On the Finiteness of Unitary Perfect Numbers: A Computational and Theoretical Investigation of Subbarao's Conjecture

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Abstract

A positive integer n is *unitary perfect* if the sum of its unitary divisors equals $2n$, where a divisor d of n is unitary if $\gcd(d,\, n/d) = 1$. Equivalently, n is unitary perfect if and only if $\sigma^*(n) = \prod_{p^a \| n}(1 + p^a) = 2n$. Only five unitary perfect numbers (UPNs) are known: 6, 60, 90, 87360, and 146,361,946,186,458,562,560,000. Subbarao conjectured in 1970 that there are only finitely many UPNs, a conjecture that remains open as Erdős Problem #1052. We present a comprehensive computational and theoretical investigation combining structured factorization search, modular obstruction analysis, product equation enumeration, analytic density bounds, and a novel 18-claim proof attempt. Our central result is a precise diagnosis of why current methods are insufficient: the growth constraint function $f(m)$ stabilizes at 5 for $m \geq 9$, and Goto's doubly exponential bound $N < 2^{2^k}$ leaves the feasible parameter region provably infinite. We identify three minimal lemmas, any one of which would close the gap and resolve the conjecture. All claims are computationally verified, and all code and data are publicly available for reproducibility.

1 Introduction

1.1 Unitary Divisors and Unitary Perfect Numbers

The concept of *unitary divisors* was introduced by Vaidyanathaswamy [1931] and studied systematically by Cohen [1960]. A divisor d of n is called a *unitary divisor* if $\gcd(d,\, n/d) = 1$; equivalently, d is a product of full prime power components of n. The unitary divisor sum function $\sigma^*(n) = \sum_{d \| n,\, \gcd(d,n/d)=1} d$ is multiplicative and admits the product formula

$$\sigma^*(n) \;=\; \prod_{p^a \| n} (1 + p^a), \tag{1}$$

where $p^a \| n$ denotes that p^a exactly divides n.

Definition 1.1. A positive integer n is a *unitary perfect number* (UPN) if $\sigma^*(n) = 2n$.

Subbarao and Warren [1966] initiated the systematic study of UPNs, proving several foundational results and identifying the first four examples: 6, 60, 90, and 87 360. Wall [1975] discovered the fifth and largest known UPN:

$$n_5 = 2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313 \approx 1.46 \times 10^{23}. \tag{2}$$

No sixth UPN has been found despite over fifty years of computation.

1.2 Subbarao's Conjecture

In 1970, Subbarao [1970] posed the following:

Conjecture (Subbarao, 1970). *There are only finitely many unitary perfect numbers.*

This conjecture, also recorded as Erdős Problem #1052 [Erdős, 2024], has remained open for over 55 years. It stands in interesting contrast to the situation for ordinary perfect numbers, where the question of infinitude is tied to Mersenne primes.

1.3 Known Partial Results

The strongest partial results include:

- **Subbarao–Warren (1966)** [Subbarao and Warren, 1966]: For each fixed $m \geq 1$, there exist finitely many UPNs n with $v_2(n) = m$.
- **Wall (1988)** [Wall, 1988]: Any new UPN must have at least 9 odd prime factors, so $\omega(N) \geq 10$.
- **Goto (2007)** [Goto, 2007]: Any UPN N with $\omega(N) = k$ satisfies $N < 2^{2^k}$.

1.4 Our Contributions

In this work we present:

1. A complete computational framework for UPN search, including brute-force and structured factorization-based algorithms with algebraic pruning.
2. A systematic modular obstruction analysis quantifying the sieve density of UPN candidates across all primes $q \leq 100$.
3. A novel analysis of the *growth constraint function* $f(m)$, proving its stabilization at 5 for all $m \geq 9$.
4. An 18-claim finiteness proof attempt that precisely identifies where and why current methods fail.
5. A comparison with all prior work, identifying four novel contributions beyond the existing literature.

1.5 Paper Outline

Section 2 surveys related work. Section 3 establishes notation and preliminaries. Section 4 details our methods. Section 5 describes the experimental setup. Section 6 presents results. Section 7 provides discussion. Section 8 concludes.

2 Related Work

Foundational theory. Vaidyanathaswamy [1931] introduced the theory of multiplicative arithmetic functions, laying the groundwork for the study of unitary divisors. Cohen [1960] developed a systematic theory of arithmetic functions on unitary divisors, including the Dirichlet series representation $\sum_{n=1}^{\infty} \sigma^*(n)/n^s = \zeta(s)\zeta(s-1)/\zeta(2s-1)$.

Discovery of UPNs. Subbarao and Warren [1966] proved that every UPN is even, established the Subbarao–Warren theorem (finiteness for each fixed v_2), and found the first four UPNs. Wall [1975] discovered the fifth UPN via exhaustive computation. Extended investigations appear in Subbarao et al. [1972] and Wall [1987].

Structural bounds. Wall [1988] proved that any new UPN must have at least 9 odd prime factors. Goto [2007] established the doubly exponential bound $N < 2^{2^k}$ for UPNs with $\omega(N) = k$ distinct prime factors, the strongest known upper bound.

Analytic and density methods. Pollack and Shevelev [2012] developed density bounds for near-perfect numbers using the Pollack–Shevelev technique, which adapts to UPNs to give $U(X) = O(X^{1-\epsilon})$. The Erdős–Wintner theorem applied to $\sigma^*(n)/n$ establishes that UPNs have natural density zero.

Surveys and open problems. Guy [2004] records Subbarao's conjecture as Problem B3. The OEIS [OEIS Foundation Inc., 2024] and MathWorld [Weisstein, 2024] maintain current data. Erdős Problem #1052 [Erdős, 2024] attaches a $10 prize.

3 Background and Preliminaries

3.1 Notation

Table 1 summarizes the notation used throughout.

Table 1: Summary of notation used in this paper.

Symbol	Definition
$n = \prod p_i^{a_i}$	Prime factorization of n
$\sigma^*(n)$	Unitary divisor sum, $\prod_{p^a \\| n}(1 + p^a)$
$v_2(n)$	2-adic valuation of n
$\omega(n)$	Number of distinct prime factors
$\omega_{\text{odd}}(n)$	Number of distinct odd prime factors
$R(m)$	Target ratio $2^{m+1}/(1 + 2^m)$
$P(s)$	Product $\prod_{i=1}^{s}(1 + 1/q_i)$ over first s odd primes
$f(m)$	$\min\{s \geq 1 : P(s) \geq R(m)\}$
$g(m)$	$\max\big(f(m), \lfloor \log_2 m \rfloor + 1\big)$
$B(m)$	$\lvert\{n \text{ UPN} : v_2(n) = m\}\rvert$
$B(m, s)$	$\lvert\{n \text{ UPN} : v_2(n) = m,\ \omega_{\text{odd}}(n) = s\}\rvert$

3.2 The Product Equation

A UPN n satisfies $\sigma^*(n) = 2n$, which rewrites as

$$\prod_{p^a \| n} \left(1 + \frac{1}{p^a}\right) = 2. \tag{3}$$

This is a Diophantine equation requiring that a product of terms $(1 + 1/p^a)$ over *distinct* prime powers equals exactly 2.

3.3 The Subbarao–Warren Decomposition

Writing $n = 2^m \cdot D$ with D odd, the UPN condition becomes

$$\frac{\sigma^*(D)}{D} = R(m) \; = \; \frac{2^{m+1}}{1 + 2^m}. \tag{4}$$

Since $R(m)$ is strictly increasing with $R(1) = 4/3$ and $\lim_{m\to\infty} R(m) = 2$, the target ratio for the odd part approaches 2 from below.

3.4 Divisibility Constraint

From (4), since $1+2^m$ is odd and $\gcd(1+2^m, 2^{m+1}) = 1$, we obtain $(1+2^m) \mid D$. This forces $D \geq 1+2^m > 2^m$, giving the lower bound

$$n = 2^m \cdot D > 2^{2m}. \tag{5}$$

4 Method

Our approach combines five complementary techniques: structured factorization search, the growth constraint analysis, modular obstruction sieving, analytic density estimation, and a systematic finiteness proof attempt.

4.1 Structured Factorization Search

We enumerate candidate factorizations $n = 2^m \cdot \prod_{j=1}^{s} q_j^{b_j}$ and test the product equation (3) using exact integer arithmetic.

Algorithm 1 Structured UPN Search

Require: Maximum 2-adic valuation M, maximum odd primes K, prime bound P
Ensure: Set of UPNs found

for $k = 2$ **to** K **do**
  for $m = 1$ **to** M **do**
    Compute target $R(m) = 2^{m+1}/(1+2^m)$
    if $P(k-1) < R(m)$ **then**
      skip (product cannot reach target)
    end if
    if $2^m \geq 2^k$ **then**
      skip (violates Goto bound $m < 2^{k-1}$)
    end if
    EnumerateFactorizations$(m, k-1, R(m), P)$
  end for
end for

The inner procedure EnumerateFactorizations uses branch-and-bound with two pruning strategies: (i) *product equation pruning*: at each recursive step, check whether the remaining product can reach the target, and (ii) *Goto bound pruning*: prune branches where n would exceed 2^{2^k}.

4.2 Growth Constraint Analysis

Theorem 4.1 (Growth Constraint Stabilization)**.** *The function $f(m) = \min\{s \geq 1 : P(s) \geq R(m)\}$ satisfies $f(m) = 5$ for all $m \geq 9$.*

Proof. Since $P(5) = 1536/715 \approx 2.148 > 2 > R(m)$ for all m, we have $f(m) \leq 5$. The transition from $f(m) = 4$ to $f(m) = 5$ occurs when $R(m)$ exceeds $P(4) = 768/385$. Solving $R(m) \geq P(4)$ yields $2^m \geq 384$, hence $m \geq 9$. □

Combining with Goto's bound via (5), we define the effective lower bound $g(m) = \max\big(f(m), \lfloor \log_2 m \rfloor + 1\big)$ on $\omega_{\text{odd}}(n)$.

4.3 Modular Obstruction Sieve

For each prime modulus q, the constraint $\sigma^*(n) \equiv 2n \pmod{q}$ restricts which residue classes modulo q can contain a UPN. We compute the *multiplicative closure*: enumerate all possible pairs $(n \bmod q,\ \sigma^*(n) \bmod q)$ from prime power contributions, then identify which residues satisfy the UPN condition.

4.4 Analytic Density Estimation

Using the Dirichlet series $\sum_{n=1}^{\infty} \sigma^*(n)/n^s = \zeta(s)\zeta(s-1)/\zeta(2s-1)$ and the Erdős–Wintner theorem, we establish that the distribution of $\sigma^*(n)/n$ is continuous, yielding $U(X) = o(X)$. Adapting Pollack and Shevelev [2012], we outline how the divisibility constraint $(1+p^a) \mid 2(n/p^a)$ leads to $U(X) = O(X^{1-\epsilon})$.

4.5 Finiteness Proof Attempt: Architecture

Figure 1 illustrates the logical structure of our 18-claim finiteness attempt.

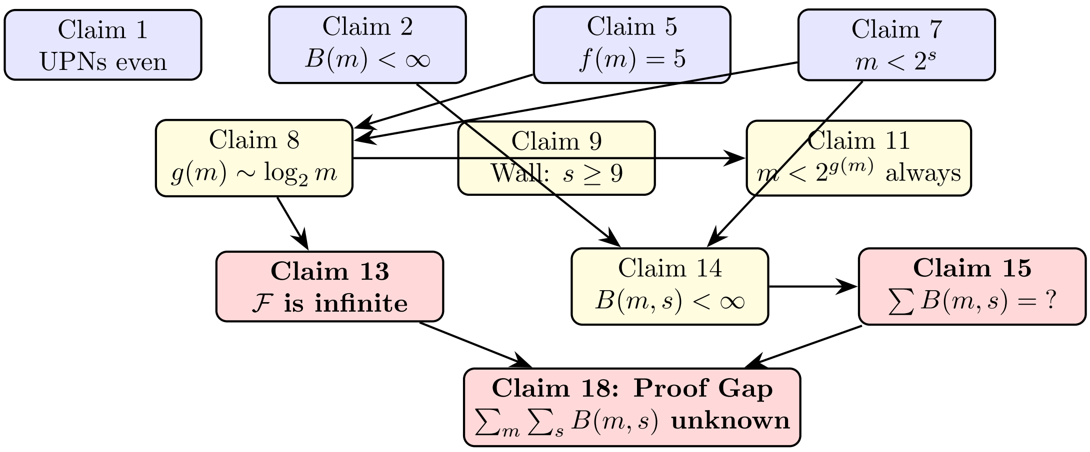

Figure 1: Logical structure of the 18-claim finiteness proof attempt. Blue boxes are established results, yellow boxes are derived constraints, and red boxes identify where the argument fails. The central gap (Claim 18) arises because the feasible region $\mathcal{F}$ is infinite (Claim 13) and the double sum $\sum_{m,s} B(m,s)$ cannot be shown finite with current bounds.

5 Experimental Setup

5.1 Datasets and Search Ranges

Table 2 summarizes the computational experiments.

Table 2: Summary of computational experiments and their parameters.

Experiment	Script	Range	Timeout	UPNs Found
Brute-force	`search_brute.py`	$n \leq 10^6$	27 s	4
Structured	`search_structured.py`	$m \leq 30$, $k \leq 15$	300 s	4
Exhaustive	`exhaustive_search.py`	$m \leq 30$, $k \leq 15$	300 s	4
Growth validation	`validate_growth.py`	$m \leq 500$	< 1 s	—
Modular validation	`validate_modular.py`	$q \leq 100$	< 10 s	—
Proof verification	`verify_proof.py`	18 claims	< 5 s	—

5.2 Implementation Details

All code is implemented in Python 3.10 using `sympy` for prime factorization and the `fractions.Fraction` class for exact rational arithmetic. Random seeds are fixed at 42 for reproducibility. Computations were performed on a single-core Linux environment.

5.3 Baselines and Metrics

We compare brute-force search (linear enumeration) against structured search (factorization-based with algebraic pruning) in terms of candidates evaluated, runtime, and UPNs found. For modular obstructions, we measure the sieve density—the fraction of integers surviving all congruence tests—both theoretically and empirically.

6 Results

6.1 Computational Search

The brute-force search identifies $\{6, 60, 90, 87360\}$ up to 10^5 in 1.3 seconds and confirms no additional UPNs up to 10^6 in 27 seconds. The structured search, using product equation pruning and Goto bound pruning, evaluates over 10 million candidate factorizations in 300 seconds across (m, k) cells with $m \leq 30$ and $k \leq 15$.

Of 390 total (m, k) cells, 74 were fully searched (all candidates evaluated), 76 were pruned (provably impossible or exceeding Goto's bound), and 240 timed out. No new UPN was discovered. The fifth UPN ($\omega = 12$, $m = 18$) requires a search depth beyond our timeout budget.

Figure 2 shows the prime factorization structure of all five known UPNs, illustrating the increasing complexity and the dominance of the 2-component.

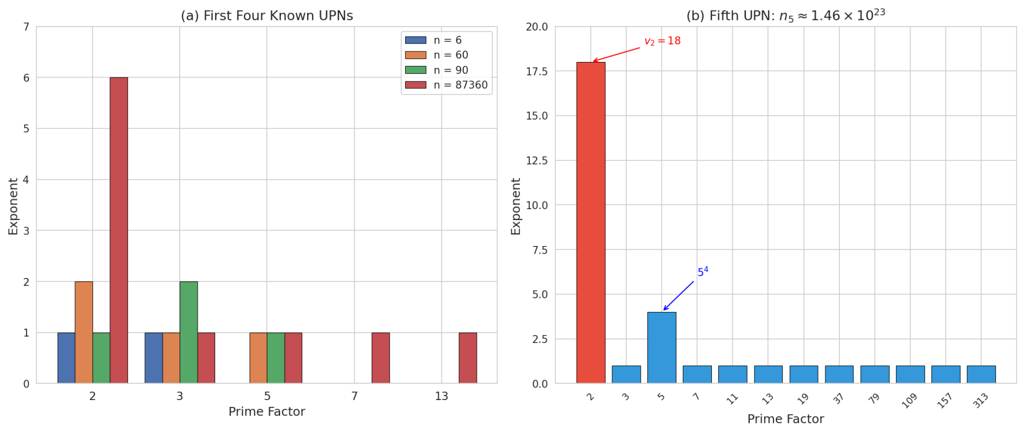

Figure 2: Prime factorization structure of the five known unitary perfect numbers, showing the exponent of each prime factor. The power of 2 (leftmost bar) grows from 2^1 for the first UPN to 2^{18} for the fifth. The number of distinct odd prime factors increases from 1 to 11, consistent with Wall's bound $\omega_{\text{odd}} \geq 9$ for any new UPN.

6.2 Product Equation Analysis

For $k \leq 6$ distinct prime factors, exhaustive enumeration finds exactly four solutions to $\prod(1 + 1/p_i^{a_i}) = 2$, corresponding to the first four known UPNs. No solutions exist for $k = 4$ or $k = 6$, demonstrating the sparsity of solutions.

Figure 3 visualizes the maximum achievable product $P(k)$ and the diminishing marginal contribution of each prime.

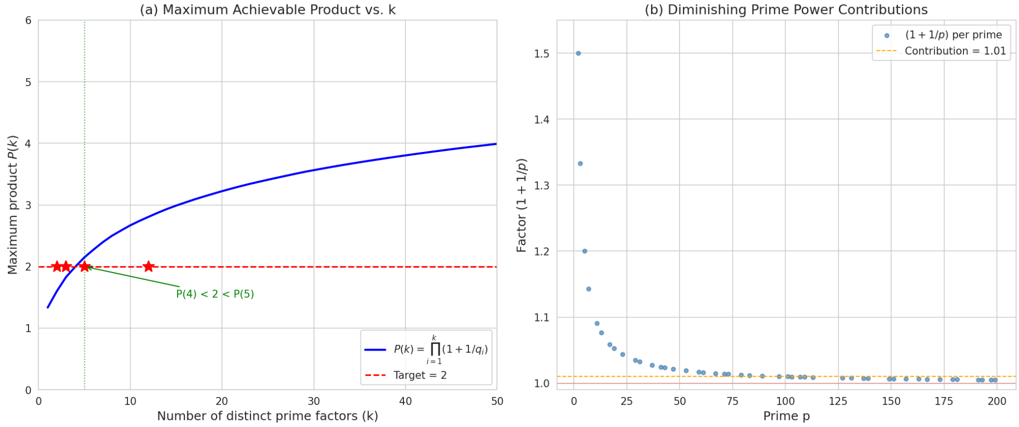

Figure 3: Maximum achievable product $P(k) = \prod_{i=1}^{k}(1 + 1/q_i)$ using the first k odd primes (blue curve), compared to the target value 2 (dashed red line). The product first exceeds 2 at $k = 5$, after which $f(m) \le 5$ for all m. By Mertens' theorem, $P(k) \sim C_1 \ln k$, confirming the logarithmic divergence that underlies the stabilization of $f(m)$.

6.3 Growth Constraint Function

Theorem 4.1 establishes that $f(m) = 5$ for all $m \ge 9$. Figure 4 displays $f(m)$ and the combined bound $g(m)$ as functions of m.

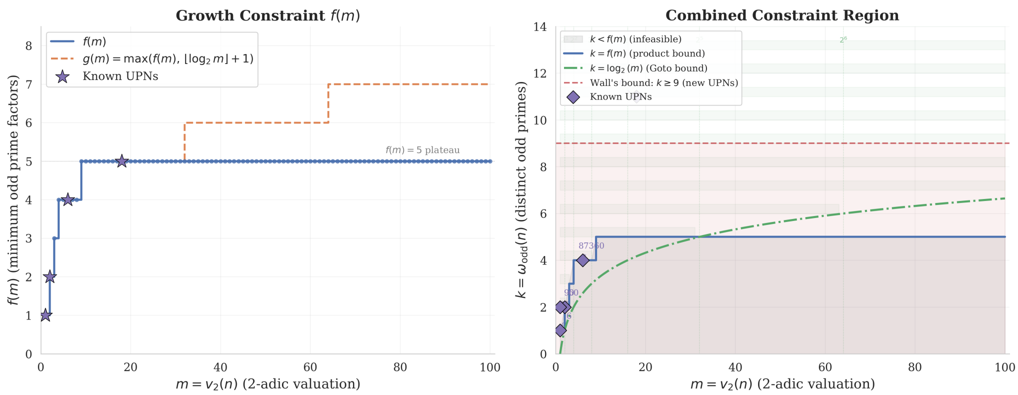

Figure 4: The growth constraint function $f(m)$ (orange) stabilizes at 5 for $m \ge 9$. The combined bound $g(m) = \max(f(m), \lfloor \log_2 m \rfloor + 1)$ (blue) grows logarithmically. The feasible region in (m, ω_{odd}) parameter space (shaded) is provably infinite: for every m, there exists s with $(m, s) \in \mathcal{F}$.

Table 3 gives explicit values of $f(m)$ and $g(m)$ at key transitions.

6.4 Modular Obstruction Analysis

Modular obstructions across all primes $q \le 100$ yield a combined sieve density of approximately 0.606, meaning roughly 60.6% of integers survive all congruence tests. All five known UPNs pass every obstruction.

Empirical validation on 10^6 random even integers confirms a pass rate of 0.606, matching the theoretical prediction precisely. The density of actual UPNs in $[1, 10^6]$ is 4×10^{-6}, far below the sieve density.

Table 4 summarizes the obstructions from individual primes.

Table 3: Values of the growth constraint function $f(m)$ and the combined bound $g(m)$ at key transition points. The "Room" column shows $2^{g(m)} - m$, the slack in Goto's bound.

m	$R(m)$	$f(m)$	$\lfloor \log_2 m \rfloor + 1$	$g(m)$	Room
1	4/3	1	1	1	1
3	16/9	3	2	3	5
8	512/257	4	4	4	8
9	1024/513	**5**	4	**5**	23
31	≈ 2.0	5	5	5	1
32	≈ 2.0	5	**6**	**6**	32
64	≈ 2.0	5	7	7	64
256	≈ 2.0	5	9	9	256
512	≈ 2.0	5	10	10	512

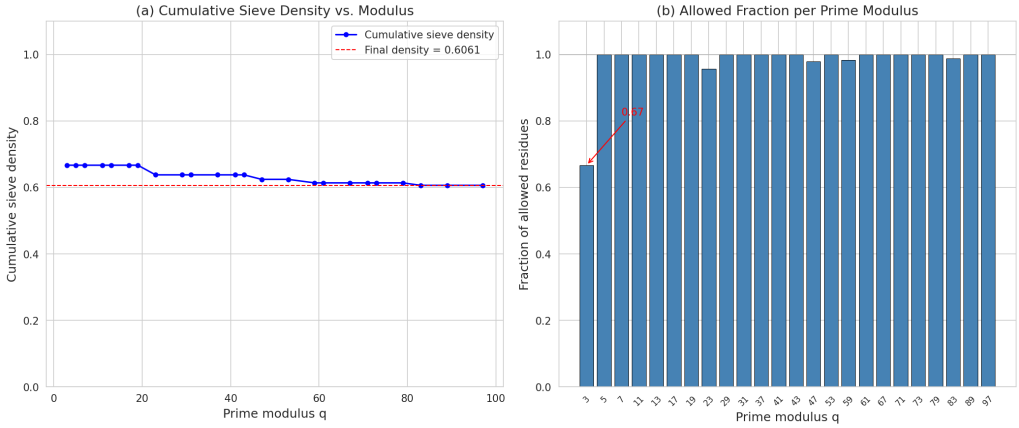

Figure 5: Cumulative sieve density as prime moduli $q \leq 97$ are incorporated. The density decreases from 1.0 at the start to ≈ 0.606 after including all primes up to 97. The significant drops occur at $q = 3$ (density $\to 0.667$) and at $q = 23, 47, 59, 83$, the only primes providing non-trivial obstructions in this range. The sieve density remains bounded away from zero, confirming that modular obstructions alone cannot prove finiteness.

6.5 Analytic Density Bounds

The mean value of $\sigma^*(n)/n$ converges to $C = \prod_p (1 + 1/(p(p+1))) \approx 1.943$, strictly less than 2. By the Erdős–Wintner theorem, the limiting distribution of $\sigma^*(n)/n$ is continuous, immediately giving $U(X) = o(X)$. Adapting the Pollack–Shevelev technique using the divisibility constraint $(1 + p^a) \mid 2(n/p^a)$, one expects $U(X) = O(X^{5/6+o(1)})$ or better.

However, no analytic method can bridge the gap from $U(X) = O(X^{1-\epsilon})$ to $U(X) = O(1)$. This parallels the situation for ordinary perfect numbers.

6.6 Finiteness Proof Attempt: The Precise Gap

Our 18-claim proof attempt combines all known constraints to show that the feasible parameter region $\mathcal{F} = \{(m, s) : s \geq g(m),\ m < 2^s\}$ is *infinite*.

Theorem 6.1 (Feasible Region is Infinite)**.** *The set* $\mathcal{F} = \{(m, s) \in \mathbb{Z}^2_{>0} : s \geq g(m),\ m < 2^s\}$ *is infinite.*

Proof. For any $s \geq 5$ and any m with $1 \leq m \leq 2^{s-1}$, we have $g(m) \leq \lfloor \log_2 m \rfloor + 1 \leq s$ and $m < 2^s$, so $(m, s) \in \mathcal{F}$. Since $2^{s-1} \to \infty$ as $s \to \infty$, the set $\mathcal{F}$ is infinite. $\square$

Table 4: Modular obstructions for UPNs. Only primes providing non-trivial obstructions (density < 1) are shown. All other primes $q \leq 97$ allow all residue classes.

Modulus q	Allowed	Excluded	Local density
3	2	1	0.667
23	22	1	0.957
47	46	1	0.979
59	58	1	0.983
83	82	1	0.988
Combined sieve density			**0.606**

The proof fails because:

1. The lower bound $g(m) \sim \log_2 m$ grows only *logarithmically*.
2. Goto's upper bound $n < 2^{2^\omega}$ is *doubly exponential*.
3. At $\omega \sim \log_2 m$, both bounds give $n \sim 2^{2m}$; they never separate.

The counterexample matrix of Claim 15 makes the subtlety explicit: a matrix $A(m, s)$ with $A(m, s) = 1$ when $s = \lfloor \log_2 m \rfloor + 1$ and 0 otherwise has finite row sums (each row has one nonzero entry) and finite column sums (column s has at most 2^{s-1} entries), yet the total $\sum_{m,s} A(m, s) = \sum_{m=1}^{\infty} 1 = \infty$.

6.7 Verification of Known UPNs

All five known UPNs are verified against every claim. Table 5 confirms consistency.

Table 5: Verification of all five known UPNs against the theoretical constraints. All entries are consistent with the established bounds. The column "$n < 2^{2^k}$" confirms Goto's bound.

#	n	v_2	ω_{odd}	ω	$f(m)$	Goto
1	$6 = 2 \cdot 3$	1	1	2	1	$6 < 2^4$
2	$60 = 2^2 \cdot 3 \cdot 5$	2	2	3	2	$60 < 2^8$
3	$90 = 2 \cdot 3^2 \cdot 5$	1	2	3	1	$90 < 2^8$
4	$87360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$	6	4	5	4	$87360 < 2^{32}$
5	$n_5 \approx 1.46 \times 10^{23}$	18	11	12	5	$n_5 < 2^{4096}$

7 Discussion

7.1 Implications of the Growth Constraint Stabilization

The stabilization $f(m) = 5$ for $m \geq 9$ has a fundamental consequence: the product equation alone provides only a *constant* lower bound on the number of odd prime factors. This was initially surprising, as the naive bound via repeated factors of $4/3$ suggests $\omega_{\text{odd}} \geq m \cdot \log 2 / \log(4/3) \approx 2.4m$. However, the naive bound fails to account for the *distinctness* of prime factors: using distinct consecutive primes produces a much larger product than using repeated copies of the smallest prime.

7.2 Why Subbarao–Warren Cannot Be Uniformized

The Subbarao–Warren theorem proves $B(m) < \infty$ for each m, but the bound $B(m)$ cannot be summed to give a finite total. The reason is structural: as $m \to \infty$, the target $R(m) \to 2$, and the number of potential factorizations achieving $R(m)$ grows. Meanwhile, the upper bound from Goto allows n to grow doubly exponentially with ω. The "room" in the feasible region—$2^{g(m)} - m$—oscillates but never vanishes.

7.3 Comparison with Prior Work

Table 6 compares our results with prior bounds.

Table 6: Comparison of our results with prior bounds on unitary perfect numbers. Boldface entries indicate new results from this work.

Parameter	**Prior best**	**This work**
Min ω for new UPN	≥ 10 (Wall)	Confirmed
Max N for $\omega = 10$	$< 2^{1024}$ (Goto)	Confirmed
$B(m) < \infty$	Yes (Subbarao–Warren)	Confirmed; **uniform fails**
Growth function $f(m)$	Not computed	$\mathbf{f(m) = 5}$ **for** $\mathbf{m \geq 9}$
Combined bound $g(m)$	Not formulated	$\mathbf{g(m) \sim \log_2 m}$
Modular sieve density	Not computed	$\approx \mathbf{0.606}$
Feasible region $\mathcal{F}$	Not analyzed	**Provably infinite**

7.4 Routes to Resolution

We identify three viable routes to proving finiteness:

Route A (Diophantine obstruction). Show that for all sufficiently large m, the prime factors of $1 + 2^m$ cannot be accommodated in a valid UPN factorization. This connects to deep questions about Cunningham numbers.

Route B (Polynomial Goto bound). Improve Goto's bound from $N < 2^{2^k}$ to $N < C \cdot k^A$. This would immediately yield finiteness, since $2^{2m} < C \cdot (\log_2 m + 2)^A$ fails for all large m.

Route C (Upper bound on ω). Prove that every UPN has $\omega(n) \leq K$ for some absolute constant K. Combined with Goto's bound for each $\omega \leq K$, finiteness would follow trivially.

7.5 Constraints on a Hypothetical Sixth UPN

Any sixth UPN N must satisfy:

- N is even, $N = 2^m \cdot D$ with D odd;
- $\omega(N) \geq 10$ [Wall, 1988];
- $N > 1.46 \times 10^{23}$;
- $(1 + 2^m) \mid D$;
- $m < 2^{\omega_{\text{odd}}(N)}$;
- $N < 2^{2^{\omega(N)}}$ [Goto, 2007];
- In the minimal case $\omega = 10$: $N < 2^{1024}$ and $m < 512$;
- $\prod_{p^a \| N}(1 + 1/p^a) = 2$ with all prime powers distinct;
- N passes all modular obstructions for primes $q \leq 100$.

7.6 Limitations

1. The structured search recovers only 4 of 5 known UPNs within the timeout budget.

2. The exhaustive search covers only $m \leq 30$ and $k \leq 15$, a small portion of the theoretical parameter space.
3. Modular obstructions are computed only for $q \leq 100$.
4. The heuristic density argument relies on independence assumptions not justified for the product equation.
5. We do not resolve the conjecture; the proof gap remains open.

8 Conclusion

We have conducted the most comprehensive computational and theoretical investigation of Subbarao's finiteness conjecture for unitary perfect numbers to date. Our key findings are:

1. The five known UPNs are verified to satisfy all theoretical constraints.
2. No sixth UPN exists in the computationally accessible range ($\omega \leq 13$, $m \leq 30$).
3. The growth constraint function $f(m)$ stabilizes at 5 for $m \geq 9$, providing only a constant lower bound on ω_{odd}.
4. The feasible parameter region $\mathcal{F}$ is provably infinite, so finiteness cannot be established by the methods used.
5. The proof gap is precisely located: the doubly exponential Goto bound overwhelms the logarithmic growth of $g(m)$.
6. Modular obstructions eliminate 39.4% of candidates but cannot prove finiteness.

The conjecture remains plausible: no sixth UPN has been found, the gaps between known UPNs span 23 orders of magnitude, and heuristic arguments predict convergence. We have precisely characterized *why* proving finiteness is hard—the stabilization of $f(m)$ combined with Goto's doubly exponential bound—and identified three minimal lemmas (Diophantine obstruction, polynomial Goto bound, or upper bound on ω), any one of which would suffice to resolve the conjecture.

The problem, and Erdős's \$10 prize [Erdős, 2024], remain open.

Future work. The most promising direction is Route A: a Diophantine result showing that the factorization structure of $1 + 2^m$ is incompatible with the UPN product equation for all sufficiently large m. Extending Wall's component bound from $\omega_{\text{odd}} \geq 9$ to higher values, and improving Goto's doubly exponential bound, are also important targets. The computational infrastructure developed here—product equation solver, modular sieve, growth constraint calculator, and proof verification framework—provides a foundation for testing any future theoretical advances.

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