In the formulation that I used,[1] I start from this problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = u_{\mathrm{D}} & \text{on } \partial\Omega \end{cases}$$

and I rewrite it in mixed form (from now on I will neglect the boundary conditions):

$$\begin{cases} \nabla \cdot \mathbf{q} = f & \text{in } \Omega \\ \mathbf{q} + \nabla u = \mathbf{0} & \text{in } \Omega \end{cases}$$

Let me introduce a function $\hat{u}$ defined on the skeleton $\mathcal{S}$ of triangulation $\mathcal{T}$. If we suppose to known in advance the function $\hat{u}$, on every cell $K_i \in \mathcal{T}$ we want to solve the following problem

$$\begin{cases} \nabla \cdot \mathbf{q}_i = f & \text{in } K_i \\ \mathbf{q}_i + \nabla u_i = \mathbf{0} & \text{in } K_i \\ u_i = \hat{u} & \text{on } \partial K_i \end{cases}$$

that can be rewritten in a weak form as
Let $\mathcal{V}_i \stackrel{\text{def}}{=} H^1(K_i) \times [H(\text{div}; K_i)]^d$*. Find* $(u_i, \mathbf{q}_i) \in \mathcal{V}_i$ *such that* $\forall (v_i, \mathbf{w}_i) \in \mathcal{V}_i$

$$\begin{cases} -\left(\nabla v_i, \mathbf{q}_i\right)_{K_i} + \langle v_i, \mathbf{n}_i \cdot \mathbf{q}_i \rangle_{\partial K_i} = (v_i, f)_{K_i} \\ -\left(\mathbf{w}, \mathbf{q}_i\right)_{K_i} + \left(\nabla \cdot \mathbf{w}, u\right)_{K_i} - \langle \mathbf{n}_i \cdot \mathbf{w}, \hat{u} \rangle_{\partial K_i} = 0 \end{cases}$$

In this formulation, all the derivatives are on the test functions. In particular, the first equation uses the fact that

$$(v_i, \nabla \cdot \mathbf{q}_i) = -\left(\nabla v_i, \mathbf{q}_i\right)_{K_i} + \langle v_i, \mathbf{n}_i \cdot \mathbf{q}_i \rangle_{\partial K_i}$$

and, in the second one, for some reasons that I do not remember (and that I suspect were not so good) I changed the signs. Moreover, I inserted $\hat{u}$ instead of u inside the boundary term because we want that $u = \hat{u}$ on the boundary.

At this point, we have n problems (where n is the number of cells) that are completely uncorrelated, beside for the presence of the variable $\hat{u}$ that instead is the same for every problem.

[1] which is freely taken from *Sevilla, Ruben & Huerta, Antonio. (2016). Tutorial on Hybridizable Discontinuous Galerkin (HDG) for Second–Order Elliptic Problems. 10.1007/978-3-319-31925-4_5*

We add now an equation to enforce normal continuity for $\mathbf{q}$: for all $\mu \in H^1(\mathcal{S})$

$$\sum_{K_i \in \mathcal{T}} \sum_{\text{face} F \text{of} K} \langle \mathbf{n}_i \cdot \mathbf{q}_i, \mu \rangle_F = 0$$

The reason why μ must be in H^1 is not completely clear to me.

In any case, now we have our problem that can be written as
Let $\mathcal{V} \stackrel{\text{def}}{=} \prod_i \mathcal{V}_i$; find $(u, \mathbf{q}) \in \mathrm{V}$ and $\hat{u} \in H^1(\mathcal{S})$ so that for every couple $(v, \mathbf{w}) \in \mathcal{V}$ and every function $\mu \in H^1(\mathcal{S})$ we have

$$\begin{cases} -(\nabla v, \mathbf{q})_{\mathcal{V}} + \langle v, \mathbf{n} \cdot \mathbf{q} \rangle_{\partial \mathcal{V}} = (v, f)_{\mathcal{V}} \\ -(\mathbf{w}, \mathbf{q})_{\mathcal{V}} + (\nabla \cdot \mathbf{w}, u)_{\mathcal{V}} - \langle \mathbf{n} \cdot \mathbf{w}, \hat{u} \rangle_{\partial \mathcal{V}} = 0 \\ \displaystyle\sum_{K_i \in \mathcal{T}} \sum_{\text{face} F \text{of} K} \langle \mu, \mathbf{n}_i \cdot \mathbf{q}_i \rangle_F = 0 \end{cases}$$

where the subscript i identify the i–th component in the space $\mathcal{V}$ and

$$(f, g)_{\mathcal{V}} \stackrel{\text{def}}{=} \sum_i (f_i, g_i)_{K_i}$$

$$\langle f, g \rangle_{\partial \mathcal{V}} \stackrel{\text{def}}{=} \sum_i \langle f_i, g_i \rangle_{\partial K_i}$$

The last step is to change $\mathbf{q}$ with

$$\hat{\mathbf{q}} \stackrel{\text{def}}{=} \mathbf{q} + \tau (u - \hat{u}) \mathbf{n}$$

for stabilizing the method. In this way we have

$$\begin{cases} -(\nabla v, \mathbf{q})_{\mathcal{V}} + \langle v, \mathbf{n} \cdot \mathbf{q} \rangle_{\partial \mathcal{V}} + \tau \langle v, u - \hat{u} \rangle_{\partial \mathcal{V}} = (v, f)_{\mathcal{V}} \\ -(\mathbf{w}, \mathbf{q})_{\mathcal{V}} + (\nabla \cdot \mathbf{w}, u)_{\mathcal{V}} - \langle \mathbf{n} \cdot \mathbf{w}, \hat{u} \rangle_{\partial \mathcal{V}} = 0 \\ \displaystyle\sum_{K_i \in \mathcal{T}} \sum_{\text{face} F \text{of} K} \langle \mu, \mathbf{n}_i \cdot \mathbf{q}_i \rangle_F + \tau \langle \mu, u_i - \hat{u} \rangle_F = 0 \end{cases}$$

1 Drift diffusion

Now I am trying to solve this

$$\begin{cases} \nabla V + \mathbf{E} = 0 \\ \nabla \cdot (\varepsilon \mathbf{E}) = n \\ \nabla n + \mathbf{W} = 0 \\ \nabla \cdot (\mu_n \mathbf{E} n - D_n \mathbf{W}) = \frac{1}{q} R_n(n) \end{cases}$$

In weak form (ν is the normal to avoid confusion with n):

$$\begin{cases} -(V, \nabla \cdot \mathbf{Q}_1) + \left\langle \widehat{V}, \mathbf{Q}_1 \cdot \nu \right\rangle + (\mathbf{E}, \mathbf{Q}_1) = 0 \\ -\left(\varepsilon \mathbf{E}, \nabla Z_1\right) + \left\langle \widehat{\varepsilon \mathbf{E} \cdot \nu}, Z_1 \right\rangle - (n, Z_1) = 0 \\ -(n, \nabla \cdot \mathbf{Q}_2) + \langle \widehat{n}, \mathbf{Q}_2 \cdot \nu \rangle + (\mathbf{W}, \mathbf{Q}_2) = 0 \\ -\left(\mu_n \mathbf{E} n - D_n \mathbf{W}, \nabla Z_2\right) + \left\langle (\mu_n n \mathbf{E} \cdot \nu - D_n \mathbf{W} \cdot \nu)^{\wedge}, Z_2 \right\rangle = \left(\frac{1}{q} R_n(n), Z_2\right) \\ \left\langle \widehat{\varepsilon \mathbf{E} \cdot \nu}, \mu_1 \right\rangle = 0 \\ \langle (\mu_n n \mathbf{E} \cdot \nu - D_n \mathbf{W} \cdot \nu)^{\wedge}, \mu_2 \rangle = 0 \end{cases}$$

For what concerns the fluxes, I made the following choice: $\widehat{V}$ and $\widehat{n}$ are independent variables, while

$$\widehat{\varepsilon \mathbf{E} \cdot \nu} \stackrel{\text{def}}{=} \varepsilon \mathbf{E} \cdot \nu + \tau \big(V - \widehat{V}\big)$$

$$(\mu_n n \mathbf{E} \cdot \nu - D_n \mathbf{W} \cdot \nu)^{\wedge} \stackrel{\text{def}}{=} \mu_n n \mathbf{E} \cdot \nu - D_n \mathbf{W} \cdot \nu + \tau \left(n - \widehat{n}\right)$$

This choice brings us to the following system

$$\begin{cases} -(V, \nabla \cdot \mathbf{Q}_1) + \left\langle \widehat{V}, \mathbf{Q}_1 \cdot \nu \right\rangle + (\mathbf{E}, \mathbf{Q}_1) = 0 \\ -\left(\varepsilon \mathbf{E}, \nabla Z_1\right) + \langle \varepsilon \mathbf{E} \cdot \nu, Z_1 \rangle + \tau \big\langle V - \widehat{V}, Z_1 \big\rangle - (n, Z_1) = 0 \\ -(n, \nabla \cdot \mathbf{Q}_2) + \langle \widehat{n}, \mathbf{Q}_2 \cdot \nu \rangle + (\mathbf{W}, \mathbf{Q}_2) = 0 \\ -\left(\mu_n \mathbf{E} n, \nabla Z_2\right) + \left(D_n \mathbf{W}, \nabla Z_2\right) + \langle \mu_n n \mathbf{E} \cdot \nu, Z_2 \rangle \\ \qquad - \langle D_n \mathbf{W} \cdot \nu, Z_2 \rangle + \tau \left\langle \left(n - \widehat{n}\right), Z_2 \right\rangle = \left(\dfrac{1}{q} R_n(n), Z_2\right) \\ \langle \varepsilon \mathbf{E} \cdot \nu, \mu_1 \rangle + \tau \big\langle V - \widehat{V}, \mu_1 \big\rangle = 0 \\ \langle \mu_n n \mathbf{E} \cdot \nu, \mu_2 \rangle - \langle D_n \mathbf{W} \cdot \nu, \mu_2 \rangle + \tau \left\langle n - \widehat{n}, \mu_2 \right\rangle = 0 \end{cases} \tag{1}$$

Now, we still have two non linear terms: $(\mu_n \mathbf{E} n, \nabla Z_2)$ and $< \mu_n n \mathbf{E} \cdot \nu, Z_2 >$.

To linearize the system, I will substitute the electric field with the one computed at the previous iteration

$$(\mu_n \mathbf{E} n, \nabla Z_2) \rightarrow (\mu_n \mathbf{E}_0 n, \nabla Z_2)$$

$$< \mu_n n \mathbf{E} \cdot \nu, Z_2 > \rightarrow < \mu_n n \mathbf{E}_0 \cdot \nu, Z_2 >$$

where $\mathbf{E}_0$ is a fixed vector obtained at the previous iteration.

Therefore, the final problem is:

$$\begin{cases} -(V, \nabla \cdot \mathbf{Q}_1) + \left\langle \widehat{V}, \mathbf{Q}_1 \cdot \nu \right\rangle + (\mathbf{E}, \mathbf{Q}_1) = 0 \\ -(\varepsilon \mathbf{E}, \nabla Z_1) + \langle \varepsilon \mathbf{E} \cdot \nu, Z_1 \rangle + \tau \left\langle V, Z_1 \right\rangle - \tau \left\langle \widehat{V}, Z_1 \right\rangle - (n, Z_1) = 0 \\ -(n, \nabla \cdot \mathbf{Q}_2) + \langle \widehat{n}, \mathbf{Q}_2 \cdot \nu \rangle + (\mathbf{W}, \mathbf{Q}_2) = 0 \\ -(\mu_n \mathbf{E}_0 n, \nabla Z_2) + (D_n \mathbf{W}, \nabla Z_2) + \langle \mu_n n \mathbf{E}_0 \cdot \nu, Z_2 \rangle \\ \qquad - \langle D_n \mathbf{W} \cdot \nu, Z_2 \rangle + \tau \langle n, Z_2 \rangle - \tau \langle \widehat{n}, Z_2 \rangle = \left(\frac{1}{q} R_n(n), Z_2 \right) \\ \langle \varepsilon \mathbf{E} \cdot \nu, \mu_1 \rangle + \tau \left\langle V, \mu_1 \right\rangle - \tau \left\langle \widehat{V}, \mu_1 \right\rangle = 0 \\ \langle \mu_n n \mathbf{E}_0 \cdot \nu, \mu_2 \rangle - \langle D_n \mathbf{W} \cdot \nu, \mu_2 \rangle + \tau \langle n, \mu_2 \rangle - \tau \langle \widehat{n}, \mu_2 \rangle = 0 \end{cases}$$

2 Another linearization

Let us suppose that the functions we are looking for (V, $\widehat{V}$, $\mathbf{E}$, n, $\widehat{n}$, and $\mathbf{W}$) can be written as

$$\begin{aligned} V &= V_0 + \delta V \\ \widehat{V} &= \widehat{V}_0 + \widehat{\delta V} \\ \mathbf{E} &= \mathbf{E}_0 + \delta \mathbf{E} \\ n &= n_0 + \delta n \\ \widehat{n} &= \widehat{n}_0 + \widehat{\delta n} \\ \mathbf{W} &= \mathbf{W}_0 + \delta \mathbf{W} \end{aligned}$$

where V_0, $\widehat{V}_0$, $\mathbf{E}_0$, n_0, $\widehat{n}_0$, and $\mathbf{W}_0$ are some arbitrary chosen functions. Then, the system 1 becomes

$$
\begin{cases}
-(V_0+\delta V,\nabla\cdot\mathbf{Q}_1)+\left\langle\widehat{V_0}+\widehat{\delta V},\mathbf{Q}_1\cdot\nu\right\rangle+(\mathbf{E}_0+\delta\mathbf{E},\mathbf{Q}_1)=0\\[2ex]
-\left(\varepsilon\left(\mathbf{E}_0+\delta\mathbf{E}\right),\nabla Z_1\right)+\left\langle\varepsilon\left(\mathbf{E}_0+\delta\mathbf{E}\right)\cdot\nu,Z_1\right\rangle\\
\qquad\qquad+\tau\left\langle V_0+\delta V-\widehat{V_0}-\widehat{\delta V},Z_1\right\rangle-(n_0+\delta n,Z_1)=0\\[2ex]
-(n_0+\delta n,\nabla\cdot\mathbf{Q}_2)+\left\langle\widehat{n_0}+\widehat{\delta n},\mathbf{Q}_2\cdot\nu\right\rangle+(\mathbf{W}_0+\delta\mathbf{W},\mathbf{Q}_2)=0\\[2ex]
-\left(\mu_n\left(n_0+\delta n\right)\left(\mathbf{E}_0+\delta\mathbf{E}\right),\nabla Z_2\right)+\left(D_n\left(\mathbf{W}_0+\delta\mathbf{W}\right),\nabla Z_2\right)\\
\qquad+\left\langle\mu_n\left(n_0+\delta n\right)\left(\mathbf{E}_0+\delta\mathbf{E}\right)\cdot\nu,Z_2\right\rangle-\left\langle D_n\left(\mathbf{W}_0+\delta\mathbf{W}\right)\cdot\nu,Z_2\right\rangle\\
\qquad\qquad+\tau\left\langle n_0+\delta n-\widehat{n_0}-\widehat{\delta n},Z_2\right\rangle=\left(q^{-1}R_n(n_0+\delta n),Z_2\right)\\[2ex]
\left\langle\varepsilon\left(\mathbf{E}_0+\delta\mathbf{E}\right)\cdot\nu,\mu_1\right\rangle+\tau\left\langle V_0+\delta V-\widehat{V_0}-\widehat{\delta V},\mu_1\right\rangle=0\\[2ex]
\left\langle\mu_n\left(n_0+\delta n\right)\left(\mathbf{E}_0+\delta\mathbf{E}\right)\cdot\nu,\mu_2\right\rangle-\left\langle D_n\left(\mathbf{W}_0+\delta\mathbf{W}\right)\cdot\nu,\mu_2\right\rangle\\
\qquad\qquad+\tau\left\langle n_0+\delta n-\widehat{n_0}-\widehat{\delta n},\mu_2\right\rangle=0
\end{cases}
\tag{2}
$$

To linearize the previous system, we approximate

$$R_n(n_0+\delta n)\approx R_n(n_0)+\frac{\mathrm{d}R_n}{\mathrm{d}n}|_{n=n_0}\delta n$$

and

$$\left(n_0+\delta n\right)\left(\mathbf{E}_0+\delta\mathbf{E}\right)\approx n_0\mathbf{E}_0+n_0\delta\mathbf{E}+\delta n\mathbf{E}_0$$

To simplify a little bit the notation, we will use the following symbols

$$R_{n_0}\overset{\text{def}}{=}R_n(n_0)$$

$$R'_{n_0}\overset{\text{def}}{=}\frac{\mathrm{d}R_n}{\mathrm{d}n}|_{n=n_0}$$

Therefore, the system 2 becomes

$$
\left\{
\begin{aligned}
&-(\delta V, \nabla\cdot\mathbf{Q}_1) + \left\langle \widehat{\delta V}, \mathbf{Q}_1\cdot\nu \right\rangle + (\delta\mathbf{E}, \mathbf{Q}_1) \\
&\qquad\qquad\qquad = (V_0, \nabla\cdot\mathbf{Q}_1) - \left\langle \widehat{V}_0, \mathbf{Q}_1\cdot\nu \right\rangle - (\mathbf{E}_0, \mathbf{Q}_1) \\
&-(\varepsilon\delta\mathbf{E}, \nabla Z_1) + \langle \varepsilon\delta\mathbf{E}\cdot\nu, Z_1\rangle + \tau\left\langle \delta V - \widehat{\delta V}, Z_1 \right\rangle - (\delta n, Z_1) \\
&\qquad\qquad = (\varepsilon\mathbf{E}_0, \nabla Z_1) - \langle \varepsilon\mathbf{E}_0\cdot\nu, Z_1\rangle - \tau\left\langle V_0 - \widehat{V}_0, Z_1 \right\rangle + (n_0, Z_1) \\
&-(\delta n, \nabla\cdot\mathbf{Q}_2) + \left\langle \widehat{\delta n}, \mathbf{Q}_2\cdot\nu \right\rangle + (\delta\mathbf{W}, \mathbf{Q}_2) \\
&\qquad\qquad\qquad = (n_0, \nabla\cdot\mathbf{Q}_2) - \left\langle \widehat{n_0}, \mathbf{Q}_2\cdot\nu \right\rangle - (\mathbf{W}_0, \mathbf{Q}_2) \\
&-(\mu_n(n_0\delta\mathbf{E} + \delta n\mathbf{E}_0), \nabla Z_2) + (D_n\delta\mathbf{W}, \nabla Z_2) - \left(q^{-1}R'_{n_0}\delta n, Z_2\right) \\
&\qquad + \langle \mu_n(n_0\delta\mathbf{E} + \delta n\mathbf{E}_0)\cdot\nu, Z_2\rangle - \langle D_n\delta\mathbf{W}\cdot\nu, Z_2\rangle + \tau\left\langle \delta n - \widehat{\delta n}, Z_2 \right\rangle \\
&\quad = \left(q^{-1}R_{n_0}, Z_2\right) + (\mu_n n_0\mathbf{E}_0, \nabla Z_2) - (D_n\mathbf{W}_0, \nabla Z_2) - \langle \mu_n n_0\mathbf{E}_0\cdot\nu, Z_2\rangle \\
&\qquad\qquad\qquad + \left\langle D_n\left(\mathbf{W}_0\right)\cdot\nu, Z_2 \right\rangle - \tau\left\langle n_0 - \widehat{n_0}, Z_2 \right\rangle \\
&\langle \varepsilon\delta\mathbf{E}\cdot\nu, \mu_1\rangle + \tau\left\langle \delta V - \widehat{\delta V}, \mu_1 \right\rangle = -\langle \varepsilon\mathbf{E}_0\cdot\nu, \mu_1\rangle - \tau\left\langle V_0 - \widehat{V}_0, \mu_1 \right\rangle \\
&\langle \mu_n(n_0\delta\mathbf{E} + \delta n\mathbf{E}_0)\cdot\nu, \mu_2\rangle - \langle D_n\delta\mathbf{W}\cdot\nu, \mu_2\rangle + \tau\left\langle \delta n - \widehat{\delta n}, \mu_2 \right\rangle \\
&\qquad\qquad = -\langle \mu_n n_0\mathbf{E}_0\cdot\nu, \mu_2\rangle + \langle D_n\mathbf{W}_0\cdot\nu, \mu_2\rangle - \tau\left\langle n_0 - \widehat{n_0}, \mu_2 \right\rangle
\end{aligned}
\right.
$$

or, equivalently,

$$
\left\{
\begin{aligned}
&-(\delta V, \nabla\cdot\mathbf{Q}_1)+\left\langle\widehat{\delta V}, \mathbf{Q}_1\cdot\nu\right\rangle+(\delta\mathbf{E}, \mathbf{Q}_1)\\
&\qquad\qquad\qquad = (V_0, \nabla\cdot\mathbf{Q}_1)-\left\langle\widehat{V}_0, \mathbf{Q}_1\cdot\nu\right\rangle-(\mathbf{E}_0, \mathbf{Q}_1)\\
&-(\varepsilon\delta\mathbf{E}, \nabla Z_1)+\langle\varepsilon\delta\mathbf{E}\cdot\nu, Z_1\rangle+\tau\left\langle\delta V, Z_1\right\rangle-\tau\left\langle\widehat{\delta V}, Z_1\right\rangle-(\delta n, Z_1)\\
&\qquad = (\varepsilon\mathbf{E}_0, \nabla Z_1)-\langle\varepsilon\mathbf{E}_0\cdot\nu, Z_1\rangle-\tau\left\langle V_0, Z_1\right\rangle+\tau\left\langle\widehat{V}_0, Z_1\right\rangle+(n_0, Z_1)\\
&-(\delta n, \nabla\cdot\mathbf{Q}_2)+\left\langle\widehat{\delta n}, \mathbf{Q}_2\cdot\nu\right\rangle+(\delta\mathbf{W}, \mathbf{Q}_2)\\
&\qquad\qquad\qquad = (n_0, \nabla\cdot\mathbf{Q}_2)-\left\langle\widehat{n}_0, \mathbf{Q}_2\cdot\nu\right\rangle-(\mathbf{W}_0, \mathbf{Q}_2)\\
&-(\mu_n(n_0\delta\mathbf{E}+\delta n\mathbf{E}_0), \nabla Z_2)+(D_n\delta\mathbf{W}, \nabla Z_2)-\left(q^{-1}R'_{n_0}\delta n, Z_2\right)\\
&\qquad+\langle\mu_n(n_0\delta\mathbf{E}+\delta n\mathbf{E}_0)\cdot\nu, Z_2\rangle-\langle D_n\delta\mathbf{W}\cdot\nu, Z_2\rangle+\tau\left\langle\delta n, Z_2\right\rangle-\tau\left\langle\widehat{\delta n}, Z_2\right\rangle\\
&\qquad\qquad = \left(q^{-1}R_{n_0}, Z_2\right)+(\mu_n n_0\mathbf{E}_0, \nabla Z_2)-(D_n\mathbf{W}_0, \nabla Z_2)\\
&\qquad\qquad\quad -\langle\mu_n n_0\mathbf{E}_0\cdot\nu, Z_2\rangle+\left\langle D_n\mathbf{W}_0\cdot\nu, Z_2\right\rangle-\tau\left\langle n_0, Z_2\right\rangle+\tau\left\langle\widehat{n}_0, Z_2\right\rangle\\
&\langle\varepsilon\delta\mathbf{E}\cdot\nu, \mu_1\rangle+\tau\left\langle\delta V, \mu_1\right\rangle-\tau\left\langle\widehat{\delta V}, \mu_1\right\rangle\\
&\qquad\qquad\qquad = -\langle\varepsilon\mathbf{E}_0\cdot\nu, \mu_1\rangle-\tau\left\langle V_0, \mu_1\right\rangle+\tau\left\langle\widehat{V}_0, \mu_1\right\rangle\\
&\langle\mu_n(n_0\delta\mathbf{E}+\delta n\mathbf{E}_0)\cdot\nu, \mu_2\rangle-\langle D_n\delta\mathbf{W}\cdot\nu, \mu_2\rangle+\tau\left\langle\delta n, \mu_2\right\rangle-\tau\left\langle\widehat{\delta n}, \mu_2\right\rangle\\
&\qquad = -\langle\mu_n n_0\mathbf{E}_0\cdot\nu, \mu_2\rangle+\langle D_n\mathbf{W}_0\cdot\nu, \mu_2\rangle-\tau\left\langle n_0, \mu_2\right\rangle+\tau\left\langle\widehat{n}_0, \mu_2\right\rangle
\end{aligned}
\right.
$$