# COM1002:
# Foundations of Computer Science

## Proof Strategies and Induction

| WEEK | 1,2,3,4,5,6,7,8 | 9 | 10 | 11 |
| --- | --- | --- | --- | --- |
| **Mon** | | Lecture<br>**Hand in Ex 6** | Lecture<br>**Hand out ex 7**<br><br>(Assessed 5%) | Lecture<br>**Hand in Ex 7** |
| **Wed** | | **Revision Lecture** | Lecture | **Revision Lecture** |
| **Thurs** | | **Revision Tutorial**<br>**QUIZ 2 (25%)**<br>Diamond 101<br>4pm-5:30pm | Tut (ex 7) | **Revision Tutorial**<br>**QUIZ 3 (25%)**<br>Diamond 101<br>4pm-5:30pm |

We are nearly there…

QUIZ 2 - overview

**6 Questions:**

Predicate Logic (x2),
Composition of Functions,
Proof in propositional logic,
Proof in predicate logic over sets
Simplication of a predicate statement

Thursday 26th November, 4pm-5pm
Computer Room 1 – Diamond
Duration: 50mins

Two attempts allowed – highest attempt taken.

# Proof in Propositional Logic

**Show that the hypotheses:**

- It is not sunny this afternoon and it is colder than yesterday.
- We will go swimming only if it is sunny.
- If we do not go swimming, then we will take a canoe trip.
- If we take a canoe trip, then we will be home by sunset.

lead to the conclusion:

*We will be home by the sunset.*

**Main steps:**

Translate the statements into proposional logic.

Write a formal proof, a sequence of steps that state hypotheses or apply inference rules to previous steps.

Show that the hypotheses:

- It is not sunny this afternoon and it is colder than yesterday. $\neg s \wedge c$
- We will go swimming only if it is sunny. $w \rightarrow s$
- If we do not go swimming, then we will take a canoe trip. $\neg w \rightarrow t$
- If we take a canoe trip, then we will be home by sunset. $t \rightarrow h$

lead to the conclusion:

- We will be home by the sunset. $h$

| Step | Reason |
|---|---|
| 1. $\neg s \wedge c$ | hypothesis |
| 2. $\neg s$ | simplification |
| 3. $w \rightarrow s$ | hypothesis |
| 4. $\neg w$ | modus tollens of 2 and 3 |
| 5. $\neg w \rightarrow t$ | hypothesis |
| 6. $t$ | modus ponens of 4 and 5 |
| 7. $t \rightarrow h$ | hypothesis |
| 8. $h$ | modus ponens of 6 and 7 |

Where:
$s$: "it is sunny this afternoon"
$c$: "it is colder than yesterday"
$w$: "we will go swimming"
$t$: "we will take a canoe trip.
$h$: "we will be home by the sunset."

| Rule of Inference | |
|---|---|
| $\dfrac{\forall x P(x)}{\therefore \quad P(c)}$ | Universal elimination |
| $\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \quad \forall x P(x)}$ | Universal introduction |
| $\dfrac{\exists x P(x)}{\therefore \quad P(c) \text{ for some element } c}$ | Existential elimination |
| $\dfrac{P(c) \text{ for some element } c}{\therefore \quad \exists x P(x)}$ | Existential introduction |

# Proof in Predicate Logic

Hypotheses: $\exists x(A(x) \wedge \neg B(x))$ and $\forall x(A(x) \rightarrow P(x))$.

Conclusion: $\exists x(P(x) \wedge \neg B(x))$.

| Step | |
|---|---|
| 1. $\exists x(A(x) \wedge \neg B(x))$ | Premise |
| 2. $A(a) \wedge \neg B(a)$ | Existential elimination (1) |
| 3. $A(a)$ | Conjunction elimination from (2) |
| 4. $\forall x(A(x) \rightarrow P(x))$ | Premise |
| 5. $A(a) \rightarrow P(a)$ | Universal elimination from (4) |
| 6. $P(a)$ | Modus Ponens from (3) and (5) |
| 7. $\neg B(a)$ | Conjunction elimination (2) |
| 8. $P(a) \wedge \neg B(a)$ | Conjunction introduction (6) and (7) |
| 9. $\exists x(P(x) \wedge \neg B(x))$ | Existential introduction from (8) |

# Inductive Proofs

# Mathematical Induction

**Mathematical induction is a form of mathematical proof.**

The **principle of mathematical induction** is a useful tool for proving that a certain predicate is true for **all natural numbers**.

# Learning Objective

By the end of this lecture, you should be able to prove by induction:

1) The sum of the first n odd numbers is $n^2$

2) The sum of the first n numbers is $\dfrac{n(n+1)(2n+1)}{6}$

3) $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = \dfrac{(n+1)(2n+1)(2n+3)}{3}$

Show that for real number $r \neq 1$,

$$\sum_{i=0}^{n-1} ar^i = \frac{a\left(1-r^n\right)}{1-r}, \quad \text{for } n \in \mathrm{N}$$

**The Principle of Mathematical Induction**

**Let $P_n$ be a statement involving the positive integer n. If:**

**1. $P_1$ is true**

**and**

**2. the truth of $P_k$ implies the truth of $P_{k+1}$, for every positive integer k,**

**then**

**$P_n$ must be true for all integers n.**

$$\left[P(1) \wedge \left(P(k) \Rightarrow P(k+1)\right)\right] \Rightarrow \forall n P(n)$$

If we have a propositional function P(n), and we want to prove that P(n) is true for any natural number n, we do the following:

- Show that **P(1) is true.** (basis step)
- Show that if P(k) then P(k + 1) for any $k \in N$. (inductive step)
- Then P(n) must be true for any $n \in N$. (conclusion)

$$\left[P(1) \wedge \left(P(k) \Rightarrow P(k+1)\right)\right] \Rightarrow \forall n P(n)$$

**Ex. Use mathematical induction to prove the following formula.**

$$S_n = 1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$$

**First, we must show that the formula works for n = 1.**

1. **For n = 1**

   $S_1 = 1 = 1^2$

**The second part of mathematical induction has two steps. The first step is to assume that the formula is valid for some integer k. The second step is to use this assumption to prove that the formula is valid for the next integer, k + 1.**

2. **Assume** $S_k = 1 + 3 + 5 + 7 + \cdots + (2k-1) = k^2$ **is true, show that** $S_{k+1} = (k + 1)^2$ **is true.**

$$S_{k+1} = 1 + 3 + 5 + 7 + \cdots + (2k - 1) + [2(k + 1) - 1]$$

$$= [1 + 3 + 5 + 7 + \cdots + (2k - 1)] + (2k + 2 - 1)$$

$$= S_k + (2k + 1)$$

$$= k^2 + 2k + 1$$

$$= (k + 1)^2$$

$\square$

**Ex. Use mathematical induction to prove the following formula.**

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**1. Show n = 1 is true.**

$$S_n = 1^2 = \frac{1(2)(3)}{6}$$

**2. Assume that $S_k$ is true.**

$$S_k = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

**Show that** $S_{k+1} = \dfrac{(k+1)(k+2)(2k+3)}{6}$ **is true.**

$$S_{k+1} = (1^2 + 2^2 + 3^2 + 4^2 + \cdots + k^2) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

**Factor out a (k + 1)**

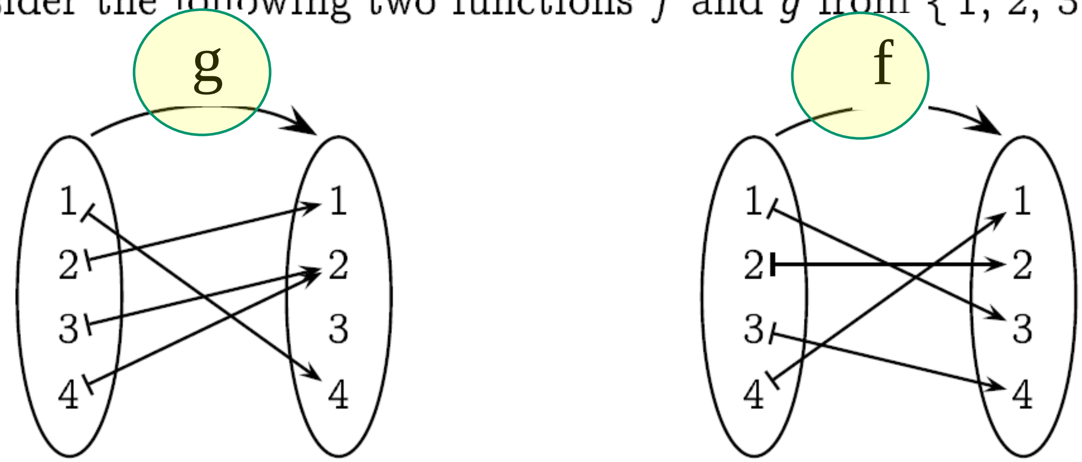
$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2 + 7k + 6]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

□

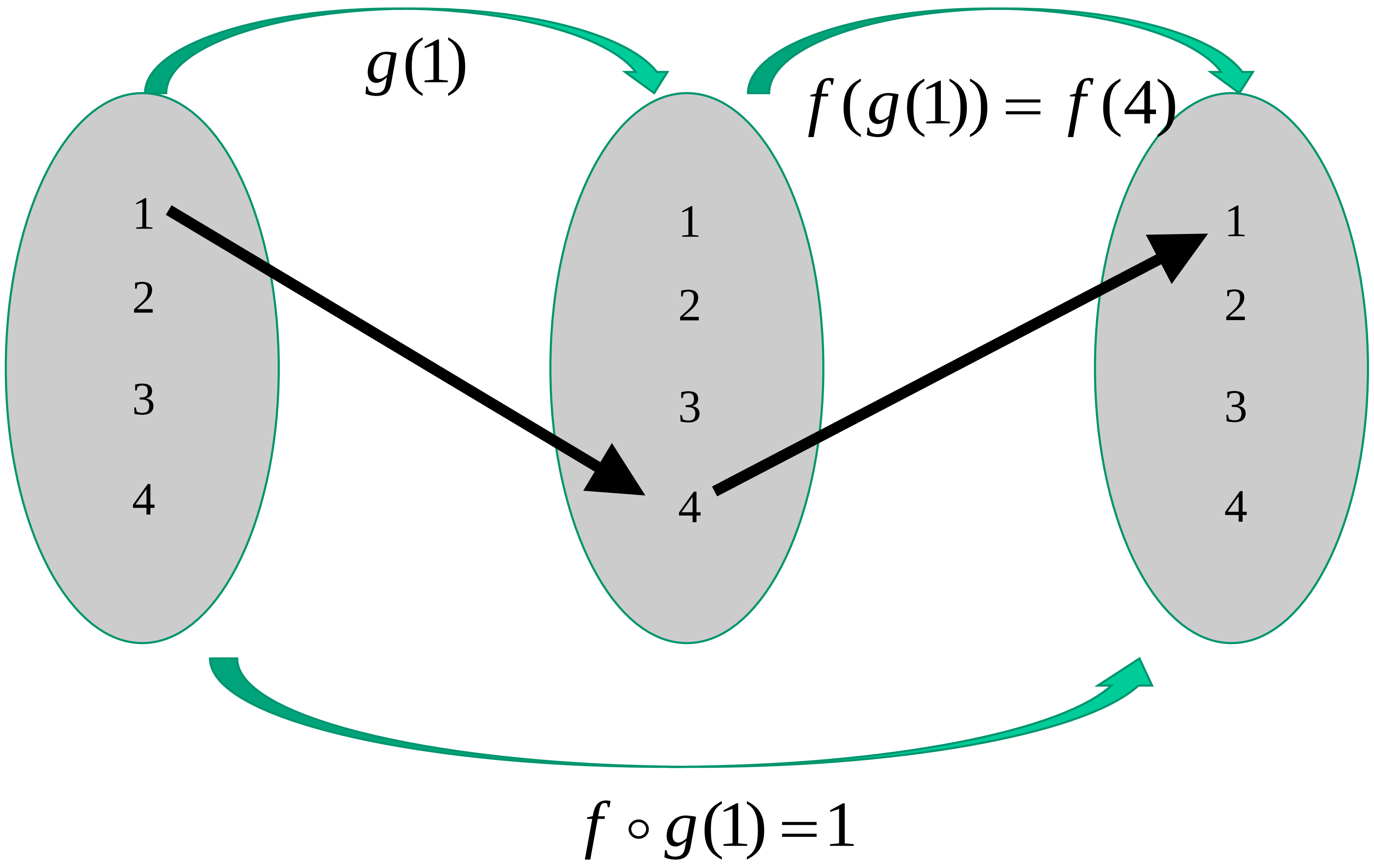
# Week 5, L2, slide 20 - Typo

Consider the following two functions $f$ and $g$ from $\{1, 2, 3, 4\}$ to itself:

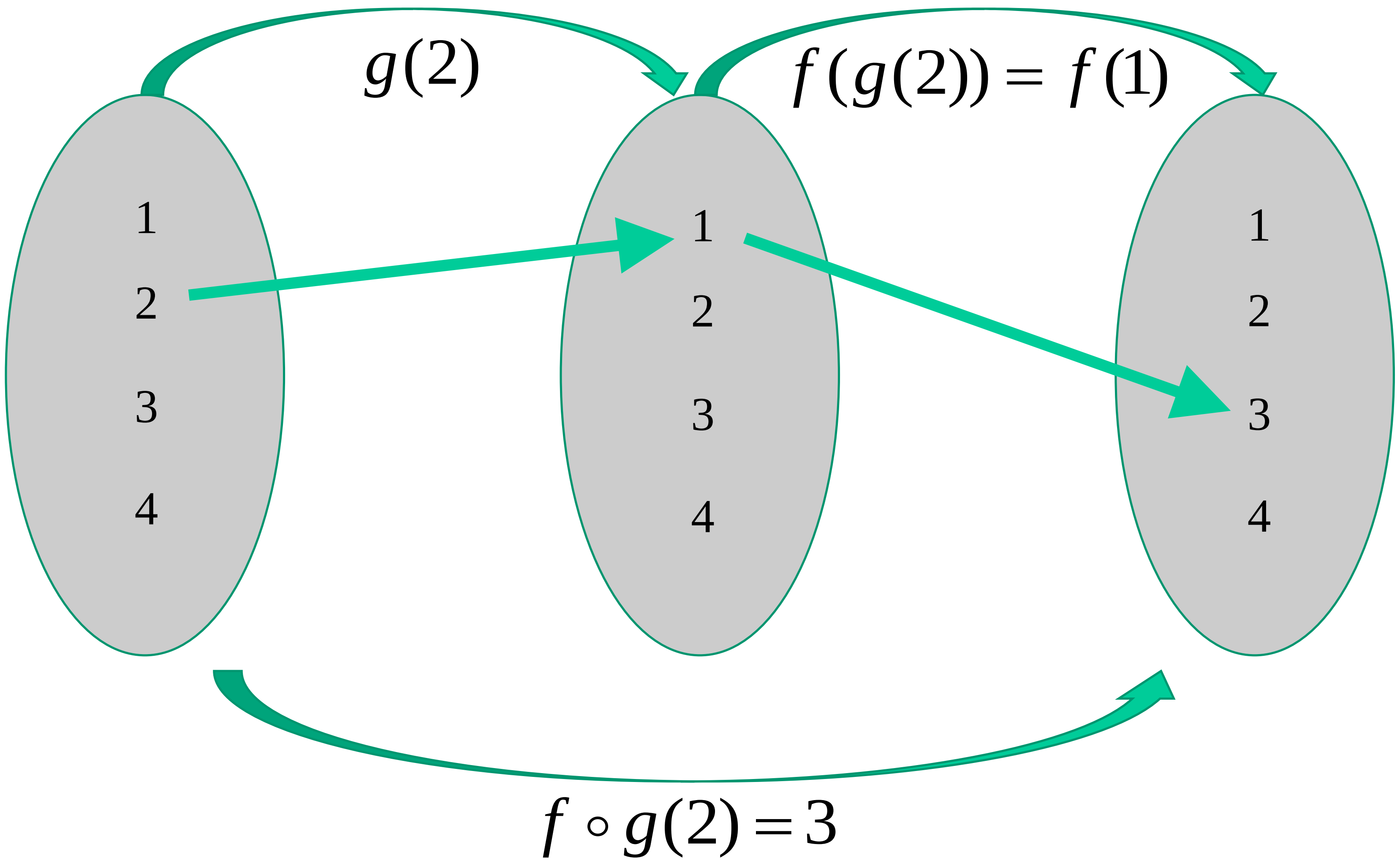



Find $f \circ g$ and $g \circ f$.

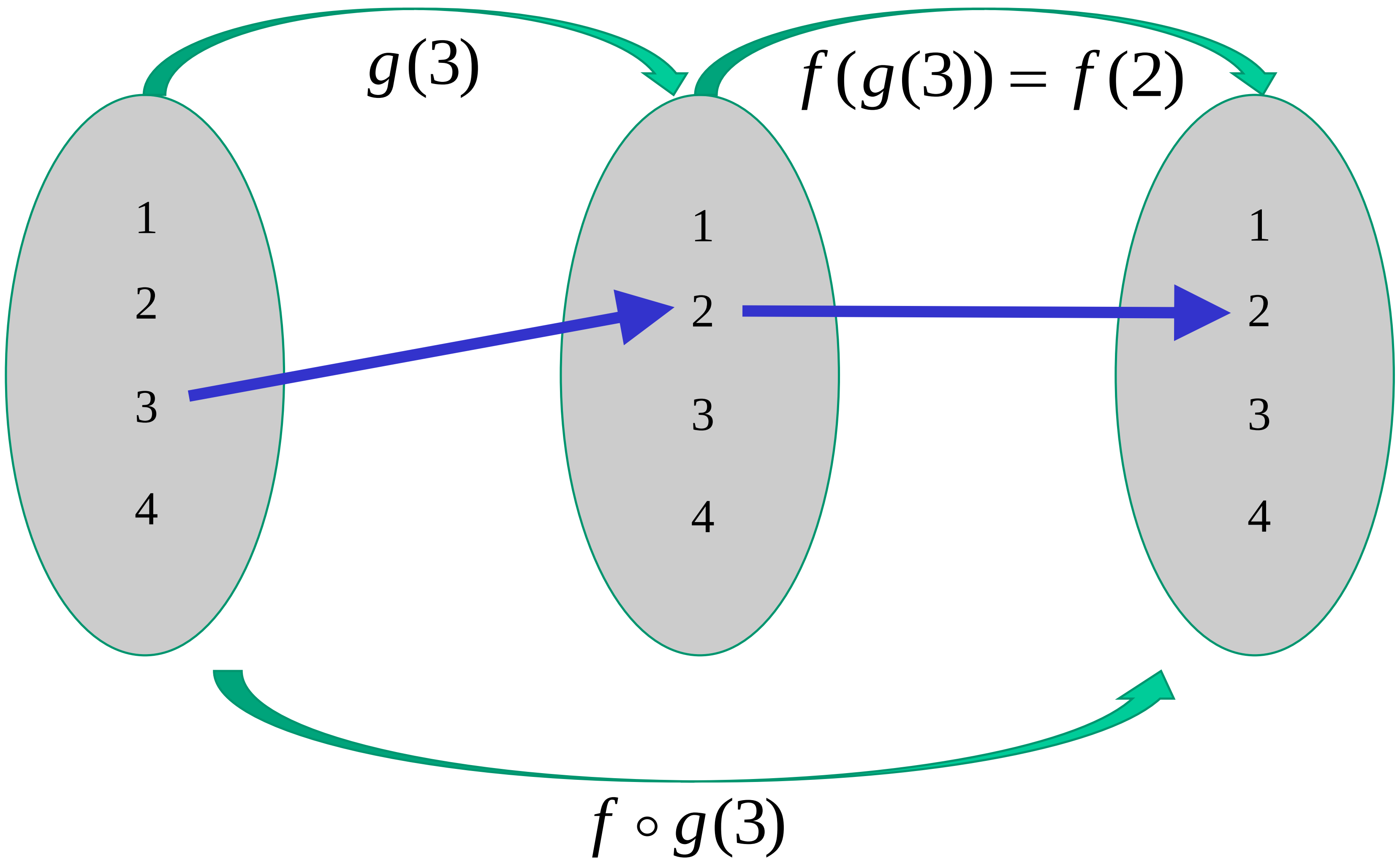
The answers on the following slides assume *g* and *f* as defined above. (I had put *g* and *f* the other way round on the handout slides)

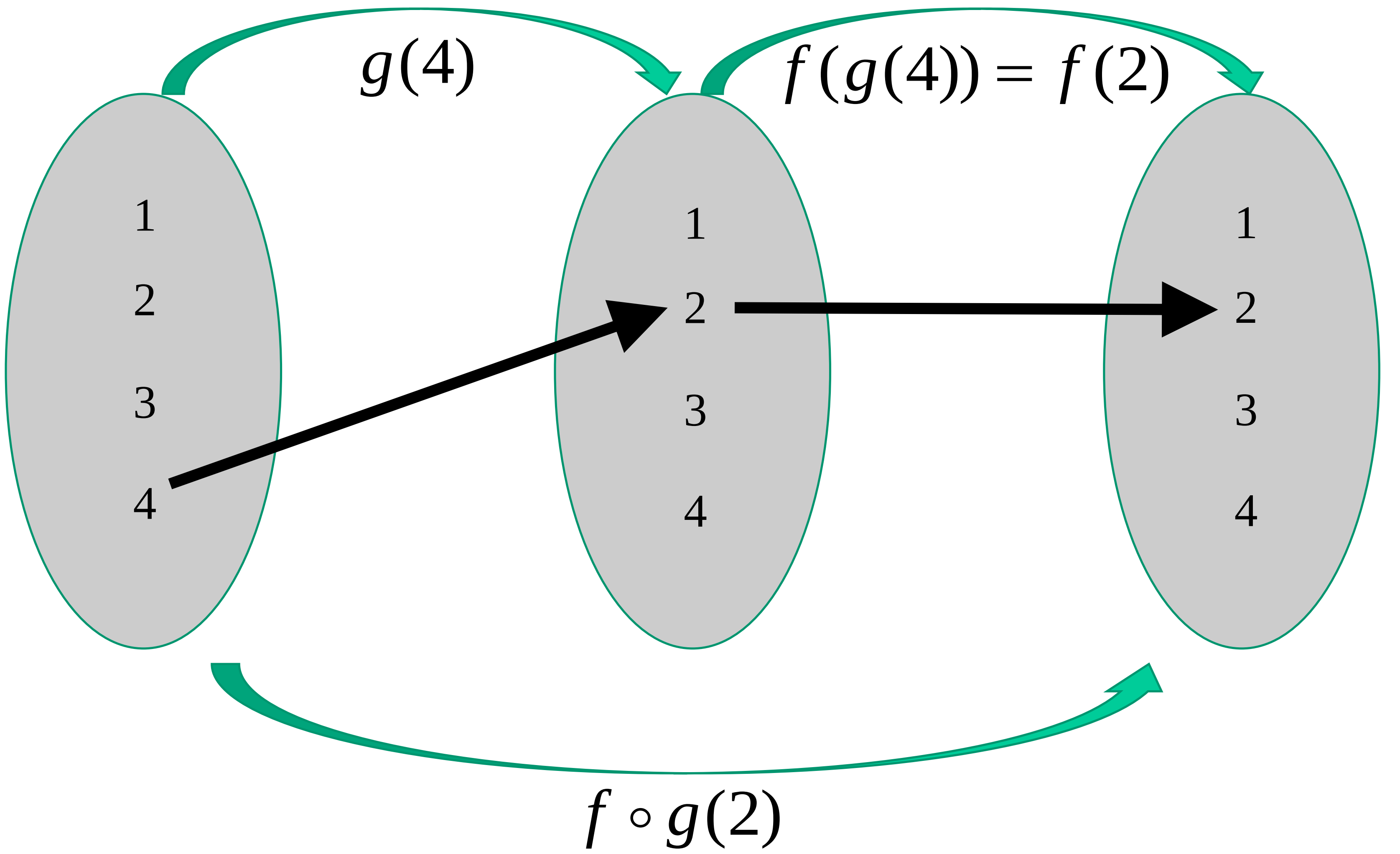


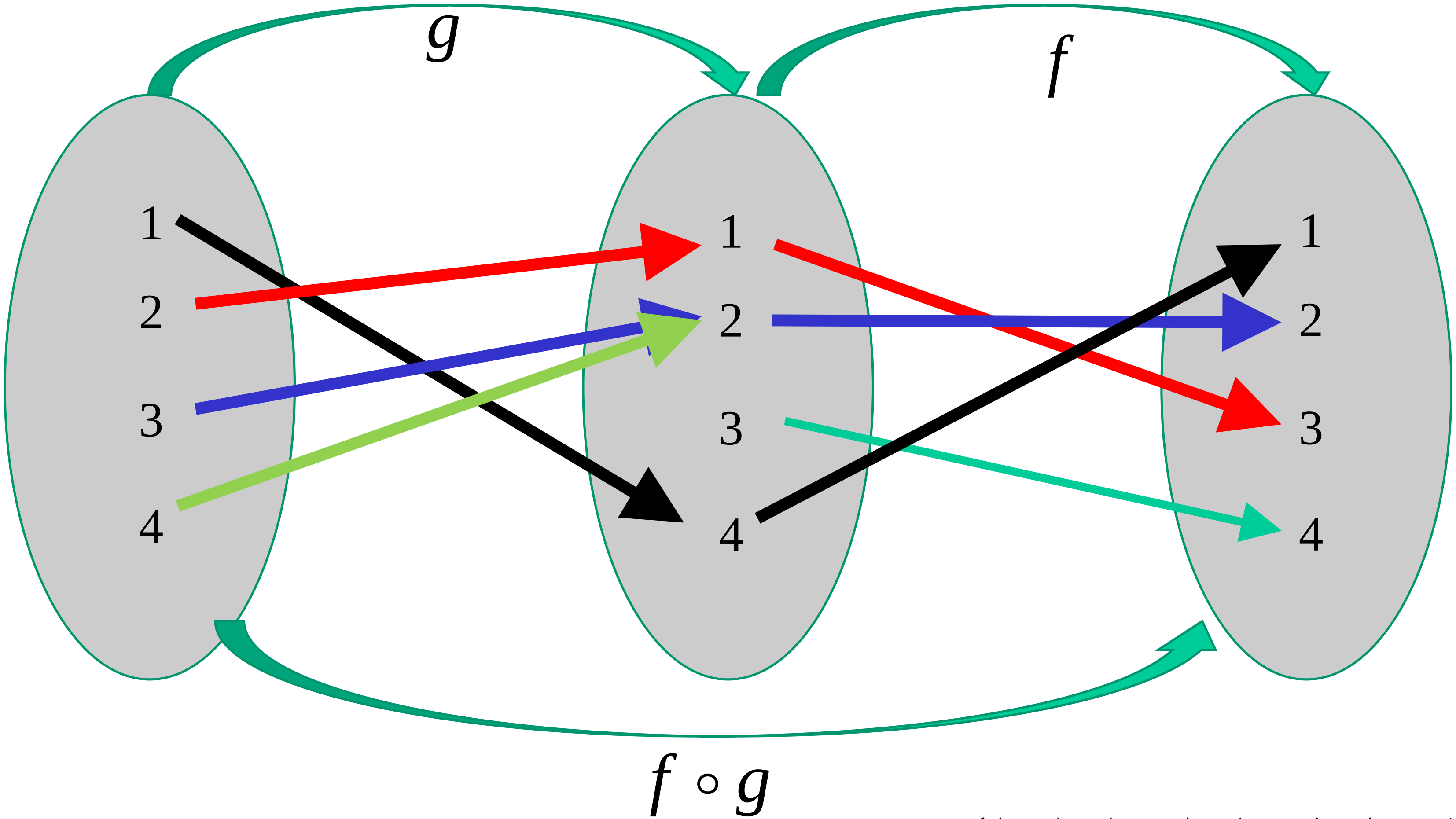
$g(1)$
$f(g(1)) = f(4)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(1) = 1$



$g(2)$
$f(g(2)) = f(1)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2) = 3$


$g(3)$
$f(g(3)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(3)$


$g(4)$
$f(g(4)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2)$




$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$$f = \{(1,3),(2,2),(3,4),(4,1)\} \qquad g = \{(1,4),(2,1),(3,2),(4,2)\}$$

$$\begin{aligned} f \circ g &= \{(1, f(g(1))),(2, f(g(2))),(3, f(g(3))),(4, f(g(4)))\} \\ &= \{(1, f(4)),(2, f(1)),(3, f(2)),(4, f(2))\} \\ &= \{(1,1),(2,3),(3,2),(4,2)\} \end{aligned}$$

$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$$f = \{(1,3),(2,2),(3,4),(4,1)\} \qquad g = \{(1,4),(2,1),(3,2),(4,2)\}$$

$$\begin{aligned} g \circ f &= \{(1, g(f(1))),(2, g(f(2))),(3, g(f(3))),(4, g(f(4)))\} \\ &= \{(1, g(3)),(2, g(2)),(3, g(4)),(4, g(1))\} \\ &= \{(1,2),(2,1),(3,2),(4,4)\} \end{aligned}$$

$$g \circ f = \{(1,2),(2,1),(3,2),(4,4)\}$$