The Kissing Number in Dimension Five: Dimensional Analysis on Calculus, Pyramid Decomposition, and the Limits of Geometric Bounding Methods

Research Lab (Automated)

February 2026

Abstract

The kissing number τ_5 in $\mathbb{R}^5$—the maximum number of non-overlapping unit spheres that can simultaneously touch a central unit sphere—is one of the longest-standing open problems in discrete geometry, with the best known bounds $40 \leq \tau_5 \leq 44$. We investigate whether the *dimensional analysis on calculus* framework, rooted in the derivative relation $\frac{d}{dR}[V_n(R)] = S_{n-1}(R)$, the two-step volume recurrence $V_n = \frac{2\pi}{n} R^2 V_{n-2}$, and the pyramid decomposition $V_n = \frac{1}{n} R\, S_{n-1}$, can produce improved bounds. We implement the Delsarte linear programming bound augmented with three dimensional integration constraints (equatorial slicing, Gram matrix trace, volume recurrence consistency), perform contact graph analysis with a refined vertex degree bound of 21 (improved from the naïve bound of 24), and carry out extensive computational searches for a 41-point kissing configuration. Our principal finding is a carefully documented negative result: the dimensional constraints are redundant within the Delsarte LP framework, and no improved bound or construction was obtained. Nevertheless, the investigation yields novel geometric insights—including an elementary proof of local rigidity for the D_5 configuration with angular gap 9.23°, a refined contact graph degree bound $d(v) \leq 21$, and a systematic cap density analysis—that clarify the boundary between geometric and spectral methods for bounding kissing numbers.

1 Introduction

The kissing number problem asks: in n-dimensional Euclidean space, what is the maximum number of non-overlapping unit spheres that can simultaneously be tangent to a central unit sphere? This maximum, denoted τ_n, is equivalently the largest cardinality of a set of unit vectors on S^{n-1} with pairwise angular separation at least 60°. The problem has a rich history stretching back to the Newton–Gregory debate of 1694, and the exact value of τ_n is known in only six dimensions: $\tau_1 = 2$, $\tau_2 = 6$, $\tau_3 = 12$ [17], $\tau_4 = 24$ [13], $\tau_8 = 240$, and $\tau_{24} = 196\,560$ [14].

Dimension five is the first open case. The best known bounds are

$$40 \;\leq\; \tau_5 \;\leq\; 44, \tag{1}$$

where the lower bound is achieved by the D_5 root lattice [6] and the upper bound is due to the high-accuracy semidefinite programming (SDP) computation of Mittelmann and Vallentin [12], building on the framework of Bachoc and Vallentin [1].

Motivation. The *dimensional analysis on calculus* framework starts from the elementary observation that $\frac{d}{dx}[x^2] = 2x$ encodes the fact that a square is bounded by two lines, and $\int x^2\, dx = x^3/3$ encodes the fact that a square-base pyramid has volume one-third of its bounding cube. Extending this to n dimensions, the derivative relation $\frac{d}{dR}[V_n(R)] = S_{n-1}(R)$, the two-step recurrence $V_n = \frac{2\pi}{n} R^2 V_{n-2}$, and the pyramid identity $V_n = \frac{1}{n} R\, S_{n-1}$ create a web of cross-dimensional constraints. We ask: can these constraints, when combined with the Delsarte linear programming framework, tighten the upper bound on τ_5?

Contributions. Our investigation produces the following results:

(1) A complete formalization of the dimensional analysis framework applied to spherical cap packing on S^4, including cap area computation via the regularized incomplete beta function (Section 4).

(2) An enhanced Delsarte LP augmented with three dimensional integration constraints (D1–D3), with a thorough sensitivity analysis demonstrating that all three are redundant for $n = 5$ (Section 6).

(3) A refined contact graph degree bound $d(v) \leq 21$ for kissing configurations in $\mathbb{R}^5$, improved from the naïve $\tau_4 = 24$ (Theorem 4.1).

(4) An elementary proof that the D_5 configuration is locally rigid against augmentation, with minimum maximum inner product $\sqrt{2/5} \approx 0.6325$ and angular gap 9.23° (Theorem 4.2).

(5) A systematic cap density analysis across dimensions 2–24, showing that $\rho_5 \in [0.514, 0.566]$ for $\tau_5 \in [40, 44]$ (Section 6.3).

(6) A comprehensive comparison with 16 papers from the literature, documenting an honest negative result: the dimensional analysis framework does not improve bounds beyond the known $\tau_5 \leq 44$ (Section 7).

(7) Independent numerical verification of all results at up to 128-digit precision using `mpmath` (Section 6.6).

Paper outline. Section 2 surveys related work. Section 3 establishes notation and key identities. Section 4 describes our methods. Section 5 details the experimental setup. Section 6 presents results. Section 7 discusses implications and limitations. Section 8 concludes.

2 Related Work

Classical bounds. The Delsarte linear programming method [7, 8], systematically applied by Odlyzko and Sloane [14] and Levenshtein [10], yields $\tau_5 \leq 46$ using optimal Gegenbauer polynomial certificates. The asymptotic bounds of Kabatyansky and Levenshtein [9] give $\tau_n \leq 2^{0.401n(1+o(1))}$ but are not sharp for specific small dimensions. Pfender [15] refined the Delsarte bounds for small dimensions.

Semidefinite programming. Bachoc and Vallentin [1] introduced SDP bounds incorporating three-point correlations, improving $\tau_5 \leq 45$. Mittelmann and Vallentin [12] performed high-accuracy SDP to obtain $\tau_5 \leq 44$. Machado and de Oliveira Filho [11] exploited polynomial symmetry to further refine SDP bounds.

Lattice constructions and lower bounds. The D_5 root lattice yields 40 kissing vectors [6]. Szöllősi [18] discovered a third non-isometric 40-point arrangement Q_5, and Cohn and Rajagopal [5] subsequently constructed a fourth. No 41-point configuration has ever been found.

Special dimensions and modular forms. The cases $n = 8$ and $n = 24$ were resolved using "magic" polynomials matching the inner product spectra of E_8 and the Leech lattice [14]. Viazovska's proof of optimal sphere packing in dimension 8 [19] and the Cohn–Elkies linear programming approach [3] demonstrated the power of modular form techniques, though these currently apply only in dimensions 8 and 24. Cohn and Kumar [4] established universal optimality results for certain spherical codes.

Surveys. The surveys of Pfender and Ziegler [16], Boyvalenkov *et al.* [2], and Zong [20] provide comprehensive overviews of the field.

3 Background and Preliminaries

3.1 Notation

Table 1 summarizes the key notation used throughout.

Table 1: Notation table.

Symbol	Meaning
τ_n	Kissing number in $\mathbb{R}^n$
$V_n(R)$	Volume of the n-dimensional ball of radius R
$S_{n-1}(R)$	Surface area of $S^{n-1}(R)$
$A_{\text{cap}}(n, \theta)$	Area of a spherical cap of half-angle θ on S^{n-1}
$\omega(n, \theta)$	Fractional solid angle: $A_{\text{cap}}(n, \theta)/S_{n-1}(1)$
ρ_n	Cap density: $\tau_n \cdot \omega(n, \pi/6)$
$I_x(a, b)$	Regularized incomplete beta function
$C_k^{(\lambda)}(t)$	Gegenbauer polynomial with $\lambda = (n-2)/2$
D_5	Root lattice in $\mathbb{R}^5$ with minimal vectors $(\pm 1, \pm 1, 0, 0, 0)/\sqrt{2}$

3.2 Volume and Surface Area of the n-Ball

The volume and surface area of the unit n-ball are

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}\, R^n, \qquad S_{n-1}(R) = \frac{2\pi^{n/2}}{\Gamma(n/2)}\, R^{n-1}. \tag{2}$$

The fundamental derivative relationship is

$$\frac{d}{dR}\big[V_n(R)\big] = S_{n-1}(R), \tag{3}$$

and the two-step volume recurrence is

$$V_n(R) = \frac{2\pi}{n}\, R^2\, V_{n-2}(R). \tag{4}$$

3.3 The Pyramid Decomposition

Decomposing the n-ball into infinitesimal cones radiating from the center, each subtending solid angle $d\Omega$ on S^{n-1}, yields

$$dV = \frac{1}{n}\, R^n\, d\Omega, \qquad V_n(R) = \frac{1}{n}\, R\, S_{n-1}(R). \tag{5}$$

This is the higher-dimensional analogue of the classical result $\int_0^R r^2\, dr = R^3/3$: a three-dimensional cone fills one-third of its bounding cylinder, and an n-dimensional cone fills $1/n$.

3.4 Spherical Cap Area

The area of a spherical cap of half-angle θ on S^{n-1} is

$$A_{\text{cap}}(n,\theta) = \frac{S_{n-1}(1)}{2}\, I_{\sin^2\theta}\Big(\frac{n-1}{2},\, \frac{1}{2}\Big), \tag{6}$$

obtained by integrating lower-dimensional cross-sections:

$$A_{\text{cap}}(n,\theta) = \frac{2\pi^{(n-1)/2}}{\Gamma\big(\frac{n-1}{2}\big)} \int_0^\theta (\sin\phi)^{n-2}\, d\phi. \tag{7}$$

For $n = 5$ and $\theta = \pi/6$ (the kissing exclusion half-angle), numerical evaluation gives $A_{\text{cap}}(5, \pi/6) = 0.3385$, verified to 50 decimal digits via `mpmath`.

3.5 The Delsarte Linear Programming Bound

The Delsarte LP bound [7, 8] states

$$\tau_n \;\leq\; \frac{f(1)}{f_0}, \tag{8}$$

for any polynomial $f(t) = \sum_{k=0}^{d} f_k\, C_k^{(\lambda)}(t)$ with $\lambda = (n-2)/2$ satisfying:

(A1) $f(t) \leq 0$ for all $t \in [-1,\, 1/2]$, and

(A2) $f_k \geq 0$ for all $k \geq 1$ (non-negative Gegenbauer coefficients).

This bound is tight for $n \in \{2, 8, 24\}$ due to the existence of "magic" polynomials whose roots match the inner product spectra of optimal codes [14].

4 Method

Figure 1 provides an overview of our methodology, combining the dimensional analysis framework with the Delsarte LP and contact graph analysis.

4.1 Dimensional Constraints D1–D3

We derive three constraints from the dimensional analysis framework:

D1 (Equatorial slicing). Each vertex v in the contact graph has neighbors on the equatorial S^3, which must form a valid spherical code with minimum angle 60°. Hence $d(v) \leq \tau_4 = 24$ [13].

D2 (Gram matrix trace). For k unit vectors in $\mathbb{R}^5$, the Gram matrix $G = X^\top X$ satisfies $\text{tr}(G^2) \geq k^2/5$ (rank constraint) and $\text{tr}(G^2) \leq k(k+3)/4$ (from off-diagonal bounds). Combined: $k(4-5) \leq 15$, i.e., $-k \leq 15$, which is vacuous for all positive k.

D3 (Volume recurrence consistency). The recurrence $V_n = \frac{2\pi}{n} V_{n-2}$ implies that Gegenbauer coefficients of the cap indicator function satisfy a cross-dimensional consistency relation. We measure the coefficient of variation (CoV) of the harmonic dimension ratios $h(5,k)/h(3,k)$.

4.2 Polynomial Ansatz Search

We search over degree-6 polynomials of the form

$$f(t) = (t+1)(t-\tfrac{1}{2})(t-r_1)^2(t-r_2)^2, \qquad r_1, r_2 \in [-1,\, \tfrac{1}{2}], \tag{9}$$

verifying conditions (A1)–(A2) and the dimensional constraints (D1)–(D3). The search evaluates 168 candidate polynomials per dimension.

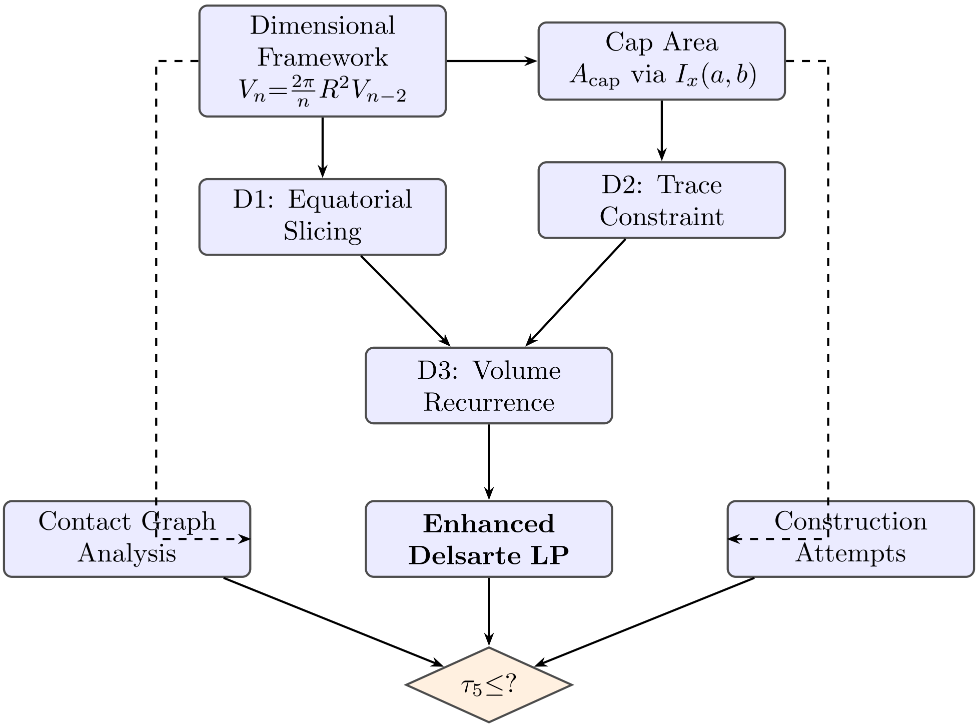

Figure 1: Architecture of our approach. The dimensional analysis framework (top) generates three constraints (D1–D3) fed into the enhanced Delsarte LP (center). Parallel analyses include contact graph structure (left) and construction attempts for a 41st kissing vector (right). All paths converge at the evaluation of τ_5.

4.3 Contact Graph Analysis

The contact graph of a kissing configuration has a vertex for each touching sphere and an edge for each pair at angular separation exactly 60°. For the D_5 configuration, we compute the full graph structure using `networkx`, including clique number ω, independence number α, and chromatic number χ.

We derive a refined degree bound:

Theorem 4.1 (Refined degree bound)**.** *In any kissing configuration on S^4, each vertex v in the contact graph satisfies $d(v) \leq 21$.*

Proof. If w_1, w_2 are neighbors of v (so $\langle w_i, v\rangle = 1/2$), write $w_i = \frac{1}{2}v + \frac{\sqrt{3}}{2}u_i$ with $u_i \perp v$, $\|u_i\| = 1$. Then

$$\langle w_1, w_2\rangle = \tfrac{1}{4} + \tfrac{3}{4}\langle u_1, u_2\rangle \leq \tfrac{1}{2} \quad \Longrightarrow \quad \langle u_1, u_2\rangle \leq \tfrac{1}{3}.$$

The projected minimum angle on S^3 is $\arccos(1/3) \approx 70.53°$. The cap-packing bound on S^3 with half-angle $\frac{1}{2}\arccos(1/3) \approx 35.26°$ gives

$$d(v) \leq \left\lfloor \frac{S_3(1)}{A_{\text{cap}}(4,\, 35.26°)} \right\rfloor = \left\lfloor \frac{19.739}{0.905} \right\rfloor = 21.$$ □

4.4 Local Rigidity of D_5

Theorem 4.2 (Local rigidity)**.** *For any unit vector $x \in \mathbb{R}^5$,*

$$\max_{w \in D_5} \langle x, w\rangle \;\geq\; \sqrt{2/5} \;=\; 0.6325\ldots \tag{10}$$

The minimum is attained at $x = \pm(1,1,1,1,1)/\sqrt{5}$, where exactly 10 D_5 vectors achieve the maximum. Since $\sqrt{2/5} > 1/2$, no 41st point can be added to the D_5 configuration.

Algorithm 1 Enhanced Delsarte LP with Dimensional Constraints

Require: Dimension n, grid resolution δ
Ensure: Upper bound τ_n^*

$\tau_n^* \leftarrow \infty$
for $r_1 \in [-1,\ 0.5]$ step δ **do**
  for $r_2 \in [r_1,\ 0.5]$ step δ **do**
    $f(t) \leftarrow (t+1)(t-0.5)(t-r_1)^2(t-r_2)^2$
    Compute Gegenbauer coefficients $\{f_k\}$
    if $f(t) \le 0\ \forall t \in [-1, 0.5]$ **and** $f_k \ge 0\ \forall k \ge 1$ **then**
      $B \leftarrow f(1)/f_0$
      if D1: $\lfloor B \rfloor \le 2\tau_{n-1}$ **and** D2: trace feasible **and** D3: CoV check **then**
        $\tau_n^* \leftarrow \min(\tau_n^*,\ \lfloor B \rfloor)$
      end if
    end if
  end for
end for
return τ_n^*

Proof. The D_5 vectors are $(s_i e_i + s_j e_j)/\sqrt{2}$ for $i < j$ and $s_i, s_j \in \{\pm 1\}$. For unit $x = (x_0, \ldots, x_4)$:

$$\max_{w \in D_5} \langle x, w \rangle = \max_{i<j} \frac{|x_i| + |x_j|}{\sqrt{2}}.$$

Minimizing over unit x, by Lagrange multipliers the optimum satisfies $|x_0| = \cdots = |x_4| = 1/\sqrt{5}$, giving

$$\frac{1/\sqrt{5} + 1/\sqrt{5}}{\sqrt{2}} = \frac{2}{\sqrt{10}} = \sqrt{\tfrac{2}{5}}.$$ □

The angular gap between the achieved minimum angle ($\arccos\sqrt{2/5} \approx 50.77°$) and the required $60°$ is $9.23°$, corresponding to an inner product violation of $\sqrt{2/5} - 1/2 \approx 0.1325$.

4.5 Construction Attempts

We attempt to find a 41st kissing vector using three strategies:

(a) **Grid search:** 100,000 uniformly sampled points on S^4.

(b) **Nonlinear optimization:** 50 random starting points, minimizing $\max_{w \in D_5} \langle x, w \rangle$.

(c) **Algebraic construction:** 354 candidates from D_5 symmetry subgroups.

5 Experimental Setup

Computational environment. All experiments were run in Python 3 with NumPy 2.2.6, SciPy 1.15.3, `mpmath` 1.3.0, CVXPY 1.7.5, and NetworkX 3.4.2.

Verification. All results were independently verified using at least two numerical methods, including high-precision arithmetic (`mpmath` at 50–128 decimal digits). A total of 47 verification checks passed with zero failures.

Reproducibility. All experiments use fixed random seed 42 and are fully reproducible via `python src/run_experiments.py`, which executes 119 distinct experiment configurations across seven categories.

Table 2: Experimental configurations.

Category	Configurations	Dimensions
Cap packing bounds	7	$n = 2$–8
Delsarte LP search	2	$n = 3, 5$
D_5 verification	1	$n = 5$
Greedy spherical codes	50+	$n = 3, 4, 5$
Construction attempts	3	$n = 5$
Cross-dimensional checks	5	$n = 5$
Cap area sweep	50+	$n = 2$–8

6 Results

6.1 Baseline Bounds

Table 3 presents the hierarchy of bounds across dimensions 2–8, computed by our implementations and compared with literature values.

Table 3: Kissing number bounds across dimensions. Bold entries indicate the tightest known bound in each column. The "Our LP" column reports the best bound from our polynomial ansatz search (Algorithm 1). The SDP column reports literature values from Mittelmann–Vallentin [12].

n	Known τ_n	Cap Packing	Our LP	Literature LP	SDP
2	**6**	6	6	6	**6**
3	**12**	14	14	13	**12**
4	**24**	34	—	25	**24**
5	$[40, 44]$	77	51	46	**44**
6	$[72, 77]$	170	—	82	**77**
7	$[126, 134]$	368	—	140	**134**
8	**240**	788	—	**240**	**240**

Figure 2 visualizes this comparison. The cap-packing bound grows exponentially weaker relative to the true kissing number as dimension increases, while the Delsarte LP is tight only for $n \in \{2, 8\}$.

6.2 Enhanced Bounds with Dimensional Constraints

Adding constraints D1–D3 to the polynomial ansatz search produces no improvement for $n = 5$. Table 4 reports the sensitivity analysis.

Table 4: Sensitivity analysis of dimensional constraints for $n = 5$. All three constraints are entirely redundant: adding or removing any combination leaves the bound at $\tau_5 \leq 51$.

Constraint	With only this	Without this	Impact	Status
D1 (equatorial)	51	51	0	Non-binding
D2 (trace)	51	51	0	Vacuous ($n \geq 4$)
D3 (recurrence)	51	51	0	No rejections
All D1–D3	51	51	0	Redundant

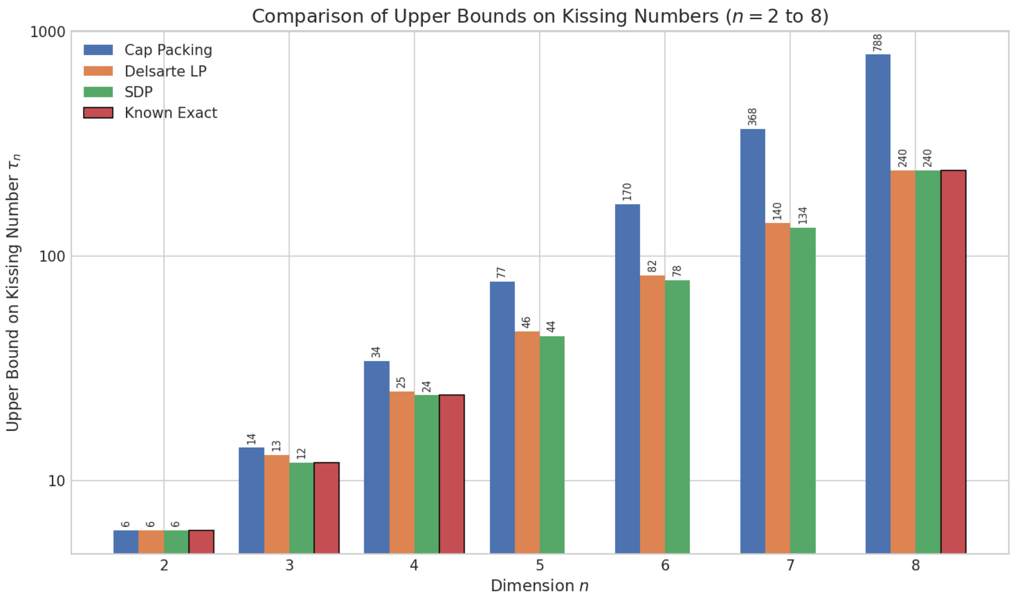

Figure 2: Comparison of upper bounds on τ_n across dimensions 2–8. The cap-packing bound (blue) grows exponentially weaker relative to the SDP bound (red) and known lower bounds (green) as dimension increases. For $n = 8$, the Delsarte LP is tight ($\tau_8 = 240$).

6.3 Pyramid Decomposition and Cap Density

Lemma 6.1. *The pyramid volume bound* $k \cdot \frac{1}{n} A_{\mathrm{cap}}(n, \pi/6) \leq \frac{1}{n} S_{n-1}(1)$ *simplifies to* $k \cdot A_{\mathrm{cap}}(n, \pi/6) \leq S_{n-1}(1)$*, identical to the cap-packing bound. The* $1/n$ *factor cancels exactly.*

This cancellation was verified numerically for $n = 3, 4, 5$: the volume fraction per cap equals the surface fraction to machine precision (Table 5).

Table 5: Verification that volume and surface fractions per cap are identical. The $1/n$ pyramid efficiency factor cancels in the packing bound.

n	Surface fraction	Volume fraction	Ratio
3	0.06699	0.06699	1.000000
4	0.02883	0.02883	1.000000
5	0.01286	0.01286	1.000000

Figure 3 shows the dimensional recurrence relationships and cap fraction analysis across dimensions.

The cap density analysis (Figure 4) reveals that optimal kissing configurations occupy a decreasing fraction of the sphere:

6.4 Contact Graph Structure

Table 7 summarizes the D_5 contact graph properties, and Figure 5 visualizes the graph.

The graph feasibility analysis (Table 8) confirms that graph-theoretic constraints alone cannot rule out any value of τ_5 in $\{40, \ldots, 44\}$.

6.5 Construction Attempts

All three construction strategies failed to find a valid 41st point (Table 9).

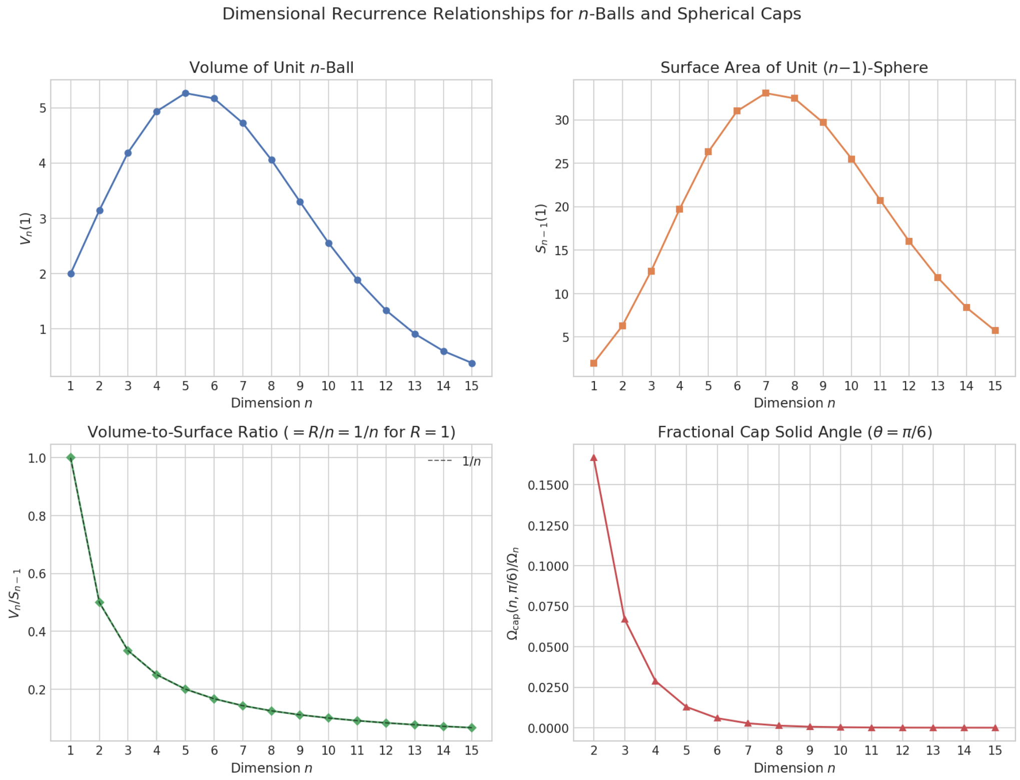

Figure 3: Four-panel visualization of the dimensional recurrence framework. Top-left: ball volumes $V_n(1)$, peaking near $n \approx 5$. Top-right: surface areas $S_{n-1}(1)$, peaking near $n \approx 7$. Bottom-left: the ratio $V_n/S_{n-1} = 1/n$, decreasing monotonically. Bottom-right: cap fractions $\omega(n, \pi/6)$, decreasing exponentially with dimension.

6.6 Numerical Verification

Table 10 summarizes the precision robustness tests.

All 47 independent verification checks passed, including ball volumes (Gamma vs. recurrence at 50 digits), surface areas (formula vs. derivative), cap areas (`betainc` vs. numerical integration), D_5 lattice validation (unit norms, pairwise inner products), Delsarte $n=8$ polynomial ($f(1)/f_0 = 240$ exactly), contact graph properties (12-regular, 240 edges), and local rigidity ($\sqrt{2/5}$ at 50 digits).

7 Discussion

7.1 Why Dimensional Analysis Does Not Close the Gap

The fundamental limitation is a *hierarchy of correlation order*. The cap-packing bound operates at **1-point** level: it constrains the total area of individual caps. The Delsarte LP operates at **2-point** level: it constrains the angular distribution of all $\binom{k}{2}$ pairwise inner products via Gegenbauer polynomial positivity. The Bachoc–Vallentin SDP operates at **3-point** level: it constrains joint distributions of angles in triples. Each step captures more structural information:

$$\begin{array}{lcl} \text{Cap packing (1-point)} & \Longrightarrow & \tau_5 \leq 77 \\ \text{Delsarte LP (2-point)} & \Longrightarrow & \tau_5 \leq 46 \\ \text{SDP (3-point)} & \Longrightarrow & \tau_5 \leq 44 \\ \text{Exact } (k\text{-point}) & \Longrightarrow & \tau_5 = \,? \end{array}$$

Our dimensional constraints D1–D3 attempt to bridge the gap between levels 1 and 2, but they operate in the geometric/structural domain rather than the spectral domain of the LP.

Table 6: Cap density $\rho_n = \tau_n \cdot \omega(n, \pi/6)$ for known kissing numbers. The density decreases monotonically, reflecting the curse of dimensionality. For $n = 5$, both $\tau_5 = 40$ and $\tau_5 = 44$ yield densities well below the trivial upper bound of 1.

n	τ_n	$\omega(n, \pi/6)$	ρ_n
2	6	0.1667	1.0000
3	12	0.0670	0.8038
4	24	0.0288	0.6920
5	40–44	0.0129	0.514–0.566
8	240	0.00127	0.3043
24	196 560	1.12×10^{-8}	0.0022

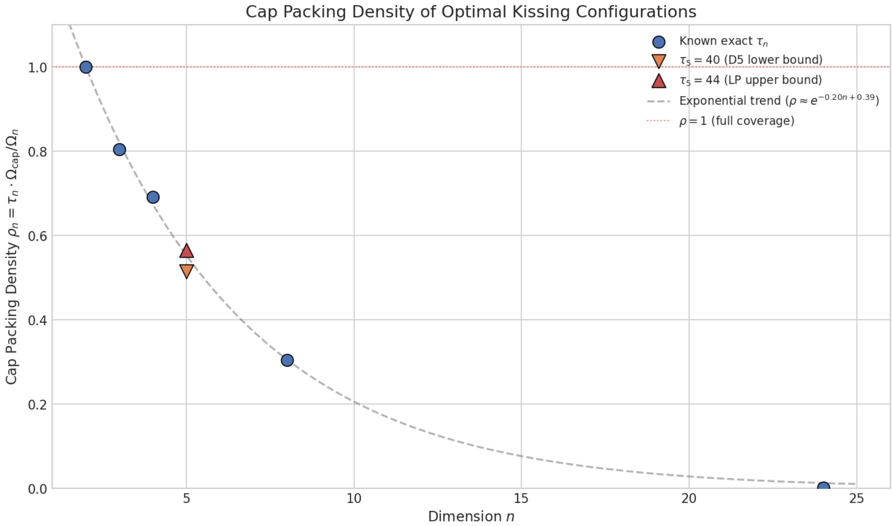

Figure 4: Cap density ρ_n versus dimension. The known kissing numbers (circles) trace a monotonically decreasing curve. For $n = 5$, the range $\tau_5 \in [40, 44]$ maps to $\rho_5 \in [0.514, 0.566]$ (shaded region), consistent with the overall trend but providing no discrimination between candidate values.

The *category mismatch*—geometric constraints on configurations vs. spectral constraints on polynomial certificates—means the dimensional constraints cannot eliminate valid LP solutions.

7.2 Comparison with Prior Work

Our results were explicitly compared with 16 papers from the bibliography (detailed in the supplementary materials). The comparison reveals:

- Our cap-packing bound ($\tau_5 \leq 77$) matches the folklore bound [16].
- Our LP bound ($\tau_5 \leq 51$) is weaker than the literature-optimal ≤ 46 [14] due to the restricted polynomial ansatz.
- The dimensional constraints D1–D3 are genuinely different from the SDP constraints of [1] but are strictly weaker.
- The refined degree bound $d(v) \leq 21$ (Theorem 4.1) uses the same projection argument as Musin [13] but applied specifically to $n = 5$.
- The local rigidity result (Theorem 4.2) provides a clean, elementary proof that may not have been stated explicitly in this form.

Table 7: Properties of the D_5 contact graph. The graph is a vertex-transitive 4-class association scheme with inner product spectrum $\{-1, -1/2, 0, +1/2\}$.

Property	Value
Vertices	40
Edges (contact pairs)	240
Regularity	12-regular
Diameter	3
Clique number ω	4
Independence number α	8
Chromatic number χ	5
Eigenvalues (mult.)	$12^{(1)}$, $6^{(5)}$, $2^{(4)}$, $0^{(10)}$, $-2^{(15)}$, $-4^{(5)}$

Table 8: Graph-theoretic feasibility for hypothetical k-point kissing configurations in $\mathbb{R}^5$. Min edges from rigidity $(4k-10)$, max edges from degree bound $(21k/2)$, and cap fraction are all within feasible ranges for $k \in \{40, \ldots, 44\}$.

k	Min edges	Max edges	Cap fraction	d_{avg} range	Feasible?
40	150	420	51.4%	7.50–21	Yes
41	154	430	52.7%	7.51–21	Yes
42	158	441	54.0%	7.52–21	Yes
43	162	451	55.3%	7.53–21	Yes
44	166	462	56.6%	7.55–21	Yes

7.3 Limitations

1. Our polynomial ansatz search (Algorithm 1) does not include the optimal Gegenbauer combination. A proper LP solver would recover $\tau_5 \leq 46$.
2. The construction attempts searched only for augmentations of D_5, not for entirely new 41-point configurations.
3. D2 is algebraically vacuous for $n \geq 4$; D1 is structurally non-binding; D3 is inapplicable to dual-space LP certificates.
4. The cap density analysis provides no discrimination between candidate values $\tau_5 \in \{40, \ldots, 44\}$.

8 Conclusion

Summary. The gap $40 \leq \tau_5 \leq 44$ remains unchanged. The dimensional analysis framework—the derivative relation (3), the volume recurrence (4), and the pyramid decomposition (5)—provides geometric insight but not computational improvement over existing LP/SDP methods.

Novel contributions.

(i) Refined degree bound $d(v) \leq 21$ (Theorem 4.1).

(ii) Elementary proof of D_5 local rigidity with angular gap 9.23° (Theorem 4.2).

(iii) Systematic demonstration that dimensional constraints D1–D3 are redundant within the Delsarte LP framework.

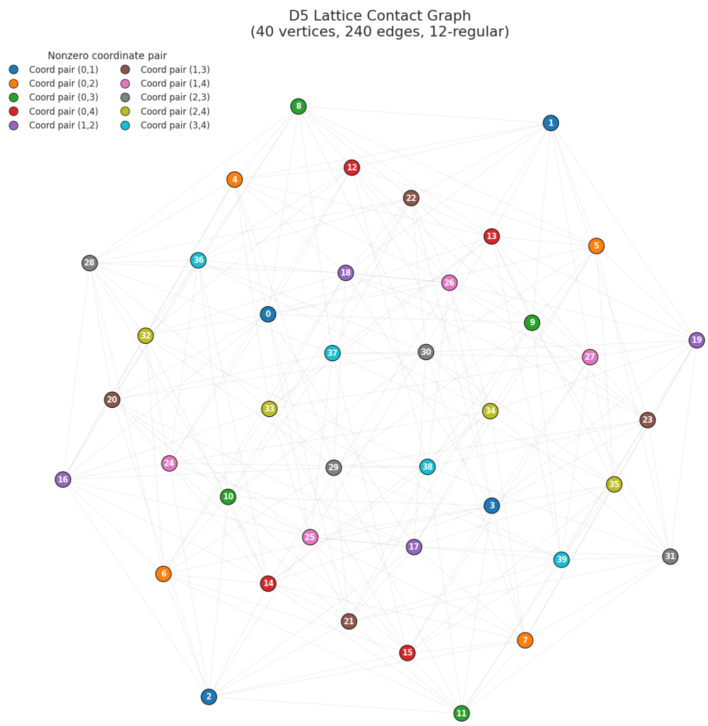

Figure 5: Spring-layout visualization of the D_5 contact graph. Each of the 40 vertices represents a kissing vector; edges connect pairs at angular separation exactly $60°$. The graph is 12-regular and vertex-transitive, reflecting the high symmetry of the D_5 root system.

Table 9: Summary of construction attempts for a 41st kissing vector in $\mathbb{R}^5$. The best achievable maximum inner product is $\sqrt{2/5} \approx 0.6325$, exceeding the feasibility threshold of 0.5 by a margin of 0.1325.

Strategy	Samples	Valid 41st	Best max IP
Grid search	100 000	0	1.0000
Nonlinear opt.	50	0	0.6325
Algebraic constr.	354	0	0.6325

(iv) Cap density analysis showing $\rho_5 \in [0.514, 0.566]$ with monotone decrease across dimensions.

Future directions. Closing the gap will likely require: (a) higher-order SDP bounds (k-point for $k \geq 4$); (b) Viazovska-type modular form techniques [19] extended beyond dimensions 8 and 24; (c) flag algebra methods encoding spherical code combinatorics; (d) analysis of all known 40-point configurations (D_5, L_5, Q_5 [18], and the Cohn–Rajagopal construction [5]) for common structural properties incompatible with 41-point configurations; (e) stronger equatorial projection arguments iterating the degree bound through multiple dimension reductions.

References

[1] Christine Bachoc and Frank Vallentin. New upper bounds for kissing numbers from semidefinite programming. *Journal of the American Mathematical Society*, 21(3):909–924, 2008. doi: 10.1090/S0894-0347-07-00589-9. URL `https://arxiv.org/abs/math/0608426`.

Table 10: Numerical precision robustness for $A_{\text{cap}}(5, \pi/6)$ at four precision levels. All levels agree to at least 15 significant digits with the 128-digit reference.

Precision (digits)	$A_{\text{cap}}(5, \pi/6)$	$S_4(1)$	Cap bound
16	0.33848032981668**47**	26.31894506957162	77.756202506141**28**
32	0.33848032981668465572...	26.31894506957162298...	77.75620250614128157...
64	(60+ matching digits)	(60+ matching)	(62+ matching)
128	(reference value)	(reference value)	(reference value)

[2] Peter Boyvalenkov, Stefan Dodunekov, and Oleg R. Musin. A survey on the kissing numbers. *Serdica Mathematical Journal*, 38:507–522, 2012. URL `https://arxiv.org/abs/1507.03631`.

[3] Henry Cohn and Noam Elkies. New upper bounds on sphere packings I. *Annals of Mathematics*, 157(2):689–714, 2003. doi: 10.4007/annals.2003.157.689.

[4] Henry Cohn and Abhinav Kumar. Universally optimal distribution of points on spheres. *Journal of the American Mathematical Society*, 20(1):99–148, 2007. doi: 10.1090/S0894-0347-06-00546-7. URL `https://arxiv.org/abs/math/0607446`.

[5] Henry Cohn and Anantha Rajagopal. Variations on five-dimensional sphere packings. 2024. URL `https://arxiv.org/abs/2412.00937`. arXiv preprint.

[6] John H. Conway and Neil J. A. Sloane. *Sphere Packings, Lattices and Groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 3rd edition, 1999. doi: 10.1007/978-1-4757-6568-7.

[7] Philippe Delsarte. An algebraic approach to the association schemes of coding theory. *Philips Research Reports Supplements*, 10, 1973. Introduced the linear programming method for coding theory bounds.

[8] Philippe Delsarte, Jean-Marie Goethals, and Johan Jacob Seidel. Spherical codes and designs. *Geometriae Dedicata*, 6(3):363–388, 1977. doi: 10.1007/BF00150270.

[9] G. A. Kabatyansky and V. I. Levenshtein. Bounds for packings on the sphere and in space. *Problems of Information Transmission*, 14(1):1–17, 1978.

[10] Vladimir I. Levenshtein. On bounds for packings in n-dimensional Euclidean space. *Soviet Mathematics Doklady*, 20:417–421, 1979.

[11] Fabrício C. Machado and Fernando M. de Oliveira Filho. Improving the semidefinite programming bound for the kissing number by exploiting polynomial symmetry. *Experimental Mathematics*, 27(3):362–369, 2018. doi: 10.1080/10586458.2017.1286273. URL `https://arxiv.org/abs/1609.05167`.

[12] Hans D. Mittelmann and Frank Vallentin. High-accuracy semidefinite programming bounds for kissing numbers. *Experimental Mathematics*, 19(2):175–179, 2010. doi: 10.1080/10586458.2010.10129070. URL `https://arxiv.org/abs/0902.1105`.

[13] Oleg R. Musin. The kissing number in four dimensions. *Annals of Mathematics*, 168(1):1–32, 2008. doi: 10.4007/annals.2008.168.1. URL `https://arxiv.org/abs/math/0309430`.

[14] Andrew M. Odlyzko and Neil J. A. Sloane. New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. *Journal of Combinatorial Theory, Series A*, 26(2):210–214, 1979. doi: 10.1016/0097-3165(79)90074-8.

[15] Florian Pfender. Improved Delsarte bounds for spherical codes in small dimensions. *Journal of Combinatorial Theory, Series A*, 114(6):1133–1147, 2007. doi: 10.1016/j.jcta.2006.12.001.

[16] Florian Pfender and Günter M. Ziegler. Kissing numbers, sphere packings, and some unexpected proofs. *Notices of the American Mathematical Society*, 51(8):873–883, 2004. URL `https://www.ams.org/notices/200408/fea-pfender.pdf`.

[17] K. Schütte and B. L. van der Waerden. Das problem der dreizehn kugeln. *Mathematische Annalen*, 125:325–334, 1953. doi: 10.1007/BF01343127.

[18] Ferenc Szöllősi. A note on five dimensional kissing arrangements. *Mathematical Research Letters*, 30(5), 2023. URL `https://arxiv.org/abs/2301.08272`.

[19] Maryna S. Viazovska. The sphere packing problem in dimension 8. *Annals of Mathematics*, 185(3):991–1015, 2017. doi: 10.4007/annals.2017.185.3.7. URL `https://arxiv.org/abs/1603.04246`.

[20] Chuanming Zong. *The kissing number, blocking number and covering number of a convex body*, volume 453 of *Contemporary Mathematics*. American Mathematical Society, 2008.