COM1002 Foundations of Computer Science

2. Sets

Course Overview

Textbooks and lecture notes:

- ❑ *A class textbook:*

*"**Modelling Computing Systems: Mathematics for Computer Science**",*
by Faron Moller & Georg Struth;

Follows chapters 1-10

Propositional Logic

Set Theory

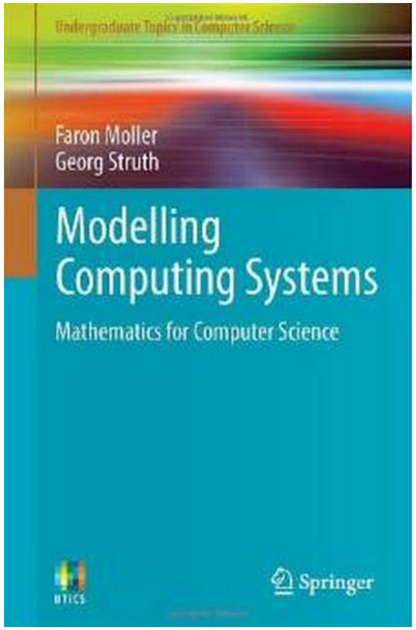
Boolean Algebra

Predicate Logic

Proof strategies

Functions and Relations

Induction and Recursion

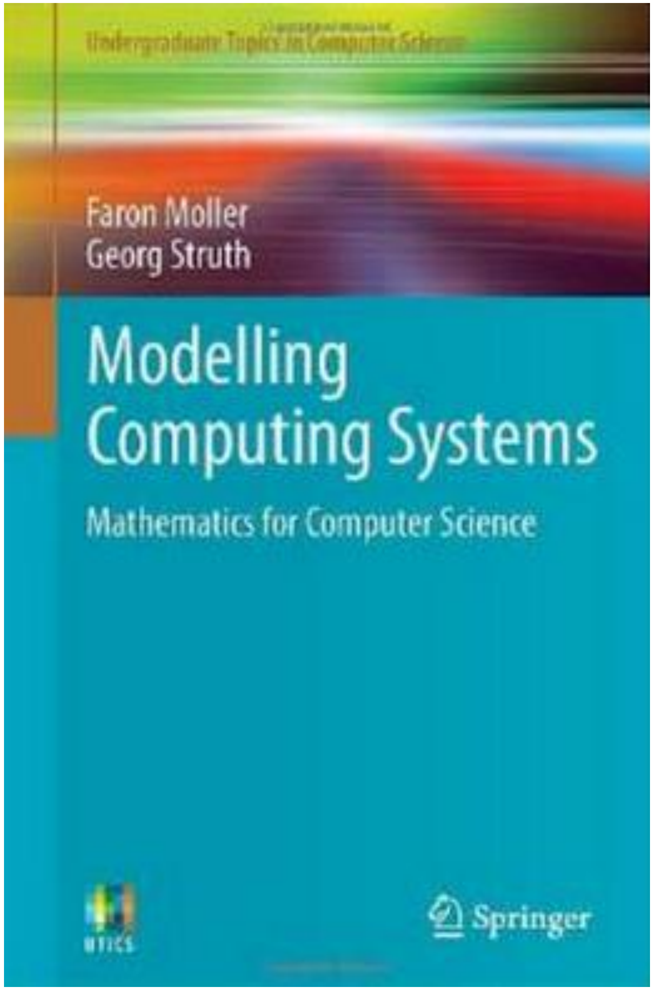


REMINDER:

If you haven't done already…

Obtain a copy of book !!!

Detailed instructions are on MOLE.



COM 1002: Foundations of Computer Science

Exercise Sheet 1

This sheet will be marked and account for 5% of your mark for this Semester. You need to hand it in in the tutorial on Thursday 15th October.

Exercise 1. Huey, Dewey and Louie have signs with their names on their bedroom doors. One day, when their uncle Donald comes to visit, they mix up the signs and their mother tells Donald that no door has its correct sign. Huey, Dewey and Louie are all sitting quietly in their own rooms. Donald is challenged to put the signs into their proper places, but is only allowed to open one door to look who is inside.

1. Give a propositional formula which models this situation. Use the propositional variable XY to express that Name X is at Y's door. For instance, HD states that the sign "Huey" is on Dewey's door. [10%]
2. Suppose Donald opens the door with the sign "Huey" and finds Louie. Can you deduce the correct door for each sign? [10%]

HINT: See exercise 1.16 in the course text book 'Modelling Computer Systems'

Exercise 2. Use truth tables to test whether the following formulas are valid.

1. $P \Rightarrow (Q \Rightarrow P)$. [10%]
2. $((P \Rightarrow (Q \Rightarrow R)) \Rightarrow ((P \Rightarrow Q) \Rightarrow (P \Rightarrow R)))$. [10%]
3. $(\neg P \Rightarrow \neg Q) \Rightarrow (Q \Rightarrow P)$. [10%]
4. $P \vee (P \wedge Q) \Leftrightarrow P$. [10%]
5. $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$. [10%]
6. $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$. [10%]

NOTE: A propositional logic formula is *valid* if, and only if, it is *TRUE* for all possible interpretations of the propositional variables (each proposition variable can be *TRUE* or *FALSE*).

Exercise 3. 1. Formalise each of the following statements in propositional logic. [10%]

(1) The only animals in this house are cats.
(2) Every animal that loves to gaze at the moon is suitable for a pet.
(3) When I detest an animal, I avoid it.
(4) No animals are carnivorous, unless they prowl at night.
(5) No cat fails to kill mice.
(6) No animal ever takes to me, except those that are in this house.
(7) Kangaroos are not suitable for pets.
(8) None but carnivora kill mice.
(9) I detest animals that do not take to me.
(10) Animals that prowl at night always love to gaze at the moon.

2. Argue that they imply *I always avoid a kangaroo.* [10%]
(Hint: Using a truth table might be very time consuming...)

1

Exercise Sheet 1 – Now on MOLE

Assessed – 5% of Mark for this Semester

Tutorials

Weeks 2-11 (no tutorial week 12)

GROUP A DIA-G04

GROUP B DIA-G05

GROUP C DIA-205

Tutorials: Group A (DIA-GO4)

Demonstrator: Hamna Afaq

Adeola, Babatunde
Bayley, Daniel
Bold, Harry
Booth, Callum
Bound, Richard
Carter, Robert
Chammas, Antoine
Cope, Rufus
Cornforth, Benjamin
Donnan, Samuel
D'Souza, Philip
Galaiya, Rajeev
Ganesh, Keerthana

Glover, Jessica
Goldsworthy, William
Guy, Sean
Halliwell, Benjamin
Hardy, Georgia
Jheeta, Gurjit
Jurewicz, Tomasz
Lewis, Elliot
McIntyre, Christopher
Mill, Charles
Moodie, David
Neal, Victoria
North, David

Reynolds, Daniel
Rochester, Euan
Sang, Sheldon
Scott, Christopher
Shuja, Muhammad
Solomon, Daniel
Thomas-Litman,
Tom
Vigar, Joshua
Walker, Jack
Walker, Matthew
Wallace, Benjamin
Wang, Wen Ting
Yip, Chung Lam

Tutorials: Group B (DIA-GO5)

Demonstrator: Maria Siregar

Andrias, Stelios
Balabanov, Stoyan
Basra, Simren
Bouchama, Sophia
Box, Kristofer
Burman, Matthew
Burrisson, Steven
Chan, Vincent
Chana, Ganga
Chishty, Muhammad
Cotcher, James
Dhinse, Manvir
Dobrea, Stefana

Furukawa, Yogen
Garside, William
Gkigkolian, Alexandros
Hanley, Adam
Hardcastle, Sarah
Holey, Matthew
Jones, Daniel
Kentish De La Iglesia, Peter
Kim, Da Eun
Knifton, George
Martin, Michael
Mirsaney, Nick
Montero Perez, Fernando

Odysseos, Christiana
Parker, Alex
Rahanu, Anika
Ramaneckaite, Greta
Ruzencevs, Andrejs
Sambrook, Patrick
Seriki, Toby
Shi, Mingqian
Shirazi Rad, Mohammad
Stevenson, Niall
Swift, Jodi
Williams, Benjamin
Zhou, Ping

Tutorials: Group C (DIA-205)

Demonstrator: Chris Quickfall

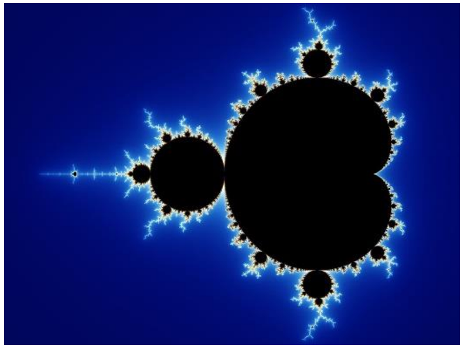
Askew, Benjamin
Baig, Zain
Bastidas, Sebastian
Burvill, Lawrence
Chan, Kar
Chen, Junjin
Chiorescu, Sanziana-Ioana
Cooper, Harrison
Da Silva Frias, Nuno
Dimitriou, Panagiotis
Gan, Benjamin
Han, Jack
Harvey, James
Ioannou, Alexis
Levene, Jake
Marketos, Antonios
Mason, Joseph
Mavrides, Nicolas
Nathan-Marsh, Afolabi
Nguyen, Manh Tri
Nigena, Idris
O'Kelly, Aidan
Page, Aiden
Parry, Owain
Raikundlia, Paras
Ratajczak, Michal
Renton, Ella
Richardson, Gregory
Roberts, James
Royce, Zachary
Sankhla, Abhinandan
Scott, Andrew
Tredget, Daniel
Trifonov, Hristiyan
Vasileiou, Antria
Wear, Matthew
Webb, James
Whittles, Alan
Williams, Ian
Williamson, Jack

What is a set ?

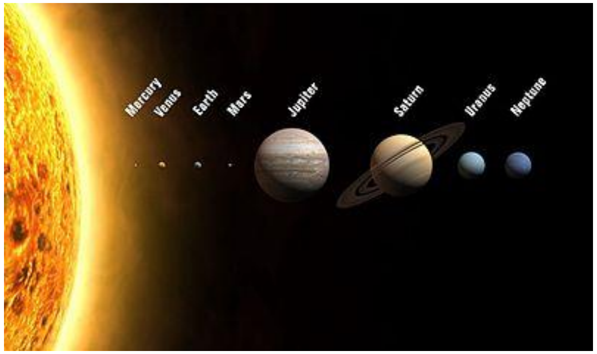
A set is a collection of elements or objects.

{1,2,3,4}

Mandelbrot set



Set of planets orbiting Earth



Set of judges on X-Factor 2014

Set of judges on X-Factor 2015

The objects in a set don’t have to be the same, e.g.

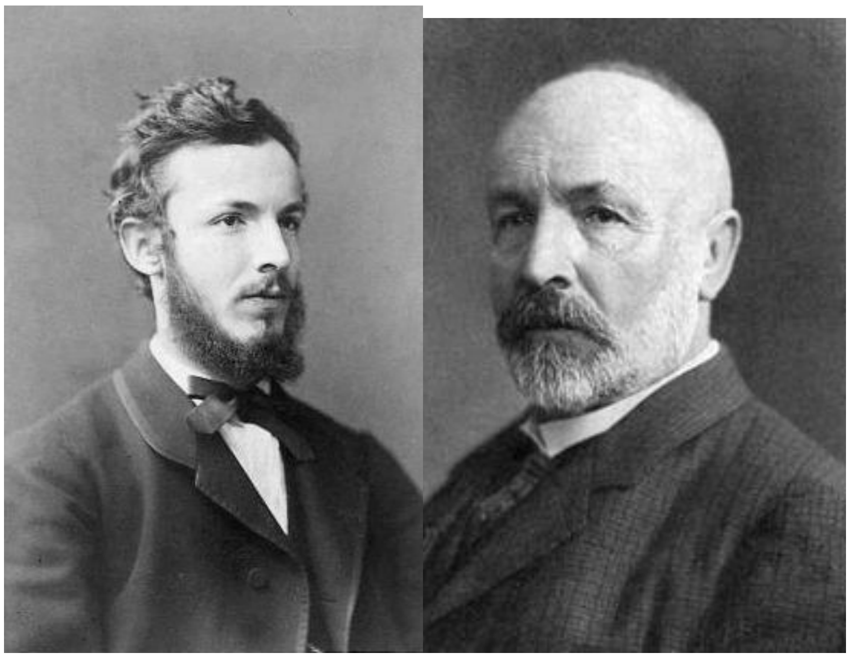
{Sinclair ZX Spectrum (48K ram), television, tape recorder, manic miner computer game tape}

1982…

COMPUTER SCIENCE
What has Set Theory ever done for us ?

- useful tool for formalising and reasoning about computation and the objects of computation.
- indivisible from Logic where Computer Science has its roots.
- it has been and is likely to continue to be a source of fundamental ideas in Computer Science from theory to practice
- the strong tradition, universality and neutrality of **Set Theory** make it firm common ground on which to provide unification between seemingly disparate areas and notations of Computer Science. Set Theory is likely to be around long after most present-day programming languages have faded from memory.
- A knowledge of Set Theory should facilitate your ability to think abstractly. It will provide you with a foundation on which to build a firm understanding and analysis of the new ideas in Computer Science that you will meet.

Set Theory: Brief Historical Perspective



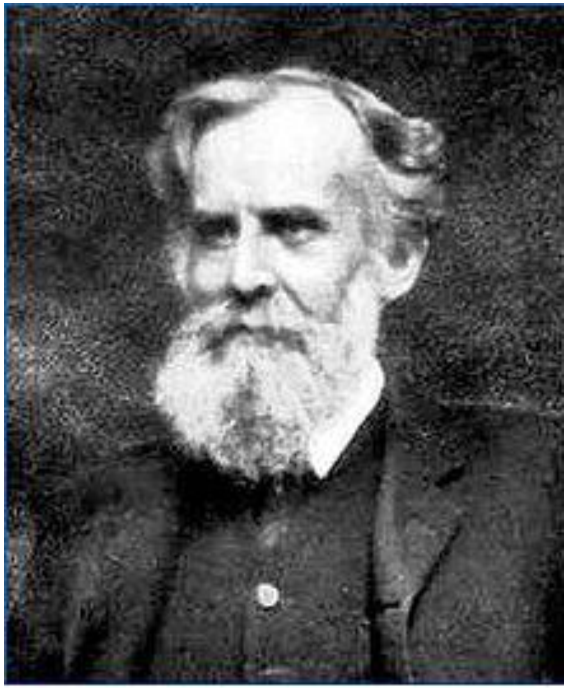
George Cantor (1845-1918)

George Cantor: inventor of Set Theory

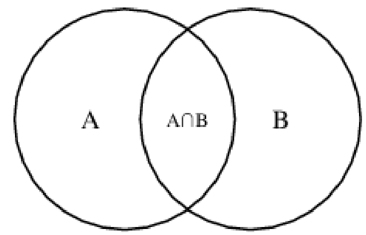
Set Theory was used to explore and answer questions about the nature of infinity: *there are infinitely many orders of infinities.*

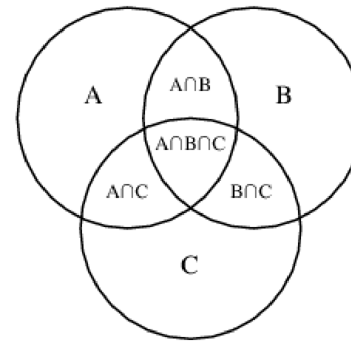
Venn Diagrams

John Venn (1834-1923)



John Venn was an English logician and philosopher noted for introducing the Venn diagram, used in the fields of set theory, probability, logic, statistics, and computer science.

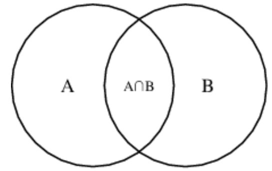


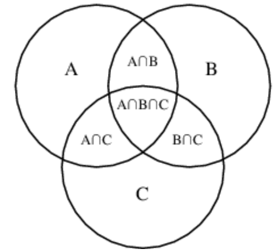


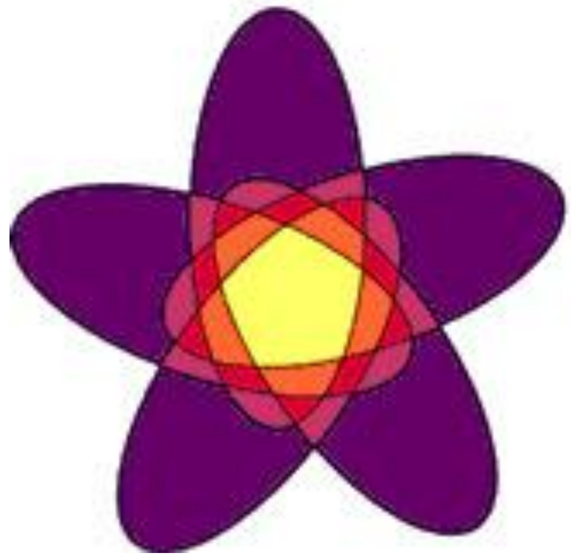
"I now first hit upon the diagrammatical device of representing propositions by inclusive and exclusive circles. Of course the device was not new then, but it was so obviously representative of the way in which any one, who approached the subject from the mathematical side, would attempt to visualise propositions, that it was forced upon me almost at once."

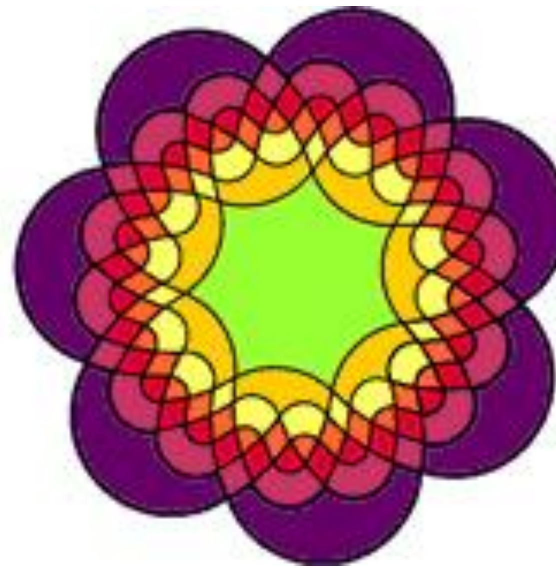
Venn diagrams and art…

Stained glass window at
Gonville and Caius College,
University of Cambridge





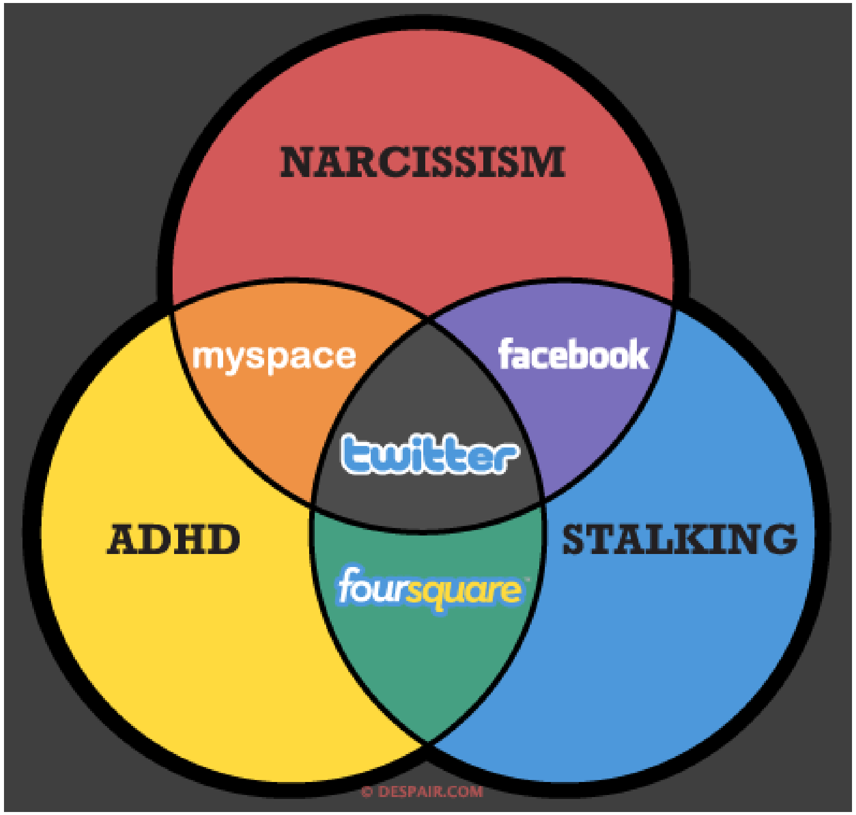




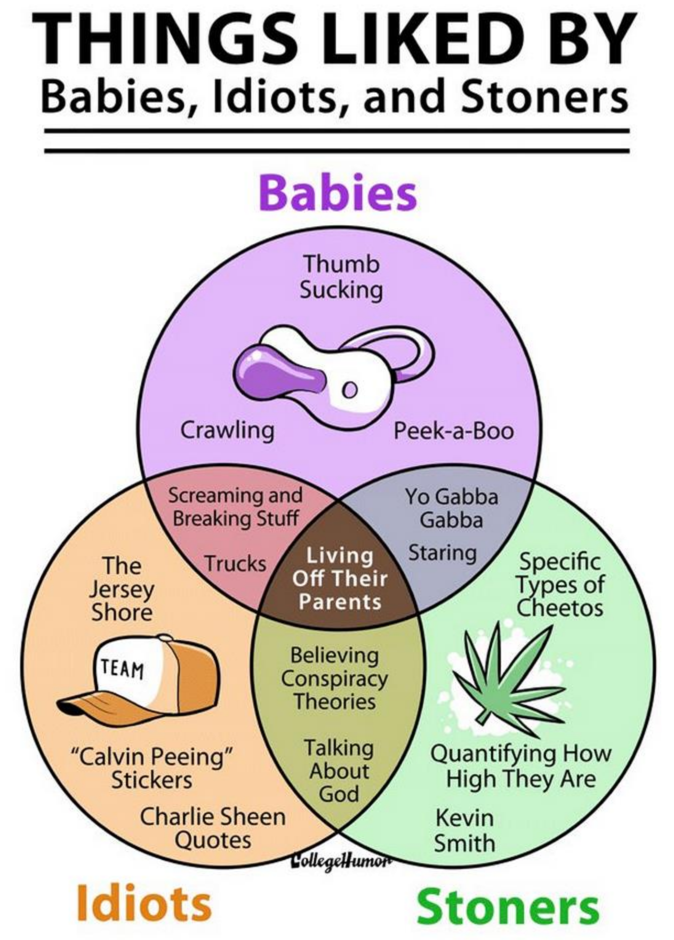
The left figure shows a 5-fold Venn diagram due to Branko Grünbaum, while the attractive 7-fold rosette illustrated in the middle figure is a 7 fold Venn diagram called "Victoria" by Ruskey.

http://mathworld.wolfram.com/VennDiagram.html

Venn diagrams in everyday life…



The social media Venn Diagram
http://blog.visual.ly/the-social-media-venn-diagram/



www.collegehumor.com/post/6458120/venn-diagram-things-liked-by-babies-idiots-and-stoners

LEARNING OUTCOME:

All babies are illogical

Nobody is despised who can manage a crocodile

Illogical persons are despised

We can solve with propositional Logic

We can also solve with Set Theory.

We will learn how to do this today!

Set Theory for Computer Science

To model data:

- ❑Identify collections of things with similar properties:
  - ❖eg numbers, strings of characters, etc;
- ❑Distinguish between different collections of things:
  - ❖eg numbers, strings of characters, etc;
- ❑Data types in programming, and sets in mathematics, are very similar concepts.

Concepts of Sets 1

Definition:

- A set is a collection of objects that share some property:
  - the property is often a logical proposition;
- The elements of a set:
  - are the objects in the collection,
  - the property must hold for all of them;
- The cardinality of a set:
  - is the number of different elements in it,
  - usually denoted |Set|, or sometimes #Set.

Concepts of Sets 2

Basic notation for sets:

- ❑A list of the elements:
  - ❖separated by commas, and
  - ❖enclosed in curly brackets;
- ❑Examples:
  - ❖{ false, true },
  - ❖{ 3, 7, 14 },
  - ❖{ red, blue, yellow }

Concepts of Sets 3

Notation for larger sets:

- Informally:
  - Could use ellipses, as
  - { 2, 3, 5, 7, 11, 13, 17 … } (prime numbers),
  - but often this is not precise enough;
- Formally, use “set builder” notation:
  - has the basic form:
  - { variable : proposition involving the variable }
  - eg { n : n is a prime number }

Concepts of Sets 4

Some common sets in mathematics:

- $\emptyset$ = { } (the empty set);
- **B** = { 0, 1 } (binary digits);
- **N** = { 0, 1, 2, 3, … } (natural numbers);
- **Z** = { … -3, -2, -1, 0, 1, 2, 3, … } (integers);
- **Q** = { m/n : m, n $\in$ **Z**, n $\neq$ 0 } (rationals);
- **R** = { x : x is a real number } (reals);

Try some of exercise 2.1/2.2

Exercise 2.1/2.2

Write out each of the following sets explicitly, by listing their elements within curly braces.

$$\{x : x \text{ is an odd integer with } 0 < x < 8\}$$

$$\{x : x \text{ is a day of the week not containing the letter n}\}$$

$$\{x : x \text{ is an integer with } x = 2y \text{ where } y \in \{1,2,3,4,5\}\}$$

$$\{x : x \text{ is an integer with } 2x = y \text{ where } y \in \{1,2,3,4,5\}\}$$

Set Membership 1

A set is defined solely by its elements, so:

- The fundamental relationship is:
  - an object x is or is not an element of a set A,
  - denoted $x \in A$ or $x \notin A$,
  - eg $7 \in \{ 3, 7, 14 \}$ and $8 \notin \{ 3, 7, 14 \}$;
- Two sets are equal if they have the same elements:
  - the order of the elements is irrelevant,
  - repetitions of elements are irrelevant,
  - eg $\{ 3, 7, 14 \} = \{ 7, 14, 3, 7, 3 \}$.

Set Membership 2

Demonstrating inequality:

- Find a "witness":
  - an element that is in one but not the other,
  - eg if the cardinalities are different;
- Note: a set is not the same as its elements:
  - eg $\emptyset$ is not the same as $\{ \emptyset \}$,
  - since $|\emptyset| = 0$, but $|\{ \emptyset \}| = 1$.

Try exercises 2.3, 2.4.

Exercise 2.3

Which of the following propositions are true ?

$$2 \in \{1,2,3\}$$

$$\{2\} \in \{1,2,3\}$$

ϕ

$$\{2\} \in \{\{1\},\{2\},\{3\}\}$$

$$\phi \in \{\ \}$$

$$\phi \in \{\phi\}$$

$\phi = \{\ \}$ The empty set. It contains no elements.

$A = \{\phi\} = \{\{\ \}\}$ The set A consists of one element, namely, the empty set.

Exercise 2.4

Which of the following sets are equal ?

$$A = \{1, \{1,2\}\}$$

$$B = \{1, \{2\}\}$$

$$C = \{1, \{1\}\}$$

$$D = \{\{1,1\},1\}$$

$$E = \{\{2,1\},1\}$$

Set Membership 3

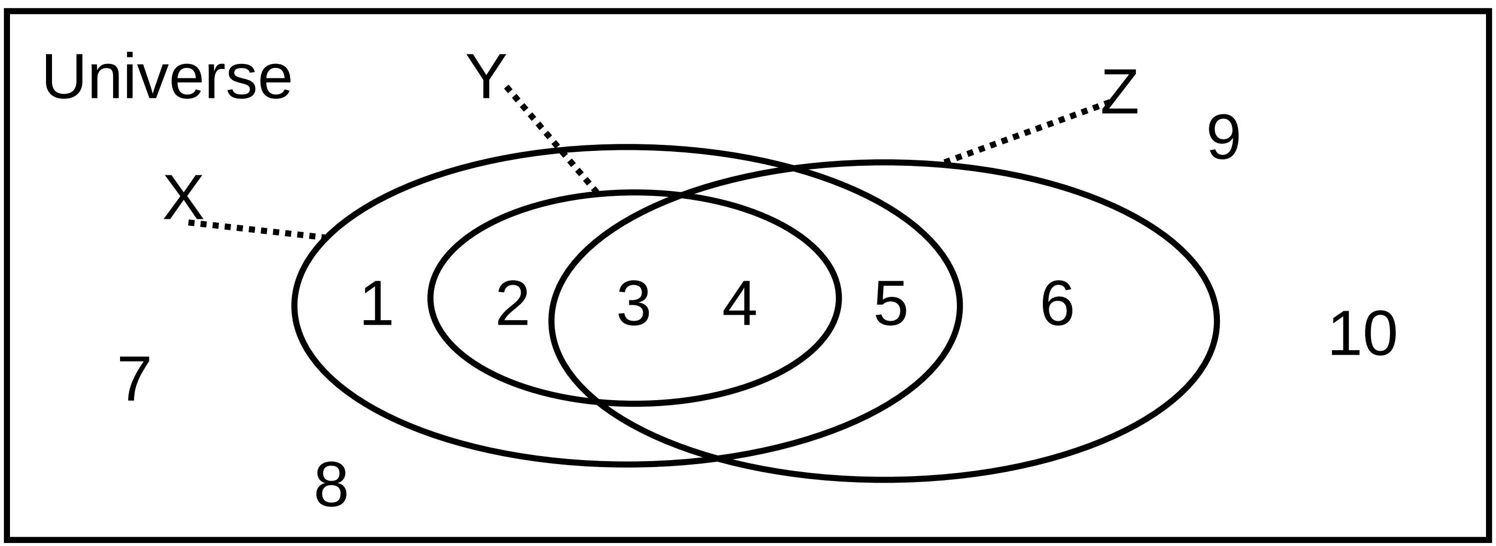
Subsets:

- If every element of A is an element of B:
  - then A is a subset of B, written $A \subseteq B$,
  - and B is a superset of A, written $B \supseteq A$,
  - hence $(A = B) \Leftrightarrow (A \subseteq B \wedge A \supseteq B)$;
- If A is a subset of B, and $A \neq B$:
  - then A is a proper subset of B, written $A \subset B$;
- Laws for subsets:
  - $\emptyset \subseteq A$, and $A \subseteq A$ (ie $\subseteq$ is reflexive), and
  - $A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$ (ie $\subseteq$ is transitive).

Set Membership 4

Venn Diagrams:

- Illustrate relationships between sets - eg:
  - X = {1, 2, 3, 4, 5}, Y = {2, 3, 4}, Z = {3, 4, 5, 6}



Operations on Sets 1

Union:

- For sets A and B:
  - written $A \cup B$,
  - elements that are in A, or B, or both,
  - $A \cup B = \{ x : x \in A \text{ or } x \in B \}$.
- So that:
  - $(x \in A \cup B) \Leftrightarrow (x \in A) \vee (x \in B)$.

Operations on Sets 2

Intersection:

- For sets A and B:
  - written $A \cap B$,
  - elements that are in both A and B,
  - $A \cap B = \{ x : x \in A \text{ and } x \in B \}$.
- So that:
  - $(x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B)$;
- Two sets A and B are disjoint if they have no elements in common:
  - ie $A \cap B = \varnothing$.

Operations on Sets 3

Difference:

- For sets A and B:
  - written $A \setminus B$,
  - elements that are in A but not in B,
  - $A \setminus B = \{ x \in A : x \notin B \}$.
- So that:
  - $(x \in A \setminus B) \Leftrightarrow (x \in A) \wedge (x \notin B)$.

Operations on Sets 4

Complement:

- For some set A and a universe set:
  - written $\overline{A}$,
  - elements that are not in A (but in the universe),
  - $\overline{A} = \{ x : x \notin A \}$, or
  - $\overline{A} = \{ x \in \text{universe} : x \notin A \}$.
- So that:
  - $(x \in \overline{A}) \Leftrightarrow (x \notin A)$.

Try part of exercise 2.15, exercise 2.16.

Exercise 2.15

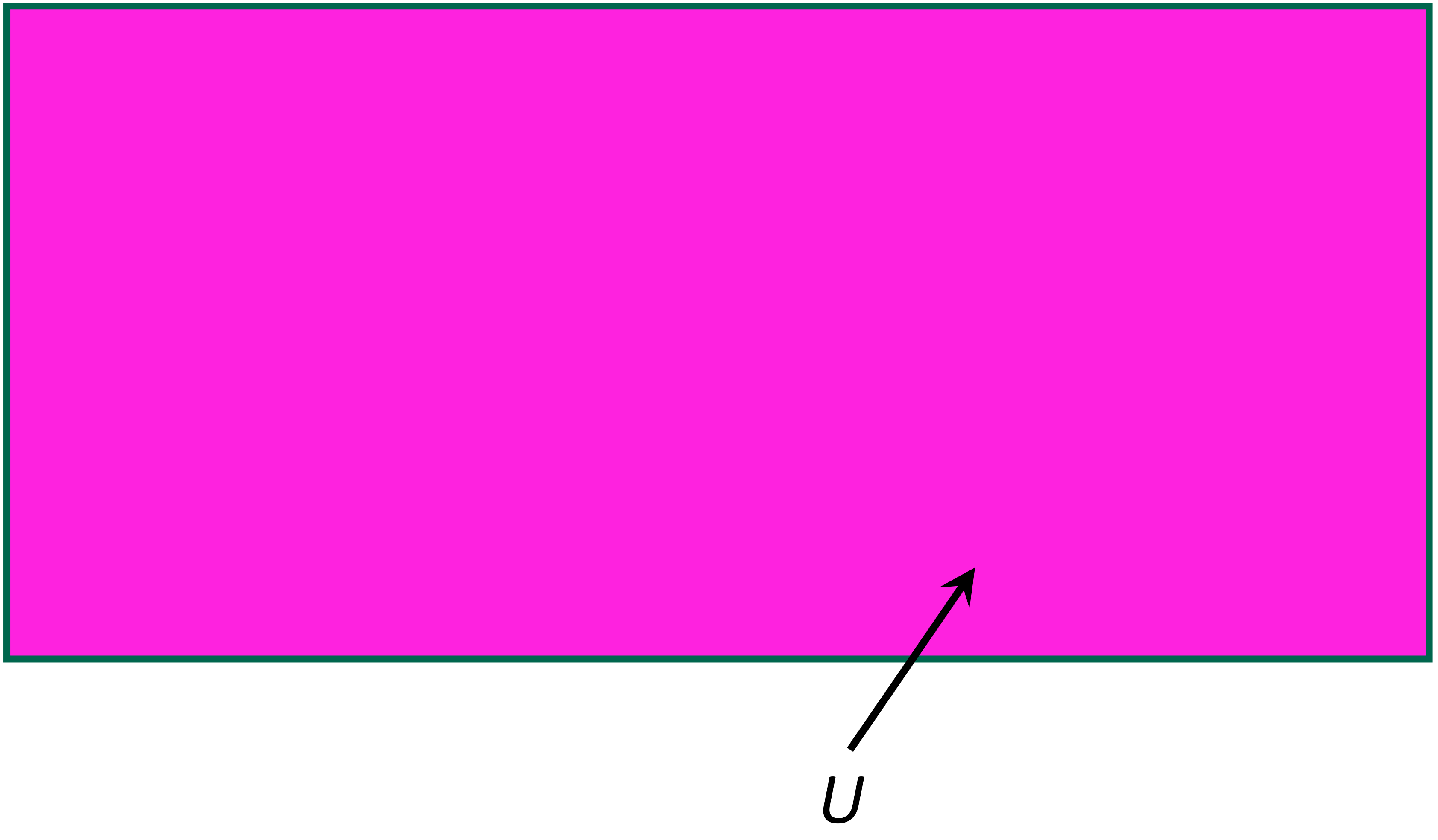
Consider the following sets:

$$U = \{1,2,3,4,5,6,7,8,9,10\}$$
$$A = \{1,3,5,7,9\}$$
$$B = \{3,4,5\}$$
$$C = \{5,6,7,8,9\}$$

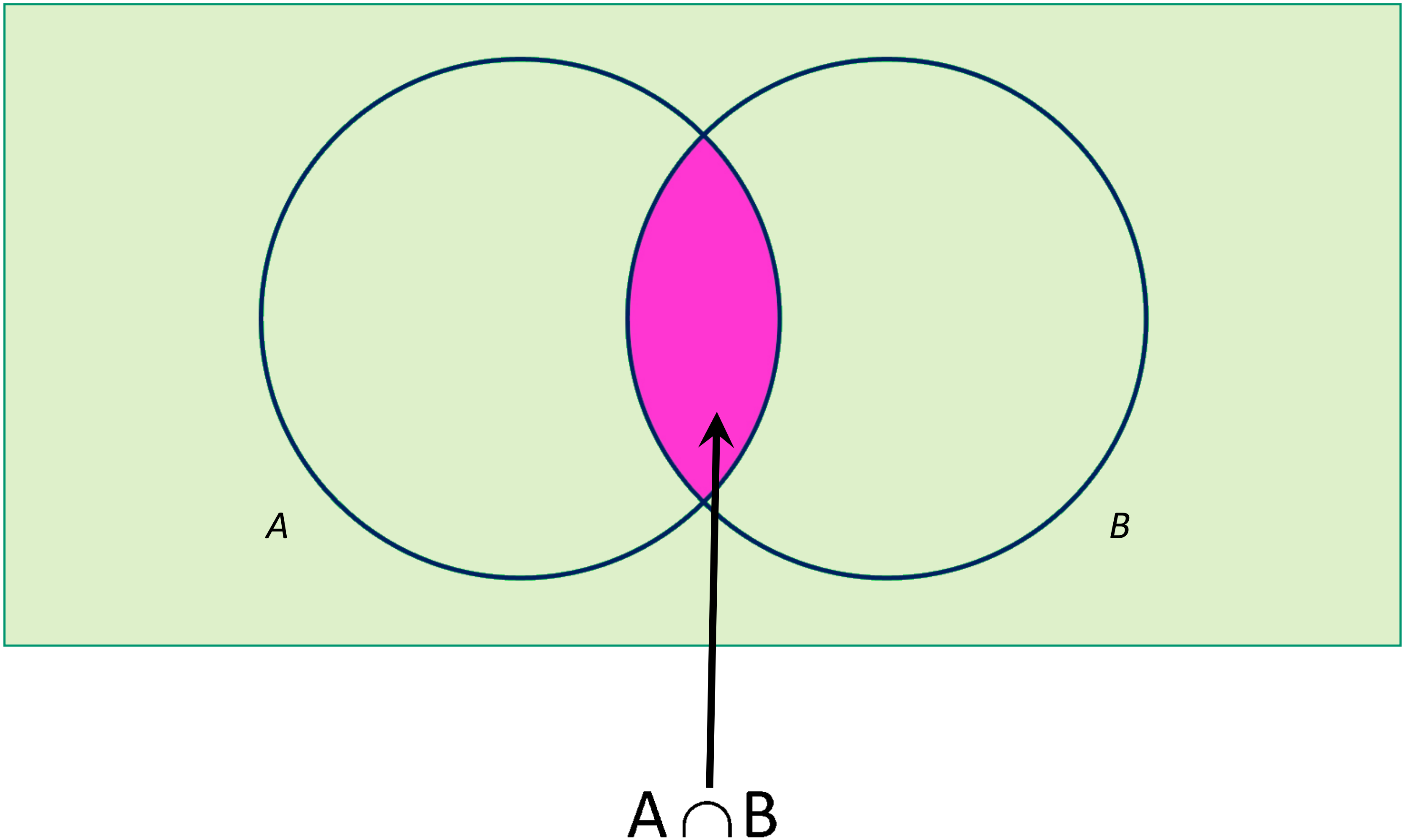
Draw a Venn diagram depicting these sets, and compute the following sets:

1. $A \cap C$
2. $(A \cap B) \cup C$
3. $A \cap (B \cup C)$
4. $(A \cup B) \setminus C$
5. $\overline{(A \cup B)} \cap C$

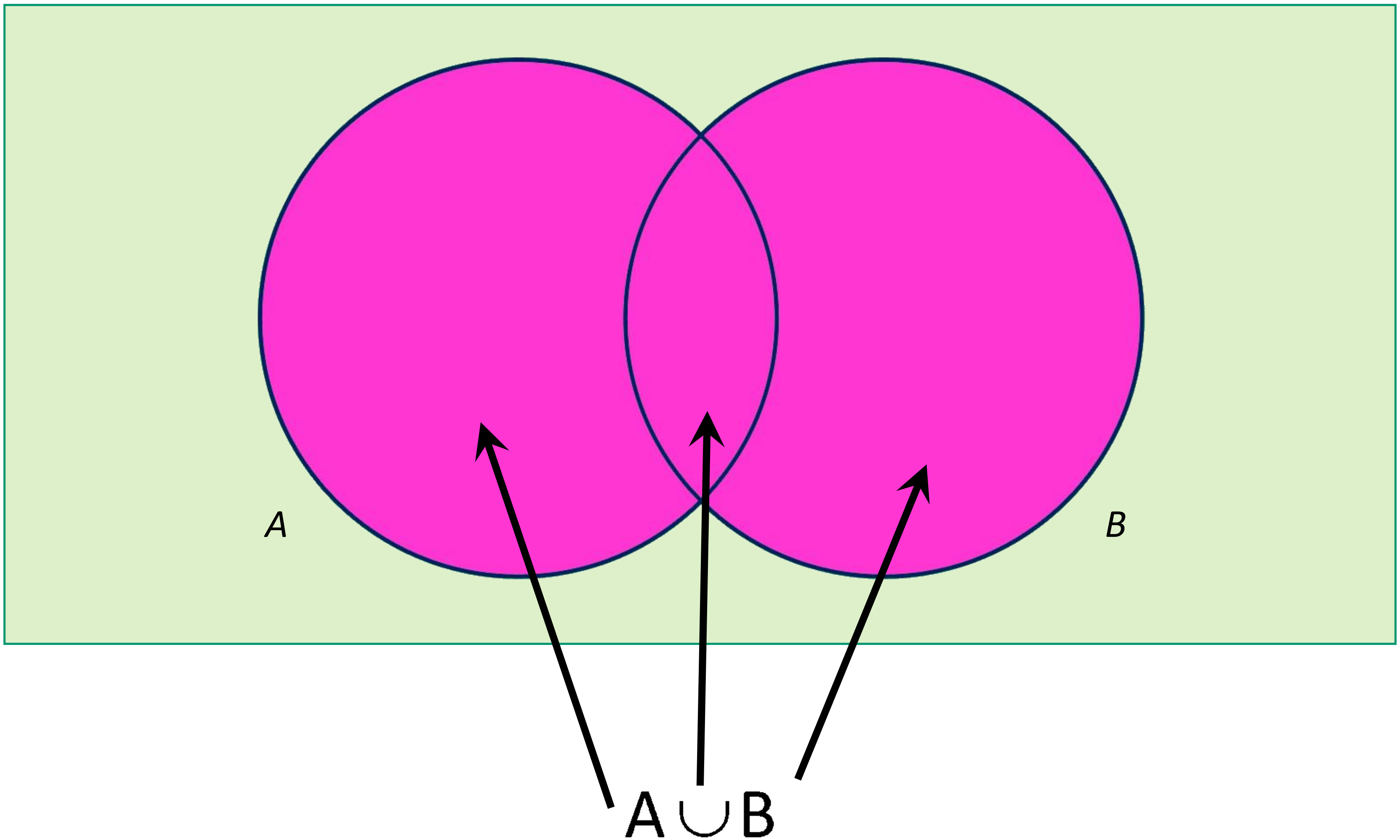
Universal Set: All members



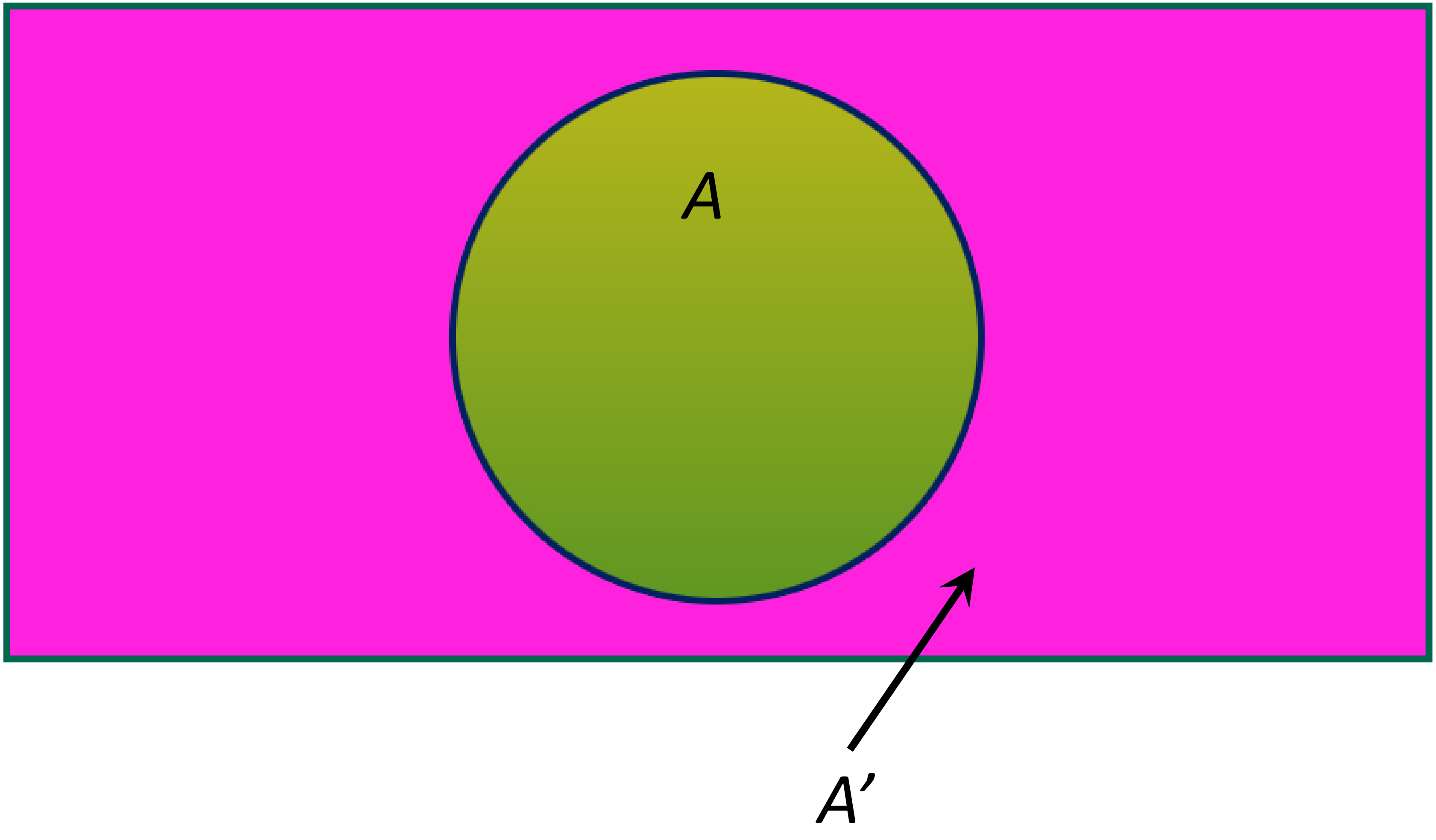
Intersection: Members of both set A and set B



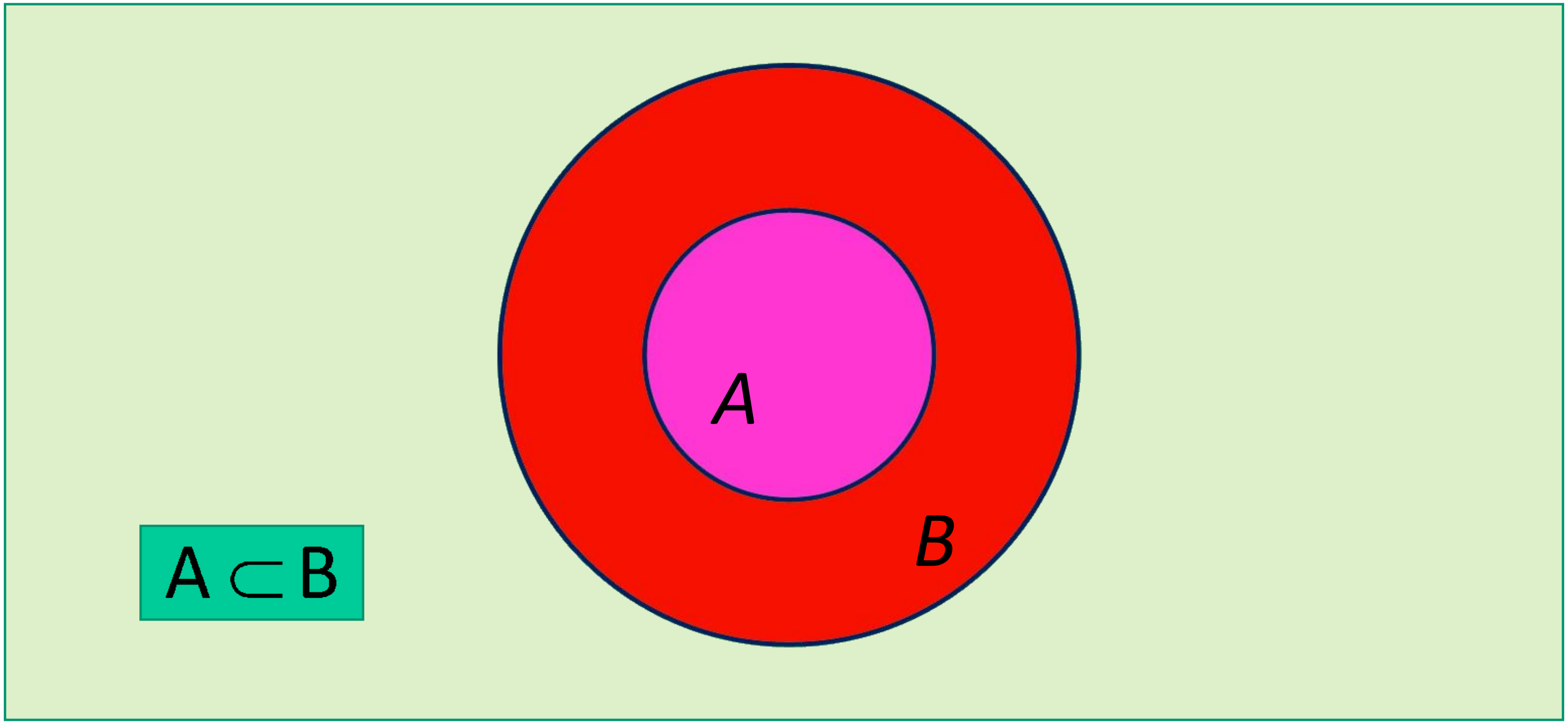
Union: Members of set A or set B or both



Complementary: Members not in the set



Subset: All members of set A are in set B

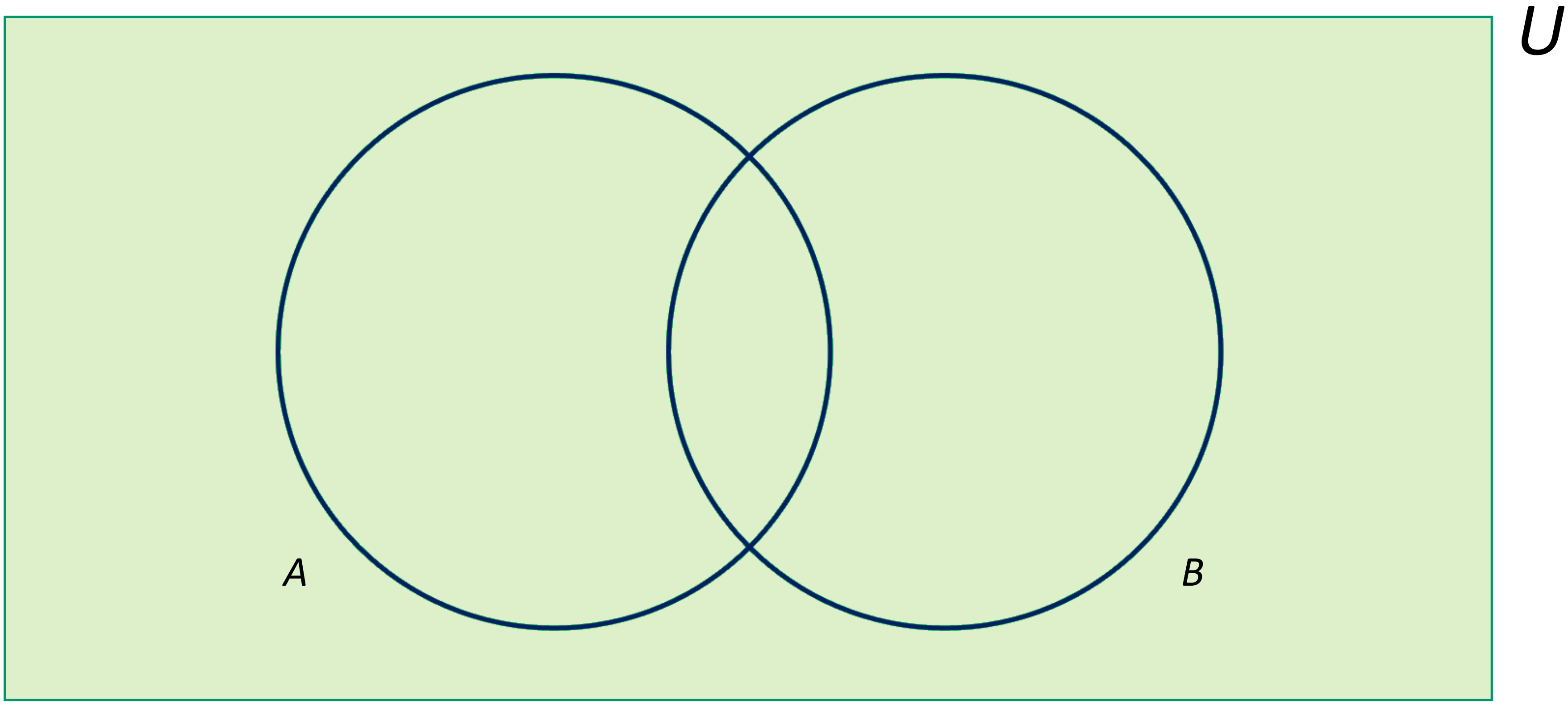


Number of elements in a set: $n(A)$

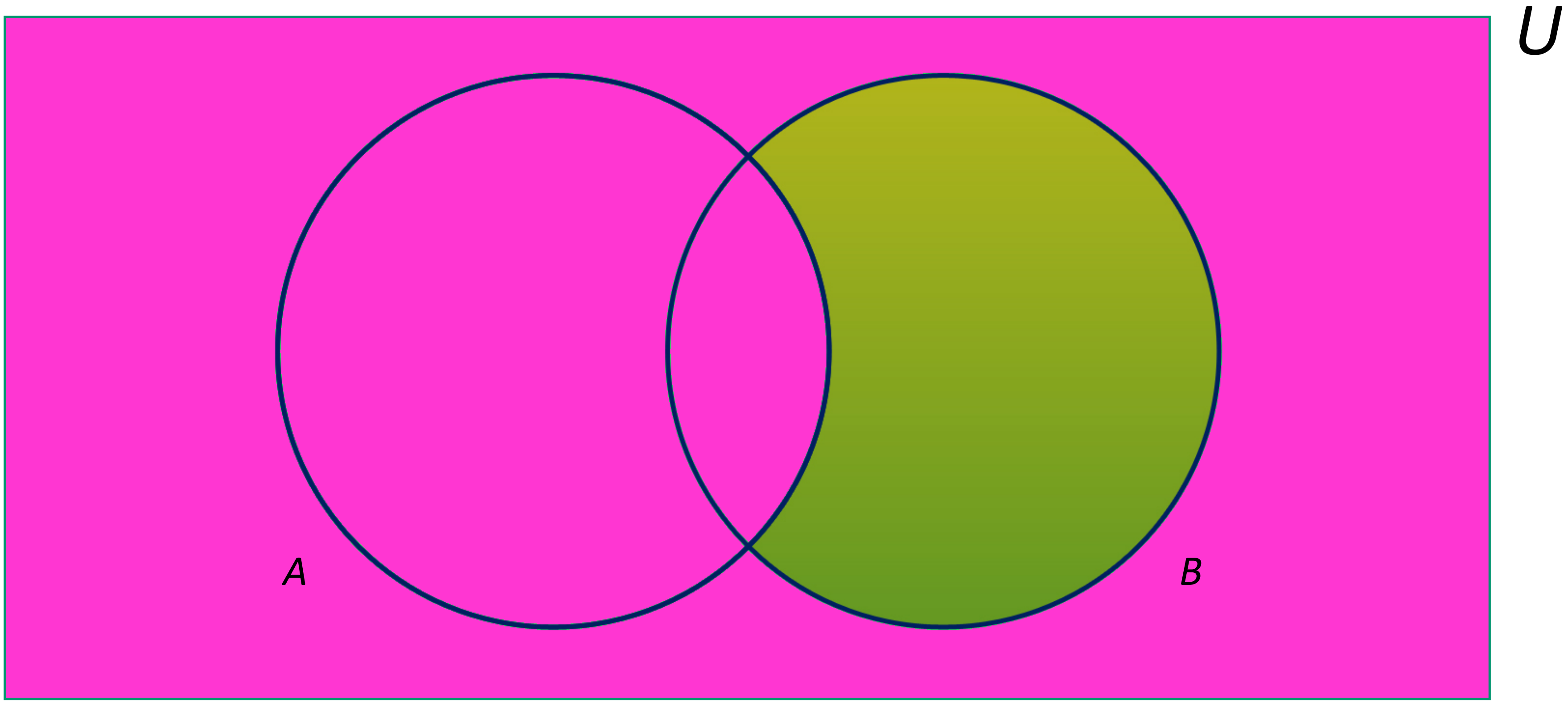
Empty set: $\emptyset$

$$\boxed{n(\emptyset)=0}$$

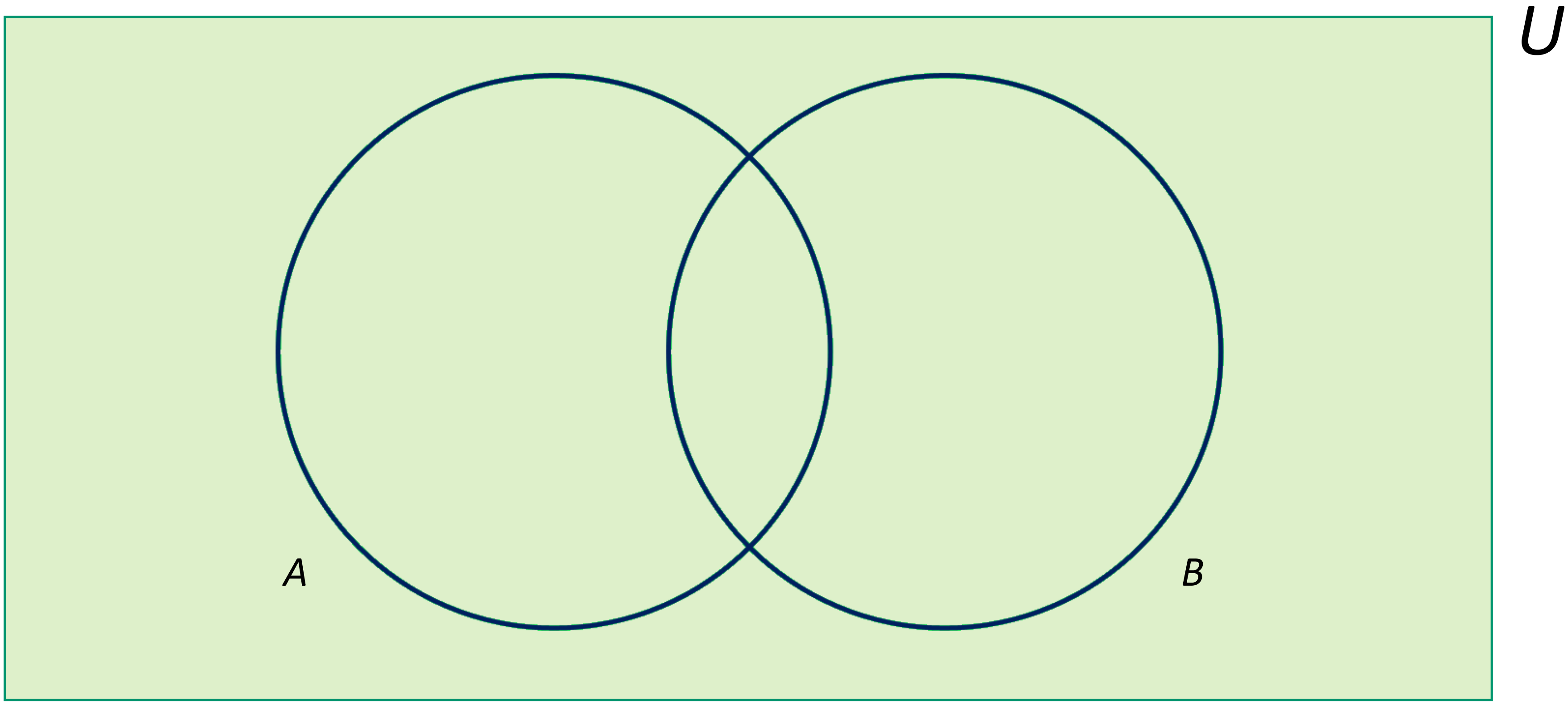
$A \cup B'$



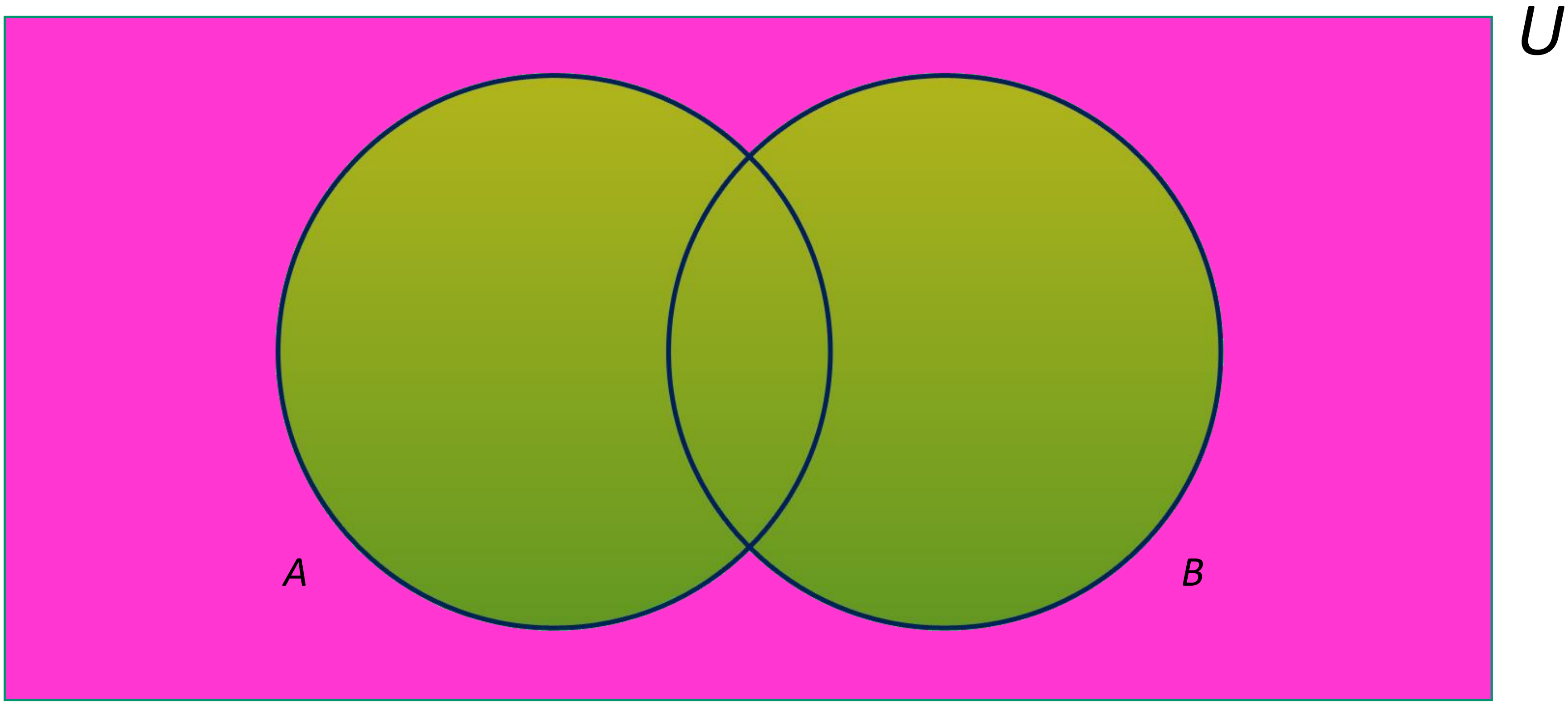
$A \cup B'$



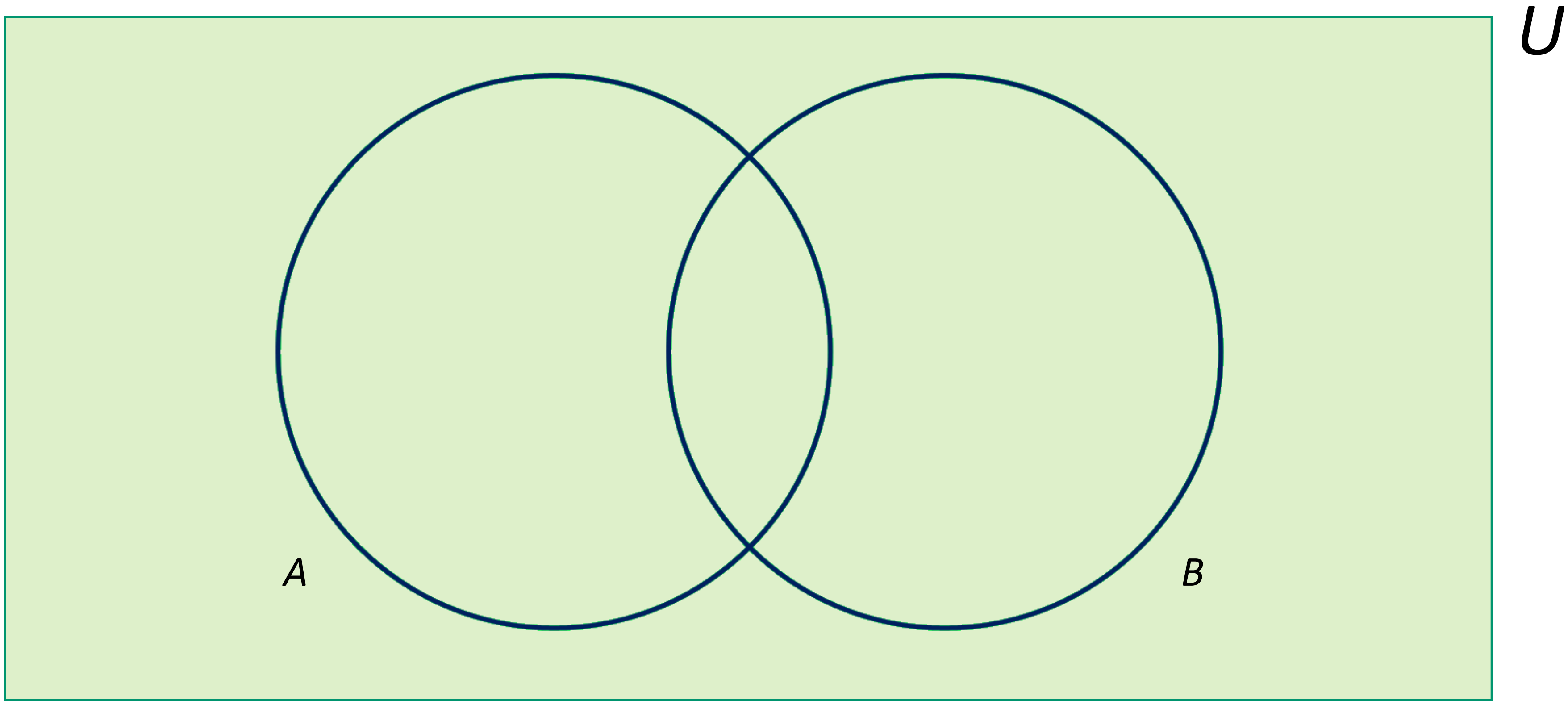
$A' \cap B'$



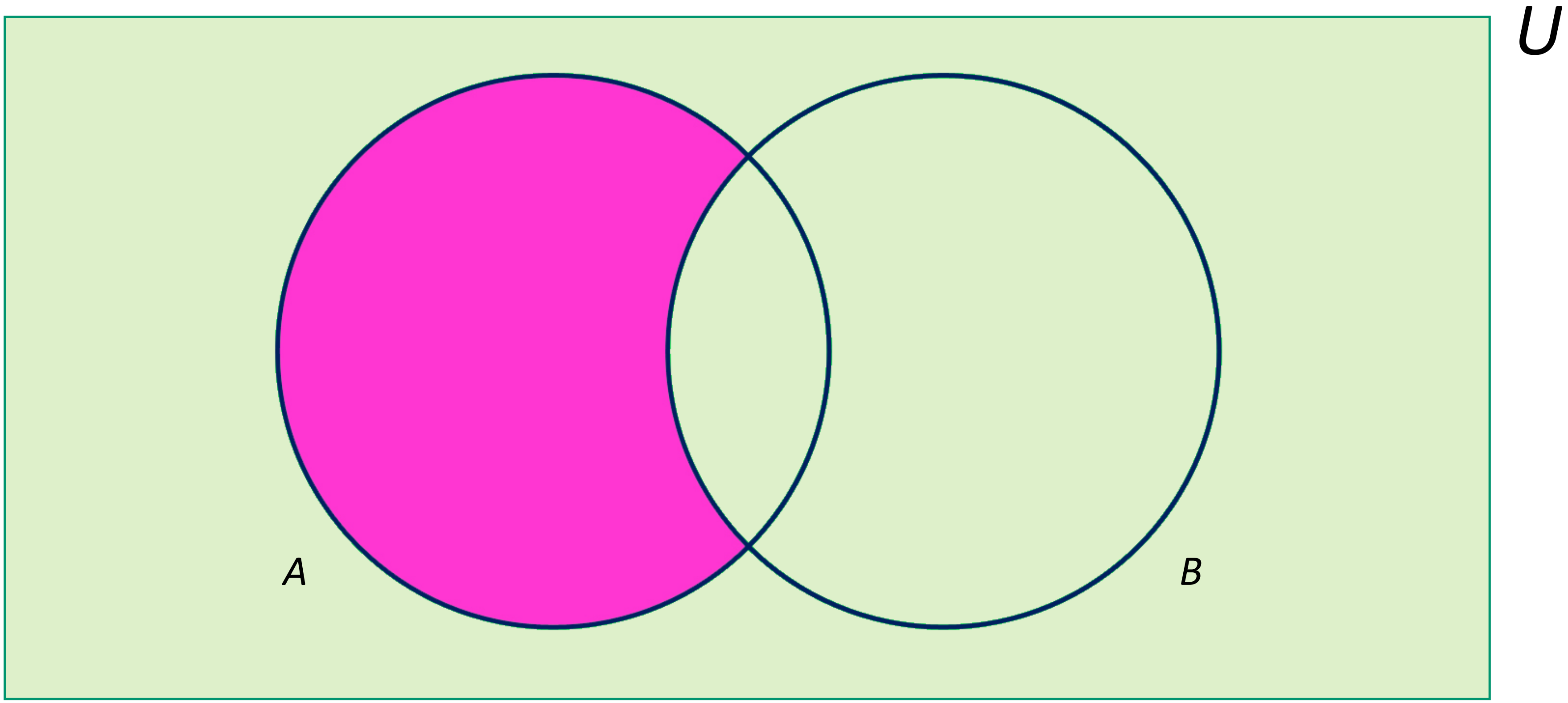
$A' \cap B'$



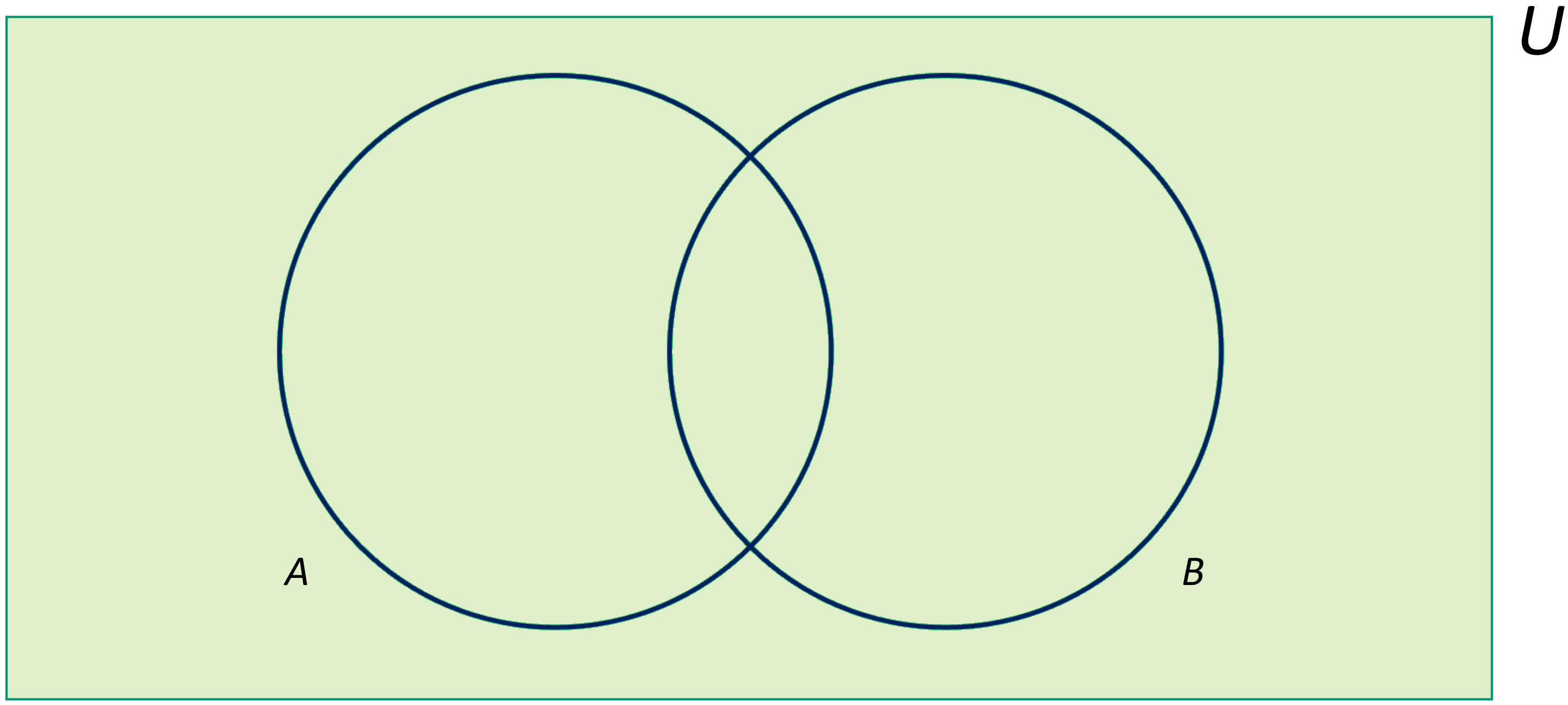
$A \cap B'$



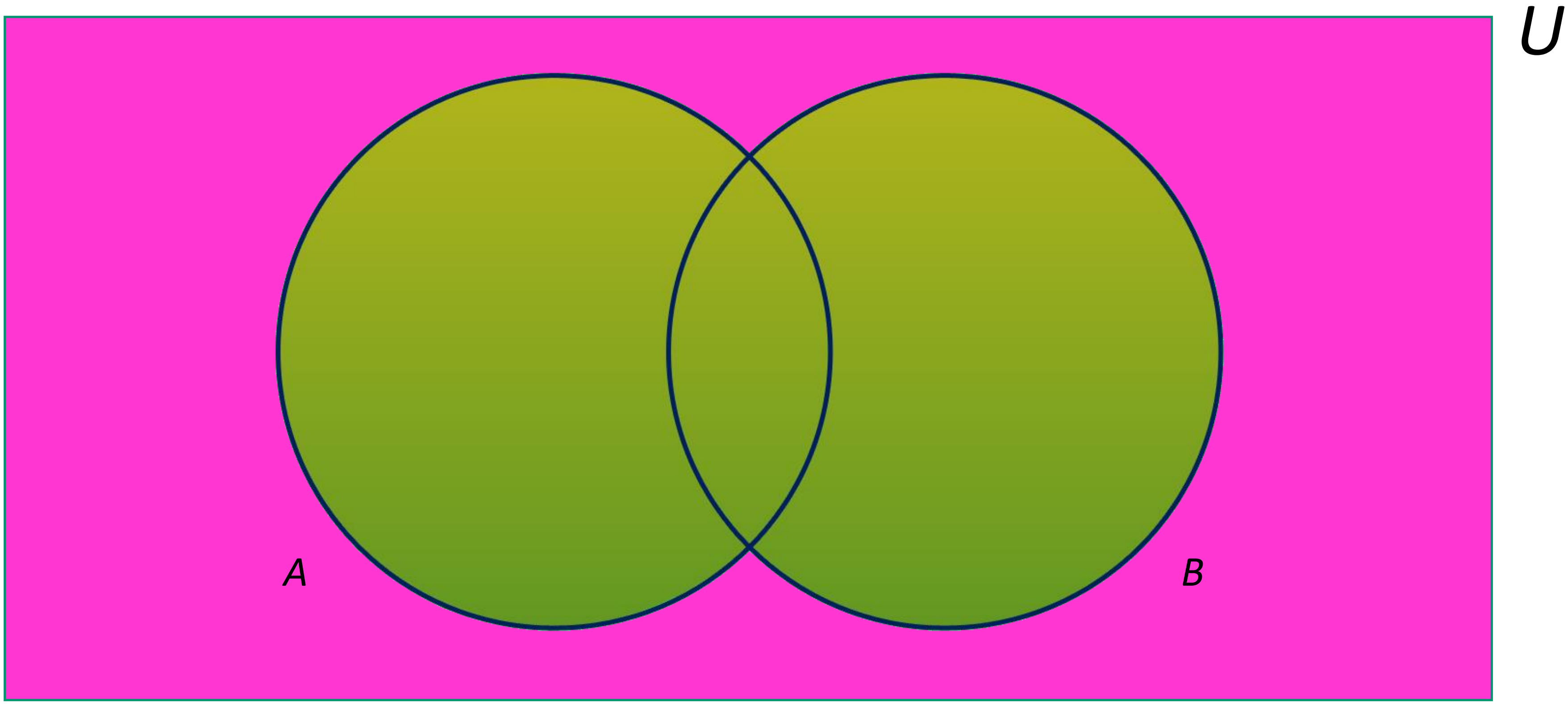
$A \cap B'$



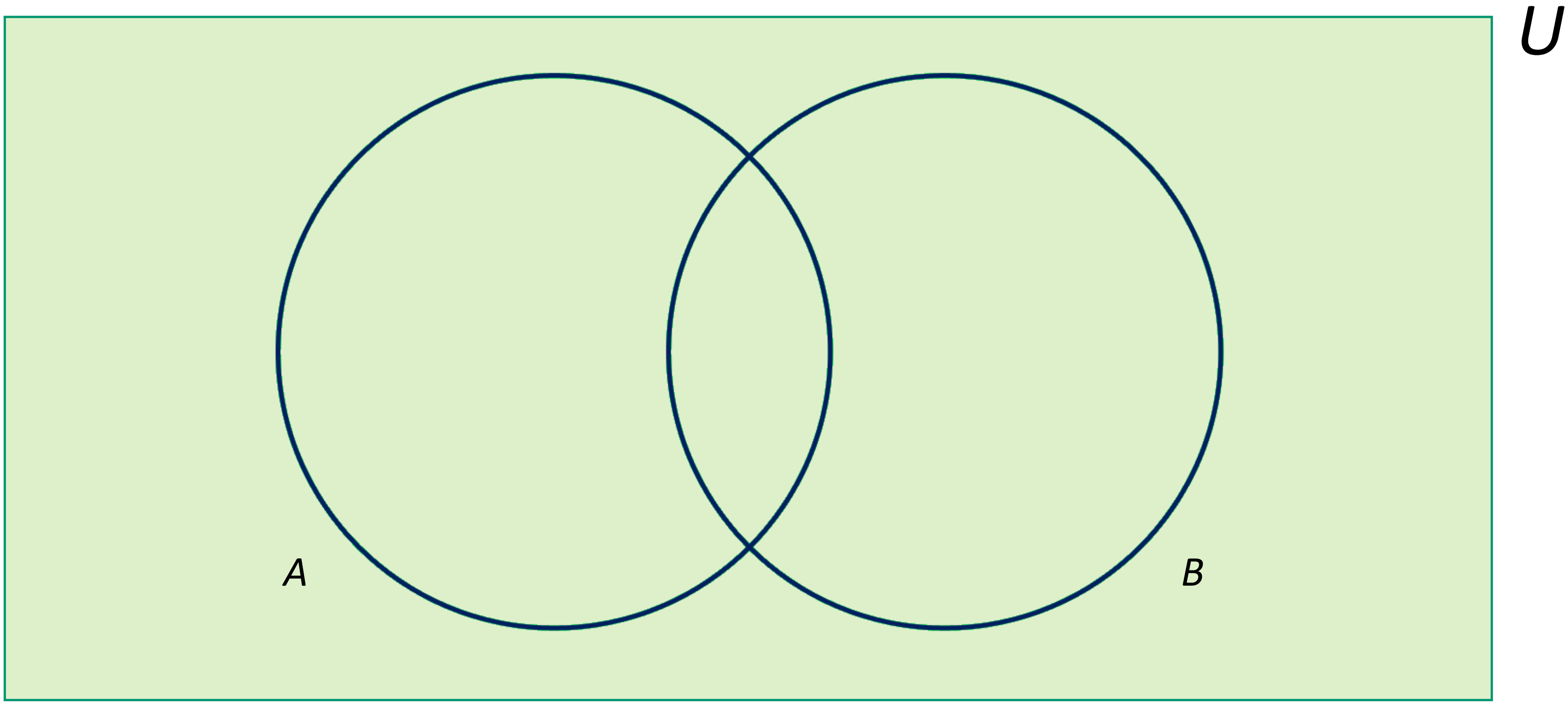
$(A \cup B)'$



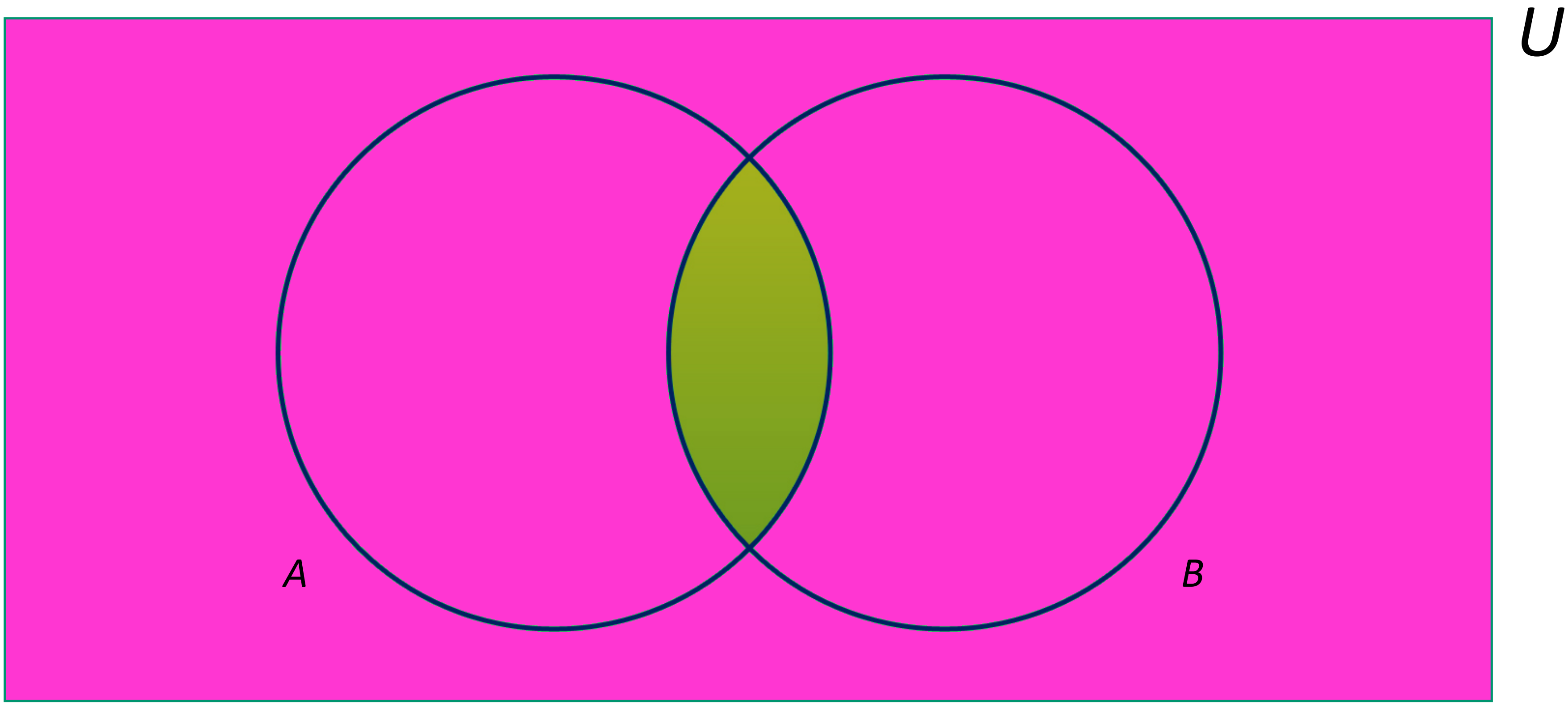
$(A \cup B)'$



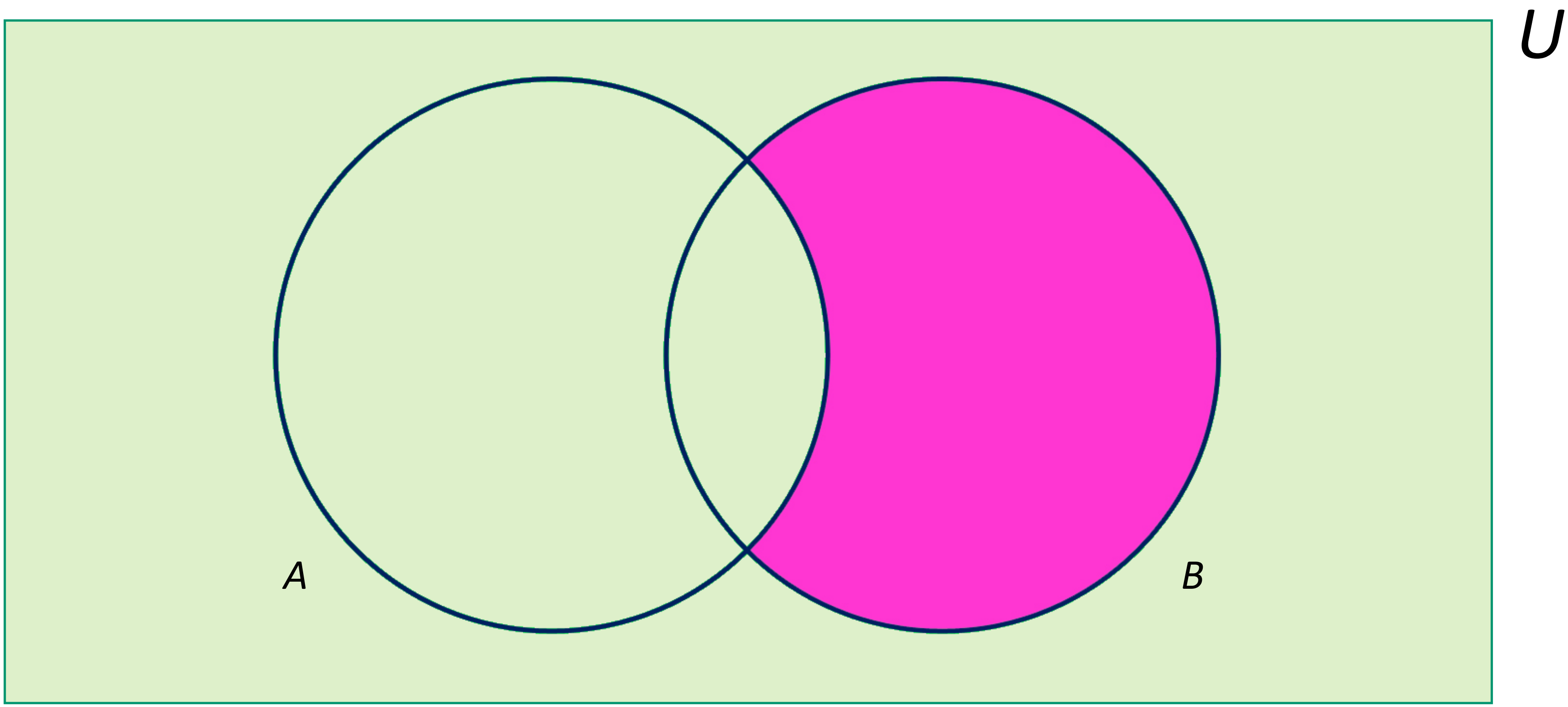
$A' \cup B'$



$A' \cup B'$

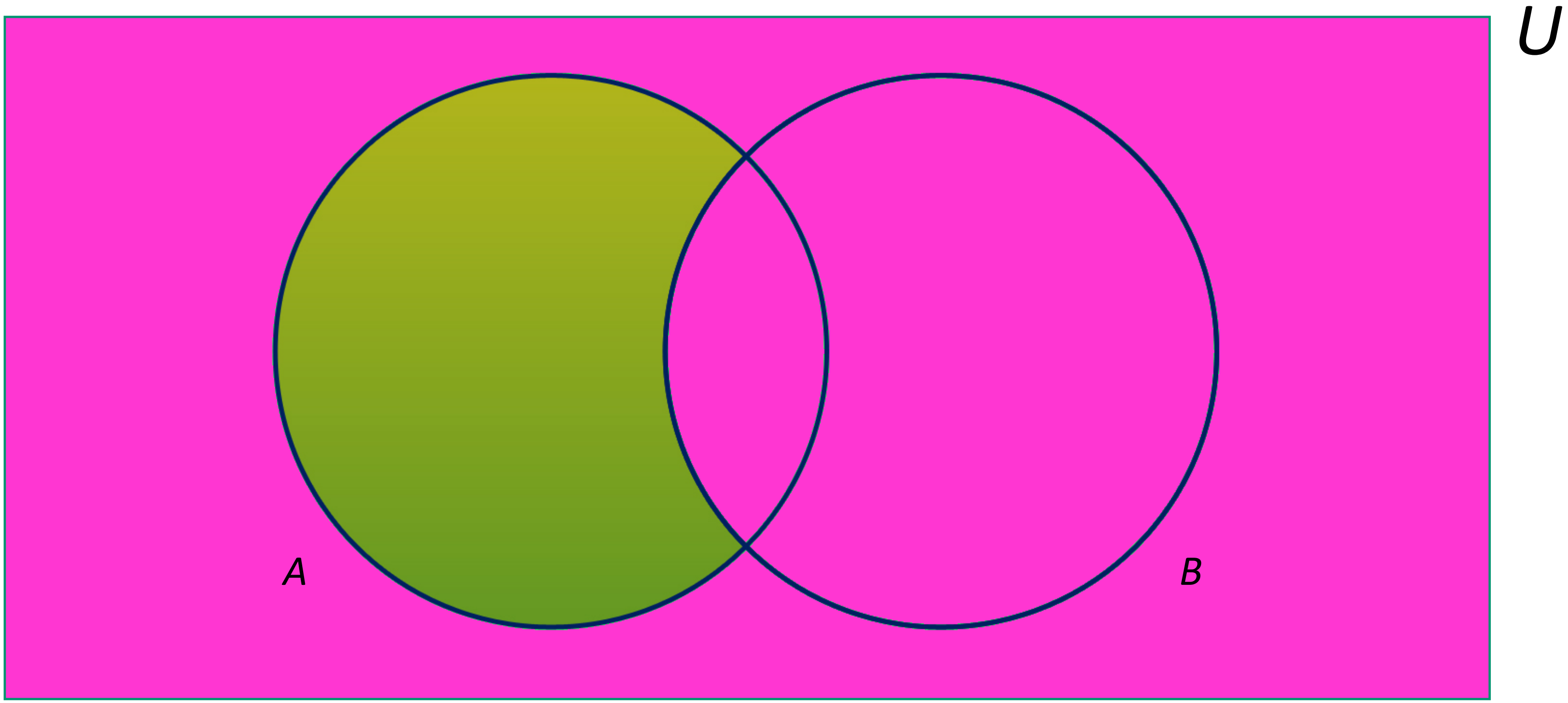


What is the shaded region?



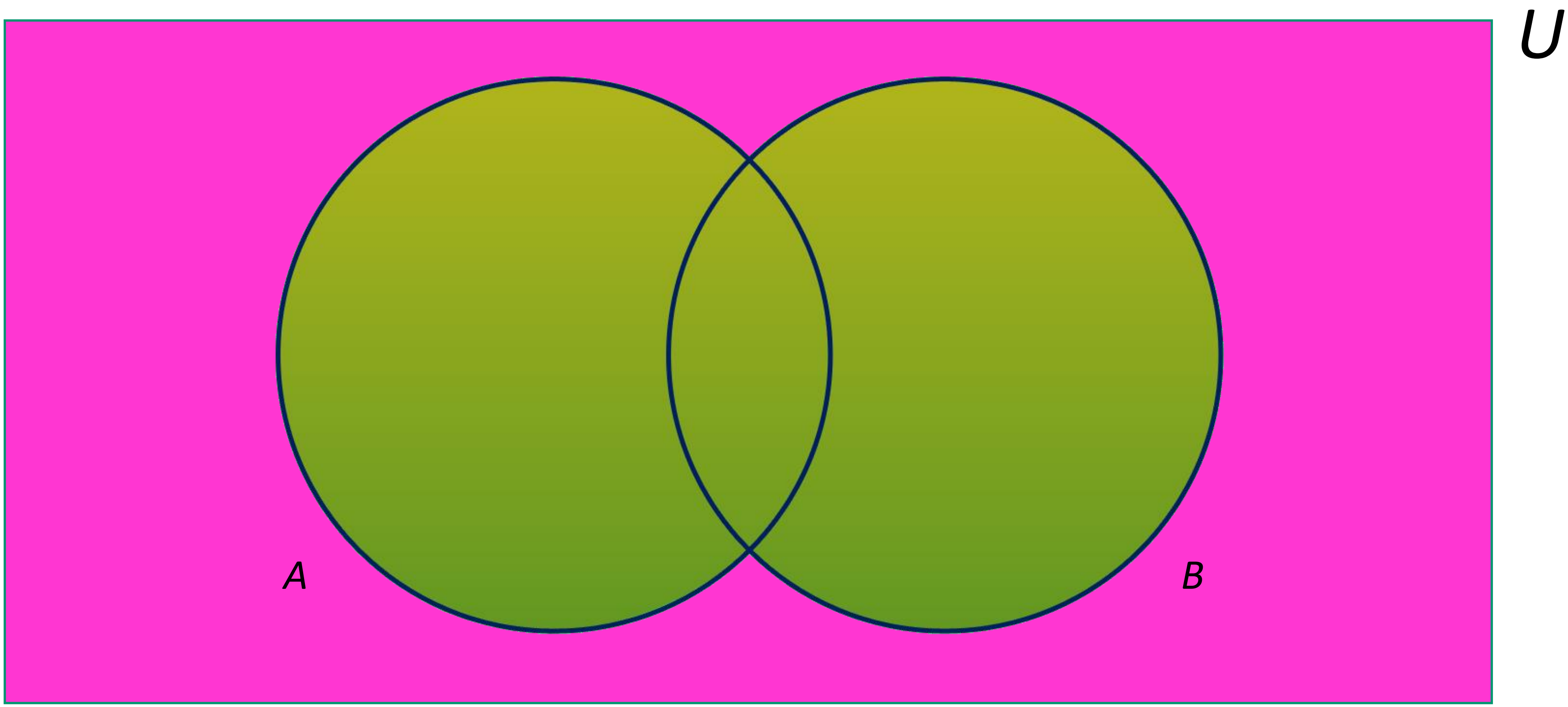
$(B \cap A')$

What is the shaded region?



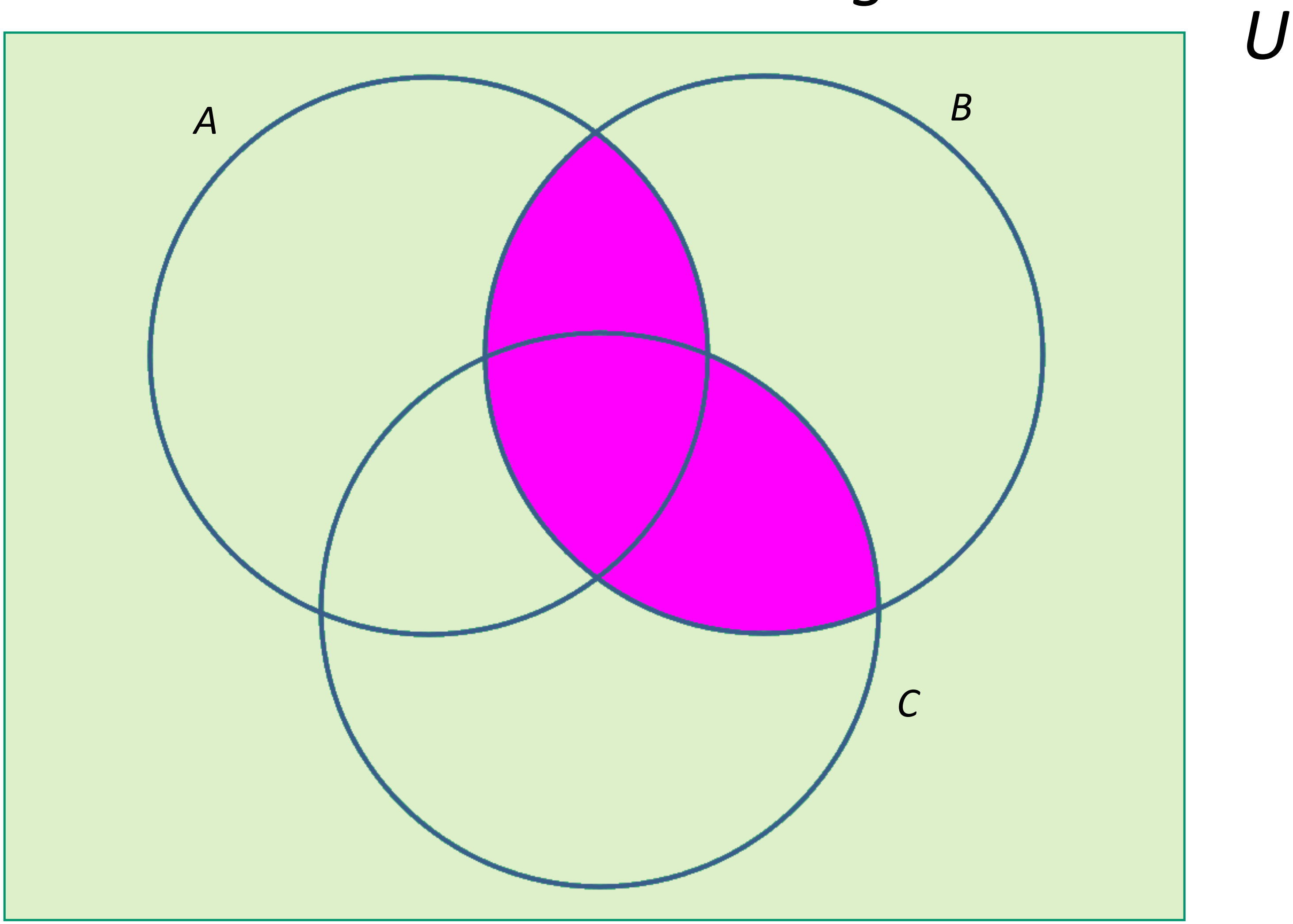
$(A' \cup B)$

What is the shaded region?



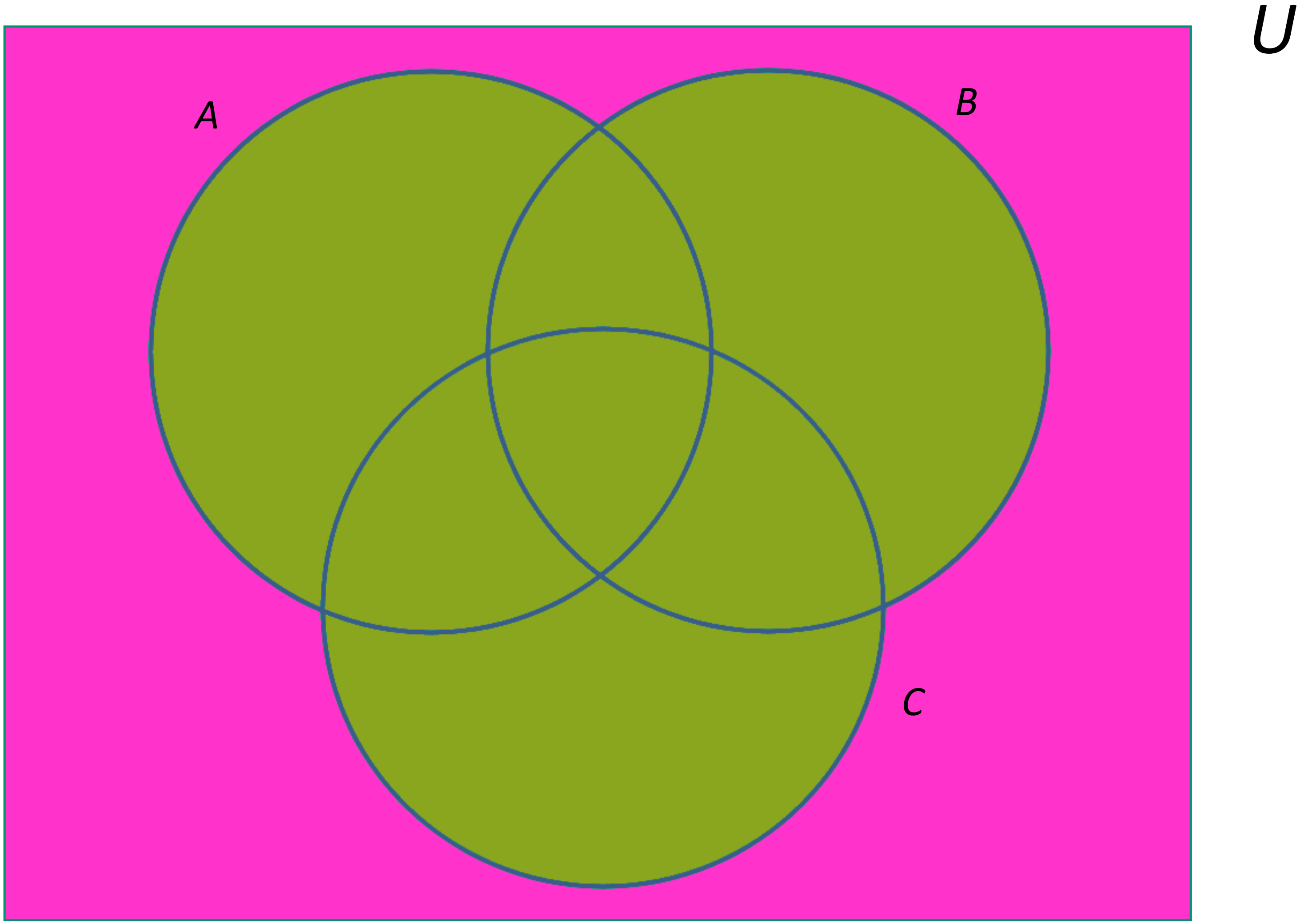
$(A \cup B)'$

What is the shaded region?



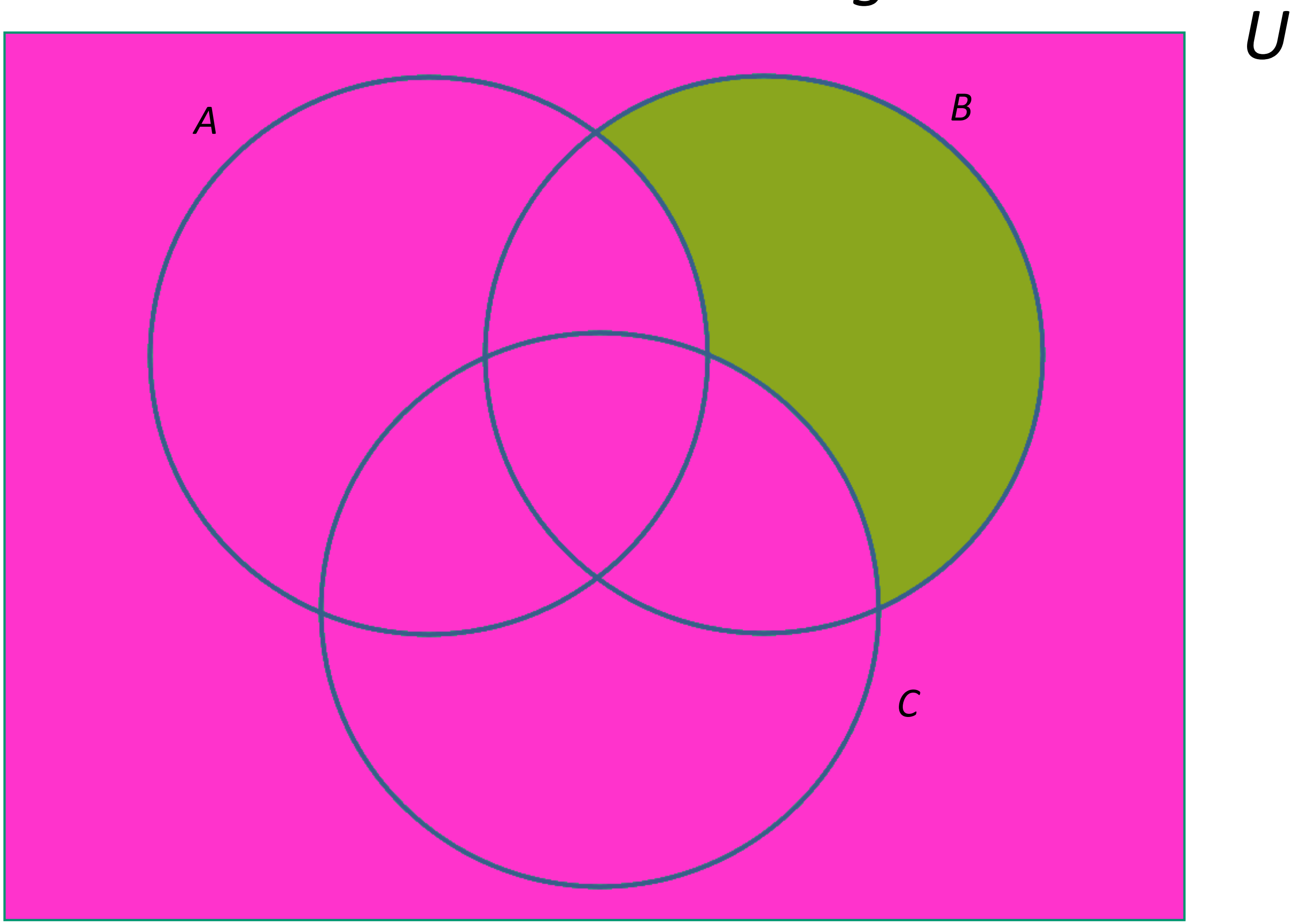
$B \cap (A \cup C)$

What is the shaded region?



$(A \cup B \cup C)'$

What is the shaded region?



$(A \cup C \cup B')$