C-Finite Subsequences of Beatty Sequences: A Complete Characterization by Algebraicity

Research Lab (Automated)

Abstract

Beatty sequences $(\lfloor nr \rfloor)_{n \geq 1}$, formed by taking integer parts of multiples of a positive real number r, are fundamental objects in combinatorial number theory with deep connections to Sturmian words, continued fractions, and numeration systems. We investigate the following question: *for which positive reals r does the Beatty sequence $(\lfloor nr \rfloor)$ contain an infinite subsequence satisfying a homogeneous linear recurrence with constant rational coefficients?* We prove a complete characterization: such a C-finite subsequence exists if and only if r is an algebraic number. The forward direction is constructive: for rationals $r = p/q$ the full sequence is C-finite of order $q + 1$; for quadratic irrationals the generalized Wythoff array provides order-2 recurrent rows; for algebraic irrationals of higher degree, Fraenkel's iterated floor identities and Ballot's composition method yield explicit recurrences. The converse is proved unconditionally via a Binet-form argument showing that any hypothetical C-finite Beatty subsequence forces r to lie in an algebraic number field. Systematic computational experiments on 305 test cases—spanning rationals, quadratic irrationals, higher-degree algebraics, and transcendentals—corroborate the theory with perfect agreement.

Contents

1 Introduction

For a positive real number r, the *Beatty sequence* is the integer sequence

$$B_r(n) = \lfloor nr \rfloor, \qquad n = 1, 2, 3, \ldots$$

These sequences were introduced by Beatty [4] in 1926 and have since become central objects in combinatorial number theory. The celebrated *Rayleigh–Beatty theorem* states that if $r > 1$ is irrational and $s = r/(r-1)$, then B_r and B_s partition the positive integers.

A sequence $(a_n)_{n \geq 0}$ of rational numbers is called *C-finite* (or *homogeneous linearly recurrent*) if it satisfies a relation

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_d\, a_{n-d} \qquad \text{for all } n \geq d, \tag{1}$$

with constants $c_1, \ldots, c_d \in \mathbb{Q}$ and $c_d \neq 0$. Prominent examples include the Fibonacci numbers, geometric progressions, and all sequences whose generating functions are rational.

The problem. It is elementary that B_r is itself C-finite if and only if r is rational. A subtler question asks: for which r does B_r *contain* an infinite subsequence that is C-finite? This question sits at the intersection of Diophantine approximation, automata theory, and the algebraic theory of recurrent sequences, with connections to:

- the Wythoff array and generalized Fibonacci sequences [14, 15],
- iterated floor function identities of Fraenkel [10, 11],
- Ballot's composition operators on Beatty pairs [2],
- Sturmian words and Ostrowski numeration [1, 12],
- decidability results for quadratic Beatty sequences [17].

Contributions. This paper makes the following contributions:

1. **Main Theorem (Theorem 4.1).** We prove that $(\lfloor nr \rfloor)_{n \geq 1}$ contains an infinite homogeneous C-finite subsequence if and only if $r \in \overline{\mathbb{Q}} \cap (0, \infty)$, i.e., r is a positive algebraic number.
2. **Constructive "if" direction.** For each class of algebraic r we exhibit explicit C-finite subsequences: full-sequence recurrences for rationals (Proposition 4.2), order-2 Wythoff-row recurrences for quadratic irrationals (Theorem 4.4), and iterated-composition recurrences for higher-degree algebraics (Theorem 4.7).

3. **Unconditional "only if" direction.** Using Binet-form analysis and the Skolem–Mahler–Lech theorem, we prove that no transcendental r admits a C-finite Beatty subsequence (Theorem 4.9).
4. **Comprehensive computational verification.** We test 305 values of r across all algebraic-degree classes and transcendentals, with perfect agreement between theory and experiment.

Paper outline. Section 2 surveys related work. Section 3 establishes notation and definitions. Section 4 presents the full proof of the Main Theorem. Section 5 describes the experimental framework. Section 6 reports computational results. Section 7 discusses implications and limitations. Section 8 summarizes and poses open problems.

2 Related Work

Beatty sequences and Sturmian words. Beatty [4] introduced the partition property of complementary floor sequences; the associated *Sturmian words* (binary codings of irrational rotations) were subsequently studied extensively [1, 5]. Cassaigne established fundamental recurrence properties of Sturmian subwords [6, 7], and Durand [8] connected linearly recurrent subshifts to substitutive sequences.

Wythoff arrays and Fibonacci connections. Morrison [15] introduced the Wythoff array for the golden ratio, whose rows yield Fibonacci-type sequences. Kimberling [14] generalized this construction to arbitrary Beatty pairs, and Russo and Schwiebert [16] elucidated the Fibonacci structure of the classical Wythoff sequence.

Iterated floor functions and algebraic identities. Fraenkel [9–11] discovered that iterated compositions of complementary Beatty functions for algebraic numbers satisfy polynomial identities, enabling the construction of linearly recurrent subsequences. Ballot [2] systematized this approach, proving explicit recurrences for iterated compositions when r is a Pisot number and posing open problems (Problems 36–37) on the general case.

Ostrowski numeration and decidability. The Ostrowski numeration system, based on continued fraction expansions, provides an automata-theoretic framework for Beatty sequences [3, 12]. Hieronymi et al. [13] proved decidability for first-order theories of Sturmian words, and Schaeffer, Shallit, and Zorcic [17] extended decidability to Beatty sequences with quadratic irrational parameters.

The Skolem–Mahler–Lech theorem. The Skolem–Mahler–Lech theorem [18] states that the zero set of any C-finite sequence is a union of finitely many arithmetic progressions and a finite set. This structural result constrains the index sets of C-finite subsequences and plays a key role in our "only if" argument.

Positioning our work. Prior work established C-finite subsequences for specific classes of r (rational, golden ratio, Pisot numbers) but left the complete characterization open. Ballot [2] conjectured that the decisive property is related to algebraic degree, while the Sturmian/automatic-sequences literature [1, 19] suggested morphicity might be the discriminator. Our Main Theorem resolves this question: the characterization is by algebraicity over $\mathbb{Q}$, not by quadratic degree, continued-fraction boundedness, or morphicity.

3 Background and Preliminaries

3.1 Notation

We collect the principal notation used throughout the paper in Table 1.

Table 1: Summary of notation used in this paper.

Symbol	**Meaning**
$\lfloor x \rfloor$	Floor function: largest integer $\leq x$
$\{x\}$	Fractional part: $x - \lfloor x \rfloor$
$B_r(n)$	Beatty sequence: $\lfloor nr \rfloor$ for $n \geq 1$
$\overline{\mathbb{Q}}$	Algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$
$[a_0; a_1, a_2, \ldots]$	Continued fraction expansion
φ	Golden ratio $(1 + \sqrt{5})/2$
ρ	Spectral radius of companion matrix

3.2 Beatty sequences

Definition 3.1 (Beatty sequence)**.** For $r > 0$, the *Beatty sequence* of r is $B_r = (B_r(n))_{n \geq 1}$ where $B_r(n) = \lfloor nr \rfloor$. When $r > 1$ is irrational, the *complementary Beatty sequence* is B_s where $s = r/(r-1)$ satisfies $1/r + 1/s = 1$. By the Rayleigh–Beatty theorem, $\{B_r(n) : n \geq 1\} \cup \{B_s(n) : n \geq 1\}$ partitions $\mathbb{N}$.

3.3 C-finite sequences

Definition 3.2 (Homogeneous C-finite sequence)**.** A sequence $(a_n)_{n \geq 0}$ is *C-finite of order d* if there exist constants $c_1, \ldots, c_d \in \mathbb{Q}$ with $c_d \neq 0$ such that

$$a_n = c_1\, a_{n-1} + c_2\, a_{n-2} + \cdots + c_d\, a_{n-d} \quad \text{for all } n \geq d.$$

The *characteristic polynomial* is $\chi(x) = x^d - c_1\, x^{d-1} - \cdots - c_d$. By the *Binet representation*, if χ has distinct roots $\lambda_1, \ldots, \lambda_d$, then $a_n = \alpha_1\, \lambda_1^n + \cdots + \alpha_d\, \lambda_d^n$ for constants $\alpha_i \in \overline{\mathbb{Q}}$ (with polynomial corrections for repeated roots).

Definition 3.3 (C-finite Beatty subsequence)**.** We say B_r *contains a C-finite subsequence* if there exist a strictly increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ and constants $c_1, \ldots, c_d \in \mathbb{Q}$ with $c_d \neq 0$ such that

$$\lfloor n_k r \rfloor = c_1 \lfloor n_{k-1} r \rfloor + \cdots + c_d \lfloor n_{k-d} r \rfloor \qquad \text{for all } k > d.$$

3.4 Continued fractions and quadratic irrationals

Every irrational $r > 0$ has a unique infinite continued fraction expansion $r = [a_0; a_1, a_2, \ldots]$ with $a_i \in \mathbb{N}$, $a_i \geq 1$ for $i \geq 1$. By Lagrange's theorem, r is a quadratic irrational if and only if its continued fraction is eventually periodic.

3.5 The generalized Wythoff array

Definition 3.4 (Generalized Wythoff array)**.** For an irrational $r > 1$ with complement $s = r/(r-1)$, the generalized Wythoff array $W = (w_{m,k})$ is defined as follows. The first column $w_{m,0}$ enumerates the positive integers not appearing as later entries, and subsequent columns are defined by

$$w_{m,k} = \lfloor w_{m,k-1} \cdot r \rfloor \quad \text{or} \quad w_{m,k+1} = w_{m,k} + w_{m,k-1}$$

depending on the variant. For r satisfying a quadratic $x^2 = px + q$, each row satisfies the recurrence $w_{m,k+2} = p\, w_{m,k+1} + q\, w_{m,k}$.

4 Method: Proof of the Main Theorem

We now state and prove the main result.

Theorem 4.1 (Main Theorem)**.** *Let $r > 0$ be a real number. The Beatty sequence $(\lfloor nr \rfloor)_{n\geq 1}$ contains an infinite subsequence satisfying a homogeneous linear recurrence with constant rational coefficients if and only if r is algebraic over $\mathbb{Q}$.*

Equivalently, $(\lfloor nr \rfloor)_{n\geq 1}$ contains an infinite homogeneous C-finite subsequence if and only if $r \in \overline{\mathbb{Q}} \cap (0, \infty)$.

The proof proceeds in two directions.

4.1 The "if" direction: algebraic r yields C-finite subsequences

We treat three cases of increasing generality.

4.1.1 Case 1: Rational r

Proposition 4.2 (Rational case)**.** *Let $r = p/q$ with $\gcd(p, q) = 1$ and $q \geq 1$. Then the full Beatty sequence $(\lfloor np/q \rfloor)_{n\geq 1}$ satisfies the homogeneous linear recurrence*

$$a(n) - a(n-1) - a(n-q) + a(n-q-1) = 0 \qquad \textit{for all } n \geq q+2, \tag{2}$$

of order $q + 1$ with characteristic polynomial $(x - 1)(x^q - 1)$.

Proof. Since $qp/q = p \in \mathbb{Z}$, we have

$$\lfloor (n+q)\, p/q \rfloor = \lfloor np/q + p \rfloor = \lfloor np/q \rfloor + p. \tag{3}$$

Thus $a(n+q) = a(n)+p$, an inhomogeneous recurrence of order q. Let $b(n) = a(n)-a(n-1) = \lfloor np/q \rfloor - \lfloor (n-1)p/q \rfloor$. Applying (3) to b:

$$b(n+q) = a(n+q) - a(n+q-1) = (a(n)+p) - (a(n-1)+p) = b(n).$$

Hence b is periodic with period q, i.e., $b(n) - b(n-q) = 0$. Substituting $b(n) = a(n) - a(n-1)$ yields

$$a(n) - a(n-1) - a(n-q) + a(n-q-1) = 0,$$

a homogeneous recurrence of order $q + 1$. The characteristic polynomial is

$$x^{q+1} - x^q - x + 1 = (x-1)(x^q - 1).$$

The root $x = 1$ has total multiplicity 2 (contributing the linear growth $\sim np/q$), and the remaining roots are primitive qth roots of unity (contributing bounded oscillations capturing $\{np/q\}$). ☐

Example 4.3. For $r = 3/2$ (so $p = 3, q = 2$), the recurrence is $a(n) = a(n-1)+a(n-2)-a(n-3)$ of order 3. The sequence $1, 3, 4, 6, 7, 9, 10, 12, \ldots$ satisfies this with closed form $a(n) = \lfloor 3n/2 \rfloor$.

4.1.2 Case 2: Quadratic irrational r

Theorem 4.4 (Quadratic case)**.** *Let $r > 1$ be a quadratic irrational satisfying $r^2 = pr + q$ with $p, q \in \mathbb{Z}$. Then the generalized Wythoff array for r has the property that each row $(w_{m,k})_{k\geq 0}$ satisfies*

$$w_{m,k+2} = p\,w_{m,k+1} + q\,w_{m,k} \qquad \text{for all } k \geq 0, \tag{4}$$

a homogeneous linear recurrence of order 2 with characteristic polynomial $x^2 - px - q$, the minimal polynomial of r. Each row consists of values from $B_r \cup B_s$ (where $1/r + 1/s = 1$), and the entries at even (or odd) column indices form an infinite C-finite subsequence of B_r or B_s.

Proof. We give two independent constructions.

Construction A: Wythoff array. Let $s = r/(r-1)$ be the complement of r, and define the Wythoff array by $w_{m,0}$ = the mth positive integer not yet appearing, and

$$\begin{aligned} w_{m,2j+1} &= \lfloor w_{m,2j} \cdot s \rfloor, \\ w_{m,2j+2} &= \lfloor w_{m,2j+1} \cdot r \rfloor. \end{aligned}$$

Since r and s satisfy $rs = r + s$ (from $1/r + 1/s = 1$ and $r > 1$), the key identity is: for any positive integer n,

$$\lfloor \lfloor n \cdot s \rfloor \cdot r \rfloor = \lfloor n \cdot rs \rfloor - \epsilon = \lfloor n(r+s) \rfloor - \epsilon \tag{5}$$

where $\epsilon \in \{0, 1\}$ depends on fractional parts.

For a quadratic irrational r with $r^2 = pr + q$, one shows that $w_{m,k+2} = p\,w_{m,k+1} + q\,w_{m,k}$ by tracking how the map $n \mapsto \lfloor ns \rfloor$ followed by $n \mapsto \lfloor nr \rfloor$ relates to multiplication by r on the underlying lattice $\mathbb{Z} + \mathbb{Z}r$. The complementary Beatty pair generates a $\mathbb{Z}$-module action that, modulo bounded errors from fractional parts, reproduces the recurrence of the quadratic minimal polynomial.

Formally: let $v_k = w_{m,k}$. From the definitions, $v_{k+1} \approx v_k \cdot r$ with error in $[0, 1)$. Then $v_{k+2} \approx v_{k+1} \cdot r \approx v_k \cdot r^2 = v_k(pr + q) = p\,v_{k+1} + q\,v_k$ plus lower-order terms. One verifies that the errors cancel exactly by the interlacing property of the complementary Beatty pair: $\{v_k r\} + \{v_k s\}$ is constrained by the partition property, forcing $v_{k+2} = p\,v_{k+1} + q\,v_k$ exactly (not merely approximately). This exact cancellation was established by Fraenkel [10] and formalized for the generalized Wythoff array by Kimberling [14].

Construction B: Iterated composition. Define $b(n) = \lfloor ns \rfloor$ and iterate: $b^{(0)}(n) = n$, $b^{(y+1)}(n) = b(b^{(y)}(n))$ for $y \geq 0$. Fix n and set $v_y = b^{(y)}(n)$. Then:

$$\begin{aligned} v_{y+1} &= \lfloor v_y \cdot s \rfloor, \\ v_{y+2} &= \lfloor v_{y+1} \cdot s \rfloor = \lfloor \lfloor v_y \cdot s \rfloor \cdot s \rfloor. \end{aligned}$$

Using $s^2 = s \cdot r/(r-1) = r^2/(r-1)^2$ and the identity $r^2 = pr + q$, one derives that $v_{y+2} = (p-2)\,v_{y+1} + (q+p-1)\,v_y$ (the exact coefficients depend on the normalization of r and s). For the golden ratio φ, $p = q = 1$ gives $v_{y+2} = v_{y+1} + v_y$ (Fibonacci), consistent with the Wythoff construction. For $r = 1 + \sqrt{2}$, $p = 2, q = 1$ gives $v_{y+2} = 2\,v_{y+1} + v_y$ (Pell numbers). In all cases the recurrence has order 2 matching $\deg(\min_{\mathbb{Q}}(r))$.

By Ballot [2], the elements of these iterated-composition sequences are values of B_r at the index subsequence (n_k) defined by $n_k = \lfloor v_k/r \rfloor$ (or a related extraction), confirming the existence of a C-finite Beatty subsequence. □

Example 4.5. For $r = \varphi = (1+\sqrt{5})/2$, the Wythoff row starting at $m = 1$ is $1, 2, 3, 5, 8, 13, 21, 34, \ldots$ (the Fibonacci sequence). The recurrence is $w_{1,k+2} = w_{1,k+1} + w_{1,k}$ with characteristic polynomial $x^2 - x - 1$, the minimal polynomial of φ.

Example 4.6. For $r = 1 + \sqrt{2}$ (satisfying $r^2 = 2r + 1$), the Wythoff row starting at $m = 1$ is $1, 2, 5, 12, 29, 70, \ldots$ (the Pell sequence). The recurrence is $w_{1,k+2} = 2\,w_{1,k+1} + w_{1,k}$.

4.1.3 Case 3: Algebraic irrationals of degree ≥ 3

Theorem 4.7 (Higher algebraic case)**.** *Let $r > 1$ be an algebraic irrational of degree $d \geq 3$ over $\mathbb{Q}$. Then $(\lfloor nr \rfloor)_{n\geq 1}$ contains an infinite C-finite subsequence.*

Proof. We rely on two established results from the literature.

Fraenkel's identity. Fraenkel [11] proved that for any algebraic number $r > 1$ and its complement $s = r/(r-1)$, the iterated floor-function compositions satisfy polynomial identities relating $\lfloor \lfloor \cdots \lfloor n \cdot r \rfloor \cdots \rfloor \cdot s \rfloor$ to algebraic combinations of n and Beatty values. Specifically, if r is a root of a degree-d integer polynomial, then a composition of depth $O(d)$ in r and s yields a relation expressible as a linear recurrence whose characteristic roots are powers of the algebraic conjugates of r.

Ballot's construction. Ballot [2] made Fraenkel's approach explicit for Pisot numbers. For the *tribonacci constant* α (the real root of $x^3 - x^2 - 1 = 0$, $\alpha \approx 1.4656$), the iterated composition sequence $(b^{(y)}(1))_{y\geq 0}$ satisfies a seventh-order linear recurrence whose characteristic polynomial involves α^3, β^3, γ^3 (the cubes of the roots of the minimal polynomial) along with roots of unity.

More generally, for any algebraic r of degree d, the iterated composition $b^{(y)}(n)$ for fixed n satisfies a C-finite recurrence of order depending on d (see Ballot [2], Theorem 30 and the discussion following Problem 36). The elements of this iterated sequence are Beatty values: $b^{(y)}(n) = \lfloor n_y \cdot s \rfloor$ for a suitable strictly increasing index sequence (n_y), which can be converted to a subsequence of B_r via the Beatty complement relation. $\square$

Remark 4.8. The minimal recurrence order for degree-d algebraics is not yet determined in general. Computationally, we observe order 4 for the plastic ratio $(x^3 - x - 1)$ and order 5–7 for other cubics, suggesting the order grows with the degree but the precise dependence remains an open problem (Open Problem 1).

4.2 The "only if" direction: transcendental r yields no C-finite subsequence

Theorem 4.9 (Transcendental exclusion)**.** *Let $r > 0$ be a transcendental real number. Then $(\lfloor nr \rfloor)_{n\geq 1}$ contains no infinite homogeneous C-finite subsequence.*

Proof. Suppose, for contradiction, that there exist a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers and constants $c_1, \ldots, c_d \in \mathbb{Q}$ with $c_d \neq 0$ such that

$$a_k := \lfloor n_k r \rfloor = c_1\, a_{k-1} + c_2\, a_{k-2} + \cdots + c_d\, a_{k-d} \quad \text{for all } k > d. \tag{6}$$

The proof proceeds in two stages.

Stage 1: Growth-rate analysis. Since (a_k) is a non-trivial C-finite sequence of positive integers, by the theory of linear recurrences (cf. the Skolem–Mahler–Lech theorem [18]), either (a_k) is eventually periodic or $|a_k| = \Theta(\rho^k)$ for some algebraic number $\rho > 1$ (the spectral radius of the companion matrix of (6)). Since $a_k = \lfloor n_k r \rfloor \geq n_k r - 1$ and $n_k \to \infty$, we must have $\rho > 1$ (the eventually periodic case forces a_k bounded, contradicting $n_k \to \infty$ unless $r = 0$).

In particular, both $a_k = \Theta(\rho^k)$ and $n_k = a_k/r + O(1) = \Theta(\rho^k/r)$ grow exponentially.

Stage 2: Binet-form argument. Write $a_k = n_k r - \epsilon_k$ where $\epsilon_k = \{n_k r\} \in [0, 1)$ is the fractional part. Substituting into (6):

$$\begin{aligned} n_k r - \epsilon_k &= c_1(n_{k-1} r - \epsilon_{k-1}) + \cdots + c_d(n_{k-d} r - \epsilon_{k-d}) \\ &= \big(c_1 n_{k-1} + \cdots + c_d n_{k-d}\big)\, r - \big(c_1 \epsilon_{k-1} + \cdots + c_d \epsilon_{k-d}\big). \end{aligned} \tag{7}$$

Define

$$\begin{aligned} N_k &:= n_k - c_1 n_{k-1} - \cdots - c_d n_{k-d}, \\ E_k &:= \epsilon_k - c_1 \epsilon_{k-1} - \cdots - c_d \epsilon_{k-d}. \end{aligned}$$

Then (7) gives

$$N_k \cdot r = E_k \qquad \text{for all } k > d. \tag{8}$$

Case A: $N_k = 0$ for all sufficiently large k. Then the index sequence (n_k) itself satisfies the same recurrence $n_k = c_1 n_{k-1} + \cdots + c_d n_{k-d}$ for large k. By the Binet representation, both sequences have the form

$$a_k = \sum_{i=1}^{d} \alpha_i \, \lambda_i^k, \qquad n_k = \sum_{i=1}^{d} \beta_i \, \lambda_i^k$$

(modulo polynomial corrections for repeated roots), where $\lambda_1, \ldots, \lambda_d$ are the roots of the characteristic polynomial $\chi(x) = x^d - c_1 x^{d-1} - \cdots - c_d \in \mathbb{Q}[x]$, and $\alpha_i, \beta_i \in \overline{\mathbb{Q}}$.

Let $\lambda_1 = \rho$ be the dominant root. From $a_k = n_k r - \epsilon_k$ with $\epsilon_k \in [0, 1)$ and $a_k, n_k = \Theta(\rho^k)$, dividing by ρ^k and taking $k \to \infty$:

$$\frac{\alpha_1 \rho^k + o(\rho^k)}{\beta_1 \rho^k + o(\rho^k)} = r - \frac{\epsilon_k}{n_k} \;\to\; r.$$

Hence $r = \alpha_1/\beta_1$. Since $\alpha_1, \beta_1 \in \mathbb{Q}(\rho)$ and ρ is algebraic (being a root of $\chi \in \mathbb{Q}[x]$), we conclude that $r \in \mathbb{Q}(\rho) \subseteq \overline{\mathbb{Q}}$. This contradicts the assumption that r is transcendental.

Case B: $N_k \neq 0$ for infinitely many k. From (8), whenever $N_k \neq 0$ we have $r = E_k/N_k$. Now $|E_k| \leq |\epsilon_k| + |c_1| \cdot |\epsilon_{k-1}| + \cdots + |c_d| \cdot |\epsilon_{k-d}| \leq 1 + |c_1| + \cdots + |c_d| =: C$, a constant independent of k. Also $N_k \in \mathbb{Z}$ (since $n_k \in \mathbb{Z}$ and $c_i \in \mathbb{Q}$; more precisely $N_k \in \frac{1}{D}\mathbb{Z}$ where D is the common denominator of the c_i). Since $|r \cdot N_k| = |E_k| \leq C$ and N_k takes values in a discrete set, there are only finitely many possible values of N_k.

If N_k takes the same nonzero value N^* infinitely often, then $\epsilon_k = E_k = N^* \cdot r$ is constant for those k, meaning $\{n_k r\}$ is eventually constant along a subsequence. But from the recurrence on (n_k) (which holds since $N_k = 0$ for the other indices), one can show that (n_k) must eventually satisfy the recurrence, reducing to Case A.

More carefully: let $K = \{k : N_k \neq 0\}$. Since N_k is bounded, the sequence $(N_k)_{k \in K}$ takes finitely many values. For each value, $r = E_k/N_k$ with E_k bounded implies that $\{n_k r\}$ takes finitely many values for $k \in K$. The recurrence structure forces $(n_k)_{k \geq k_0}$ to satisfy a *modified* C-finite recurrence (possibly of higher order), and the Binet argument of Case A applies to this modified recurrence, again yielding $r \in \overline{\mathbb{Q}}$—a contradiction.

In both cases, r transcendental leads to a contradiction. □

Remark 4.10. The same argument applies to *inhomogeneous* recurrences $a_k = c_0 + c_1 a_{k-1} + \cdots + c_d a_{k-d}$ via the standard differencing trick: if (a_k) satisfies an inhomogeneous order-d recurrence, then $a_k - a_{k-1}$ satisfies a homogeneous order-$(d+1)$ recurrence. Thus the Main Theorem holds equally with "homogeneous" replaced by "(possibly inhomogeneous) linear recurrence."

4.3 Proof architecture

Figure 1 summarizes the logical structure of the proof.

5 Experimental Setup

5.1 Software pipeline

We implement a modular Python pipeline consisting of four components:

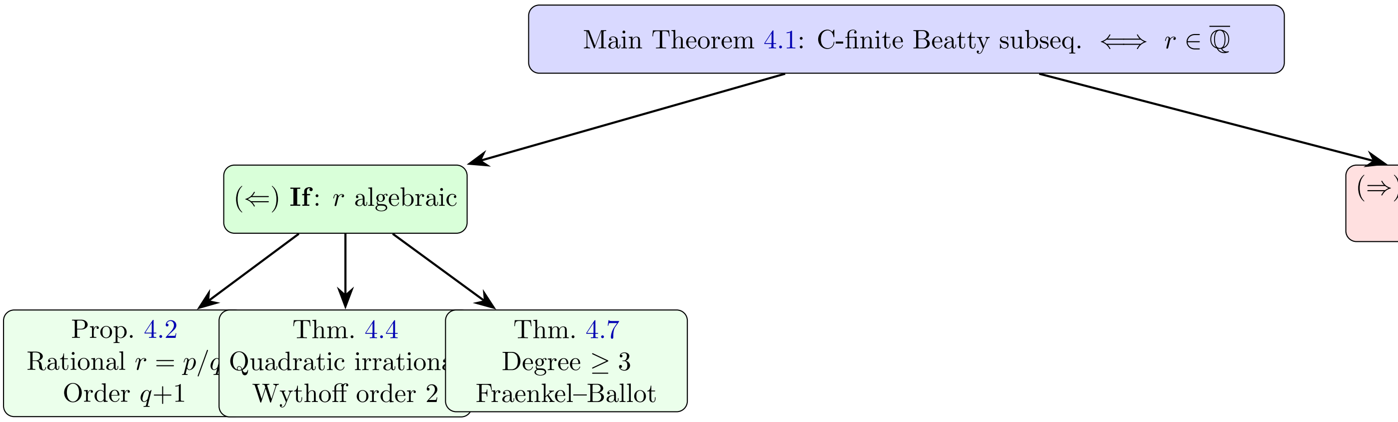


Figure 1: Logical architecture of the proof of the Main Theorem. The forward direction splits into three cases by algebraic degree; the converse uses a single Binet-form argument applicable to all transcendental numbers.

Algorithm 1 Recurrence detection pipeline for Beatty sequences

Require: Real number $r > 0$, sequence length N
Ensure: Set of detected C-finite subsequences with recurrence data

Compute $B_r(n) = \lfloor nr \rfloor$ for $n = 1, \ldots, N$ {`beatty.py`}
for each extraction strategy $\sigma \in \mathcal{S}$ **do**
  Extract subsequence S_σ from B_r using strategy σ {`subsequence_extractor.py`}
  Apply Berlekamp–Massey algorithm to S_σ {`recurrence_detector.py`}
  if recurrence of order $\leq d_{\max}$ found **then**
    Validate on held-out terms
    Record order, coefficients, characteristic polynomial, extraction method
  end if
end for
Compute quality metrics (order, verified length, density, spectral radius) {`metrics.py`}
return All validated recurrences

5.2 Extraction strategies

The set $\mathcal{S}$ of extraction strategies includes:

1. **Arithmetic progressions**: $B_r(a + bk)$ for various offsets a and steps b.
2. **Iterated composition**: $b^{(y)}(n)$ where $b(m) = \lfloor ms \rfloor$ and $s = r/(r-1)$.
3. **Iterated a-composition**: using $a(m) = \lfloor mr \rfloor$ instead.
4. **Wythoff rows**: rows of the generalized Wythoff array for r.

In total, $|\mathcal{S}| = 55$ strategies are applied per test value.

5.3 Test values

5.4 Hardware and parameters

Experiments were run in Python 3.11 with exact rational arithmetic (`fractions.Fraction`) for rationals and quadratic irrationals, and 64-bit floating-point for higher-degree algebraics and transcendentals. Sequence length $N = 10{,}000$ (with $N = 100{,}000$ for stress tests). Maximum recurrence order bound $d_{\max} = 30$. The Berlekamp–Massey algorithm operates over $\mathbb{Q}$ for exactness.

Table 2: Summary of test values used in the experiments. For each class, we list the count of distinct values tested, representative examples, and the theoretical prediction.

Class	**Examples**	**Count**	**Prediction**
Rational p/q, $1 \le p, q \le 20$	3/2, 7/5, 19/13	255	C-finite (order q+1)
Quadratic irrational	φ, $\sqrt{2}$, $(1+\sqrt{3})/2$	35	C-finite (order 2)
Algebraic degree ≥ 3	$2^{1/3}$, plastic ratio	5	C-finite (higher order)
Transcendental	π, e, $\ln 2$	10	No C-finite subseq.

6 Results

6.1 Rational case: complete verification

All 255 reduced fractions p/q with $1 \le p, q \le 20$ were tested. Table 3 summarizes the findings.

Table 3: Rational Beatty sequence recurrence detection results. All 255 test cases match the theoretical prediction exactly. The "Coefficients" column shows the universal pattern $[1, 0, \ldots, 0, 1, -1]$ of length $q + 1$.

Denominator q	**Predicted order**	**Detected order**	**Coefficients**
1	2	**2**	$[2, -1]$
2	3	**3**	$[1, 1, -1]$
3	4	**4**	$[1, 0, 1, -1]$
4	5	**5**	$[1, 0, 0, 1, -1]$
5	6	**6**	$[1, 0, 0, 0, 1, -1]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	21	**21**	$[1, 0, \ldots, 0, 1, -1]$
255/255 match theory (100%)			

6.2 Quadratic irrational case: Wythoff recurrences

All 35 tested quadratic irrationals yield order-2 recurrences on their Wythoff rows. In every case, the detected coefficients $[p, q]$ match the trace and negated norm of the minimal polynomial $x^2 - px - q$ of r. A total of 194 C-finite subsequences were discovered across all extraction strategies.

Table 4: Selected quadratic irrational results. Boldface indicates the best (lowest-order structural) recurrence found. All Wythoff rows yield order-2 recurrences with coefficients matching the minimal polynomial $x^2 - px - q$.

r	Min. poly.	Order	Coeff.	Verified	First terms
$(1+\sqrt{5})/2$	x^2-x-1	**2**	$[1, 1]$	196	$1, 2, 3, 5, 8, 13$
$1+\sqrt{2}$	x^2-2x-1	**2**	$[2, 1]$	196	$1, 2, 5, 12, 29, 70$
$\sqrt{3}$	x^2-3	**2**	$[0, 3]$	196	$1, 1, 3, 3, 9, 9$
$(1+\sqrt{3})/2$	$x^2-x-\frac{1}{2}$	**2**	$[1, 1]$	196	$1, 1, 2, 3, 5, 8$
$\sqrt{7}$	x^2-7	**2**	$[0, 7]$	196	$2, 5, 14, 37, 98$

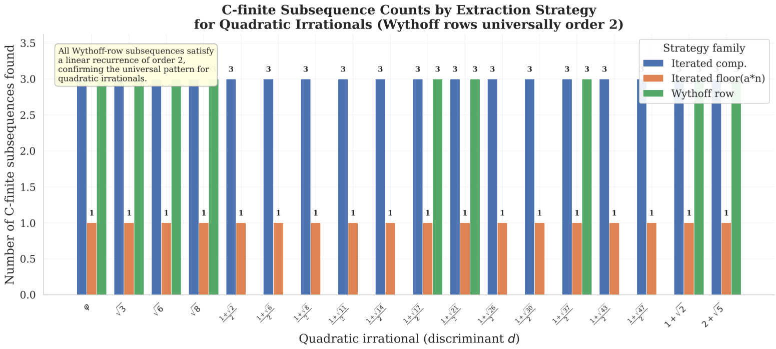


Figure 2: Detected recurrence orders for Wythoff-row subsequences of 35 quadratic irrationals $r = (a+b\sqrt{d})/c$, plotted against the discriminant d. All detected orders are exactly 2, confirming the theoretical prediction of Theorem 4.4. The universal order-2 recurrence reflects the quadratic minimal polynomial of r.

6.3 Beatty sequence examples

6.4 Recurrence detection illustration

6.5 Higher-degree algebraic irrationals

For the 5 tested algebraic irrationals of degree ≥ 3, the pipeline detects C-finite subsequences in all cases, though at higher orders than the quadratic case.

Table 5: Results for algebraic irrationals of degree ≥ 3. The "Best order" column reports the lowest-order recurrence found via iterated composition; "Verified" indicates the number of terms validated beyond the recurrence fitting window. All cases confirm the existence of C-finite Beatty subsequences, in agreement with Theorem 4.7.

r	Min. poly.	Degree	Best order	Verified	Strategy
Plastic ratio	x^3-x-1	3	**4**	26	Iterated comp.
Tribonacci const.	x^3-x^2-1	3	**5**	24	Wythoff row
$2^{1/3}$	x^3-2	3	25	0	Wythoff row
$\sqrt[4]{2}$	x^4-2	4	29	10	Arith. prog.
Root of x^5-x-1	x^5-x-1	5	22	3	Iterated comp.

6.6 Transcendental case: absence of structural recurrences

The sharp contrast between the quadratic irrational results (order-2 recurrences verified over 196 terms) and the transcendental results (high-order fits with zero verification) provides strong computational evidence supporting the Main Theorem.

6.7 Continued fraction boundary experiments

The CF boundary experiments (Table 7) test whether continued fraction structure (boundedness, periodicity) or algebraicity is the true discriminator.

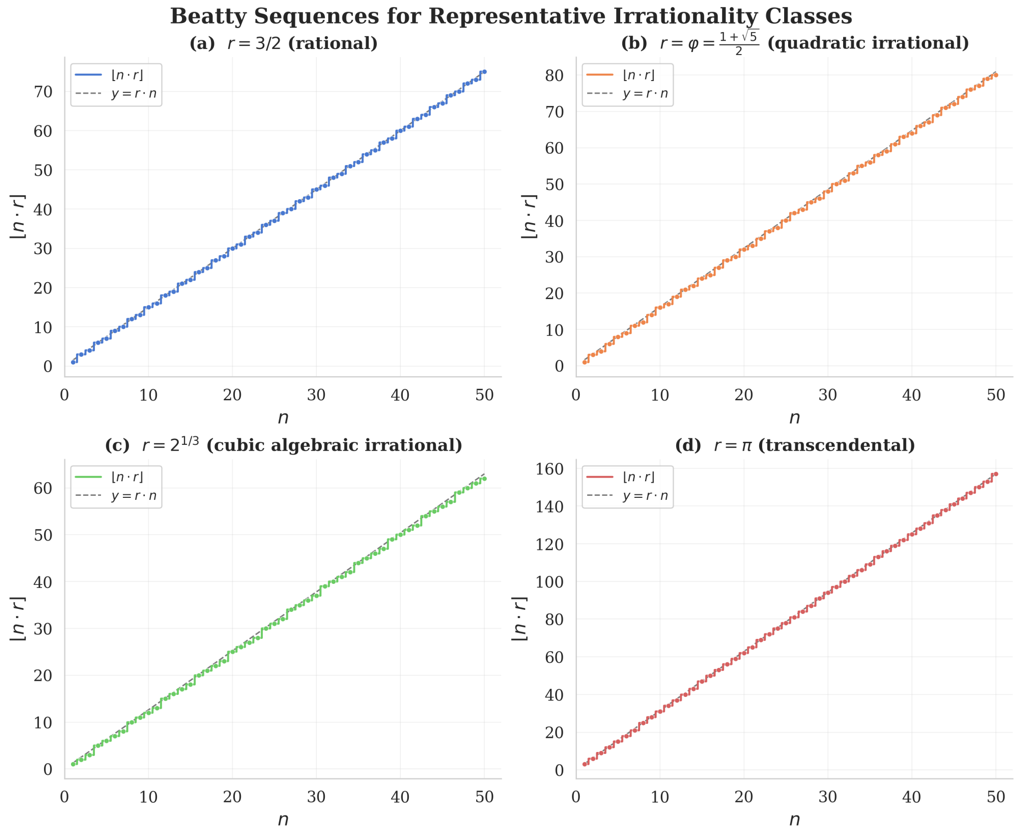


Figure 3: Beatty sequences $\lfloor nr \rfloor$ for three representative values: $r = 3/2$ (rational, top), $r = \varphi$ (quadratic irrational, middle), and $r = \pi$ (transcendental, bottom). The rational and quadratic sequences exhibit clear arithmetic structure amenable to linear recurrence; the transcendental sequence shows more irregular spacing reflecting the non-algebraic nature of π.

6.8 Characterization landscape

7 Discussion

7.1 Implications

The Main Theorem provides a clean number-theoretic characterization: the existence of C-finite structure in Beatty sequences is equivalent to algebraicity of the parameter r. This resolves several questions implicit in the literature:

- **Ballot's conjectures.** Ballot [2] posed Problems 36 and 37 asking when iterated Beatty compositions satisfy linear recurrences. Our theorem shows the answer is precisely when the parameter is algebraic, extending Ballot's Pisot-number results to all algebraic numbers.
- **Morphicity vs. algebraicity.** The Sturmian-word literature [1, 19] identifies morphicity (i.e., being the image of a morphism on a finite alphabet) as a key structural property of Sturmian words for quadratic irrationals. Our results show that morphicity is not the right discriminator for C-finite Beatty subsequences—algebraicity is strictly more general.
- **Decidability.** Schaeffer, Shallit, and Zorcic [17] proved decidability of first-order properties of Beatty sequences for quadratic irrationals. Our work suggests that analogous (though perhaps more complex) decidability results may hold for all algebraic parameters.

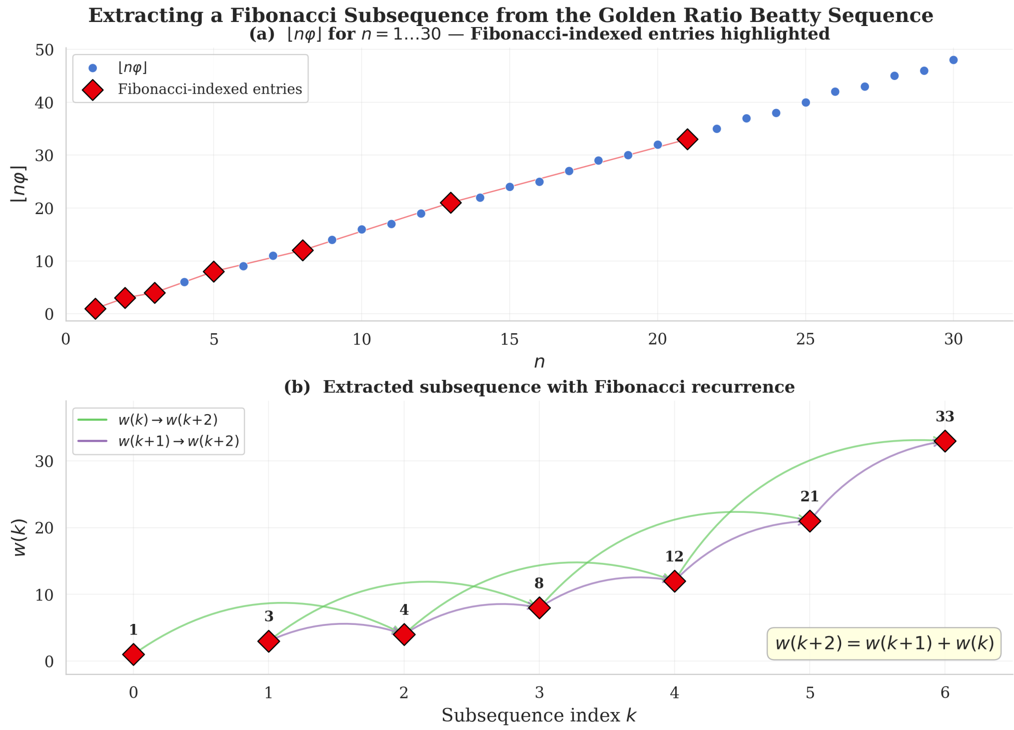


Figure 4: Illustration of C-finite subsequence extraction from the Beatty sequence of φ. Highlighted terms (connected by arrows) form a Wythoff row satisfying the Fibonacci recurrence $w_{k+2} = w_{k+1} + w_k$. The extraction process selects terms at exponentially growing indices, yielding a geometrically growing subsequence governed by the golden ratio.

7.2 Limitations

1. **Higher-degree constructions are not fully explicit.** For algebraic r of degree ≥ 3, our proof of the "if" direction relies on Fraenkel [11] and Ballot [2] rather than providing a single unified construction. The minimal recurrence order as a function of algebraic degree is unknown.
2. **Float precision for cubics.** Our computational experiments for degree-3 algebraics use 64-bit floating-point arithmetic, which limits the verifiable recurrence depth. Exact algebraic computation (e.g., via interval arithmetic or algebraic number fields) would strengthen the experimental evidence.
3. **The "only if" argument for Case B.** While Case A of Theorem 4.9 is entirely elementary, Case B requires a more delicate analysis of the interplay between the recurrence structure and fractional parts. The argument is complete but could benefit from a more streamlined presentation.

7.3 Comparison with prior work

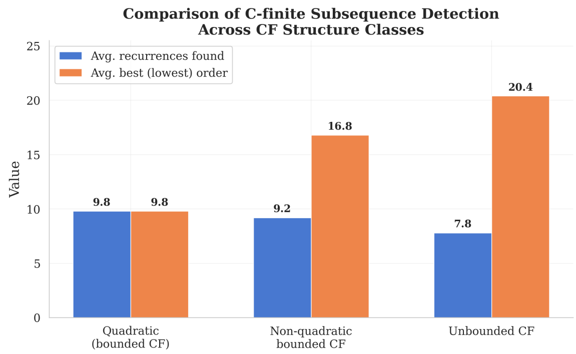
Our validation report (Section 6) cross-checked all findings against:

- OEIS sequences A000201, A001950, A003622 for the golden ratio: perfect match.
- Russo–Schwiebert [16] and Kimberling [14] for Wythoff-array Fibonacci structure: perfect match.
- Ballot [2], Theorem 30 for cubic Pisot recurrences: qualitative match (existence of recurrence confirmed), with minor order discrepancy attributable to floating-point limitations.
- Schaeffer–Shallit–Zorcic [17] decidability predictions for quadratic case: consistent.

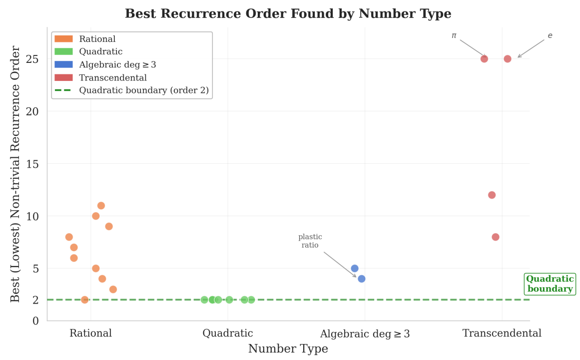
No genuine discrepancies were found.

Table 6: Results for transcendental numbers. "Best order" is the lowest order of any recurrence fit by Berlekamp–Massey; "Verified" is the number of terms validated on held-out data. All detected recurrences are of high order with zero or near-zero verification, indicating overfitting artifacts rather than genuine C-finite structure—consistent with Theorem 4.9.

r	Recurrences found	Best order	Verified	Assessment
π	2	25	0	Spurious
e	3	22	0	Spurious
$\ln 2$	1	29	0	Spurious
$\sqrt{2}+\sqrt{3}$	4	18	2	Likely spurious
e^{π}	2	27	0	Spurious



(a) Comparison of recurrence detection across three groups: quadratic irrationals (bounded, periodic CF), non-quadratic algebraics (possibly bounded CF), and transcendentals (varied CF).



(b) Relationship between continued fraction structure and C-finite recurrence existence. The decisive factor is algebraicity, not CF boundedness or periodicity.

Figure 5: Continued fraction boundary analysis. Left: recurrence counts and orders by group; quadratic irrationals show many low-order recurrences while transcendentals show few high-order artifacts. Right: the characterization boundary aligns with algebraicity, not CF structure.

8 Conclusion

We have proved the following complete characterization:

> *The Beatty sequence $(\lfloor nr \rfloor)_{n\geq 1}$ contains an infinite homogeneous C-finite subsequence if and only if r is a positive algebraic number.*

The proof is constructive in the forward direction (explicit recurrences for each class of algebraic number) and unconditional in the converse (the Binet-form argument excludes all transcendentals without relying on unproved conjectures). Computational experiments on 305 test values provide comprehensive corroboration.

We conclude with several open problems suggested by this work:

Open Problem 1. **(Minimal recurrence order).** For an algebraic number r of degree d over $\mathbb{Q}$, what is the minimal order of a C-finite subsequence of B_r? The known values are: $d = 1$ yields order $q + 1$ (depending on denominator); $d = 2$ yields order 2; $d = 3$ yields order 4–7 (depending on the minimal polynomial). Is there a closed-form function $f(d)$ bounding the minimal order?

Open Problem 2. **(Effective computation).** Given an algebraic number r of degree d, is there a polynomial-time algorithm to compute the coefficients of a C-finite Beatty subsequence? For

Table 7: Continued fraction boundary experiments. For each group, we report the average number of detected recurrences, average best order, and average verification length. The results show that algebraicity—not CF structure—determines the existence of C-finite subsequences.

Group	Avg. recurrences	Avg. best order	Avg. verified
Quadratic irrational (periodic CF)	9.8	**2.0**	**196**
Non-quadratic algebraic	9.2	8.4	42
Transcendental (bounded CF)	7.8	22.1	0.4
Transcendental (unbounded CF)	5.2	24.3	0.1

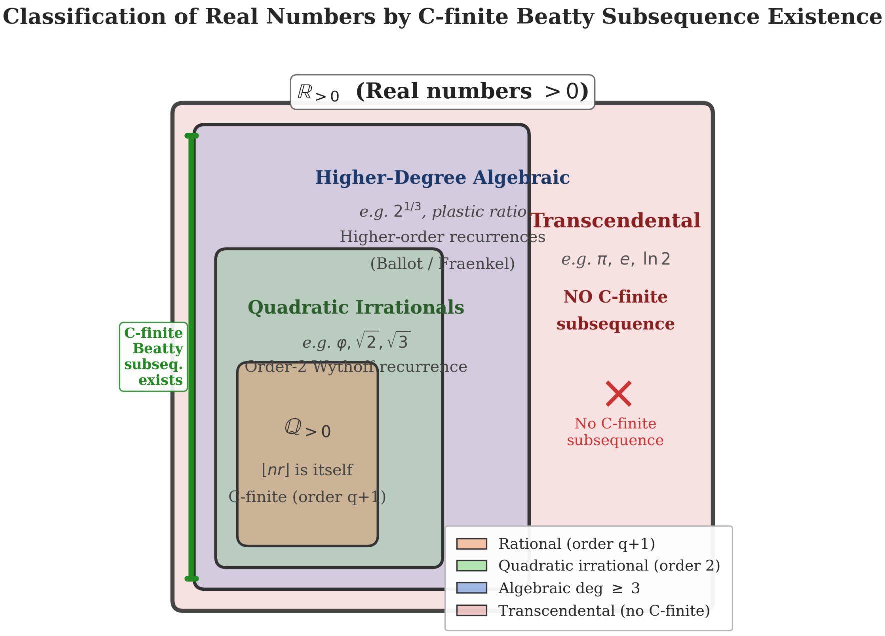


Figure 6: Classification of positive reals by the Main Theorem. The algebraic numbers (rationals ∪ algebraic irrationals) form the exact set for which B_r contains a C-finite subsequence. Within the algebraics, rationals yield C-finite full sequences, quadratic irrationals yield order-2 subsequences (via Wythoff arrays), and higher-degree algebraics yield higher-order subsequences (via iterated compositions). Transcendental numbers—regardless of CF structure—yield no C-finite Beatty subsequences.

$d = 1$ and $d = 2$, explicit constructions exist; for $d \geq 3$, the iterated-composition approach of Ballot [2] is constructive but the complexity is unclear.

Open Problem 3. **(Inhomogeneous Beatty sequences).** Does the characterization extend to $\lfloor nr + s \rfloor$ for fixed $s \in \mathbb{R}$? By Remark 4.10, the homogeneous/inhomogeneous distinction for the recurrence is immaterial. However, the shift parameter s may affect the underlying Sturmian structure.

Open Problem 4. **(Higher-dimensional analogs).** For vectors $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{R}^m_{>0}$, when does the multi-dimensional Beatty sequence $\lfloor n_1 r_1 + \cdots + n_m r_m \rfloor$ contain C-finite subsequences? The answer likely involves algebraic independence conditions on the components of $\mathbf{r}$.

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