"We get along so much better after I disabled the 'Comments' function of our relationship!"

COM1002

Foundations of Computer Science

Lecture 9: Functions

26th Oct 2015

Paul Watton

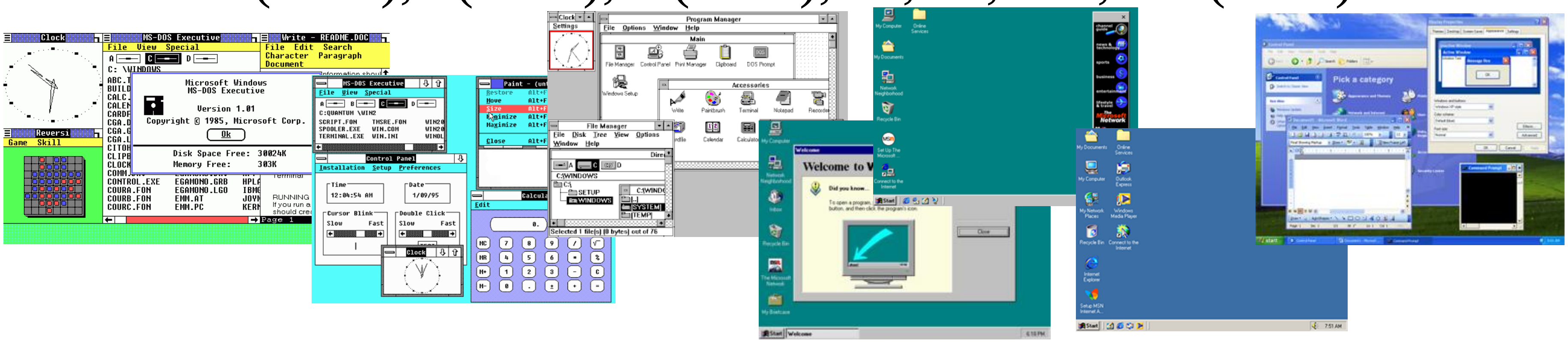
Motivation

T**o study the behaviour of software:**

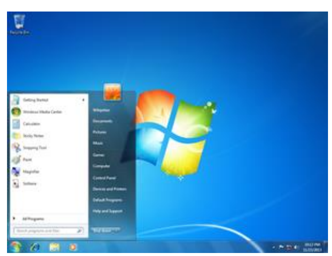
- software can be modelled as functions;
- the behaviour of functions can be specified in predicate logic:
- verifying the behaviour of software:
  - involves reasoning about functions.

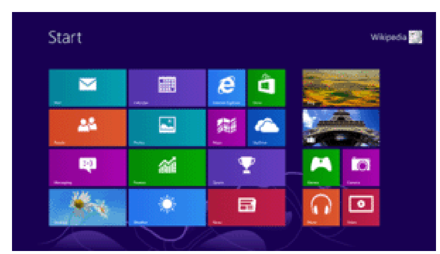
Verifying and testing the behaviour of software is important…

Windows: 1 (1985); 2(1987); 3 (1990); 95; 98; ME; XP (2001)



Windows: Vista (2007); 7(2009); 8 (2012); 10 (2015).





before it creates havoc with the world…

WEEK	4	5	6	7
Mon	Lecture **Handout Ex 3** (Assessed 5%)	Lecture **Hand in Ex 3** **Handout Ex4 (online)**	Lecture	Lecture **Handout Ex5** (Assessed 5%)
Wed	Lecture	Lecture	**Revision Lecture**	Lecture
Thurs	Tut (ex 3)	Tut (ex 4)	Tutorial (Revision) **QUIZ 1 (25%)** Diamond 101 4pm-5:30pm	Tut (ex 5)

WEEK	8	9	10	11
Mon	Lecture **Hand in Ex 5** **Handout Ex6** (Assessed 5%)	Lecture **Hand in Ex 6**	Lecture **Hand out ex 7** (Assessed 5%)	**Revision Lecture** **Hand in Ex 7**
Wed	Lecture	**Revision Lecture**	Lecture	**Revision Lecture**
Thurs	Tut (ex 6)	**Revision Tutorial** **QUIZ 2 (25%)** Diamond 101 4pm-5:30pm	Tut (ex 7)	**Revision Tutorial** **QUIZ 3 (25%)** Diamond 101 4pm-5:30pm

Mole Quiz 1

TOPICS

C1: Propositional Logic

C2: Set Theory

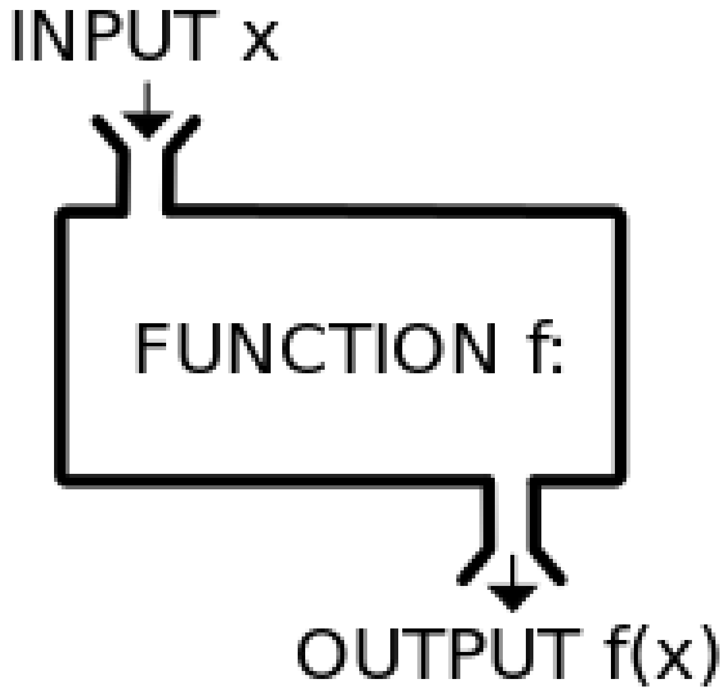
Duration (50 minutes)

Open Book

Equations will be provided.

More details in Wednesday's lecture (28^{th} Oct)

What is a function?



A function *f* takes an input *x*, and returns a **single output** *f*(*x*). One metaphor describes the function as a "machine" or "black box" that for each input returns a corresponding output.

Learning Objective

To be able to describe and discuss functions we need to learn relevant terminology...

domain **co-domain** **graph** **image**

Total function **Partial function** **bijection**

injection **surjection** **pre-image**

one-to-one **composite functions**

By the end of this lecture you will understand and be able to apply the above terminology…

domain co-domain graph image

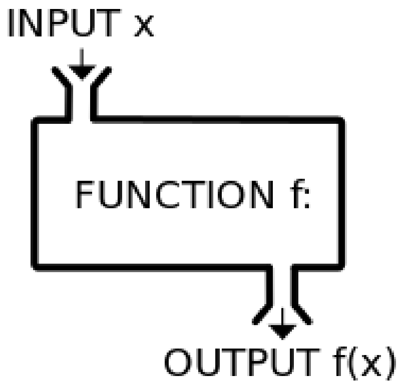
Total function Partial function bijection

injection surjection pre-image

one-to-one composite functions

If you don’t master the terminology, it is difficult to think about and discuss functions and consequently…

Basic Concepts 1



Any Function:

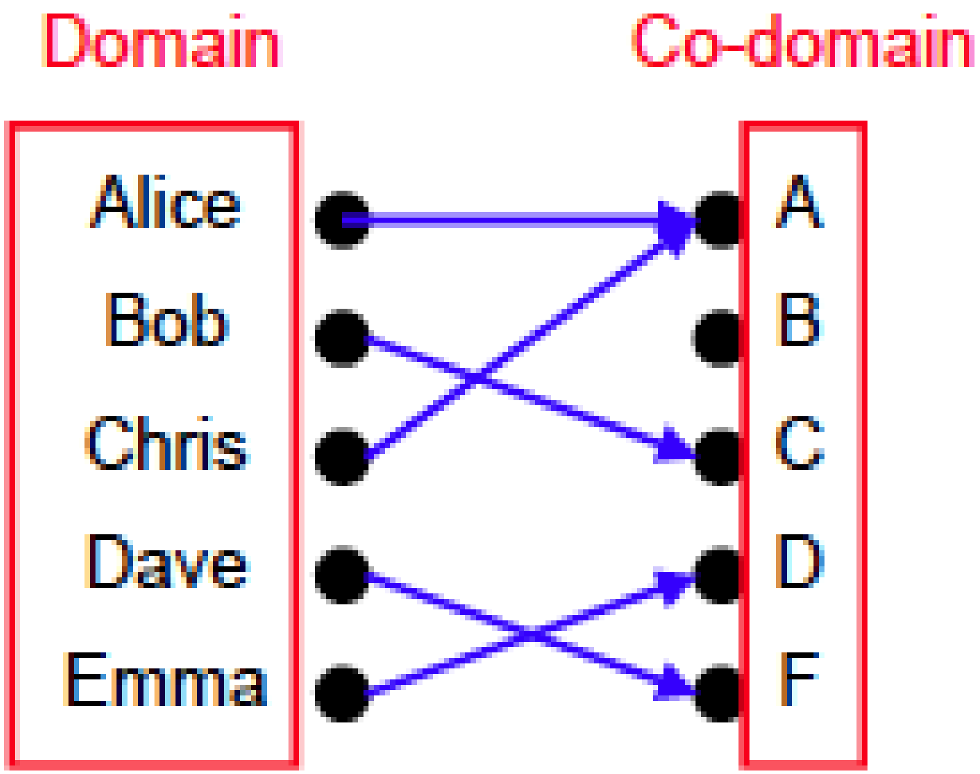
Takes some **input**:

- the **argument** or parameter to the function,
- which must be an **element** of some **set**,
- known as the **domain** of the function:
  - sometimes denoted **dom (f)** for a function **f**;

Produces an **output**:

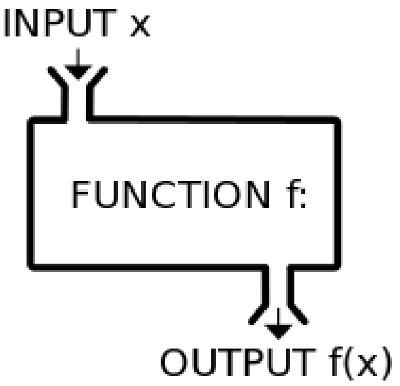
- known as the **value** of the function
- also an **element** of some **set**,
- called the **co-domain** of the function.

Examples



A class grade function

Basic Concepts 2

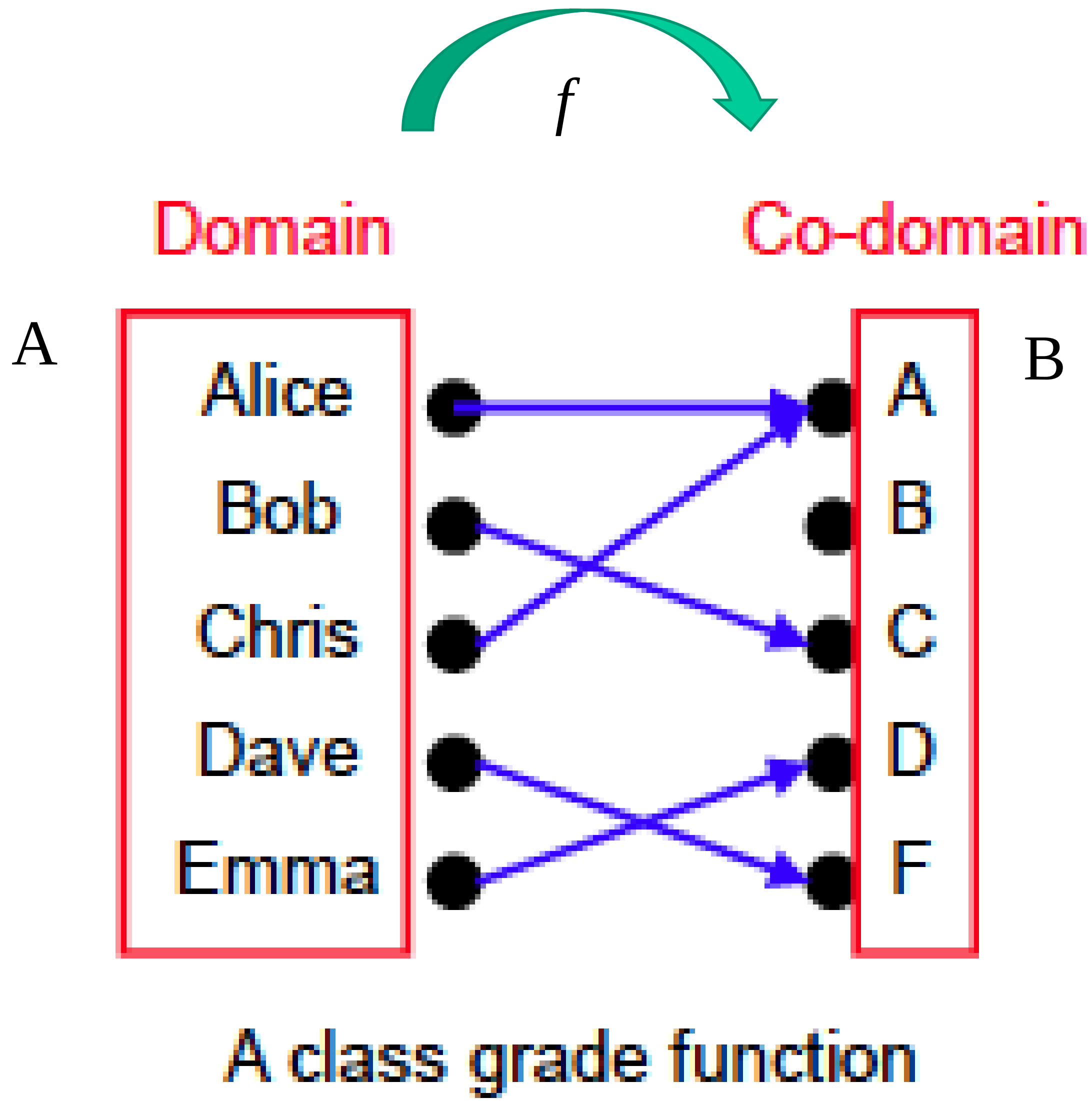


For some function f:

- If the **domain** is the set **A**;
- And the **co-domain** is the set **B**;
- Then the function is said to be **from A to B**;
- and is denoted $f : A \rightarrow B$.

Applying a function:

- If **f** is applied to an argument **a**,
- it produces the value denoted **f(a)**.

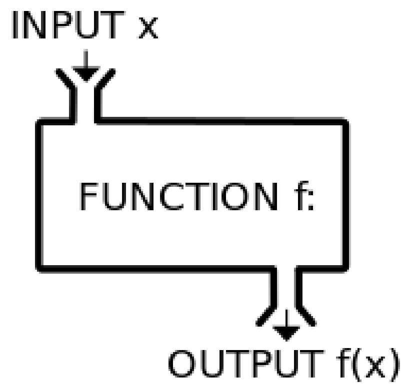


A={Alice, Bob, Chris, Dave, Emma}

B={A,B,C,D,E}

What is f(Emma) ?

Basic Concepts 3

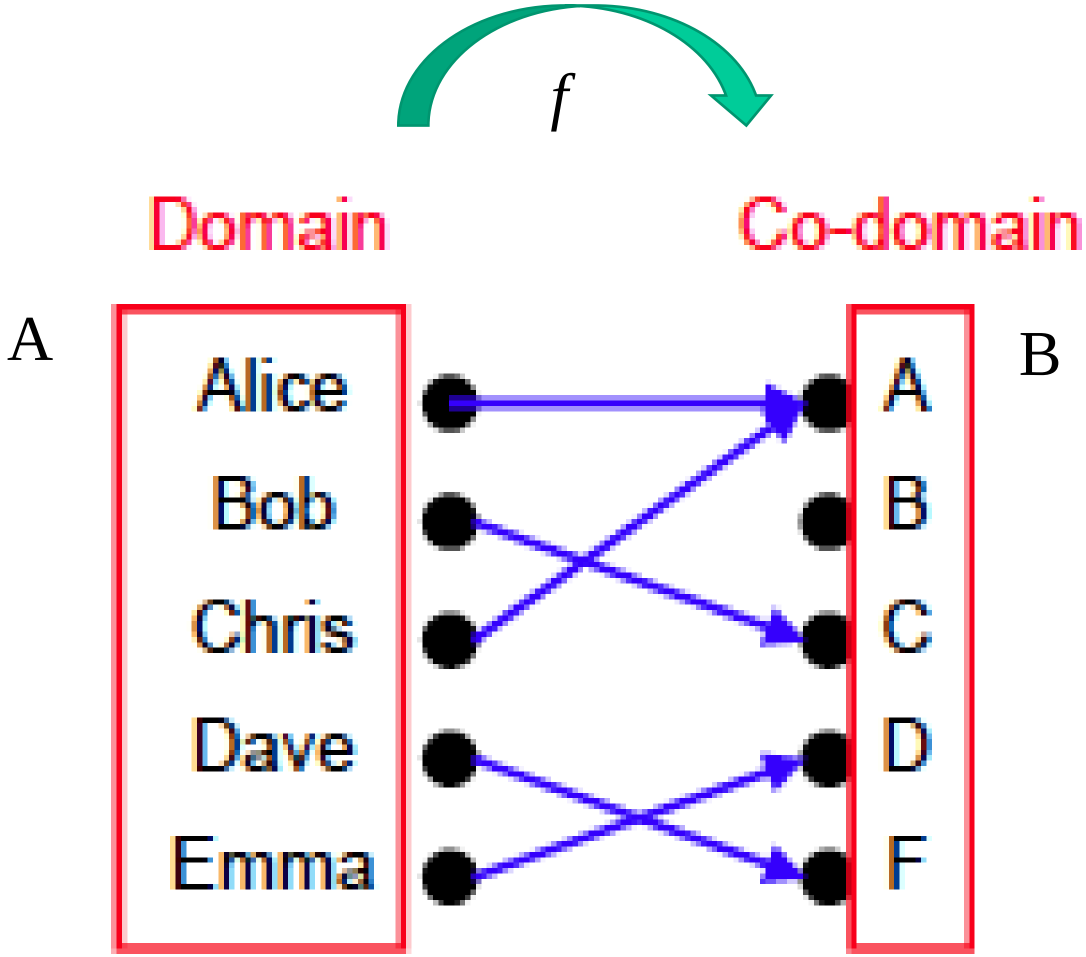


The operation of a function:

- Given an **argument** a,
- A function f **maps** a into b = f(a),
- And so creates pairs:
  - of the form **(argument, result)**,
  - denoted $a \mapsto b$, or $f : a \mapsto b$;
- A set of these is called a **map**, or mapping,
  - from A to B,
  - and the pairs are sometimes called **maplets**.

The **graph** of a function is defined as:

$$\text{graph}(f) = \{ (a, b) \in A \times B : b = f(a) \}.$$

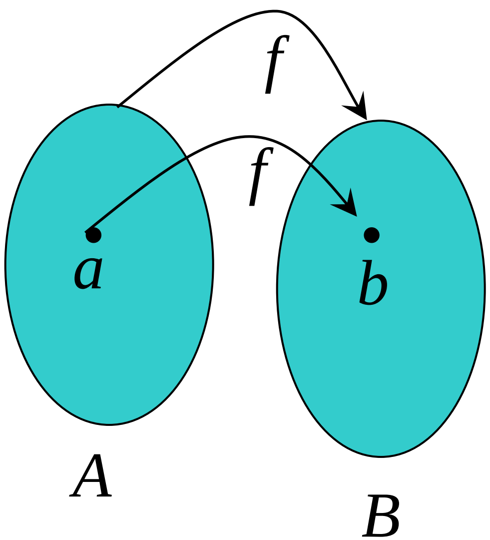


A class grade function

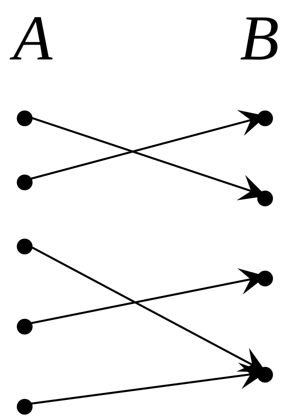
graph(f)={(Alice,A), (Bob,C), (Chris,A), (Dave,F), (Emma,D)}

Given any sets A, B, a function *f* from *(or "mapping")* A to B (f:A→B) is an assignment of **exactly one** element ***f(x)***$\in$***B*** to each element ***x***$\in$***A***.

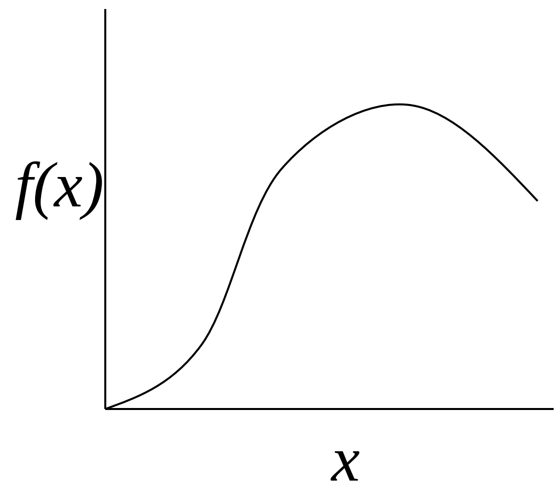
Mappings of functions can be represented graphically in several ways:



Venn diagrams

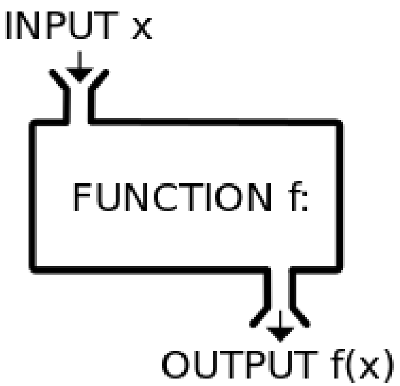


Graph



Plot

Basic Concepts 4



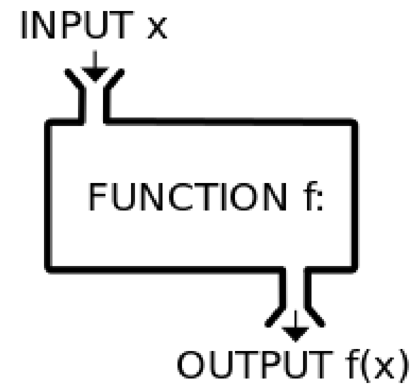
Key characteristic of a function:

- For any function **f**;
- And for a given argument **a**;
- **Only a single value f(a) can be produced**.

In terms of maps:

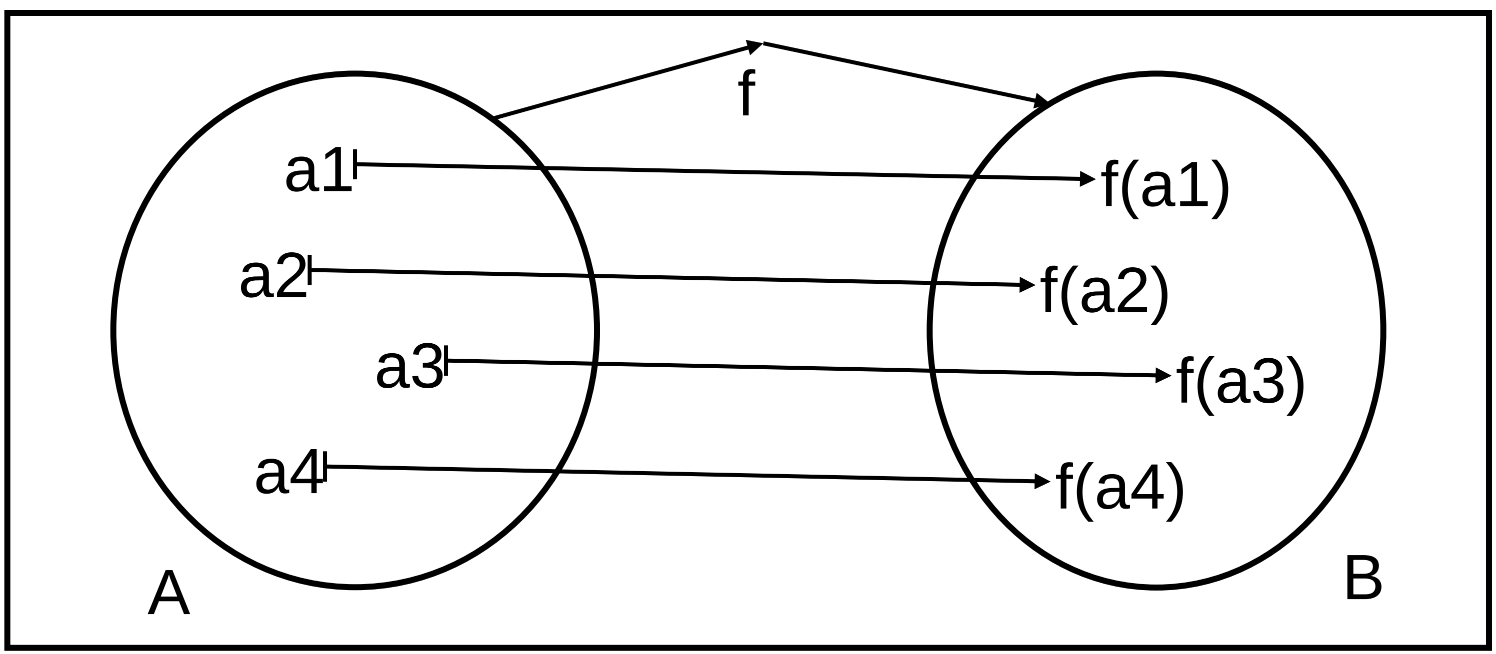
- If f is a set of pairs a $\mapsto$ b,
- There is only one pair with any given a:
  - ie $(f{:}a \mapsto b) \wedge (f{:}a \mapsto c) \Rightarrow b = c$.

Basic Concepts 5

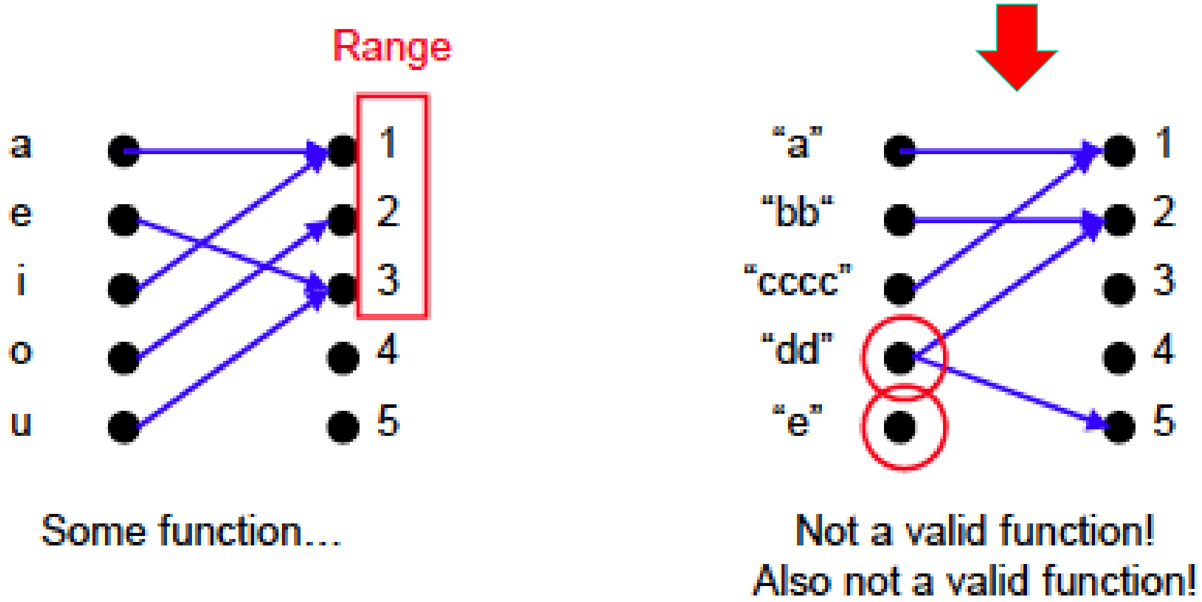


Illustrating maps of functions:

Can use an **extended Venn diagram**.



More examples



What is the difference between co-domain and range?

Exercises

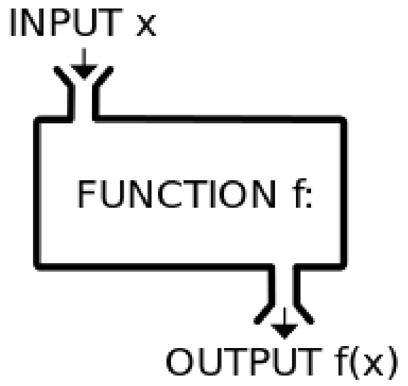
Why is f not a function from R to R if

(a) $f(x) = 1/x$

(b) $f(x) = x^{½}$

(c) $f(x) = \pm(x^2 + 1)^{½}$

Basic Concepts 6



Total and **partial** functions:

For a given **domain**:

- a **total function** is defined for **all elements**,
- a **partial function** is only defined for **some**;

For a given **co-domain**:

- the **range** is the **set of values** of a function:
  - range (f) = { f(a) : a ∈ dom (f) }.

Basic Concepts 7

Image and **Pre-Image**:

For any function $\mathbf{f : A \rightarrow B}$;

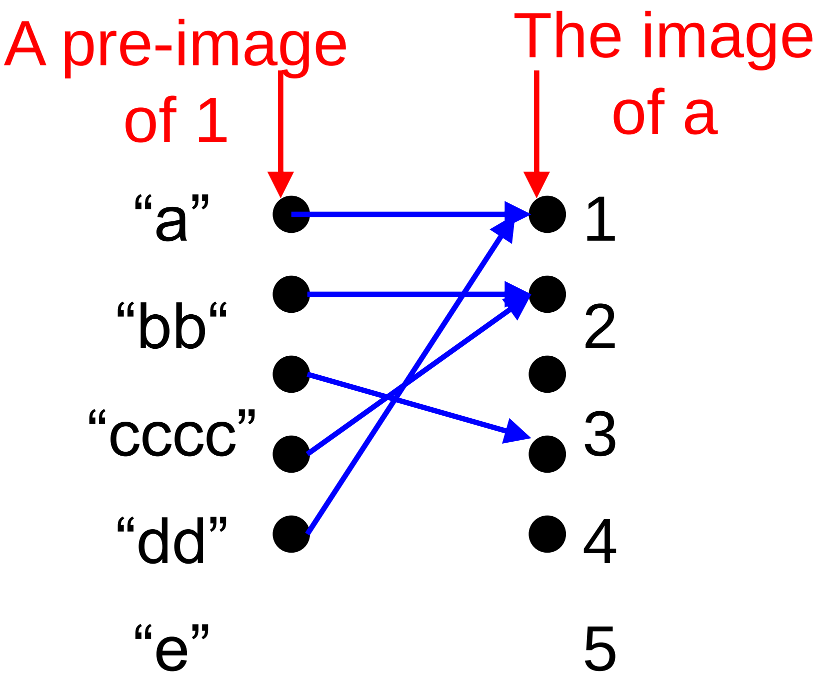
For any **subset S** of the domain of f:

- the **image** of the function is the **set of values produced from that subset**,
- for $S \subseteq A$, $f(S) = \{ f(a) : a \in S \}$;

For any **subset** of the **co-domain** of f:

- the **pre-image of the function** is the set of arguments which produce that subset,
- for $T \subseteq B$, $f^{-1}(T) = \{ a \in A : f(a) \in T \}$.

A string length function



What is the pre-image of the subset T={1,2} of the codomain ?

Basic Concepts 8

Total and Partial Functions

- For a **total function** $f : A \rightarrow B$:
  - $f^{-1}(B) = A$;
  - this does not hold for a **partial function**,
    - and many functions in software are partial.

- *Try Exercises 6.2 and 6.3:*

Exercises 6.2

Exercise 6.2

Indicate which of the following are functions from the set Humans of all humans to itself. For each that isn't a function, indicate why it fails to be a function.

1. $\mathsf{Mother}(x)$ represents the mother of x.
2. $\mathsf{Parent}(x)$ represents the parent of x.
3. $\mathsf{Child}(x)$ represents the child of x.
4. $\mathsf{FirstBornChild}(x)$ represents the first-born child of x.

Each student in a class of 12 is assigned a particular grade – an integer percentage between 0 and 100 – which appears on a list of a bulletin board

Andrews	75	Evans	78	Parker	64
Archer	92	Fletcher	46	Smith	59
Collins	64	Greene	68	Taylor	100
Davies	88	Lewis	54	Williams	78

Formally this table is a function score : Class $\rightarrow$ Marks, where

$$\begin{aligned}\mathsf{Class} \;=\; \{&\text{Andrews, Archer, Collins, Davies, Evans, Fletcher,}\\ &\text{Greene, Lewis, Parker Smith, Taylor, Williams}\}\end{aligned}$$

and

$$\mathsf{Marks} = \{0, 1, 2, 3, 4, \ldots, 100\}.$$

For example, $\mathsf{score}(\text{Greene}) = 68$, that is, the function score maps the value Greene to the value 68.

What is the range of the function score ?

Basic Concepts 9

Multiple arguments:

The domain of a function f could be a cartesian product:

- eg A_1 x A_2 x A_3 x … x A_n,
- so f takes n arguments, and has ***arity*** n;

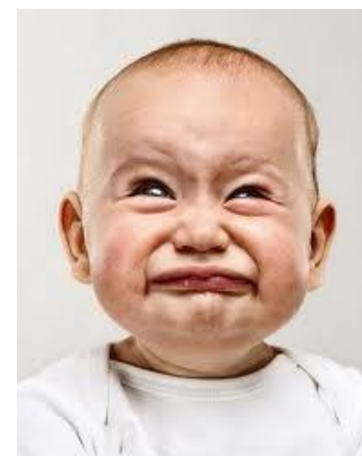
Binary functions:

- those with arity 2,
- often written in infix form, as x *f* y:
  - eg 2 + 2 rather than + (2, 2).

Injections

In mathematics we mean…

Formally: given $\boldsymbol{f{:}A \to B}$



“x is injective” $:\equiv (\forall\ x,y\text{: } (x \neq y) \rightarrow (f(x) \neq f(y)))$

or, equivalently…

“x is injective” $:\equiv (\forall\ x,y\text{: } (f(x)=f(y)) \rightarrow (x = y))$

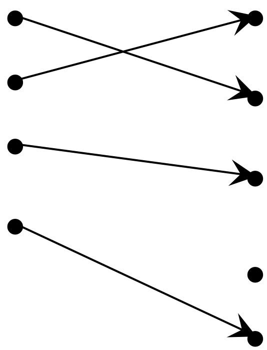
Classes of Functions 1

Injections:

- If $a1 \mapsto b1$, $a2 \mapsto b2$ are in graph (f):
  - for f to be a function, $a1 = a2 \Rightarrow b1 = b2$,
  - but we can have $a1 \neq a2 \wedge b1 = b2$;
- **A function f is an injection, or one-to-one:**
  - if different inputs produce different results,
  - ie if $b1 = b2 \Rightarrow a1 = a2$.

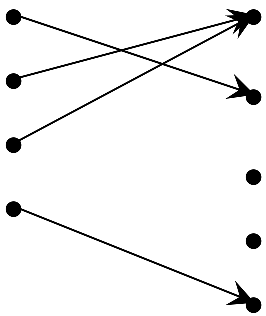
One-to-One (Injection) Illustration

Graph representations of functions that are (or not) one-to-one:



One-to-one

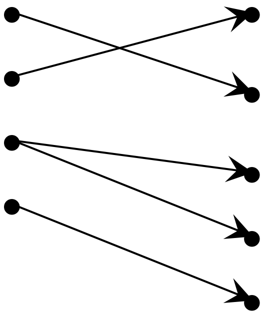
(injection)



Not one-to-one

(not an injective

function)



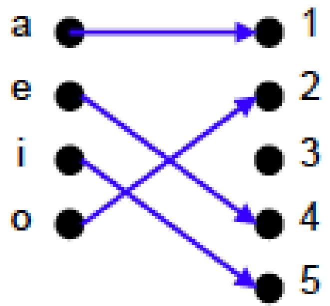
Not even a

function!

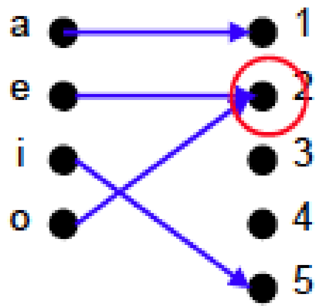
One-to-one functions

A function is one-to-one if each element in the co-domain has a unique pre-image

– Meaning no 2 values map to the same result



A one-to-one function



A function that is not one-to-one

The term injective is synonymous with one-to-one

NOTE: there can be un-used elements in the co-domain

Exercises 6.5: Injective functions

Indicate which of the following functions are one-to-one. For those that are not one-to-one, indicate the reason that they fail to be one-to-one.

1. The function $\mathsf{score} : \mathsf{Class} \to \mathsf{Marks}$ from Example 6.1.
2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
3. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.

board.

Andrews	75	Evans	78	Parker	64
Archer	92	Fletcher	46	Smith	59
Collins	64	Greene	68	Taylor	100
Davies	88	Lewis	54	Williams	78

Formally this table is a function $\mathsf{score} : \mathsf{Class} \to \mathsf{Marks}$, where

Classes of Functions 2

Surjections:

For any function $f : A \rightarrow B$:

- the *range* is a subset of the **co-domain**,
- ie *range*(f) $\subseteq$ B;

A function **f is a surjection**, or onto:

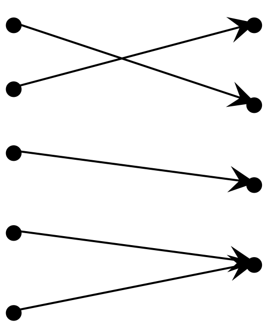
- if all elements of the co-domain can be produced,
- ie if range (f) = B.

A function $f{:}A{\rightarrow}B$ is *onto* or *surjective* or *a surjection* iff its **range is equal to its codomain** ($\forall b{\in}B, \exists a{\in}A: f(a){=}b$).

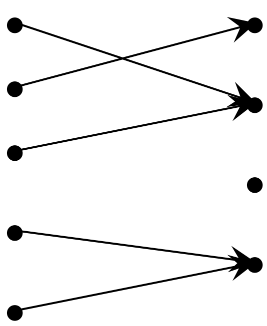
An *onto* function maps the set *A* onto (over, covering) the *entirety* of the set *B*, not just over a piece of it.

Illustration of Surjective (onto)

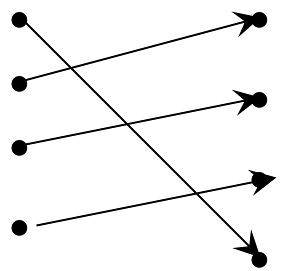
Some functions that are or are not *onto* their codomains:



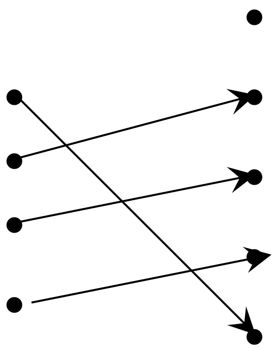
Onto
(but not 1-1)



Not Onto
(nor 1-1)



Both 1-1
and onto



1-1 but
not onto

Exercises 6.6: Surjective functions

Definition 6.5

*A function $f : A \to B$ is **onto**, or **surjective**, if, and only if, its range is equal to its codomain, $\mathit{range}(f) = B$; that is, every value $b \in B$ is the image of some value $a \in A$:*

$$\forall\, b \in B\; \exists\, a \in A\; \big(f(a) = b\big).$$

Exercise 6.6 (Solution on page 432)

Indicate which of the following functions are onto. For those that are not onto, indicate the reason that they fail to be onto.

1. The function $\mathsf{score} : \mathsf{Class} \to \mathsf{Marks}$ from Example 6.1.
2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
3. The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.

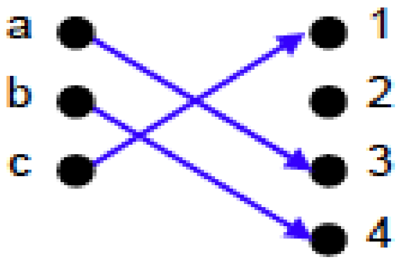
Classes of Functions 3

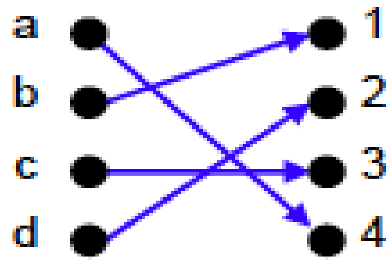
Bijections:

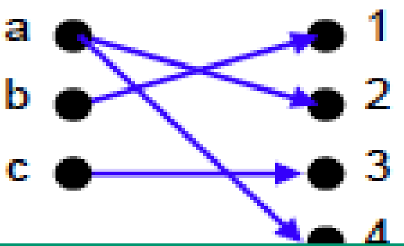
- A function f is a bijection if:
  - it is an **injection and a surjection**,
  - ie it is one-to-one and onto;
- Every bijection has an inverse:
  - for $f : A \to B$ denoted $f^{-1} : B \to A$,
  - f^{-1} is also a bijection, and $(f^{-1})^{-1} = f$;
- **Bijections are fundamental for coding data.**

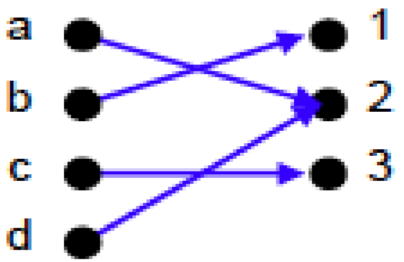
Exercise

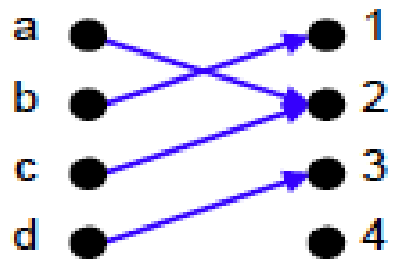
- Are the following functions onto, one-to-one, both, or neither?











1-to-1 and onto function are called **bijective.**

Example 6.8

Let $A = \{a, b, c, \ldots, z\}$ be the set consisting of the usual 26 characters of the alphabet. We can use a bijection $f : A \to A$ as the basis of a simple encryption scheme. For example, suppose we take the bijection f defined as follows:

$$
\begin{array}{ccccccccccccc}
a & b & c & d & e & f & g & h & i & j & k & l & m \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
y & k & t & e & c & s & w & u & b & m & z & v & l \\
 & & & & & & & & & & & & \\
n & o & p & q & r & s & t & u & v & w & x & y & z \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
q & g & d & p & o & j & f & r & h & n & a & x & i
\end{array}
$$

To encode a message we apply the function f to each letter of the message. For example, the message

WE ATTACK AT DAWN

would be encoded as

NC YFFYTZ YF EYNQ

It is important that the function f is a bijection. No two letters can be mapped to the same letter, as otherwise it would be impossible to decode since different messages would give rise to the same encrypted text.

In order to decode messages that we receive which are encoded as above, we simply apply the inverse function f^{-1} to each of the letters of the encrypted text.

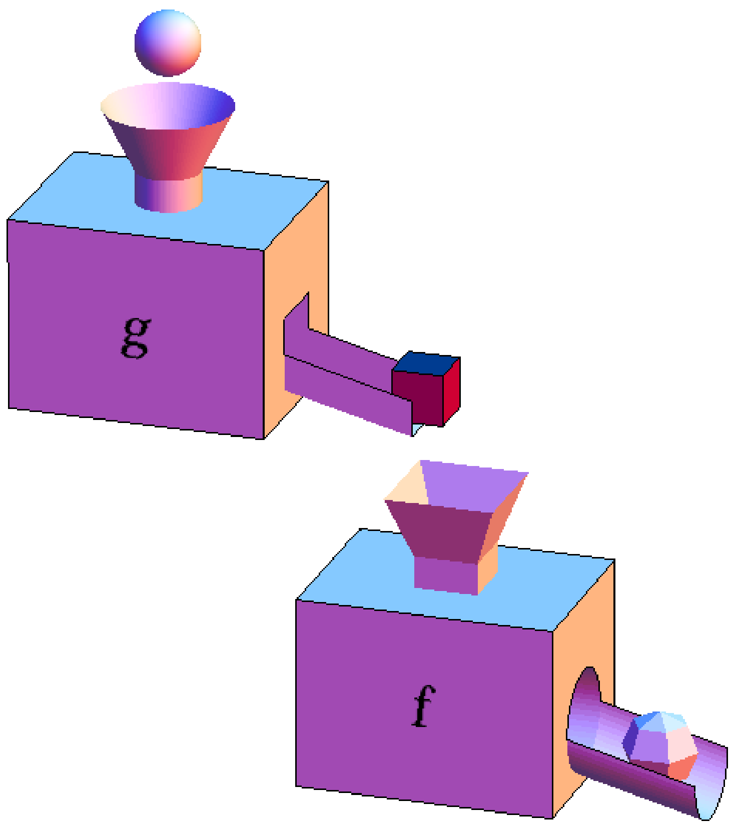
This encryption method is insecure; it is very easy to decode encrypted messages even if you don't know the function f with which they are encrypted. However, the idea of using a bijection f to encode messages, thus allowing such messages to be decoded with the inverse function f^{-1}, is fundamental.

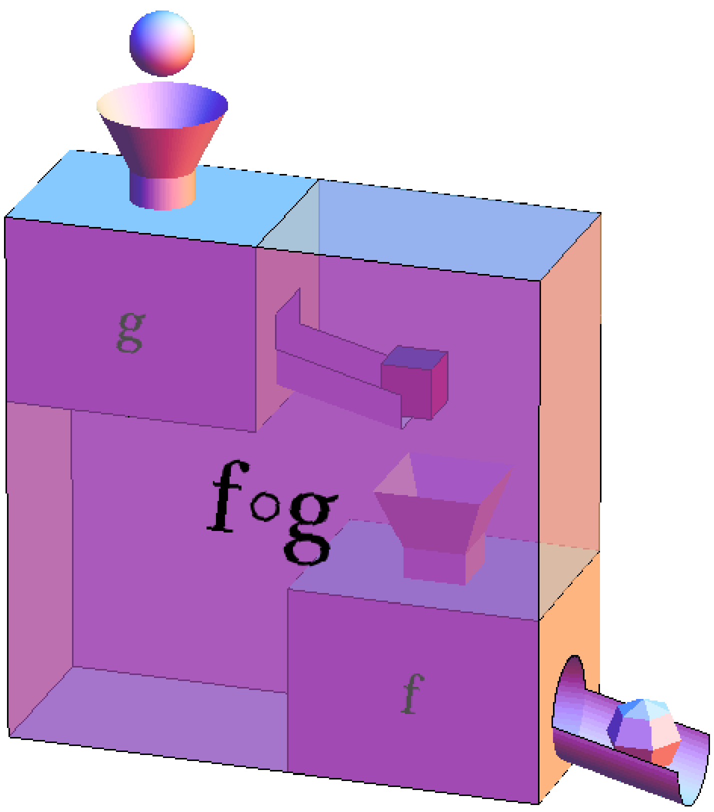
What is the inverse of the function f?

a	b	c	d	e	f	g	h	i	j	k	l	m
$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$
y	k	t	e	c	s	w	u	b	m	z	v	l

n	o	p	q	r	s	t	u	v	w	x	y	z
$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$	$\mapstodown$
q	g	d	p	o	j	f	r	h	n	a	x	i

Composite Functions



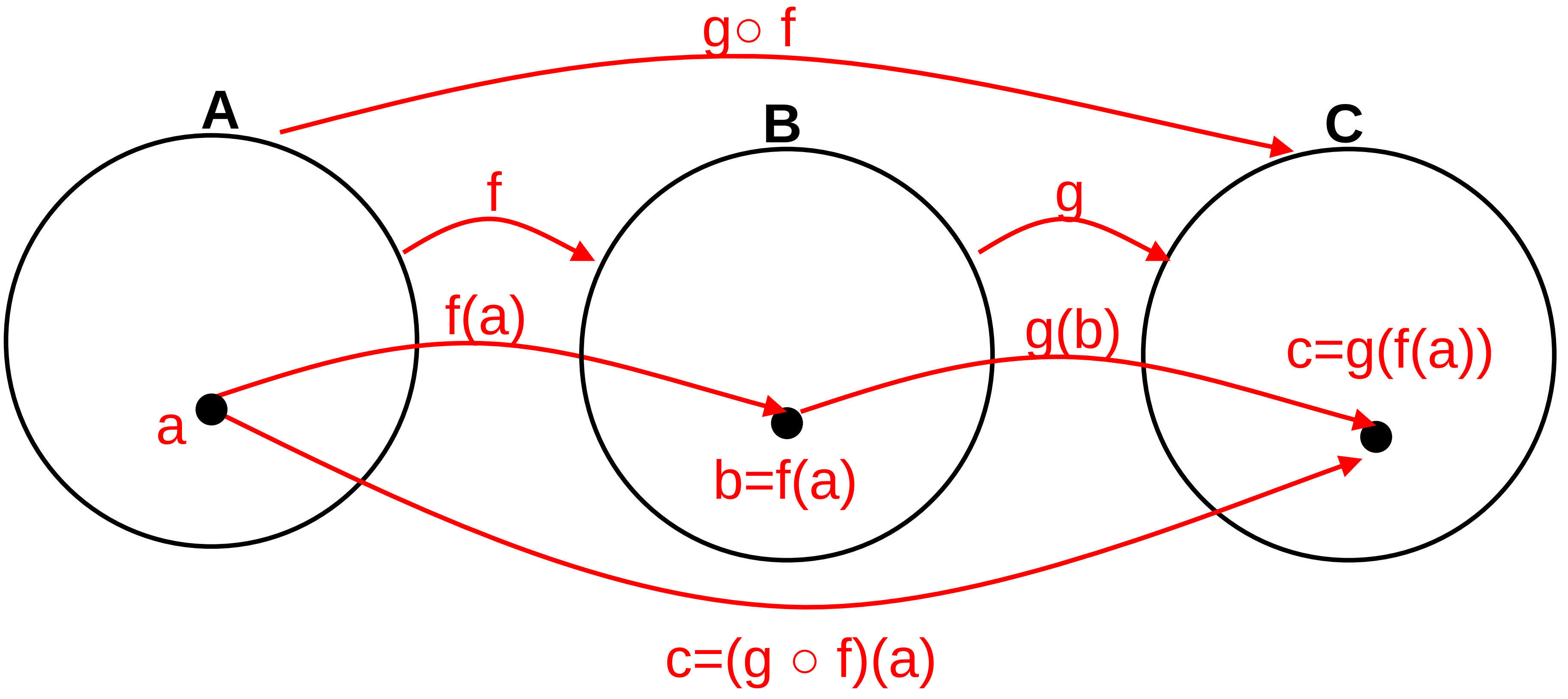


Composing Functions 1

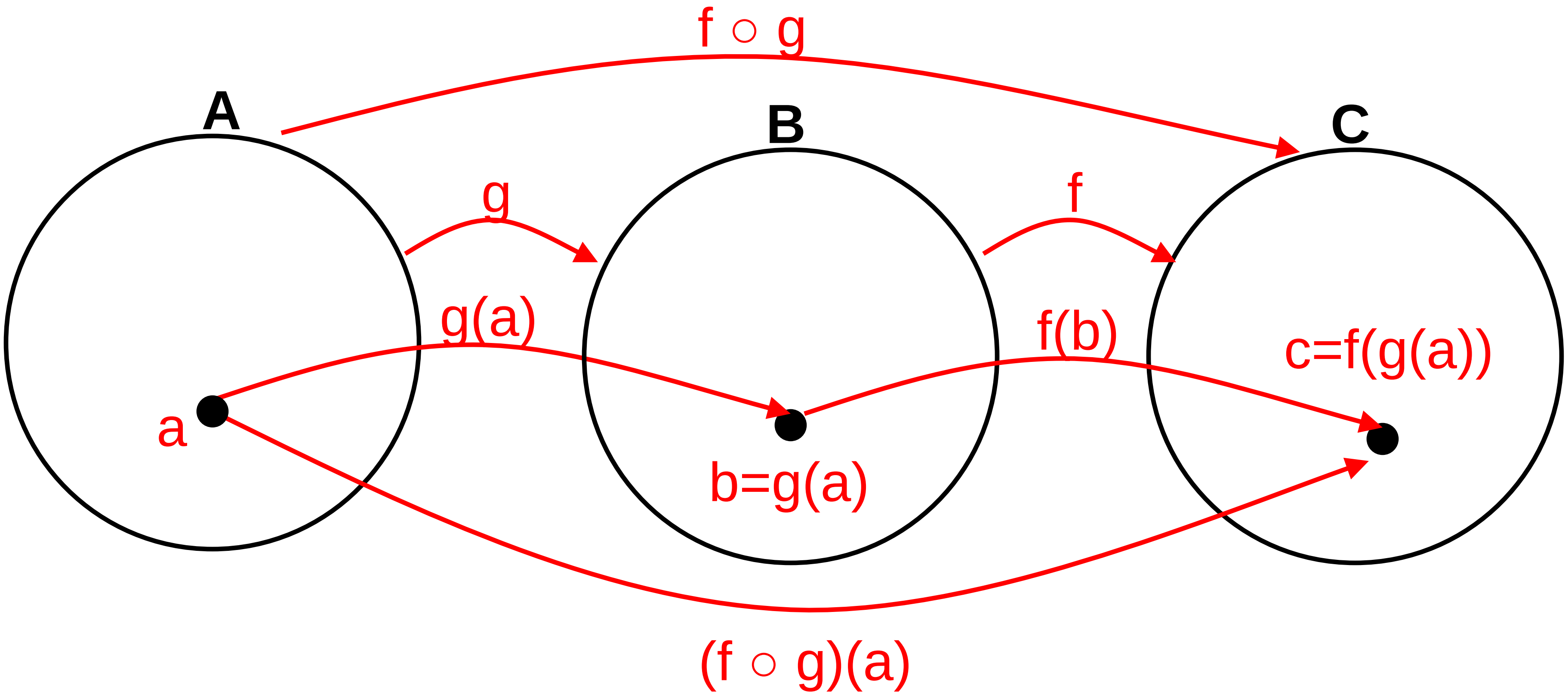
For two (or more) functions:

- Assume $f : A \to B$ and $g : B \to C$;
- For some argument $a \in A$ we can:
  - apply f to give $f(a) \in B$,
  - then apply g to give $g(f(a)) \in C$;
- This is equivalent to:
  - composing f and g to give $g \circ f : A \to C$,
  - and applying it, as $(g \circ f)(a) = g(f(a))$.

Compositions of functions: g○f

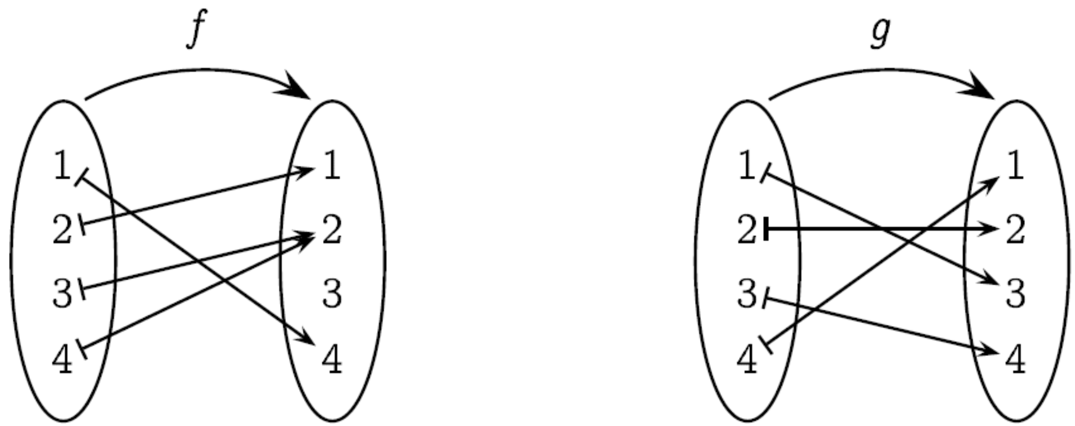


Compositions of functions: $f \circ g$



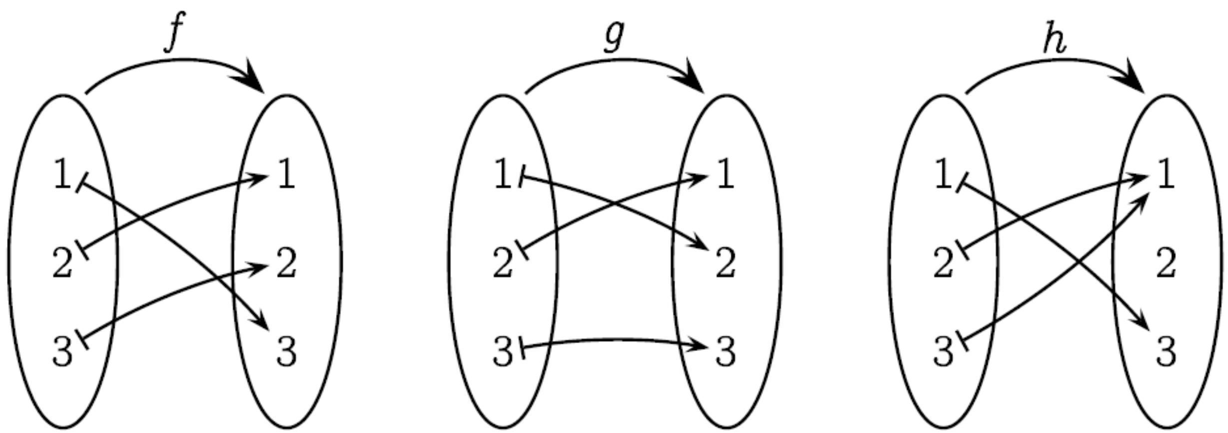
Exercise 6.9 (Solution on page 433)

Consider the following two functions f and g from $\{1, 2, 3, 4\}$ to itself:



Find $f \circ g$ and $g \circ f$.

6. Consider the following three functions f, g and h from $\{1, 2, 3\}$ to itself:



Find $f \circ g \circ h$

7. Find $g \circ f$ and $f \circ g$, where $f(x) = x^2+1$ and $g(x) = x-2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.