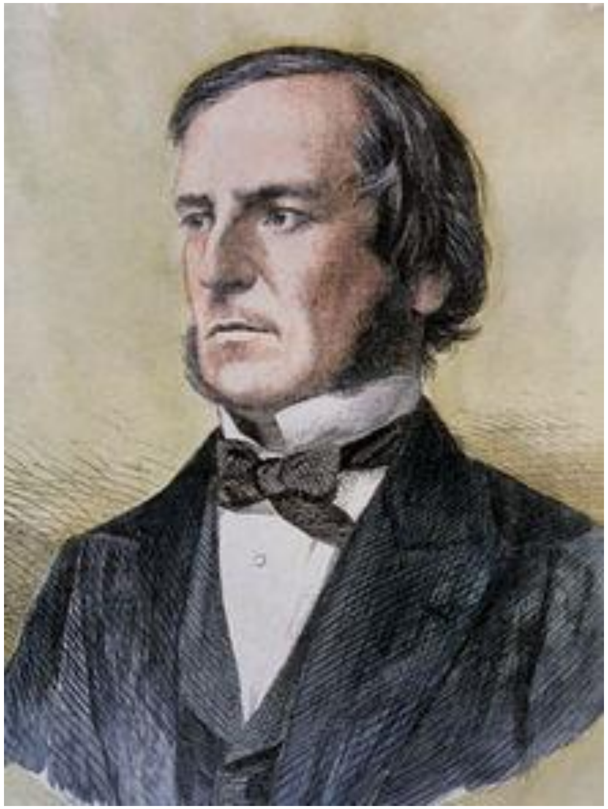
COM1002: Foundations of mathematics - Recap

George Boole 2 November 1815 – 8 December 1864)



George Boole's 200th Birthday

WEEK	4	5	6	7
Mon			Lecture	Lecture **Handout Ex5** (Assessed 5%)
Wed		Lecture	**Revision Lecture**	Lecture
Thurs		Tut (ex 4)	Tutorial **QUIZ 1 (25%)** Computer Room1, Diamond 4pm-6:00pm	Tut (ex 5)

COM1002 – Instructions for MOLE quiz 1.

- **I will summarize now**

Quiz Overview

The quiz consists of eight questions, and you are expected to attempt all of them. They are all presented to you at once, so that you can decide on the order in which you wish to attempt them.

Each question is a multiple-choice one, where you are presented with a problem that needs to be solved, and asked to select the correct solution from amongst the ones presented as possible answers.

Examples of the kinds of problems that are included are:

- identifying the location of an error in a truth table;
- identifying the correct Venn diagram for a set model of some system, or vice-versa;
- identifying the correct set model or propositional logic model for some set of statements;
- identifying the laws used in a proof in propositional logic.

Open Book Examination

The quiz is slightly unusual for an invigilated university examination, in that it is an "*open book*" examination. This means that you are allowed to bring to the quiz your copies of the textbook, your lecture notes, worked exercises and similar written material.

Similarly, during the course of the quiz you are permitted to refer to the copies of this material that are available on MOLE – although, of course, the more time you spend referring to this material, the less time you will have available for solving the problems that are presented in the quiz questions.

A consequence of this "open book" format is that the normal examination rules about bringing personal belongings into the laboratory where the quiz takes place will not need to apply, and in this respect the quiz will operate like a conventional laboratory session.

Accessing the QUIZ

You will be given precise instructions for accessing the quiz at the start of it, but in essence it will become available on the home page of the MOLE course for COM1002 at 4:10pm on Thursday 5th November until then it will not be visible.

Once you start it, you will then have 50 minutes in which to complete it.

If you are entitled to extra time, please contact me ASAP

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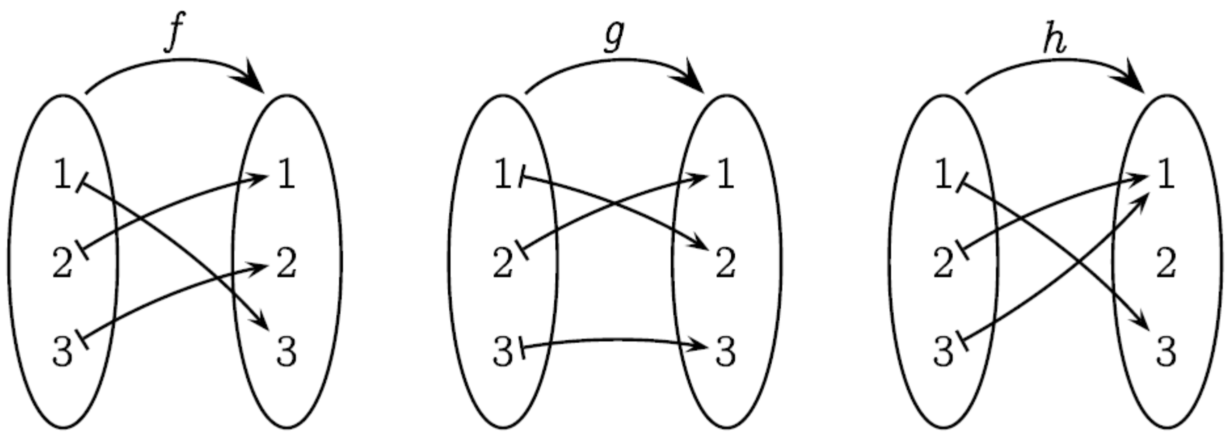
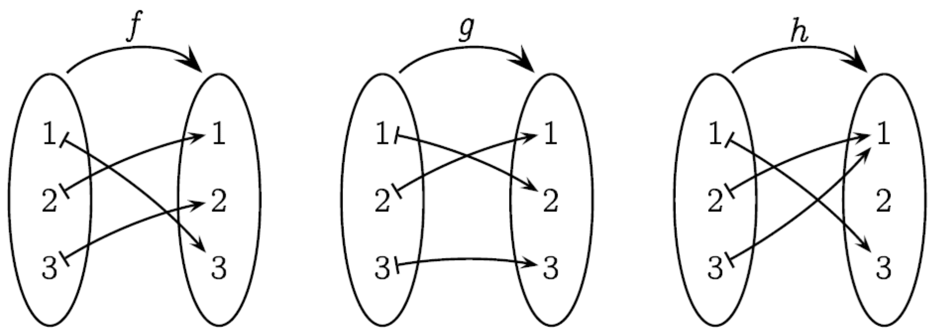
6. Consider the following three functions f, g and h from $\{1, 2, 3\}$ to itself:



Find $f \circ g \circ h$

7. Find $g \circ f$ and $f \circ g$, where $f(x) = x^2+1$ and $g(x) = x-2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

6. Consider the following three functions f, g and h from $\{1, 2, 3\}$ to itself:



$$f \circ g \circ h = f(g(h(x)))$$

$$f(g(h(1))) = f(g(3)) = f(3) = 2$$

$$f(g(h(2))) = f(g(1)) = f(2) = 1$$

$$f(g(h(3))) = f(g(1)) = f(2) = 1$$

$$f \circ g \circ h : \{1,2,3\} \rightarrow \{2,1,1\}$$

Example: Solve with Set Theory and Solve with Propositional Logic

(1) No one, who is going to a party, ever fails to brush their hair;
(2) No one looks fascinating, if they are untidy;
(3) Opium-eaters have no self-control;
(4) Every one, who has brushed their hair, looks fascinating;
(5) No one wears deely boppers, unless they are going to a party;
(6) A person is always untidy, if they have no self-control.

Univ. "persons"; P = going to a party; B = having brushed one's hair; C = having self-control; F = looking fascinating; O = Opium Eaters; T = tidy; D = wearing deely boppers.

p	$\neg$ p
F	T
T	F

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \Rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Exercise 1.23

Construct truth tables for the following formulae:

$$\neg\left(P \Leftrightarrow \neg Q\right)$$

$$\left(P \wedge Q\right) \vee \left(\neg P \wedge \neg Q\right)$$

$$\left(P \wedge Q\right) \Rightarrow \left(\neg R \wedge \neg Q\right)$$

Equivalent Propositions

If p and q are logically equivalent propositional statements, then:

- their truth tables must be the same, and
- the proposition $p \Leftrightarrow q$ must be true;

A proposition that must be true is **valid** and is called a **tautology**:

A proposition that must be false is called a **contradiction**.

Commutativity:	$p \vee q \Leftrightarrow q \vee p$ and $p \wedge q \Leftrightarrow q \wedge p$;
Associativity:	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$;
Idempotence:	$p \vee p \Leftrightarrow p$ and $p \wedge p \Leftrightarrow p$
Distributivity:	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$;
De Morgan's Laws:	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$;
Double Negation Law:	$\neg(\neg p) \Leftrightarrow p$;
Implication Law:	$p \Rightarrow q \Leftrightarrow \neg p \vee q$;
Contrapositive Law:	$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$;
Equivalence Law:	$p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$.
tautology Laws:	$p \vee \mathrm{true} \Leftrightarrow \mathrm{true}$ and $p \wedge \mathrm{true} \Leftrightarrow p$;
Contradiction Laws:	$p \vee \mathrm{false} \Leftrightarrow p$ and $p \wedge \mathrm{false} \Leftrightarrow \mathrm{false}$;
Excluded Middle Laws:	$p \vee \neg p \Leftrightarrow \mathrm{true}$ and $p \wedge \neg p \Leftrightarrow \mathrm{false}$;
Absorption Laws:	$p \vee (p \wedge q) \Leftrightarrow p$ and $p \wedge (p \vee q) \Leftrightarrow p$;

What laws are being used in the following proof

1. $p \wedge (\neg p \vee q) \Leftrightarrow p \wedge (\neg p \vee q)$

(distributivity) $\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$

(excluded middle) $\Leftrightarrow \text{false} \vee (p \wedge q)$

(commutativity laws) $\Leftrightarrow (p \wedge q) \vee \text{false}$

(Contradiction Laws) $\Leftrightarrow (p \wedge q)$

What laws are being used in the following proof

$\neg(p \Rightarrow q)$	$\Leftrightarrow$	$\neg(p \Rightarrow q)$
(implication laws)	$\Leftrightarrow$	$\neg(\neg p \vee q)$
(De Morgan laws)	$\Leftrightarrow$	$\neg\neg p \wedge \neg q$
(double negation)	$\Leftrightarrow$	$p \wedge \neg q$

What laws are being used in the following proof

$$p \Rightarrow (q \wedge r) \quad \Leftrightarrow \quad p \Rightarrow (q \wedge r)$$

(implication) $\Leftrightarrow \quad \neg p \vee (q \wedge r)$

(distributivity) $\Leftrightarrow \quad (\neg p \vee q) \wedge (\neg p \vee r)$

(implication) $\Leftrightarrow \quad (p \Rightarrow q) \wedge (p \Rightarrow r)$

What laws are being used in the following proof

$$
\begin{aligned}
(p \vee q) \Rightarrow r \quad & \Leftrightarrow \quad (p \vee q) \Rightarrow r \\
\text{(implication)} \quad & \Leftrightarrow \quad \neg(p \vee q) \vee r \\
\text{(De Morgan)} \quad & \Leftrightarrow \quad (\neg p \wedge \neg q) \vee r \\
\text{(Distributivity)} \quad & \Leftrightarrow \quad (\neg p \wedge r) \vee (\neg q \wedge r) \\
\text{(Implication)} \quad & \Leftrightarrow \quad (p \Rightarrow r) \wedge (q \Rightarrow r)
\end{aligned}
$$

Algebraic Laws for Sets

Commutativity: $A \cup B = B \cup A$ and $A \cap B = B \cap A$

Associativity: $A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$;

Idempotence: $A \cup A = A$ and $A \cap A = A$;

Distributivity: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

De Morgan's Laws: $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Double Complement Law: $\overline{\overline{A}} = A$

Universe Laws: $A \cup \mathcal{U} = \mathcal{U}$ and $A \cap \mathcal{U} = A$;

Empty Set Laws: $A \cup \varnothing = A$ and $A \cap \varnothing = \varnothing$;

Complement Laws: $A \cup \overline{A} = \mathcal{U}$ and $A \cap \overline{A} = \varnothing$;

Absorption Laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$.

Exercise 2.28

State the algebraic laws being used in the following proof

$$
\begin{aligned}
A \cap (\overline{A} \cup B) &= (A \cap \overline{A}) \cup (A \cap B) && \text{Distributive law} \\
&= \varnothing \cup (A \cap B) && \text{Complement law} \\
&= (A \cap B) \cup \varnothing && \text{Commutative law} \\
&= (A \cap B) && \text{empty set law}
\end{aligned}
$$

Kangaroo Puzzle in Set Theory

- The only animals in this house are cats.
- Every animal that loves to gaze at the moon is suitable for a pet. When I detest an animal, I avoid it.
- No animals are carnivorous, unless they prowl at night.
- No cat fails to kill mice.
- No animal ever takes to me, except those that are in this house. Kangaroos are not suitable for pets.
- None but carnivora kill mice.
- I detest animals that do not take to me.
- Animals that prowl at night always love to gaze at the moon.

Kangaroo Puzzle in Set Theory

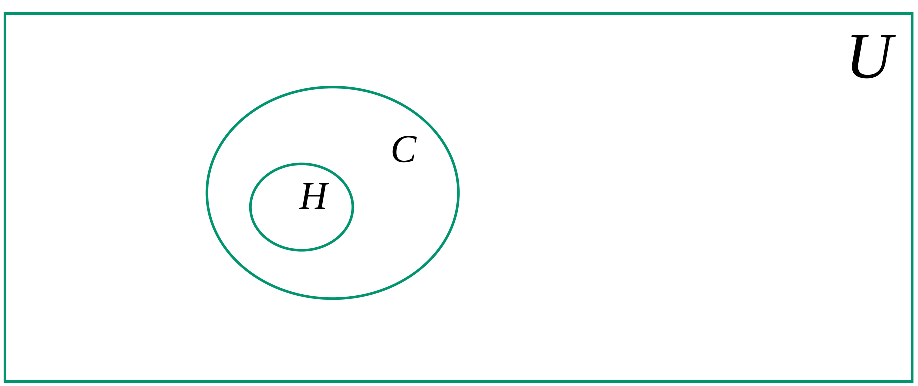
$$U = \{x : x \text{ is an animal}\}$$

The only animals in this house are cats.

If it is an animal in house **then** it is a cat.

$$H = \{x : x \text{ is in house}\}$$

$$C = \{x : x \text{ is a cat}\}$$



$$H \subseteq C$$

$$H \Rightarrow C$$

$$\overline{C} \subseteq \overline{H}$$

Kangaroo Puzzle in Set Theory

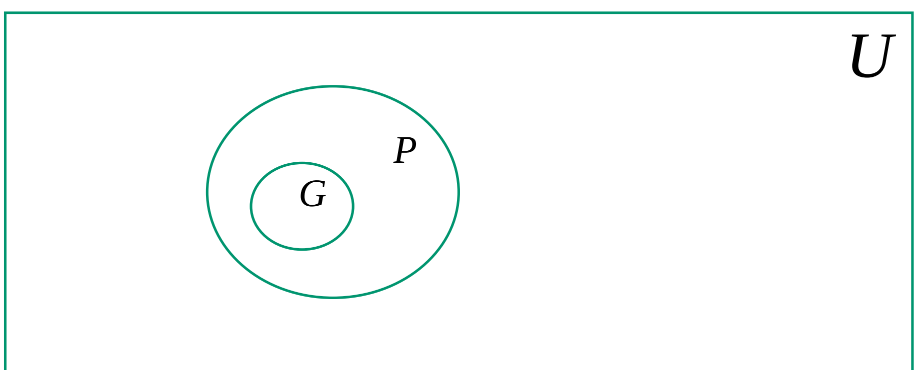
$$U = \{x : x \text{ is an animal}\}$$

2. Every animal that loves to gaze at the moon is suitable for a pet.

If it gazes at the moon **then** is suitable for a pet.

$$G = \{x : x \text{ likes to gaze at the moon }\}$$

$$P = \{x : x \text{ is suitable for a pet}\}$$



$$G \Rightarrow P$$

$$G \subseteq P \qquad \overline{P} \subseteq \overline{G}$$

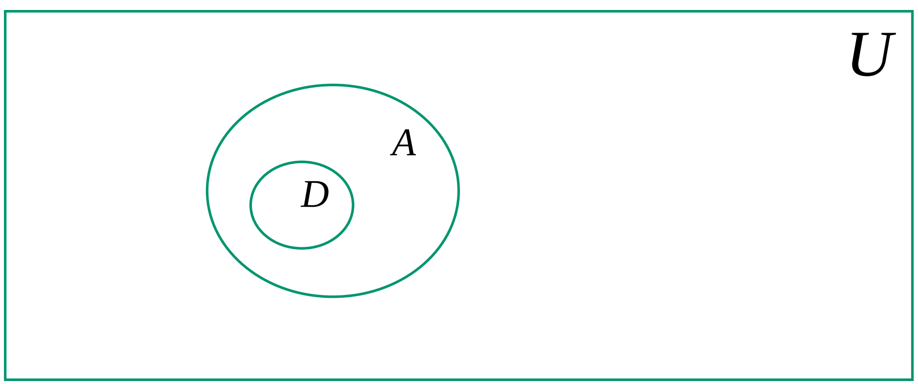
Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

3. when I detest an animal, I avoid it.

$$D = \{x : \text{I detest } x\}$$

$$A = \{x : \text{I avoid } x\}$$



$$D \Rightarrow A$$

$$D \subseteq A \qquad \overline{A} \subseteq \overline{D}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

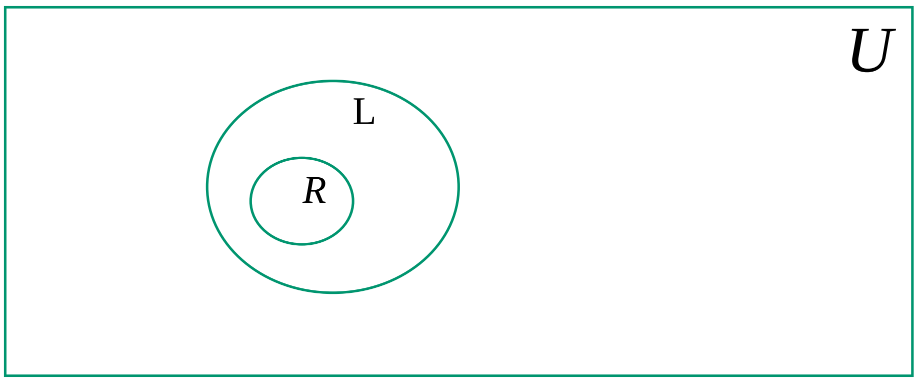
4. No animals are carnivorous, unless they prowl at night.

(EQUIVALENTLY:
If animal does not prowl at night **then** it is not carnivorous
If animal is carnivorous **then** it prowls at night

$$R = \{x : x \text{ is carnivorous }\}$$

$$L = \{x : x \text{ prowls at night }\}$$

$$R \Rightarrow L$$



$$R \subseteq L \qquad \overline{L} \subseteq \overline{R}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

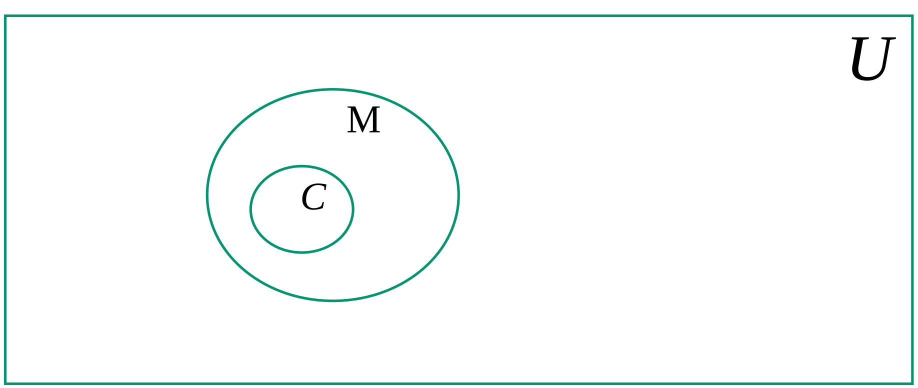
5. No cat fails to kill mice.

All cats kill mice.

If it is a cat **then** it kills mice

$$C = \{x : x \text{ is a cat }\}$$

$$M = \{x : x \text{ kills mice }\}$$



$$C \Rightarrow M$$

$$C \subseteq M$$

$$\overline{M} \subseteq \overline{C}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

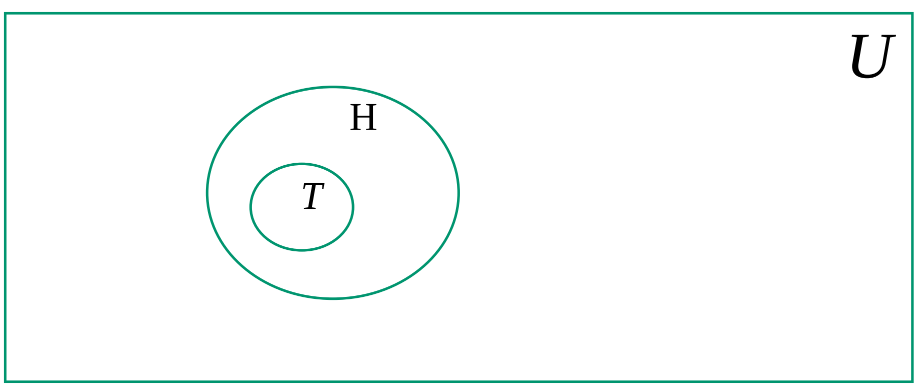
6. No animal ever takes to me, except those that are in this house.

animals that are not in this house do not take to me

If animal is **not** in this house **then** it does **not** take to me

If animal takes to me **then** it is in this house

$$T = \{x : x \text{ takes to me}\}$$
$$H = \{x : x \text{ is in this house}\}$$



$$T \Rightarrow H$$

$$T \subseteq H$$

$$\overline{H} \subseteq \overline{T}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

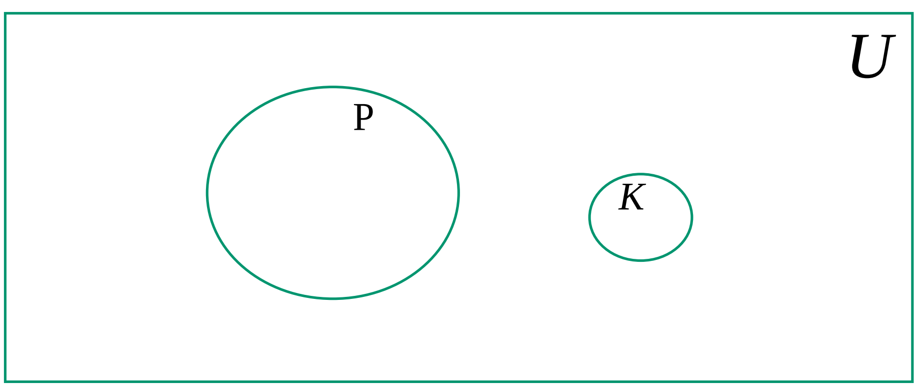
7. Kangaroos are not suitable for pets.

(**IF** it is a kangaroo **THEN** it is not suitable to be a pet)

$$K = \{x : x \text{ is a kangaroo }\}$$

$$P = \{x : x \text{ is suitable for a pet}\}$$

$$\overline{P} = \{x : x \text{ is not suitable for a pet}\} = \{x : x \notin P\} = U / P$$



$$K \subseteq \overline{P}$$

$$K \Rightarrow \neg P$$

$$P \subseteq \overline{K}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

8. None but carnivora kill mice.

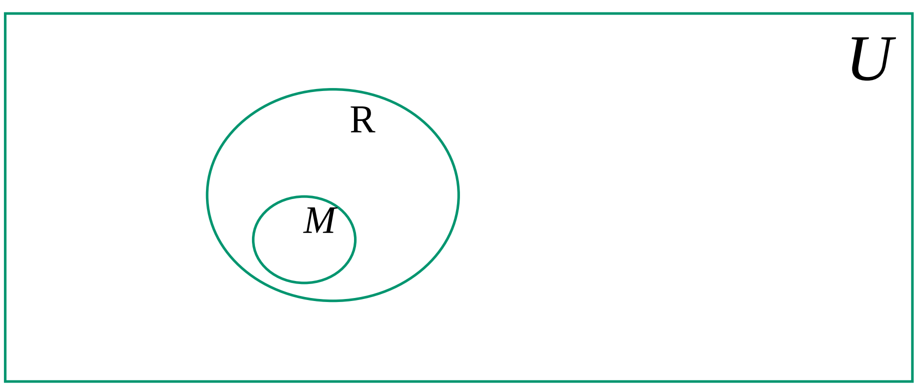
(**IF** it is not a carnivore **THEN** it does not kill mice)

(**IF** it kills mice **THEN** it is a carnivore)

$$R = \{x : x \text{ is carnivorous}\}$$

$$M = \{x : x \text{ kills mice}\}$$

$$M \Rightarrow R$$



$$M \subseteq R \qquad \overline{R} \subseteq \overline{M}$$

Kangaroo Puzzle in Set Theory

$$U = \{x : x \text{ is an animal}\}$$

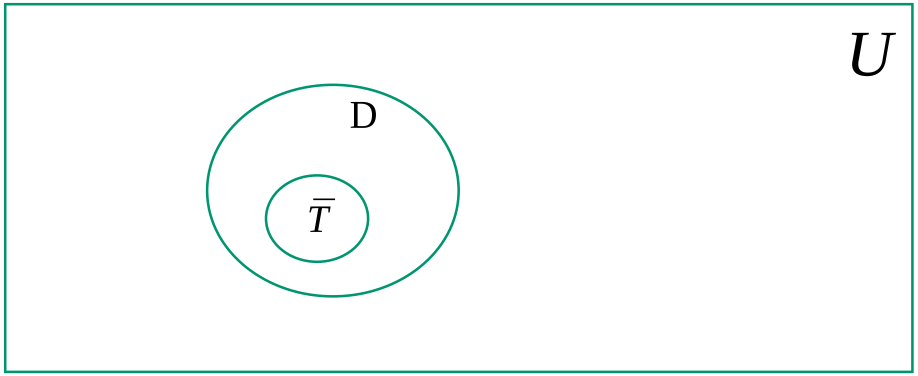
9. I detest animals that do not take to me.

(**IF** it does not take to me **THEN** I detest it)

$$T = \{x : x \text{ takes to me }\}$$

$$D = \{x : \text{I detest } x \}$$

$$\bar{T} \Rightarrow D$$



$$\bar{T} \subseteq D \qquad \bar{D} \subseteq T$$

Kangaroo Puzzle in Set Theory

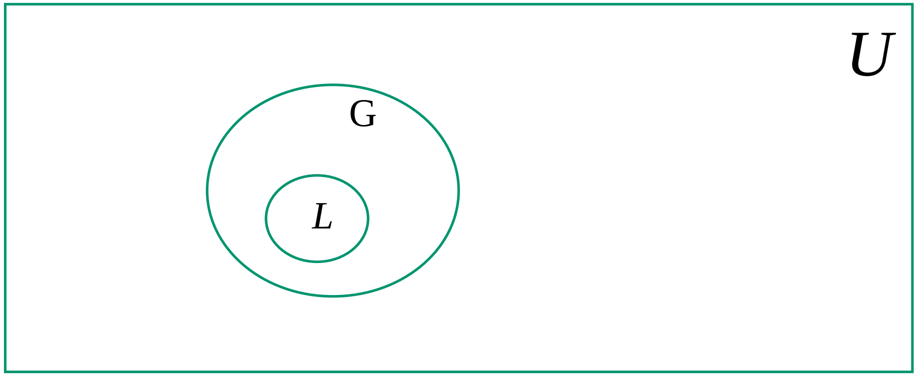
$$U = \{x : x \text{ is an animal}\}$$

10. Animals that prowl at night always love to gaze at the moon.

If it prowls at night **then** it loves to gaze at the moon.

$$L = \{x \in U : x \text{ prowls at night}\}$$

$$G = \{x \in U : x \text{ likes to gaze at the moon}\}$$



$$L \Rightarrow G$$

$$L \subseteq G \qquad \overline{G} \subseteq \overline{L}$$

Kangaroo Puzzle in Set Theory

Solution in Set Theory K

$$K \subseteq \overline{P} \subseteq \overline{G} \subseteq \overline{L} \subseteq \overline{R} \subseteq \overline{M} \subseteq \overline{C} \subseteq \overline{H} \subseteq \overline{T} \subseteq D \subseteq A$$

If it is a kangaroo **then** I avoid it

$K \subseteq \overline{P}$	$M \subset R$	$\overline{T} \subseteq D$	$L \subseteq G$	$T \subseteq H$
$P \subseteq \overline{K}$	$\overline{R} \subseteq \overline{M}$	$\overline{D} \subseteq T$	$\overline{G} \subseteq \overline{L}$	$\overline{H} \subseteq \overline{T}$
$H \subseteq C$	$G \subseteq P$	$D \subseteq A$	$R \subseteq L$	$C \subseteq M$
$\overline{C} \subseteq \overline{H}$	$\overline{P} \subseteq \overline{G}$	$\overline{A} \subseteq \overline{D}$	$\overline{L} \subseteq \overline{R}$	$\overline{M} \subseteq \overline{C}$

Consistency/ Inconsistency

A set of propositional statements is inconsistent if at least one of them must be **false**.

A set of propositional statements is consistent if it is possible for them all to be **true**.