Dr. Richard Stevens (r.j.a.m.stevens@utwente.nl)
Yang Liu (y.liu-4@utwente.nl)
Davide Selvatici (d.selvatici@utwente.nl)
Physics of Fluids
Department of Applied Physics
University of Twente

**Turbulence**
**Assignment 2**

VERY IMPORTANT: Submit this assignment both digitally on canvas before **23:59h on December 20st, 2023**. You are allowed to work in pairs: 2 people submit 1 solution. **Show all of your steps in the derivations, state the assumptions you make, and provide source code of numerically solved tasks!** Ensure that your solutions are clear and legible. Unfortunately, if the grader cannot understand your process, you will not receive any points. Reminder: Physical quantities must have units and figures must have appropriate labels and axes (i.e. logarithmic, if applicable).

# 1 The model spectrum (2.5 points)

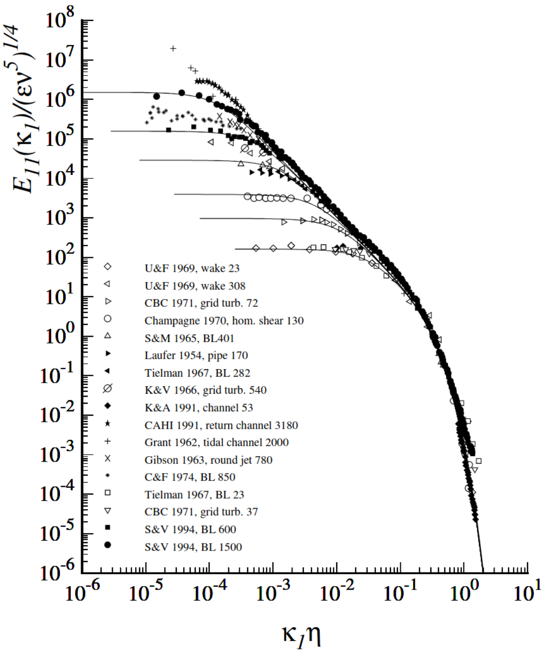


Figure 1: Measurements of one-dimensional longitudinal velocity spectra (symbols), and model spectra (Eq. 6.246) for $Re_\lambda$ = 30, 70, 130, 300, 600 and 1500 (lines). The experimental data are taken from Saddoughi and Veeravalli (1994) where references to the different experiments are given. For each experiment, the final number in the key is the value of $Re_\lambda$.

**a)** A model for the energy-spectrum is given in section 6.5.3 of Pope:

$$E(\kappa) = C\varepsilon^{2/3}\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta), \tag{1}$$

where

$$f_L(\kappa L) = \left(\frac{\kappa L}{[(\kappa L)^2 + c_L]^{1/2}}\right)^{5/3+p_0}, \qquad f_\eta(\kappa\eta) = \exp\{-\beta\{[(\kappa\eta)^4 + c_\eta^4]^{1/4} - c_\eta\}\}. \tag{2}$$

Duplicate three of the lines in the above figure (Fig. 6.14 of Pope). You have to plot both $E$ and $E_{11}$. Use the parameter given in Pope, Sec. 6.5.3: $\beta = 5.2$, $C = 1.5$, $c_L = 6.78$, $c_\eta = 0.4$, $p_0 = 2$. Describe the physical meaning of your plots throughout its whole range.
*Hint 1: Use relations, given in Pope (page 200, 222 onwards), to derive relations between normalized quantities $E'(\kappa', Re_\lambda)$ and $E'_{11}(\kappa', Re_\lambda)$.*
*Hint 2: Remember that the Taylor Reynolds number $Re_\lambda$ and the integral-scale Reynolds number $Re_L$ are related as: $Re_\lambda = \sqrt{\frac{20}{3} Re_L}$*

**b)** Integrate your spectra numerically to calculate the turbulent kinetic energy $k$ and the TKE dissipation rate $\varepsilon$, and estimate the corresponding Reynolds number. Compare the numerically-integrated Reynolds numbers to the initial Reynolds numbers that you picked create the spectrum in **(a)**.

## 2 Dissipation rates in Rayleigh-Bénard convection (3.5 points)

In Rayleigh-Bénard (RB) convection a layer of fluid is heated from below and cooled from above. Normally one considers a RB system in which the temperature of the upper and lower plate is constant and the sidewall is adiabatic and the walls are impermeable. The control parameters of the system are the Rayleigh number $Ra = \beta g \Delta L^3/(\nu\kappa)$, which indicates the dimensionless temperature difference between the two horizontal plates, and the Prandtl number $Pr = \nu/\kappa$, which is a measure of how important viscosity and conduction are in relation to each other, and is a property of the fluid. Here $\beta$ is the thermal expansion coefficient, $g$ the gravitational acceleration, $\Delta$ the temperature difference between the plates, $\nu$ the kinematic viscosity, and $\kappa$ the thermal diffusivity. The heat transport in non dimensional form is given by the Nusselt number $Nu$, which is defined as:

$$Nu \equiv \frac{\langle \theta u_3 \rangle_A - \kappa \partial_3 \langle \theta \rangle_A}{\kappa \Delta L^{-1}} \tag{3}$$

As RB is a closed system, due to energy conservation we can think that for the statistically stationary state, there must be a balance between the energy that is fed into the system and the energy that is dissipated inside the system. Indeed, we can derive two exact relations for the kinetic and thermal energy-dissipation rates $\epsilon_u$ and $\epsilon_\theta$, respectively, namely:

$$\epsilon_u \equiv \left\langle \nu[\partial_i u_j(\mathbf{x},t)]^2 \right\rangle_V = \frac{\nu^3}{L^4}(Nu-1)RaPr^{-2}, \tag{4}$$

$$\epsilon_\theta \equiv \left\langle \kappa[\partial_i \theta(\mathbf{x},t)]^2 \right\rangle_V = \kappa \frac{\Delta^2}{L^2} Nu. \tag{5}$$

These relations can be derived from the Boussinesq equations and the corresponding boundary conditions, assuming only statistical stationarity.

**a)** Obtain the expression for the kinetic dissipation rate. Start from the Boussinesq equations for $u_i$:

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i p + \nu \partial_j^2 u_i + \beta g_i \theta, \tag{6}$$

*Hint 1 : Note that (6) is a momentum balance, we are looking for an energy dissipation rate.*
*Hint 2 : Use Gauss theorem.*
*Hint 3 : Assume $Nu(z) = const. \Rightarrow \langle Nu \rangle_z = Nu$.*

**b)** Obtain the expression for the thermal dissipation rate. Start from the Boussinesq equations for $\theta$:

$$\partial_t \theta + u_j \partial_j \theta = \kappa \partial_j^2 \theta, \tag{7}$$

*Hint : Keep the previous exercise in mind.*

**c)** In the Nusselt number, two different terms can be distinguished, a convective and a conductive term, being $\langle \theta u_3 \rangle_A$ and $\kappa \partial_3 \langle \theta \rangle_A$, respectively. In fully turbulent flow, which of these terms will be dominant?

## 3 Reynolds decomposition & turbulent kinetic energy equation (4 points)

The Navier-Stokes equation for large-eddy simulations of a wind farm in stably stratified Ekman boundary layer is given by equations 8, 9, and 10.

$$\partial_i u_i = 0, \tag{8}$$

$$\begin{aligned}\partial_t u_i + u_j \partial_j u_i = &-\partial_i p - \partial_i \tau_{ij}^{\mathsf{sgs}} + g\beta(\theta - \theta_0)\delta_{i3} + f(U_g - u)\delta_{i2} \\ &- f(V_g - v)\delta_{i1} + F_i\end{aligned}, \tag{9}$$

$$\partial_t \theta + u_j \partial_j \theta = -\partial_j q_j, \tag{10}$$

where $u_i$ and $\theta$ are the filtered velocity and potential temperature respectively, $g$ is the acceleration due to gravity, $\beta = 1/\theta_0$ is the buoyancy parameter with respect to the reference potential temperature $\theta_0$, $\delta_{ij}$ is the Kronecker delta, and $f$ is the Coriolis parameter. The boundary layer is driven by a mean pressure gradient $\nabla p_\infty$, represented by the (constant) geostrophic wind with $U_g = -\frac{1}{\rho f}\frac{\partial p_\infty}{\partial y}$ and $V_g = \frac{1}{\rho f}\frac{\partial p_\infty}{\partial x}$ as its components. $F_i = (F_x, F_y, 0)$ represents the turbine forces. The molecular viscosity is neglected as it is a high Reynolds number flow, which is a common practice in atmospheric boundary layer simulations. $\tau_{ij}^{\mathsf{sgs}}$ is the traceless part of the sub-filter scale stress tensor, and $q_j$ is the sub-filter scale heat flux tensor.

**a)** Derive the equation for kinetic energy $k(x, y, z, t) = \frac{1}{2}u_i u_i$ from the momentum conservation equation (9).

**b)** Take the time average of (9), and derive an equation for the mean kinetic energy $\frac{1}{2}\overline{u_i}\,\overline{u_i}$. *Hint: Decompose **all** variables into a mean and fluctuating component, e.g. $\theta = \overline{\theta} + \theta'$, where an overbar denotes a time-average.*

**c)** From your answers to (a) and (b), derive an equation for turbulent kinetic energy $\frac{1}{2}u_i'u_i'$.

**d)** Describe the physical meaning of each term in the turbulent kinetic energy equation.

**Please continue to the next page!**

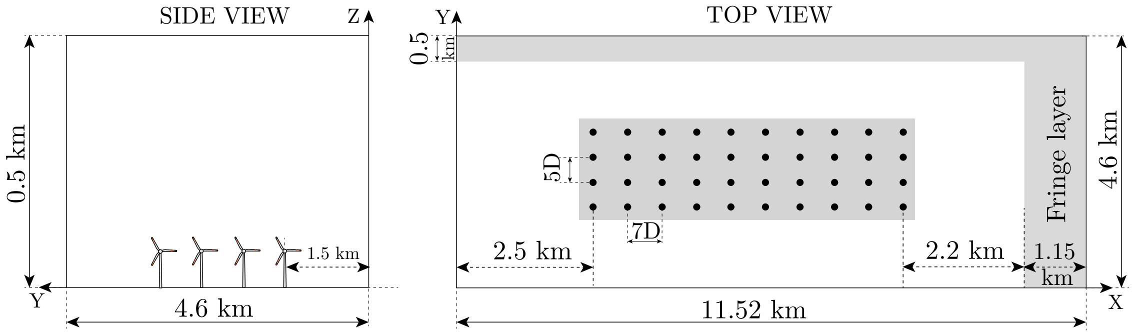


Figure 2: Schematic showing the wind farm layout. Position of the wind turbines is represented by black circles.

On Canvas you can find data from LES of a wind farm with 40 turbines (10 rows and 4 columns, Fig. 2). The turbine diameter is $D = 90\,\mathrm{m}$ and the hub height is $z_h = 90\,\mathrm{m}$. The turbines are separated by a distance of $s_x = 7D$ and $s_y = 5D$ in the streamwise and spanwise directions respectively. The computational domain is $11.52\,\mathrm{km} \times 4.6\,\mathrm{km} \times 0.5\,\mathrm{km}$. The domain is discretized by $1280 \times 512 \times 100$ grid points in the streamwise, spanwise, and vertical directions respectively. This corresponds to a grid spacing of $9\,\mathrm{m} \times 9\,\mathrm{m} \times 5\,\mathrm{m}$ (sufficient resolution of all the relevant length scales). The HDF5 data files contain:

- tavg_u.h5 - Streamwise velocity
- tavg_v.h5 - Spanwise velocity
- tavg_txxs11.h5, etc. - Dissipation terms
- tavg_fu.h5 - Turbine force in x -direction multiplied by u-velocity
- tavg_fv.h5 - Turbine force in y - direction multiplied by v - velocity
- tavg_production.h5 - Turbulence production term

The last four of these should correspond to terms in your TKE equation derived in part (d).

**e)** Plot a colored contour plot of the streamwise velocity $u_x$ at $z = 0.1$ km and through the middle of turbine column 2, i.e. at $y = 1.95\,\mathrm{km}$. What do you observe? Explain the significance.

**f)** Calculate the total power produced by individual turbines by performing numerical integration (use trapezoidal rule) over a control volume of size $7D \times 5D \times 2D$ around each turbine. The control volume must be chosen such that the turbine is situated in the middle of it, for example, in x-direction, consider $3.5D$ in front and $3.5D$ behind the turbines and so on. Calculate the mean power production of each row and normalize that power by the power produced by the first row. Plot a graph of normalized power vs row. Explain what you observe.

**g)** Using the data files provided show that, unlike in a textbook shear flow, the turbulence production and dissipation are not equal. In fact you will find that the production is more than dissipation. Explain why the production and dissipation are not equal in a wind farm. Normalize the mean turbulence production and dissipation of each row by the mean power produced by the first row of turbines, and plot a graph of normalized power, dissipation, production versus rows.