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Turbulence
Assignment 1

VERY IMPORTANT: Submit this assignment digitally on canvas before **23:59h on December 1st, 2023**. You are allowed to work in pairs: 2 people submit 1 solution. **Show all of your steps in the derivations, state the assumptions you make, and provide source code of numerically solved tasks!** Ensure that your solutions are clear and legible. Unfortunately, if the grader cannot understand your process, you will not receive any points. Reminder: Physical quantities must have units and figures must have appropriate labels and axes (i.e. logarithmic, if applicable)..

1 Turbulent statistics (2.0 points)

Turbulence is characterized by the chaotic and irregular motion of flow. We have provided measurement data of a turbulent flow which you will use to calculate various statistics. Download `velocity_A1_Ex1.h5` from Canvas. The file contains data of a three-dimensional velocity field.

Reading in the data from `.h5`-files is fairly simple. We have provided some code snippets on the online cheatsheet to help you get started.

Note: The velocity field in the h5-files is given as $\mathbf{u}(x, y, z) = (u, v, w) = (\texttt{vx}, \texttt{vy}, \texttt{vz})$ *such that* $(N_x, N_y, N_z) = (192, 96, 257)$ *and* $(L_x, L_y, L_z) = (2.0, 1.0, 1.0)$. *Here, the* z *axis is aligned with the vertical direction whereas* x *and* y *are the horizontal axes.* N *indicates the number of grid points along an axis and* L *indicates the length of the domain along the axis.*

a) Make 2D contour plots of all velocity components at the horizontal plane at $z = 0.5$. Determine whether the fields satisfy periodic boundary conditions.

b) Compute the fluctuations $u'(z)$, $v'(z)$ and $w'(z)$ at the location $(x = 1.0, y = 0.5)$ and plot their profiles as function of height z. The fluctuations are defined relative to the horizontal (xy-)average, such that for example

$$u'(z) = u - \overline{u}(z), \qquad \text{where} \quad \overline{u}(z) = \frac{1}{L_x L_y} \iint_A u(x, y, z)\, \mathrm{d}x\mathrm{d}y$$

Throughout this problem, we shall treat the horizontal average as the relevant mean flow.

c) Calculate the root-mean-square (RMS) of u, v, w and the turbulent kinetic energy at each of the horizontal planes in the velocity data and plot their profiles as function of height z.

d) Evaluate the PDFs of u' at the horizontal planes given by $z = 0.5$ and $z = 0.06$ and plot it. Compare the both PDFs with the PDF of Gaussian distribution.

e) Write down expressions for the standard deviation, skewness, and kurtosis of u solely containing averages of powers of u (so that no expression contains a fluctuation term such as u'). Using these expressions, calculate these quantities for u' in the two horizontal planes $z = 0.5$ and $z = 0.06$. Discuss how to interpret the values of these statistics in the context of your answer for (d).

2 Stochastic model for large scale convection roll (1.5 point)

In Rayleigh-Bénard convection, a layer of fluid heated from below and cooled from above with temperature difference ΔT, there is a large scale circulation (LSC) of the fluid. The LSC can be described as $T(t, \theta) = T_0 + \delta(t) \cdot \cos(\theta - \theta_0(t))$, where $\delta(t)$ is the amplitude/strength of the LSC and $\theta_0(t)$ the orientation of the LSC in the azimuthal plane θ (Brown and Ahlers, 2007). The change in δ is determined by the balance between the buoyancy and the drag forces and is driven by stochastic fluctuations $f_\delta(t)$ which phenomenologically represent the action of the small-scale turbulent background. Brown and Ahlers (2007) propose the simple model

$$\dot{\delta} = \frac{\delta}{\tau_\delta}\left(1 - \sqrt{\frac{\delta}{\delta_0}}\right) + f_\delta(t), \tag{1}$$

where δ_0 is the steady-state amplitude, and τ_δ is a decay time scale. The azimuthal acceleration of the LSC is damped by the rotational inertia of the LSC, and again a stochastic fluctuation term $f_{\dot{\theta}}(t)$ is added to obtain:

$$\ddot{\theta}_0 = -\frac{\dot{\theta}_0 \delta}{\tau_{\dot{\theta}} \delta_0} + f_{\dot{\theta}}(t), \tag{2}$$

where $\tau_{\dot{\theta}} = L^2/(2\nu R_e)$.

Simulate the behaviour of the LSC for one day.

- Numerically solve the ODE system (1)-(2) for $\delta(t)$ and $\theta_0(t)$.
- Model the stochastic terms $f_\delta(t)$ and $f_{\dot{\theta}}(t)$ with Gaussian distributed white noise. Use a standard deviation of $\sqrt{D/h}$ where $D_\delta = 3.5 \times 10^{-5}\,\mathrm{K}^2/\mathrm{s}$, $D_{\dot{\theta}} = 2.5 \times 10^{-5}\,\mathrm{rad}^2/\mathrm{s}^3$, and h is the simulation time step.
- Take $\delta_0 = 0.25K$ (experimental value) for the simulation.
- Use the decay time scales

$$\tau_\delta = \frac{L^2}{18\nu R_e^{1/2}} \qquad\qquad \tau_{\dot{\theta}} = \frac{L^2}{2\nu R_e} \tag{3}$$

calculated from the physical parameters

L	ν	R_e
$0.2476\,\mathrm{m}$	$6.69 \times 10^{-3}\,\mathrm{cm}^2/\mathrm{s}$	3700

Table 1: cell height L, viscosity ν, equilibrium Reynolds number R_e

Think about the meaning of the initial condition for $\dot{\theta}_0(t = 0)$ and $\theta_0(t = 0)$. Choose on your own and explain your choice.

Hint(s): You should use the intrinsic standard function `randn` *of NumPy/Julia/Matlab that generates normally-distributed random numbers to model turbulence $f_\delta(t)$ and $f_{\dot{\theta}}(t)$ and use Euler's method to integrate the equations instead of using standard ODE solvers.*

3 Lagrangian description of flow (1.5 points)

Consider a two-dimensional flow given by the stream function:

$$\psi(x, y, t) = -2x^2 \sin(2t) + y^2 \cos(t) \tag{4}$$

a) Calculate the velocity field:

$$\mathbf{U}(x, y, t) = \frac{\partial \psi}{\partial y}\mathbf{i} - \frac{\partial \psi}{\partial x}\mathbf{j} \tag{5}$$

b) At $t = 0$ we put dye in a circle, with unit radius, around the origin. The dye can be viewed as a passive scalar. Draw the time evolution of this circle. Give snapshots of what has been the circle at $t = 0$ for $t = 0.1$, $t = 1$, and $t = 10$.

c) Now we assume that the dye is no longer a passive tracer, but it has instead a density which is much larger than the fluid. The dynamics of small dye particles of such kind can be described by the following equation of motion:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(t) \tag{6}$$

$$\frac{d\mathbf{v}(t)}{dt} = \frac{1}{\tau}\left(\mathbf{U}(\mathbf{x}(t), t) - \mathbf{v}(t)\right) \tag{7}$$

where τ represents the typical response time of the dye-particle to the fluid flow fluctuations. We have the following initial condition for the dye

$$\mathbf{v}(0) = \mathbf{U}(\mathbf{x}(0), 0) \tag{8}$$

What is the difference in the dye distribution at $t = 10$ for values of $\tau = 10^{-1}$, $\tau = 10^0$, $\tau = 10^1$, and $\tau = 10^2$? Give the corresponding snapshots. For comparison reasons you should fix the plot ranges.

Hint: You should make sure that your Δt in your integration is small enough, such that the final solution is independent of the time step Δt.

4 Linear stability of a system of ODEs (2.5 points)

Stability analysis is widely used in fluid dynamics. Roughly speaking, it involves 3 basic steps:

1. Determining stationary solution(s) $\boldsymbol{s}$ of the equations,
2. Prescribing a specific form of small perturbation to $\boldsymbol{s}$,
3. Predicting the evolution of such perturbations.

The aim of the present exercise is to apply this strategy to a problem in which a *linear* stability analysis is feasible.

To begin with, it is convenient to introduce:

$$\boldsymbol{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

which allows us to write a set of ordinary differential equations in a compact form as:

$$\frac{d\boldsymbol{f}}{dt} = \boldsymbol{F}(\boldsymbol{f}). \tag{9}$$

Then, let $\boldsymbol{f}^*(t)$ equal:

$$\boldsymbol{f}^*(t) = \boldsymbol{s}(t) + \boldsymbol{e}(t),$$

with $\boldsymbol{s}(t)$ a stationary solution (i.e., a solution for which $d\boldsymbol{s}/dt = \boldsymbol{F}(\boldsymbol{s}) = 0$) and $\boldsymbol{e}$ a small perturbation.

To describe the temporal behavior of $\boldsymbol{e}$, we differentiate $\boldsymbol{f}^* = \boldsymbol{s} + \boldsymbol{e}$:

$$\frac{d\boldsymbol{f}^*}{dt} = \frac{d\boldsymbol{s}}{dt} + \frac{d\boldsymbol{e}}{dt}.$$

Now, by expanding $\boldsymbol{F}(\boldsymbol{f}^*) = \boldsymbol{F}(\boldsymbol{s} + \boldsymbol{e})$ around $\boldsymbol{s}$ and taking only the linear term one finds:

$$\boldsymbol{F}(\boldsymbol{s} + \boldsymbol{e}) = J\boldsymbol{e}, \tag{10}$$

where J is the Jacobian *matrix*, a multivariate analog of $F'(s)$ in

$$F(s + e) = F(s) + eF'(s) + O(e^2), \tag{11}$$

with stationary solution $F(s) = 0$. The matrix elements of J are *functions* of f, g, and h and are evaluated at $\boldsymbol{s}$:

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial f} & \frac{\partial F_1}{\partial g} & \frac{\partial F_1}{\partial h} \\ \frac{\partial F_2}{\partial f} & \frac{\partial F_2}{\partial g} & \frac{\partial F_2}{\partial h} \\ \frac{\partial F_3}{\partial f} & \frac{\partial F_3}{\partial g} & \frac{\partial F_3}{\partial h} \end{bmatrix}_{(s_1, s_2, s_3)}$$

Here, if $\boldsymbol{e}$ is an eigenvector of J, then $J\boldsymbol{e} = \lambda\,\boldsymbol{e}$ and hence eq. (9) becomes

$$\frac{d\boldsymbol{e}}{dt} = \lambda\,\boldsymbol{e}.$$

Therefore, the central aspect of the linear stability analysis is the following: if $\lambda > 0$, then the perturbation grows exponentially fast with time.

In this exercise we study the following non-linear ordinary differential equations analytically:

$$\frac{df}{dt} = -g - h \tag{12}$$

$$\frac{dg}{dt} = f + \alpha g \tag{13}$$

$$\frac{dh}{dt} = \beta + (f - \gamma)h, \tag{14}$$

where f, g and h depend only on t. The equations are relatively simple, consisting of a single non-linear term only, but nevertheless show interesting features. In this exercise we will study these equations in some detail. For now we will take for the parameters $\alpha = 0.25$, $\beta = 0.5$ and let $\gamma > 0$.

a) Find analytically the stationary solutions, $\boldsymbol{s}$, and show that $\boldsymbol{s}$ does not exist when γ is lower than some critical value. How many solutions for $\boldsymbol{s}$ can you find when γ equals the critical value? And how many when γ is greater than the critical value?

b) Construct the Jacobian matrix.

c) Determine the value(s) for $\boldsymbol{s}$ with the critical value of γ you found in (a). Do the same when $\gamma = 1$. What are the corresponding eigenvalues of the associated Jacobian matrix?

d) Discuss the stability based on results you found in (c).

5 Numerical study (2.5 points)

In the previous exercise we studied a system of ODE's algebraically, where we have learned about some of its local behaviour. Now we will numerically integrate these ODE's and study how trajectories $\boldsymbol{f}(t)$ move throughout space.

a) Integrate the system of equations (12) to (14), with initial condition $\boldsymbol{f}(0) = (1, 0, -2)^T$ for three different cases: $\gamma = 2.0$, $\gamma = 3.2$ and $\gamma = 3.8$. Keep α, β the same, i.e. $\alpha = 0.25$, $\beta = 0.5$. Make a parametric plot ($f(t)$,$g(t)$) for these three cases and describe the solutions behaviour. You might want to skip the initial transient.

b) Now take $\alpha = 0.1$, $\beta = 0.1$, and $\gamma = 18$, and make a 3D parametric curve of $\boldsymbol{f}(t)$. Also plot the three time series $f(t)$, $g(t)$, and $h(t)$ separately. Which of these signals is the most intermittent? Do you observe something unusual about the trajectory $\boldsymbol{f}(t)$? This is known as a strange attractor.

c) Make a Probability Density Function (PDF) of $f(t)$, obtained in (b), for:

- $0 < t < 100$,
- $30 < t < 100$,
- $30 < t < 500$,
- $30 < t < 1000$.

What are the differences between the histograms? Elaborate on your answer in terms of statistical stationarity.

d) Simulate the system again with the same conditions as in (b), but now using initial conditions $\boldsymbol{f}_1(0) = (1, 10^{-8}, -2)^T$. Make a parametric plot of the new $\boldsymbol{f}_1$, and plot the difference $\boldsymbol{f}_1 - \boldsymbol{f}$ for each component separately between the new results and the simulated values of $\boldsymbol{f}$ computed in (b). Additionally repeat the plots from part (d). What do you notice about the PDFs?

e) Compute the power spectrum of $f(t)$ as computed in (b). Normalize your result such that Parseval's theorem is satisfied, that is

$$\int_0^\infty |f(t)|^2 dt = \int_0^\infty |\hat{f}(k)|^2 dk \tag{15}$$

What is the spectrum at low frequencies: sharp peaks or broad noise? Does it provide an indication of chaos?