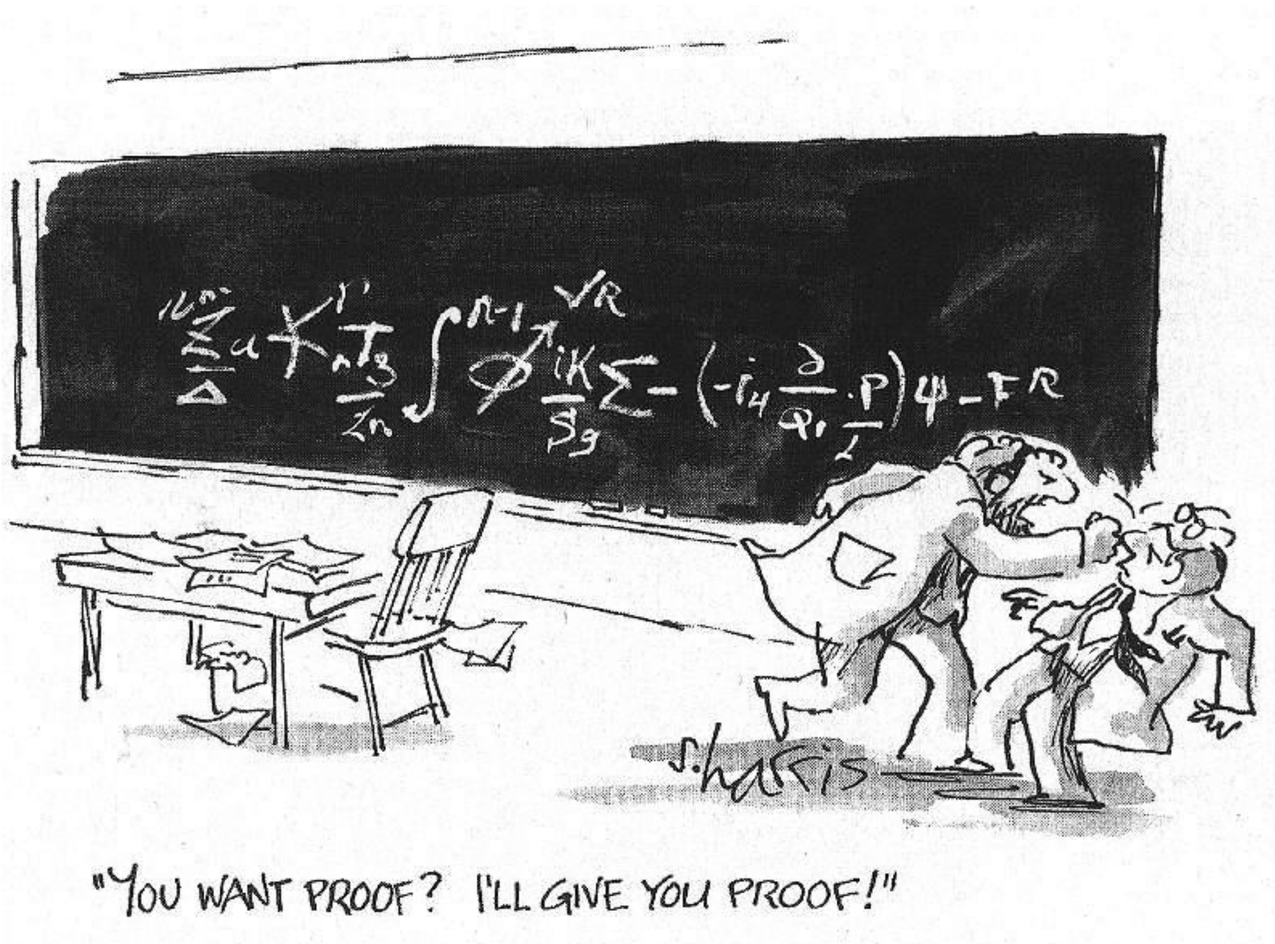
COM1002: Foundations of Computer Science



Proof Strategies

Paul Watton

WEEK	8	9	10	11
Mon	Lecture **Hand in Ex 5** **Handout Ex6** (Assessed 5%)	Lecture **Hand in Ex 6**	Lecture **Hand out ex 7** (Assessed 5%)	Lecture **Hand in Ex 7**
Wed	Lecture	**Revision Lecture**	Lecture	**Revision Lecture**
Thurs	Tut (ex 6)	**Revision Tutorial** **QUIZ 2 (25%)** Diamond 101 4pm-5:30pm	Tut (ex 7)	**Revision Tutorial** **QUIZ 3 (25%)** Diamond 101 4pm-5:30pm

We are nearly there…

Learning Objective 1: To understand which rule of inference is used in each argument below?

Modus Ponens ***Conjunction elimination***

Hypothetical syllogism ***Disjunctive syllogism***

Disjunction introduction ***Modus Tollens***

Alice is a waitress. Therefore, Alice is either a waitress or an astronaut.

Jerry is a waiter and an astronaut. Therefore, Jerry is a waiter.

If it is rainy, then I will not get out of bed. It is rainy. Therefore, I will not get out of bed.

If it snows today, I will miss my lectures. I did not miss my lectures today. Therefore, it did not snow today.

If it is Friday, I will drink too much beer. If I drink too much beer, then I will have a hangover the next morning. If it is Friday I will have a hangover the next morning.

I don’t get up or I go to my lecture. I didn’t go to my lecture. Therefore, I didn’t get up.

Learning Objective 2:

Show that the hypotheses:

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming, then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

lead to the conclusion:

We will be home by sunset.

Learning Objective 3:

In each of following cases, some **premises** (statements that are assumed to be true) are given, as well as a statement to prove. Use inference rules to prove the statement from the premises.

(i)

$$\begin{cases} \neg A \Rightarrow (C \wedge D) \\ A \Rightarrow B \\ \neg B \end{cases}$$

Prove C

(ii)

$$\begin{cases} (P \wedge Q) \\ P \Rightarrow \neg(Q \wedge R) \\ S \Rightarrow R \end{cases}$$

Prove $\neg$S

(iii)

$$\begin{cases} \neg(A \vee B) \Rightarrow C \\ \neg A \\ \neg C \end{cases}$$

Prove B

Proof strategies: Motivation

A method to evaluate predicates:

- and to do it systematically;

- for (parts of) specifications:

  to determine the conditions where they are true:

- and in software verification:

  to show that a software implementation matches its specification.

Types of Proof Strategy - 1

Common Features:

A proof starts with some **assumptions**:

- sometimes called premises, conditions, or pre-conditions;

A proof has a **goal**:

Strategies attempt to**:**

- produce intermediate assumptions which are closer to the goal or simplify the goal.

- *Introduction strategies* & *Elimination strategies*

Types of Proof Strategy - 2

Intermediate Assumptions:

- ❑ May involve introducing a logical element:

  e.g. assuming P, and **proving** Q from it

  - **introduces** $\Rightarrow$ between P and Q;
  - **Proves:** P $\Rightarrow$ Q

  e.g. proving Q(x) for an arbitrary x

  - **introduces** a universal quantification;

- ❑ These are called *introduction strategies*.

Types of Proof Strategy 3

Intermediate Assumptions:

May involve eliminating a logical element;
So as to reduce the "distance" to the goal:

These are called *elimination strategies*.

Types of Proof Strategy 4

Proofs Typically Involve:

- Applying *elimination strategies* to the assumptions: either the initial or intermediate ones;
- Applying *introduction strategies* to the goal:
  or to intermediate assumptions;
- So as to get the intermediate steps closer to each other.

Strategy to prove $P \Rightarrow Q$

Implication Introduction:

Strategy 1:

- **assume** P,
- use this assumption in a proof of Q,
- this proves $P \Rightarrow Q$ (ie it **introduces** $\Rightarrow$);

Strategy 2:

- **assume** $\neg Q$
- use this assumption in a proof of $\neg P$;
- this proves $\neg Q \Rightarrow \neg P$ and thus proves $P \Rightarrow Q$

Modus Ponens and Modus Tollens

Elimination strategies for implication:

If $P \Rightarrow Q$ and P is **true**, then Q is **true**:

– known as **modus ponens**,

If $P \Rightarrow Q$ and $\neg Q$, then $\neg P$ follows:

– known as **modus tollens.**

Strategies for Connectives 3

Negation:

Introduction strategy:

- assume P,
- from this, derive a contradiction, i.e. $Q \wedge \neg Q$
- this proves $\neg$ P (ie it introduces $\neg$);

Elimination strategy:

- assume $\neg$ P, and derive a contradiction, $Q \wedge \neg Q$
- which proves P (ie it eliminates $\neg$).

Strategies for Connectives 4

Conjunction:

Introduction strategy:

- prove P,
- prove Q,
- which proves $P \wedge Q$ (ie it introduces $\wedge$);

Elimination strategy:

- Given $P \wedge Q$ is true
- then P and Q are true (can eliminate $\wedge$).

Strategies for Connectives 5

Equivalence:

Introduction strategy:

- ❖ prove $P \Rightarrow Q$,
- ❖ prove $Q \Rightarrow P$,
- ❖ which proves $P \Leftrightarrow Q$ (ie it introduces $\Leftrightarrow$);

❑ *Elimination strategy*:

- ❖ Given $P \Leftrightarrow Q$,
- ❖ we can infer both $P \Rightarrow Q$ and $Q \Rightarrow P$ (eliminates $\Leftrightarrow$).

Strategies for Connectives 6

Disjunction:

- *Introduction strategy*:
  - prove one of P or Q,
  - which proves $P \vee Q$ (ie it introduces $\vee$);
- Often used with case analysis:
  - for the two cases of some property R,
  - prove P from R, and Q from $\neg R$,
  - which proves $P \vee Q$.

Strategies for Connectives 7

Universal Quantification:

Treat as a (potentially infinite) conjunction;

Universal Introduction strategy:

- ❖ $P(a)$ for any arbitrary a,
- ❖ therefore $\forall x\ P(x)$ (ie it introduces $\forall$);

Universal Elimination strategy:

- ❖ $\forall x\ P(x)$,
- ❖ therefore $P(a)$ if a is in the universe for P.

Strategies for Connectives 8

Existential Quantification:

Treat as a (potentially infinite) disjunction;

Existential Introduction strategy:

- ❖ $P(a)$ for some element a,
- ❖ therefore $\exists x\ P(x)$ (ie it introduces $\exists$);

Existential Elimination strategy:

- ❖ $\exists x\ P(x)$,
- ❖ therefore $P(a)$ for some element a

Strategies for Connectives 9

Uniqueness:

- Either the *separate aspect approach*:
  - using the appropriate strategies,
  - prove existence, ie $\exists x\ P(x)$, and
  - then prove uniqueness, ie $\forall y \forall z\ ((P(y) \wedge P(z)) \Rightarrow y = z)$;
- Or the *combined approach*:
  - prove $\exists x\ (P(x) \wedge \forall y\ (P(y) \Rightarrow y = x))$,
  - using the appropriate strategies.

$p \Rightarrow (p \vee q)$	*Disjunction introduction*
$((p \Rightarrow q) \wedge (r \Rightarrow q) \wedge (p \vee r)) \Rightarrow q$	*Disjunction elimination*
$(p \wedge q) \Rightarrow p$	*Conjunction elimination*
$(p \wedge q) \Rightarrow q$	*Conjunction elimination*
$(p) \wedge (q) \Rightarrow (p \wedge q)$	*Conjunction introduction*
$((p \Rightarrow q) \wedge p) \Rightarrow q$	*Modus Ponens*
$((p \Rightarrow q) \wedge \neg q) \Rightarrow \neg p$	*Modus Tollens*
$(\neg p \wedge (p \vee q)) \Rightarrow q$	*Disjunctive syllogism*
$((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$	*Hypothetical syllogism*

Which rule of inference is used in each argument below?

Alice is a waitress. Therefore, Alice is either a waitress or an astronaut.

Disjunction introduction

Jerry is a waiter and an astronaut. Therefore, Jerry is a waiter.

Conjunction elimination

If it is rainy, then I will not get out of bed. It is rainy. Therefore, I will not get out of bed.

Modus Ponens

If it snows today, I will miss my lectures. I did not miss my lectures today. Therefore, it did not snow today.

Modus Tollens

If it is Friday, I will drink too much beer. If I drink too much beer, then I will have a hangover the next morning. If it is Friday I will have a hangover the next morning.

Hypothetical syllogism

I don’t get up or I go to my lecture. I didn’t go to my lecture. Therefore, I didn’t get up.

Disjunctive syllogism

$$DG = \text{Don't get up}$$

$$L = \text{Go to Lecture}$$

I don’t get up or I go to my lecture.

$$\left(DG \vee L\right)$$

Disjunctive syllogism

$$\left(\neg p \wedge \left(p \vee q\right)\right) \Rightarrow q$$

$$\left(\left(p \vee q\right) \wedge \neg q\right) \Rightarrow p$$

I didn’t go to my lecture.

$$\neg L$$

Therefore, I didn’t get up.

$$\left(\left(DG \vee L\right) \wedge \neg L\right) \Rightarrow DG$$

EXAMPLE: In each of following cases, some **premises** (statements that are assumed to be true) are given, as well as a statement to prove. Use inference rules to prove the statement from the premises.

(i)
$$\begin{cases} \neg A \Rightarrow (C \wedge D) \\ A \Rightarrow B \\ \neg B \end{cases}$$

Prove C

(ii)
$$\begin{cases} (P \wedge Q) \\ P \Rightarrow \neg(Q \wedge R) \\ S \Rightarrow R \end{cases}$$

Prove $\neg S$

(iii)
$$\begin{cases} \neg(A \vee B) \Rightarrow C \\ \neg A \\ \neg C \end{cases}$$

Prove B

(i)

$$\begin{cases} \neg A \Rightarrow (C \wedge D) \\ A \Rightarrow B \\ \neg B \end{cases}$$

Prove C

1.	$A \to B$	Premise
2.	$\sim B$	Premise
3.	$\sim A$	Modus tollens (1,2)
4.	$\sim A \to (C \wedge D)$	Premise
5.	$C \wedge D$	Modus ponens (3,4)
6.	C	Decomposing a conjunction (5)

(ii) $$\begin{cases} (P \wedge Q) \\ P \Rightarrow \neg(Q \wedge R) \\ S \Rightarrow R \end{cases}$$

Prove $\neg S$

1.	$P \wedge Q$	Premise
2.	P	Conjunction elimination (1)
3.	Q	Conjunction elimination (1)
4.	$P \Rightarrow \neg(Q \wedge R)$	Premise
5.	$\neg(Q \wedge R)$	Modus Ponens (3,4)
6.	$\neg Q \vee \neg R$	DeMorgan (5)
7.	$\neg R$	Disjunctive syllogism (3,6)
8.	$S \Rightarrow R$	Premise
9.	$\neg S$	Modus Tollens (7,8)

(iii)

$$\begin{cases} \neg(A \vee B) \Rightarrow C \\ \neg A \\ \neg C \end{cases}$$

Prove B

1.	$\sim (A \vee B) \rightarrow C$	Premise
2.	$\sim C$	Premise
3.	$A \vee B$	Modus tollens (1,2)
4.	$\sim A$	Premise
5.	B	Disjunctive syllogism (3,4)

Negate each quantified statement, simplifying so that only the simple statements are negated. Show each step of your work.

(a) $\forall x(\sim P(x) \wedge \sim Q(x))$

(b) $\exists x(Q(x) \rightarrow \sim P(x))$