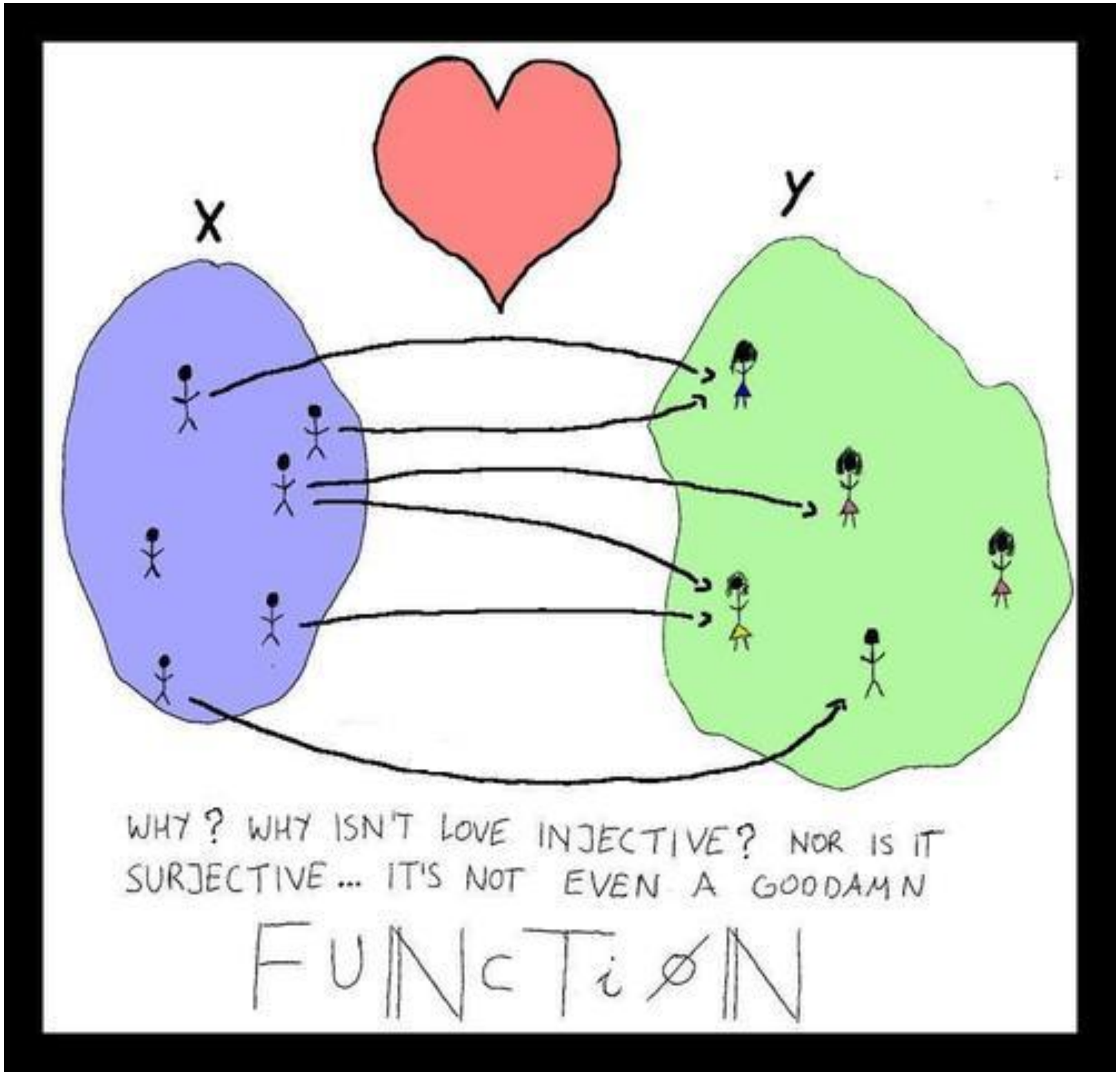


COM1002

Foundations of Computer Science

Lecture 10: Functions

28th Oct 2015

Paul Watton

WEEK	4	5	6	7
Mon			Lecture	Lecture **Handout Ex5** (Assessed 5%)
Wed		Lecture	**Revision Lecture**	Lecture
Thurs		Tut (ex 4)	Tutorial **QUIZ 1 (25%)** Computer Room1, Diamond 4pm-6:00pm	Tut (ex 5)

Teaching and Assessment

Assessment:

- ❑For this semester:
  - ❖five examples sheets are assessed
    (5% each)
  - ❖three online exam-type “quizzes” (25% each)

COM1002 – Instructions for MOLE quiz 1.

- **I will summarize now**

This MOLE quiz for COM1002 counts formally as an invigilated university examination, even though it is not taking place during the normal examination period (which is weeks 13 to 15 of a semester). It is worth 25% of the marks for this module. It will operate under the rules that apply to invigilated university examinations, which are available at <www.sheffield.ac.uk/ssid/exams>, and which you need to read before the quiz takes place.

Quiz Overview

The quiz consists of eight questions, and you are expected to attempt all of them. They are all presented to you at once, so that you can decide on the order in which you wish to attempt them.

Each question is a multiple-choice one, where you are presented with a problem that needs to be solved, and asked to select the correct solution from amongst the ones presented as possible answers.

Examples of the kinds of problems that are included are:

- identifying the location of an error in a truth table;
- identifying the correct Venn diagram for a set model of some system, or vice-versa;
- identifying the correct set model or propositional logic model for some set of statements;
- identifying the laws used in a proof in propositional logic.

Open Book Examination

The quiz is slightly unusual for an invigilated university examination, in that it is an "*open book*" examination. This means that you are allowed to bring to the quiz your copies of the textbook, your lecture notes, worked exercises and similar written material.

Similarly, during the course of the quiz you are permitted to refer to the copies of this material that are available on MOLE – although, of course, the more time you spend referring to this material, the less time you will have available for solving the problems that are presented in the quiz questions.

A consequence of this "open book" format is that the normal examination rules about bringing personal belongings into the laboratory where the quiz takes place will not need to apply, and in this respect the quiz will operate like a conventional laboratory session.

Accessing the QUIZ

You will be given precise instructions for accessing the quiz at the start of it, but in essence it will become available on the home page of the MOLE course for COM1002 at 4:10pm on Thursday 5th November until then it will not be visible.

Once you start it, you will then have 50 minutes in which to complete it.

If you are entitled to extra time, please contact me…

P.Watton@Sheffield.ac.uk

Examination Regulations

An important feature of the quiz is that, like any examination, your attempt at it must be your own, and so the normal examination regulations will definitely apply that during the quiz you must not communicate with others, or even attempt to do so.

During the quiz you are forbidden to use your machine to access **email, instant messaging, Facebook, or any other applications that might allow you to communicate with others**. This therefore means that while you are working on the quiz, the only application that you are allowed to have open on the machine where you are working is the browser that you are using to access MOLE, and possibly a text editor if you want to do rough working and would prefer to type this rather than hand-write it – although you are strongly advised to hand-write any rough working.

Examination stationery will be provided, as it would be for a normal closed-book examination, and you are required to use only this supplied stationery for any hand-written rough working. However, nothing that is written on this examination stationery will be considered in the process of marking the quiz, which is done automatically by MOLE.

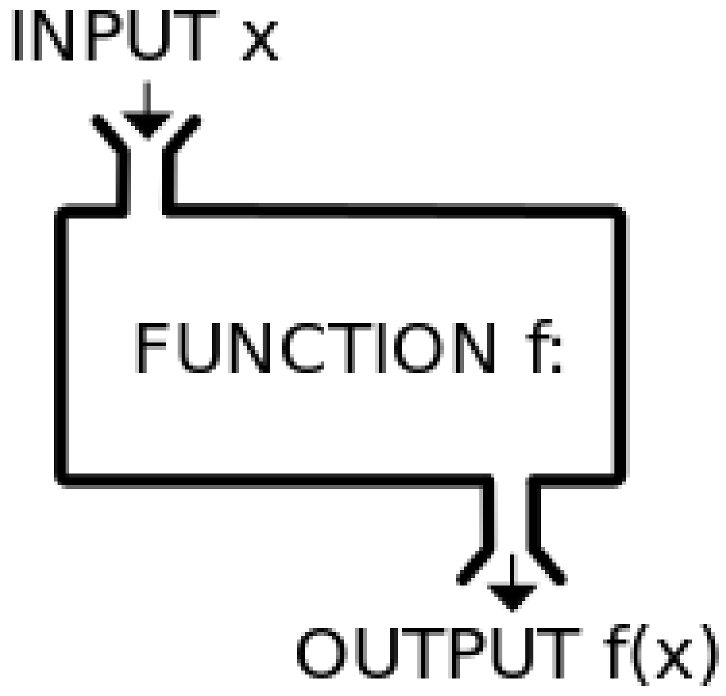
Examination Regulations

- For accessing the quiz via MOLE you will be required to use the machine in the laboratory.

- Because such quizzes have only recently been brought into the scope of the university's regulations for invigilated examinations, the issue of whether or not you ought to be allowed to use your own laptop (or similar device) in addition is one that needs to be discussed, since it has only recently been recognised as an issue.

- Until this discussion has taken place, therefore, the current university regulations have to apply, and these mean that for this quiz you are forbidden to use any other electronic devices.

- Hence**, if you have any such device (and in particular this includes a mobile phone) in your personal belongings, then you must ensure that it is completely switched off before you bring your belongings into the laboratory.**



A function *f* takes an input *x*, and returns a **single output** *f*(*x*). One metaphor describes the function as a "machine" or "black box" that for each input returns a corresponding output.

C6: Functions = Learning Objective

To be able to describe and discuss functions we need to learn relevant terminology...

domain **co-domain** **graph** **image**

Total function **Partial function** **bijection**

injection **surjection** **pre-image**

one-to-one **composite functions**

Recap: Classes of Functions

a function f is an injection, or one-to-one:

- if different inputs produce different results,
- ie if b1 = b2 $\Rightarrow$ a1 = a2.

a function **f is a surjection**, or onto:

- if all elements of the co-domain can be produced,
- i.e. if range (f) = B.

a function f is a bijection if:

- it is an **injection and a surjection**,

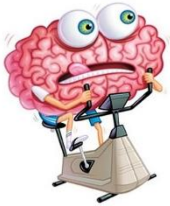
Classes of Functions 2

Formally: given $\boldsymbol{f}$**:**$\boldsymbol{A \rightarrow B}$

"f is injective" $:\equiv (\forall\ x,y\text{: } (x \neq y) \rightarrow (f(x) \neq f(y)))$
or, equivalently…
"f is injective" $:\equiv (\forall\ x,y\text{: } (f(x)=f(y)) \rightarrow (x = y))$

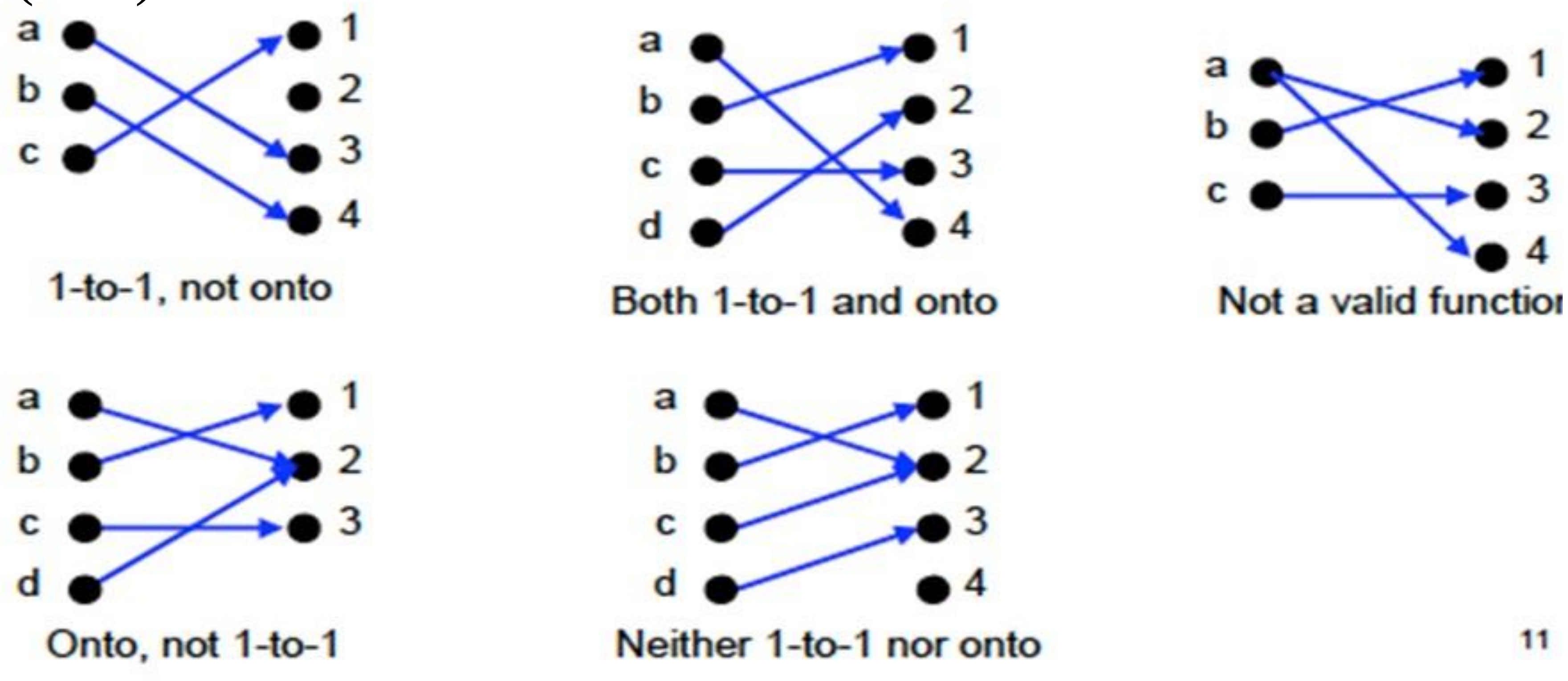
A function $f{:}A \rightarrow B$ is *onto* or *surjective* or *a surjection* iff its **range is equal to its codomain** ($\forall b \in B$, $\exists a \in A$: $f(a)=b$).

An *onto* function maps the set A onto (over, covering) the *entirety* of the set B, not just over a piece of it.



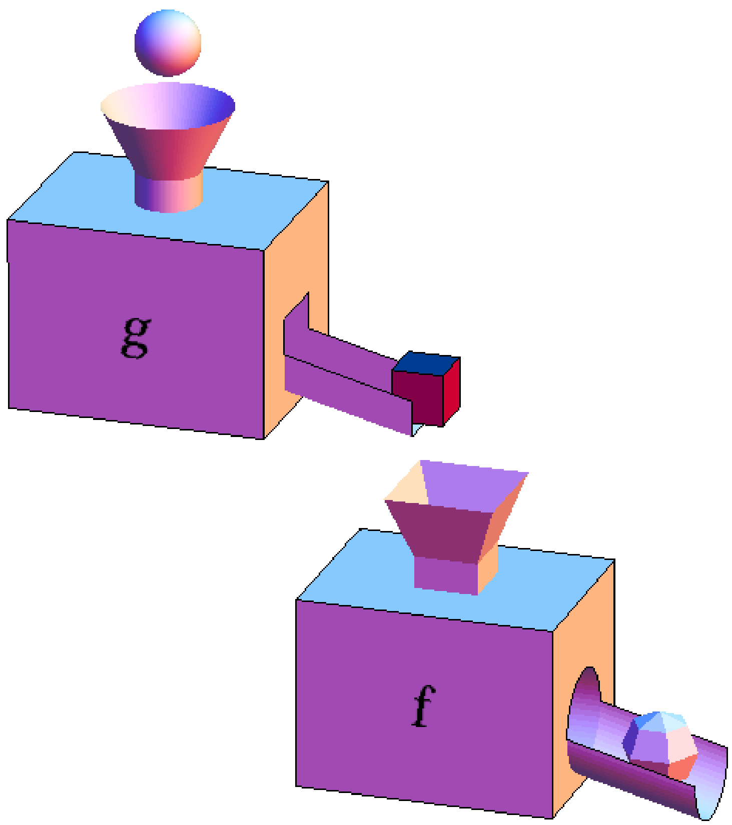
Warm-up

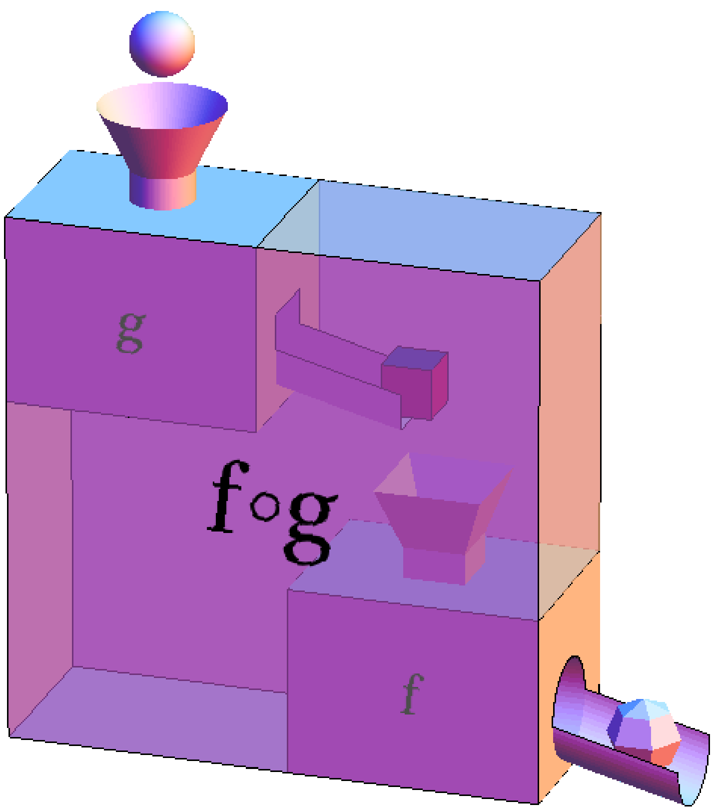
Are the following functions injective (one to one), surjective (onto) or neither ?



1-to-1 and onto function are called **bijective.**

Composite Functions



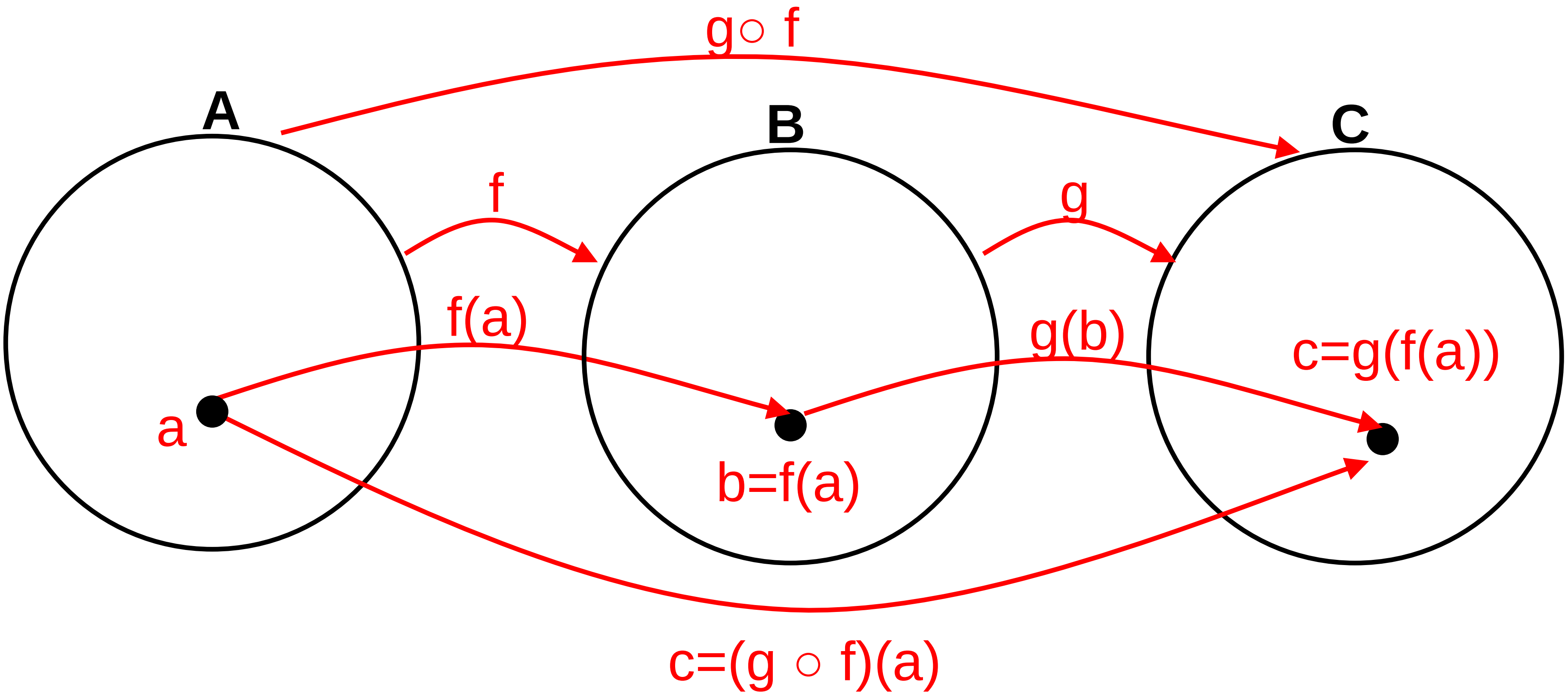


Composing Functions 1

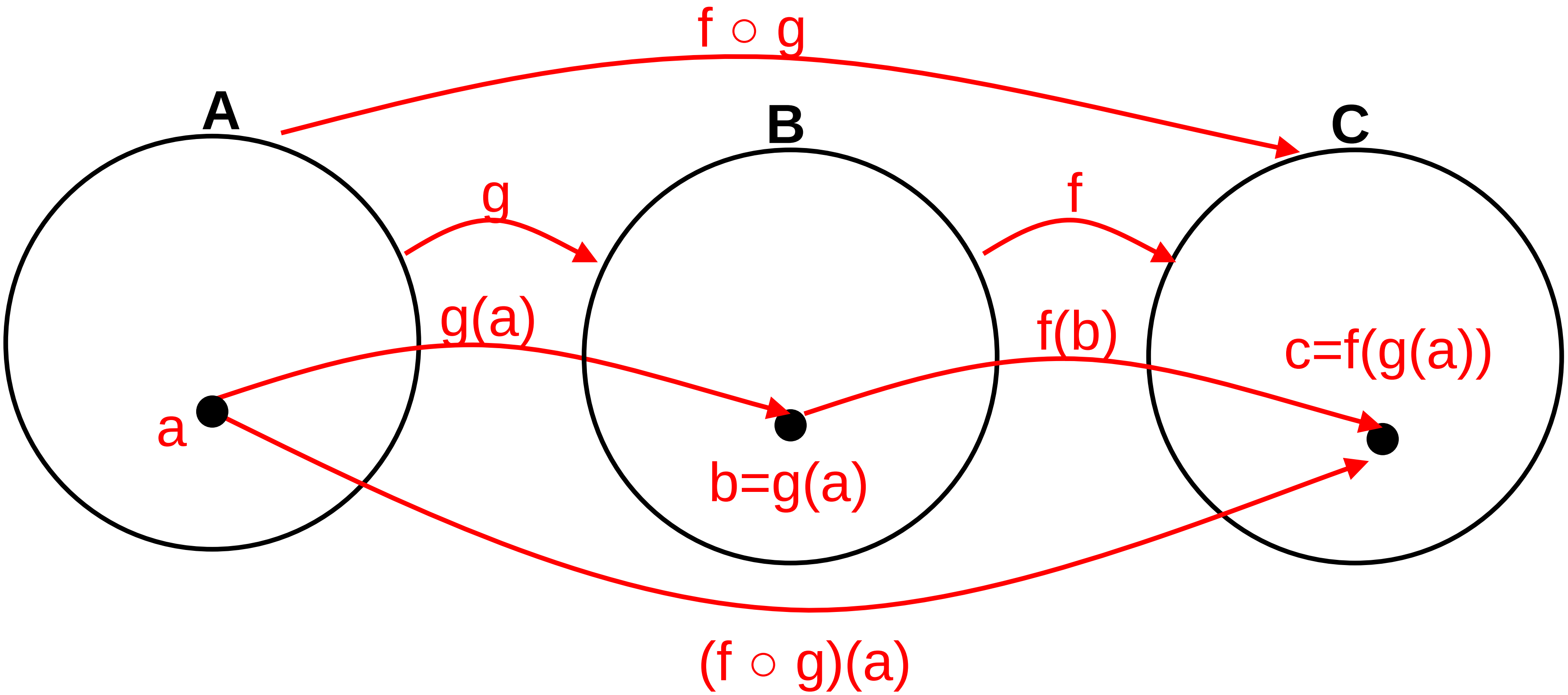
For two (or more) functions:

- Assume $f : A \to B$ and $g : B \to C$;
- For some argument $a \in A$ we can:
  - apply f to give $f(a) \in B$,
  - then apply g to give $g(f(a)) \in C$;
- This is equivalent to:
  - composing f and g to give $g \circ f : A \to C$,
  - and applying it, as $(g \circ f)(a) = g(f(a))$.

Compositions of functions: g○f

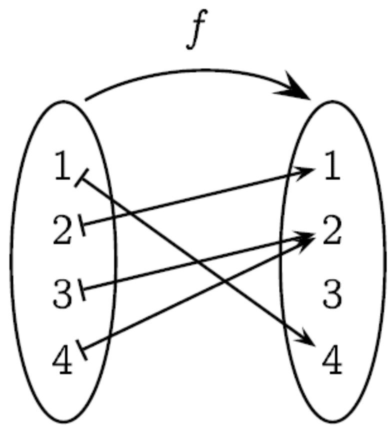


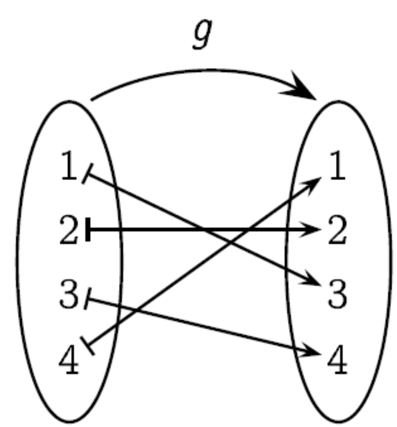
Compositions of functions: f○g



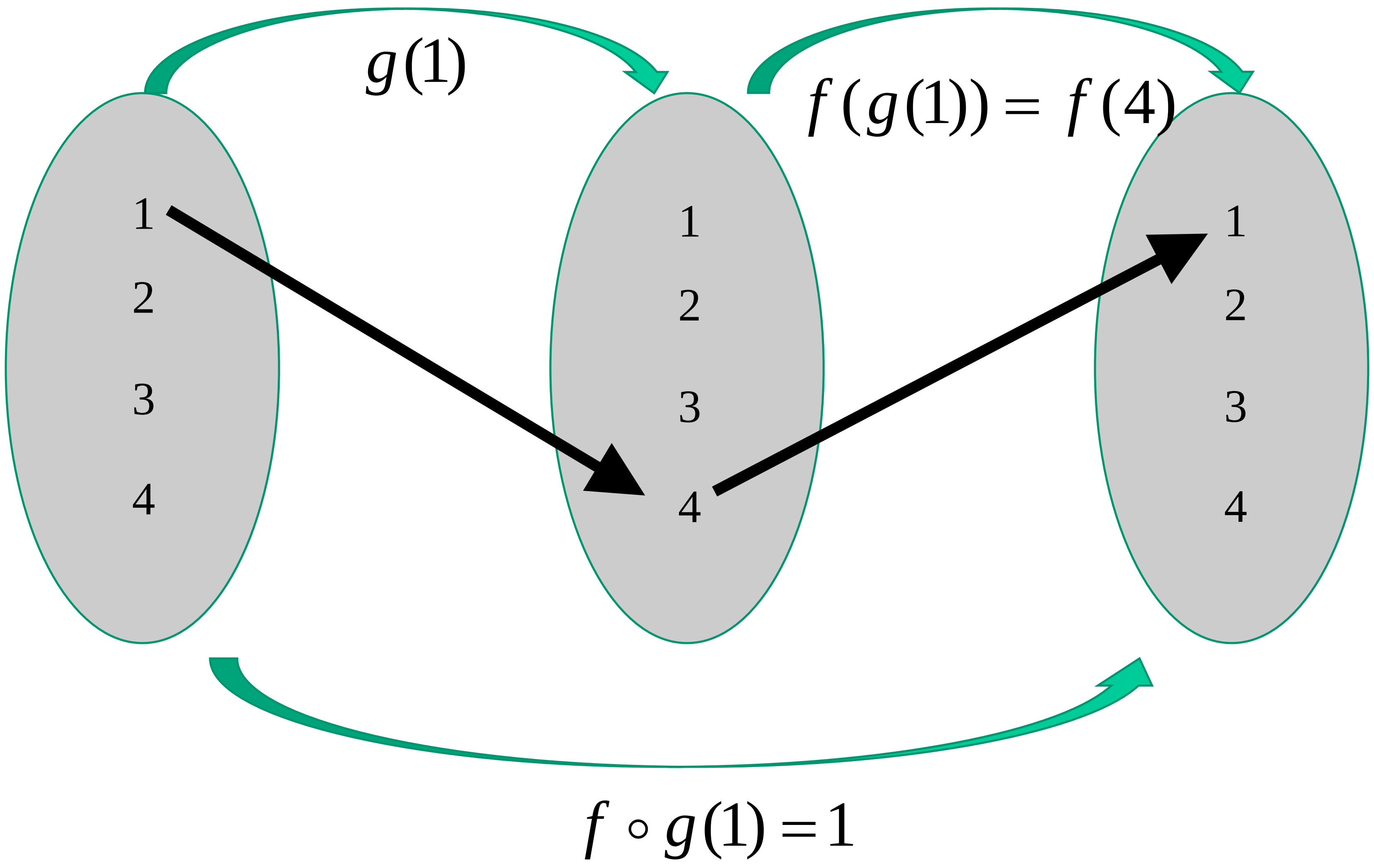
Exercise 6.9 (Solution on page 433)

Consider the following two functions f and g from $\{1, 2, 3, 4\}$ to itself:

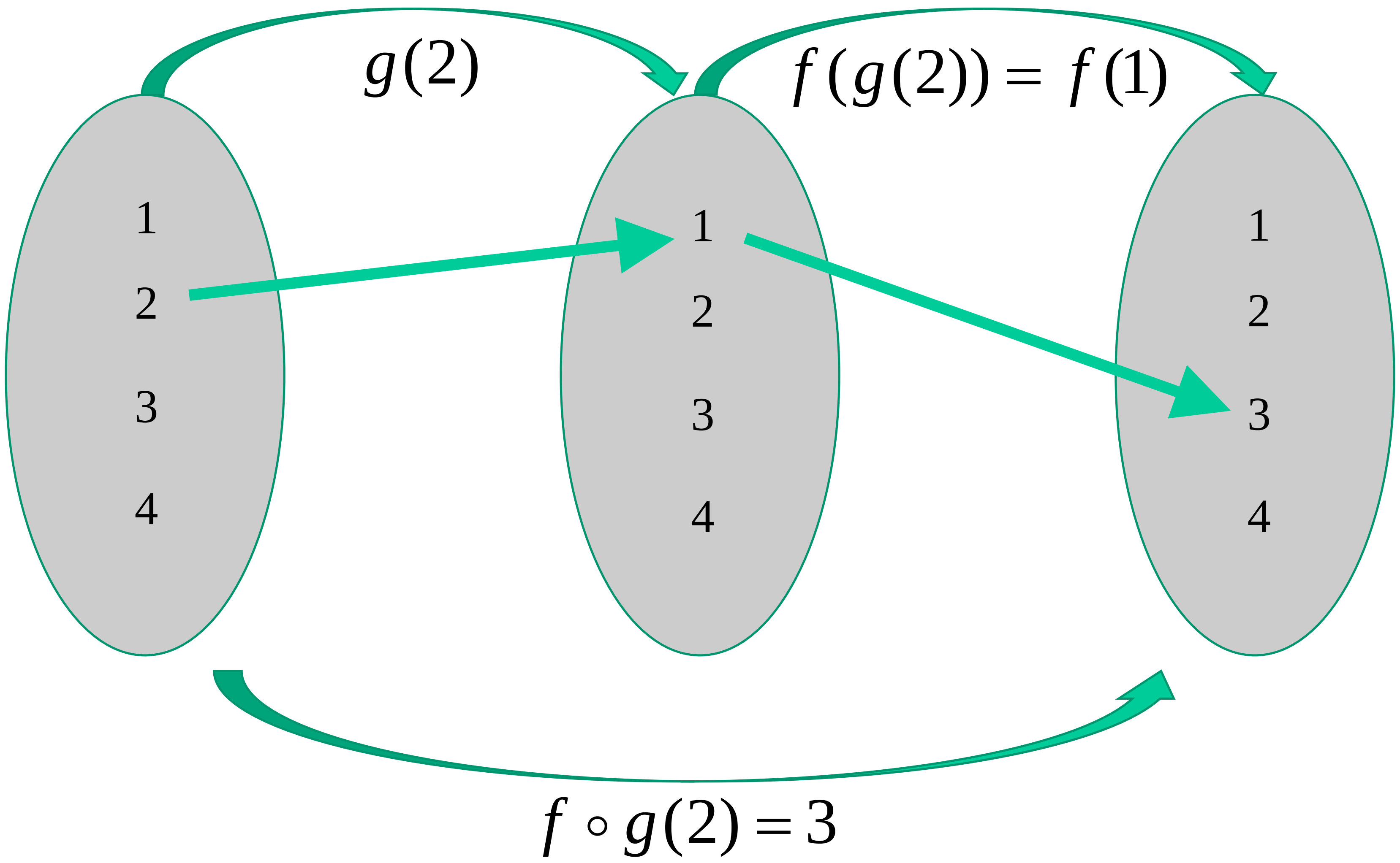




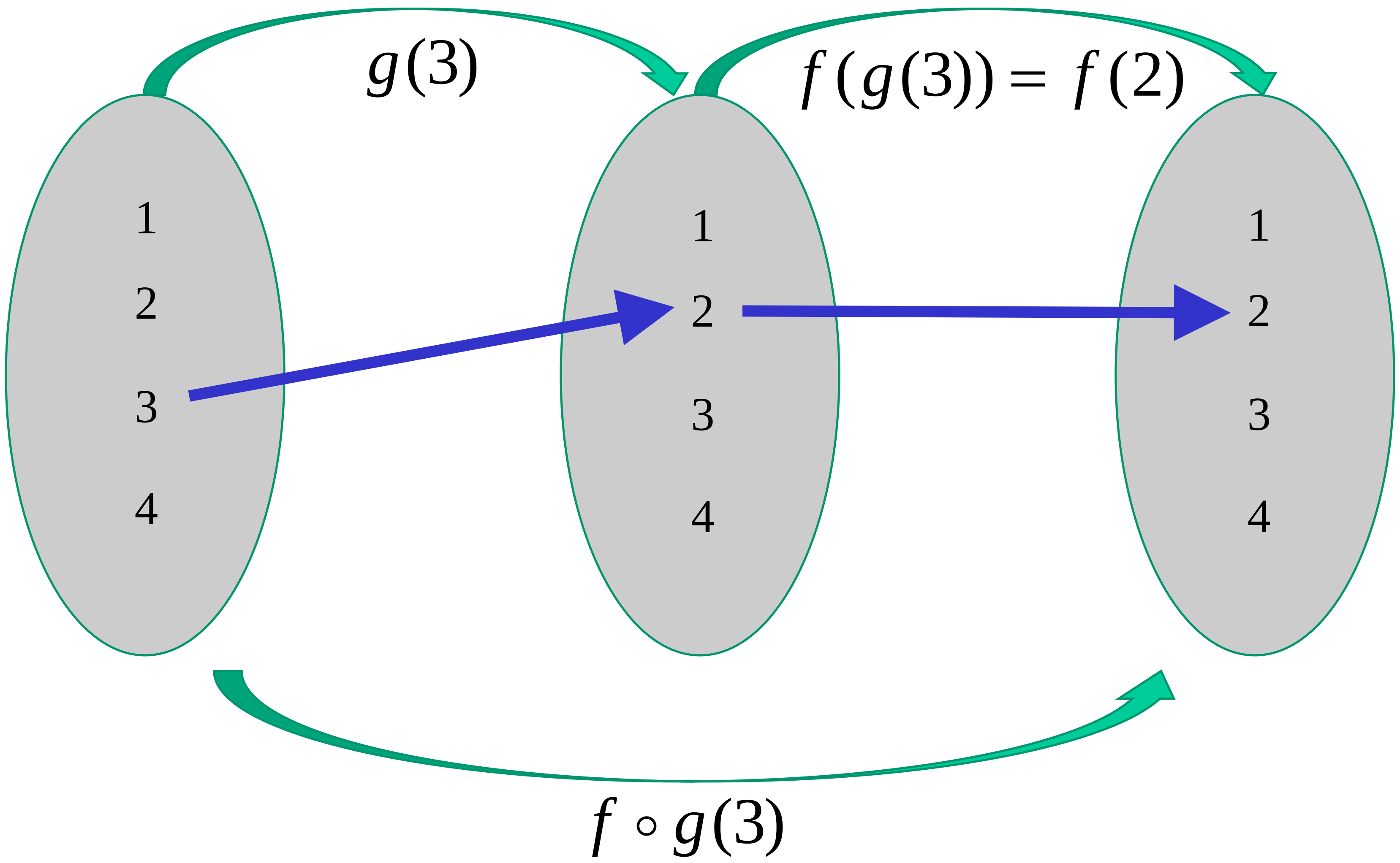
Find $f \circ g$ and $g \circ f$.

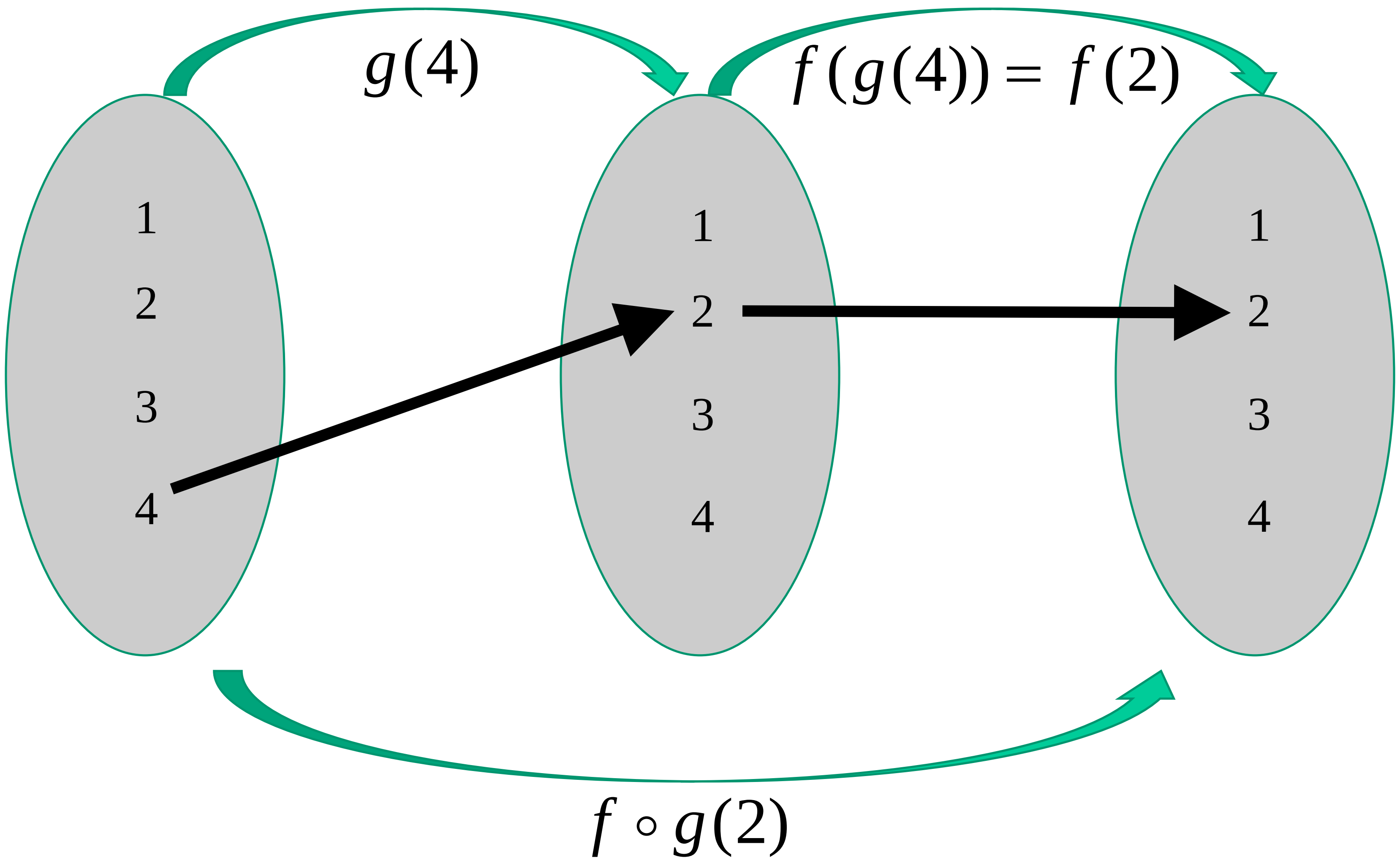


$g(1)$
$f(g(1)) = f(4)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(1) = 1$

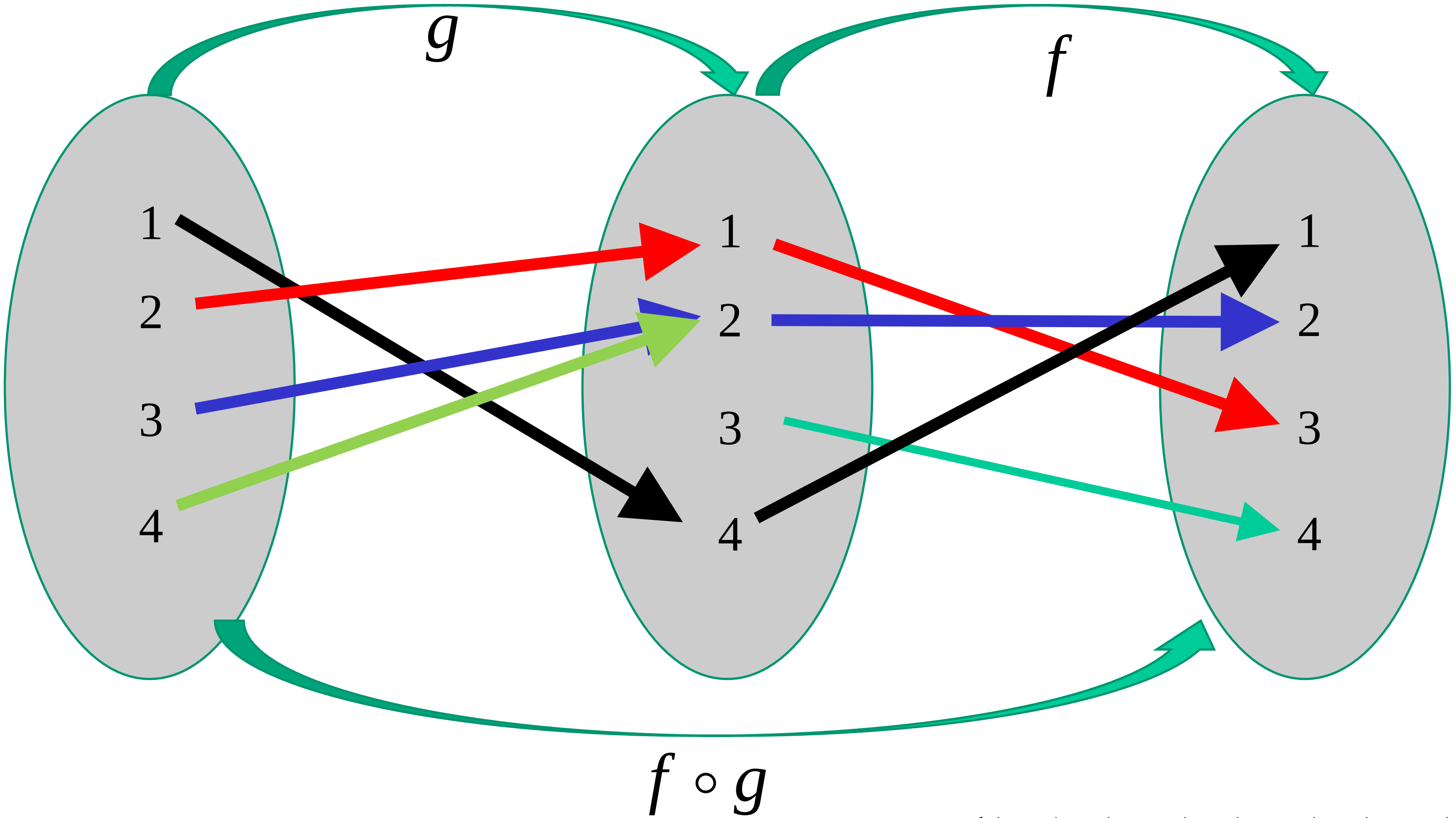


$g(2)$
$f(g(2)) = f(1)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2) = 3$


$g(3)$
$f(g(3)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(3)$



$g(4)$
$f(g(4)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2)$



$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$$f = \{(1,3),(2,2),(3,4),(4,1)\} \qquad g = \{(1,4),(2,1),(3,2),(4,2)\}$$

$$\begin{aligned} f \circ g &= \{(1, f(g(1))),(2, f(g(2))),(3, f(g(3))),(4, f(g(4)))\} \\ &= \{(1, f(4)),(2, f(1)),(3, f(2)),(4, f(2))\} \\ &= \{(1,1),(2,3),(3,2),(4,2)\} \end{aligned}$$

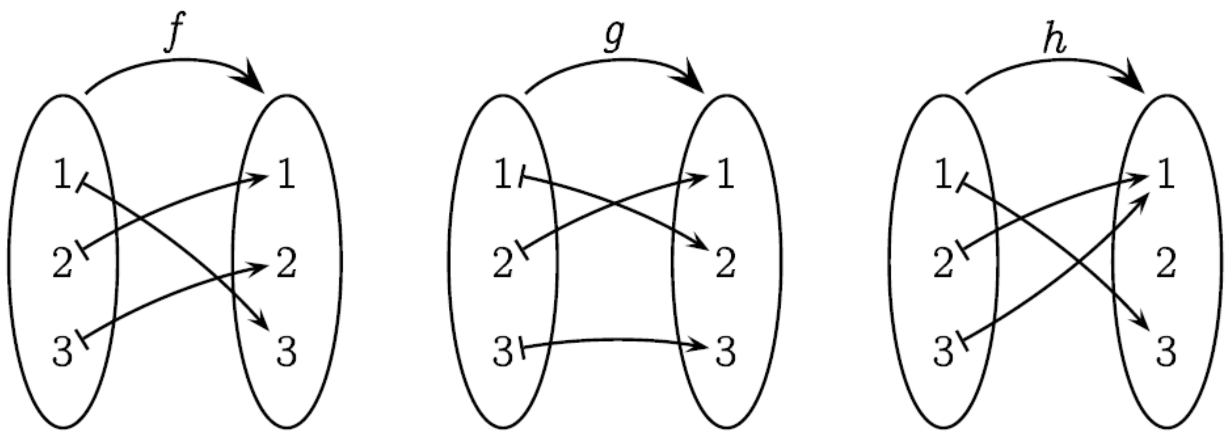
$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$$f = \{(1,3),(2,2),(3,4),(4,1)\} \qquad g = \{(1,4),(2,1),(3,2),(4,2)\}$$

$$\begin{aligned} g \circ f &= \{(1, g(f(1))),(2, g(f(2))),(3, g(f(3))),(4, g(f(4)))\} \\ &= \{(1, g(3)),(2, g(2)),(3, g(4)),(4, g(1))\} \\ &= \{(1,2),(2,1),(3,2),(4,4)\} \end{aligned}$$

$$g \circ f = \{(1,2),(2,1),(3,2),(4,4)\}$$

6. Consider the following three functions f, g and h from $\{1, 2, 3\}$ to itself:



Find $f \circ g \circ h$

7. Find $g \circ f$ and $f \circ g$, where $f(x) = x^2+1$ and $g(x) = x-2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

Composing Functions 2

Self-composition:

- Requires a function $f : A \to A$;
- So $f \circ f : A \to A$:
  - ie $(f \circ f)(a) = f(f(a))$,
  - and $f \circ f$ is usually written f^2;

More generally:

- $f^{n+1} = f \circ f^n$, with special cases:
- $f^1 = f$, and $f^0 = id_A$ (the identity function for A).

Theorems 6.9 and 6.10, try Exercise 6.10.

Theorem 6.9
and Proof Strategies for Implication (5.2)

If $f : A \to B$ and $g : B \to C$ are both injective (one - to - one) then so is $g \circ f : A \to C$.

Proof:

Outline: The statement is of the form '*If X then Y*.' To prove this statement is true, we assume *X is true* and then *prove Y*. This proves 'If X then Y.'

Assumptions

"f is injective" :≡ $(\forall\, x, y: (f(x)=f(y)) \to (x=y))$ *(1)*

"g is injective" :≡ $(\forall\, x, y: (g(x)=g(y)) \to (x=y))$ *(2)*

Need to show

$$(g \circ f)(x) \text{ is injective, i.e.}$$

$$\forall x, y\left[\left((g \circ f)(x) = (g \circ f)(y)\right) \Rightarrow (x = y)\right]$$

Proof

$\boxed{\text{Let } x, y \in A \text{ be arbitrary}}$

$$[(g \circ f)(x) = (g \circ f)(y)] \Leftrightarrow [g(f(x)) = g(f(y))]$$

(2) g is injective

$$\Rightarrow [f(x) = f(y)]$$

(1) f is injective

$$\Rightarrow [x = y]$$

Hence $[(g \circ f)(x) = (g \circ f)(y)] \Rightarrow [x = y]$

$\boxed{\text{This is true for arbitrary } x, y \in A \text{ and therefore}}$

$$\forall x, y[((g \circ f)(x) = (g \circ f)(y)) \Rightarrow (x = y)]$$

$\square$

Theorem 6.10

and proof strategies for implication 5.2

If $f : A \to B$ and $g : B \to C$ are both surjective (onto) then so is $g \circ f : A \to C$.

RECALL DEFINTION OF SURJECTIVE

A function f:$A\to B$ is *onto* or *surjective* or *a surjection* iff its **range is equal to its codomain** ($\forall b \in B$, $\exists a \in A$: $f(a)=b$).

Proof:

Outline: The statement is of the form '*If X then Y*.' To prove this statement is true, we assume *X is true* and then *prove Y*. This proves 'If X then Y.'

Assumptions

f:$A \rightarrow B$ is *surjective* **iff** $(\forall b \in B, \exists a \in A$: $f(a)=b)$.

g:$B \rightarrow C$ is *surjective* **iff** $(\forall c \in C, \exists b \in B$: $f(b)=c)$.

Need to show

$(g \circ f : A \rightarrow C)$ is *surjective*, i.e. $\forall c \in C, \exists a \in A : g \circ f(a) = c$

Consider arbitrary $c \in C$,

Since g is surjective, $c = g(b)$ for some $b \in B$.

Since f is surjective, $b = f(a)$ for some $a \in A$.

Hence

$c = g(b) = g\left(f(a)\right) = \left(g \circ f\right)(a)$ for some $a \in A$.

c is arbitrary and thus the above holds for any c. Hence

$\forall c \in C, \exists a \in A : g \circ f(a) = c$ and thus $(g \circ f : A \rightarrow C)$ is *surjective*.

$\square$

Exercise 6.10

Prove that if $f : A \to B$ and $g : B \to C$ are both bijections then so is $g \circ f : A \to C$.

Proof

If $f : A \to B$ and $g : B \to C$ are both bijections , iff they are both injections and they are both surjection s.

By Theorem 6.9, if $f : A \to B$ and $g : B \to C$ are both injections then $g \circ f : A \to C$ is an injection.

By Theorem 6.10, if $f : A \to B$ and $g : B \to C$ are both surjection s then $g \circ f : A \to C$ is a surjection .

$g \circ f : A \to C$ is an injection and a surjection and thus it is a bijection.

$\square$

Functions for Counting 1

Given two sets A and B:

- An injection from A to B shows $|A| \leqslant |B|$
  - the function is one-to-one, so
  - B is at least as large as A;
- A surjection from A to B shows $|A| \geqslant |B|$
  - the function is onto, so
  - A is at least as large as B;
- A bijection from A to B shows $|A| = |B|$
  - the function is both one-to-one and onto, so
  - B has the same size (cardinality) as A.

Functions for Counting 2

Counting elements in a set involves:

- Constructing a counting function:
  - from the natural numbers to the set,
  - and using successive numbers,
  - and which is a **bijection;**
- Example:
  - for the set { Joel, Felix, Oscar, Amanda},
  - construct {1 $\mapsto$ Joel, 2 $\mapsto$ Felix, 3 $\mapsto$ Oscar, 4 $\mapsto$ Amanda};

  This is the basis of the cardinality function.

Functions for Counting 3

Laws for these relationships:

- ❑Will be skipped here.

Finite and Infinite sets:

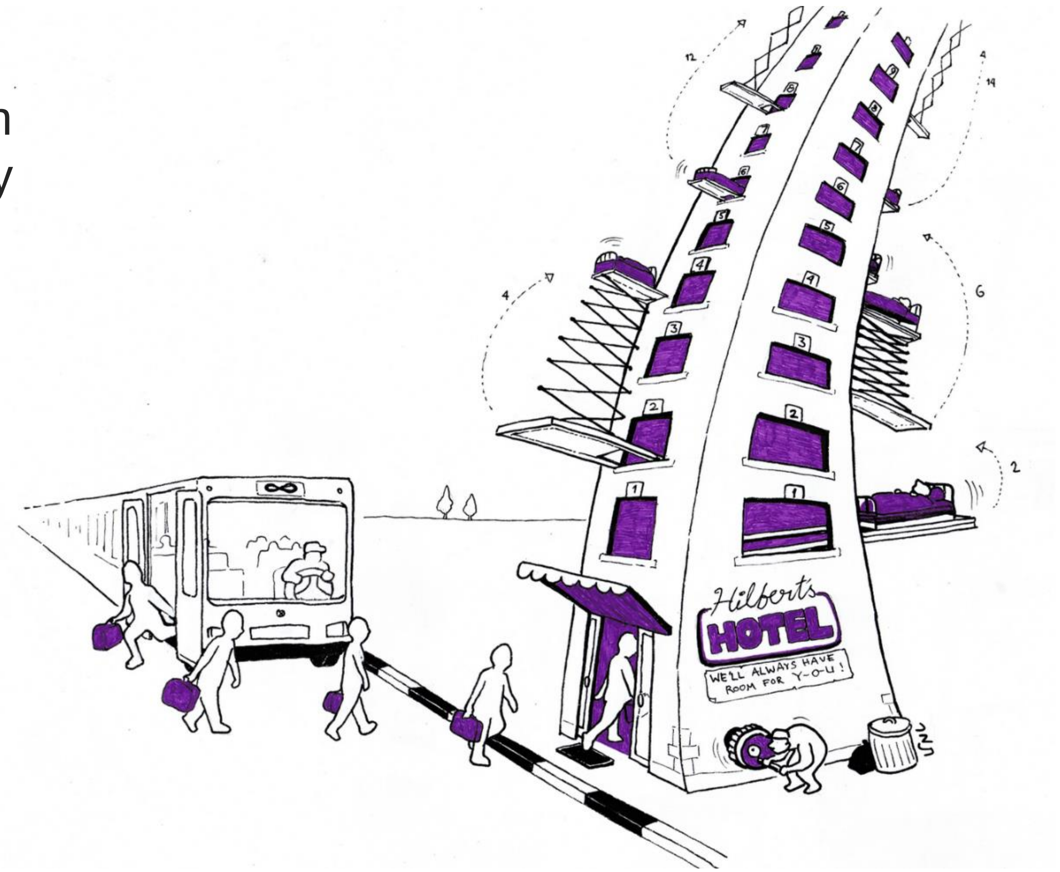
- sets can be finite, or countably infinite:
  - if there is a bijection $f : N \to A$;
- and hence countable or uncountable.

Hilbert's paradox of the Grand Hotel:

The nature of countably infinite sets…

(for those that want to read more…not assessed!!!)

Hilbert's paradox of the Grand Hotel is a thought experiment which illustrates a counterintuitive property of infinite sets. It is demonstrated that a fully occupied hotel with infinitely many rooms may still accommodate additional guests, even infinitely many of them, and that this process may be repeated infinitely often. The idea was introduced by David Hilbert in a 1924 lecture and was popularized through George Gamow's 1947 book *One Two Three... Infinity*.



https://en.wikipedia.org/wiki/Hilbert%27s_paradox_of_the_Grand_Hotel