

***Spontaneous* synchronisation for the Kuramoto model**

We consider a large population of interacting oscillators living in the unit circle, under the dynamics of the so called *mean-field Kuramoto model*, defined in Lecture 11:

$$\frac{d\theta_i}{dt}(t) = \omega_i + \frac{K}{N} \sum_{j=1}^N \sin(\theta_j(t) - \theta_i(t)) \quad (1)$$

with $i = 1, \dots, N$ and K the *coupling constant*. As we have seen in the lecture, for small coupling K each oscillator rotates with its natural frequency ω_i , whereas for large coupling K almost all angles θ_i will be entrained by the mean field and the oscillators *synchronize*. In this project we study first a simple case where the natural frequencies of the rotators are known and in the second part, when they are sampled from a probability distribution $g(\omega)$.

Lyapunov function for studying the stability of the *equilibria*

If we define

$$\bar{\omega} := \frac{1}{N} \sum_{i=1}^N \omega_i \quad (2)$$

the mean natural frequency, it is very reasonable to think that it is the common frequency of the *phase locked* component.

i) Prove indeed that:

$$\frac{d}{dt} \left(\frac{1}{N} \sum_{i=1}^N \theta_i(t) \right) = \bar{\omega}$$

Comment the result.

We consider now the new variables ϕ_i in the reference frame rotating with $\omega = \bar{\omega}$, i.e.:

$$\phi_i(t) := \theta_i(t) - \bar{\omega}t$$

so that the Kuramoto model becomes:

$$\frac{d\phi_i}{dt}(t) = (\omega_i - \bar{\omega}) + \frac{K}{N} \sum_{j=1}^N \sin(\phi_j(t) - \phi_i(t)) \quad (3)$$

ii) Show that the function \mathcal{H} :

$$\mathcal{H} := -\frac{K}{2N} \sum_{i,j} \cos(\phi_i - \phi_j) - \sum_{i=1}^N (\omega_i - \bar{\omega}) \phi_i \quad (4)$$

is a *Lyapunov function*, i.e. it is that for any solution ϕ_i of the ODE:

$$\dot{\mathcal{H}} = \sum_{i=1}^N \frac{\partial \mathcal{H}}{\partial \phi_i} \frac{d\phi_i}{dt} \leq 0$$

Therefore, by the properties of the Lyapunov function, the dynamics in (3) converge to a minimum of \mathcal{H} .

iii) Find the equations for the *extreme points* of \mathcal{H} (i.e. $\nabla \mathcal{H} = 0$). Is the configuration with $\omega_i = \bar{\omega}$, for all i an asymptotically stable equilibrium?

iv) If we introduce the order parameter:

$$r e^{i\Psi} := \frac{1}{N} \sum_{j=1}^N e^{i\phi_j}$$

show that the stationary solutions of the Kuramoto model satisfies:

$$\Delta(\omega_i) = K r \sin(\phi_i - \Psi)$$

with $\Delta(\omega_i) = \omega_i - \bar{\omega}$.

BONUS Just a curiosity. The Lyapunov function can be interpreted as the Hamiltonian the *mean-field* Heisenberg *XY* ferromagnet with coupling strength K . The first term aims at keeping all spins parallel. The second term represents a kind of random field: the two terms counteract each other (*frustration*). Show that the ground state of \mathcal{H} is rotationally invariant also with the present of the random field. Therefore, what do you expect for the *Jacobian* matrix of the system calculated at the fixed points? Is $(1, 1, \dots, 1)$ an eigenvector?

2. Numerical studies for the Kuramoto Model

Normal distribution of the natural frequencies

We consider now a population of $N = 1000$ of Kuramoto oscillators, with a uniformly distributed initial phase $\theta_i(0)$ and natural frequencies ω sampled from a standard normal distribution $N(0, 1)$. Therefore:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2}$$

From the theory, we know the order parameter r satisfies the consistency equation:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr \sin \theta) d\theta \quad (5)$$

that can be integrated numerically for any given value of K . We know as well that the *critical value* of the coupling constant is:

$$K_c = 2\sqrt{2/\pi}$$

1. Integrate (1) using *Euler method* (see Ch. 20 of J. Butcher, *Numerical Methods in ODE*) for $t \in [0, T]$, with $T = 100$, a time step of $dt = 10^{-2}$. Perform the simulation for different values of K (e.g. $K \in [0.5]$ with $dK = 0.2$). For each K calculate the value of the limiting r_∞ (e.g. $r(T)$). Plot r_∞ in function of K . Comment the graph obtained. Is it in good agreement with the numerical solutions of (5)? What can be improved?
2. Run now the model for the two values $K = 1$ and $K = 2$. Make a graph of $r(t)$ in function of t . Comment the findings. Is it in agreement with what the theory predicts?

Natural frequencies uniformly distributed

Let us consider now the case:

$$g(\omega) = \begin{cases} \frac{1}{2\gamma}, & |\omega| \leq \gamma \\ 0, & |\omega| > \gamma \end{cases}$$

with $\gamma = 1/2$.

- iii) Also in this case the initial phases $\theta_i(0)$ are uniformly distributed. However, this time:

$$N = 2000, \quad dt = 0.05, \quad T = 200, \quad dK = 0.03, \quad K_{\max} = 1.5$$

For each K calculate the value of the limiting r_∞ (e.g. $r(T)$). Plot r_∞ in function of K . Comment the graph obtained. Is it different from the case of the *unimodal* normal distribution? Which is the K_c in this case?

- iv) Consider now $K = 1$. For a fixed realisation of the natural frequencies ω_i , perform 10 different simulations obtained sampling different initial conditions $\theta_i(0)$. Then, plot 10 curves $r(t)$ of the coherence parameter in function of the time for 10 different simulations.
- v) Consider again $K = 1$. For a fixed realisation of the initial conditions $\theta_i(0)$, perform 10 different simulations obtained sampling different realisation of the natural frequencies ω_i . What can you say about the relaxation time towards the synchronised state? Are there any differences with respect of point iv)? Try to find some plausible explanation.

3. BONUS: Kuramoto model on Watts-Strogatz Model

Consider now a Kuramoto model on the Watts-Strogatz model $WS(N; 2r; p)$ defined as follow:

$$\dot{\theta}_i = \omega_i + \frac{K}{2r} \sum_{j \in \mathcal{G}(i)} \sin(\theta_j - \theta_i)$$

with $i \in \{1, \dots, N\}$; $\mathcal{G}(i)$ is the set of the connected neighbors of the site i in the realisation of the WS model, and ω_i are sampled from a standard normal distribution.

- a) Try to design and perform a simulation study in order to study the relation between the coherence parameter r_∞ and the coupling constant K , for different values of the *rewiring parameter* p (e.g. $p \in \{0, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$). Interpret your results and try to explain the peculiarity of the case $p = 0$. Suggested values: $k = 3; N = 2000, T = 200$.
- b) Try to estimate the relation between the critical coupling K_c and the *rewiring probability* p .