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Local and Global Self-Entrainments in Oscillator Lattices

Hidetsugu SAKAGUCHI, Shigeru SHINOMOTO and Yoshiki KURAMOTO

Department of Physics, Kyoto University, Kyoto 606

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By computer simulations of an active rotator model, it is found that 1-, 2- and 3-dimensional oscillator lattices with distributed natural frequencies exhibit peculiar clustering patterns due to local entrainment. A simple theory suggests that some of such self-entrained clusters may or may not develop into macroscopic size depending on system dimension, and this fact consistently explains our numerically obtained order parameter curves.

A simplified description of a large population of limit-cycle oscillators is possible by employing an active rotator model[1)]

$$\dot{\phi}_i=\omega_i+\sum_{j=1}^{N}\Gamma_{ij}(\phi_j-\phi_i)\,, \qquad (i=1,2,\cdots,N) \tag{1}$$

where ϕ_i represent oscillator phases (mod. 2π) and ω_i random parameters. As a convenient form of Γ_{ij}, we usually take

$$\Gamma_{ij}(\phi_j-\phi_i)=K_{ij}\sin(\phi_j-\phi_i-\alpha)\,. \qquad (K_{ij}>0) \tag{2}$$

Although very special, a uniform-coupling model in which $K_{ij}=K/N$ is highly instructive. In this case, the long-time behavior of some macrovariables are obtained analytically, as some of the present authors discussed previously.[2),3)] A remarkable finding through such analyses is the existence of a critical K value above which a macroscopic number of rotators come to have an identical frequency. A natural question which arises then will be whether the same conclusion remains true when the interaction is made more realistic, i.e., short-ranged. Related to this question, it would also be interesting to study how the patterns of mutual entrainment vary with system dimension and coupling strength (see Ref. 4) for preliminary study in this direction). The present article approaches these problems by a combination of computer simulations and simple analytic argument.

In order to define our system more precisely, we introduce a d-dimensional hypercubic lattice, let each lattice point be occupied by a rotator obeying Eq. (2) and assume K_{ij} to be nonvanishing only if i and j are nearest neighbors. For the sake of simplicity, the present study will be restricted to the case of vanishing α. Thus our equations of motion become

$$\dot{\phi}_i=\omega_i+K\sum_{\langle j\rangle}\sin(\phi_j-\phi_i)\,, \tag{3}$$

where $\sum_{\langle j\rangle}$ represents the summation over all the nearest neighbors of the i-site. The natural frequencies ω_i are given randomly from site to site according to a probability distribution $g(\omega)$. For the purpose of statistical consideration, which will be touched upon later, we introduce an ensemble of such heterogeneous systems, in much the same way as we do in usual statistical mechanical theories of random lattices. As a result of mutual coupling, rotator frequencies generally deviate from their natural frequencies. It is therefore suitable to define real frequencies of individual rotators, and they are given by

$$\tilde{\omega}_i=\lim_{T\to\infty}\frac{1}{T}(\phi_i(t+T)-\phi_i(t)), \tag{4}$$

which we expect to be independent of t.

As a preliminary consideration, imagine a pair of coupled rotators of phases ϕ_1 and ϕ_2. From Eq. (3)

$$\dot{\phi}_{1,2}=\omega_{1,2}+K\sin(\phi_{2,1}-\phi_{1,2}) \qquad \text{or}$$

$$\dot{\phi}_+=\omega_1+\omega_2, \qquad \dot{\phi}_-=\omega_1-\omega_2-2K\sin\phi_-,$$

where $\phi_\pm=\phi_1\pm\phi_2$. Thus the condition for their mutual entrainment (i.e., $\tilde{\omega}_1=\tilde{\omega}_2$) is given by $2K\geq|\omega_1-\omega_2|$, under which $\tilde{\omega}_{1,2}=(\omega_1+\omega_2)/2$. When $2K<|\omega_1-\omega_2|$, we have $\tilde{\omega}_{1,2}=(1/2)\{\omega_1+\omega_2\pm\sqrt{(\omega_1-\omega_2)^2-4K^2}\}$. Such a simple consideration may help to understand qualitatively the simulation results of rotator lattices to be presented below.

Our computer simulations were carried out on one-, two- and three-dimensional hypercubic lattices with periodic boundary conditions. We always let $g(\omega)$ be a normal distribution centered about 0 and with variance 1. Note that taking different values for its mean and variance would produce nothing new because of certain symmetries inherent in our system. Instead of solving Eq. (3) accurately by, i.e., the Runge-Kutta-Gill method, we employed a simple Euler scheme and solved a set of first-order difference equations

$$\phi_i(t+\Delta t)-\phi_i(t)=\Delta t\cdot\{\omega_i+K\sum_{\langle j\rangle}\sin(\phi_j(t)-\phi_i(t))\}$$

with $\Delta t=0.05\sim0.10$. Some numerical evidences that such a seemingly crude approximation works exceedingly well, especially for the calculation of macroscopic quantities, were reported in Refs. 3) and 5). Also in the present study, we tested the correctness of our Euler scheme by comparing with the results from the Runge-Kutta-Gill method. It turned out that, although the values of $\phi_i(t)$ may deviate fairly rapidly from their true values, the errors in more relevant quantities such as $\tilde{\omega}_i$ and order parameters as defined later are completely negligible for our present purposes. In each run the total number of time steps was taken to be 9000, out of which the initial 3000 steps were discarded as transient for the calculation of long-time averages. For the most of our calculations, we started with $\phi_i=0(i=1, 2, \cdots, N)$ initially. Some of such calculations were checked by choosing another initial condition in which ϕ_i are taken to be completely random, and no significant change was seen in the quantities of our present concern.

Figure 1 shows a few spatial patterns of $\tilde{\omega}$ in a linear chain of rotators, each pattern corresponding to different coupling strength. A remarkable feature seen there is that the system appears to be composed of sharply bounded clusters in each

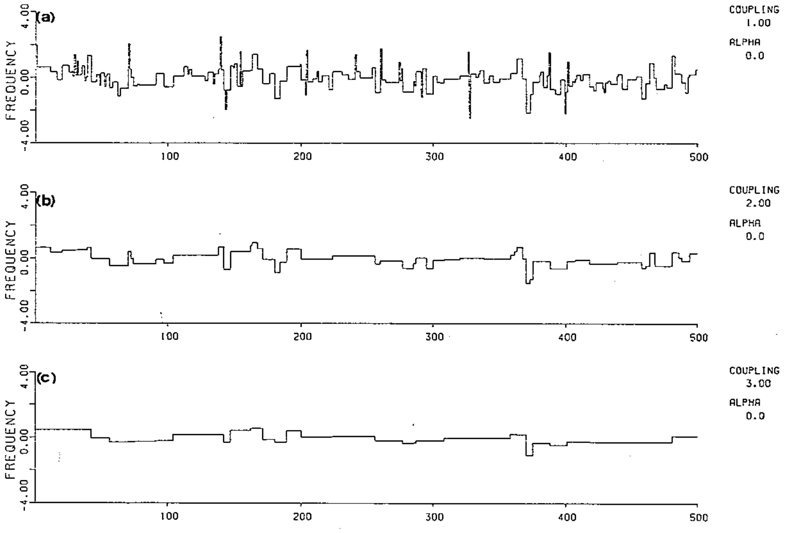

Fig. 1. Spatial distribution of real frequencies $\tilde{\omega}$ in one-dimensional lattice with $N=500$: (a) $K=1.0$, (b) 2.0, (c) 3.0.

of which the rotators are mutually entrained. This kind of frequency plateaux and resulting staircase structures have already been reported by Ermentrout and Kopell in a rotator chain with constant gradient in natural frequency.[6] Although in Fig. 1 the mean size of the self-entrained clusters apparently increases with K, we have some reasons to expect that the entire system with N supposed to be infinite is unable to form a single cluster while K remains finite. One obvious reason for this inability of perfect entrainment is that the Gaussian tail of $g(\omega)$ gives a finite chance, however small it may be, that the natural frequency of a given rotator happens to be so high (or low) that it can hardly satisfy the above-mentioned condition for entrainment to any of its nearest neighbors. Such a chance could perfectly be eliminated if we modify $g(\omega)$ by introducing low and high frequency cutoffs and, at the same time, letting K be sufficiently large. As we see later, however, there is another more important reason why such a modification is still insufficient to guarantee perfect entrainment.

Figure 2 illustrates some two-dimensional patterns of entrainment. They were obtained by drawing bonds between nearest neighbor sites whenever the corresponding rotators are mutually entrained. We regarded them as mutually entrained if and only if their difference in calculated ω is smaller than π/T, where T is the total length of the run subtracted by an initial transient period. The same criterion was also used in three-dimensional simulations to be described shortly. Note, however, that, as a result of incompleteness of this criterion, which is due to finiteness of T, the patterns in Fig. 2 include a few of seemingly impossible ways of bonding, e.g., the part in which only one out of the four sides of a unit cell is left open. As is anticipated, the average size of the self-entrained clusters increases with K. However, it remains still unclear if the size of the largest cluster, denoted by N_s, can attain $O(N)$ in the limit $N\to\infty$. In

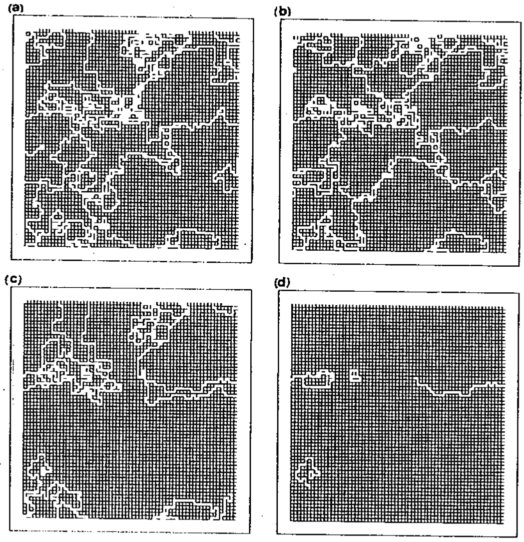

Fig. 2. Patterns of synchronized clusters in two-dimensional square lattice with $N=64^2$: (a) $K=1.5$, (b) 1.6, (c) 1.7, (d) 1.8.

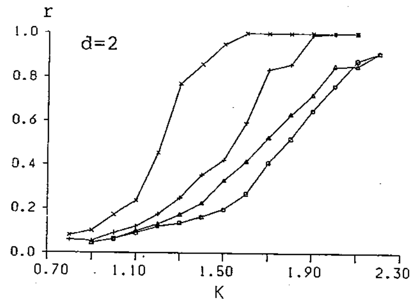

Fig. 3. Order parameter curves in two-dimensional systems. The symbols $\times$, $+$, $\triangle$ and $\square$ represent the data corresponding to the system sizes $N=32^2$, 64^2, 96^2 and 128^2, respectively.

terms of the order parameter r defined by

$$r=\lim_{N\to\infty}\frac{N_s}{N} \tag{5}$$

we are asking if r can become nonzero under finite K. Figure 3 shows numerically obtained order parameter curves as a function of K and their system-size dependence. Each curve was obtained from the average over six samples. Phase-transitionlike behavior exhibited by systems of smaller N seems to subside as N becomes larger, although these results alone would still be too short for us to claim the absence of macroscopic entrainment in two-dimensional systems of infinite size.

In Fig. 4 cross sections of a few three-dimensional patterns of entrainment are shown. By comparison with Fig. 2, it is noticed that the patterns here are characterized by the coexistence of relatively large clusters and individually rotating elements rather than of clusters of comparable sizes. Figure 5 gives order parameter curves in three-dimensional lattices. Their striking feature as compared to Fig. 3 is that the curves in Fig. 5 suggest their convergence to a unique nontrivial curve in large N limit, although we are again unable to draw any definite conclusion about the existence of a genuine phase transition.

We now show by a simple consideration that if the occurrence of perfect entrainment under finite K is postulated, then this leads to a self-contradiction in lower dimensional systems, i.e., $d\leq 2$, but no contradiction if $d>2$. This result would be relevant to our main theme because impossibility of perfect entrainment for any large K possibly implies the absence of macroscopic entrainment over the entire region of K; conversely, if perfect entrainment is possible at sufficiently large K, which will actually be the case in higher dimensional systems, then this seems to imply the existence of some K value at which a macroscopic cluster first appears. Indeed, imperfect bulk entrainment (i.e., $0<r<1$) could become perfect one by increasing K sufficiently, especially when the tails of $g(\omega)$ have been cut off, which means that if the one is present (absent) in a given system, the other will also be present (absent). Our argument continues as follows. Perfect entrainment implies that the phase difference

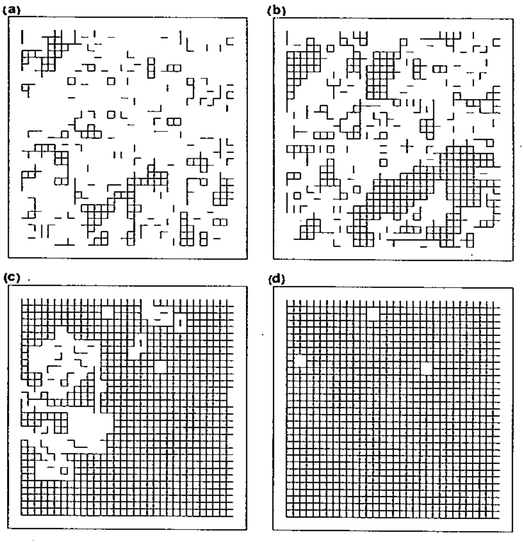

Fig. 4. Cross sections of patterns of synchronized clusters in three-dimensional cubic lattice with $N=32^3$: (a) $K=0.5$, (b) 0.55, (c) 0.6, (d) 0.65.

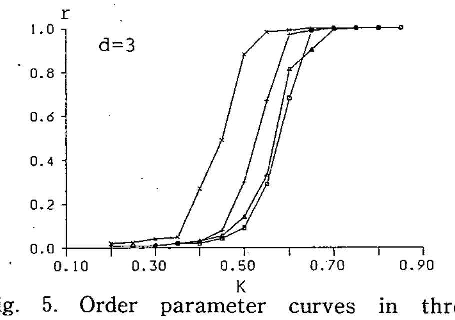

Fig. 5. Order parameter curves in three-dimensional systems. The symbols ×, +, △ and □ represent the data corresponding to the system sizes $N=8^3$, 16^3, 24^3 and 32^3, respectively.

between any two rotators, in particular those forming a nearest-neighbor pairs, is bounded as $t\rightarrow\infty$, so that, by letting K be larger and larger, this difference can be made arbitrarily small. Then, one is allowed to linearize Eq. (3) in $\phi_j-\phi_i$ to obtain

$$\dot{\phi}_i=\omega_i+K\sum_{\langle j\rangle}(\phi_j-\phi_i)\,. \qquad (6)$$

For a finite periodic lattice, one may find from Eq. (6) a solution of perfect entrainment by setting $\dot{\phi}_i=\bar{\omega}$ for all i. The resulting N linear equations generally give $N-1$ independent phase values and $\bar{\omega}$. The latter is obviously given by the simple average of ω_i's, i.e., the first moment of $g(\omega)$, which we set to zero. The solution thus obtained turns out consistent with the underlying assumption that $\phi_j-\phi_i$ are small, provided K is made sufficiently large under fixed N. The question which follows is whether such a consistency is preserved when N is brought to infinity and K is fixed but at an arbitrarily large value. As is seen below, breakdown occurs if $d\leq 2$. We express our solution in terms of spatial Fourier amplitudes defined by

$$(\phi_q,\,\omega_q)=\frac{1}{\sqrt{N}}\sum_j(\phi_j,\,\omega_j)\exp(i\boldsymbol{q}\cdot\boldsymbol{r}_j)\,,\qquad K_q=K\sum_j(1-\exp(i\boldsymbol{q}\cdot\boldsymbol{\delta}))\,,$$

where $\boldsymbol{q}$ has components $q_\alpha=2\pi n_\alpha/L$, n_α being an integer, L the size of our hypercubic system; $\boldsymbol{r}_i$ are position vectors of lattice points, and $\boldsymbol{\delta}$ the nearest-neighbor position vectors relative to the origin. The solution of perfect entrainment is then given by

$$\phi_q=K_q^{-1}\omega_q\,. \qquad (\boldsymbol{q}\neq 0) \qquad (7)$$

We now consider statistical averages, denoted by $\langle\cdots\rangle$, over the aforementioned ensemble whose basic properties are $\langle\omega_i\rangle=0$ and $\langle\omega_i\omega_j\rangle=\delta_{ij}$, or $\langle\omega_q\rangle=0$ and $\langle\omega_q\omega_{-q'}\rangle=\delta_{q,q'}$. Thus, for $\boldsymbol{q}\neq 0$, we have

$$\langle\phi_q\rangle=0\,,\qquad \langle|\phi_q|^2\rangle=|K_q|^{-2}\,. \qquad (8)$$

The phase difference between two rotators has the property

$$\langle|\phi_i-\phi_j|^2\rangle=\frac{1}{N}\sum_q|K_q|^{-2}(1-\cos(\boldsymbol{q}\cdot\boldsymbol{r}_{ij}))\,, \tag{9}$$

where $\boldsymbol{r}_{ij}=\boldsymbol{r}_j-\boldsymbol{r}_i$. For fixed distance $|\boldsymbol{r}_{ij}|$, the above quantity may or may not diverge as $L\to\infty$ depending on system's dimension. Dangerous contribution as we let $L\to\infty$ (and also possibly $|\boldsymbol{r}_{ij}|\to\infty$) comes from small $q=|\boldsymbol{q}|$. If we note that K_q behaves like $K(aq)^2$ for small q, where $a=|\boldsymbol{\delta}|$, the small-q contribution to the right-hand side of Eq. (9) is estimated in order of magnitude as

$$\frac{|\boldsymbol{r}_{ij}|^2}{K^2}\int_{q_c}^{q_0}dq\cdot q^{d-3}\,, \tag{10}$$

where $q_c\sim L^{-1}$ and $q_0\sim|\boldsymbol{r}_{ij}|^{-1}$. Thus the divergence as $L\to\infty$ occurs if $d\le 2$. Since this result remains true even if i and j are nearest neighbor sites, our starting assumption of perfect entrainment, which necessitates $|\phi_i-\phi_j|<\pi/2$ for any nearest neighbor pair, breaks down if $d\le 2$.

Granted that perfect entrainment is possible for $d>2$, the phase difference becomes still indefinitely large with distance $|\boldsymbol{r}_{ij}|$ if $d\le 4$, as is easily seen by considering the contribution from $q\sim q_0$. This fact suggests that in space of intermediate dimensions $2<d\le 4$, especially $d=3$, the order parameter r can be nonzero, whereas another order parameter σ defined by

$$\sigma=\frac{1}{N}\sum_{j=1}^{N}\exp(i\phi_j) \tag{11}$$

vanishes identically; one may alternatively say that in such a system "frequency order" is present but "phase order" is absent. If this is true, our rotator systems with short-range interaction make a remarkable contrast to those with uniform coupling for which the emergence of r and σ as we change K is known to be simultaneous.[2)]

Finally, it should be remarked that some of the present results may drastically be changed if the effect of finite α is included. Preliminary study in this direction suggests that inclusion of α facilitates frequency order but too large α may cause complicated behaviors such as spatio-temporal chaos and initial-condition dependence of macrostates. The details will be reported elsewhere.

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