6. A fair coin is tossed twice.
   Let E_1 be the event that both tosses have the same outcome, that is, $E_1 = (HH, TT)$.
   Let E_2 be the event that the first toss is a head, that is, $E_2 = (HH, HT)$.
   Let E_3 be the event that the second toss is a head, that is, $E_3 = (TH, HH)$.
   Show that E_1, E_2, and E_3 are pairwise independent but not mutually independent.

Consider the sample space made by tossing a coin twice, the observation of results can be distinguished into four elements:

$$S = \{ HH, HT, TH, TT \}$$

Let the events: $E_1 = \{HH, TT\}$; $E_2 = \{HH, HT\}$; $E_3 = \{TH, HH\}$

We can represent these Events through Venn-diagram:

```
```{r}

# Install and load the VennDiagram package

# install.packages("VennDiagram")

library(VennDiagram)


# Define the sets

E1 <- c("HH", "TT")

E2 <- c("HH", "HT")

E3 <- c("TH", "HH")


# Create the Venn diagram

venn.plot <- venn.diagram(

  x = list(E1 = E1, E2 = E2, E3 = E3),

  category.names = c("E1", "E2", "E3"),

  filename = NULL,
```
```

```
  fill = c("orange", "blue", "green"), # Define fill colors

  alpha = c(0.15, 0.15, 0.15), # Define transparency levels

  cat.fontface = "bold", # Set category label font face

  cat.fontsize = 14, # Set category label font size

  cat.dist = 0.2, # Set category label distance

  cat.default.pos = "text", # Set category label position

  cex = 0, # Set label size into zero

  fontface = "bold", # Set label font face

  fontfamily = "sans", # Set label font family

  label.col = "black", # Set label color

  label.fontface = "bold", # Set label font face

  margin = 0.05, # Set margin

  category.cex = 1.5, # Set category label size

  category.fontface = "bold" # Set category label font face

)

# Add labels for elements inside circles

grid.text("HH", x = unit(0.50, "npc"), y = unit(0.60, "npc"), gp = gpar(fontsize
= 12, col = "black", fontface = "bold"))

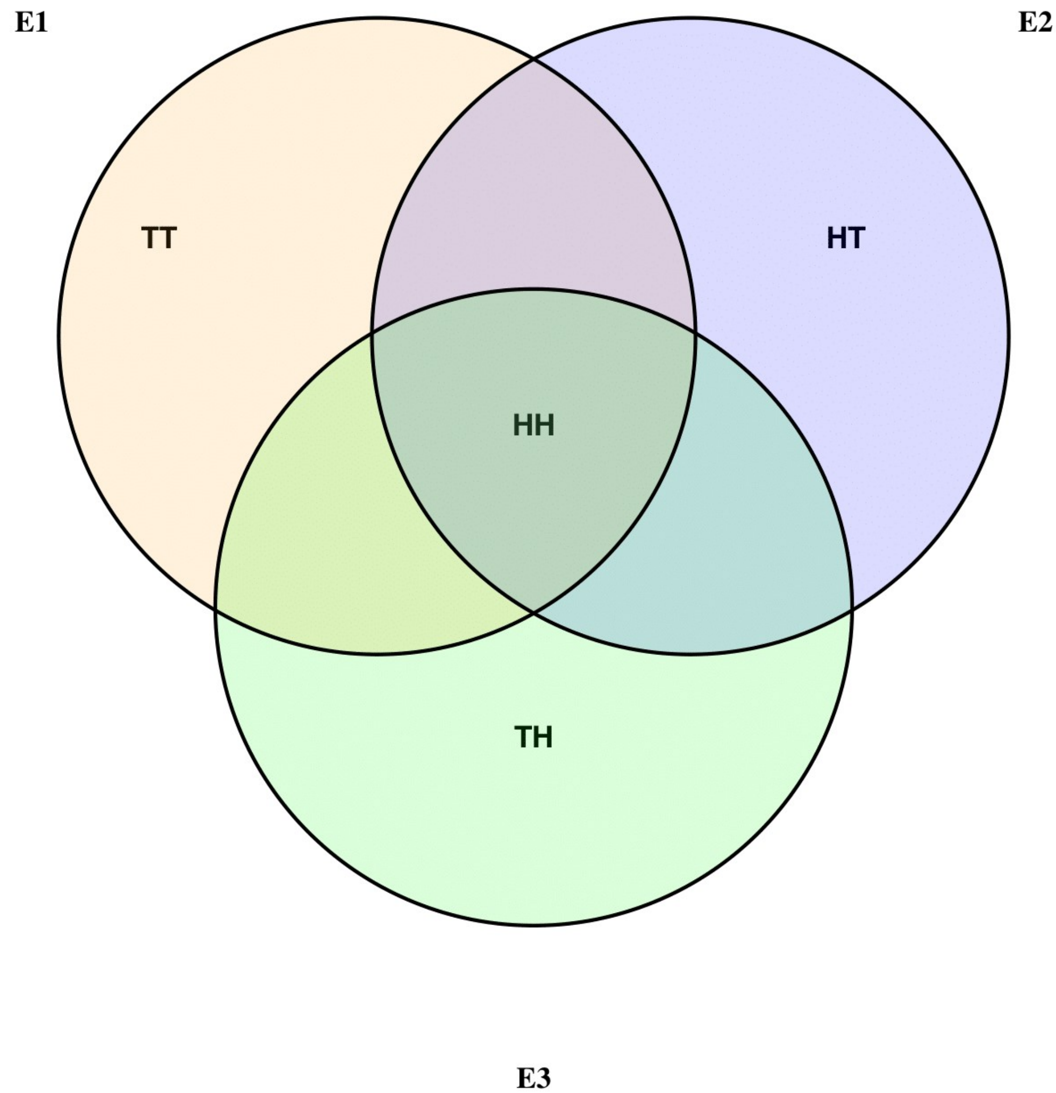
grid.text("TT", x = unit(0.2, "npc"), y = unit(0.75, "npc"), gp = gpar(fontsize
= 12, col = "black", fontface = "bold"))

grid.text("HT", x = unit(0.75, "npc"), y = unit(0.75, "npc"), gp = gpar(fontsize
= 12, col = "black", fontface = "bold"))

grid.text("TH", x = unit(0.50, "npc"), y = unit(0.35, "npc"), gp = gpar(fontsize
= 12, col = "black", fontface = "bold"))

# Display the Venn diagram

grid.draw(venn.plot)
```



Calculate the probability of each event such that whenever, $P(A) = \frac{\text{Desired outcome for A}}{\text{Sample Space}}$

$$P(E_1) = \frac{\{HH, TT\}}{\{\ HH, HT, TH, TT\ \}} = \frac{2}{4} = \frac{1}{2}$$

$$P(E_2) = \frac{\{HH, HT\}}{\{\ HH, HT, TH, TT\ \}} = \frac{2}{4} = \frac{1}{2}$$

$$P(E_3) = \frac{\{HH, TH\}}{\{\ HH, HT, TH, TT\ \}} = \frac{2}{4} = \frac{1}{2}$$

Thus, $P(\mathrm{E}_1) = P(E_2) = P(E_3) = 1/2$

These events $E_1, E_2,$ and E_3 are said to be pairwise independent if and only if the following are establishing.

Want to show:

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

$$P(E_1 \cap E_3) = P(E_1)P(E_3)$$

$$P(E_2 \cap E_3) = P(E_2)P(E_3)$$

Consider each equation, by given $P(\mathrm{E}_1) = P(E_2) = P(E_3) = 1/2$:

$$P(E_1 \cap E_2) = \frac{\{HH,TT\} \cap \{HH,HT\}}{\{\,HH,HT,TH,TT\,\}} = \frac{\{HH\}}{\{\,HH,HT,TH,TT\,\}} = \frac{1}{4} = \frac{1}{2} * \frac{1}{2} = P(E_1)P(E_2)$$

$$P(E_1 \cap E_3) = \frac{\{HH,TT\} \cap \{HH,TH\}}{\{\,HH,HT,TH,TT\,\}} = \frac{\{HH\}}{\{\,HH,HT,TH,TT\,\}} = \frac{1}{4} = \frac{1}{2} * \frac{1}{2} = P(E_1)P(E_3)$$

$$P(E_2 \cap E_3) = \frac{\{HH,HT\} \cap \{HH,TH\}}{\{\,HH,HT,TH,TT\,\}} = \frac{\{HH\}}{\{\,HH,HT,TH,TT\,\}} = \frac{1}{4} = \frac{1}{2} * \frac{1}{2} = P(E_2)P(E_3)$$

Hence the following equations and between two events are indeed pairwise independent.

Moreover, the probability among these events $E_1, E_2,$ and E_3 states that all involved individual probabilities must equal to the intersections of all events, to make these events mutually independent from another.

We want to show, that in the given experiment of tossing two coins, the following probability can be expressed as:

$$P(E_1)P(E_2)P(E_3) = P(E_1 \cap E_2 \cap E_3)$$

By assumption, we already know that any random pairs from the left-hand expressions are pairwise independent.

$$P(E_1 \cap E_2)P(E_3) = P(E_1 \cap E_2 \cap E_3)$$

Expand the definition of probability in right-hand expression:

$$P(E_1 \cap E_2)P(E_3) = \frac{\{HH, TT\} \cap \{HH, TH\} \cap \{HH, TH\}}{\{\ HH, HT, TH, TT\ \}}$$

Through law of associativity of events, $\{HH, TT\} \cap \{HH, TH\} \cap \{HH, TH\} = \{HH\}$

$$P(E_1 \cap E_2)P(E_3) = \frac{\{HH\}}{\{\ HH, HT, TH, TT\ \}}$$

Therefore, $P(E_1 \cap E_2 \cap E_3) = 1/4$

Conversely, expand the definition of probability of the left-hand expression, substitute $P(E_1 \cap E_2)P(E_3)$:

$$\left(\frac{\{HH\}}{\{\ HH, HT, TH, TT\ \}}\right)\left(\frac{\{HH, HT\}}{\{\ HH, HT, TH, TT\ \}}\right) = \frac{1}{4}$$

Simplify

$$\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

$$\frac{1}{8} = \frac{1}{4}$$

Contradiction, since we want to show that the equation $P(E_1)P(E_2)P(E_3) = P(E_1 \cap E_2 \cap E_3)$, however, the proof suggests otherwise, therefore, the given experiment, tossing two fair coins, having the three events $E_1, E_2,$ and E_3; these events are pairwise independent but they are not mutually independent. Q.E.D.