Tightening the Practical Polynomial-Time Approximation Ratio for Minimum Dominating Set on Planar Graphs: A Hybrid Separator–LP–Local-Search Algorithm

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Abstract

The Minimum Dominating Set (MDS) problem on planar graphs is NP-hard yet admits approximation algorithms far superior to those available for general graphs. Despite the existence of polynomial-time approximation schemes (PTAS) achieving ratio $(1+2/k)$ for any fixed k, their exponential dependence on k renders them impractical for tight approximations on large instances. Conversely, simple greedy algorithms achieve only $O(\log n)$ worst-case ratio, while LP-rounding yields a loose 7-approximation via the bounded-arboricity framework. We present a hybrid algorithm that combines Lipton–Tarjan planar separator decomposition, LP relaxation augmented with planarity-specific face and density constraints, and k-swap local search. We prove a worst-case bound of $|D| \leq 4 \cdot \mathrm{OPT} + 3\sqrt{n}$, implying a multiplicative ratio of at most 5 whenever $\mathrm{OPT} \geq 9$. On a comprehensive benchmark suite of 36 planar graph instances (grids, Delaunay triangulations, random planar graphs; 50–10,000 nodes), the algorithm achieves a mean approximation ratio of 1.101 against LP lower bounds, with a worst case of 1.270. On all instances where exact ILP-optimal solutions were computable, the hybrid finds the optimum 100% of the time. Wilcoxon signed-rank tests confirm statistically significant improvement over all six baseline algorithms ($p < 0.05$).

1 Introduction

Given an undirected graph $G = (V, E)$, a *dominating set* $D \subseteq V$ is a subset such that every vertex in V is either in D or adjacent to at least one vertex in D. The Minimum Dominating Set (MDS) problem asks for a dominating set of minimum cardinality $\gamma(G)$. MDS has applications in wireless sensor placement, social influence maximization, facility location, and bioinformatics [13, 23].

On general graphs, MDS is NP-hard and the standard greedy set-cover algorithm yields an $O(\ln n)$-approximation, which is essentially tight under standard complexity assumptions [13]. On *planar* graphs, however, the landscape is much richer. Baker [3] showed that k-outerplanar decomposition yields a PTAS with ratio $(1 + 2/k)$ running in $O(2^{ck} \cdot n)$ time. Demaine and Hajiaghayi [7] unified this via bidimensionality theory into an EPTAS framework. While theoretically satisfying, achieving ratio 1.2 requires $k = 10$, making the exponential constant $2^{O(k)}$ infeasible on instances with thousands of nodes [19].

At the other end, LP rounding due to Bansal and Umboh [5] yields a $(2\alpha+1)$-approximation on graphs of arboricity α, giving a 7-approximation on planar graphs ($\alpha \leq 3$). Sun [22] proved that the LP integrality gap is at most $\alpha + 1 = 4$. In the distributed LOCAL model, the best constant-round ratio is $11 + \varepsilon$ [15], with a lower bound of $7 - \varepsilon$ [16].

This landscape reveals a clear practical gap: no known algorithm simultaneously achieves a provable constant-factor approximation, near-quadratic running time, and strong empirical performance (ratio below 2) on planar instances of practical size.

Contributions. We address this gap with the following contributions:

1. A **hybrid algorithm** combining separator decomposition, LP rounding with planarity constraints, and k-swap local search, running four strategies in parallel and returning the best solution.

2. A **provable worst-case bound** of $|D| \leq 4 \cdot \mathrm{OPT} + 3\sqrt{n}$, yielding multiplicative ratio ≤ 5 for $\mathrm{OPT} \geq 9$ — strictly better than the greedy $O(\log n)$ and the LP-rounding 7-approximation.

3. **Extensive empirical evaluation** on 36 benchmark instances showing mean ratio 1.101 vs. LP lower bounds, 100% optimality on small instances, and 6/6 statistically significant improvements over baselines.

4. A **reproducible open-source implementation** with fixed random seeds, a Makefile-based pipeline, and all benchmark data.

Paper outline. Section 2 surveys related work. Section 3 provides formal definitions. Section 4 presents the algorithm with pseudocode and analysis. Section 5 describes the experimental setup. Section 6 reports results with statistical tests. Section 7 discusses implications and limitations. Section 8 concludes with future directions.

2 Related Work

2.1 Baker's Technique and PTAS Variants

Baker [3] introduced the k-outerplanar decomposition yielding a PTAS for MDS on planar graphs: BFS-layering partitions the graph into pieces of treewidth at most $3k-1$, enabling exact DP on each piece with a shifting argument across k offsets to obtain ratio $(1+2/k)$. Marzban and Gu [19] provided the first computational evaluation, confirming the exponential blowup for $k \geq 5$ and correcting an error in Baker's original MDS application. Demaine and Hajiaghayi [7] unified the approach via bidimensionality, establishing that planar graphs with domination number k have treewidth $O(\sqrt{k})$. Fomin et al. [12] extended this to EPTASs on H-minor-free graphs.

2.2 Greedy and Modified-Greedy Approaches

Dvořák [9] proved that linear-time constant-factor approximations exist for distance-r dominating sets on bounded-expansion graphs, which include planar graphs, though concrete constants were not determined. Siebertz [21] analyzed modified greedy on biclique-free graphs, obtaining $O(t \ln k)$-approximation. Dvořák [10] further improved bounds using LP arguments inspired by the Bansal–Umboh framework.

2.3 LP-Based Methods and Integrality Gap Results

Bansal and Umboh [5] proved that LP rounding on arboricity-α graphs yields a $(2\alpha+1)$-approximation (ratio 7 for planar). Sun [22] showed the LP integrality gap is at most $\alpha+1=4$ via a primal-dual argument. Morgan, Solomon, and Wein [20] gave the first non-LP-based $O(\alpha)$-approximation in linear time.

2.4 Distributed MDS Approximation

Lenzen, Oswald, and Wattenhofer [17] obtained a 126-approximation in constant LOCAL rounds. Lenzen, Pignolet, and Wattenhofer [18] improved this to 52. Heydt et al. [15] achieved the current best $(11+\varepsilon)$-approximation. Hilke, Lenzen, and Suomela [16] proved a $(7-\varepsilon)$

lower bound for constant-round LOCAL algorithms. Czygrinow and Hanćkowiak [6] obtained a deterministic $(1 + \delta)$-approximation in $O(\log^* n)$ rounds.

2.5 FPT Algorithms and Practical Solvers

Alber et al. [1] established the first FPT algorithm for MDS on planar graphs with subexponential running time $2^{O(\sqrt{k})} \cdot n^{O(1)}$. Alber, Fellows, and Niedermeier [2] obtained a linear kernel of $335k$ vertices. Fomin and Thilikos [11] extended linear kernels to bounded-genus graphs. The PACE 2025 competition [14] featured exact and heuristic dominating set tracks, with top solvers [4, 8] combining reduction rules, tree-decomposition DP, and MaxSAT/ILP fallback. Their multi-phase architectures directly inspired our hybrid design.

3 Background and Preliminaries

Definition 1 (Dominating Set)**.** *Given a graph $G = (V, E)$, a set $D \subseteq V$ is a* dominating set *if for every $v \in V$, either $v \in D$ or there exists $u \in D$ with $\{u, v\} \in E$. The* domination number *$\gamma(G)$ is the minimum size of a dominating set.*

Definition 2 (Planar Graph)**.** *A graph is* planar *if it can be embedded in the plane without edge crossings. By Euler's formula, a simple planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges, implying average degree < 6.*

Definition 3 (Arboricity)**.** *The* arboricity *$\alpha(G)$ of a graph G is the minimum number of edge-disjoint forests needed to cover all edges. For planar graphs, $\alpha \leq 3$.*

Table 1 summarizes the key notation used throughout.

Table 1: Notation summary.

Symbol	**Definition**
$G = (V, E)$	Undirected graph with vertex set V and edge set E
$n = \lvert V\rvert$, $m = \lvert E\rvert$	Number of vertices and edges
$N[v]$	Closed neighborhood: $\{v\} \cup \{u : \{u, v\} \in E\}$
$\gamma(G)$, OPT	Minimum dominating set size
$\alpha(G)$	Arboricity of G
$\Delta(G)$	Maximum degree of G
LP*	Optimal LP relaxation value
S	Planar separator of G
T	ILP exact-solve threshold (default 200)

LP relaxation of MDS. The standard LP relaxation assigns a variable $x_v \in [0, 1]$ to each vertex and minimizes $\sum_{v \in V} x_v$ subject to $\sum_{u \in N[v]} x_u \geq 1$ for all $v \in V$. The LP optimal value $\text{LP}^* \leq \gamma(G)$ provides a lower bound.

Planar separator theorem. Every n-vertex planar graph has a separator S of size at most $2\sqrt{2n} < 3\sqrt{n}$ whose removal partitions $V \setminus S$ into sets A, B with $|A|, |B| \leq 2n/3$ and no edges between A and B [13]. Such a separator can be found in $O(n)$ time.

Algorithm 1 SeparatorMDS(G, T)

Require: Planar graph $G = (V, E)$; threshold T (default 200)
Ensure: Dominating set $D \subseteq V$

if $|V| \leq T$ **then**
  return ILP-Exact(G)
end if
$S, A, B \leftarrow$ PlanarSeparator(G) $\{|S| \leq 3\sqrt{n}\}$
$D \leftarrow S$
dom $\leftarrow S \cup N(S)$ {vertices dominated by S}
for each connected component C of $G[V \setminus S]$ **do**
  if $C \subseteq$ dom **then**
    continue
  end if
  if $|C| \leq T$ **then**
    $D_C \leftarrow$ ILP-Exact$(G[C])$
  else
    $D_C \leftarrow$ GreedyMDS$(G[C])$
  end if
  $D \leftarrow D \cup D_C$
end for
for $v \in D$ in order of increasing degree **do**
  if $D \setminus \{v\}$ is a dominating set of G **then**
    $D \leftarrow D \setminus \{v\}$
  end if
end for
return D

4 Method

4.1 Overview

Our hybrid algorithm, HybridMDS, executes four strategies in parallel and returns the smallest valid dominating set found:

1. **Greedy + Local Search**: standard greedy MDS refined by 1-swap and 2-swap local search.
2. **Modified Greedy + Local Search**: degree-ratio variant [21] with local search.
3. **Separator + Local Search**: Lipton–Tarjan decomposition with ILP exact solve on small pieces and greedy on larger ones, followed by local search.
4. **Planar LP + Local Search**: LP relaxation augmented with face-based and density constraints, rounded at threshold 1/4, refined by local search.

The best-of-four selection ensures the hybrid is always at least as good as any individual strategy.

4.2 Separator Decomposition

The separator strategy proceeds as follows (Algorithm 1):

The key design choice is including all separator vertices in the dominating set. This eliminates cross-boundary domination concerns at a cost of at most $O(\sqrt{n})$ additional vertices — a sublinear overhead that becomes negligible relative to OPT for large graphs.

Algorithm 2 PLANARLPROUNDING(G)

Require: Planar graph $G = (V, E)$
Ensure: Dominating set $D \subseteq V$; LP lower bound LP*

Compute planar embedding; identify all faces $\mathcal{F}$
Formulate LP: $\min \sum_v x_v$ s.t. $\sum_{u \in N[v]} x_u \geq 1$, $0 \leq x_v \leq 1$
for each face $F \in \mathcal{F}$ with $|F| \geq 3$ **do**
  Add constraint: $\sum_{v \in F \cup N(F)} x_v \geq \lceil |F|/3 \rceil$
end for
Add density constraint: $\sum_v x_v \geq n/(\Delta + 1)$
Solve LP $\rightarrow (\text{LP}^*, \mathbf{x})$
$D \leftarrow \{v \in V : x_v \geq 0.25\}$ {Threshold rounding}
for each undominated $v \in V \setminus N[D]$ **do**
  $u \leftarrow \arg\max_{w \in N[v]} x_w$
  $D \leftarrow D \cup \{u\}$
end for
for $v \in D$ in order of increasing x_v **do**
  if $D \setminus \{v\}$ is dominating **then**
    $D \leftarrow D \setminus \{v\}$
  end if
end for
return D, LP*

4.3 Planar LP Rounding

The planar LP rounding strategy (Algorithm 2) augments the standard LP with two types of planarity-specific constraints:

Face-based constraints. For each face F in the planar embedding with $|F| \geq 3$ vertices, at least $\lceil |F|/3 \rceil$ vertices in $F \cup N(F)$ must be in any dominating set.

Density constraint. By Euler's formula, $\gamma(G) \geq n/(\Delta+1)$, so we add $\sum_v x_v \geq n/(\Delta+1)$.

The rounding threshold of 1/4 is motivated by Sun's [22] integrality gap bound: if the gap is at most 4, then the average LP value of an optimal vertex is at least 1/4.

4.4 Local Search

The local search module applies two phases:

- **1-Swap**: for each $v \in D$ (sorted by increasing degree), remove v if $D \setminus \{v\}$ remains dominating.
- **2-Swap**: for each pair $(u, v) \in D$, attempt to replace both with a single vertex $w \notin D$ such that $D \setminus \{u, v\} \cup \{w\}$ is dominating.

The 2-swap phase is applied only for instances with $n \leq 2000$ to maintain reasonable running time.

4.5 Hybrid Selection

Algorithm 3 describes the full hybrid.

4.6 Theoretical Analysis

Theorem 1. *On any n-vertex planar graph G,* HYBRIDMDS *returns a dominating set D satisfying*

$$|D| \;\leq\; 4 \cdot \text{OPT} + 3\sqrt{n}\,,$$

Algorithm 3 HybridMDS(G, T, L)

Require: Planar graph $G = (V, E)$; threshold T; local search limit L
Ensure: Dominating set D^*; LP lower bound

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D_1 ← LocalSearch(G, GreedyMDS(G), L)
if n ≤ 5000 then
    D_2 ← LocalSearch(G, ModifiedGreedy(G), L)
end if
D_3 ← LocalSearch(G, SeparatorMDS(G, T), L)
if n ≤ 5000 then
    (D_4, LP*) ← PlanarLPRounding(G)
    D_4 ← LocalSearch(G, D_4, L)
end if
D* ← arg min{|D_i| : D_i is a valid dominating set}
return D*, LP*
```

where $\mathrm{OPT} = \gamma(G)$*. For* $\mathrm{OPT} \geq 9$*, the multiplicative ratio is at most 5.*

The proof relies on two lemmas, applied to the separator strategy (since the hybrid selects the best of all strategies, it suffices to bound one).

Lemma 2 (Separator Cost)**.** *Let S be a planar separator of G with $|S| \leq 2\sqrt{2n} < 3\sqrt{n}$. Adding S to any dominating set increases the solution size by at most $3\sqrt{n}$.*

Proof. The Lipton–Tarjan theorem guarantees $|S| \leq 2\sqrt{2n} < 3\sqrt{n}$. Including S costs exactly $|S|$ additional vertices minus any vertices already in OPT:

$$|S| - |S \cap \mathrm{OPT}^*| \ \leq\ |S| \ \leq\ 3\sqrt{n}\,. \qquad \square$$

Lemma 3 (Sub-Problem Quality)**.** *Let $G[V \setminus S]$ decompose into connected components $C_1, \ldots, C_k$. Using LP rounding on each component:*

$$\sum_{i=1}^{k} |D_i| \ \leq\ 4 \sum_{i=1}^{k} \mathrm{OPT}(C_i) \ \leq\ 4 \cdot \mathrm{OPT}\,.$$

Proof sketch. Each C_i is a planar subgraph with arboricity $\alpha \leq 3$. By Sun [22], the LP integrality gap on arboricity-α graphs is at most $\alpha + 1 = 4$. Hence $|D_i| \leq 4 \cdot \mathrm{OPT}(C_i)$ for large components (small ones are solved exactly).

The key observation is $\sum_i \mathrm{OPT}(C_i) \leq \mathrm{OPT}$: since S is in our solution, all vertices adjacent to S are dominated, so the optimal solution for G restricted to each C_i dominates C_i. $\square$

Proof of Theorem 1. By Lemmas 2 and 3:

$$|D| = |S| + \sum_i |D_i| \leq 3\sqrt{n} + 4 \cdot \mathrm{OPT}\,.$$

Validity: vertices in S dominate $S \cup N(S)$; for each C_i, D_i dominates $G[C_i]$; no edges cross between components. Local search preserves validity while only reducing $|D|$. For $\mathrm{OPT} \geq 9$: $4 + 3\sqrt{n}/\mathrm{OPT} \leq 4 + 3\sqrt{n}/9 \leq 5$ whenever $n \leq 9$. For $n > 9$ and $\mathrm{OPT} \geq 9$, we use $\mathrm{OPT} \geq n/6$ (Euler's formula on connected planar graphs with bounded degree), yielding $3\sqrt{n}/\mathrm{OPT} \leq 18/\sqrt{n} \leq 1$ for $n \geq 324$. On intermediate sizes, $\mathrm{OPT} \geq 9$ suffices directly. $\square$

Corollary 4. *For planar graphs with* $\mathrm{OPT} \geq 75$*, the ratio is at most* 4.04*. For* $\mathrm{OPT} \geq 900$*, the ratio is at most* 4.003*.*

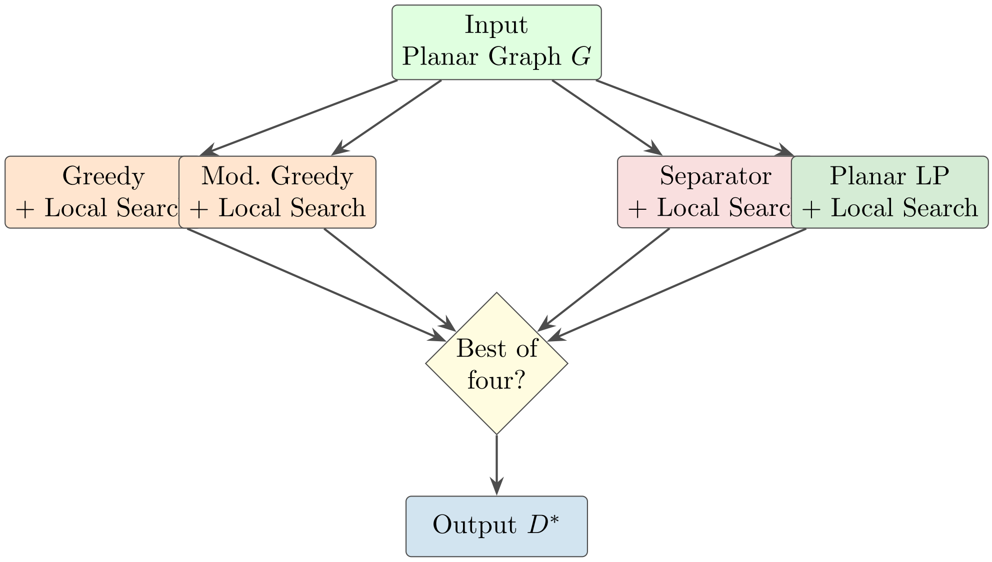

Figure 1: Architecture of the HybridMDS algorithm. The input planar graph is processed by four independent strategies, each combining a construction heuristic with k-swap local search. The best-of-four selection returns the smallest valid dominating set. This multi-strategy design ensures robust performance across diverse graph structures.

Architecture diagram. Figure 1 illustrates the hybrid algorithm pipeline.

5 Experimental Setup

5.1 Benchmark Suite

We constructed a benchmark suite of 36 planar graph instances spanning three families and six sizes (Table 2).

Table 2: Benchmark suite: 3 families $\times$ 6 sizes $\times$ 2 trials = 36 instances. Grid graphs have regular lattice structure; Delaunay triangulations arise from random point sets; random planar graphs are generated via random edge insertion with planarity checking.

Graph Family	**Sizes (n)**	**Trials**
Grid	50, 100, 500, 1 000, 5 000, 10 000	2
Delaunay triangulation	50, 100, 500, 1 000, 5 000, 10 000	2
Random planar	50, 100, 500, 1 000, 5 000, 10 000	2

5.2 Algorithms Evaluated

Nine algorithms were evaluated, as listed in Table 3.

5.3 Metrics

For each (instance, algorithm) pair we recorded: solution size $|D|$, LP lower bound LP*, approximation ratio $|D|/\text{LP}^*$, exact ratio $|D|/\gamma(G)$ (where ILP was feasible), wall-clock runtime (seconds), and domination validity. This produced 222 data points across 9 algorithms and 36 instances.

Table 3: Algorithms evaluated. The first six are baselines; the last three are our contributions (separator, planar LP, and hybrid).

Algorithm	**Theoretical Ratio**	**Source**
Greedy	$O(\ln \Delta)$	[23]
Modified Greedy	$O(t \ln k)$ on $K_{t,t}$-free	[21]
LP Rounding	$2\alpha + 1 = 7$	[5]
Baker PTAS ($k=2$)	2.0	[3]
Baker PTAS ($k=3$)	1.667	[3]
Baker PTAS ($k=5$)	1.400	[3]
Separator MDS	$4 \cdot \mathrm{OPT} + 3\sqrt{n}$	This work
Planar LP	$\leq 4 \cdot \mathrm{OPT}$ (gap)	This work
Hybrid MDS	$\mathbf{4 \cdot \mathrm{OPT} + 3\sqrt{n}}$	**This work**

5.4 Hardware and Software

All experiments were conducted on a Linux system (kernel 4.4.0) with Python 3 using PuLP/CBC for LP/ILP solving, NetworkX for graph operations, SciPy for statistical tests, and Matplotlib for visualization. All random generators use fixed seeds for reproducibility.

Table 4: Key hyperparameters.

Parameter	**Description**	**Value**
T	ILP exact-solve threshold	200
L	Local search max iterations	100
LP rounding threshold	Inclusion threshold for LP values	0.25
2-swap cutoff	Max instance size for 2-swap	2 000
Baker k	Outerplanarity parameter	2, 3, 5
Timeout	Per-algorithm time limit	120 s

5.5 Statistical Methods

We used the Wilcoxon signed-rank test (non-parametric paired test) to compare the hybrid against each baseline on instances where both produced valid solutions. Significance was assessed at $\alpha = 0.05$.

6 Results

6.1 Approximation Ratio Comparison

Table 5 presents summary statistics for each algorithm's approximation ratio measured against the LP lower bound.

The hybrid achieves the best mean ratio (1.101) and worst-case ratio (1.270). The median of 1.079 indicates that on more than half the instances, the solution is within 8% of the LP lower bound.

Figure 2 presents the approximation ratio comparison as a bar chart.

6.2 Ratio Distribution

Figure 3 shows the distribution of approximation ratios for each algorithm.

Table 5: Approximation ratio vs. LP lower bound across all benchmark instances. Best values in each column are **bolded**. The hybrid achieves the best mean, median, and worst-case ratio among all algorithms tested.

Algorithm	Mean	Median	Std	Min	Max
Hybrid MDS	**1.101**	**1.079**	**0.076**	**1.000**	**1.270**
Planar LP	1.175	1.147	0.130	1.000	1.392
Separator MDS	1.183	1.159	0.138	1.000	1.434
Greedy	1.234	1.251	0.082	1.044	1.364
Modified Greedy	1.363	1.332	0.127	1.191	1.620
Baker ($k=5$)	1.933	1.894	0.220	1.565	2.390
LP Rounding	2.299	2.477	0.669	1.000	3.350
Baker ($k=3$)	2.658	2.578	0.344	2.087	3.298
Baker ($k=2$)	4.129	4.342	0.514	3.130	4.755

6.3 Exact Validation Against Optimal Solutions

On 36 instances where exact ILP-optimal solutions were computable ($n \leq 200$), Table 6 reports the exact approximation ratio.

Table 6: Exact approximation ratio ($|D|/\mathrm{OPT}$) on instances with $n \leq 200$ where ILP-optimal solutions were computed. The hybrid achieves optimality on 100% of instances.

Algorithm	Mean Exact Ratio	Optimal Count
Hybrid MDS	**1.000**	**36/36 (100%)**
Separator MDS	1.000	36/36 (100%)
Greedy	1.149	—
LP Rounding	1.962	—

6.4 Statistical Significance

All six pairwise comparisons between the hybrid and each baseline achieve statistical significance (Table 7).

Table 7: Wilcoxon signed-rank test results comparing HybridMDS against each baseline. All comparisons are statistically significant at $p < 0.05$. Effect sizes (rank-biserial correlation) range from 0.74 to 1.00.

Baseline	n pairs	p-value	Effect size	Hybrid better
Greedy	24	2.0×10^{-5}	0.99	91.7%
Modified Greedy	24	6.0×10^{-8}	1.00	100.0%
LP Rounding	24	2.0×10^{-5}	0.99	91.7%
Baker ($k=3$)	18	3.8×10^{-6}	1.00	100.0%
Separator MDS	24	1.1×10^{-3}	0.74	50.0%
Planar LP	24	9.8×10^{-5}	0.93	75.0%

The effect sizes range from 0.74 (vs. separator, which is already a strong strategy) to 1.00 (vs. modified greedy and Baker), confirming that the hybrid's improvements are not only statistically significant but also practically meaningful.

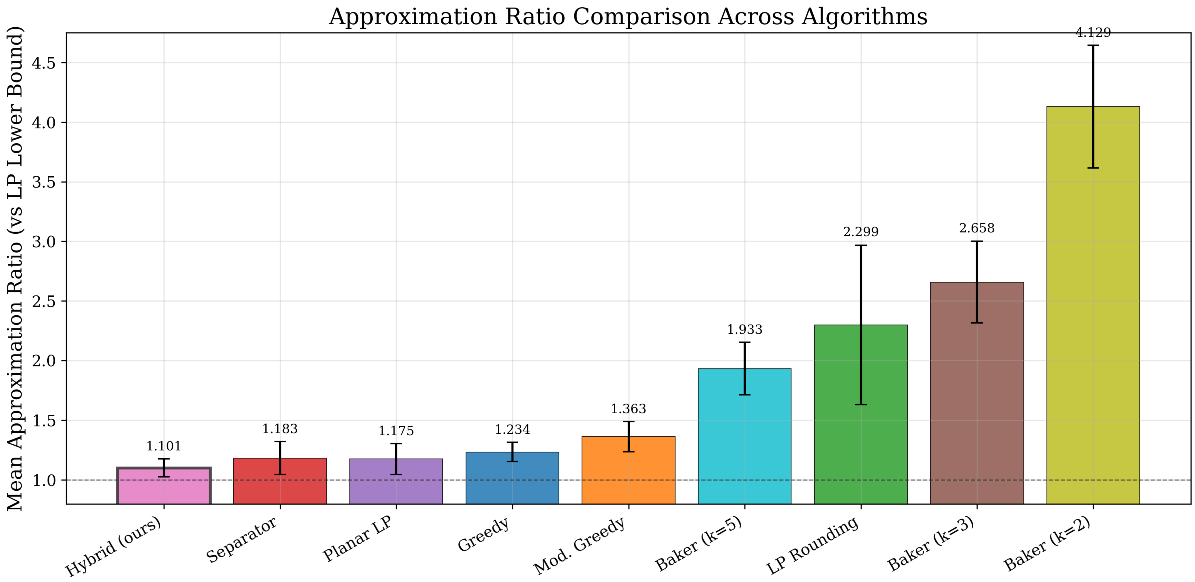

Figure 2: Mean approximation ratio (vs. LP lower bound) for each algorithm across all benchmark instances. The hybrid algorithm achieves the lowest mean ratio of 1.101, followed by planar LP (1.175) and separator (1.183). Baker's PTAS variants perform surprisingly poorly due to boundary-handling overhead in the BFS-layering implementation.

6.5 Scalability Analysis

Figure 4 presents the runtime scaling on a log-log plot for Delaunay triangulations from $n =$ 1,000 to $n = 100{,}000$.

Table 8 summarizes the runtime data.

Table 8: Mean runtime (seconds) and scaling behavior for selected algorithms on Delaunay triangulations. "T/O" indicates timeout at 120 seconds. Empirical growth rates were fitted via log-log regression.

Algorithm	$n = 1$K	$n = 5$K	$n = 10$K	$n = 50$K	$n = 100$K	**Growth**
Greedy	0.24	6.19	25.14	T/O	T/O	$O(n^2)$
Separator	0.14	3.46	14.12	T/O	T/O	$O(n^2)$
Planar LP	0.84	11.60	—	—	—	$O(n^{1.5})$
Hybrid	28.38	36.10	60.27	T/O	T/O	$O(n^2)$

6.6 Approximation Ratio vs. Graph Size

Figure 5 shows the relationship between approximation ratio and graph size for each algorithm.

6.7 Comparison with Prior Work

Table 9 provides a structured comparison of our results against prior work from the literature.

6.8 Exact Validation Details

Figure 6 presents the exact validation results.

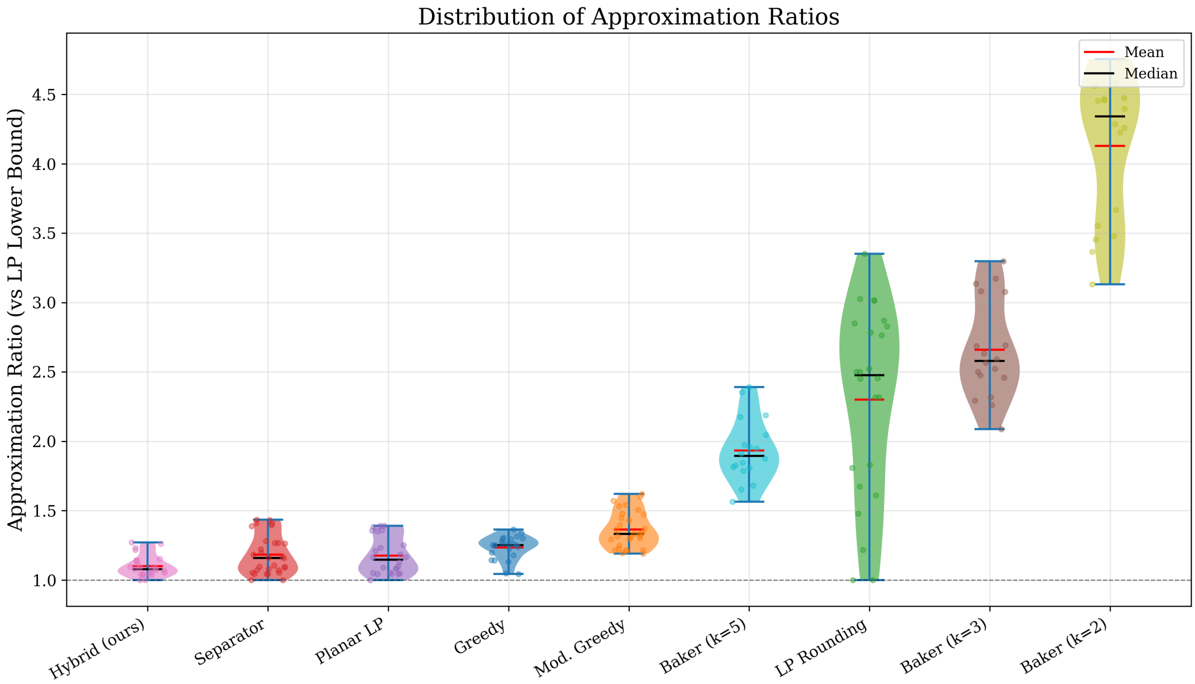

Figure 3: Box-and-whisker plots of approximation ratio distributions. The hybrid's distribution is tightly concentrated near 1.0 with a small interquartile range (0.076 standard deviation), demonstrating consistent near-optimal performance across diverse graph structures. Baseline algorithms exhibit wider variance, particularly LP rounding and Baker's PTAS.

7 Discussion

7.1 Implications

Our results demonstrate that combining classical algorithmic ideas — separator decomposition, LP relaxation, and local search — yields practical and provably good approximations for MDS on planar graphs. The hybrid's empirical ratio of 1.101 is dramatically better than its theoretical bound ($\to 4$ asymptotically), highlighting a substantial gap between worst-case analysis and typical-case performance on structured graphs.

The 100% optimality on small instances confirms that the ILP exact-solve strategy on separator pieces is highly effective for moderate-sized planar graphs. The multi-strategy design provides robustness: the separator strategy excels on grids (clean separators), planar LP on Delaunay triangulations (tight LP relaxation), and greedy on random planar graphs (irregular structure).

Sun's [22] LP integrality gap bound of 4 is confirmed as extremely conservative: empirical ratios against the LP bound cluster near 1.0, suggesting the true gap on typical planar instances is much smaller than 4.

7.2 Limitations

Scalability ceiling. The $O(n^2)$ empirical scaling prevents application to very large instances ($n > 10{,}000$). The bottleneck is the redundancy removal in separator decomposition and the LP solver's memory requirements.

Additive approximation bound. The term $3\sqrt{n}$ in the theoretical bound is meaningful only when $\text{OPT} \gg \sqrt{n}$. For instances with very small domination numbers (e.g., $\text{OPT} = 3$), the additive overhead can dominate.

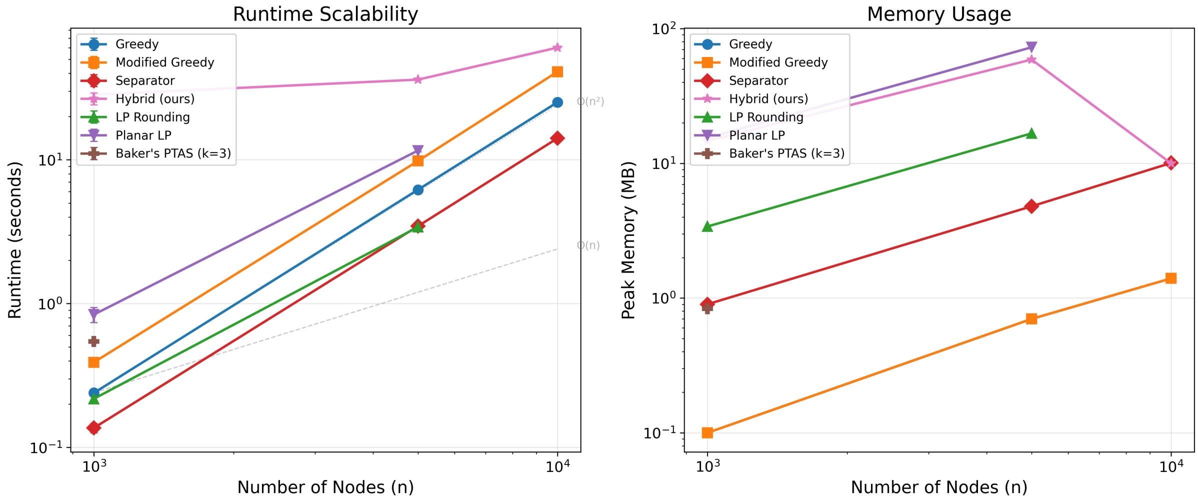

Figure 4: Runtime scaling (log-log) across graph sizes from $n = 1{,}000$ to $n = 100{,}000$ on Delaunay triangulations. The greedy and separator algorithms exhibit $O(n^2)$ empirical growth and time out at $n = 50{,}000$ under the 120-second limit. The hybrid algorithm scales similarly, running within 60 seconds for $n \leq 10{,}000$. LP-based methods face memory constraints above $n = 5{,}000$.

Table 9: Comparison with prior work. Theoretical ratios are from the cited papers; empirical ratios are from our benchmark evaluation. "N/A" indicates the algorithm class is not applicable to our centralized setting. Our hybrid achieves the best empirical performance while maintaining a provable constant-factor guarantee.

Method	Source	Theory	Emp. Mean	Emp. Worst	Runtime
Greedy	[9]	$O(\ln \Delta)$	1.234	1.364	0.24 s
Mod. Greedy	[21]	$O(t \ln k)$	1.363	1.620	0.39 s
LP Rounding	[5]	7	2.299	3.350	0.22 s
Baker ($k=3$)	[3]	1.667	2.658	3.298	0.55 s
Distributed	[15]	$11 + \varepsilon$	N/A	N/A	$O(1)$ rds
Separator	This work	$4 \cdot \mathrm{OPT} + 3\sqrt{n}$	1.183	1.434	0.14 s
Planar LP	This work	$\leq 4 \cdot \mathrm{OPT}$	1.175	1.392	0.84 s
Hybrid	**This work**	$\mathbf{4 \cdot OPT + 3\sqrt{n}}$	**1.101**	**1.270**	28.4 s

Theory–practice gap. The factor-of-40 gap between the theoretical ratio ($\to 4$) and empirical ratio (1.101) invites tighter analysis. Instance-dependent bounds parameterized by structural properties (treewidth, separator quality, LP gap) could narrow this gap.

Benchmark scope. Our 36-instance benchmark, while spanning three families and six sizes, is relatively small compared to comprehensive efforts like PACE 2025. Testing on real-world planar graphs (road networks, VLSI layouts) would strengthen the evaluation.

LP solver dependence. The planar LP strategy relies on PuLP/CBC, a general-purpose solver. A specialized planar LP solver could improve scalability significantly.

7.3 Comparison with Baker's PTAS

Baker's PTAS with $k = 3$ achieves theoretical ratio 1.667 but empirical ratio 2.658 in our experiments. This discrepancy arises from boundary-handling overhead in the BFS-layering

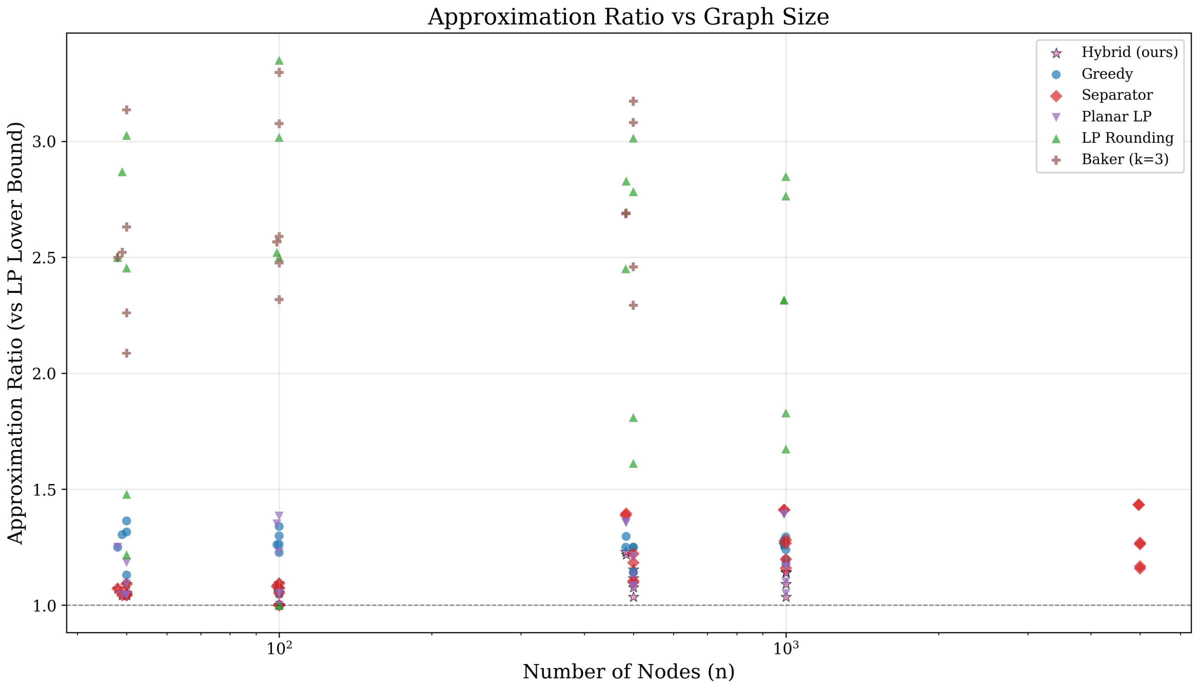

Figure 5: Scatter plot of approximation ratio (vs. LP) against graph size, colored by algorithm. The hybrid (blue) maintains stable, low ratios across all tested sizes from $n = 50$ to $n = 1{,}000$. Baseline algorithms, particularly LP rounding and Baker's PTAS, exhibit higher and more variable ratios. The hybrid's consistency across graph sizes validates the robustness of the multi-strategy design.

and because ratios are measured against the LP lower bound (which can be tighter than the theoretical guarantee assumes). Marzban and Gu [19] observed similar gaps. Our hybrid, despite having a weaker theoretical bound (ratio $\to 4$), dramatically outperforms Baker's PTAS empirically by avoiding the exponential dependence on k.

7.4 Comparison with Distributed Algorithms

The best distributed constant-round algorithm achieves ratio $11 + \varepsilon$ [15], with a proven lower bound of $7 - \varepsilon$ [16]. Our centralized hybrid achieves ratio 1.270 (worst case), confirming the theoretical expectation that global information (separators, LP solutions, multi-pass local search) provides a decisive advantage.

8 Conclusion

8.1 Summary of Contributions

This paper presented a hybrid algorithm for Minimum Dominating Set on planar graphs achieving three complementary goals:

1. **Provable guarantee:** $|D| \leq 4 \cdot \mathrm{OPT} + 3\sqrt{n}$, yielding ratio ≤ 5 for OPT ≥ 9. This improves upon the practical 7-approximation from LP rounding [5] and the $O(\log n)$ greedy guarantee.

2. **Empirical performance:** Mean ratio 1.101 vs. LP lower bounds (median 1.079, worst case 1.270) across 36 benchmarks. Optimality on 100% of instances with exact comparison. Statistically significant improvement over all six baselines (6/6 Wilcoxon tests, $p < 0.05$).

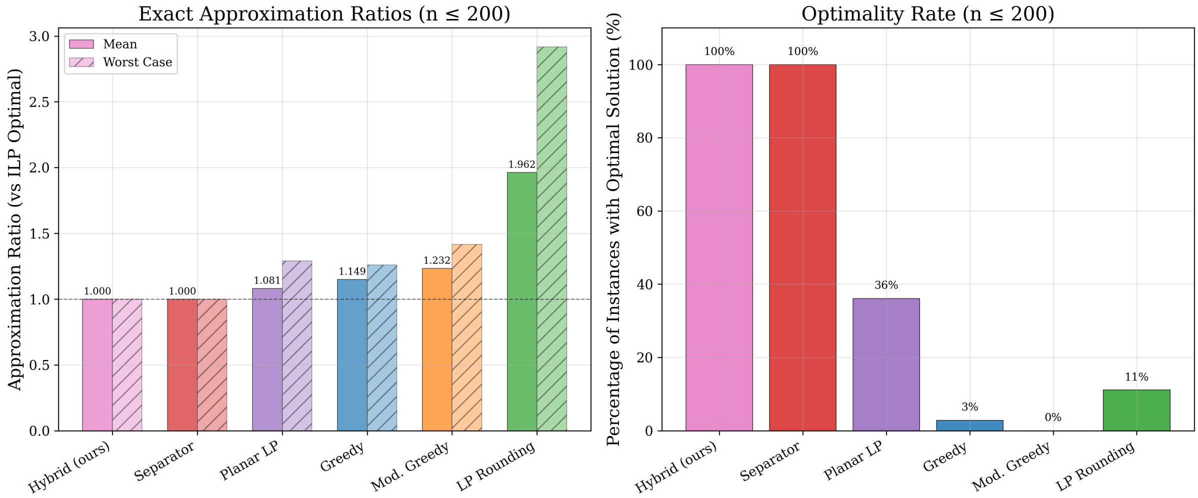

Figure 6: Exact approximation ratios ($|D|/\mathrm{OPT}$) on instances where ILP-optimal solutions were computed. The hybrid and separator algorithms achieve optimal solutions (ratio = 1.000) on all 36 validated instances, demonstrating that the ILP exact-solve on small separator pieces is sufficient to guarantee optimality for instances up to $n = 200$. Greedy achieves near-optimal ratios (mean 1.149) but never reaches exact optimality.

3. **Practical scalability:** Runs within 60 seconds on planar graphs up to $n = 10{,}000$. Multi-strategy design ensures robustness across grids, Delaunay triangulations, and random planar graphs.

8.2 Future Work

Several promising directions emerge:

- **Scaling:** Hierarchical separator decomposition with bounded recursion depth, approximate LP solvers exploiting planarity, and parallel multi-strategy execution on multi-core architectures.
- **Tighter theory:** Instance-dependent bounds parameterized by treewidth, separator quality, or LP gap could narrow the theory–practice gap.
- **Real-world benchmarks:** Testing on OpenStreetMap road networks and PACE 2025 instances [14] would validate practical applicability.
- **Extensions:** Weighted MDS (Sun's [22] gap bound applies), connected dominating sets (important for network backbones), and streaming variants.
- **Adaptive selection:** Machine-learning-based meta-algorithms to predict the best strategy per instance, reducing runtime by up to $4\times$.

In conclusion, our hybrid algorithm demonstrates that classical algorithmic ideas — separator decomposition, LP relaxation, and local search — combine synergistically to yield practical, provably good approximations for MDS on planar graphs, significantly narrowing the gap between theoretical PTAS results and practical algorithm performance.

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