COM1002:
Foundations of Computer Science

Week9 Lecture 1

Revision: MOLE QUIZ 2

WEEK	1,2,3,4,5,6,7,8	9	10	11
Mon			Lecture **Hand out ex 7** (Assessed 5%)	
Wed		**Revision Lecture** **Hand in Ex 6**	Lecture	
Thurs		**Revision Tutorial** **QUIZ 2 (25%)** Diamond 201 4pm-5pm 5pm-6pm (lecture clashes)	Tut (ex 7)	**Hand in ex 7** **QUIZ 3 (25%)** Diamond 201 4pm-5pm 5pm-6pm (lecture clashes)

Please don’t enter room until 4pm if there are signs which say do not enter (tutorial class 3pm-4pm)

QUIZ 2 - overview

6 Questions:
Composition of Functions
Simplification of a predicate statement
Proof in propositional logic, Proof in predicate logic over sets
Predicate Logic (x2) (*Express English statements as predicates*)

Thursday 26th November, 4pm-5pm
Computer Room 1 – Diamond; Duration: 50mins

Two attempts allowed – highest attempt taken.

Please notify me at start of QUIZ if you are entitled to extra time.

For documented medical related conditions (dyslexia, anxiety) a printed-handout of the QUIZ can be provided. However, answers to the quiz are still entered on MOLE.

Three functions f, g and h are defined over the set of single digit numbers denoted by D = { n $\in$ N : n < 10 }, where the sets of pairs forming the graphs of each of these functions are as given below.

f = {(0, 2), (1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (6, 8), (7, 9), (8, 0), (9, 1)}

g = {(0, 0), (1, 2), (2, 4), (3, 6), (4, 8), (5, 0), (6, 2), (7, 4), (8, 6), (9, 8)}

h = {(0, 1), (1, 4), (2, 7), (3, 0), (4, 3), (5, 6), (6, 9), (7, 2), (8, 5), (9, 8)}

What is the graph of $f \circ h \circ g$?

$$f \circ h \circ g(x) = f\Big(h\big(g(x)\big)\Big)$$

$$\begin{aligned} f \circ h \circ g(0) &= f\big(h(g(0))\big) \\ &= f\big(h(0)\big) \\ &= f(1) \\ &= 3 \end{aligned}$$

$$\begin{aligned} f \circ h \circ g(1) &= f\big(h(g(1))\big) \\ &= f\big(h(2)\big) \\ &= f(7) \\ &= 9 \end{aligned}$$

$$\begin{aligned} f \circ h \circ g(2) &= f\big(h(g(2))\big) \\ &= f\big(h(4)\big) \\ &= f(3) \\ &= 5 \end{aligned}$$

$$f \circ h \circ g = \{(0,1),(1,0),(2,5),(3,?),(4,?),(5,?),(6,?),(7,?),(8,?),(9,?)\}$$

The remaining maplets of the graph are left as an exercise…

Example: A horse that is registered for today's race is not a throughbred. Every horse registered for today's race has won a race this year.

P(x): x is registered for today's race.
Q(x): x is a thoroughbred.
R(x): x has won a race this year.
U = all horses

$$\exists x\left(P(x) \wedge \neg Q(x)\right)$$

$$\forall x\left(P(x) \Rightarrow R(x)\right)$$

Express the statements as predicates and show that we can conclude that:

A horse that has won a race this year is not a thoroughbred.

$$\exists x\left(R(x) \wedge \neg Q(x)\right)$$

$\exists x \big(P(x) \wedge \neg Q(x) \big)$	Premise
$\big(P(a) \wedge \neg Q(a) \big)$	Existential Elimination
$P(a)$	Conjunction elimination
$\neg Q(a)$	Conjunction elimination
$\forall x (P(x) \Rightarrow R(x))$	Premise
$P(a) \Rightarrow R(a)$	Universal Elimination
$R(a)$	Modus Ponens 3,6
$R(a) \wedge \neg Q(a)$	Conjunction introduction
$\exists x (R(x) \wedge \neg Q(x))$	Existential Introduction

Negating Nested Quantifiers

Let $P(x, f)$ denote that person x has taken flight f.
Let $Q(f, a)$ denote that flight f is operated by airline a.
Formulate: “There is no person who has taken a flight on every airline in the world.”

$$\neg\exists x\, \forall a\, \exists f\, (P(x, f) \wedge Q(f, a))$$

Now use De Morgan’s Laws to move the negation as far inwards as possible.

$$\begin{array}{ll}
\neg\exists x\, \forall a\, \exists f\, (P(x, f) \wedge Q(f, a)) & \\
\forall x\, \neg\forall a\, \exists f\, (P(x, f) \wedge Q(f, a)) & \text{by De Morgan's for } \exists \\
\forall x\, \exists a\, \neg\exists f\, (P(x, f) \wedge Q(f, a)) & \text{by De Morgan's for } \forall \\
\forall x\, \exists a\, \forall f\, \neg(P(x, f) \wedge Q(f, a)) & \text{by De Morgan's for } \exists \\
\forall x\, \exists a\, \forall f\, (\neg P(x, f) \vee \neg Q(f, a)) & \text{by De Morgan's for } \wedge
\end{array}$$

Negate the statements:

$$\left[\forall x\left(\left[\exists z \neg R(x,z)\right] \vee \left[\exists z\left(P(x,z) \wedge Q(x,z)\right)\right]\right)\right]$$

$$\left[\forall x\left(\left[\exists z \neg R(x,z)\right] \Rightarrow \left[\exists z\left(P(x,z) \vee Q(x,z)\right)\right]\right)\right]$$

Give examples of usage of:

(a) existence-elimination/introdction
(b) universal-elimination/introduction
(c) and-introduction
(d) implication-introduction
(e) or introduction
(f) modus ponens
(g) modus Tollens
(h) application of distributivity

Existential quantification

$$\exists x P(x)$$

The existential quantifier always means “at least one”, which means that there may be one or more of the specified thing in existence.

Unique existential quantification

$$\exists! x P(x)$$

Sometimes, it may be useful to say that there is only one. In these cases, an existential quantifier is written as $\exists!$, which means “there exists exactly one”

Universal quantification

$$\forall x P(x)$$

In predicate logic, a **universal quantification** is a type of **quantifier**, which is interpreted as "for all". It expresses that a propositional function can be satisfied by every member of a domain of discourse.

Existential quantification

Let the Universe of Discourse be the Set of first year students on COM1002

Let L(x) denote ‘x likes Brussel sprouts’

NB: Overseas students:

Eating Brussel sprouts is a strange English tradition at Christmas time which involves eating a vegetable nobody likes.

What do the following Predicate expressions mean ?

$$\exists x L(x) \qquad \exists x \neg L(x)$$

$$\neg\exists x \neg L(x) \qquad \neg\exists x L(x)$$

Universal quantification

Let L(x) denote ‘x likes Brussel sprouts’

What do the following Predicate expressions mean ?

$$\forall x L(x)$$

$$\forall x \neg L(x)$$

$$\neg \forall x L(x)$$

$$\neg \forall x \neg L(x)$$

The Frog Puzzle

1. All frogs are green.
2. Some frogs are green.
3. Not all frogs are green.
4. Some frogs are not green.
5. No frogs are green.
6. All frogs are not green.
7. Only frogs are green.
8. All and only frogs are green.
9. Kermit is a green frog.

Match up the pairs

$$\neg\forall x[F(x) \Rightarrow G(x)]$$

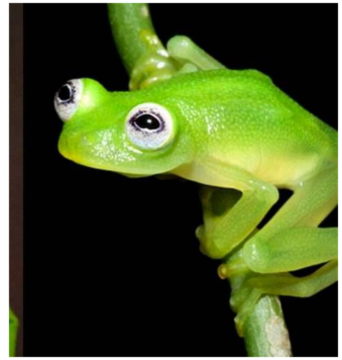
$$\forall x[G(x) \Leftrightarrow F(x)]$$

$$\exists x[F(x) \wedge G(x)]$$

$$\neg\exists x[F(x) \wedge G(x)] \quad \forall x[F(x) \Rightarrow G(x)] \quad \exists x[F(x) \wedge \neg G(x)]$$

$$\exists x[F(x) \wedge K(x)] \quad \forall x[G(x) \Rightarrow F(x)] \quad \forall x[F(x) \Rightarrow \neg G(x)]$$

All frogs are green. $\forall x[F(x) \Rightarrow G(x)]$



Some frogs are green. $\exists x[F(x) \wedge G(x)]$

Not all frogs are green. $\neg\forall x[F(x) \Rightarrow G(x)]$

Some frogs are not green. $\exists x[F(x) \wedge \neg G(x)]$

No frogs are green. $\neg\exists x[F(x) \wedge G(x)]$

All frogs are not green. $\forall x[F(x) \Rightarrow \neg G(x)]$

Only frogs are green. $\forall x[G(x) \Rightarrow F(x)]$

All and only frogs are green. $\forall x[G(x) \Leftrightarrow F(x)]$

Kermit is a green frog. $\exists x[F(x) \wedge K(x)]$

Rules for Quantification 1

Negation:

- General rules:
  - $\neg \forall x\, P(x) \Leftrightarrow \exists x \neg P(x)$, and
  - $\neg \exists x\, P(x) \Leftrightarrow \forall x \neg P(x)$;
- These are related to De Morgan's Laws:
  - compare $\forall x\, P(x)$ with $P(a) \wedge P(b) \wedge \ldots$,
  - and $\exists x\, P(x)$ with $P(a) \vee P(b) \vee \ldots$.

Rules for Quantification 2

Conjunction and Disjunction:

Equivalences:

- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$,
- $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$,
- from the same comparisons;

Examples of non-equivalences:

- assume x is an arbitrary integer: we have
- $(\exists x\ \text{Prime}(x)) \wedge (\exists x\ \text{Square}(x))$,
- but $\exists x\ (\text{Prime}(x) \wedge \text{Square}(x))$ is false;

Rules for Quantification 4

Re-ordering Quantifiers:

Equivalences:

- $\forall x\ \forall y\ P(x, y) \Leftrightarrow \forall y\ \forall x\ P(x, y)$, and
- $\exists x\ \exists y\ P(x, y) \Leftrightarrow \exists y\ \exists x\ P(x, y)$;

A non-equivalence:

- $\forall x\ \exists y\ P(x, y)$ is not equivalent to $\exists y\ \forall x\ P(x, y)$.






P for the predicate "is a politician"
H for the predicate "is honest,"

Express with Predicates:
Some politicians are honest.
Some politicians are dishonest.
All politicians are dishonest.
All dishonest people are politicians.

$$\exists x\left[P(x) \wedge H(x)\right]$$

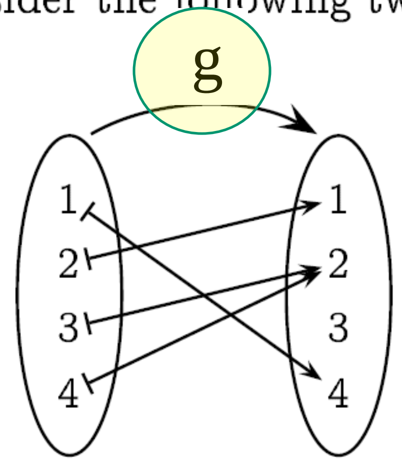
$$\exists x\left[P(x) \wedge \neg H(x)\right]$$

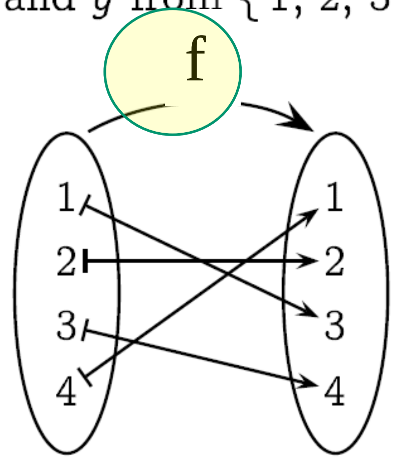
$$\forall x\left[P(x) \Rightarrow \neg H(x)\right]$$

$$\forall x\left[\neg H(x) \Rightarrow P(x)\right]$$

Week 5, L2, slide 20 - Typo

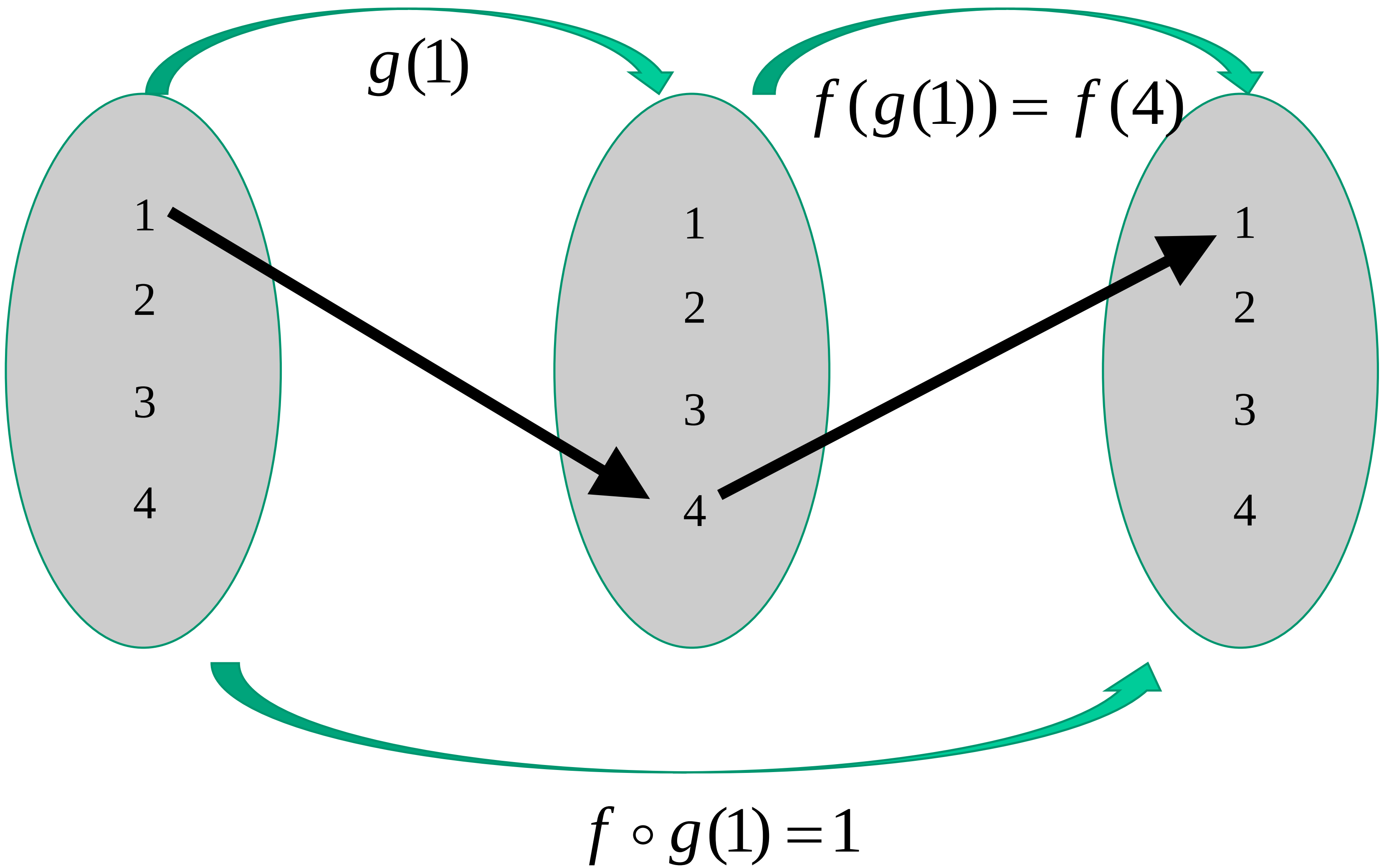
Consider the following two functions f and g from $\{1, 2, 3, 4\}$ to itself:



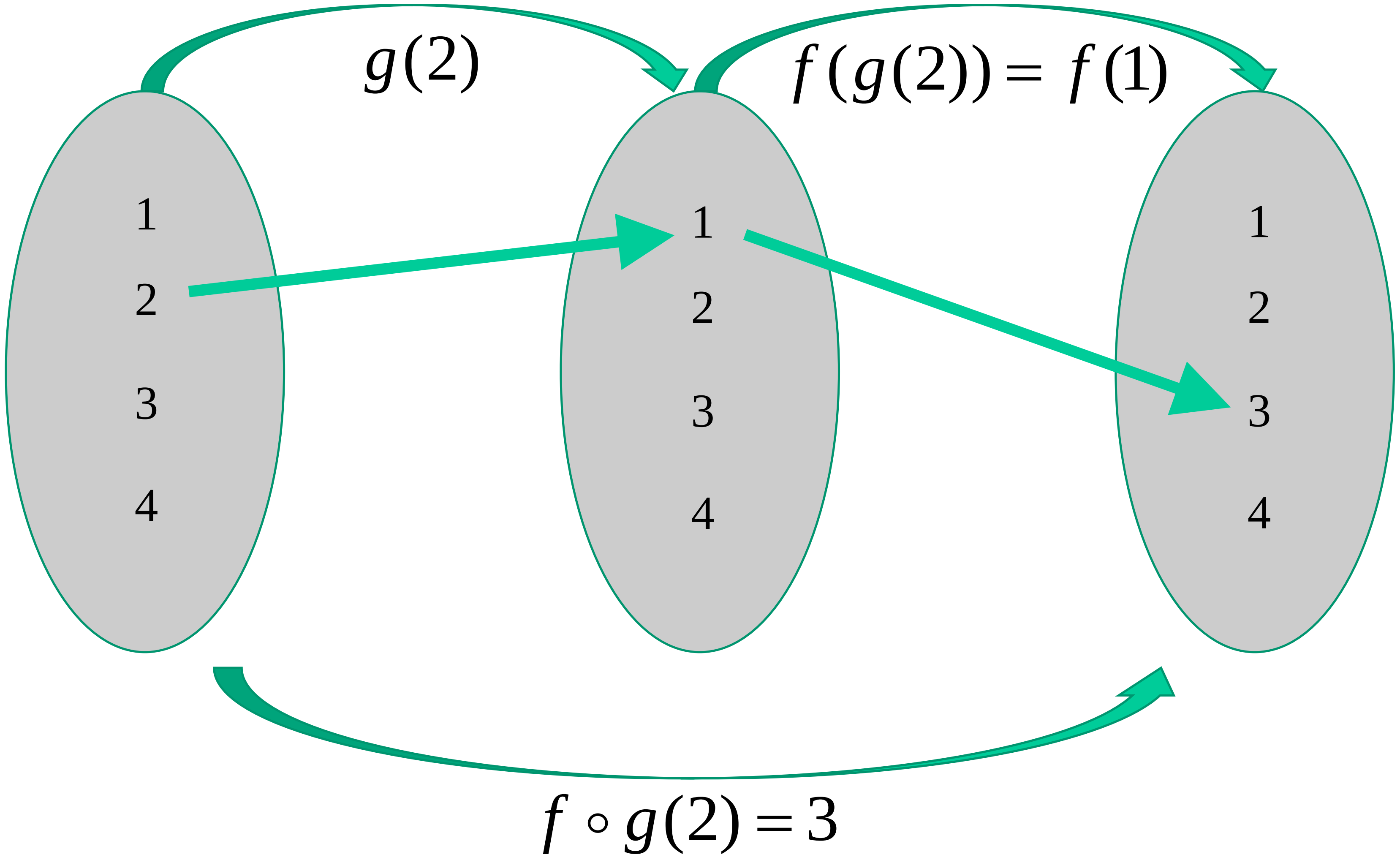


Find $f \circ g$ and $g \circ f$.

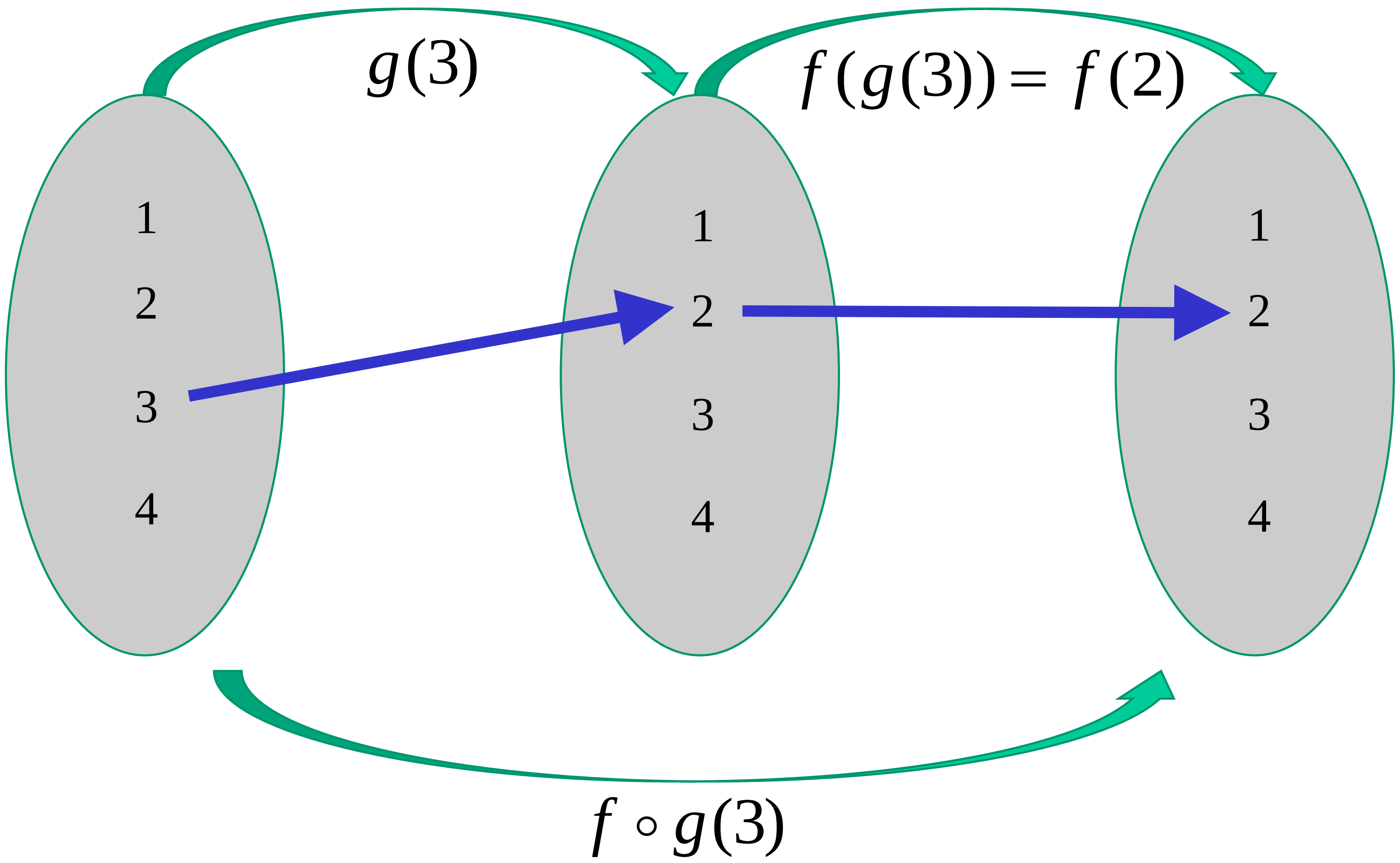
The answers on the following slides assume *g* and *f* as defined above.
(I had put *g* and *f* the other way round on the handout slides)

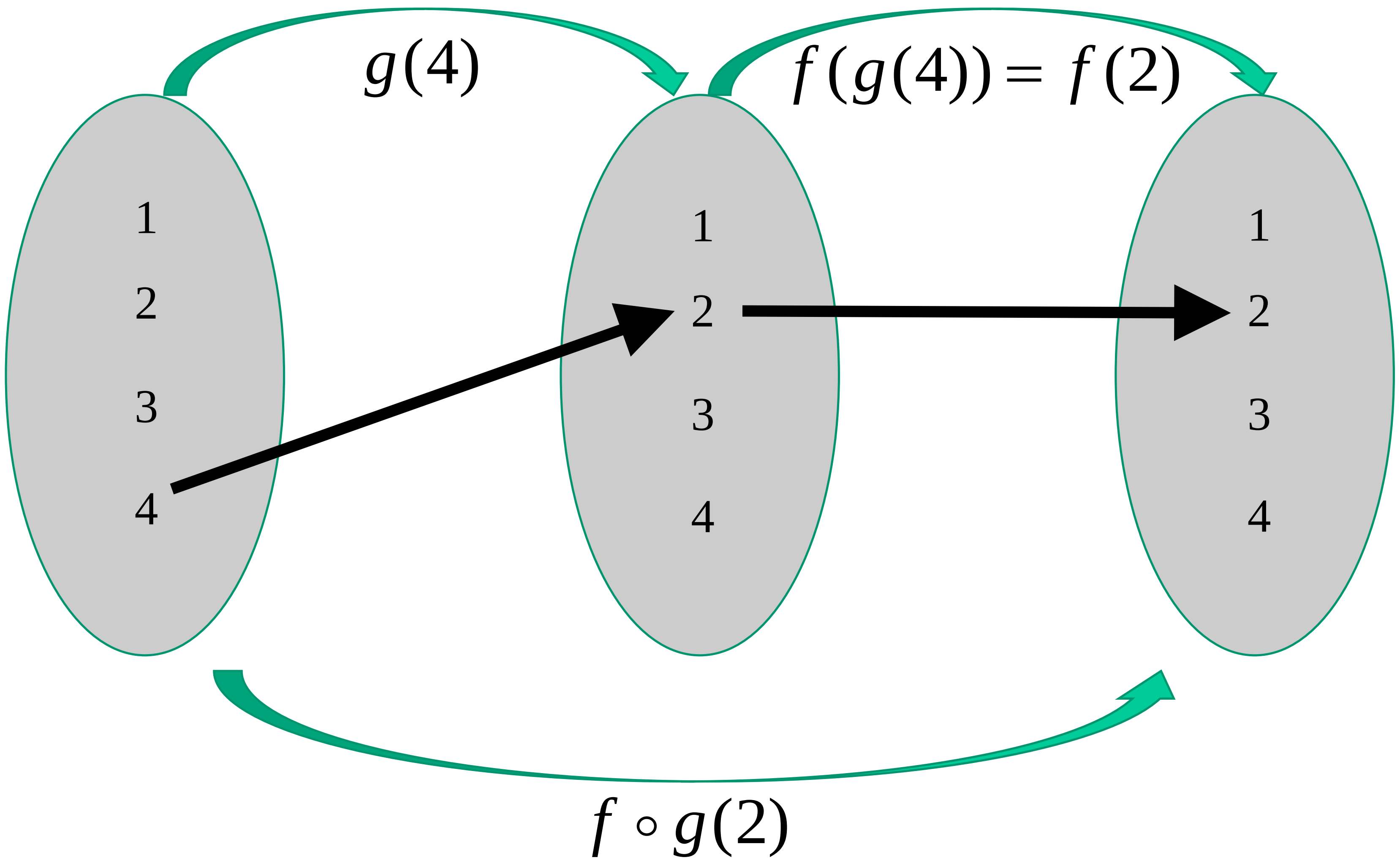


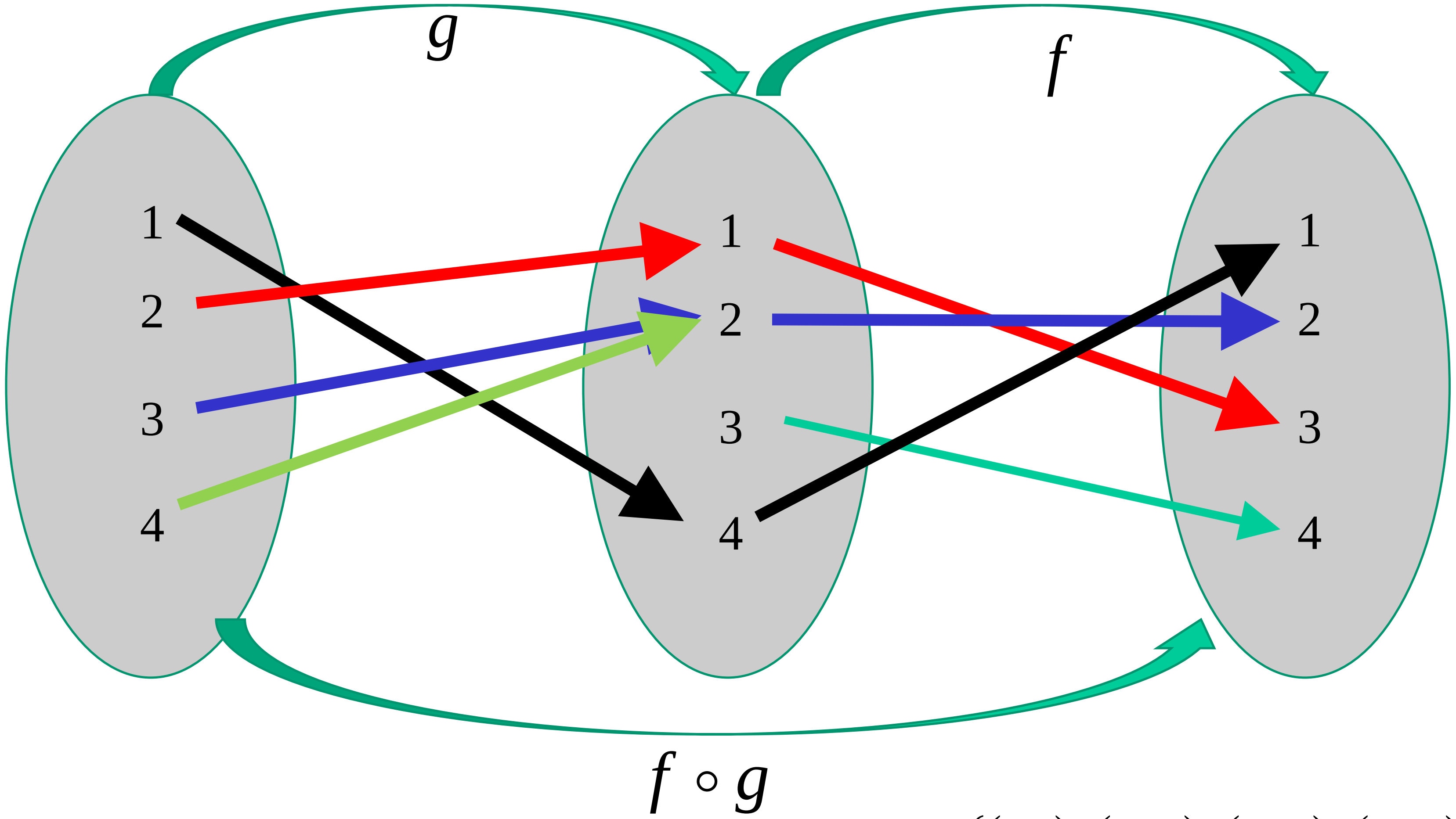
$g(1)$
$f(g(1)) = f(4)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(1) = 1$



$g(2)$
$f(g(2)) = f(1)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2) = 3$


$g(3)$
$f(g(3)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(3)$


$g(4)$
$f(g(4)) = f(2)$
1
2
3
4
1
2
3
4
1
2
3
4
$f \circ g(2)$



$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$$f = \{(1,3),(2,2),(3,4),(4,1)\} \qquad g = \{(1,4),(2,1),(3,2),(4,2)\}$$

$$\begin{aligned} f \circ g &= \{(1, f(g(1))),(2, f(g(2))),(3, f(g(3))),(4, f(g(4)))\} \\ &= \{(1, f(4)),(2, f(1)),(3, f(2)),(4, f(2))\} \\ &= \{(1,1),(2,3),(3,2),(4,2)\} \end{aligned}$$

$$f \circ g = \{(1,1),(2,3),(3,2),(4,2)\}$$

$f=\{(1,3),(2,2),(3,4),(4,1)\}$ $\quad$ $g=\{(1,4),(2,1),(3,2),(4,2)\}$

$$
\begin{aligned}
g\circ f &= \{(1,g(f(1))),(2,g(f(2))),(3,g(f(3))),(4,g(f(4)))\}\\
&= \{(1,g(3)),(2,g(2)),(3,g(4)),(4,g(1))\}\\
&= \{(1,2),(2,1),(3,2),(4,4)\}
\end{aligned}
$$

$$g\circ f=\{(1,2),(2,1),(3,2),(4,4)\}$$