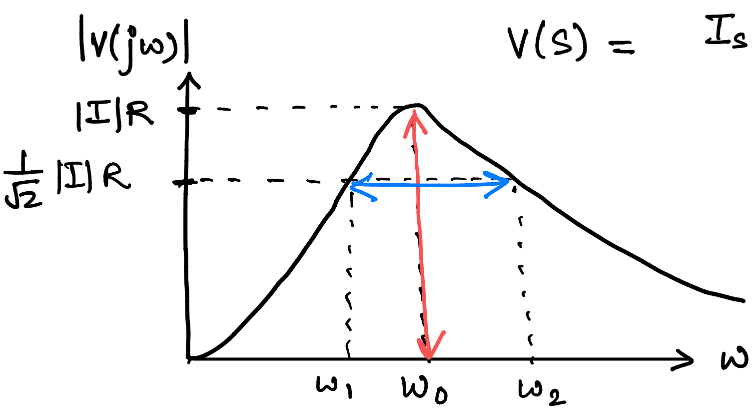
Assume $I_s = I_0 e^{\sigma t} \cos(\omega t + \phi)$

$$Y(s) = \frac{1}{Z(s)} = \frac{1}{R} + \frac{1}{(1/Cs)} + \frac{1}{sL}$$

$$= \frac{s^2 + s(1/RC) + 1/LC}{(1/C)\, s}$$

$$\left.\begin{aligned} V_{out} &= V \\ I_{in} &= I_s \end{aligned}\right\} \quad \text{Transfer function} = \frac{V(s)}{I_s(s)} = Z(s) = \frac{1}{Y(s)}$$

$$V(s) = I_s(s) \cdot Z(s)$$



ω_1, ω_2: half power freq.

hight depends on R

width depends on L and C as well.

At resonance :

$$I_S = I_R$$
$$I_C = -I_L \quad \Rightarrow \quad I_C + I_L = 0$$

Quality Factor :

$$Q = 2\pi \frac{\text{max. energy stored}}{\text{total energy lost per period}}$$

$$= 2\pi \frac{[w_L(t) + w_C(t)]_{max}}{P_R T}$$

$P_R =$ average power lost in resistor $= \frac{1}{2}|I|^2 R$ watts

$T =$ time period (sec)

$w_L(t) = \frac{1}{2} L\, i_L^2(t)$ Joules

$w_C(t) = \frac{1}{2} C\, v_C^2(t)$ Joules

$$Q = \omega_0 RC = R\sqrt{\frac{C}{L}}$$

$\alpha =$ damping factor $= \frac{1}{2RC} = \frac{\omega_0}{2Q_0}$

$\omega_d =$ damped freq. $= \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1-(1/2Q_0)^2}$

Half Power Bandwidth (BW)

$$BW = \omega_2 - \omega_1 = \omega_0 / Q_0$$

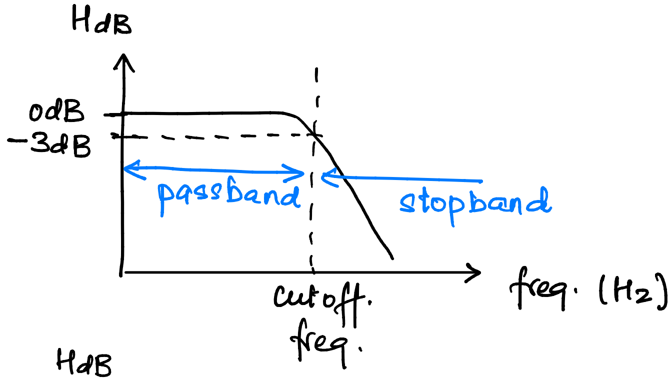
where $\omega_0 = \sqrt{\omega_1 \omega_2}$

$$BW \propto \frac{1}{Q_0}$$

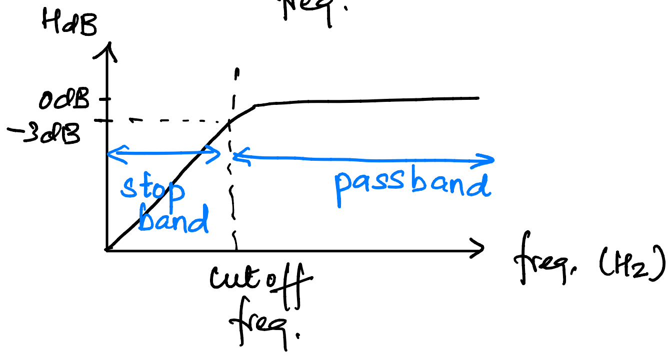
$Q_0 > 5$: High-Q circuit

Filter Design :

① Low pass filter

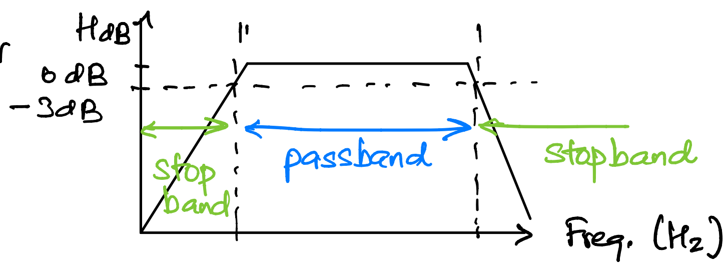


② High pass filter

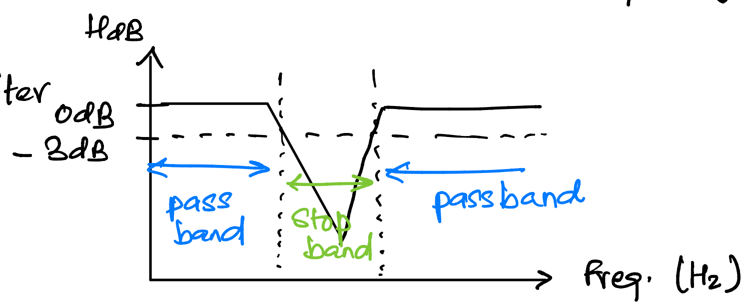


$$\frac{V_{out}(s)}{V_{in}(s)} = H(s)$$

③ Band pass filter



④ Band stop filter

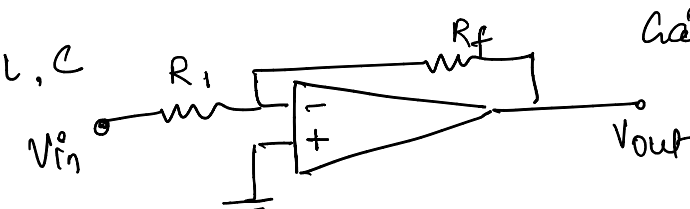


Passive Filter : gain ≤ 1 ie $H_{dB} \leq 0dB$

using R, L, C

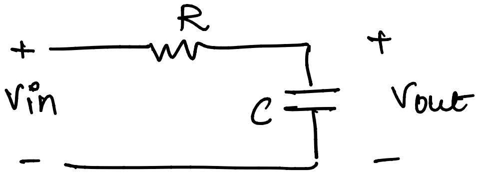
Active Filter : opamp and R, L, C

Gain > 1



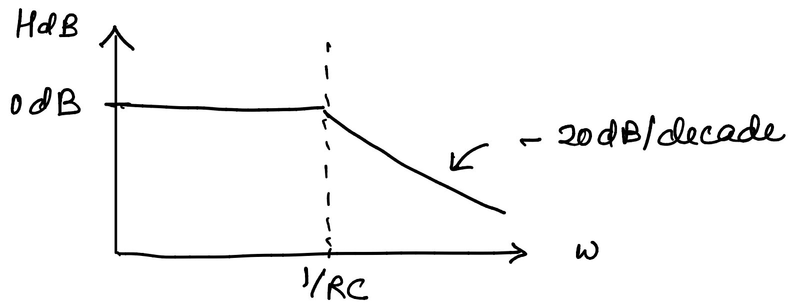
Filter can have gain > 1 ie $H_{dB} > 0dB$

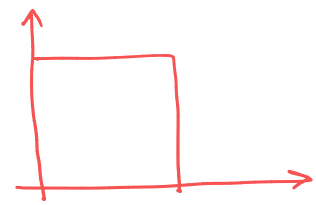
1st order LPF using R, L, C,



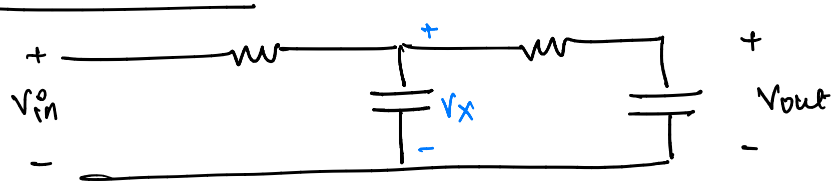
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1/Cs}{1/Cs + R} = \frac{1}{1+RCs}$$

$$= \frac{1}{1 + s/(1/RC)}$$





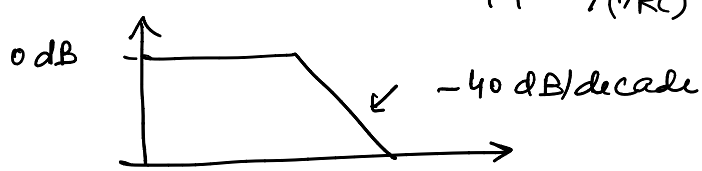
2nd order LPF :-



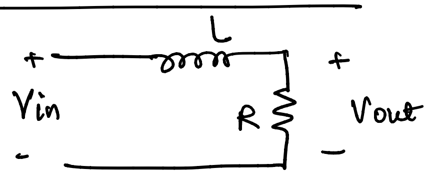
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$$= \frac{V_{out}(s)}{V_x} \cdot \frac{V_x}{V_{in}}$$

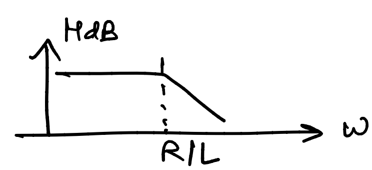
$$\Rightarrow \quad H(s) = \frac{1}{1 + s/(1/RC)} \cdot \frac{1}{1 + s/(1/RC)}$$

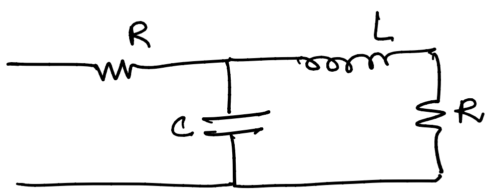


Another 1st order LPF :-

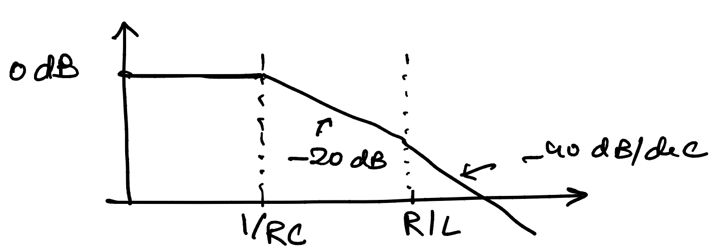


$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R+Ls} = \frac{1}{1 + s/(R/L)}$$



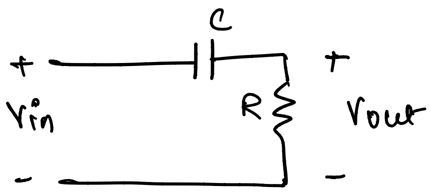


$$H(s) = \frac{1}{1 + s/(1/RC)} \cdot \frac{1}{1 + s/(R/L)}$$



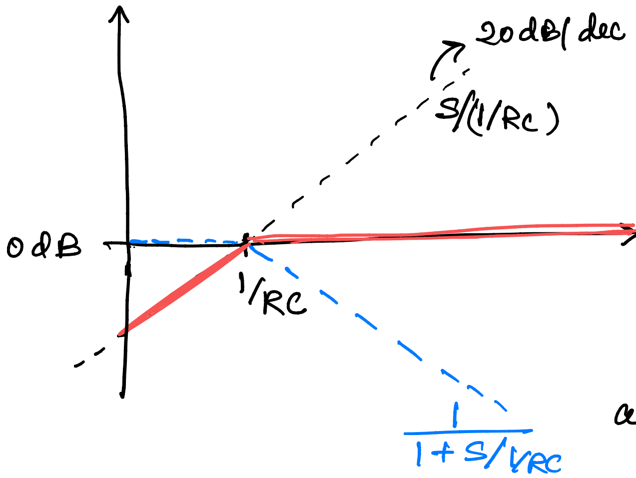
lets assume $1/RC < R/L$

High Pass Filter :—



$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R + 1/Cs} = \frac{RCs}{RCs + 1}$$

$$= \frac{s/(1/RC)}{1 + s/(1/RC)}$$



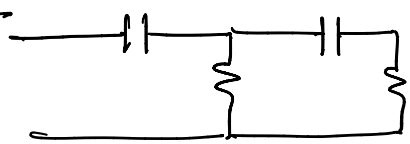
zero: $s = 0$

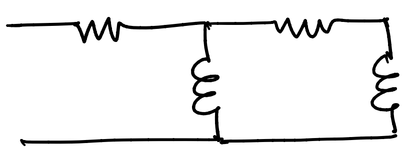
Poles: $s = -1/RC$

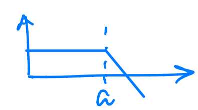
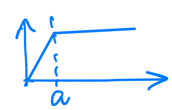
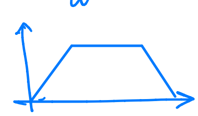
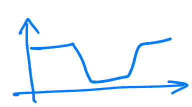
$s/(1/RC)$

at $\omega = 1/RC$ $\quad H_{dB} = 0\,dB$

2nd order:



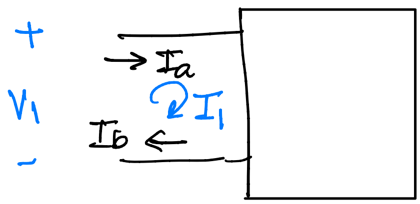


General transfer functions for nth order filters:	Simplest form
LPF : $H(s) = \dfrac{Ka_0}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$	$\dfrac{1}{1 + s/a}$ 
HPF : $H(s) = \dfrac{Ks^n}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$	$\dfrac{s/a}{1 + s/a}$ 
BPF : $H(s) = \dfrac{Ks^{n/2}}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$, n is even	$\dfrac{s}{s^2 + as + b}$ 
BSF : $H(s) = \dfrac{K(s^2 + \omega_0^2)^{n/2}}{s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0}$, n is even	$\dfrac{s^2 + \omega_0^2}{s^2 + as + b}$ 

Two-Port Network

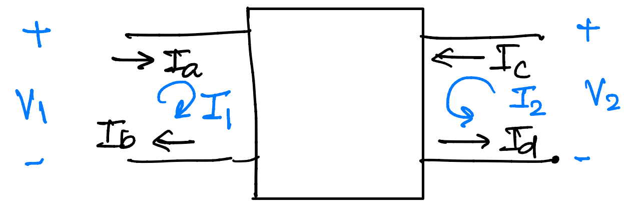
Chapter 16 of 9th edition

Port: A pair of terminals at which a signal can enter or leave a network.



$I_a = I_b$

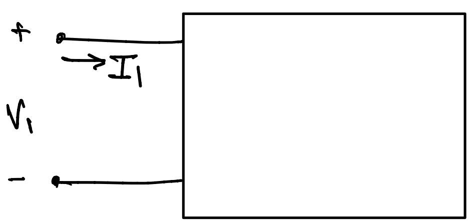
One port network



$I_a = I_b$
$I_c = I_d$

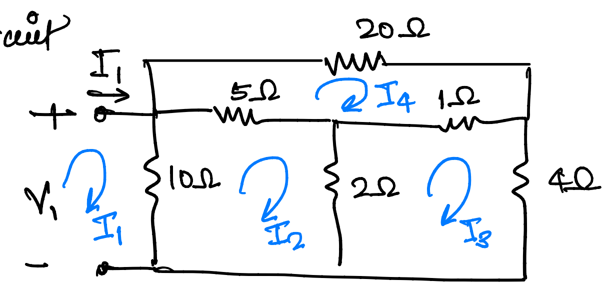
Two port network

One Port Network :



Input impedance : $Z_{in} = \dfrac{V_1}{I_1}$

Consider a circuit



Write KVL equations:

$$-V_1 + 10\,(I_1 - I_2) = 0$$
$$10\,(I_1 - I_2) + 5\,(I_2 - I_4) + 2\,(I_2 - I_3) = 0$$
$$2\,(I_3 - I_2) + 1\,(I_3 - I_4) + 4I_3 = 0$$
$$20\,I_4 + 1\,(I_4 - I_3) + 5\,(I_4 - I_2) = 0$$

Simplifying gives

$$\begin{cases} V_1 = 10I_1 - 10I_2 \\ 0 = -10\,I_2 - 17I_2 - 2I_3 - 5I_4 \\ 0 = -2I_2 + 7I_3 - I_4 \\ 0 = -5I_2 - I_3 + 26\,I_4 \end{cases}$$

$$\begin{bmatrix} V_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & -5 & -1 & 26 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

Cramer's rule:

$$I_1 = \frac{\begin{vmatrix} V_1 & -10 & 0 & 0 \\ 0 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & -5 & -1 & 26 \end{vmatrix}}{\begin{vmatrix} 10 & -10 & 0 & 0 \\ -10 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & -5 & -1 & 26 \end{vmatrix}}$$

$$\Rightarrow \quad I_1 = \frac{V_1 \begin{vmatrix} 17 & -2 & -5 \\ -2 & 7 & -1 \\ -5 & -1 & 26 \end{vmatrix}}{\begin{vmatrix} 10 & -10 & 0 & 0 \\ -10 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & -5 & -1 & 26 \end{vmatrix}} \qquad \begin{matrix} \Delta_{11} \\ \\ \Delta z \end{matrix}$$

$$\Rightarrow \quad I_1 = V_1 \frac{\Delta_{11}}{\Delta z}$$

$$\Rightarrow \quad \boxed{Z_{in} = \frac{V_1}{I_1} = \frac{\Delta z}{\Delta_{11}}}$$

This can be generalized for other circuits as well. Lets say that the circuit has N meshes. Using similar approach we can show that

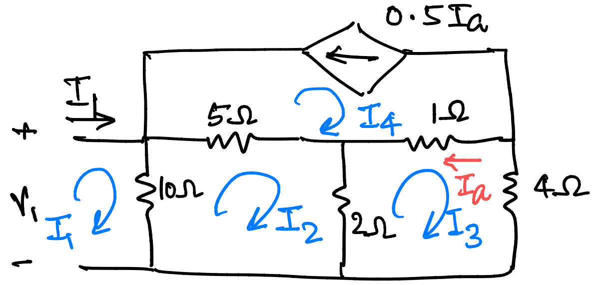
$$Z_{in} = \frac{\Delta z}{\Delta_{11}}$$

where

$$\Delta z = \begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & Z_{N3} & \cdots & Z_{NN} \end{vmatrix}$$

$$\Delta_{11} = \begin{vmatrix} Z_{22} & Z_{23} & \cdots & Z_{2N} \\ \vdots & \vdots & & \vdots \\ Z_{N2} & Z_{N3} & \cdots & Z_{NN} \end{vmatrix}$$

Example



writing KVL:

$$-V_1 + (I_1 - I_2)10 = 0$$
$$10(I_2 - I_1) + 5(I_2 - I_4) + 2(I_2 - I_3) = 0$$
$$2(I_3 - I_2) + 1(I_3 - I_4) + 4I_3 = 0$$

$$I_4 = -0.5\,I_a = -0.5(I_4 - I_3) \Rightarrow 1.5I_4 - 0.5I_3 = 0$$

Simplifying gives:
$$\begin{bmatrix} V_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 0 & 0 \\ -10 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}$$

$$\Delta Z = \begin{vmatrix} 10 & -10 & 0 & 0 \\ -10 & 17 & -2 & -5 \\ 0 & -2 & 7 & -1 \\ 0 & 0 & -0.5 & 1.5 \end{vmatrix}$$

$$= 10 \begin{vmatrix} 17 & -2 & -5 \\ -2 & 7 & -1 \\ 0 & -0.5 & 1.5 \end{vmatrix} - (-10) \begin{vmatrix} -10 & 0 & 0 \\ -2 & 7 & -1 \\ 0 & -0.5 & 1.5 \end{vmatrix}$$

$$= 10 \times 159 + 10(-100)$$
$$= 590$$

$$\Delta_{11} = \begin{vmatrix} 17 & -2 & -5 \\ -2 & 7 & -1 \\ 0 & -0.5 & 1.5 \end{vmatrix} = 159$$

$$Z_{in} = \frac{\Delta_Z}{\Delta_{11}} = \frac{590}{159} = 3.7\,\Omega$$

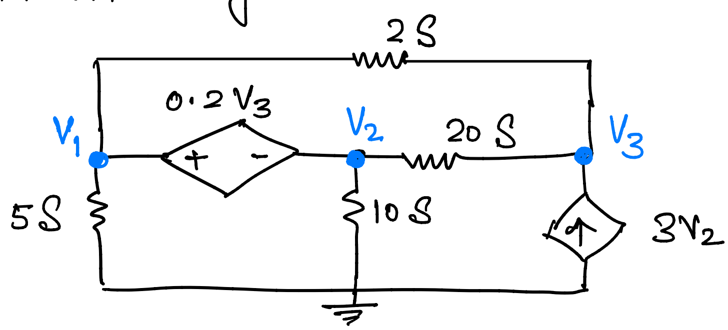
Similar analysis can be done using admittances, KCL equations:

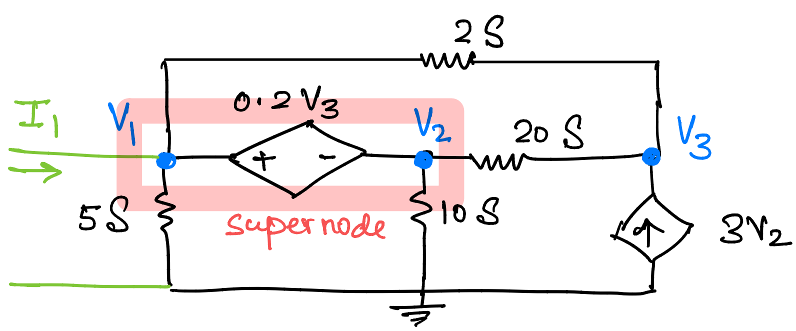
Consider a circuit of 3 nodes:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

then $Y_{in} = \dfrac{1}{Z_{in}} = \dfrac{\Delta Y}{\Delta_{11}}$

<u>Example:</u> find Y_{in} using ΔY and Δ_{11}.





writing KCL:

at node 1: $\quad -I_1 + V_1(5) + (V_1 - V_3)2 + V_2(10) + (V_2 - V_3)20 = 0$

at node 3: $\quad -3V_2 + (V_3 - V_1)2 + (V_3 - V_2)20 = 0$

at super node: $\quad V_1 - V_2 = 0.2V_3$

simplifying $\Rightarrow$

$$\begin{cases} I_1 = 7V_1 + 30V_2 - 22V_3 \\ 0 = -2V_1 - 23V_2 + 22V_3 \\ 0 = V_1 - V_2 - 0.2V_3 \end{cases}$$

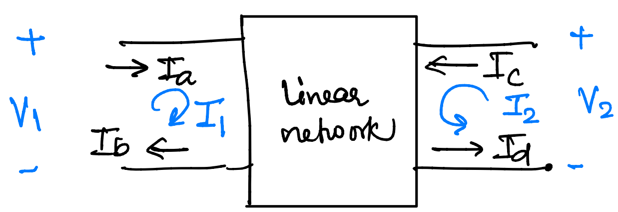
$$\Delta_Y = \begin{vmatrix} 7 & 30 & -22 \\ -2 & -23 & 22 \\ 1 & -1 & -0.2 \end{vmatrix} = 284.2$$

$$\Delta_{11} = \begin{vmatrix} -23 & 22 \\ -1 & -0.2 \end{vmatrix} = 26.6$$

$$Y_{in} = \frac{\Delta Y}{\Delta_{11}} = 10.68 \quad S$$

Two port networks

Admittance Parameters



Assumptions:
1. Linear circuit (R, L, C)
2. no independent sources

Applying superposition: $I_1 = (Y_{11})V_1 + (Y_{12})V_2$
$I_2 = (Y_{21})V_1 + (Y_{22})V_2$

$Y_{11}, Y_{12}, Y_{21}, Y_{22}$ are called admittance parameters.

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{[I]} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{[Y]} \underbrace{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}_{[V]}$$

Matrix Eqn: $[I] = [Y][V]$

$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$ Short circuit input impedance

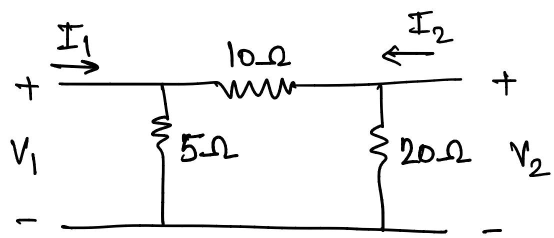
$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$ Short circuit transfer impedance

$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$ Short circuit transfer impedance

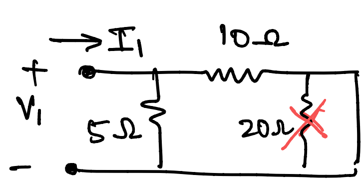
$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$ Short circuit output impedance

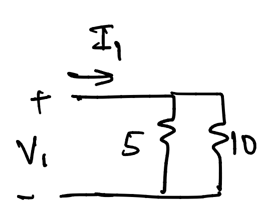
short circuit admittance parameters

Example: Find Y parameters of the network



$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$

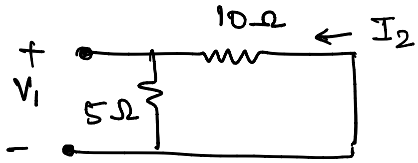


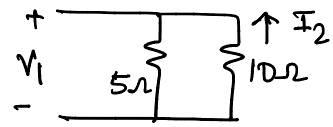


$\frac{I_1}{V_1} = Y_{eq.} = \frac{1}{10} + \frac{1}{5}$

$$Y_{11} = \frac{1}{10} + \frac{1}{5} = \frac{3}{10} = 0.1\,S$$

$$Y_{21} = \frac{I_2}{V_1}\Big/_{V_2=0}$$

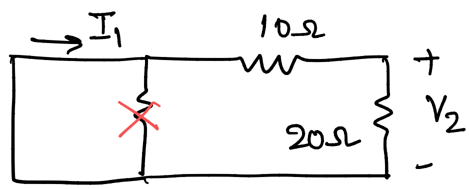




$$V_1 = -10 I_2$$

$$Y_{21} = \frac{-1}{10} = -0.1\,S$$

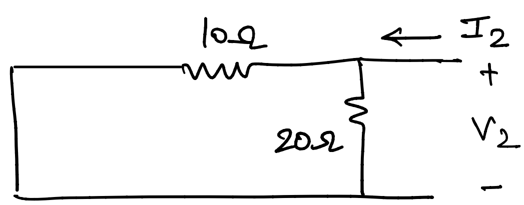
$$Y_{12} = \frac{I_1}{V_2}\Big/_{V_1=0}$$

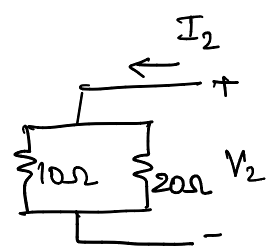


$$V_2 = -10 I_1$$

$$Y_{12} = -\frac{1}{10} = -0.1\,S$$

$$Y_{22} = \frac{I_2}{V_2}\Big/_{V_1=0}$$





$$Y_{22} = \frac{1}{10} + \frac{1}{20} = \frac{3}{20} = 0.15\,S$$
