Lecture 11: Syncronization in Complex Systems

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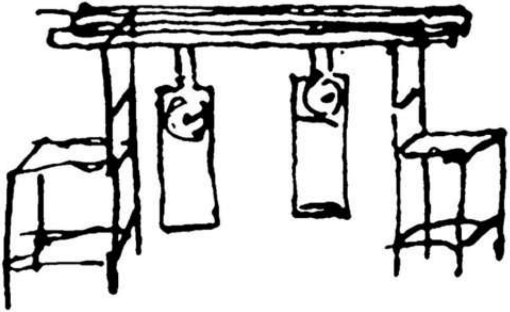
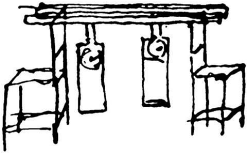


27 November 2019

Overview

Christiaan Huygens (1629-1695): *sympathy* of two clocks



"... It is quite worth noting that when we suspended two clocks so constructed from two hooks imbedded in the same wooden beam, the motions of each pendulum in opposite swings were so much in agreement that they never receded the least bit from each other and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it reestablished itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible"

Huygens experiment

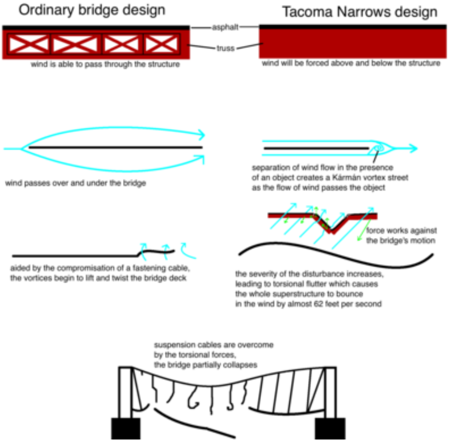
The sympathy of two pendulum clocks

Crowd synchrony on the Millennium Bridge

[S. Strogatz, D. Abrams, A. McRobie, B. Eckhard, E. Ott, Nature (2005)]

This is not due to the *wind*...

.. like in the Tacoma Narrows Bridge (1940)



This is not due to the *wind*...

- No resonance near vortex shedding frequency
- no vibrations of empty bridge
- No swaying with few people
- nor with people standing still
- but onset above a critical number of people in motion

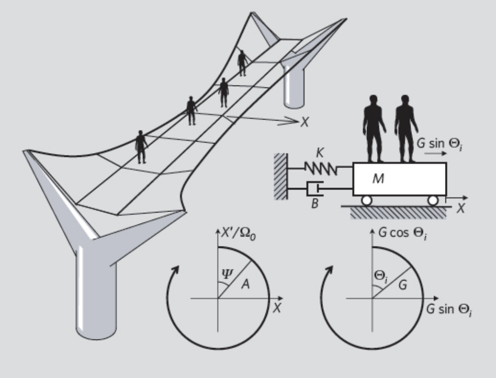
Crowd synchrony on the Millennium Bridge

[S. Strogatz, D. Abrams, A. McRobie, B. Eckhard, E. Ott, Nature (2005)]

The bridge is modeled as a weakly damped and driven harmonic oscillator:

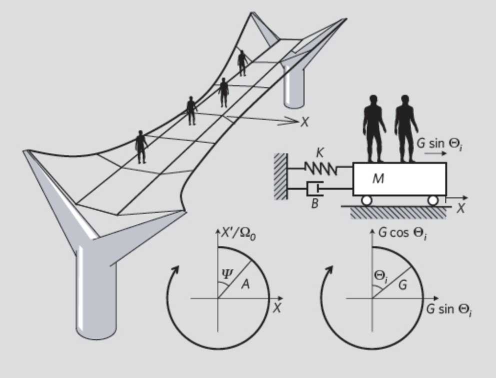
$$M\frac{d^2X}{dt^2} + B\frac{dX}{dt} + KX = G\sum_{i=1}^{N}\sin\Theta_i$$

where $X(t)$ is the displacement of the lateral mode; M, B and K are its mass, damping and stiffness.

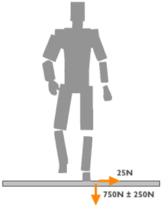


Crowd synchrony on the Millennium Bridge

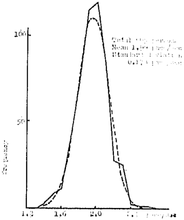
Each pedestrian i gives an alternating sideways force $G \sin \Theta_i$ to the bridge, where G is the maximum force and the phase increases by 2π during a full left/right walking cycle.



Crowd synchrony on the Millennium Bridge



- Each pedestrian: downward force $\approx 800N$;
- sideways force $\approx 25N$;
- People walk at a rate (Ω_i) of about 2 steps per second



Crowd synchrony on the Millennium Bridge

- The bridge's movement, is assumed to alter each pedestrian's gait according to:
$$\frac{d\Theta_i}{dt} = \Omega_i + CA\sin(\Psi - \Theta_i + \alpha)$$
- C quantifies pedestrians' sensitivity to bridge vibrations of amplitude $A(t)$ and phase $\Psi(t)$
- $\Omega_0 := \sqrt{K/M}$ is the bridge's resonant frequency
- α is the phase parameter

Experiment: crowd synchrony on the Millennium Bridge

- A controlled experiment was performed on the Millennium Bridge.
- As more and more people walk on to the deck there is no hint of instability for small crowd
- There is a critical size N_c after which wobbling and synchrony erupt simultaneously
- N_c can be predicted using the theory of Kuramoto model studied in this lecture:

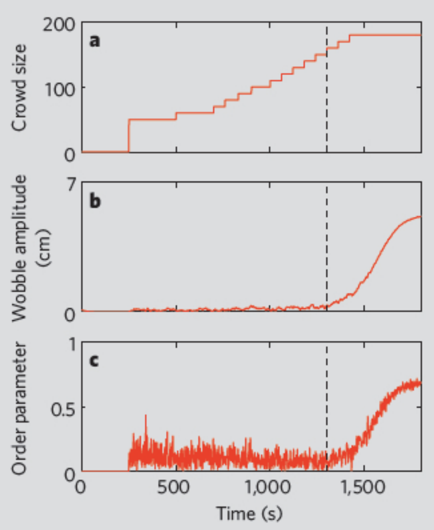
$$N_c = \frac{2\zeta}{\pi}\left(\frac{K}{GCP(\Omega_0)}\right)$$

with

$$\zeta := \frac{B}{\sqrt{4MK}}$$

the damping ratio.

Results experiment



From *micro* to *macro* (synchronization)

- microscopic model for the unit
- interaction between units
- dynamics
- macroscopic model for collective behaviour of large class of interacting units

- (non-equilibrium) phase transitions describe sudden changes in physical systems, e.g. transition between incoherence and coherence
- for detecting phase transitions we need to specify an order parameter

Collective coherent (synchronous) behaviour...

is a tendency of systems towards order (vs entropy) and appears in many real-world examples such as:

- pacemaker cell beats, firing of neuron assemblies
- flashing of fireflies, fish swarms swimming together, bird flocks
- applauding audience

Example: flashing of fireflies

[S. H. Strogatz, *Spontaneous synchronization in nature* (1997)]

Example: flashing of fireflies

[S. H. Strogatz, *Spontaneous synchronization in nature* (1997)]

- The example is about fireflies living in the forests of Southeast Asia.
- When the night falls, these fireflies start to flash incoherently.
- However, after a while these pulses become synchronized.
- Each firefly isolated from the population flashes at its own natural frequency.
- However, within the population it *corrects* its flashing rhythm to that of the other fireflies.
- This suggests that the fireflies can be viewed as a system of coupled oscillators.
- The fireflies reach a globally synchronized state only by their mutual interactions.

The Kuramoto Model

[Y. Kuramoto. *Self-entrainment of a population of coupled non-linear oscillators* (1975)]

- We consider a large population of N oscillators living in the unit circle.
- Without interactions the oscillators rotate independently around the unit circle at their natural frequencies ω_i:

$$\dot{\theta}_i(t) = \omega_i$$

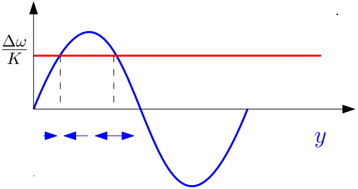
  with $i = 1, \dots, N$
- The natural frequencies ω_i are distributed according to a given probability density $g(\omega)$.
- In order to capture spontaneous synchronization, Kuramoto proposed the following mean-field Kuramoto model:

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))$$

  with K the coupling constant.

Simple case: N=2

- $$\begin{cases} \dot{\theta}_1(t) = \omega_1 + \frac{K}{2}\sin(\theta_2(t) - \theta_1(t)) \\ \dot{\theta}_2(t) = \omega_2 + \frac{K}{2}\sin(\theta_1(t) - \theta_2(t)) \end{cases}$$
- If $y := \theta_2 - \theta_1$:
$$\dot{y} = \Delta\omega - K\sin(y)$$
with $\Delta\omega := \omega_2 - \omega_1$.
- The stationary solutions are the equilibria $y^\star$ such that:
$$\sin(y^\star) = \Delta\omega / K$$



for $K > K_c = |\Delta\omega|$

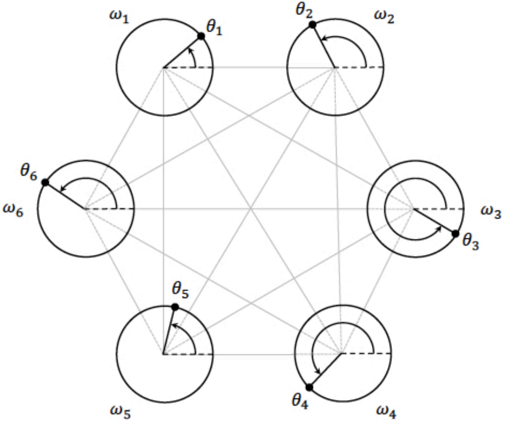
Simple case: N=2

- That means that the system will evolve into a state:

$$\begin{cases} \dot{\theta}_1(t) & = & \omega_1 + \frac{K}{2}\sin(y^\star) = \omega_1 + \frac{\omega_2 - \omega_1}{2} = \bar{\omega} \\ \dot{\theta}_2(t) & = & \omega_2 - \frac{K}{2}\sin(y^\star) = \omega_2 - \frac{\omega_2 - \omega_1}{2} = \bar{\omega} \end{cases}$$

The Kuramoto Model: N interacting oscillators on a *complete graph*

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j(t) - \theta_i(t))$$



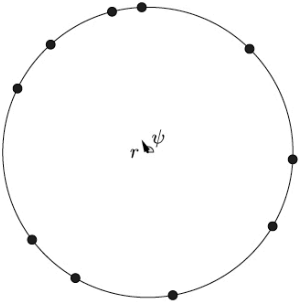
The Kuramoto Model: interacting oscillators on a *complete graph*

- In case that the frequency density $g(\omega)$ is symmetric and unimodal, Kuramoto noticed that the model exhibits the temporal analog of a phase transition
- If the spread of the natural frequencies assigned to the population is too large compared to K, then the oscillators are not able to synchronize and rotate near their own frequencies.
- For increasing K, this remains the case until K exceeds a critical threshold K_c.
- Then a small fraction of synchronized oscillators starts to emerge and becomes of macroscopic size when $K > K_c$
- The critical coupling K_c separates the two regimes in which the system will be either in an incoherent state $(K < K_c)$ or in a partially synchronized state $(K > K_c)$.

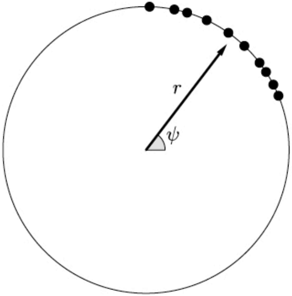
The Kuramoto Model: *order parameter*

To investigate the macroscopic synchronization behavior we introduce an order parameter:

$$r e^{i\Psi} := \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j}$$

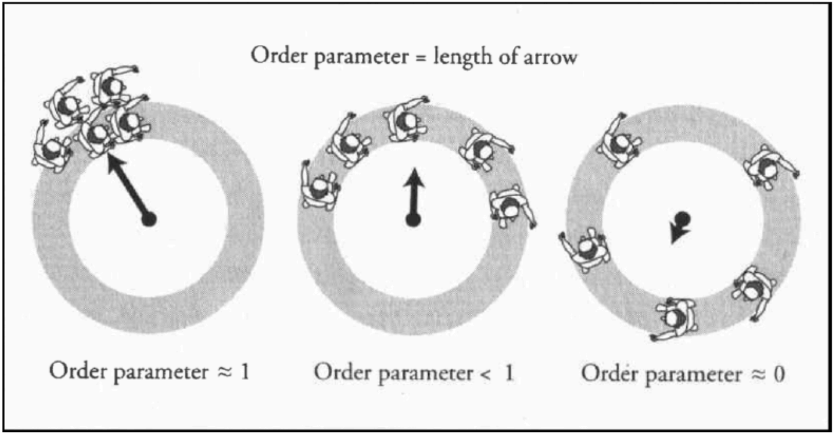


(a) $r = 0.095$



(b) $r = 0.929$

The Kuramoto Model: *order parameter*



The Kuramoto Model: *order parameter*

Since trivially

$$re^{i(\Psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}$$

with *imaginary* part:

$$r\sin(\Psi-\theta_i) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j-\theta_i)$$

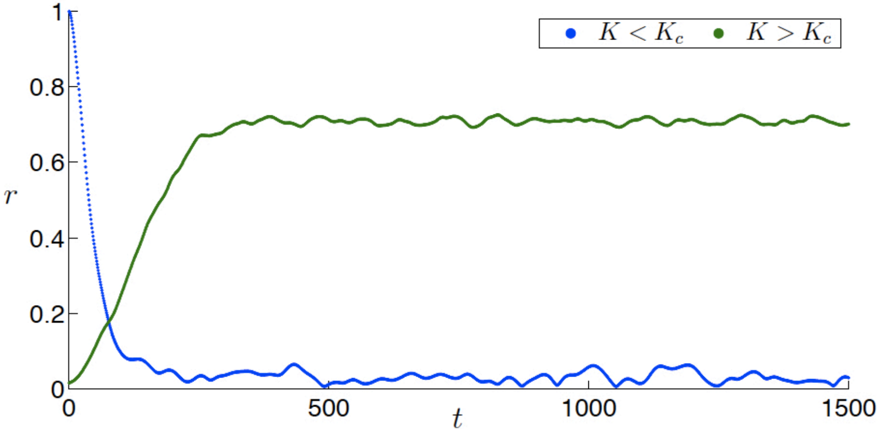
that gives, for the mean field Kuramoto model:

$$\dot{\theta}_i(t) = \omega_i + K r \sin(\Psi - \theta_i)$$

with $i = 1, \ldots, N$.

The interaction term pulls the phases θ_i towards the average phase Ψ with a strength proportional to the phase coherence r

The Kuramoto Model: simulations



- $N = 1000$; $K_c \approx 1.6$; $K = 1, 2$

Continuum limit ($N \to \infty$)

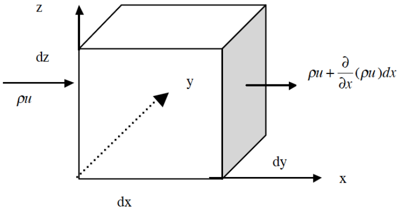
- Study of the relation between r and K

[J. Acebron et al., *The Kuramoto model*]

- In this limit, we define the quantity $\rho(\theta, \omega, t)d\theta$ as the fraction of oscillators with natural frequency ω and phase between θ and $\theta + d\theta$/
- Since the number of oscillators is conserved, we have that the continuity equation (i.e. divergence theorem):

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v)$$

is satisfied, with v the velocity field of an oscillator with phase θ and frequency ω.



Continuum limit ($N \to \infty$)

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- Since the number of oscillators is conserved, we have that the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v)$$

is satisfied, with v the velocity field of an oscillator with phase θ and frequency ω. We know that

$$v(\theta, \omega, t) = \omega + K r(t) \sin(\Psi(t) - \theta)$$

with the continuous order parameter:

$$r(t)e^{i\Psi(t)} = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, \omega, t) g(\omega) d\omega d\theta = \langle e^{i\theta} \rangle$$

Continuum limit ($N \to \infty$)

Remark. The definition of the continuum model seems really natural. However, the convergence of the finite N Kuramoto model as $N \to \infty$ has not been rigorously proven. There are rigorous results when noise is added to the model!

Kuramoto's analysis of the continuum limit ($N \to \infty$)

- We are interested at the stationary solutions of:

$$\boxed{\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v)}$$

- A trivial solution is:

$$\rho(\theta, \omega) = 1/2\pi, \ r = 0$$

corresponding to the incoherent state.

Kuramoto's analysis of the continuum limit ($N \to \infty$)

- Assume r, Ψ to be constant and let put ourselves in the frame rotating at frequency Ψ. Then, $\Psi = 0$, so that:

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i)$$

- The phase-locked oscillators satisfies $\dot{\theta}_i = 0$ in the rotating frame (stick to phase θ_i), so that:

$$\omega_i = Kr\sin(\theta_i)$$

  so that $|\omega_i| \leq Kr$

- Oscillators with $|\omega_i| > Kr$ can't be locked and they rotate around the circle
- Being in this case r and Ψ constant, Kuramoto assumed that the drifting oscillators have a stationary distribution on the circle:

$$\rho(\theta, \omega)v(\theta, \omega) = C(\omega)$$

  for $|\omega| > Kr$, with $C(\omega)$ a constant.

Kuramoto's analysis of the continuum limit ($N \to \infty$)

- Study of the relation between r and K

- Hence, for each $K > K_c$ the stationary density of the partially syncronized state is:

$$\rho(\theta, \omega) = \begin{cases} \delta(\theta - \arcsin(\omega/Kr)), & |\omega| \leq Kr \\ \frac{C(\omega)}{|\omega - Kr\sin\theta|}, & |\omega| > Kr \end{cases}$$

- By the normalization condition $\int_0^{2\pi} \rho(\theta, \omega, t) d\theta = 1$, the constant $C(\omega)$ is:

$$C(\omega) = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}$$

Kuramoto's analysis of the continuum limit ($N \to \infty$)

We can now calculate the order parameter by its definition in the continuum case:

$$
\begin{aligned}
r &= \langle e^{i\theta} \rangle_{\text{lock}} + \langle e^{i\theta} \rangle_{\text{drift}} \\
&= \int_{-\pi/2}^{\pi/2} \int_{-Kr}^{Kr} e^{i\theta} \delta(\theta - \sin^{-1}(\frac{\omega}{Kr})) g(\omega) d\theta d\omega \\
&+ \int_{-\pi}^{\pi} \int_{|\omega| > Kr} e^{i\theta} \frac{C g(\omega)}{|\omega - Kr \sin(\theta)|} d\theta d\omega
\end{aligned}
$$

Kuramoto's analysis of the continuum limit ($N \to \infty$)

$$
\begin{aligned}
r &= \int_{-\pi/2}^{\pi/2}\int_{-Kr}^{Kr} e^{i\theta}\delta(\theta - \sin^{-1}(\frac{\omega}{Kr}))g(\omega)d\theta d\omega \\
&+ \int_{-\pi}^{\pi}\int_{|\omega|>Kr} e^{i\theta}\frac{Cg(\omega)}{|\omega - Kr\sin(\theta)|}d\theta d\omega
\end{aligned}
$$

If we <u>assume</u> $g(\omega) = g(-\omega)$, then the symmetry $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$ implies that the second term is zero. Hence:

$$
\begin{aligned}
r &= \int_{|\omega|<Kr} \cos(\sin^{-1}(\frac{\omega}{Kr}))g(\omega)d\omega \\
&= \int_{-\pi/2}^{\pi/2} \cos(\theta)g(Kr\sin(\theta))Krcos\theta d\theta
\end{aligned}
$$

that is

$$
r = Kr\int_{-\pi/2}^{\pi/2} \cos^2(\theta)g(Kr\sin(\theta))d\theta
$$

Kuramoto's analysis of the continuum limit ($N \to \infty$)

- Therefore we have to solve the following self-consistency equation:

$$r = Kr \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr\sin(\theta)) d\theta$$

for the case $r > 0$ (remember we set $\Psi = 0$) we have:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr\sin(\theta)) d\theta$$

- In the limit $r \to 0$, we get the Kuramoto's formula for the critical coupling:

$$K_c = \frac{2}{\pi g(0)}$$

- with a Taylor expansion around $r = 0$:

$$r \approx \sqrt{\frac{16}{\pi K_c^4(-g'')}} (K - K_c)^{1/2}$$

as $K \to K_c$.

Kuramoto's analysis of the continuum limit ($N \to \infty$)

- In case of Cauchy distribution (i.e. *Lorentzian*) of the natural frequencies $g(\omega)$:

$$g(\omega) = \frac{\Delta}{\pi(\Delta^2 + \omega^2)}$$

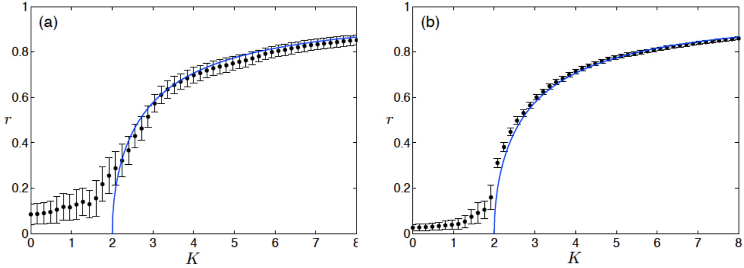
  with Δ a *scale* parameter.

- In this case we can compute r:

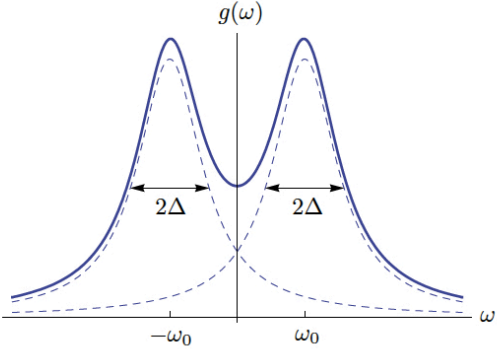
$$r = \sqrt{1 - \frac{K_c}{K}}$$

  for $K \geq K_c$, with $K_c = 2\Delta$.

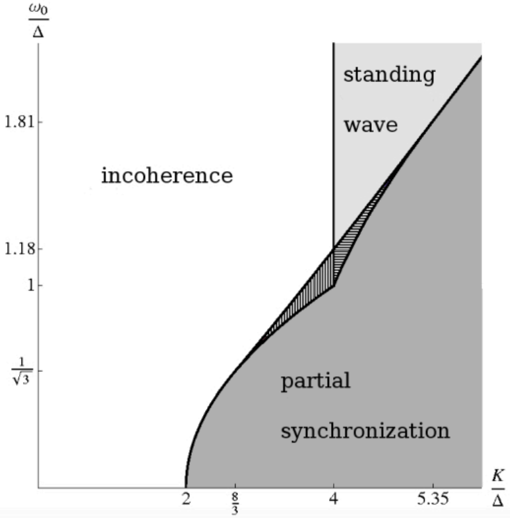
- For $\Delta = 1$ and $N = 100$, $N = 1000$:



What happened for *bimodal* distribution?

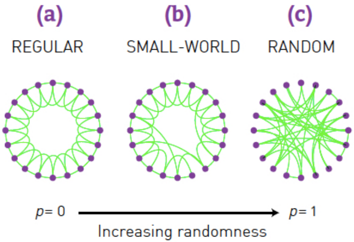


...more complex phase diagram!



What happened for *small world networks* (i.e. Strogatz-Watts model)?

The models has been defined in lecture 5:

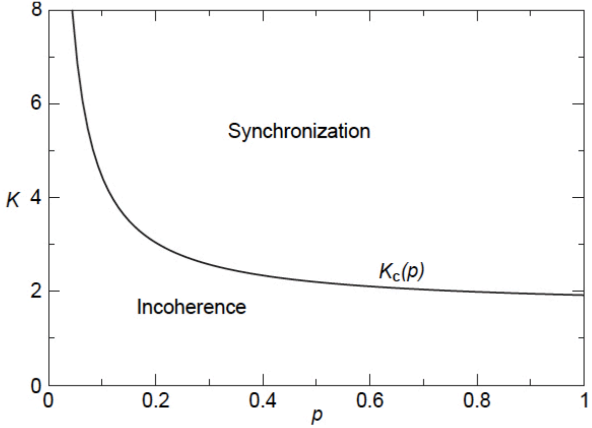


$$\dot{\theta}_i = \omega_i + \frac{K}{k} \sum_{j \in \mathcal{G}(i)} \sin(\theta_j - \theta_i)$$

where $\mathcal{G}(i)$ is the set of the connected neighbors of the site i and k the average degree of a vertex.

What happened for *small world networks* (i.e. Strogatz-Watts model)?

[H. Hong et al.,*Synchronization on small-world networks* (2002)]



The phase boundary is well described by the equation:

$$K_c(p) \approx 1.64 + 0.28\, p^{-1}$$

Kuramoto model with disorder

[Giacomin et al., *Coherence stability and effect of random natural frequencies in populations of coupled oscillators* (2014)]

Classical model of synchronization was introduced by Kuramoto (1975) with noise (Giacomin et al.) has the same phase diagram

Microscopic description:

- at each site $i \in \{1, ..., N\}$ we put a rotator $X_i \in [0, 2\pi)$ (unit)
- system of N rotators $\Theta(t) = (\theta_i(t))_{i=1}^N$ evolves in time (dynamics) according to

$$d\theta_i(t) = \left(\omega_i + \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_k(t) - \theta_i(t)) \right) dt + dB_i(t)$$

and $\omega_i > 0$ is the self-frequency, $K > 0$ is a coupling parameter (interaction) and $(B_i(t))_{t,i}$ a sequence of independent Brownian motions

Kuramoto model without disorder

Remarks:

- $\Theta = (\theta_1(t), ..., \theta_N(t))_{t \geq 0}$ is a Markov process taking values on a given measurable space
- empirical measure process $(\rho_N(t))_{t \geq 0}$ is a measure-valued Markov process

$$\rho_N(t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{\theta_k(t)}$$

- order parameter

$$\int e^{i\theta} d\rho_N(t) = \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k(t)} = r_N e^{\Psi_N}$$

for $h(\theta) = e^{i\theta}$ we get the measure of coherence (r_N, Ψ_N) given by:

$$r_N e^{\Psi_N} = \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k}$$

Macroscopic limit

- infinite volume limit (WLLN)

$$\rho_N(t) \overset{N\to\infty}{\to} \rho(t)$$

is solution of an PDE (McKean-Vlasov equation) of the form

$$\frac{d}{dt}\rho(t) = L\rho(t) \tag{1}$$

where L is a non-linear operator

- stationary solutions for the PDE are solutions of $Lq_* = 0$
- set of stationary solutions $\overset{1:1}{\leftrightarrow}$ solutions (r, Ψ) of

$$r = \int_0^{2\pi} e^{i(x-\Psi)} q_*(x)dx =: \Phi(r) \tag{2}$$

Phase diagram

We call a solution corresponding to $r = 0$ **incoherent** and a solution corresponding to $r > 0$ **coherent**.
Kuramoto, Strogatz, Bonila, Giacomin et al proved:

- phase transition:
  - for $K \leq K_c$ the only solution to $r = \Phi(r)$ is the incoherent solution $r = 0$, Ψ general
  - for $K > K_c$ we have $(0, \Psi)$ and (r_+, Ψ_+) as possible solutions, (incoherent and coherent)
- stability:
  - for $K < K_c$, $(0, \Psi)$ is globally stable
  - for $K = K_c$, $(0, \Psi)$ is neutrally stable
  - for $K > K_c$, $(0, \Psi)$ is unstable, (r_+, Ψ) is globally stable