Chapter 4
Magnetostatics I

June 4, 2021

Magnetostatics is reviewed, first as a subject independent of electrical phenomena. The vector potential and gauge degree of freedom are introduced, resulting in the Poisson equation for each component of the vector potential. The solution leads to the Biot–Savart Law.

Contents

1 Introduction

1.1 Magnetostatics as a stand-alone subject

There are several ways to approach magnetism. Historical interest started with permanent magnets in nature. These are complicated systems, with magnetic fields generated by atomic currents which are hard to measure and control. Modern treatments would start with moving charges or currents, which are easy to measure and control. Here we first deal with magnetostatics as such a stand-alone subject, not related to electric phenomena (except for similarities in the mathematical formalism). Later parts of the course will emphasize the covariant approach, in which electric and magnetic phenomena are unified.

1.2 Current and moving charge

The magnetic field is defined operationally by sensing the force on one moving point charge, through the formula

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1}$$

But it is easier to deal with a wire segment $d\boldsymbol{\ell}$ carrying a current I. The corresponding force is

$$\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B} \tag{2}$$

Problem 1
Consider a wire segment with cross-section area A and length $d\boldsymbol{\ell}$. There are n charge carriers per unit volume, each of charge q, and moving with (average) velocity v.
(a) What is the current I?
(b) Starting from (1), find the total force on the charge carriers, and hence derive (2). §

1.3 Force between parallel currents

The classic experiment (**Figure 1a**) deals with two long parallel wires a distance r apart, carrying currents I_1 and I_2, say in the same direction. It is found that the force F per unit length ℓ on each wire varies as

$$\frac{F}{\ell} = \kappa \frac{I_1 I_2}{r} \tag{3}$$

where κ is a constant. The force is attractive if the currents are in the same direction. With reference to (2), this means that the field $\mathbf{B}$ due to the first wire must circulate around it in the sense of the right-hand rule (**Figure 1b**).

1.4 Definition of ampere

The proportionality constant κ in (3) is fixed by defining the unit of current: If two equal currents at a separation of $r = 1$ m cause an attractive force of 2×10^{-7} N per unit length of wire, then the current is defined as one *ampere*. In other words, in the MKSA system of units, we have *exactly*

$$\kappa = 2 \times 10^{-7} \text{ N A}^{-2}$$

The coulomb is defined in terms of the ampere, not the other way round.

Update in 2019
The definition of electrical and magnetic units was changed in May 2019, so the above is not strictly valid. But unless you work to extreme accuracy, you can in effect still use the above definition.

2 Laws of magnetostatics

2.1 Ampere's law

Integral form
Static magnetic fields are caused by currents, and the essential law is due to Ampere:

$$\oint_\Gamma \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S} \tag{4}$$

On the RHS, I is the current passing through a surface S, equal to the current density $\mathbf{J}$ integrated over the surface. On the LHS, the line integral is taken over the boundary of the same surface: $\Gamma = \partial S$, and using the right-hand rule to relate the positive sense of $d\mathbf{S}$ and the positive sense of $d\boldsymbol{\ell}$. The constant is

$$\mu_0 = 4\pi \times 10^{-7} \text{ T m A}^{-1}$$

Problem 2
Consider an infinite long wire carrying a current I_1. Apply (4) to a circular path Γ of radius r around the wire, and hence show that the tangential magnetic field is given by

$$B = \frac{\mu_0 I_1}{2\pi r}$$

The force F acting on a length ℓ of another current I_2 in a parallel wire at a distance r is then given by

$$\frac{F}{\ell} = I_2 \, B = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Finally, comparing with the definition of the ampere, derive the value of μ_0 as an *exact* statement.[1]
§

Differential form
Use Stokes' theorem to convert the LHS of (4) to a surface integral:

$$\oint_\Gamma \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S (\boldsymbol{\nabla} \times \mathbf{B}) \cdot d\mathbf{S}$$

Then since (4) must hold for any surface S, we find

$$\boxed{\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J}} \tag{5}$$

Conservation of charge
For any integral relationship such as (4), a subtlety must be checked. The same closed loop Γ can be the boundary to different surfaces S_1 and S_2. Consider the hemisphere in **Figure 2**, say the southern hemisphere of the earth. The equator is Γ. We can take S_1 to be the flat equatorial plane, and S_2 to be the surface of the hemisphere. The two surface integrals must be equal.

The two surfaces together constitute a *closed* surface S, and with the usual convention that $d\mathbf{S}$ is in the direction of the *outward* normal, we find, schematically

$$\oint_S = \int_{S_1} - \int_{S_2}$$

so

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \tag{6}$$

This is the condition of *conservation of charge* in time-independent situations: there can be no net flow of charge coming out of a closed surface. In short, Ampere's law requires (or proves) the conservation of charge.

The differential form provides a simpler proof: Take divergence of (5). Since the divergence of a curl is zero, we get

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{J} = 0} \tag{7}$$

which is the differential form of (6), and can be obtained from it by application of Gauss' theorem.

[1] Exact within the pre-2019 definition of electrical units.

Although the derivation using the differential form looks simpler, the integral formalism gives a more physical picture.

Need for generalization
The condition (6) or (7) would not hold if there is time dependence. **Figure 3** shows a capacitor being charged, and again two surfaces S_1 and S_2, which together form a closed surface S that encloses one plate. There is a net flow of current into S if the plate is being charged. So

Ampere's law in this form cannot be correct when there is time dependence.

Maxwell eventually supplied the missing ingredient — an additional time-dependent term. But for the moment we only consider situations that are either static or so slowly changing that the time-dependent term can be neglected.

2.2 No magnetic monopole

Consider the case of a long current-carrying wire. Ampere's law, applied to the loop Γ in **Figure 4**, determines the tangential component B_t around the circular loop, but does not determine the radial component B_r. We need an additional condition.

The condition is that *there is no magnetic charge or monopole*, which, in analogy to Gauss' law for electrostatics, gives, in integral form

$$\oint \mathbf{B} \cdot d\mathbf{S} \quad = \quad 0$$

or in differential form

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{B} = 0} \tag{8}$$

This is an *experimental* law, based on the fact that no monopole has ever been found.[2] This leads to a question: Are monopoles allowed theoretically? If they are, is there any reason why they have not been found? We return to this briefly in the next Chapter, and for an extensive discussion later, it being one of the most intriguing questions left unanswered by Maxwell.

Unless otherwise specified in speculative discussions, we shall henceforth assume that there are no monopoles, and that (8) holds.

Problem 3
Use the above condition to show that the radial component B_r in **Figure 4** is zero. §

[2] Very tight limits have been set.

3 Vector potential

3.1 Definition

Since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\mathbf{B}$ can be written as a curl, thus introducing a vector potential $\mathbf{A}$:

$$\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}} \tag{9}$$

The vector potential plays a role in magnetostatics similar to Φ in electrostatics.

Why should electrostatics be described by *one* potential function Φ while magnetostatics is described by *three* potential functions A_i, $i = 1, 2, 3$? The heuristic answer (which can be made precise below; see (14)) is that static electric fields are caused by the charged density ρ (one function) whereas static magnetic fields are caused by the current density $\mathbf{J}$ (three functions). A corollary follows: Since $\mathbf{J}$ is just a case of ρ that is moving, so $\mathbf{A}$ must also be a case of Φ that is moving — a heuristic statement that will be made precise through the covariant formulation of electrodynamics.

3.2 Gauge degree of freedom

Consider a simple example, with a constant magnetic field B_0 in the z-direction:

$$\mathbf{B} \quad = \quad B_0 \, \hat{\mathbf{e}}_z$$

Problem 4
Show that all of the following vector potentials give rise to the magnetic field above:

$$\begin{aligned} \mathbf{A} &= B_0 \, x \, \hat{\mathbf{e}}_y \\ \mathbf{A} &= -\, B_0 \, y \, \hat{\mathbf{e}}_x \\ \mathbf{A} &= \frac{1}{2} B_0 \, (x \, \hat{\mathbf{e}}_y - y \, \hat{\mathbf{e}}_x) \end{aligned}$$

This shows that some aspects of the vector potential are not physical — just like a constant added to Φ. §

To discuss this more generally, notice that the curl of a gradient is zero, so we are free to change $\mathbf{A}$ by

$$\mathbf{A}(\mathbf{r}) \quad \mapsto \quad \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla}\Lambda(\mathbf{r}) \tag{10}$$

since

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Lambda) \quad = \quad 0$$

Problem 5
Determine the functions Λ for transforming between the different choices of the vector potential $\mathbf{A}$ in Problem 4. §

The gauge degree of freedom turns out to be extremely important. At the simplest level, we shall

$B_0 x \hat{e}_y + \nabla\Lambda = -B_0 y \hat{e}_x$

$\nabla\Lambda = -B_0(x\hat{e}_y + y\hat{e}_x)$

$\Lambda = -B_0 xy$

see immediately below that it helps to solve the potential from the current. At a more sophisticated level, it leads to the question: Is the vector potential physical? This question will be considered at length later. There are many subtleties: $\mathbf{A}$ is physical to the extent that (a) it is needed in quantum mechanics (the magnetic field is not enough), and (b) a charge that travels in a region where $\mathbf{B} = 0$ but $\mathbf{A} \neq 0$ can, in certain circumstances, detect an effect, but (c) despite these, and the condition $\mathbf{B} = \nabla \times \mathbf{A}$, monopoles are not completely ruled out, i.e., there is an escape from $\nabla \cdot \mathbf{B} = 0$. Let these tantalizing questions provide some impetus for some advanced parts of this course.

3.3 Vector potential and current

Equation in any gauge
From (9) and (5) we find

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) \;=\; \mu_0 \mathbf{J}$$

Using the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \;=\; \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

we get

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \;=\; \mu_0 \mathbf{J} \tag{11}$$

These are three equations for the three unknown functions A_x, A_y, A_z. For example

$$\begin{aligned} & \partial_x \left(\partial_x A_x + \partial_y A_y + \partial_z A_z\right) + \left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) A_x \\ = \; & \mu_0 \, J_x \end{aligned}$$

These equations are *coupled*: all three unknown functions appear in each equation.

Decoupling
We can eliminate the $\boldsymbol{\nabla} \cdot \mathbf{A}$ term in (11) by a choice of gauge. Let

$$\boldsymbol{\nabla} \cdot \mathbf{A} \;=\; f$$

and make the change as in (10); then

$$\boldsymbol{\nabla} \cdot \mathbf{A} \;\mapsto\; f + \nabla^2 \Lambda$$

which can be made to vanish by a choice of Λ.

Problem 6
If f is given, how would you determine Λ in order to satisfy the above condition? §

Henceforth we assume this gauge change has been implemented, and $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. Then (11) becomes

$$\boxed{-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}} \tag{12}$$

$-\nabla^2 \Lambda = f$

or writing out the components

$$\begin{aligned} -\nabla^2 A_x &= \mu_0 J_x \\ -\nabla^2 A_y &= \mu_0 J_y \\ -\nabla^2 A_z &= \mu_0 J_z \end{aligned} \tag{13}$$

(By the way, the equations would be much more complicated in non-cartesian coordinates.) Now the three unknowns A_x, A_y, A_z are decoupled, and the three equations in (13) can be solved independently. In fact, each is the same (apart from the multiplicative constant) as the Poisson equation in electrostatics. We therefore obtain (repeating the electrostatic equation to emphasize the parallel)

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\mathbf{r} - \mathbf{s}|} \, \rho(\mathbf{s}) \, d^3 s \\ A_i(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{1}{|\mathbf{r} - \mathbf{s}|} \, J_i(\mathbf{s}) \, d^3 s \end{aligned} \tag{14}$$

We can next take the curl of the above and obtain $\mathbf{B}$, which in principle solves the problem of magnetostatics — obtaining the field from the currents.

Problem 7
Check that the vector potential given by (14) satisfies $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. Hint: Take divergence with respect to $\mathbf{r}$, change differentiation to $\mathbf{s}$, then integrate by parts. It is of course assumed that the current vanishes at infinity. §

3.4 Point source

It is useful to record the corresponding formulas for a point source, namely a point charge q at $\mathbf{s}(t)$ moving with velocity $\mathbf{v} = d\mathbf{s}(t)/dt$, in which case (14) reduces to

$$\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \frac{q}{\xi} \\ A_i(\mathbf{r}) &= \frac{\mu_0}{4\pi} \frac{q v_i}{\xi} \end{aligned} \tag{15}$$

where

$$\boldsymbol{\xi} \;=\; \mathbf{r} - \mathbf{s} \tag{16}$$

is the separation between the source point and the observation point.

These formulas are placed on record in anticipation of the Lienard–Wiechert potential to be derived for a moving point source. However, the formula for $\mathbf{A}$ due to a wire segment or a point charge should be seen only as a formal expression, since a wire segment or a single moving point charge by itself is not a static situation.

4 Biot–Savart law

The Biot–Savart law gives the magnetic field directly as an integral over the current density, much as Coulomb's law gives the electric field as an integral over the charge density.

4.1 Derivation

Basic formula
Start from (14) and take the curl. Everywhere below, the summation convention over repeated indices is adopted.

$$\begin{aligned} B_k(\mathbf{r}) &= \epsilon_{kji} \frac{\partial}{\partial r_j} A_i(\mathbf{r}) \\ &= \frac{\mu_0}{4\pi} \int \epsilon_{kji} \left(\frac{\partial}{\partial r_j} \frac{1}{|\mathbf{r}-\mathbf{s}|} \right) J_i(\mathbf{s})\, d^3s \\ &= -\frac{\mu_0}{4\pi} \int \epsilon_{kji} \frac{r_j - s_j}{|\mathbf{r}-\mathbf{s}|^3} J_i(\mathbf{s})\, d^3s \end{aligned}$$

or in vector form

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} \int \frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3} \times \mathbf{J}(\mathbf{s})\, d^3s \\ &= \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{s}) \times \frac{\boldsymbol{\xi}}{\xi^3}\, d^3s \qquad (17) \end{aligned}$$

where we adopt the shorthand (16).

It is useful to display the mathematical parallel with electrostatics;

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{s}) \frac{\boldsymbol{\xi}}{\xi^3}\, d^3s \qquad (18)$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{s}) \times \frac{\boldsymbol{\xi}}{\xi^3}\, d^3s \qquad (19)$$

Note the cross product in the magnetic formula.

Formula for a wire
If the current is confined to a thin wire, then

$$\mathbf{J}\, d^3s = I\, d\boldsymbol{\ell}$$

where I is the constant current, and $d\boldsymbol{\ell}$ is an element of the wire, in the direction of the current. Thus, the Biot–Savart law is often written in the form

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{d\boldsymbol{\ell} \times \boldsymbol{\xi}}{\xi^3}} \qquad (20)$$

reduced to a one-dimensional integral. Moreover, the wire must form a closed loop (with an infinite wire regarded as a limiting case) because a finite wire cannot carry a steady current.

Caveat
In (18), we can peel away the integral sign and regard the integrand as the electric field produced by a small charge element $\rho\, d^3s$. But we cannot do the same in (19), at least not uniquely so, since there is no such thing as a small element of current $\mathbf{J}\, d^3s$ independent of all other currents; that would not be consistent with time-independence — that bit of current by itself would not conserve charge.

4.2 Long wire

Consider a thin wire lying on the z-axis, carrying a current I in the $+z$ direction. The magnetic field is measured at a distance r from the wire, say (without loss of generality) on the x-axis (**Figure 5**). Then

$$\begin{aligned} \mathbf{s} &= s\,\hat{\mathbf{e}}_z \\ \mathbf{r} &= r\,\hat{\mathbf{e}}_x \\ \boldsymbol{\xi} &= r\,\hat{\mathbf{e}}_x - s\,\hat{\mathbf{e}}_z \\ \xi^2 &= r^2 + s^2 \\ d\boldsymbol{\ell} &= ds\,\hat{\mathbf{e}}_z \\ d\boldsymbol{\ell} \times \boldsymbol{\xi} &= r\, ds\,\hat{\mathbf{e}}_y \end{aligned}$$

Putting this into (20), we find that $\mathbf{B}$ is along $\hat{\mathbf{e}}_y$, with magnitude given by

$$B = \frac{\mu_0 I}{4\pi} \int \frac{r\, ds}{(r^2+s^2)^{3/2}}$$

For an infinite wire, the integral is to be taken from $s = -\infty$ to $s = \infty$. The dimension of the integral must be r^{-1} and a simple evaluation give $2r^{-1}$, hence

$$B = \frac{\mu_0 I}{2\pi r}$$

in agreement with previous results.

Problem 8
Carry out the integral above. Hint: change to the angle θ as illustrated in **Figure 5**. §

4.3 Small loop

Next consider a small loop of radius a lying in the x–y plane and carrying a current in the counter-clockwise sense, i.e., in the direction of positive ϕ, where ϕ is defined as below. We want to find the magnetic field $\mathbf{B}$ at a point on the z-axis. By symmetry, the field can only have a z-component (**Figure 6**). We have

$$\begin{aligned} \mathbf{s} &= a\cos\phi\,\hat{\mathbf{e}}_x + a\sin\phi\,\hat{\mathbf{e}}_y \\ \mathbf{r} &= z\,\hat{\mathbf{e}}_z \\ \boldsymbol{\xi} &= -a\cos\phi\,\hat{\mathbf{e}}_x - a\sin\phi\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z \\ \xi^2 &= a^2 + z^2 \end{aligned}$$

$$
\begin{aligned}
d\boldsymbol{\ell} &= d\mathbf{s} \\
&= (-a\sin\phi\,\hat{\mathbf{e}}_x + a\cos\phi\,\hat{\mathbf{e}}_y)\,d\phi \\
d\boldsymbol{\ell}\times\boldsymbol{\xi} &= a^2\hat{\mathbf{e}}_z\,d\phi + \ldots
\end{aligned}
$$

where the last line has omitted the x and y components, in the knowledge that these will integrate to zero. Putting this into (20), we find that $\mathbf{B}$ is along $\hat{\mathbf{e}}_z$, with magnitude given by

$$
\begin{aligned}
B &= \frac{\mu_0 I}{4\pi}\int \frac{a^2\,d\phi}{(a^2+z^2)^{3/2}} \\
&= \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}} \qquad (21)
\end{aligned}
$$

For the observation point far away, the loop contributes only as its area πa^2, and we define the *magnetic moment* μ as the product of the current and the area of the loop:

$$
\mu = I(\pi a^2) \qquad (22)
$$

in terms of which we get, for $z \gg a$

$$
B \approx \frac{\mu_0\,\mu}{2\pi z^3} \qquad (23)
$$

on the symmetry axis.

4.4 Solenoid

A solenoid can be regarded as a stack of small loops.

Problem 9
A solenoid is in the form of a cylinder of radius a and length L (**Figure 7a**). It is wound tightly with n turns of wire per unit length, and the wire carries a current I.
(a) For a long, thin solenoid ($L \gg a$) and excluding a part near the ends, the magnetic field lines are uniform and along the axis, and there is negligible field outside the solenoid. By applying Ampere's law in integral form to the loop shown in **Figure 7b**, find the field B.
(b) Next, by integrating the result (21) over z, find the field B at the center of a finite solenoid.
(c) More generally, find the field B at position Z on the axis. Plot B versus Z, including the transition region just inside and just outside the solenoid. §

A Supplement: Models for magnetism — Part 1

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

This supplement starts the discussion of an apparently trivial question: What causes magnetism? Within classical electrodynamics, we can imagine three different models:

1. There are only point particles carrying net charges, whose movements give rise to currents, which are the source of magnetic field.
2. There are point particles carrying net charges *and* intrinsic magnetic moments. The movement of the point charge gives rise to magnetic fields as before, but even if the point particles do not move, the intrinsic magnetic moments are additional sources of magnetic field.
3. There may also be magnetic monopoles.

The three models are different, and we need to be careful which model is assumed, and which one describes reality better.

A.1 Point charges only

Model 1 is assumed unless otherwise specified. In this model, the only dynamical variable for a particle is the position $\mathbf{r}$, whose evolution is given by Newton's second law, with the force given by the Lorentz force law. A consequence is that the magnetic field does no work, because the force is perpendicular to the velocity.

The moving (and structureless) point charges produce magnetic fields as discussed in this Chapter.

A.2 Point charges with intrinsic magnetic moment

However, the real world is not like that: an electron (or any other elementary particle) is not just a point charge, but also has an *intrinsic* magnetic moment. Magnetic moments will be discussed in more detail in the next Chapter.

But even without those details, it is easy to see that there are two ways to think about such magnetic moments.

- We do not think of an electron as a *point* charge. Rather, think of it as a tiny ball of charge which is rotating. We apply Model 1 not to the whole electron, but to each of its infinitesimal parts, which are moving because of the rotation. Everything in Model 1 seems to work. Indeed, we often adopt this point of view heuristically.
- Or, we try to develop a theory of point particles which possess intrinsic magnetic moments. In this case, the force acting on the particle is

not just the Lorentz force law, but contains an additional term due to the magnetic moment. (We shall get a glimpse of what such a term looks like in the next Chapter.)

Is there any reason why we need to adopt the more complicated second approach? Yes!

First, a rigid ball contradicts relativity. If the ball has a radius a, and you push its left edge, the right edge should not "know" until a time $2a/c$ later. During this short time interval, the left edge would move while the right edge stays stationary — the ball has deformed and is not rigid. If you try to get rid of this problem by taking the limit $a \to 0$, then the finite value of angular momentum means that the tangential velocity on the surface exceeds c.

Second, and more seriously, if the magnetic moment is due to a spinning "ball", then one can certainly apply torques to the "ball" and speed up or slow down its spin; in other words, the magnitude of spin (and therefore of the magnetic moment) is allowed to change. But the spin and the magnetic moment of an electron (and of other elementary particles) have *fixed* magnitudes; only the *directions* can change. This property cannot be accommodated by an extension of Model 1.

Therefore we need Model 2, in which the dynamic variables for a point particle are its position $\mathbf{r}$ (3 numbers) and the direction of its spin $\mathbf{S}$ (2 numbers, because it has a fixed magnitude). The equations of motion should look like[3]

$$m\frac{d^2\mathbf{r}}{dt^2} = \ldots \tag{24}$$

$$\frac{d\mathbf{S}}{dt} = \ldots \tag{25}$$

- On the RHS of (24), in addition to the Lorentz force, we need a term to describe the force acting on the magnetic dipole.
- On the RHS of (25) we need the torque $\boldsymbol{\tau}$, and moreover we need to ensure that

  $$\boldsymbol{\tau} \cdot \mathbf{S} = 0$$

  so that the magnitude of $\mathbf{S}$ is unchanged.
- That would only solve one half of the problem, how the particle responds to fields. There is another half: on the RHS of the inhomogeneous Maxwell equations we need to add terms other than $\mathbf{J}$ (defined as the motion of the charges) as sources of the fields.
- We need to make sure that the modified force law and the modified Maxwell's equations are mutually consistent in an action-reaction sense.
- We need to elevate all these considerations to a relativistic language.

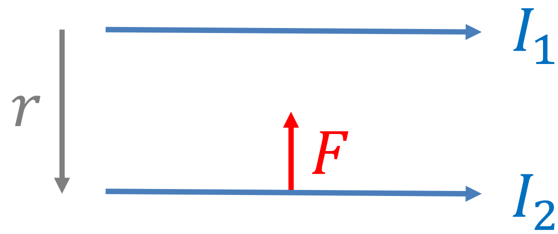
We leave some aspects of these generalizations to later Supplements. Here we only note one property: In Model 2, the magnetic field can do work. In real life, a static magnetic field can do work on an electron.

A.3 Magnetic monopoles

Finally, Model 3 allows another possibility: that there may be magnetic monopoles. This possibility will occupy much of the course towards the end. So far, Model 3 is only relevant in the theoretical domain — no monopoles have been observed so far.

[3]For simplicity restricting to the non-relativistic domain

(a)



$$\frac{F}{l} = \kappa \frac{I_1 I_2}{r}$$

(b)

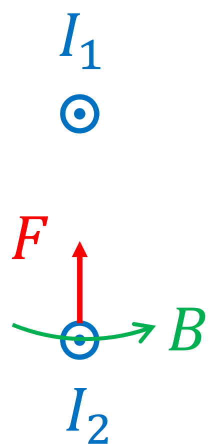


Figure 1

Force between parallel wires

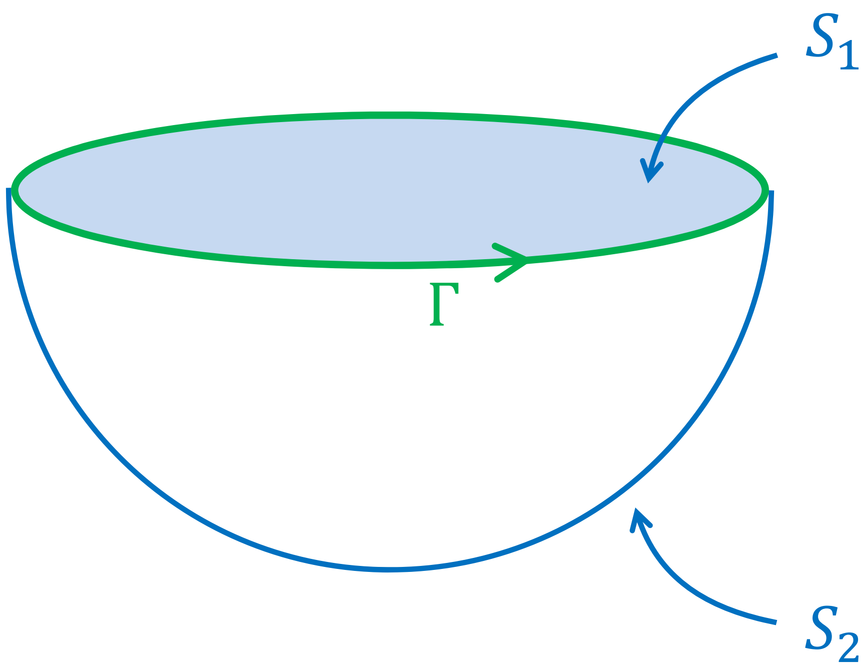


Figure 2

The same path Γ is the boundary of both S_1 and S_2

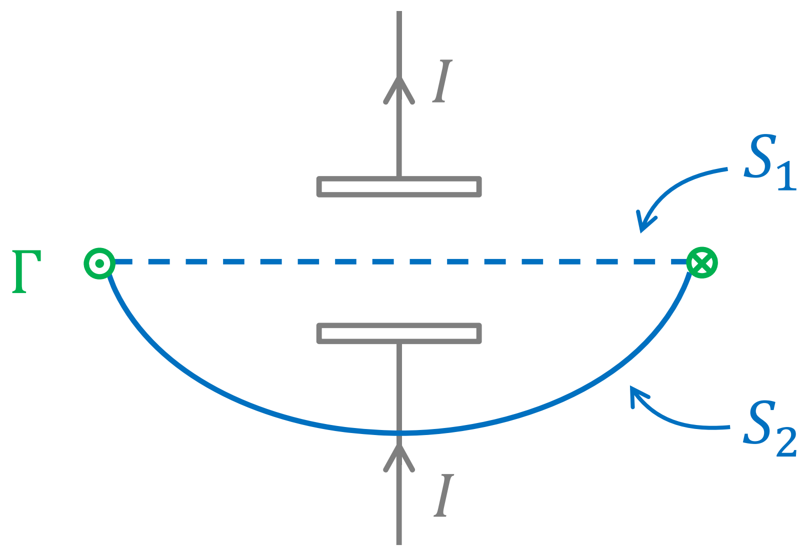


Figure 3

Which surface to use when applying Ampere's law?

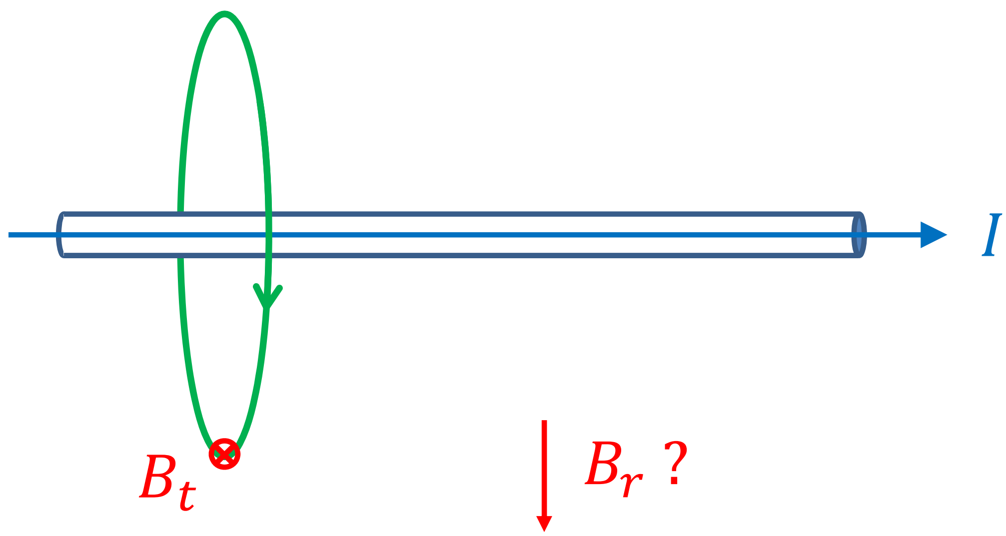


Figure 4

Applying Ampere's law to the loop determines B_t but not B_r

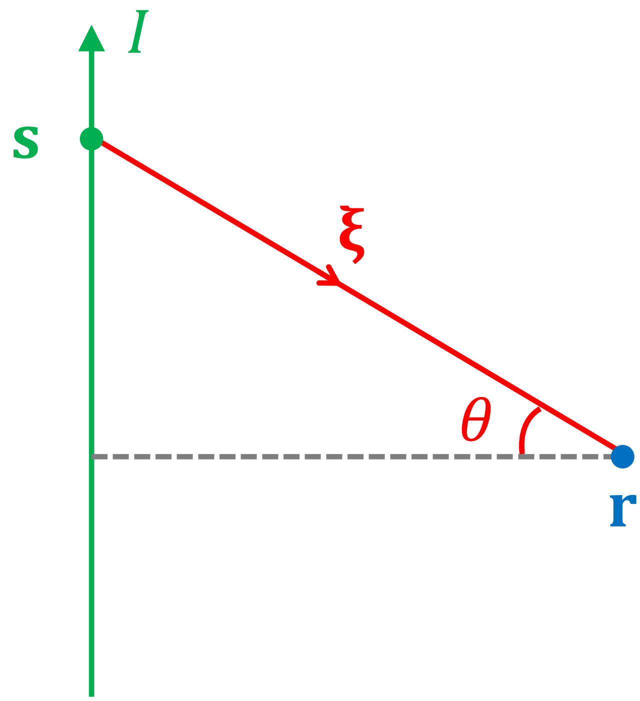


Figure 5

Field **B** at **r** caused by current at **s**

$$\boldsymbol{\xi} = \mathbf{r} - \mathbf{s}\,; \qquad\qquad \mathrm{d}\boldsymbol{l} = \mathrm{d}\boldsymbol{s}$$

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{s} \times \boldsymbol{\xi}}{\xi^3}$$

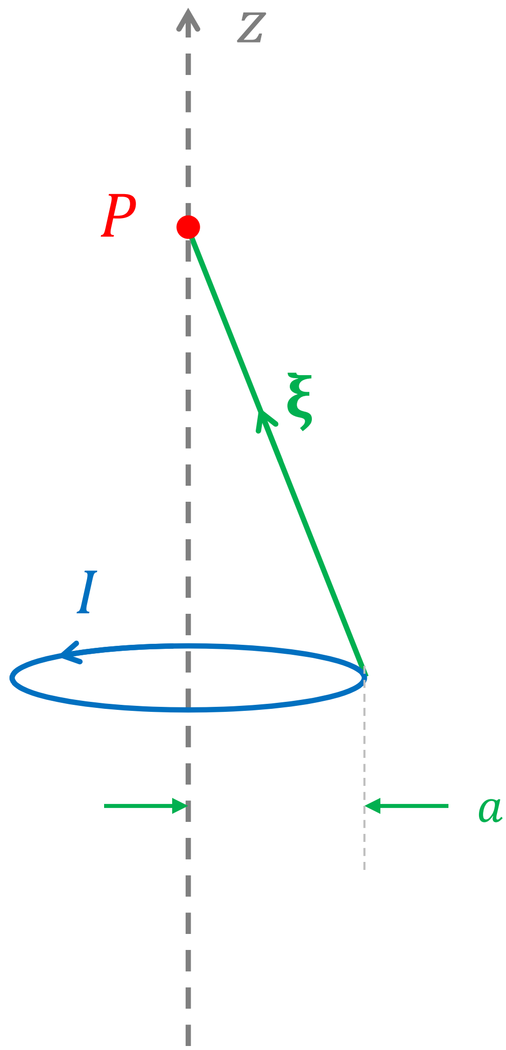


Figure 6

Field **B** at the point P caused by a ring

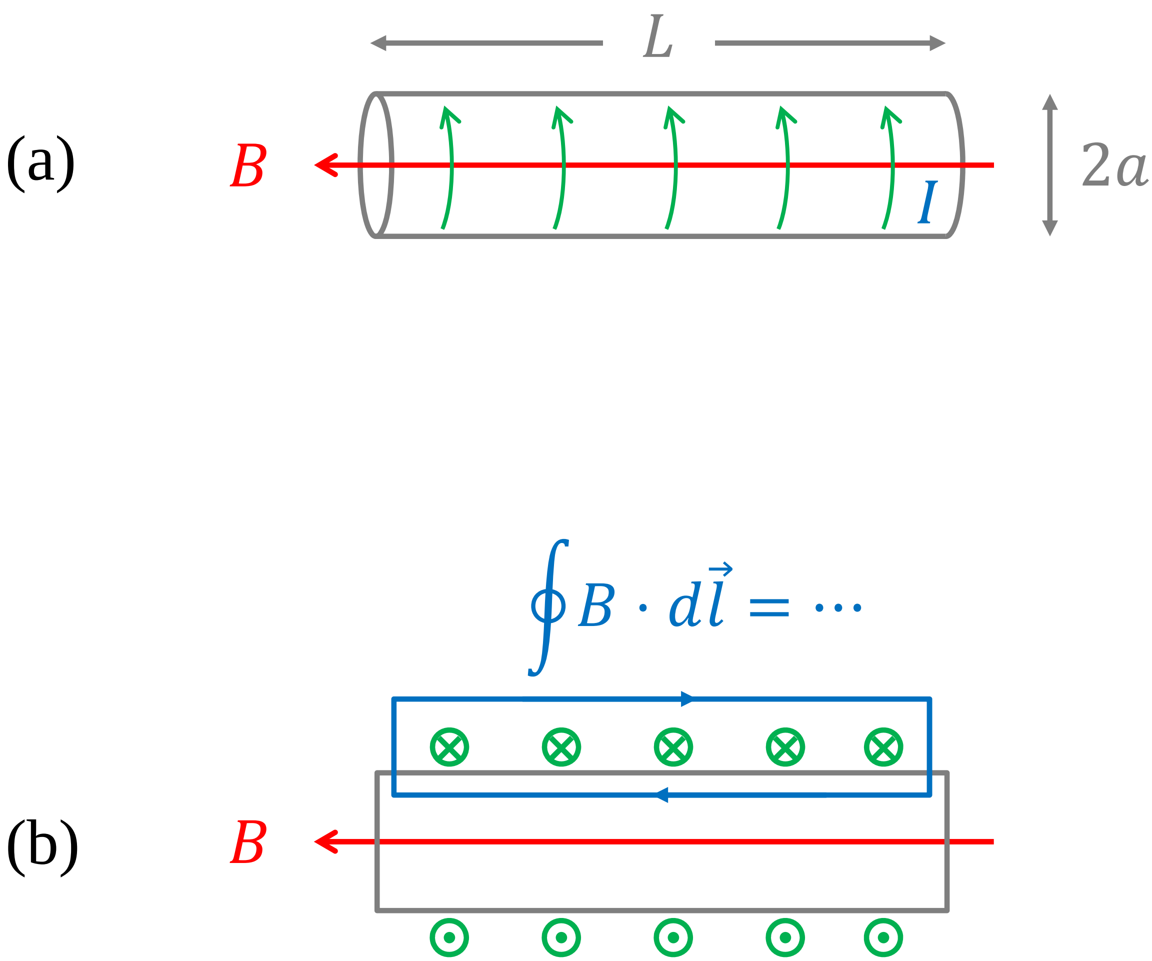


Figure 7

Field caused by solenoid