University of Utrecht
Mathematics Department
Introduction to Complex Systems
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Project 3

Spontaneous synchronisation for the Kuramoto model

We consider a large population of interacting oscillators living in the unit circle, under the dynamics of the so called *mean-field Kuramoto model*, defined in Lecture 11:

$$\frac{d\theta_i}{dt}(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)) \tag{1}$$

with $i = 1, \ldots, N$ and K the *coupling constant*. As we have seen in the lecture, for small coupling K each oscillator rotates with its natural frequency ω_i, whereas for large coupling K almost all angles θ_i will be entrained by the mean field and the oscillators *synchronize*. In this project we study first a simple case where the natural frequencies of the rotators are known and in the second part, when they are sampled from a probability distribution $g(\omega)$.

Lyapunov function for studying the stability of the *equilibria*

If we define

$$\bar{\omega} := \frac{1}{N}\sum_{i=1}^{N}\omega_i \tag{2}$$

the mean natural frequency, it is very reasonable to think that it is the common frequency of the *phase locked* component.

i) Prove indeed that:

$$\frac{d}{dt}\left(\frac{1}{N}\sum_{i=1}^{N}\theta_i(t)\right) = \bar{\omega}$$

Comment the result.

We consider now the new variables ϕ_i in the reference frame rotating with $\omega = \bar{\omega}$, i.e.:

$$\phi_i(t) := \theta_i(t) - \bar{\omega}t$$

so that the Kuramoto model becomes:

$$\frac{d\phi_i}{dt}(t) = (\omega_i - \bar{\omega}) + \frac{K}{N}\sum_{j=1}^{N}\sin(\phi_j(t) - \phi_i(t)) \tag{3}$$

ii) Show that the function $\mathcal{H}$:

$$\mathcal{H} := -\frac{K}{2N}\sum_{i,j}\cos(\phi_i - \phi_j) - \sum_{i=1}^{N}(\omega_i - \bar{\omega})\phi_i \tag{4}$$

is a *Lyapunov function*, i.e. it is that for any solution ϕ_i of the ODE:

$$\dot{\mathcal{H}} = \sum_{i=1}^{N}\frac{\partial\mathcal{H}}{\partial\phi_i}\frac{d\phi_i}{dt} \leq 0$$

Therefore, by the properties of the Lyapunov function, the dynamics in (3) converge to a minimum of $\mathcal{H}$.

iii) Find the equations for the *extreme points* of $\mathcal{H}$(i.e. $\nabla\mathcal{H}$=0). Is the configuration with $\omega_i = \bar{\omega}$, for all i an asymptotically stable equilibrium?

iv) If we introduce the order parameter:

$$re^{i\Psi} := \frac{1}{N}\sum_{j=1}^{N}e^{i\phi_j}$$

show that the stationary solutions of the Kuramoto model satisfies:

$$\Delta(\omega_i) = Kr\sin(\phi_i - \Psi)$$

with $\Delta(\omega_i) = \omega_i - \bar{\omega}$.

BONUS Just a curiosity. The Lyapunov function can be interpreted as the Hamiltonian the *mean-field* Heisenberg XY ferromagnet with coupling strength K. The first term aims at keeping all spins parallel. The second term represents a kind of random field: the two terms counteract each other (*frustration*). Show that the ground state of $\mathcal{H}$ is rotationally invariant also with the present of the random field. Therefore, what do yo expect for the *Jacobian* matrix of the system calculated at the fixed points? Is $(1, 1, \ldots, 1)$ an eigenvector?

2. Numerical studies for the Kuramoto Model

Normal distribution of the natural frequencies

We consider now a population of $N = 1000$ of Kuramoto oscillators, with a uniformly distributed initial phase $\theta_i(0)$ and natural frequencies ω sampled from a standard normal distribution $N(0, 1)$. Therefore:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2}$$

From the theory, we know the order parameter r satisfies the consistency equation:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr \sin\theta) d\theta \tag{5}$$

that can be integrated numerically for any given value of K. We know as well that the *critical value* of the coupling constant is:

$$K_c = 2\sqrt{2/\pi}$$

1. Integrate (1) using *Euler method* (see Ch. 20 of J. Butcher, *Numerical Methods in ODE*) for $t \in [0, T]$, with $T = 100$, a time step of $dt = 10^{-2}$. Perform the simulation for different values of K (e.g. $K \in [0.5]$ with $dK = 0.2$). For each K calculate the value of the limiting r_∞ (e.g. $r(T)$). Plot r_∞ in function of K. Comment the graph obtained. Is it in good agreement with the numerical solutions of (5)? What can be improved?

2. Run now the model for the two values $K = 1$ and $K = 2$. Make a graph of $r(t)$ in function of t. Comment the findings. Is it in agreement with what the theory predicts?

Natural frequencies uniformly distributed

Let us consider now the case:

$$g(\omega) = \begin{cases} \frac{1}{2\gamma}, & |\omega| \leq \gamma \\ 0, & |\omega| > \gamma \end{cases}$$

with $\gamma = 1/2$.

iii) Also in this case the initial phases $\theta_i(0)$ are uniformly distributed. However, this time:

$$N = 2000, \;\; dt = 0.05, \;\; T = 200, \;\; dK = 0.03, \;\; K_{\max} = 1.5$$

For each K calculate the value of the limiting r_∞ (e.g. $r(T)$). Plot r_∞ in function of K. Comment the graph obtained. Is it different from the case of the *unimodal* normal distribution? Which is the K_c in this case?

iv) Consider now $K = 1$. For a fixed realisation of the natural frequencies ω_i, perform 10 different simulations obtained sampling different initial conditions $\theta_i(0)$. Then, plot 10 curves $r(t)$ of the coherence parameter in function of the time for 10 difference simulations.

v) Consider again $K = 1$. For a fixed realisation of the initial conditions $\theta_i(0)$, perform 10 different simulations obtained sampling different realisation of the natural frequencies ω_i, What can you say about the relaxation time towards the synchronised state? Are there any differences with respect of point iv)? Try to find some plausible explanation.

3. BONUS: Kuramoto model on Watts-Strogatz Model

Consider now a Kuramoto model on the Watts-Strogatz model $WS(N; 2r; p)$ defined as follow:

$$\dot{\theta}_i = \omega_i + \frac{K}{2r} \sum_{j \in \mathcal{G}(i)} \sin(\theta_j - \theta_i)$$

with $i \in \{1, \ldots, N\}$; $\mathcal{G}(i)$ is the set of the connected neighbors of the site i in the realisation of the WS model, and ω_i are sampled from a standard normal distribution.

a) Try to design and perform a simulation study in order to study the relation between the coherence parameter r_∞ and the coupling constant K, for different values of the *rewiring parameter* p (e.g. $p \in \{0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.1\}$). Interpret your results and try to explain the peculiarity of the case $p = 0$.
Suggested values: $k = 3; N = 2000, T = 200$.

b) Try to estimate the relation between the critical coupling K_c and the *rewiring probability* p.