COM1002 Foundations of Computer Science

Lecture 3: Monday 5th October

Dr Paul Watton

A logician said to his son…

"If you don't eat your vegetables, you can't have any ice cream."

Upon hearing this, the son choked down a plate of broccoli, and his father, duly impressed, sent him to bed without any ice cream.

LEARNING OUTCOME

Who understands why this a joke…?

Algebra of Propositional Logic

By the end of this lecture, we will be able to use the algebraic laws of propositional logic to show that the following statement is true…

$$(P \to L) \wedge (H \to L) \quad \Leftrightarrow \quad (P \vee H) \to L$$

Exercises and Tutorials

- Tutorials will begin this THURSDAY 8th Oct
- I will upload an **exercise sheet** to MOLE this evening. (*hand in beginning of Tutorial 15th Oct*)
- 3 tutorial groups – details to be on MOLE tomorrow
- Exercise sheet will be assessed
- REMINDER: COM1002 Semester 1
  - Three MOLE Quizzes: 25%, 25%, 25% (weeks 6,9,11)
  - Five assessed exercise sheets: 5 * 5%

Modelling with Prop. Logic 1

An Aeroplane Controller Example:

- Some current code:
  - ```
    if CabinPressure < MinPressure
    then PrepareForLanding;
    if FlightHeight < MinHeight
    then PrepareForLanding;
    ```
- A programmer proposes optimisation:
  - ```
    if ((CabinPressure < MinPressure)
        and (FlightHeight < MinHeight))
    then PrepareForLanding;
    ```
- Is this a valid optimisation?

Solution:

- Define appropriate variables:
  - let `P` be `CabinPressure < MinPressure`,
    let `H` be `FlightHeight < MinHeight`, and
    let `L` be the action `PrepareForLanding`;
- Model the current behaviour:
  - as; $(P \rightarrow L) \wedge (H \rightarrow L)$
- Consider two optimisations
  - as $(P \wedge H) \rightarrow L$ $\qquad (P \vee H) \rightarrow L$
- Which of these formulae are correct ?

Solution (using truth tables)

P	H	L	$(P \vee H)$	$(P \wedge H)$	$P \rightarrow L$	$H \rightarrow L$	$(P \vee H) \rightarrow L$	$(P \wedge H) \rightarrow L$	$(P \rightarrow L) \wedge (H \rightarrow L)$
T	T	T	T	T	T	T	**T**	T	**T**
T	T	F	T	T	F	F	**F**	F	**F**
T	F	T	T	F	T	T	**T**	T	**T**
T	F	F	T	F	F	T	**F**	T	**F**
F	T	T	T	F	T	T	**T**	T	**T**
F	T	F	T	F	T	F	**F**	T	**F**
F	F	T	F	F	T	T	**T**	T	**T**
F	F	F	F	F	T	T	**T**	T	**T**

Algebraic Laws 1

These define equivalence of propositions:

Commutativity:

$p \vee q \Leftrightarrow q \vee p \qquad p \wedge q \Leftrightarrow q \wedge p$;

Associativity:

$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$

$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$

Idempotence:

$p \vee p \Leftrightarrow p$ and $p \wedge p \Leftrightarrow p$

If the expressions are equivalent what do we know about their Truth Tables ?

Algebraic Laws 2

Distributivity:

$$p \lor (q \land r) \Leftrightarrow (p \lor q) \land (p \lor r)$$

$$p \land (q \lor r) \Leftrightarrow (p \land q) \lor (p \land r)$$

De Morgan's Laws:

$$\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$$

$$\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$$

Double Negation Law: $\neg(\neg p) \Leftrightarrow p$;

Algebraic Laws 3

Tautology Laws:

$p \vee \text{true} \Leftrightarrow \text{true}$

$p \wedge \text{true} \Leftrightarrow p$;

Excluded Middle Laws:

$p \vee \neg\, p \Leftrightarrow \text{true}$

$p \wedge \neg\, p \Leftrightarrow \text{false}$;

Contradiction Laws:

$p \vee \text{false} \Leftrightarrow p$

$p \wedge \text{false} \Leftrightarrow \text{false}$;

Absorption Laws:

$p \vee (p \wedge q) \Leftrightarrow p$

$p \wedge (p \vee q) \Leftrightarrow p$;

Algebraic Laws 4

Implication Law:

$$p \Rightarrow q \Leftrightarrow \neg p \vee q;$$

Contrapositive Law:

$$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p;$$

Equivalence Law:

$$p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p).$$

Commutativity:	$p \vee q \Leftrightarrow q \vee p \qquad p \wedge q \Leftrightarrow q \wedge p$
Associativity:	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$
	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$
Idempotence:	$p \vee p \Leftrightarrow p$ **and** $p \wedge p \Leftrightarrow p$
Distributivity:	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws:	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
	$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
Double Negation Law:	$\neg(\neg p) \Leftrightarrow p;$
Tautology Laws:	$p \vee \mathtt{true} \Leftrightarrow \mathtt{true}$
	$p \wedge \mathtt{true} \Leftrightarrow p$
Contradiction Laws:	$p \vee \mathtt{false} \Leftrightarrow p \quad p \wedge \mathtt{false} \Leftrightarrow \mathtt{false}$
Excluded Middle Laws:	$p \vee \neg p \Leftrightarrow \mathtt{true}$
	$p \wedge \neg p \Leftrightarrow \mathtt{false}$
Absorption Laws:	$p \vee (p \wedge q) \Leftrightarrow p \quad p \wedge (p \vee q) \Leftrightarrow p$
Implication Law:	$p \Rightarrow q \Leftrightarrow \neg p \vee q$
Contrapositive Law:	$p \Rightarrow q \Leftrightarrow \neg q \Rightarrow \neg p$
Equivalence Law:	$(p \Leftrightarrow q) \Leftrightarrow (p \Rightarrow q) \wedge (q \Rightarrow p)$

SHOW: $(P \to L) \wedge (H \to L) \quad \Leftrightarrow \quad (P \vee H) \to L$

Solution (using algebraic laws) is straightforward

$$(P \to L) \wedge (H \to L) \Leftrightarrow (P \to L) \wedge (H \to L)$$

Implication Law

$$\Leftrightarrow (\neg P \vee L) \wedge (\neg H \vee L)$$

Distributivity Law

$$\Leftrightarrow (\neg P \wedge \neg H) \vee L$$

De Morgan Law

$$\Leftrightarrow \neg (P \vee H) \vee L$$

Implication Law

$$\Leftrightarrow (P \vee H) \to L$$

Exercise 1.27:

Give derivations of the following equivalences:

1. $p \wedge (\neg p \vee q) \quad \Leftrightarrow \quad p \wedge q$
2. $\neg(p \Rightarrow q) \quad \Leftrightarrow \quad p \wedge \neg q$
3. $p \Rightarrow (q \vee r) \quad \Leftrightarrow \quad (p \Rightarrow q) \vee (p \Rightarrow r)$
4. $p \Rightarrow (q \wedge r) \quad \Leftrightarrow \quad (p \Rightarrow q) \wedge (p \Rightarrow r)$
5. $(p \wedge q) \Rightarrow r \quad \Leftrightarrow \quad (p \Rightarrow r) \vee (q \Rightarrow r)$
6. $(p \vee q) \Rightarrow r \quad \Leftrightarrow \quad (p \Rightarrow r) \wedge (q \Rightarrow r)$

Exercise 1.27(1): $p \wedge (\neg p \vee q) \Leftrightarrow p \wedge q$

1. $p \wedge (\neg p \vee q) \Leftrightarrow p \wedge (\neg p \vee q)$

(distributivity law) $\Leftrightarrow (p \wedge \neg p) \vee (p \wedge q)$

(excluded middle law) $\Leftrightarrow \text{false} \vee (p \wedge q)$

(commutativity laws) $\Leftrightarrow (p \wedge q) \vee \text{false}$

(Contradiction Laws) $\Leftrightarrow (p \wedge q)$

Exercise 1.27(2): $\neg(p \Rightarrow q) \quad \Leftrightarrow \quad p \wedge \neg q$

$$\neg(p \Rightarrow q) \quad \Leftrightarrow \quad \neg(p \Rightarrow q)$$

(implication laws) $\quad \Leftrightarrow \quad \neg(\neg p \vee q)$

(De Morgan laws) $\quad \Leftrightarrow \quad \neg\neg p \wedge \neg q$

(double negation) $\quad \Leftrightarrow p \wedge \neg q$

Exercise 1.27(4): $p \Rightarrow (q \wedge r) \quad \Leftrightarrow \quad (p \Rightarrow q) \wedge (p \Rightarrow r)$

$$p \Rightarrow (q \wedge r) \quad \Leftrightarrow \quad p \Rightarrow (q \wedge r)$$

(implication) $\quad \Leftrightarrow \quad \neg p \vee (q \wedge r)$

(distributivity) $\quad \Leftrightarrow \quad (\neg p \vee q) \wedge (\neg p \vee r)$

(implication) $\quad \Leftrightarrow \quad (p \Rightarrow q) \wedge (p \Rightarrow r)$

Exercise 1.27(6): $(p \vee q) \Rightarrow r \quad \Leftrightarrow \quad (p \Rightarrow r) \wedge (q \Rightarrow r)$

$$
\begin{array}{rcl}
(p \vee q) \Rightarrow r & \Leftrightarrow & (p \vee q) \Rightarrow r \\
\text{(implication)} & \Leftrightarrow & \neg(p \vee q) \vee r \\
\text{(De Morgan)} & \Leftrightarrow & (\neg p \wedge \neg q) \vee r \\
\text{(Distributivity)} & \Leftrightarrow & (\neg p \wedge r) \vee (\neg q \wedge r) \\
\text{(Implication)} & \Leftrightarrow & (p \Rightarrow r) \wedge (q \Rightarrow r)
\end{array}
$$

Exercise 1.27(3): $p \Rightarrow (q \vee r) \quad \Leftrightarrow \quad (p \Rightarrow q) \vee (p \Rightarrow r)$

$$p \Rightarrow (q \vee r) \quad \Leftrightarrow \quad p \Rightarrow (q \vee r)$$

(implication) $\Leftrightarrow \neg p \vee (q \vee r)$

(imdempotence) $\Leftrightarrow (\neg p \vee \neg p) \vee (q \vee r)$

(associativity) $\Leftrightarrow \neg p \vee (\neg p \vee (q \vee r))$

(associativity) $\Leftrightarrow \neg p \vee ((\neg p \vee q) \vee r)$

(commutativity) $\Leftrightarrow \neg p \vee (r \vee (\neg p \vee q))$

(associativity) $\Leftrightarrow (\neg p \vee r) \vee (\neg p \vee q)$

(implication) $\Leftrightarrow (p \Rightarrow r) \vee (p \Rightarrow q)$

Exercise 1.27(5): $(p \wedge q) \Rightarrow r \quad \Leftrightarrow \quad (p \Rightarrow r) \vee (q \Rightarrow r)$

$(p \wedge q) \Rightarrow r$	$\Leftrightarrow$	$(p \wedge q) \Rightarrow r$
(implication)	$\Leftrightarrow$	$\neg(p \wedge q) \vee r$
(De Morgan)	$\Leftrightarrow$	$(\neg p \vee \neg q) \vee r$
(Indempotence)	$\Leftrightarrow$	$(\neg p \vee \neg q) \vee (r \vee r)$
(associativity)	$\Leftrightarrow$	$((\neg p \vee \neg q) \vee r) \vee r$
(associativity)	$\Leftrightarrow$	$(\neg p \vee (\neg q \vee r)) \vee r$
(associativity)	$\Leftrightarrow$	$\neg p \vee ((\neg q \vee r) \vee r)$
(commutativity)	$\Leftrightarrow$	$\neg p \vee (r \vee (\neg q \vee r))$
(associativity)	$\Leftrightarrow$	$(\neg p \vee r) \vee (\neg q \vee r)$
(implication)	$\Leftrightarrow$	$(p \Rightarrow r) \vee (q \Rightarrow r)$

Algebra of Propositional Logic

By the end of this lecture, we will be able to use the algebraic laws of propositional logic to show that the following statement is true.

$$(P \to L) \wedge (H \to L) \quad \Leftrightarrow \quad (P \vee H) \to L$$

Can we now do this ?

Kangaroo Puzzle: Exercise 1

- The only animals in this house are cats.
- Every animal that loves to gaze at the moon is suitable for a pet.
- When I detest an animal, I avoid it.
- No animals are carnivorous, unless they prowl at night.
- No cat fails to kill mice.
- No animal ever takes to me, except those that are in this house.
- Kangaroos are not suitable for pets.
- None but carnivora kill mice.
- I detest animals that do not take to me.
- Animals that prowl at night always love to gaze at the moon.

Argue that the above statements imply I always avoid a kangaroo

Kangaroo Puzzle: TIPS

- The only animals in this house (H) are cats (C).
- Every animal that loves to gaze at the moon (G) is suitable for a pet (P).
- When I detest (D) an animal, I avoid (A) it.
- **No** animals are carnivorous (V), **unless** they prowl at night (P).
- **No** cat (C) fails to kill mice (M).
- **No** animal ever takes to me (T), **except** those that are in this house (H).
- Kangaroos (K) are **not** suitable for pets (P).
- **None** but carnivore (V) kill mice (M).
- I detest (D) animals that do **not** take to me (T).
- Animals that prowl at night (P) always love to gaze at the moon (G).

Argue that the above statements imply I always avoid a kangaroo