Project: Modeling Capacitor Discharge using a Adams-Bashforth-Moulton Predictor-Corrector Method

Numerical Solution of Ordinary Differential Equations

Due: 11:59PM, 2025-03-24

1 Project Overview

In this project, you will implement and analyze a numerical method for approximating the solution to ordinary differential equations (ODEs). Specifically, you will construct a predictor-corrector method, which combines the advantages of both explicit and implicit multi-step methods. This method will utilize a three-step Adams-Bashforth method as the predictor and a two-step Adams-Moulton method as the corrector. To initialize the multi-step method, you will use the Runge-Kutta 4th order method to generate the first three approximations.

2 Background: Modeling Capacitor Discharge in an RC Circuit

Consider a simple RC circuit consisting of a resistor (R) and a capacitor (C) connected in series. Initially, the capacitor is charged to a voltage V_0. At time $t = 0$, the capacitor is allowed to begin to discharge through the resistor. We want to model the voltage across the capacitor, $V(t)$, as a function of time.

The voltage drop across the discharging capacitor is given by:

$$V + RC\frac{dV}{dt} = 0$$

Rearranging this equation to solve for $\frac{dV}{dt}$, we obtain:

$$\frac{dV}{dt} = -\frac{1}{RC}V$$

This is a linear, first-order ordinary differential equation with solution:

$$V(t) = V_0 e^{-\frac{t}{RC}}$$

where the initial condition $V(0) = V_0$ represents the voltage across the capacitor when it starts to discharge.

Suppose the resistor in our circuit has resistance $R = 12\text{k}\Omega = 12 \times 10^3\ \Omega$ and the capacitor has capacitance of $C = 100\,\mu\text{F} = 100 \times 10^{-6}\,\text{F}$. (Note that the product of ohms and farads is seconds, so $\Omega \cdot \text{F} = s$). Lastly, suppose the initial voltage is $V_0 = 12\ V$.

3 Methods to Implement

3.1 Runge-Kutta 4th Order Method (RK4)

Since both Adams-Bashforth and Adams-Moulton methods are multi-step methods, they require values from previous time steps (obtained by approximation or given) to approximate the next estimate of $y(t_{i+1})$. We will use the Runge-Kutta 4th order method to calculate the first few starting values. Given an initial value problem of the form:

$$y' = f(t, y), \quad y(t_0) = \alpha$$

and a step size h, the RK4 method to advance from time t_n to $t_{n+1} = t_n + h$ is given by:

$$\begin{aligned}
k_1 &= hf(t_n, w_n) \\
k_2 &= hf(t_n + \frac{h}{2}, w_n + \frac{k_1}{2}) \\
k_3 &= hf(t_n + \frac{h}{2}, w_n + \frac{k_2}{2}) \\
k_4 &= hf(t_n + h, w_n + k_3) \\
w_{n+1} &= w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

You will use RK4 to compute w_1 and w_2 starting from the initial condition w_0 at t_0.

3.2 Three-Step Adams-Bashforth Method (AB3)

Adams-Bashforth methods are explicit multi-step methods. The three-step Adams-Bashforth method (AB3) uses values from the previous three time steps to predict the solution at the next time step. The formula for AB3 is:

$$w_{n+1}^{(p)} = w_n + \frac{h}{12}\left(23f(t_n, w_n) - 16f(t_{n-1}, w_{n-1}) + 5f(t_{n-2}, w_{n-2})\right)$$

This formula will be used as the *predictor* step to obtain an initial estimate $w_{n+1}^{(p)}$ of the solution at t_{n+1}.

3.3 Two-Step Adams-Moulton Method (AM2)

Adams-Moulton methods are implicit multi-step methods. The two-step Adams-Moulton method (AM2) uses values from the previous time step and the *current* time step to refine the solution. The formula for AM2 is:

$$w_{n+1} = w_n + \frac{h}{12}\left(5f(t_{n+1}, w_{n+1}) + 8f(t_n, w_n) - f(t_{n-1}, w_{n-1})\right)$$

This formula will be used as the *corrector* step. Notice that w_{n+1} appears on both sides of the equation, making it implicit. In a typical predictor-corrector implementation, we use the predicted value $w_{n+1}^{(p)}$ obtained from the AB3 method to approximate $w_{n+1}^{(c)}$ on the right-hand side in the first iteration of the corrector step. The updated corrector method is then:

$$w_{n+1}^{(c)} = w_n + \frac{h}{12}\left(5f(t_{n+1}, w_{n+1}^{(p)}) + 8f(t_n, w_n) - f(t_{n-1}, w_{n-1})\right)$$

3.4 Predictor-Corrector Algorithm

To solve the ODE using the AB3-AM2 predictor-corrector method, you will follow these steps:

1. **Initialization:** Use the RK4 method to compute w_1, w_2, w_3 starting from the initial condition $w_0 = y(t_0) = \alpha$ at t_0. Set $t_1 = t_0 + h, t_2 = t_0 + 2h, t_3 = t_0 + 3h$.
2. **Iteration (for $n \geq 3$):** For each subsequent time step $t_{n+1} = t_n + h$:
   i. **Predictor Step:** Use the AB3 method to compute the predicted value $w_{n+1}^{(p)}$ using w_n, w_{n-1}, w_{n-2} and their corresponding function values $f(t_n, w_n), f(t_{n-1}, w_{n-1}), f(t_{n-2}, w_{n-2})$.

ii. **Corrector Step:** Use the AM2 method with the predicted value $w_{n+1}^{(p)}$ to compute the corrected value $w_{n+1}^{(c)}$ using $w_n, w_{n-1}, w_{n+1}^{(p)}$ and their corresponding function values $f(t_n, w_n)$, $f(t_{n-1}, w_{n-1})$, $f(t_{n-2}, w_{n-2})$, $f(t_{n+1}, w_{n+1}^{(p)})$.

iii. Set $w_{n+1} = w_{n+1}^{(c)}$.

iv. Update time: $t_{n+1} = t_n + h$.

4 Project Assignment

1. **Implement RK4:** Write a function that implements the Runge-Kutta 4th order method. This function should take as input the function $f(t, y)$, the current time t_n, the current solution w_n, and the step size h, and return the solution at the next time step w_{n+1}.

2. **Implement Predictor-Corrector Method:** Integrate the RK4 starting method, AB3 predictor, and AM2 corrector into a single function that solves an ODE over a specified time interval. This function should take as input the function $f(t, y)$, the initial condition (t_0, w_0), the step size h, and the final time T. It should return arrays of time points and the corresponding approximate solutions.

3. **Testing and Analysis:**

   - Use the values of R, C, and V_0 to model the discharge of the RC circuit described in section 2.
   - Using your predictor-corrector method, determine how long (how much time t) does it take for the capacitor to discharge 50% of its initial voltage? What about 99%?
   - How do your answers to the previous question differ from the values found using the analytical solution: $V(t) = V_0 e^{-\frac{t}{RC}}$
   - Compare the numerical solutions obtained from the predictor corrector method with the analytical solution, and with the solution obtained by just using RK4.
   - Investigate the behavior of the predictor corrector method, the RK4 method, and the analytical solution for different step sizes h. Present your results in a table.
   - GRADUATE STUDENTS: Consider the computational costs and local truncation error of each method.

5 Submission

You will need to submit the following:

- **Source Code:** Submit all code files implementing RK4 and the AB3-AM2 predictor-corrector method. Code should be well-commented and easy to understand. Your submission should include a brief README describing how to run your code.
- **Report:** A written report (in PDF format) that includes:
  - GRADUATE STUDENTS ONLY: A clear description of the implemented methods (AB3-AM2 and RK4).
  - Results of your testing and analysis described in the previous section, including tables comparing your numerical solutions (AB3-AM2 and RK4) to the analytical solution.
  - Discussion of your findings, including observations on accuracy, the effect of step size, and any challenges encountered in implementation or curiosities discovered when assessing the model.
  - GRADUATE STUDENTS: Comparison of the computational costs and local truncation errors of each method.

6 Grading

50 points Complete, functioning, and correct code that meets the specifications outlined above.

50 points Report which includes the following:

- Clear descriptions of the numerical methods implemented.
- Presentation of test results with appropriate visualizations.
- Insightful discussion of the method's performance and analysis.

Any work you submit for grading, must be completely your own work. You should not base your code in any way on anyone else's code (including mine). Any outside sources used for general background should be cited.
