Chapter 12
Radiation by a moving charge

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The Lienard–Wiechert potential is derived and used to analyze radiation by a moving charge. This formalism goes beyond the earlier one restricted to small sources with harmonic time dependence. Special attention is paid to radiation by relativistic particles.

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1 Green's function

1.1 Introduction

Definition of Green's function

Recall that time-dependent fields are governed by

$$D\begin{pmatrix}\Phi\\ \mathbf{A}\end{pmatrix} = \begin{pmatrix}\rho/\epsilon_0\\ \mu_o\mathbf{J}\end{pmatrix} \tag{1}$$

where

$$D = -\nabla^2 + c^{-2}\partial_t^2 \tag{2}$$

This Chapter is essentially about solving this equation. Because no particular time-dependence is assumed, this is a PDE in 4D.

As usual for inhomogeneous linear equations, define a Green's function G that solves (1) when the RHS is replaced by a δ-function:

$$\boxed{D\,G(\mathbf{r},t) = \delta^3(\mathbf{r})\,\delta(t)} \tag{3}$$

in terms of which

$$\begin{aligned}&\Phi(\mathbf{r},t)\\ =\;& \frac{1}{\epsilon_0}\int G(\mathbf{r}-\mathbf{r}',t-t')\,\rho(\mathbf{r}',t')\,d^3r'dt'\\ &A_i(\mathbf{r},t)\\ =\;& \mu_0\int G(\mathbf{r}-\mathbf{r}',t-t')\,J_i(\mathbf{r}',t')\,d^3r'dt'\end{aligned} \tag{4}$$

The spatial coordinate of the source is now written as $\mathbf{r}'$ rather than $\mathbf{s}$, to correspond to the time variable t'.

These integral formulas can be written in the shorthand

$$\boxed{\begin{pmatrix}\Phi\\ \mathbf{A}\end{pmatrix} = G\otimes\begin{pmatrix}\rho/\epsilon_0\\ \mu_0\mathbf{J}\end{pmatrix}} \tag{5}$$

where convolution is defined in 1D as

$$\begin{aligned}f &= g\otimes h\\ f(x) &= \int g(x-x')\,h(x')\,dx'\end{aligned}$$

and analogously in higher dimensions.

Thus, the problem is reduced to that of finding G, after which it is only a matter of integration.

The rest of this Section will refer either only to Φ or only to A_i, with the understanding that the other components of (1) are similar.

Relation with static Green's function

Although the name "Green's function" is used generically whenever the RHS of any inhomogeneous linear equation is replaced by a δ-function, the Green's function G here is of course not the same as that introduced in statics, which for the present purpose will be denoted as $G_S(\mathbf{r})$. The relationship between G and G_S is easily found, as follows.

Suppose the charge density is time-independent: $\rho(\mathbf{r}',t') = \rho(\mathbf{r}')$. Then doing the time integral in (4), we see that

$$\boxed{\int G(\mathbf{r},t)\,dt = G_S(\mathbf{r}) = \frac{1}{4\pi\,r}} \qquad (6)$$

where the known result for G_S has been inserted. This is a useful check on the results in this Chapter.

Unphysical problem
The Green's function G_S in electrostatics is the potential due to a point charge, and the analog of (4) is the law of superposition. But (3) is not a physical problem — the RHS represents a charge (or current) density that appears only for one instant, and hence does not satisfy the conservation of charge. So the Green's function G in this case is only a mathematical device.

Result for the Green's function
In order not to be distracted by the intermediate steps in the derivation, we first give the result for G, leaving the derivation to Sections 1.2 and 1.3:

$$\boxed{G(\mathbf{r},t) = \frac{1}{4\pi\,r}\,\delta(t - r/c)} \qquad (7)$$

Several features can be noted.

- The δ-function in (7) means that if there is a source at the origin, appearing only for an instant at $t = 0$, then at a position $\mathbf{r}$, the field also appears only for an instant, at $t = r/c$.
- The influence spreads out as r^{-1} in the potential and field,[1] and therefore eventually as r^{-2} in the energy flux.
- The r^{-1} dependence is the same as the static case, and in fact the relationship (6) is readily checked.

The derivation of (7) is next presented. The techniques are similar to those for the static case, which should first be reviewed.

Explicitly covariant form
The Green's function can also be written as

$$\boxed{G(\mathbf{r},t) = \frac{1}{2\pi c}\,\delta(t^2 - r^2/c^2)\,\Theta(t)} \qquad (8)$$

The argument of the δ-function involves a relativistic invariant, i.e., it is nonzero only on the light cone. The Θ-function is 1 (0) if the argument is positive (negative) and selects the future light cone.[2] This alternate form is conceptually appealing, but will not be used for the rest of this Chapter.

[1]Roughly speaking, the potential goes as $\exp(ikr)/r$, hence the field has both r^{-1} and r^{-2} terms, with the latter negligible far away.

[2]Thus, this factor does not break invariance.

1.2 Momentum representation

In momentum representation,[3] the equation (1) becomes

$$\left[k^2 - (\omega/c)^2\right]\,\tilde{G}(\mathbf{k},\omega) \;=\; 1 \qquad (9)$$

giving trivially

$$\boxed{\tilde{G}(\mathbf{k},\omega) = \frac{1}{k^2 - (\omega/c)^2}} \qquad (10)$$

1.3 Coordinate representation

Next transform back to coordinate space.

$$G(\mathbf{r},t) \;=\; \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2 - (\omega/c)^2}\,\exp(-i\omega t)\exp(i\mathbf{k}\cdot\mathbf{r}) \qquad (11)$$

In such multiple integrals it is convenient to write the differentials in front, right next to the corresponding integral signs, so that it is clear which limits refer to which variable.

The dot product in the exponential can be written as

$$\mathbf{k}\cdot\mathbf{r} \;=\; kru$$

where $u = \cos\theta$ and θ is the angle between $\mathbf{k}$ and $\mathbf{r}$. Then

$$\frac{d^3k}{(2\pi)^3} \;=\; \frac{1}{8\pi^3}\int_0^\infty dk\,k^2\int_{-1}^{1} 2\pi\,du$$

and the u-integral gives

$$\int_{-1}^{1} du\,e^{ikru} \;=\; \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

Hence we get

$$G(\mathbf{r},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\cdot\frac{1}{8\pi^3}\times \int_0^\infty\frac{dk\,k^2}{k^2 - (\omega/c)^2}\cdot\frac{2\pi}{ikr}\left(e^{ikr} - e^{-ikr}\right) \qquad (12)$$

Consider the integral over k, namely the second line in (12), to be denoted as I_k. The integrand is even in k, and the integral can be extended to the full real line and then multiplied by 1/2, giving

$$I_k \;=\; \frac{\pi}{ir}\int_{-\infty}^{\infty}\frac{dk\,k}{k^2 - (\omega/c)^2}\left(e^{ikr} - e^{-ikr}\right)$$

[3]In these contexts, we shall loosely use the term "momentum" interchangeably with "wave number", and "energy" with "angular frequency", all these pairs differing only by a factor of $\hbar$.

A decision has to be made about the treatment of the two poles, at $k = \pm(\omega/c)$. We choose to go *over* the right pole and *under* the left pole (**Figure 1**). For the term involving e^{ikr}, close the contour in the upper half plane. Only the pole at $k = \omega/c$ is enclosed and its contribution is

$$2\pi i \cdot \frac{1}{k+\omega/c} \cdot k\, e^{ikr}\Big|_{k=\omega/c} \quad = \quad i\pi e^{i\omega r/c}$$

The e^{-ikr} term gives the same contribution. (For this term, the contour has to be closed in the lower half plane, thus enclosing only the left pole, so that $k = -\omega/c$ at the pole, and $e^{-ikr} = e^{i\omega r/c}$. The contour is traversed in the negative sense, which cancels the negative prefactor.) Therefore

$$I_k \quad = \quad \frac{2\pi^2}{r}\, e^{i\omega r/c} \tag{13}$$

When this is put back into (12) we get

$$\begin{aligned} G(\mathbf{r},t) &= \frac{1}{4\pi r}\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-r/c)} \\ &= \frac{1}{4\pi r}\,\delta(t-r/c) \end{aligned} \tag{14}$$

which is the answer claimed at the beginning.

Problem 1
Choose other ways of handling the poles and evaluate G. Explain why these choices are not adopted. §

2 Lienard–Wiechert potential

2.1 Source term

Consider a single charge[4] q moving with position $\boldsymbol{\xi}(t)$. The charge and current densities are then

$$\begin{aligned} \rho(\mathbf{r}',t') &= q\,\delta^3(\mathbf{r}'-\boldsymbol{\xi}(t')) \\ J_i(\mathbf{r}',t') &= qv_i(t')\,\delta^3(\mathbf{r}'-\boldsymbol{\xi}(t')) \end{aligned} \tag{15}$$

where the velocity is

$$\mathbf{v}(t) \quad = \quad \frac{d\boldsymbol{\xi}(t)}{dt} \tag{16}$$

2.2 Evaluating the potentials

Doing the spatial integral
When (15) is put into (4) with the explicit form of G inserted, we get,

$$\begin{aligned} A_i(\mathbf{r},t) & \\ &= \frac{\mu_0}{4\pi}\int \frac{qv_i(t')}{|\mathbf{r}-\mathbf{r}'|}\,\delta^3(\mathbf{r}'-\boldsymbol{\xi}(t')) \\ &\quad \times \delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)\, d^3r'\, dt' \\ &= \frac{\mu_0}{4\pi}\int \frac{qv_i(t')}{|\mathbf{r}-\boldsymbol{\xi}(t')|} \\ &\quad \times \delta(t-t'-|\mathbf{r}-\boldsymbol{\xi}(t')|/c)\, dt' \end{aligned} \tag{17}$$

where the spatial integral has been done, cancelling the spatial δ-function and turning $\mathbf{r}' \mapsto \boldsymbol{\xi}(t')$. Note the meaning of the three spatial variables (**Figure 2**):

- The variable $\mathbf{r}$ denotes the observation point.
- The variable $\mathbf{r}'$ is a dummy variable in the source distribution; it has no particular value. This variable has been eliminated in the final form of (17).
- The variable $\boldsymbol{\xi}(t')$ is a given function of the time t', representing the motion of the charge.

Retarded time
The δ-function forces t' to the unique value given by the implicit relation:

$$\boxed{t = t' + |\mathbf{r}-\boldsymbol{\xi}(t')|/c} \tag{18}$$

The interpretation is as follows, in terms of the field (or signal) emitted and received.

- The signal is received at time t at position $\mathbf{r}$.
- The last term in (18) represents the time taken for the signal to travel from the source point $\boldsymbol{\xi}$ to the observation point $\mathbf{r}$.
- Hence t' is the time of emission of the signal. It is also called the *retarded time*. It is a function of t, $\mathbf{r}$ and the given motion of the particle, i.e., the function $\boldsymbol{\xi}$.

In the rest of the Chapter, t' is understood to be the retarded time, and any such expression will be indicated by the subscript $_{\text{ret}}$.

Compression ratio
The following ratio appears in the evaluation of (17).

$$\boxed{\kappa \equiv \int \delta(t-t'-|\mathbf{r}-\boldsymbol{\xi}(t')|/c)\, dt'}$$

Note that this is not unity, because t' appears in $\boldsymbol{\xi}$ as well, and this is a crucial factor below. To understand this, multiply and divide by dt, so that

$$\kappa \quad = \quad \int \delta(t-t'-|\mathbf{r}-\boldsymbol{\xi}(t')|/c)\, dt \cdot \frac{dt'}{dt}$$

The t integral (in contrast to the t' integral) is easy and equal to unity, so we get

$$\boxed{\kappa = \frac{dt'}{dt}\Big|_{\text{ret}}} \tag{19}$$

[4] As usual, we denote an arbitrary charge by q, reserving e for the elementary charge.

where this is to be evaluated at the point where the argument of the δ-function vanishes, i.e., at the retarded time. The factor κ has the interpretation of a *compression ratio*: For example, if 10 units of emission time ($\Delta t' = 10$) is compressed into 1 unit of reception time ($\Delta t = 1$), then $\kappa = 10$. This ratio is closely related to the ratio of frequencies in the Doppler effect.

Simple form for the LW potential

Putting these together, and now also showing the result for Φ in an obviously parallel way, we get

$$\boxed{\begin{aligned} \Phi(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{R}\kappa\right]_{\text{ret}} \\ A_i(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\left[\frac{qv_i}{R}\kappa\right]_{\text{ret}} \end{aligned}} \tag{20}$$

in which

$$\boxed{\mathbf{R} = \mathbf{r} - \boldsymbol{\xi}(t')}$$

is the displacement from the source point to the observation point.

The answer is easy to remember:

- Start with the formulas in statics.
- Multiply by the compression ratio κ.
- Evaluate everything at the retarded time.

The limit of a slowly moving charge is recovered trivially.

Evaluating the compression factor

Since (18) is explicit if considered as a function $t(t')$ but implicit if considered as the inverse function $t'(t)$, it is more convenient to evaluate $\kappa^{-1} = dt/dt'$. Start with the colinear case: the particle travels along $+x$ and the observation point is also along $+x$; then (18) simplifies to

$$\begin{aligned} t &= t' + [x - \xi(t')]/c \\ \kappa^{-1} = \frac{dt}{dt'} &= 1 - v/c = 1 - \beta \end{aligned} \tag{21}$$

where $v = d\xi/dt$ is the velocity and $\beta = v/c$ as usual. Therefore

$(1-\beta)(1+\beta)$

$$\kappa = \frac{1}{1-\beta} \approx 2\gamma^2 \tag{22}$$

where the final form applies in the ultra-relativistic case, with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{\mathcal{E}}{mc^2} \tag{23}$$

expressed in terms of the energy $\mathcal{E}$ and rest mass m of a relativistic particle.

In the backward direction, the sign of β is reversed, and (21) becomes

$$\kappa = \frac{1}{1+\beta} \approx \frac{1}{2} \tag{24}$$

The essential conclusion is:

There is a huge enhancement in the forward direction if $\beta \to 1$.

The above analysis needs to be extended to the general case where there is an angle θ between the direction of observation and the velocity (the above being special cases for $\theta = 0$ and $\theta = \pi$); the result is

$$\kappa = \frac{1}{1 - \hat{\mathbf{n}}\cdot\boldsymbol{\beta}} = \frac{1}{1 - \beta\cos\theta} \tag{25}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\hat{\mathbf{n}}$ is the unit vector in the direction of observation

$$\hat{\mathbf{n}} = \frac{\mathbf{R}}{R}$$

Problem 2

Prove (25). §

Problem 3

Show that (25) is sharply peaked in the forward direction if $\beta \to 1$. In fact, show that it drops to half the maximum value for

$$\theta \approx \gamma^{-1}$$

so that the peak is narrow if $\beta \to 1$. §

So far we have only derived the enhancement factor for the potentials. We need to extend the discussion to the fields and then the radiation flux, and it will be seen that higher powers of κ will emerge.

2.3 Radiation field

Simplify calculation

The somewhat complicated algebra for the radiation field can be simplified with a few tricks.

- Recall we only need $\mathbf{A}$ and can forget about Φ. After $\mathbf{A}$ is obtained, $\mathbf{E}$ is the *transverse* part of $-\partial\mathbf{A}/\partial t$, and $\mathbf{B}$ has a magnitude E/c.
- The time derivative of (20) contains many pieces, since all three variables $R, \hat{\mathbf{n}}, \boldsymbol{\beta}$ depend on t (via their arguments t'). But notice that the derivative of R and $\hat{\mathbf{n}}$ have the following properties. (a) They involve an extra power of R^{-1} and do not contribute in the radiation zone. (b) The result will depend on β but not its time derivative; in other words it has nothing to do with acceleration. We know that radiation must involve acceleration — a uniformly moving charge does not radiate. Therefore we only need to differentiate $\boldsymbol{\beta}$.
- Henceforth, the notation $[\ \ ]_{\text{ret}}$ will not be explicitly shown, and will be understood.

With this in mind, we write $\boldsymbol{A}$ as

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 c\, q}{4\pi R}\,\frac{\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}} \\ \frac{\partial \mathbf{A}}{\partial t} &= \frac{\mu_0 c\, q}{4\pi R}\left[\frac{\partial}{\partial t'}\left(\frac{\boldsymbol{\beta}}{1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta}}\right)\right]\frac{dt'}{dt} \qquad (26)\end{aligned}$$

Differentiate with respect to time
The last factor above is just κ.

Problem 4
Consider the square bracket above.
(a) Show that it is equal to

$$\kappa^2\left[\dot{\boldsymbol{\beta}} - \dot{\boldsymbol{\beta}}(\hat{\mathbf{n}}\cdot\boldsymbol{\beta}) + \boldsymbol{\beta}(\hat{\mathbf{n}}\cdot\dot{\boldsymbol{\beta}})\right]$$

(b) Then show that this can be written as

$$\kappa^2\left[\dot{\boldsymbol{\beta}} + \hat{\mathbf{n}}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\right] \qquad (27)$$

by using the identity for double cross products. §

Take transverse part
Next we have to take the transverse part, i.e., project out the component along $\hat{\mathbf{n}}$. The last term above is already transverse, and it remains only to do the projection for the first term.

Problem 5
Show that for any vector $\mathbf{w}$, the projection of the transverse part can be accomplished by

$$\mathbf{w} \quad\mapsto\quad -\,\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\mathbf{w})$$

This can be demonstrated either (a) directly by using the cross product formula, or (b) by noticing that the operation of $\hat{\mathbf{n}}\times$ has the effect of removing the longitudinal component while turning the transverse component 90 degrees. §

Using this result in (26) and (27), we find

$$\begin{aligned}\mathbf{E} &= \frac{\mu_0 c\, q}{4\pi}\left[\frac{1}{R}\kappa^3\,\boldsymbol{\Sigma}\right]_{\mathrm{ret}} \qquad (28)\\ \boldsymbol{\Sigma} &= \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\boldsymbol{\beta}}) - \hat{\mathbf{n}}\times(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}) \\ &= \hat{\mathbf{n}}\times\left[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right] \qquad (29)\end{aligned}$$

There are *three* powers of κ in (28): two from (27) and one from the final factor in (26).

Energy flux
The Poynting vector S is then given by

$$\begin{aligned}R^2\, S &= R^2\cdot\frac{1}{\mu_0 c}E^2 \\ &= \frac{1}{4\pi}\left(\frac{q^2}{4\pi\epsilon_0\, c}\right)\kappa^6\,\boldsymbol{\Sigma}^2 \qquad (30)\end{aligned}$$

This is just $dP/d\Omega$. All this is understood to be evaluated at the retarded time.

But P (and its differential version for each solid angle) is the energy flow per unit observation time t, whereas it is often convenient to express the result in terms of P', the energy flow per unit emission time, through the relationship

$$\begin{aligned}P\,dt &= P'\,dt' \\ P' &= \frac{dt}{dt'}\,P = \kappa^{-1}P\end{aligned}$$

so that finally

$$\boxed{\frac{dP'}{d\Omega} = \frac{1}{4\pi}\left(\frac{q^2}{4\pi\epsilon_0\, c}\right)\kappa^5\,\boldsymbol{\Sigma}^2} \qquad (31)$$

with *five* powers of κ. While we shall consider details in a number of examples, it is important to note the following.

- The factor κ leads to a large enhancement in the forward direction in the relativistic limit.
- This enhancement is made much more prominent by the presence of five powers.

3 Applications

3.1 Recover dipole radiation

As the first application, consider a point charge q executing simple harmonic oscillation along the z-axis, with angular frequency ω, and amplitude z_0 small enough that the multipole expansion applies, i.e.,

$$z_0 \ll \lambda = \frac{2\pi c}{\omega}$$

The typical velocity is $v \sim \omega z_0$, so

$$\beta \quad\sim\quad \frac{\omega z_0}{c} \ll 1$$

Dominance by the dipole term is the same as the motion being non-relativistic. This then means that

$$\kappa \quad\approx\quad 1$$

and the times t, t' are the same except for a uniform shift.

Moreover, in the formula for $\boldsymbol{\Sigma}$,

$$\hat{\mathbf{n}} - \boldsymbol{\beta} \quad\approx\quad \hat{\mathbf{n}}$$

since the first term has magnitude unity. This then gives

$$\boldsymbol{\Sigma} \quad=\quad \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\dot{\boldsymbol{\beta}}) \qquad (32)$$

with magnitude being the transverse part of $\dot{\boldsymbol{\beta}}$. Thus we have

$$\begin{aligned} \frac{dP'}{d\Omega} &= \frac{1}{4\pi}\left(\frac{q^2}{4\pi\epsilon_0\, c}\right)\dot{\beta}^2 \sin^2\theta \\ &= \frac{1}{4\pi}\left(\frac{q^2}{4\pi\epsilon_0\, c^3}\right) a^2 \sin^2\theta \end{aligned} \tag{33}$$

where θ is the angle between $\hat{\mathbf{n}}$ and $\dot{\boldsymbol{\beta}}$, and $\mathbf{a} = c\dot{\boldsymbol{\beta}}$ is the acceleration.

Problem 6
Check that (33) agrees with the result in the last Chapter based on the multipole expansion for a harmonic source. §

3.2 Linacs

Next consider a charge q in a linear accelerator (linac). In that case, the $\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}$ term in (29) vanishes, and the magnitude of $\boldsymbol{\Sigma}$ is

$$\Sigma = \dot{\beta}\sin\theta \tag{34}$$

where θ is the angle measured from the line of motion. Thus

$$\begin{aligned} \frac{dP'}{d\Omega} &= C\, f(\theta) \\ C &= \frac{1}{4\pi}\frac{q^2}{4\pi\epsilon_0\, c}\dot{\beta}^2 \\ f(\theta) &= \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \end{aligned} \tag{35}$$

Although the numerator $\sin^2\theta$ implies that the flux in the strictly forward direction is zero, it is still true that, for $\beta \to 1$, (a) the energy is concentrated *near* the forward direction, and (b) the total power is large. Both of these properties are made precise in the following problem.

Problem 7
In the limit $\beta \to 1$, show that
(a) $f(\theta)$ has a width $\Delta\theta \sim \gamma^{-1}$, and
(b) the integral over solid angles is $\int f(\theta)\, d\Omega \approx (8\pi/3)\gamma^6$. §

$\sim \int \theta^3 \left(\frac{\gamma^2}{1+(\gamma\theta)^2}\right)^5 d\theta$

sub $u=\gamma\theta$

Power
Thus we find

$$P' = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0\, c}\cdot\dot{\beta}^2\gamma^6 \tag{36}$$

Problem 8
As the charged particle is accelerated (say along z), its energy $\mathcal{E}$ increases. Show that

$$\frac{d\mathcal{E}}{dz} = mc\cdot\gamma^3\dot{\beta}$$

The factor of γ^3 indicates that it is very difficult to increase β. §

Thus we can write the power as

$$P' = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0\, m^2c^3}\left(\frac{d\mathcal{E}}{dz}\right)^2 \tag{37}$$

To evaluate precisely the total energy radiated as the charge accelerates from rest to the final state would require detailed knowledge of the field $\mathbf{E}$ along the linac, which would take us too far afield. For the purpose of an estimate, we replace

$$\frac{d\mathcal{E}}{dz} \mapsto \frac{\mathcal{E}}{L}$$

to describe a linac in which the final energy $\mathcal{E}$ is achieved in a length L. Furthermore, the total energy $\Delta\mathcal{E}$ radiated is estimated by multiplying by the total emission time $\Delta t'$, which for a highly relativistic linac would be $\approx L/c$. Some arithmetic then leads to

$$\begin{aligned} \frac{\Delta\mathcal{E}}{\mathcal{E}} &\approx \frac{2}{3}\frac{q^2}{4\pi\epsilon_0\, mc^2}\frac{1}{L}\frac{\mathcal{E}}{mc^2} \\ &= \frac{2}{3}\frac{r_e}{L}\gamma \end{aligned} \tag{38}$$

It is true that this expression scales with $\gamma = \mathcal{E}/mc^2$, which can be large. For example, for SLAC, which is an electron accelerator ($q = -e$), the final energy is $\mathcal{E} = 18$ GeV, so $\gamma \approx 3.6 \times 10^4$. But the other factor is tiny: the classical radius of the electron is $r_e \approx 3 \times 10^{-15}$ m, while the length of the linac is certainly macroscopic; for SLAC it is about $L = 3$ km. Thus

$$\frac{r_e}{L} \approx 10^{-18}$$

In short, radiative loss is not a practical problem in linacs. The reason is that one cannot have much $\dot{\beta}$ when β is limited to unity.

3.3 Circular accelerators

Now consider a point charge in a circular accelerator of radius L, at very high velocity. Take the accelerator to be in the z–x plane, and suppose instantaneously the particle is travelling along $+z$ (**Figure 3**), and has a centripetal acceleration towards $-x$. Then

$$\begin{aligned} \boldsymbol{\beta} &= \beta\,\hat{\mathbf{e}}_z \\ \dot{\boldsymbol{\beta}} &= -\frac{c}{L}\hat{\mathbf{e}}_x \end{aligned}$$

the latter equation being written in the $\beta \to 1$ limit. In the former equation we do not yet set $\beta = 1$, in order to be careful about $1-\beta$ terms.

The full arithmetic is somewhat messy, and we just sketch the main points to emphasize the ideas, omitting the evaluation of a numerical factor. From (31), the total power is

$$\begin{aligned} P' &= \frac{1}{2}\left(\frac{q^2}{4\pi\epsilon_0\, c}\right) I \\ I &= \int \kappa^5\, \mathbf{\Sigma}^2\, \sin\theta\, d\theta \end{aligned} \tag{39}$$

where the integrand is understood to be averaged over ϕ.

The integral I is dimensionless, and in the ultra-relativistic limit it goes as a power of γ. The main point is to determine the power. We note that (from Problem 7), (a) $\theta \sim \gamma^{-1}$, and (b) $\kappa \sim \gamma^2$. Also, (c), a careful evaluation shows that $\mathbf{\Sigma}$ contains terms like γ^{-2} and also $\theta^2 \sim \gamma^{-2}$. Thus we get

$$I \sim (\gamma^2)^5 \cdot (\gamma^{-2})^2 \cdot (\gamma^{-2}) \sim \gamma^4 \tag{40}$$

giving,

$$P' = C\,\frac{1}{2}\left(\frac{q^2}{4\pi\epsilon_0\, c}\right)\frac{c^2}{L^2}\gamma^4 \tag{41}$$

The numerical prefactor $C = 4/3$ is determined by a detailed evaluation. Thus the energy loss per cycle is

$$\begin{aligned} \Delta\mathcal{E} &= P' \cdot \frac{2\pi L}{c} \\ &= \frac{4\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0\, L}\right)\gamma^4 \end{aligned} \tag{42}$$

The fractional energy loss per cycle is then

$$\begin{aligned} \frac{\Delta\mathcal{E}}{\mathcal{E}} &= \frac{4\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0\, L}\right)\gamma^4\frac{1}{\gamma\, mc^2} \\ &= \frac{4\pi}{3}\left(\frac{q^2}{4\pi\epsilon_0\, mc^2}\right)\frac{1}{L}\gamma^3 \\ &= \frac{24\pi}{3}\,\frac{r_e}{L}\gamma^3 \end{aligned} \tag{43}$$

There is an extra power of γ^2 compared with a linac, so energy loss is much more significant.

Problem 9
Evaluate the energy loss per cycle for
(a) the Cornell electron synchrotron, which has an energy of 10 GeV in a circular ring of radius $L = 100$ m. Answer: about 9 MeV; and
(b) the proposed new electron-positron collider in Beijing, which will store electrons (also positrons) of energy 250 GeV in a circular accelerator of circumference 55 km [1]. §

This is a significant energy loss (about 0.1 percent per cycle). In order to limit the energy loss, high energy machines must have very large radii L.

Synchrotron radiation
In recent years, the energy loss has been turned around to an advantage — the synchrotron radiation is now used as a powerful source of X-rays for various purposes.

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or from
`https://www.scribd.com/doc/48520397/Jackson-Classical-Electrodynamics-3rd-edition`
The first edition (1962) can be downloaded from
`https://archive.org/details/ClassicalElectrodynamics`

A Details of evaluation for circular orbit

In this Appendix, details are provided for the evaluation of the power emitted by an ultra-relativistic point charge in a circular orbit. Start with the expression

$$\begin{aligned} P' &= \frac{1}{2}\left(\frac{q^2}{4\pi\epsilon_0 c}\right) I \\ I &= \int \kappa^5\, \mathbf{\Sigma^2}\, \sin\theta\, d\theta \end{aligned} \tag{44}$$

for a charge q moving at velocity nearly equal to c, in a circular orbit of radius L. All the calculation below is just to evaluate the numerical factor $C = 4/3$ cited in the main text.

The enhancement factor κ

$$\begin{aligned} \kappa &= [1 - \beta\cos\theta]^{-1} \\ &= \left[1 - \beta(1 - \theta^2/2)\right]^{-1} \\ &= \left[(1-\beta) - \beta\theta^2/2\right]^{-1} \end{aligned} \tag{45}$$

In the first term, we use

$$\begin{aligned}\gamma^{-2} &= (1+\beta)(1-\beta) \approx 2(1-\beta) \\ 1-\beta &= \gamma^{-2}/2\end{aligned}$$

and in the last term we can approximate $\beta = 1$. The error is of order $(1-\beta)\theta^2 \sim \gamma^{-2}\theta^2$. Then

$$\begin{aligned}\kappa &= \left[\gamma^{-2}/2 + \theta^2/2\right]^{-1} \\ &= \frac{2\gamma^2}{1+\gamma^2\theta^2} = \frac{2\gamma^2}{1+u^2}\end{aligned} \tag{46}$$

where

$$u = \gamma\theta \tag{47}$$

should be regarded as $O(\gamma^0)$.

Thus

$$\boxed{\kappa^5 = 32\gamma^{10}\,(1+u^2)^{-5}} \tag{48}$$

Azimuthal angle

The integrand (in particular $\boldsymbol{\Sigma}^2$) is understood to be averaged over azimuthal angle ϕ. First adopt the shorthand

$$s = \sin\phi \quad , \quad c = \cos\phi$$

(There should be no confusion with the velocity of light, also represented as c.) Second, upon averaging

$$\begin{aligned} s^2, c^2 &\mapsto 1/2 \\ s^4, c^4 &\mapsto 3/8 \\ s^2c^2 &\mapsto 1/8\end{aligned}$$

Evaluation of $\boldsymbol{\Sigma}^2$

$$\begin{aligned}\mathbf{n} &= \theta c\,\hat{\mathbf{e}}_x + \theta s\,\hat{\mathbf{e}}_y + (1-\theta^2/2)\,\hat{\mathbf{e}}_z \\ \mathbf{n}-\boldsymbol{\beta} &= \theta c\,\hat{\mathbf{e}}_x + \theta s\,\hat{\mathbf{e}}_y + (1-\theta^2/2-\beta)\,\hat{\mathbf{e}}_z \\ &= \theta c\,\hat{\mathbf{e}}_x + \theta s\,\hat{\mathbf{e}}_y + (\gamma^{-2}-\theta^2)/2\,\hat{\mathbf{e}}_z \\ \dot{\boldsymbol{\beta}} &= -\frac{c}{L}\,\hat{\mathbf{e}}_x\end{aligned}$$

There will therefore eventually be a factor of

$$\boxed{\frac{c^2}{L^2}} \tag{49}$$

which we drop for the time being, to be inserted back at the end. With this understanding

$$\begin{aligned}&(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}} \\ =\; &-\left[\theta c\,\hat{\mathbf{e}}_x + \theta s\,\hat{\mathbf{e}}_y + (\gamma^{-2}-\theta^2)/2\,\hat{\mathbf{e}}_z\right] \\ &\times\hat{\mathbf{e}}_x \\ =\; &\theta s\,\mathbf{e}_z - (\gamma^{-2}-\theta^2)/2\,\hat{\mathbf{e}}_y\end{aligned}$$

Thus

$$\begin{aligned}&\boldsymbol{\Sigma} \\ =\; &\mathbf{n}\times\left((\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right) \\ =\; &[\theta c\,\hat{\mathbf{e}}_x + \theta s\,\hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z] \\ &\times\left[\theta s\,\mathbf{e}_z - (\gamma^{-2}-\theta^2)/2\,\hat{\mathbf{e}}_y\right]\end{aligned}$$

where in the first factor one can approximate $\cos\theta = 1$.

<u>x component</u>

$$\begin{aligned}\Sigma_x &= \theta^2 s^2 + (\gamma^{-2}-\theta^2)/2 \\ \Sigma_x^2 &= \theta^4 s^4 + \theta^2 s^2(\gamma^{-2}-\theta^2) \\ &\quad + (1/4)(\gamma^{-2}-\theta^2)^2 \\ &\mapsto (3/8)\theta^4 + (1/2)\theta^2(\gamma^{-4}-\theta^2) \\ &\quad + (1/4)(\gamma^{-2}-\theta^2)^2 \\ &= \gamma^{-4}\left[(3/8)u^4 + (1/2)u^2(1-u^2)\right. \\ &\qquad \left. + (1/4)(1-u^2)^2\right] \\ &= (1/8)\gamma^{-4}\left[3u^4 + 4(u^2-u^4)\right. \\ &\qquad \left. +2(1-2u^2+u^4)\right] \\ &= (1/8)\gamma^{-4}\left(2+u^4\right)\end{aligned}$$

$$\boxed{\boldsymbol{\Sigma}_x^2 = (1/8)\gamma^{-4}\left(2+u^4\right)} \tag{50}$$

<u>y component</u>

$$\begin{aligned}\Sigma_y &= -\theta^2 sc \\ \Sigma_y^2 &= \theta^4 s^2 c^2 \\ &\mapsto (1/8)\theta^4 = (1/8)\gamma^{-4}u^4\end{aligned}$$

$$\boxed{\boldsymbol{\Sigma}_y^2 = (1/8)\gamma^{-4}\left(u^4\right)} \tag{51}$$

<u>z component</u>

$$\begin{aligned}\Sigma_z &= -\theta c(\gamma^{-2}-\theta^2)/2 \\ &= O(\gamma^{-3}) \\ \Sigma_z^2 &= 0\end{aligned} \tag{52}$$

to the order required.

$$\boxed{\boldsymbol{\Sigma}_z^2 = 0} \tag{53}$$

<u>All components</u>

Adding the three components together, we get

$$\boxed{\boldsymbol{\Sigma}^2 = (1/4)\gamma^{-4}\left(1+u^4\right)} \tag{54}$$

Integration over angles

$$\sin\theta\,d\theta = \theta\,d\theta$$

and using the variable u

$$\boxed{\sin\theta\, d\theta = \gamma^{-2}\, u\, du} \tag{55}$$

The integral
Thus I in (44) becomes

$$\begin{aligned} I &= \int 32\gamma^{10}(1+u^2)^{-5} \cdot \\ &\quad \cdot \frac{c^2}{L^2}\frac{1}{4}\gamma^{-4}\left(1+u^4\right)\cdot\gamma^{-2}\, u\, du \end{aligned}$$

where the three factors (separated by $\cdot$) come from κ^5, $\mathbf{\Sigma}^2$ with the factor $(c/L)^2$ restored and $\sin\theta\, d\theta$. Sorting out the factors, and introducing

$$v = u^2$$

we get

$$I = 4\frac{c^2}{L^2}\gamma^4 \int \frac{1+v^2}{(1+v)^5} dv \tag{56}$$

Since $u = \gamma\theta$ and we take the limit $\gamma \to \infty$, the integral goes from u (hence v) from 0 to ∞. Introduce

$$1+v = t$$

Then the integral in (56) is

$$\begin{aligned} & \int_1^\infty \frac{1+(t-1)^2}{t^5}\, dt \\ =& \int_1^\infty \frac{2-2t+t^2}{t^5}\, dt \\ =& \frac{2}{4}-\frac{2}{3}+\frac{1}{2}=\frac{1}{3} \end{aligned}$$

which then gives

$$\boxed{I = \frac{4}{3}\frac{c^2}{L^2}\gamma^4} \tag{57}$$

Power and energy
The power (energy per unit emitting time) is then

$$\boxed{P' = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c}\frac{c^2}{L^2}\gamma^4} \tag{58}$$

and the energy loss per cycle is

$$\Delta\mathcal{E} = P'\cdot\frac{2\pi L}{c} = \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0 L}\gamma^4 \tag{59}$$

The fractional energy loss per cycle is then

$$\begin{aligned} \frac{\Delta\mathcal{E}}{\mathcal{E}} &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0 L}\gamma^4\cdot\frac{1}{\gamma m c^2} \\ &= \frac{4\pi}{3}\frac{q^2}{4\pi\epsilon_0 m c^2}\frac{1}{L}\gamma^3 \end{aligned}$$

which can be written neatly as

$$\boxed{\frac{\Delta\mathcal{E}}{\mathcal{E}} = \frac{4\pi}{3}\frac{r_e}{L}\gamma^3} \tag{60}$$

The prefactor $4\pi/3$ agrees with Jackson [2]; see for example equation (14.32) in the Second Edition. In making that comparison, we need to replace $4\pi\epsilon_o \mapsto 1$ in order to go to the sytem of units used by Jackson.

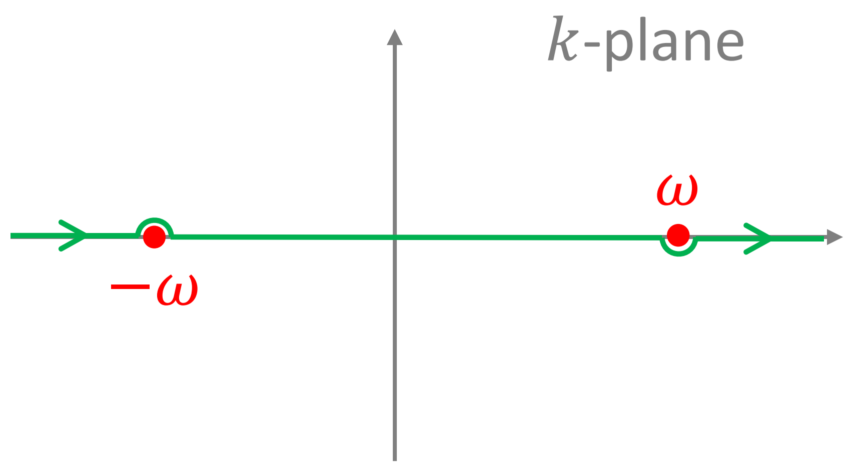

Figure 1 Contours in k plane

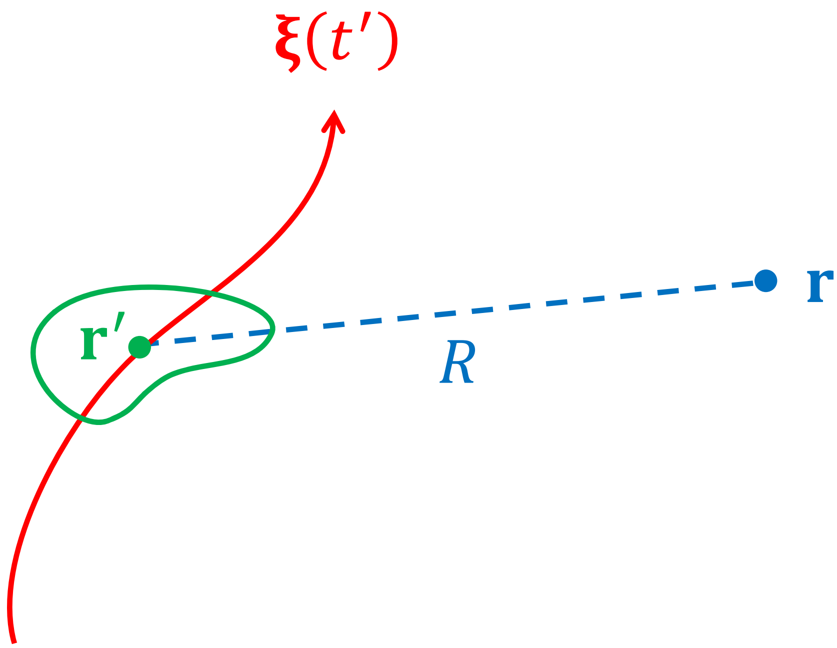

Figure 2 The two dots show the emitting point at $\mathbf{r}' = \boldsymbol{\xi}(t')$ and the observation point $\mathbf{r}$ at t

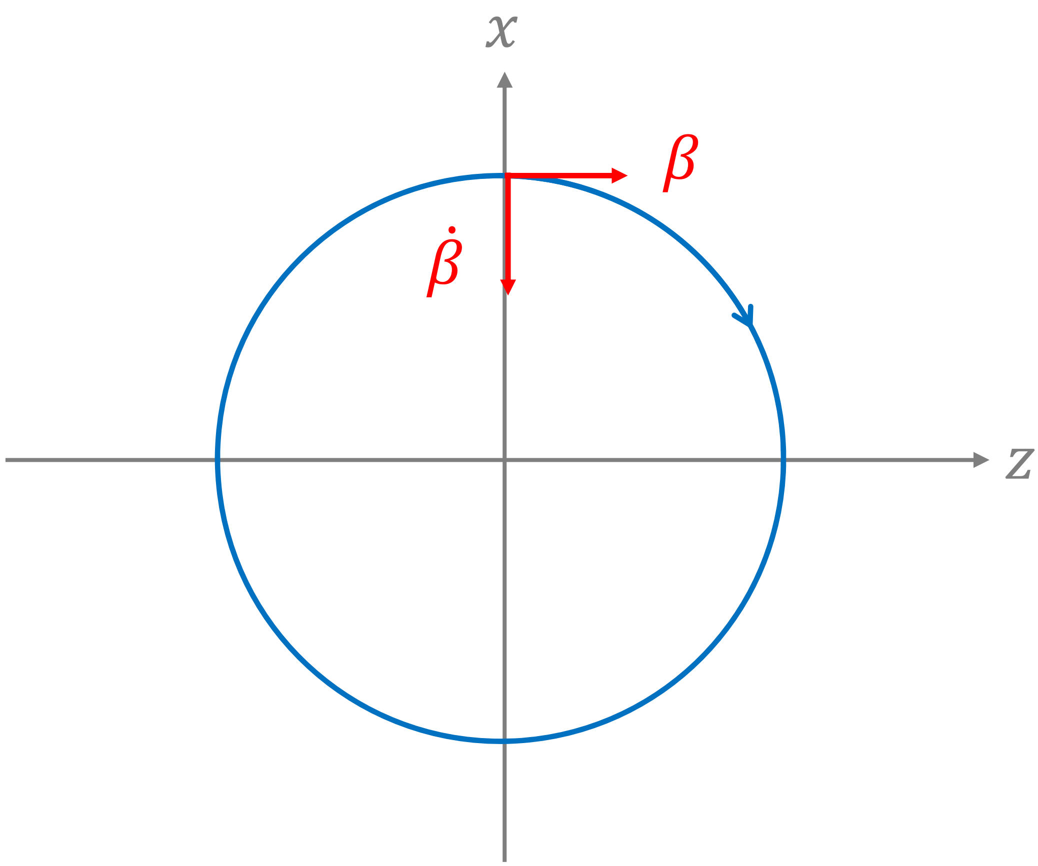

Figure 3   Charged particle in a circular accelerator