1: A dimensional analysis on the Anti-Derivative

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1 Introduction

In this paper, I hope to explore a new way of thinking about calculus. Rather than purely thinking about these operations (the derivative and the anti-derivative) in a numerical or graphical sense, thinking about them in a dimensional sense might give way to a range of new ideas as well as allow for a deeper understanding of these operations; such an understanding would be beneficial to students more curious about the why than the how. Perhaps information uncovered in this paper may lead to more advanced research in other fields, namely topology.

2 Background

The derivative is defined as the slope of a tangent line at a function $f(x)$. This makes perfect sense as we can easily visualize how the rate of change of different points of a graph, gives us a new function. The Anti-Derivative However,

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gives us the area under $f(x)$. Knowing that the Anti-Derivative(Integral) is the opposite of the derivative makes it even more complicated, as it's hard to relate(especially geometrically) why the opposite of slope is area.

3 Crossing into higher dimensions

Throughout this paper, I will only be using the power rule and reverse power rule to keep the operations simple. When taking the anti-derivative, we may use this equation $\int ax^n dx = \frac{ax^{n+1}}{n+1}$. We notice that x will always go up 1 degree higher. We can construct each of x's degrees as its current dimension. For example, x^2 is a square consisting of sides of length x; x^3 is a cube consisting of sides with length x. One way to convey this geometrically is to represent each "dimensional crossing" as the minimum number of new lines added to make the function or object $n+1$ dimensional.

Suppose we have the function

$$f(x) = 2x$$

This can be read as "two x" or "two lines" - the coefficient needs to be separated from the main function to indicate how many of the n-dimensional geometries are present.

Figure 1: A representation of the function $f(x) = 2x$ as two lines in 1 dimension

Knowing that there must be an increase of 1 dimension higher, we may add a new side of x along the new dimension y. This must be done to each of the individual x's as there were originally 2. We obtain a triangular shape

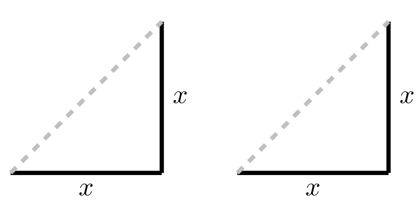

Figure 2: A representation of the function $2(\frac{x^2}{2})$ as two planes

by adding the minimum number of lines required to create an n-dimensional object, ensuring that the lines are connected in a way that takes up space in that dimension. To calculate the area of a triangle, we use the formula $\frac{bh}{2}$. For our current object, the area is $2(\frac{x^2}{2})$, which simplifies to x^2.

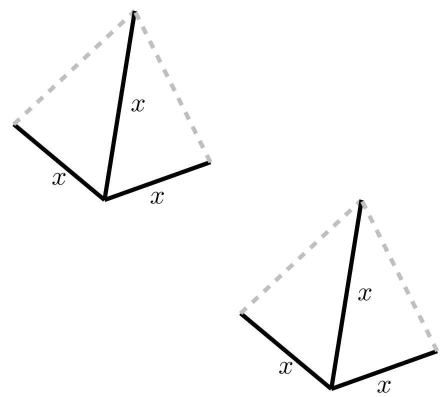

Figure 3: Two triangular pyramids formed by expanding $2(\frac{x^2}{2})$ into the third dimension.

When we expand into the third dimension, we add one new expansion of length x along the new axis, connecting each endpoint along the expansion to all the other points with a straight line. Following this rule, we obtain two triangular pyramids from the previous function $(2(\frac{x^2}{2}))$. By using the formula $\frac{bh}{3}$ with a base of $\frac{x^2}{2}$, we find that each triangular pyramid takes up a 3-dimensional space of $\frac{x^3}{6}$, or a combined space of $\frac{x^3}{3}$. This function matches the calculated antiderivative of $\frac{x^3}{3}$. Therefore, we can conclude that the antiderivative of a function gives us the space in the $n+1$ dimension that an object occupies.

3.1 Polynomials

The same principles apply to polynomials, but instead of taking the whole function as the object, the object would have to be split into parts. For example, $f(x) = x^2 + 4x + 3$ can be read as "a square, 4 lines, and 3 points".

4 Reverting to lower dimensions

The derivative of a function is commonly understood as the slope of a tangent line at any point within that function. In the geometric context, taking the derivative is analogous to moving down one dimension. This process essentially destroys the main line of the expansion and erases stray connections to endpoints.

4.1 Expansions

When taking the derivative in this geometric format, we observe an interesting pattern in each function's coefficient. Consider the function $f(x) = x^2$ shown in Figure 2. Taking the derivative (moving down one dimension) brings us back to the original function's coefficient, which in this case is 2. Surprisingly, 2 is also the number of ways to manipulate a line x to form a square.

Figure 4: possible expansions for x^2

As shown in Figure 4, a line can be manipulated in two ways (horizontally and vertically) to form a square. This can be accomplished by infinitely stacking the lines close together until a width of size x is formed. This process is analogous to the dx commonly seen at the end of a derivation and applies to all dimensions. Similarly, the function x^3 with the derivative $3x^2$ can be represented with infinitely stacked squares stretched along three axes (x, y, and z).

Postulate 4.1 *From the basic monomial ax^n, let ψ denote the number of $n-1$ expansions needed to form the current n dimensional object.$\psi = an$.*

Another way to think of ψ is the number of ways an n dimensional object can move/translate itself along a $n-1$ dimensional plane.

Note 1: The equation applies to monomials only, but the underlying concept applies to polynomials

5 Graphs

Interestingly, approaching the functions in this manner gives us more insight into the properties of higher-dimensional objects. The integral of $x^3 (\frac{x^4}{4})$ reads as "one cube provides 1/4 of the needed expansions to make a 4-dimensional object".
Looking at the graphs of higher dimensional functions provides further insight into the characteristics of these objects.

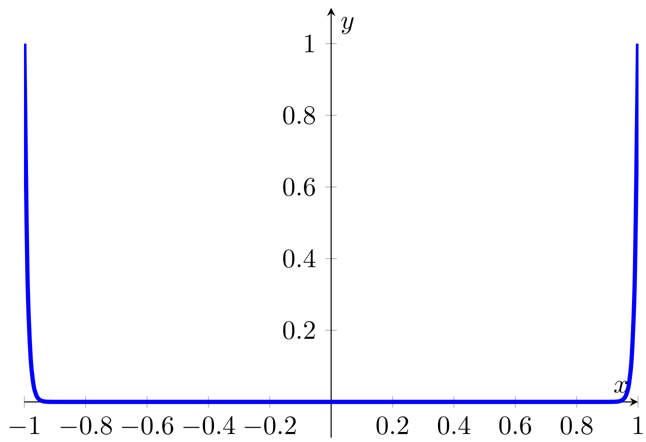

Figure 5: Graph of $f(x) = x^{100}$

A 100-dimensional object occupies negligible space when its side lengths are less than 1. This makes sense as it would take 100 99-dimensional expansions with each taking 99 98-dimensional expansions–9,900 98-dimensional expansions in total. This pattern continues until you reach the 0th dimension. With this knowledge, we can formulate a more comprehensive definition of ψ.

Postulate 5.1

$$\psi_s = a\frac{n!}{s!}$$

where "s" represents the dimension being targeted.

In the case of the function x^{100}, $\psi_{98} = 1\frac{100!}{98!} = 9900$. This reads as "it takes 9,900 98-dimensional expansions to form a 100-dimensional object."

6 Conclusion

Armed with this knowledge, we start to see relationships between the space objects take up and their expansions. The surface area of a cube is just double its 2-dimensional expansions($2\psi x^{\psi-1}$ or $2\psi_2 x^{\psi_2 - 1}$), the perimeter of a square is double its 1-dimensional expansions, and the number of points to form a line is double its 0-dimensional expansions. Perhaps more could be done to uncover more relationships between more complex shapes such as circular or curved ones. A deeper look may also provide insight into how objects act in dimensions greater than the third.
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[1] This is the first chapter, for I hope to publish future chapters discussing its applications with circular geometry and perhaps fractal dimensions