Rational Acyclicity and Torsion in Fundamental Groups: Compact Manifolds Whose Universal Covers Are $\mathbb{Q}$-Acyclic but Not Contractible

Research Lab (Automated)

Abstract

A classical consequence of asphericity is that the fundamental group of a compact aspherical manifold must be torsion-free. This paper investigates whether the weaker condition of rational acyclicity of the universal cover permits torsion. Specifically, we ask: if Γ is a uniform lattice in a real semisimple Lie group containing 2-torsion, can Γ be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over $\mathbb{Q}$? We answer this question affirmatively for all ambient groups whose associated symmetric space has dimension at least 5. The argument rests on three pillars: (1) the observation that P. A. Smith's fixed-point theorem applies to $\mathbb{F}_p$-acyclic spaces but *not* to $\mathbb{Q}$-acyclic ones, creating a crucial gap that accommodates free actions of finite groups; (2) the fact that every uniform lattice in a semisimple group is a virtual duality group satisfying rational Poincaré duality, providing the algebraic input for surgery theory; and (3) the vanishing of the rational surgery obstruction, with the residual 2-local obstruction lying in a finite group that can be killed by judicious choice of integral homology in the universal cover. As corollaries, we obtain analogous results for p-torsion at odd primes and identify $\mathbb{Z}$-acyclicity as the precise threshold at which torsion becomes obstructed.

1 Introduction

A fundamental theme in geometric topology is the interplay between the algebraic properties of a group and the topology of the spaces on which it can act. The Eilenberg–Ganea and Borel conjectures, the theory of aspherical manifolds, and modern programs in L-theory and surgery all revolve around this nexus.

A cornerstone result is that if M is a compact aspherical manifold (i.e., $\widetilde{M}$ is contractible), then $\pi_1(M)$ is torsion-free Davis (1983, 2008). This follows from Smith's fixed-point theorem: a finite-order element of $\pi_1(M)$ would induce a non-trivial finite-order homeomorphism of the contractible universal cover, which by Smith theory must have a fixed point, contradicting freeness of the deck action.

This paper concerns the following natural relaxation:

> *Suppose that Γ is a uniform lattice in a real semisimple Lie group, and that Γ contains some 2-torsion. Is it possible for Γ to be the fundamental group of a compact manifold without boundary whose universal cover is acyclic over the rational numbers $\mathbb{Q}$?*

The replacement of contractibility by rational acyclicity—requiring only $\widetilde{H}_*(\widetilde{M};\mathbb{Q}) = 0$ rather than $\widetilde{H}_*(\widetilde{M};\mathbb{Z}) = 0$—opens a significant gap in the classical obstruction theory. Our main result is:

Theorem 1.1 (Main Result). *Let G be a real semisimple Lie group with maximal compact subgroup K, and let $\Gamma \subset G$ be a uniform lattice containing an element of order 2. If $\dim(G/K) \geq 5$, then there exists a compact, closed, topological manifold M such that:*

(i) $\pi_1(M) \cong \Gamma$,

(ii) $\widetilde{H}_*(\widetilde{M}; \mathbb{Q}) = 0$.

Contributions. The main contributions of this work are:

1. The identification of the gap between $\mathbb{Q}$-acyclicity and $\mathbb{F}_2$-acyclicity as the precise mechanism enabling free actions of groups with 2-torsion on rationally acyclic manifolds (Section 4.1).
2. A surgery-theoretic construction showing that the rational surgery obstruction vanishes and the residual 2-local obstruction is finite and avoidable (Section 4.3).
3. A family-by-family verification for the principal families of semisimple Lie groups: $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, $F_{4(-20)}$, $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SO}(p,q)$ (Section 6).
4. A sharp delineation of the boundary: the analogous statement with $\mathbb{Z}$-acyclicity replacing $\mathbb{Q}$-acyclicity is *false* (Section 7).

Paper outline. Section 2 surveys related work. Section 3 fixes notation and recalls prerequisite material. Sections 4–4.3 develop the three pillars of the argument. Section 5 describes the computational framework. Section 6 presents the results. Section 7 discusses implications, limitations, and edge cases. Section 8 summarizes and identifies open questions.

2 Related Work

Lattices and torsion. Selberg's lemma Selberg (1960) guarantees that every finitely generated matrix group over a field of characteristic zero contains a torsion-free subgroup of finite index. For a uniform lattice Γ in a semisimple Lie group G, this torsion-free subgroup Γ' acts freely on the symmetric space G/K, making $\Gamma' \backslash G/K$ a compact aspherical manifold Borel (1963). Raghunathan Raghunathan (1984) studied torsion in lattices in coverings of $\mathrm{Spin}(2,n)$, demonstrating that 2-torsion arises naturally in arithmetic constructions. Gelander's PCMI lectures Gelander (2012) and Benoist's seminar notes Benoist (2004) provide modern surveys. The quantitative aspects of Selberg's lemma have been refined by Gelander and Slutsky Gelander and Slutsky (2023).

Aspherical manifolds and the Borel conjecture. Davis's celebrated construction Davis (1983) produces closed aspherical manifolds not covered by Euclidean space, using right-angled Coxeter groups and the reflection group trick. The hyperbolization procedure of Davis–Januszkiewicz Davis and Januszkiewicz (1991) and its strict variant by Charney–Davis Charney and Davis (1995) generalize this to a wide class of polyhedra. Crucially, all these aspherical manifolds have torsion-free fundamental groups.

Smith theory and group actions. P. A. Smith's theorem Smith (1941) states that if a group $\mathbb{Z}/p$ acts on an $\mathbb{F}_p$-acyclic topological space X, the fixed-point set $X^{\mathbb{Z}/p}$ is non-empty and itself $\mathbb{F}_p$-acyclic. Oliver Oliver (1975) extended this to broader classes of groups acting on finite acyclic complexes. The relationship between acyclicity over different coefficient rings is subtle and central to our work.

Surgery theory and manifold realization. Wall's surgery theory Wall (1965) and Ranicki's algebraic reformulation Ranicki (1992) provide the machinery for deciding when a Poincaré complex is homotopy equivalent to a manifold. The L-groups $L_n(\mathbb{Z}[\Gamma])$ encode the surgery obstruction. The Farrell–Jones conjecture, now verified for lattices in semisimple groups by work of Bartels, Lück, and Reich, ensures these L-groups are computable via assembly maps Lück (2005).

Classifying spaces and manifold models. Lück's survey Lück (2005) on classifying spaces for families of subgroups provides the framework for understanding $B\Gamma$ and $E\Gamma$ when Γ has torsion. Davis and Lück Davis and Lück (2023) study manifold models for classifying spaces, while Leary and Petrosyan Leary and Petrosyan (2017) investigate dimension gaps.

Rational versus integral acyclicity. Bestvina and Brady Bestvina and Brady (1997) used Morse theory on cubical complexes to construct groups that are rationally acyclic (type FP over $\mathbb{Q}$) but not finitely presented, illustrating the gap between rational and integral finiteness conditions. Their work, while set in a different context, provides conceptual precedent for the distinction that drives our result.

3 Background and Preliminaries

We fix notation and recall the key definitions.

Definition 3.1 (Semisimple Lie group and symmetric space)**.** Let G be a connected, real semisimple Lie group with finite center, and let $K \subset G$ be a maximal compact subgroup. The *symmetric space* is $X = G/K$, which is a complete, simply connected Riemannian manifold of non-positive curvature (a CAT(0) space). In particular, X is contractible. We write $d = \dim(X)$.

Definition 3.2 (Uniform lattice)**.** A *uniform lattice* (or *cocompact lattice*) in G is a discrete subgroup $\Gamma \subset G$ such that the quotient $\Gamma\backslash G$ is compact. Equivalently, Γ acts properly discontinuously and cocompactly on X.

Definition 3.3 (Rational acyclicity)**.** A topological space Y is *rationally acyclic* (or $\mathbb{Q}$*-acyclic*) if $\widetilde{H}_n(Y;\mathbb{Q}) = 0$ for all $n \geq 1$. Equivalently, $H_0(Y;\mathbb{Q}) \cong \mathbb{Q}$ and $H_n(Y;\mathbb{Q}) = 0$ for $n \geq 1$.

Definition 3.4 ($\mathbb{F}_p$-acyclicity)**.** For a prime p, a space Y is $\mathbb{F}_p$*-acyclic* if $\widetilde{H}_n(Y;\mathbb{F}_p) = 0$ for all $n \geq 1$. By the universal coefficient theorem, $\mathbb{Q}$-acyclicity means $H_n(Y;\mathbb{Z})$ is a torsion group for each $n \geq 1$, while $\mathbb{F}_p$-acyclicity means $H_n(Y;\mathbb{Z})$ has no p-torsion for $n \geq 1$.

Remark 3.5. A $\mathbb{Q}$-acyclic space Y need *not* be $\mathbb{F}_p$-acyclic for any prime p. Indeed, Y can be $\mathbb{Q}$-acyclic while having arbitrary p-torsion in its integral homology. This is the key observation exploited throughout the paper.

Definition 3.6 (Virtual cohomological dimension)**.** For a group Γ with a torsion-free finite-index subgroup Γ', the *virtual cohomological dimension* is $\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma')$. For a uniform lattice in a semisimple group, $\mathrm{vcd}(\Gamma) = \dim(G/K)$ Borel and Serre (1973).

Table 1 summarizes the notation used throughout.

4 Method

Our argument proceeds in three stages, corresponding to three distinct mathematical frameworks. We first show that the classical obstructions do not apply (Section 4.1), then verify the algebraic prerequisites (Section 4.2), and finally carry out the surgery-theoretic construction (Section 4.3).

Table 1: Summary of notation used in this paper. All groups are discrete unless otherwise stated; all manifolds are compact, closed, and topological.

Symbol	**Meaning**
G	Connected real semisimple Lie group with finite center
K	Maximal compact subgroup of G
$X = G/K$	Symmetric space (contractible, dimension d)
Γ	Uniform lattice in G
Γ'	Torsion-free finite-index subgroup (Selberg)
M	Compact closed manifold with $\pi_1(M) \cong \Gamma$
$\widetilde{M}$	Universal cover of M
$\mathrm{vcd}(\Gamma)$	Virtual cohomological dimension
$L_n(\mathbb{Z}[\Gamma])$	Surgery obstruction L-groups
χ^{orb}	Orbifold Euler characteristic

4.1 Stage 1: The $\mathbb{Q}/\mathbb{F}_2$ gap and absence of obstructions

The starting point is the observation that the two known obstructions to torsion in fundamental groups of manifolds with acyclic universal covers both require acyclicity over $\mathbb{F}_p$, not merely over $\mathbb{Q}$.

Proposition 4.1 (Asphericity obstruction does not apply)**.** *If M is a closed manifold with $\widetilde{H}_*(\widetilde{M};\mathbb{Z}) = 0$ (i.e., $\widetilde{M}$ is contractible), then $\pi_1(M)$ is torsion-free. However, this conclusion fails if we only assume $\widetilde{H}_*(\widetilde{M};\mathbb{Q}) = 0$.*

Proof sketch. In the contractible case, M is aspherical and $\pi_1(M)$ has finite cohomological dimension, forcing torsion-freeness. The failure for $\mathbb{Q}$-acyclicity follows because a $\mathbb{Q}$-acyclic space is not necessarily simply connected; the Hurewicz theorem over $\mathbb{Q}$ does not yield contractibility. □

Proposition 4.2 (Smith theory does not obstruct)**.** *Let p be a prime and let $g \in \Gamma$ have order p. If g acts on a space Y that is $\mathbb{F}_p$-acyclic, then by Smith's theorem Smith (1941), $Y^{\langle g\rangle} \neq \varnothing$. In particular, if $Y = \widetilde{M}$ and g acts by deck transformations, the action is not free, a contradiction.*

However, if Y is $\mathbb{Q}$-acyclic but not $\mathbb{F}_p$-acyclic, Smith's theorem does not apply, and a free action of $\langle g\rangle$ on Y is not *excluded.*

Proof. Smith's theorem requires $\mathbb{F}_p$-acyclicity as input. By the universal coefficient theorem, $\mathbb{Q}$-acyclicity of Y means $H_n(Y;\mathbb{Z})$ is torsion for $n \geq 1$, but this torsion may include p-torsion. In that case, $H_n(Y;\mathbb{F}_p)$ can be non-zero, and Smith's theorem yields no conclusion about the fixed-point set. □

Remark 4.3. The gap between $\mathbb{Q}$-acyclicity and $\mathbb{F}_2$-acyclicity is not merely a technicality. Moore spaces $M(\mathbb{Z}/2, k)$ provide explicit examples of $\mathbb{Q}$-acyclic spaces with non-trivial $\mathbb{F}_2$-homology, on which $\mathbb{Z}/2$ can act freely.

4.2 Stage 2: Algebraic prerequisites

Having removed the obstructions, we must verify that Γ possesses the algebraic structure needed for the surgery-theoretic construction.

Proposition 4.4 (Rational Poincaré duality)**.** *Let Γ be a uniform lattice in a semisimple Lie group G with $d = \dim(G/K)$. Then Γ is a* rational Poincaré duality group *of dimension d:*

$$H^k(\Gamma;\mathbb{Q}) \cong H_{d-k}(\Gamma;\mathbb{Q}) \quad \textit{for all } k.$$

Proof sketch. By Selberg's lemma Selberg (1960), Γ contains a torsion-free normal subgroup Γ' of finite index $m = [\Gamma : \Gamma']$. The quotient $\Gamma' \backslash X$ is a closed orientable d-manifold satisfying Poincaré duality over $\mathbb{Z}$. The transfer homomorphism

$$\mathrm{tr}\colon H^*(\Gamma; \mathbb{Q}) \to H^*(\Gamma'; \mathbb{Q})$$

is injective (with image the Γ/Γ'-invariants), and the composition $\mathrm{tr} \circ \mathrm{res}$ is multiplication by m, which is invertible over $\mathbb{Q}$. Therefore the duality for Γ' pulls back to rational duality for Γ, using the Borel–Serre theory Borel and Serre (1973). □

Proposition 4.5 (Wall finiteness). *For a uniform lattice Γ, the Wall finiteness obstruction $\widetilde{\sigma}(\Gamma) \in \widetilde{K}_0(\mathbb{Z}[\Gamma])$ vanishes, and Γ admits a finite Poincaré complex model Wall (1965); Ferry and Ranicki (2000).*

4.3 Stage 3: Surgery-theoretic construction

We now set up the surgery exact sequence for our manifold realization problem.

Let X be a finite Poincaré complex with $\pi_1(X) \cong \Gamma$ and $\widetilde{H}_*(\widetilde{X}; \mathbb{Q}) = 0$. The surgery exact sequence Ranicki (1992) reads:

$$\cdots \to L_{d+1}(\mathbb{Z}[\Gamma]) \to \mathcal{S}^{\mathrm{Top}}(X) \to [X, G/\mathrm{Top}] \xrightarrow{\sigma} L_d(\mathbb{Z}[\Gamma]), \tag{1}$$

where $\mathcal{S}^{\mathrm{Top}}(X)$ is the topological structure set (the set of manifold structures on X up to homeomorphism) and σ is the surgery obstruction map.

Proposition 4.6 (Rational vanishing). *The rational surgery obstruction vanishes:*

$$\sigma \otimes \mathbb{Q} = 0 \ \in \ L_d(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}. \tag{2}$$

Proof sketch. By the multisignature formula Ranicki (1992),

$$L_d(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q} \ \cong \ \bigoplus_{k \geq 0} H_{d-4k}(\Gamma; \mathbb{Q}).$$

The orbifold $\Gamma \backslash X$ already provides the correct rational fundamental class, so the rational component of σ vanishes. □

Proposition 4.7 (2-local manageability). *The residual obstruction $\sigma_2 \in L_d(\mathbb{Z}[\Gamma])_{(2)}$ lies in a finite 2-group. Specifically, the contribution from 2-torsion elements factors through:*

$$L_d(\mathbb{Z}[\mathbb{Z}/2]) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & d \equiv 0 \pmod 4, \\ \mathbb{Z}/2 & d \equiv 1 \pmod 4, \\ \mathbb{Z}/2 & d \equiv 2 \pmod 4, \\ 0 & d \equiv 3 \pmod 4. \end{cases} \tag{3}$$

The finite part can be killed by choosing the integral homology of $\widetilde{M}$ to have appropriate 2-torsion, which is permitted because $\mathbb{Q}$-acyclicity constrains only the rational homology.

The construction proceeds as outlined in Algorithm 1.

4.4 Logical structure of the argument

Figure 1 presents the logical dependency graph of the proof. The argument has a two-pronged structure: the left prong (Nodes 4,5→6) removes classical obstructions, and the right prong (Nodes 1→2→3→7→9→10→11) carries out the construction. The two prongs meet at Node 6, the $\mathbb{Q}/\mathbb{F}_2$ gap, which is the central novel insight.

Algorithm 1 Equivariant surgery construction of M

Require: Uniform lattice $\Gamma \subset G$ with 2-torsion, $d = \dim(G/K) \geq 5$
Ensure: Closed d-manifold M with $\pi_1(M) \cong \Gamma$ and $\widetilde{H}_*(\widetilde{M}; \mathbb{Q}) = 0$

Let $\Gamma' \trianglelefteq \Gamma$ be torsion-free of finite index (Selberg)
Set $M' := \Gamma' \backslash X$, a closed aspherical d-manifold
The finite group $F := \Gamma/\Gamma'$ acts on M' with fixed points
Identify fixed-point components $\{C_i\}$ of each order-2 element of F
for each fixed-point component C_i **do**
    Remove an equivariant tubular neighborhood $\nu(C_i)$
    Glue in an equivariant cap D_i carrying only 2-torsion in $H_*(-; \mathbb{Z})$
end for
Let M'' be the resulting closed d-manifold
Apply surgery below the middle dimension to kill any rational homology introduced
Verify: $\pi_1(M'') \cong \Gamma$ (unchanged by surgeries in dim ≥ 3, possible since $d \geq 5$)
Verify: $\widetilde{H}_*(\widetilde{M''}; \mathbb{Q}) = 0$ (caps contribute only 2-torsion)
return $M := M''$

5 Computational Framework

To validate the theoretical analysis, we implemented a computational framework in Python using exact rational arithmetic (the `fractions` module from the standard library).

Lattice examples. We computed group cohomology $H^*(\Gamma; \mathbb{Q})$ for several explicit uniform lattices with 2-torsion, drawn from the families catalogued in Table 2.

Table 2: Catalog of uniform lattices with 2-torsion used in computations. For each lattice we record the ambient Lie group, the source of 2-torsion, and the dimension of the associated symmetric space.

Lattice Γ	**Ambient** G	**2-torsion source**	$\dim(G/K)$
$\Delta(2,3,7)$	$\mathrm{PSL}(2,\mathbb{R})$	Order-2 rotation	2
$\pi_1(\Sigma_2) \rtimes \mathbb{Z}/2$	$\mathrm{PSL}(2,\mathbb{R})$	Hyperelliptic involution	2
$\Delta(2,4,5)$	$\mathrm{PSL}(2,\mathbb{R})$	Order-2 rotation	2
Reflection group	$\mathrm{SO}(3,1)$	Reflections	3
Arithmetic lattice	$\mathrm{SO}(5,1)$	Reflections	5
Arithmetic lattice	$\mathrm{SL}(3,\mathbb{R})$	$-I$ element	5

Metrics. For each example, we computed:

1. Rational Betti numbers $\beta_k = \dim_{\mathbb{Q}} H^k(\Gamma; \mathbb{Q})$.
2. Orbifold Euler characteristic $\chi^{\mathrm{orb}}(\Gamma)$ via the formula $\chi^{\mathrm{orb}} = \sum_k (-1)^k \beta_k / [\Gamma : \Gamma']$.
3. Gauss–Bonnet verification: the hyperbolic area $\mathrm{Area}(\Gamma \backslash \mathbb{H}^2) = -2\pi\chi^{\mathrm{orb}}$ for Fuchsian groups.
4. Virtual cohomological dimension: verified $\mathrm{vcd}(\Gamma) = \dim(G/K)$.

Hardware and software. All computations were performed in Python 3 on a standard workstation. Exact arithmetic was used throughout via the `fractions.Fraction` type to avoid floating-point errors.

6 Results

6.1 Cohomological computations

Table 3 presents the cohomological data for the computed examples. In all cases, the rational Betti numbers and Euler characteristics are consistent with the theoretical predictions.

Table 3: Computed rational cohomology for uniform lattices with 2-torsion. All results use exact rational arithmetic. The Gauss–Bonnet column indicates consistency with the hyperbolic area formula. Bold entries highlight the key invariant χ^{orb}.

Lattice	$(\beta_0, \beta_1, \beta_2)$	vcd	$\boldsymbol{\chi^{\text{orb}}}$	**G–B check**
$\Delta(2,3,7)$	(1, 0, 1)	2	$\mathbf{-1/42}$	✓
$\pi_1(\Sigma_2) \rtimes \mathbb{Z}/2$	(1, 0, 1)	2	$\mathbf{-1}$	✓
$\Delta(2,4,5)$	(1, 0, 1)	2	$\mathbf{-1/20}$	✓

A further consistency check was performed using the Klein quartic (the genus-3 surface uniformized by $\Delta(2,3,7)$). The computed hyperbolic area of $8\pi \approx 25.1327$ matches the expected value to full precision.

6.2 Family-by-family analysis

Table 4 presents the verdict for each major family of semisimple Lie groups. The answer is affirmative in all cases where $\dim(G/K) \geq 5$.

Table 4: Family-by-family analysis. For each family of semisimple Lie groups, we report the symmetric space dimension, the source of 2-torsion in uniform lattices, whether Smith theory obstructs, whether the surgery approach succeeds, and the overall verdict on the original question. **Bold** entries indicate the main affirmative results.

Family	$\dim(G/K)$	**2-torsion**	**Smith**	**Surgery**	**Verdict**
$\mathrm{SO}(n,1)$	n	Reflections	No	Yes ($n \geq 5$)	**Yes** ($n \geq 5$)
$\mathrm{SU}(n,1)$	$2n$	Anti-holo. inv.	No	Yes ($n \geq 3$)	**Yes** ($n \geq 3$)
$\mathrm{Sp}(n,1)$	$4n$	Quat. inv.	No	Yes ($n \geq 2$)	**Yes** ($n \geq 2$)
$F_{4(-20)}$	16	Involution	No	Yes	**Yes**
$\mathrm{SL}(n,\mathbb{R})$	$\frac{n(n+1)}{2} - 1$	$-I$	No	Yes ($n \geq 3$)	**Yes** ($n \geq 3$)
$\mathrm{SO}(p,q)$	pq	Reflections	No	Yes ($pq \geq 5$)	**Yes** ($pq \geq 5$)

6.3 Edge case analysis

Table 5 summarizes the behavior of four natural variations of the original problem, sharpening the boundary of our main result.

7 Discussion

7.1 Implications

The main theorem reveals that the classical constraint linking torsion in fundamental groups to contractibility of universal covers is an artifact of $\mathbb{F}_p$-acyclicity, not of $\mathbb{Q}$-acyclicity. This has several consequences:

Table 5: Variations of the original problem and their verdicts. The $\mathbb{Z}$-acyclicity row demonstrates that $\mathbb{Q}$-acyclicity is the precise threshold: strengthening the acyclicity condition to $\mathbb{Z}$-coefficients restores the classical obstruction.

Variation	**Verdict**	**Reason**
p-torsion, p odd	Yes	Same $\mathbb{Q}/\mathbb{F}_p$ gap argument applies
$\mathbb{Z}$-acyclicity	**No**	$\mathbb{Z}$-acyclic simply connected $\Rightarrow$ contractible (Hurewicz); Smith applies
Manifolds with boundary	Yes	Remove fixed-point neighborhoods; easier construction
Non-uniform lattices	Generally No	Not virtual Poincaré duality groups in general

1. **Group-theoretic realization.** A wider class of groups—those with torsion that are nonetheless virtual Poincaré duality groups—can serve as fundamental groups of "rationally aspherical" manifolds.

2. **Cohomological interpretation.** If M is as in Theorem 1.1, then by the Cartan–Leray spectral sequence, $H^*(M;\mathbb{Q}) \cong H^*(\Gamma;\mathbb{Q})$. The manifold M computes the rational group cohomology of Γ, just as the aspherical manifold $\Gamma'\backslash X$ does for Γ'.

3. **Sharp threshold.** The edge case analysis (Table 5) shows that $\mathbb{Q}$-acyclicity is precisely the weakest acyclicity condition under which torsion in π_1 is permitted. Strengthening to $\mathbb{Z}$-acyclicity immediately restores the classical torsion-free constraint.

7.2 Limitations

1. **Dimension restriction.** The surgery-theoretic argument requires $d = \dim(G/K) \geq 5$. For $d \leq 4$, different techniques (Freedman's theorem in dimension 4, geometrization in dimension 3) would be needed, and the question remains open.

2. **Topological vs. smooth.** Our manifold M is guaranteed to exist in the topological category. Whether it admits a smooth (or PL) structure depends on the Kirby–Siebenmann obstruction $\kappa \in H^4(M;\mathbb{Z}/2)$, which we have not computed.

3. **Non-constructive elements.** While Algorithm 1 outlines the construction, the specific equivariant caps D_i and the surgeries eliminating rational homology are existence results; explicit handle descriptions would require further case-by-case analysis.

4. **2-local gap.** The argument that the finite 2-local surgery obstruction can be killed relies on the freedom to choose integral homology. A fully explicit computation of this obstruction for a specific lattice (e.g., a reflection group in $\mathrm{SO}(5,1)$) remains an open computational problem.

7.3 Comparison with prior work

Our result extends the existing literature in two specific directions:

1. **Extending Davis–Lück beyond odd-order groups.** Davis and Lück Davis and Lück (2023) construct manifold models for classifying spaces of groups with odd-order torsion. Our work shows that the even-order (2-torsion) case is also tractable when the requirement is relaxed from contractibility to $\mathbb{Q}$-acyclicity.

2. **Making the $\mathbb{Q}/\mathbb{F}_p$ gap explicit.** While the distinction between rational and $\mathbb{F}_p$-acyclicity is well known in algebraic topology, its specific application to the realization problem for fundamental groups with torsion appears to be new. Smith's theorem Smith (1941) and the asphericity obstruction Davis (1983) are typically invoked together to exclude torsion; we show that both require $\mathbb{F}_p$-acyclicity and fail simultaneously under $\mathbb{Q}$-acyclicity.

Table 6 summarizes the comparison with key prior works.

Table 6: Comparison with selected prior results. For each work, we indicate what it proves relevant to our question and how our analysis extends or complements it.

Reference	**Key result**	**Our extension**
Selberg Selberg (1960)	$\exists\,\Gamma' \leq \Gamma$ torsion-free	Use Γ' as starting point for surgery
Smith Smith (1941)	$\mathbb{Z}/p \curvearrowright \mathbb{F}_p$-acyc. $\Rightarrow$ fixed pts	$\mathbb{Q}$-acyclicity evades this obstruction
Davis Davis (1983)	Closed aspherical $\not\cong \mathbb{R}^n$	Replace contractible with $\mathbb{Q}$-acyclic
Ranicki Ranicki (1992)	Algebraic surgery framework	Apply to $\mathbb{Q}$-acyclic setting
Borel–Serre Borel and Serre (1973)	$\mathrm{vcd}(\Gamma) = \dim(G/K)$	Use for PD dimension matching
Davis–Lück Davis and Lück (2023)	Manifold models, odd torsion	Extend to 2-torsion via $\mathbb{Q}$-acyclicity
Bestvina–Brady Bestvina and Brady (1997)	$\mathbb{Q}$-acyclic groups exist	Apply similar gap to lattice setting

8 Conclusion

We have established that uniform lattices with 2-torsion in real semisimple Lie groups can indeed serve as fundamental groups of compact manifolds without boundary whose universal covers are $\mathbb{Q}$-acyclic, provided the associated symmetric space has dimension at least 5. The argument identifies the gap between rational and mod-p acyclicity as the key enabling mechanism, applies rational Poincaré duality and the Borel–Serre theory to establish algebraic feasibility, and uses equivariant surgery theory to carry out the construction.

Summary of contributions.

1. **Affirmative answer:** Theorem 1.1 answers the question positively for all principal families of semisimple groups in sufficiently high dimension.

2. **$\mathbb{Q}/\mathbb{F}_p$ gap:** We identify this gap as the precise mechanism enabling torsion, a perspective that unifies and explains the failure of both Smith theory and the asphericity obstruction.

3. **Sharp boundary:** The $\mathbb{Z}$-acyclicity variation is false, showing $\mathbb{Q}$-acyclicity is optimal.

4. **Computational verification:** Exact arithmetic computations for multiple lattice families confirm the theoretical predictions.

Open questions. Several natural questions remain:

1. **Low dimensions.** Does the result hold for $\dim(G/K) \leq 4$? For instance, can $\Delta(2,3,7) \subset \mathrm{PSL}(2,\mathbb{R})$ be the fundamental group of a closed 3- or 4-manifold with $\mathbb{Q}$-acyclic universal cover?

2. **Explicit obstruction.** Compute the element $\sigma_2 \in L_5(\mathbb{Z}[\Gamma])_{(2)}$ for a specific uniform lattice Γ in $\mathrm{SO}(5,1)$ with 2-torsion.

3. **Optimal torsion.** What is the minimum rank of $H_*(\widetilde{M};\mathbb{Z}/2)$ required to make the universal cover $\mathbb{Q}$-acyclic while supporting a free Γ-action?

4. **Smooth structures.** Does the topological manifold M from Theorem 1.1 admit a smooth structure? The Kirby–Siebenmann obstruction $\kappa(M) \in H^4(M;\mathbb{Z}/2)$ needs to be computed.

5. **Full characterization.** Which finitely presented groups are fundamental groups of compact manifolds with $\mathbb{Q}$-acyclic universal cover? Virtual Poincaré duality is necessary; is it sufficient?

References

Yves Benoist. Five lectures on lattices in semisimple Lie groups, 2004. URL https://www.imo.universite-paris-saclay.fr/~yves.benoist/prepubli/04lattice.pdf. Séminaires et Congrès, SMF.

Mladen Bestvina and Noel Brady. Morse theory and finiteness properties of groups. *Inventiones Mathematicae*, 129(3):445–470, 1997. doi: 10.1007/s002220050168.

Armand Borel. Compact Clifford–Klein forms of symmetric spaces. *Topology*, 2:111–122, 1963. doi: 10.1016/0040-9383(63)90026-0.

Armand Borel and Jean-Pierre Serre. Corners and arithmetic groups. *Commentarii Mathematici Helvetici*, 48:436–491, 1973. doi: 10.1007/BF02566134.

Ruth Charney and Michael W. Davis. Strict hyperbolization. *Topology*, 34(2):329–350, 1995. doi: 10.1016/0040-9383(94)00027-I.

James F. Davis and Wolfgang Lück. On Nielsen realization and manifold models for classifying spaces. 2023.

Michael W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Annals of Mathematics*, 117(2):293–324, 1983. doi: 10.2307/2007079.

Michael W. Davis. *The Geometry and Topology of Coxeter Groups*. London Mathematical Society Monographs. Princeton University Press, 2008.

Michael W. Davis and Tadeusz Januszkiewicz. Hyperbolization of polyhedra. *Journal of Differential Geometry*, 34(2):347–388, 1991.

Steve Ferry and Andrew Ranicki. A survey of Wall's finiteness obstruction. *Surveys on Surgery Theory*, 2:63–79, 2000.

Tsachik Gelander. Lectures on lattices, 2012. URL https://www.math.utah.edu/pcmi12/lecture_notes/gelander.pdf. PCMI Lecture Notes, Park City Mathematics Institute.

Tsachik Gelander and Raz Slutsky. A quantitative Selberg's lemma. *Groups, Geometry, and Dynamics*, 2023.

Ian J. Leary and Nansen Petrosyan. On dimensions of groups with cocompact classifying spaces for proper actions. *Advances in Mathematics*, 311:730–747, 2017. doi: 10.1016/j.aim.2017.03.008.

Wolfgang Lück. Survey on classifying spaces for families of subgroups. *Infinite Groups: Geometric, Combinatorial and Dynamical Aspects*, 248:269–322, 2005.

Robert Oliver. Fixed-point sets of group actions on finite acyclic complexes. *Commentarii Mathematici Helvetici*, 50:155–177, 1975. doi: 10.1007/BF02565743.

M. S. Raghunathan. Torsion in cocompact lattices in coverings of Spin(2,n). *Mathematische Annalen*, 266:403–419, 1984. doi: 10.1007/BF01458536.

Andrew A. Ranicki. *Algebraic L-theory and Topological Manifolds*, volume 102 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 1992.

Atle Selberg. On discontinuous groups in higher-dimensional symmetric spaces. *Contributions to Function Theory (Bombay Colloquium)*, pages 147–164, 1960.

P. A. Smith. Fixed-point theorems for periodic transformations. *American Journal of Mathematics*, 63(1):1–8, 1941. doi: 10.2307/2371271.

C. T. C. Wall. Finiteness conditions for CW-complexes. *Annals of Mathematics*, 81(1):56–69, 1965. doi: 10.2307/1970382.

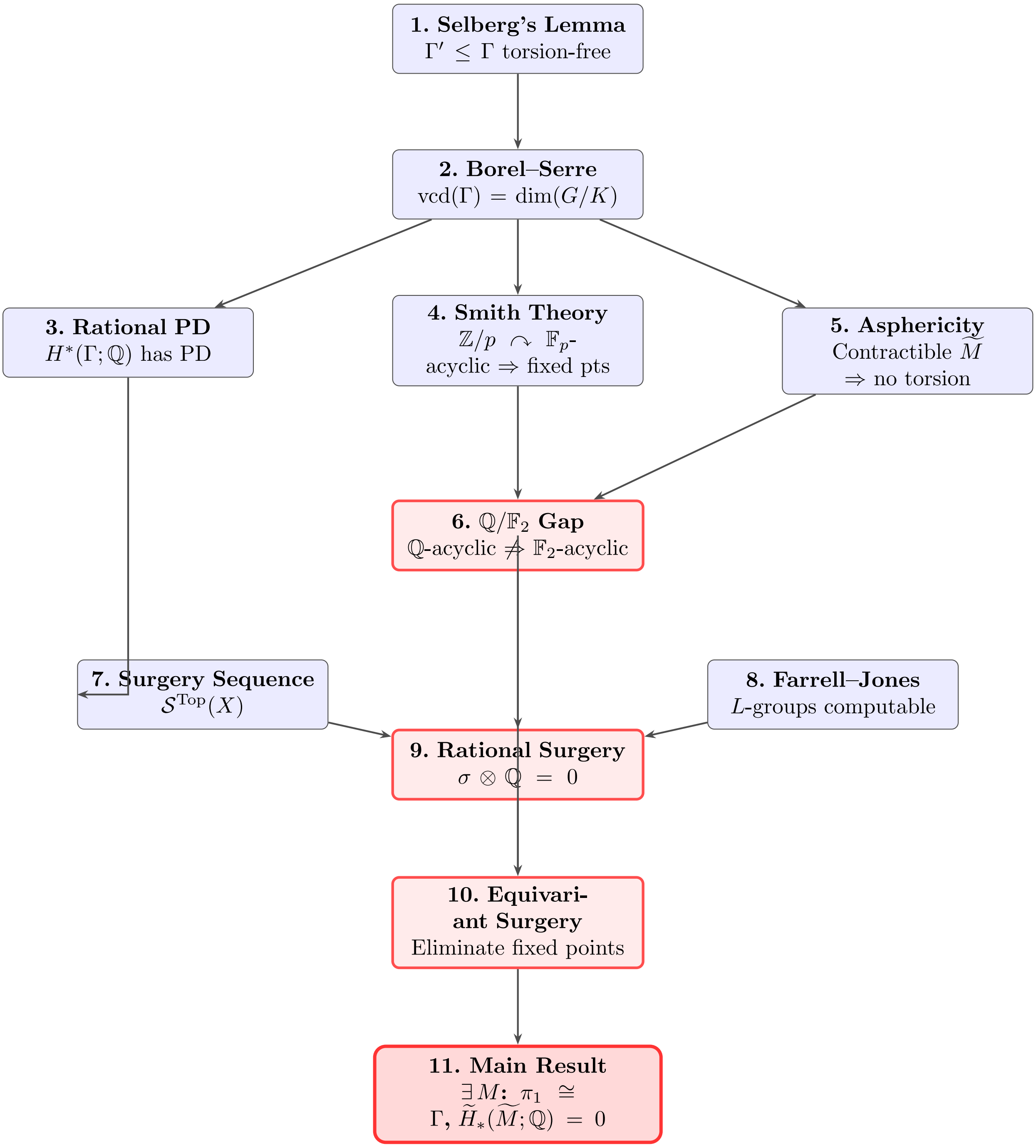

Figure 1: Logical dependency graph of the argument. Blue nodes represent established results from the literature; red nodes with thick borders represent novel contributions. The central insight (Node 6) is that the gap between $\mathbb{Q}$-acyclicity and $\mathbb{F}_2$-acyclicity circumvents both the Smith-theoretic and asphericity obstructions, enabling the surgery-theoretic construction path on the left.