Quantum Optics and Laser 2021/22
Homework - Quantum Description of Light
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1 Fluctuation of the coherent state

Following the definition of the quadrature operators X_1 and X_2 and of their fluctuations, demonstrate that when the light in a single mode is in a coherent state $|\alpha\rangle$, then such fluctuations are the same as for the vacuum state.

Solution. A coherent state $|\alpha\rangle$ is defined to be the (unique) eigenstate of the annihilation operator $\hat{a}$ with corresponding eigenvalue α, namely

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \qquad\qquad \langle\alpha|\,\hat{a}^\dagger = \alpha^\star\,|\alpha\rangle$$

where one equation is simply given by taking the conjugate of the other (and vice-versa). The expectation values of quadrature operators defined as

$$\hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^\dagger) \qquad\qquad \hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^\dagger). \tag{1}$$

are given by

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha+\alpha^\star) \qquad\qquad \langle\alpha|\hat{X}_2|\alpha\rangle = \frac{1}{2i}(\alpha-\alpha^\star)$$

from which it follows directly

$$\langle\alpha|\hat{X}_1|\alpha\rangle^2 = \frac{1}{4}\left(\alpha^2+\alpha^{*2}+|\alpha|^2\right) \qquad\qquad \langle\alpha|\hat{X}_2|\alpha\rangle^2 = -\frac{1}{4}\left(\alpha^2+\alpha^{*2}-|\alpha|^2\right). \tag{2}$$

Furthermore,

$$\begin{aligned}\hat{X}_1^2 &= \frac{1}{4}(\hat{a}+\hat{a}^\dagger)(\hat{a}+\hat{a}^\dagger)\\ &= \frac{1}{4}(\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}) \underset{(a)}{=} \frac{1}{4}\left(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1\right)\end{aligned}$$

where in (a) we used the commutator $\left[\hat{a},\hat{a}^\dagger\right]=1$; analogously, $\hat{X}_2 = -\frac{1}{4}\left(\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^\dagger\hat{a}-1\right)$. From this,

$$\langle\alpha|\hat{X}_1^2|\alpha\rangle^2 = \frac{1}{4}\left(\alpha^2+\alpha^{*2}+2|\alpha|^2+1\right) \qquad\qquad \langle\alpha|\hat{X}_2^2|\alpha\rangle^2 = -\frac{1}{4}\left(\alpha^2+\alpha^{*2}-2|\alpha|^2-1\right). \tag{3}$$

and since quantum fluctuations of the quadrature operators can be characterised by the variance

$$\left\langle\left(\Delta\hat{X}_*\right)^2\right\rangle = \left\langle\hat{X}_*^2\right\rangle - \left\langle\hat{X}_*\right\rangle^2, \tag{4}$$

substituting Eq. (2) and Eq. (3) in Eq. (4) we finally get to

$$\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_\alpha = \frac{1}{4} = \left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_\alpha$$

which are exactly the same values we obtain for the fluctuations of the quadrature operators in the case of the vacuum state: indeed both saturate the uncertainty product given by

$$\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle \left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle \geq \frac{1}{16}.$$

2 Photon statistics

Consider a beam of blue light ($\lambda = 405\,\mathrm{nm}$).

(a) What is the power that carry one billion photons per second?

(b) What is the fluctuation of the arrival number for time bins of 10 ps duration considering unitary detection efficiency?

(c) How much momentum is carried by one photon?

Solution.

(a) The power is given by the relation

$$P = \Phi h\nu$$

with $\Phi = 1\,\frac{\mathrm{photon}}{\mathrm{ns}}$ the mean photon flux and $\nu = c/\lambda$ the corresponding frequency. Substituting for the given values one obtains $P \approx 0.5\,\mathrm{nW}$.

(b) Given that the beam is coherent, we can exploit the fact that under the assumption that the registration of photons are statistically independent, the probability distribution of the number of photons $p(n)$ follows a Poisson distribution. The fluctuation (i.e. variance) is thus equal to its mean $\sigma_n^2 = \bar{n}$ and $\bar{n} = \Phi T = \frac{PT}{h\nu}$ where $T = 10\,\mathrm{ps}$ is the considered time interval. Substituting one gets to $\sigma_n^2 = 0.01$.

(c) From $p = h/\lambda$ one gets $p \approx 10\,\mathrm{neV}$.

3 Temporal evolution of a single mode state

Consider a single-mode cavity field. At time $t = 0$ it is expressed as the super-position of two Fock states, as:

$$|\psi(0)\rangle = \frac{|n\rangle + \mathrm{e}^{i\phi}\,|n+1\rangle}{\sqrt{2}}, \tag{5}$$

where the phase is $\phi \neq 0$ and $n > 0$. Describe the time evolution of this state, that is $|\psi(t)\rangle$ at a later time $t > 0$.

Solution. Given the (normalised) state at $t = 0$ in Eq. (5) with Hamiltonian $\hat{H}$, its time evolution is given by

$$
\begin{aligned}
|\psi(t)\rangle &= \mathrm{e}^{\frac{-i\hat{H}t}{\hbar}} |\psi(0)\rangle \\
&\underset{(b)}{=} \frac{1}{\sqrt{2}} \left(\exp\left(-i\omega t\left(n+\frac{1}{2}\right)\right)|n\rangle + \mathrm{e}^{i\phi} \exp\left(-i\omega t\left(n+\frac{3}{2}\right)\right)|n+1\rangle \right),
\end{aligned}
$$

where in (b) we substituted for the values of energy $\mathcal{E}_n = \hbar\omega\left(n+\frac{1}{2}\right)$.

4 Photon absorption in Fock basis

Consider the superposition of the vacuum and 20 photon number state

$$
|\psi\rangle = \frac{|0\rangle + |20\rangle}{\sqrt{2}}. \tag{6}
$$

(a) Calculate the average photon number for this state.

(b) Now, the annihilation operator is applies to the state, so a single photon is absorbed. Calculate the new value for the average photon number.

(c) The annihilation operator is applies again to the resulting state. Calculate the new value for the average photon number.

(d) Analyse two results and the meaning of the average photon number.

Solution.

(a) The average photon number $\bar{n}$ for the state in Eq. (6) is given by

$$
\bar{n} = \langle\psi|N|\psi\rangle = \langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle
$$

which can be easily calculated to be

$$
\bar{n} = \frac{1}{2}(0+20) = 10.
$$

(b) If we assume that a single photon is absorbed, the normalised state will become $|\psi'\rangle = |19\rangle$, and consequently the average photon number will be $\bar{n}' = 19$.

(c) If the process is repeated, the average photon number will be $\bar{n}'' = 18$.

(d) From the previous answers, it is clear to see that indeed the number operator counts the number of particles (photons in this case) in a system. Once the photon is absorbed, the uncertainty given by the superpostion of states is removed: from the moment we detect the presence of a photon, the presence of the vacuum state $|0\rangle$ is excluded. All we are left with is the ket $|20\rangle$, from which a photon is absorbed: is thus reasonable to conclude the results found analytically in points (b) and (c).