Chapter 2
Electrostatics

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A quick review is given of electrostatics, introducing the Green's function and also the elementary form of the multipole expansion in rectangular coordinates.

Contents

1 Laws of electrostatics

The laws of electrostatics should be familiar, and what follows is a quick review.

1.1 Coulomb's law

Force and electric field
The force between two stationary point charges is

$$\mathbf{F} \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{q_1\,q_2}{r^2}\,\hat{\mathbf{e}}_r = \frac{1}{4\pi\epsilon_0}\,\frac{q_1\,q_2}{r^3}\,\mathbf{r}$$

where $\mathbf{r}$ is the separation and $\hat{\mathbf{e}}_r = \mathbf{r}/r$ is the corresponding unit vector. The electric field caused by a stationary point charge q at the origin is

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^2}\,\hat{\mathbf{e}}_r = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r^3}\,\mathbf{r}$$

More generally, if there is a stationary point charge q at $\mathbf{s}$, then the electric field at $\mathbf{r}$ is given by

$$\mathbf{E}(\mathbf{r}) \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{q}{|\mathbf{r}-\mathbf{s}|^3}\,(\mathbf{r}-\mathbf{s}) \qquad (1)$$

The field due to a collection of point charges is given by the appropriate sum (see below).

System of units
In the MKSA system of units, $[q]$ = C and the coefficient in Coulomb's law has the empirical value

$$\frac{1}{4\pi\epsilon_0} \;=\; 9.0\times10^9\ \mathrm{N\ m^2\ C^{-2}}$$

In electrostatic units,

$$\frac{1}{4\pi\epsilon_0} \mapsto 1$$

Gravity
Everything in this Chapter also applies to Newtonian gravity, replacing charges by masses, and

$$\frac{1}{4\pi\epsilon_0} \;\mapsto\; -\,G$$

The minus sign arises because masses always attract, opposite to the case of like charges. The electric field $\mathbf{E}$ is mapped to $\mathbf{g}$, the gravitational force per unit test mass, often referred to as the acceleration due to gravity, with magnitude 9.8 m s^{-2} on the surface of the earth.

1.2 Gauss' law

Coulomb's law implies Gauss' law, which in integral form[1] and then differential form is given by

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \quad (2)$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (3)$$

where q is the total charge inside the surface, and ρ is the charge density.

Gauss' law applied to a stationary charge leads to Coulomb's law. But when charges are not stationary, Coulomb's law no longer holds, while Gauss' law remains valid.

1.3 Conservative force

Electrostatic forces are conservative: the work done around a closed loop is zero, so

$$\oint \mathbf{E} \cdot d\mathbf{r} = 0 \quad (4)$$

$$\boldsymbol{\nabla} \times \mathbf{E} = 0 \quad (5)$$

This condition does not hold when there are changing magnetic fields.

1.4 Potential and Poisson's equation

Definition of potential
The condition (4) or (5) allows a scalar potential to be defined

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi \quad (6)$$

$$\Phi(\mathbf{r}) = -\int_O^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad (7)$$

where O is an arbitrary reference point, and the integral is independent of path. Thus Φ is the work done in bringing a unit charge from O to $\mathbf{r}$, and is just the electric potential or voltage.[2] The point O is arbitrary; therefore Φ is specified up to an additive constant. The unit is $[\Phi] = \mathrm{V}$. The minus signs follow the analogous formulas relating force and potential energy; thus $\mathbf{E}$ points from high values of Φ to low values of Φ.

Poisson's equation
Combining (6) with Gauss' law, we find

$$\boxed{-\nabla^2\Phi(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}} \quad (8)$$

Problems in electrostatics are reduced to solving Φ from this equation, and then $\mathbf{E}$ from Φ. It is more convenient to deal with Φ (one function) than with $\mathbf{E}$ (three functions).

The reason for moving the minus sign to the LHS is that $-\nabla^2$ (a) is a positive operator,[3] and (b) corresponds to k^2 in momentum space, as we shall see below.

Potential due to a point charge
The potential due to a point charge q at the origin is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

More generally, the potential due to a point charge at $\mathbf{s}$ (the source position) is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r}-\mathbf{s}|} \quad (9)$$

This can be obtained by integrating (along a radius) the electric field from Coulomb's law; or it can be regarded as the solution to Poisson's equation for ρ being a δ-function. The latter point of view will next be given.

2 Superposition and Green's function

EM in vacuum is a *linear* theory, and fields due to different sources can be added or superposed. The law of superposition is an experimental result; it is not even exactly true. For very high fields, there is some nonlinearity. A direct manifestation is Delbruck scattering [1] — the scattering of a photon by the static Coulomb field around a heavy nucleus. Born and Infeld proposed a nonlinear theory of EM in vacuum [2, 3], of course deviating from the standard theory only for strong fields.[4] Thus it is incor-

[1] Surface and volume integrals will be written with a single integral sign $\int$. The conventional notation with multiple $\int$ symbols can hardly be contemplated when we imagine that spacetime may have 10 dimensions.

[2] To be precise, the potential is defined only when there is no changing magnetic flux, but the voltage is always operationally defined as the reading of a voltmeter. In electrostatics, the two are identical. There are subtleties when changing magnetic fluxes are involved, as in an inductor.

[3] In the sense that $\int \Phi(-\nabla^2\Phi)\, d^3r \geq 0$ for any Φ that vanishes sufficiently rapidly at spatial infinity.

[4] The Born–Infeld theory has nice properties, but there is no evidence that it is realized. However, it has been revived in the context of string theory.

rect to refer to a *principle* of superposition. Nevertheless, for most situations not involving material media, superposition can be assumed to hold.

2.1 Formula for the potential

If there are charges q_n at positions $\mathbf{s}_n$, then superposing the result in (9), we find

$$\Phi(\mathbf{r}) \quad = \quad \frac{1}{4\pi\epsilon_0} \sum_n \frac{q_n}{|\mathbf{r}-\mathbf{s}_n|} \tag{10}$$

For a continuous distribution with density ρ, the charge in a small volume is

$$q_n \quad \mapsto \quad \rho(\mathbf{s})\, d^3s$$

and the continuous version of (10) is

$$\Phi(\mathbf{r}) \quad = \quad \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{s})\, d^3s}{|\mathbf{r}-\mathbf{s}|} \tag{11}$$

which in principle solves Poisson's equation, at least for the boundary condition that $\Phi(\mathbf{r}) \to 0$ as $\mathbf{r} \to \infty$.

By the way, there is no worry about the denominator causing any singularity. Define $\boldsymbol{\xi} = \mathbf{r} - \mathbf{s}$. Then the integral in (11) goes as

$$\int \frac{1}{\xi}\, d^3\xi \quad \sim \quad \int \frac{1}{\xi}\, \xi^2\, d\Omega\, d\xi$$

which has no divergence at $\xi = 0$.

2.2 Green's function

We now introduce a general technique for solving inhomogeneous linear equations such as (8): first look for a solution where the inhomogeneity is a δ-function — or in physical terms, where ρ is a point charge. Thus define a function of two spatial variables $G(\mathbf{r}, \mathbf{s})$ which satisfies

$$-\nabla^2_{\mathbf{r}}\, G(\mathbf{r}, \mathbf{s}) \quad = \quad \delta^3(\mathbf{r}-\mathbf{s}) \tag{12}$$

The differential operator on the LHS acts on the variable $\mathbf{r}$. In physical terms and up to multiplicative constants, this is the potential at $\mathbf{r}$ caused by a point charge at $\mathbf{s}$. The differential equation (12) is to be solved with the same boundary conditions on $\mathbf{r}$ as for (8), which we here assume is that the function vanishes as $\mathbf{r} \to \infty$.

If G is found, then the solution to (8) is obtained as an integral:

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int G(\mathbf{r}, \mathbf{s})\, \rho(\mathbf{s})\, d^3s} \tag{13}$$

and this is readily checked by applying the operator $-\nabla^2_{\mathbf{r}}$ to the above, giving a factor $\delta^3(\mathbf{r}-\mathbf{s})$, which converts $\rho(\mathbf{s}) \mapsto \rho(\mathbf{r})$. It remains to solve for G.

Some simplifications are possible: G does not depend on $\mathbf{r}$ and $\mathbf{s}$ separately, but only on their difference:

$$G(\mathbf{r}, \mathbf{s}) \quad = \quad G(\mathbf{r}-\mathbf{s})$$

Therefore we only need to consider $\mathbf{s} = 0$ (in physical terms the potential due to a point charge at the origin).

Thus, we need to solve

$$\boxed{-\nabla^2\, G(\mathbf{r}) = \delta^3(\mathbf{r})} \tag{14}$$

2.3 Position representation

By symmetry or isotropy of space, G can depend only on $r = |\mathbf{r}|$. First consider (14) for $r > \delta > 0$. Then

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) \quad = \quad 0$$

where ∇^2 has been written in spherical coordinates and terms involving angular coordinates are dropped. The bracket must be a constant, so

$$\begin{aligned} r^2 \frac{dG}{dr} \quad &= \quad A \\ G \quad &= \quad -\frac{A}{r} \end{aligned}$$

To evaluate A, integrate (14) over a volume V being a small sphere of radius r centered at the origin.

$$\begin{aligned} -\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\, G)\; dV \quad &= \quad 1 \\ -\int_S \boldsymbol{\nabla}\, G \cdot \hat{\mathbf{n}}\, dS \quad &= \quad 1 \end{aligned}$$

where we have used Gauss' theorem to convert the volume integral to an integral over the spherical surface S at radius r, and $\hat{\mathbf{n}}$ is the unit outward normal. But

$$\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} G \quad = \quad \frac{dG}{dr} = \frac{A}{r^2}$$

whereas the surface integral contributes a factor $4\pi r^2$.[5] Thus $-4\pi A = 1$, and finally

$$\boxed{G(\mathbf{r}) = \frac{1}{4\pi r}} \tag{15}$$

It is straightforward to check that (13) and (15) reproduce (11).

[5] Note that in this evaluation we only make use of the value of $G(r)$ for $r > \delta > 0$.

2.4 Momentum representation

Fourier transform
As a reminder, any function $f(x)$ of position can be represented as

$$\begin{aligned} f(x) &= \int \tilde{f}(k)\, e^{ikx}\, \frac{dk}{2\pi} \\ \tilde{f}(k) &= \int f(x)\, e^{-ikx}\, dx \end{aligned} \tag{16}$$

- Note that the two exponentials appear with opposite signs.
- The variable k is the wave number (or, up to a factor of $\hbar$, the momentum, hence the heuristic name) associated with the "wave" $\exp(ikx)$.
- In physics, the convention is to associate 2π with the k factor as shown. Mathematicians like to write $dk/\sqrt{2\pi}$ and $dx/\sqrt{2\pi}$ in (16) for a more symmetric expression.
- The generalization to higher dimensions is obvious.
- In 1D, if $f(x) = \delta(x)$ then $\tilde{f}(k) = 1$. In 3D, if $f(\mathbf{r}) = \delta^3(\mathbf{r})$, then $\tilde{f}(\mathbf{k}) = 1$.

Fourier transforms are especially convenient for handling derivatives. Under the transform,

$$\begin{aligned} \boldsymbol{\nabla} &\mapsto i\mathbf{k} \\ -\nabla^2 &\mapsto k^2 \end{aligned} \tag{17}$$

in the sense that

$$\begin{aligned} \boldsymbol{\nabla} f(\mathbf{r}) &= \boldsymbol{\nabla} \int \tilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \\ &= \int i\mathbf{k}\, \tilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \\ -\nabla^2 f(\mathbf{r}) &= -\nabla^2 \int \tilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \\ &= \int k^2\, \tilde{f}(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \end{aligned}$$

Green's function in momentum space
Transforming (14) to momentum space, we have

$$k^2\, \tilde{G}(k) = 1$$

giving

$$\boxed{\tilde{G}(k) = \frac{1}{k^2}} \tag{18}$$

Back to coordinate space
It is now a matter of transforming back to coordinate space:

$$\begin{aligned} G(r) &= \int \frac{1}{k^2}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, \frac{d^3k}{(2\pi)^3} \\ &= \frac{1}{8\pi^3} \int_0^\infty \int_{-1}^1 \frac{1}{k^2}\, e^{ikru}\, 2\pi du\, k^2 dk \\ &= \frac{1}{4\pi^2} \int_0^\infty \frac{e^{ikr} - e^{-ikr}}{ikr}\, dk \\ &= \frac{1}{2\pi^2 r} \int_0^\infty \frac{\sin kr}{k}\, dk \end{aligned}$$

In the above $u = \cos\theta$ and θ is the angle between $\mathbf{k}$ and $\mathbf{r}$. The final integral is $\pi/2$ independent of r (see Appendix A). Thus we recover (15).

A rather elaborate account of the Green's function method has been presented, in anticipation of a similar problem of the form

$$-\left(\nabla^2 + \omega^2/c^2\right) G(\mathbf{r}) = \delta^3(\mathbf{r})$$

which will appear in the discussion of radiation due to a point source (and by superposition any source) oscillating at a definite frequency.

3 Multipoles

3.1 General formula

Suppose all the charge is contained in a region of dimension a, i.e., in the integral (11), all $|\mathbf{s}| < a$, and we observe at a distance $r \gg a$. Then the charge distribution is sensed only in an aggregate way, and the formula for Φ can be much simplified. An example is observing the charge distribution in an atom or molecule ($a \sim 0.1$ nm) from a distance of say 1 μm. Incidentally, all the formalism below applies to gravity as well.

The key idea is to expand $|\mathbf{r} - \mathbf{s}|^{-1}$ in powers of the small parameter $s/r \sim a/r$.

$$\begin{aligned} &|\mathbf{r} - \mathbf{s}|^{-1} = \left(r^2 - 2\mathbf{r}\cdot\mathbf{s} + s^2\right)^{-1/2} \\ &= \frac{1}{r}\left(1 - \frac{2\mathbf{r}\cdot\mathbf{s}}{r^2} + \frac{s^2}{r^2}\right)^{-1/2} \\ &= \frac{1}{r}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{2\mathbf{r}\cdot\mathbf{s}}{r^2} + \frac{s^2}{r^2}\right) + \right. \\ &\quad \left. + \frac{(-1/2)(-3/2)}{2!}\left(-\frac{2\mathbf{r}\cdot\mathbf{s}}{r^2}\right)^2 + \ldots\right] \\ &= \frac{1}{r}\left[1 + \frac{\mathbf{r}\cdot\mathbf{s}}{r^2} - \frac{1}{2}\frac{s^2}{r^2} + \frac{3}{2}\frac{(\mathbf{r}\cdot\mathbf{s})^2}{r^4} + \ldots\right] \\ &= \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{s}}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{s})^2 - r^2 s^2}{2r^5} + \ldots \end{aligned} \tag{19}$$

The three terms are respectively of order $1/r$, $s/r^2 \sim a/r^2$ and $s^2/r^3 \sim a^2/r^3$.

Put the first term into (11) and pull everything independent of $\boldsymbol{s}$ outside the integral:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\, \frac{1}{r} \int \rho(\mathbf{s})\, d^3s$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r} \tag{20}$$

where q is the total charge. The leading term is equivalent to regarding the charge distribution as a point. We next work out a few correction terms.

3.2 Electric dipole

Formula for the potential
For the next term, write

$$\mathbf{r}\cdot\mathbf{s} = r_j s_j$$

with summation over repeated indices understood. Again put this into (11) and take everything independent of $\mathbf{s}$ outside the integral:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{r_j}{r^3}\int \rho(\mathbf{s})\, s_j\, d^3s$$

so that

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p_j r_j}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}} \tag{21}$$

where

$$\boxed{\mathbf{p} = \int \rho(\mathbf{s})\,\mathbf{s}\, d^3s} \tag{22}$$

is called the *dipole moment* of the charge distribution. If $\rho(\mathbf{s})$ describes discrete charges, then

$$\begin{aligned}\rho(\mathbf{s}) &= \sum_n q_n\,\delta^3(\mathbf{s}-\mathbf{s}_n)\\ \mathbf{p} &= \sum_n q_n\,\mathbf{s}_n\end{aligned} \tag{23}$$

Center of charge
Suppose we shift to a new origin O' at position $\mathbf{s}_0$, so that the displacement from O' is

$$\mathbf{s}' = \mathbf{s}-\mathbf{s}_0$$

Then the dipole moment about the new origin is

$$\begin{aligned}\mathbf{p}' &= \int \rho\,\mathbf{s}'\,d^3s = \int \rho\,(\mathbf{s}-\mathbf{s}_0)\,d^3s\\ &= \mathbf{p}-\mathbf{s}_0\,q\end{aligned}$$

where $q=\int\rho\,d^3s$ is the total charge. For any system with $q\neq 0$, we can choose

$$\mathbf{s}_0 = q^{-1}\,\mathbf{p}$$

which is in fact the center of charge; this makes $\mathbf{p}'=0$. In other words, for a system with a net charge, it is always possible — and usually convenient — to choose the origin so that there is no dipole moment. In practice, this means we discuss dipole moments *only for neutral systems*. In the analogous problem of gravity, the total mass must be nonzero, and we always choose the origin to eliminate the dipole.

Canonical model
The simplest example is a system of two point charges on the z-axis, with $+q$ at $z=+a/2$ and $-q$ at $z=-a/2$. From (23),

$$\mathbf{p} = qa\,\hat{\mathbf{e}}_3 \tag{24}$$

where $\hat{\mathbf{e}}_j$ are the unit vectors in the three directions. All dipoles can be thought of in terms of this model (if necessary rotated), since Φ depends only on $\mathbf{p}$, independent of how the dipole is actually constituted. This canonical model consists of two poles, hence the name *dipole* for this term in the expansion.

Problem 1
For a dipole in the z-direction, we have

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{p\,z}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{p\,\cos\theta}{r^2} \tag{25}$$

Using rectangular coordinates, calculate the components of $\mathbf{E}$. Note that the magnitude of $\mathbf{E}$ scales as r^{-3}. See also Problem 6. §

Problem 2
The electric dipole moment p of elementary particles is usually expressed in terms of a distance a, where $p=e\,a$, and e is the elementary unit of charge. What would you guess to be the order of magnitude of a for a neutron? Actually the value is very small, and at the time of writing there is only an experimental upper limit. Find out what it is, and compare with the naive guess. The huge discrepancy reveals some symmetry principle which (nearly?) "forbids" such a moment. §

Field pattern
The field pattern is shown schematically in **Figure 1**. See also Section 3.4. In anticipation of a contrast with magnetic dipoles, note that the total flux through the x–y plane (perpendicular to the dipole) is nonzero.

3.3 Electric quadrupole

Formula for the potential
The next term in (19) goes as $r^2s^2/r^5\sim s^2/r^3$. We can write the numerator as

$$\begin{aligned}&3(\mathbf{r}\cdot\mathbf{s})^2 - r^2s^2\\ =\;& 3(s_ir_i)(s_jr_j) - s^2\delta_{ij}r_ir_j\\ =\;& (3s_is_j - s^2\delta_{ij})\,r_ir_j\end{aligned}$$

where the convention of summing over repeated indices is adopted. Putting this into (11) gives the following contribution to the potential:

$$\boxed{\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{q_{ij}r_i r_j}{2r^5}} \tag{26}$$

where

$$\boxed{q_{ij} = \int \rho(\mathbf{s})\,(3s_i s_j - s^2\delta_{ij})\,d^3s} \tag{27}$$

is called the *quadrupole moment* of the charge distribution.

Typical magnitude
For a total charge q confined to a typical linear dimension a, we expect $q_{ij} \sim qa^2$ — though there is a caveat about nearly spherical distributions; see below.

Traceless
Since q_{ij} is symmetric, one would expect six independent components. But there is one constraint. Take the trace (i.e., set $i = j$ and sum); since $\delta_{ii} = 3$, from (27) we find

$$q_{ii} \quad = \quad q_{11} + q_{22} + q_{33} = 0 \tag{28}$$

so that there are only five independent degrees of freedom.

Alternate form for potential
In (26), we can replace

$$r_i r_j \mapsto \frac{1}{3}\left(3r_i r_j - r^2\delta_{ij}\right) \tag{29}$$

so that the same type of factor appears as in (27). The extra term introduced is proportional to q_{ii}, which is zero.

Problem 3
Evaluate the 6 quantities on the RHS of (29) on the unit sphere, expressing the result in terms of the angles θ, ϕ. Show that each of these has average value zero over the unit sphere. If you have studied spherical harmonics, these are related to Y_{2m}, $m = 0, \pm 1, \pm 2$. §

Spherical symmetry
For a system with spherical symmetry, we can make some further deductions. First, reflection symmetry about any one axis implies $q_{ij} = 0$ for $i \neq j$. Secondly, since the three axes are equivalent, $q_{11} = q_{22} = q_{33}$, which together with (28) shows that they are all zero. Thus a spherical distribution of charge has no quadrupole moment.

This should have been obvious: application of Gauss' law together with symmetry implies that the potential is exactly the same as that for a point charge. Not only is the quadrupole moment zero, but all higher moments are also zero.

Therefore for a *nearly* spherical distribution, the order of magnitude of the quadrupole moment should be

$$|q_{ij}| \quad \sim \quad |\epsilon| qa^2$$

where ϵ is a small dimensionless number that describes the departure from sphericity. For example, for a spheroid with semi axes a, a, b and mean radius $R = (a^2 b)^{1/3}$,

$$\epsilon \quad = \quad \frac{a-b}{R}$$

which describes the oblateness (if $\epsilon > 0$).

Axial symmetry
If there is rotational symmetry about the z-axis, the matrix $[q]$ is diagonal, and takes the form[6]

$$[q] \quad = \quad \text{diag}\,(-Q/2, -Q/2, Q) \tag{30}$$

Problem 4
This problem is phrased for gravity, for a nearly spherical object such as the sun, which has mass M, mean radius R, no dipole moment, and quadrupole moment given by (30), with

$$Q = -2\epsilon M R^2$$

where the conventional quadrupole parameter ϵ is dimensionless, and small if the object is nearly spherical. Show that

$$\Phi \quad = \quad -\frac{GM}{r}\left[1 - \epsilon\left(\frac{R}{r}\right)^2 P_2(\cos\theta)\right]$$

where

$$P_2(t) \quad = \quad \frac{1}{2}(3t^2 - 1)$$

The sun is expected to be slightly oblate: $\epsilon > 0$, $Q < 0$. §

The sun is slightly oblate because of its own rotation. The quadrupole parameter is $\epsilon \sim 10^{-7}$ and is one of the causes for the advance of perihelion for a planet (e.g., Mercury) moving in the equatorial plane ($\theta = \pi/2$) [5].

Problem 5
Point charges are placed on the z-axis, at coordinates z as shown

[6] The parameter Q is defined to be positive for a prolate shape.

charge	z
q	$2a$
$-q$	$-2a$
$-2q$	a
$2q$	$-a$

Table 1. Distribution of point charges

Find the leading term for Φ at (a) $(r,0,0)$ and (b) $(0,0,r)$. §

Nuclear quadrupole moment

The nuclear quadrupole moment[7] is usually expressed in terms of the number Q as defined in (30), but with numerical values usually given in units of eb, where

$$e\text{b} \quad = \quad e \text{ barn} = 1.6 \times 10^{-19} \text{ C} \times 10^{-28} \text{ m}^2$$

The values are [6]

- Typically a fraction of eb, e.g. 0.40 for ^{59}Co.
- Nearly zero for nuclei with a closed shell, which should be theoretically spherical.
- Quite large for some rare earths, e.g., +0.80 for ^{176}Lu.

3.4 Summary of potential and field

Potential

Summarizing the above, we have

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \times \left(q\frac{1}{r} + p_i\frac{r_i}{r^3} + \frac{1}{6}q_{ij}\frac{3r_ir_j - r^2\delta_{ij}}{r^5} + \dots \right) \tag{31}$$

Recalling the identities[8]

$$\partial_i \frac{1}{r} \quad = \quad -\frac{r_i}{r^3}$$
$$\partial_i\partial_j \frac{1}{r} \quad = \quad \frac{3r_ir_j - r^2\delta_{ij}}{r^5}$$

we can write the potential as

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \times \left(q - p_i\partial_i + \frac{1}{6}q_{ij}\partial_i\partial_j + \dots \right)\frac{1}{r} \tag{32}$$

Field

Apply another derivative to get $E_k = -\partial_k\Phi$:

$$E_k = \frac{1}{4\pi\epsilon_0} \times \left(-q\partial_k + p_i\partial_i\partial_k - \frac{1}{6}q_{ij}\partial_i\partial_j\partial_k + \dots \right)\frac{1}{r} \tag{33}$$

[7] I am grateful to D Kiang for drawing my attention to this example.

[8] These formulas apply away from $r = 0$. Otherwise there are extra terms involving $\delta^3(\mathbf{r})$; see e.g., Ref. [4].

In particular, for a dipole,

$$E_k \quad = \quad \frac{1}{4\pi\epsilon_0}\, p_i\partial_k\partial_i\frac{1}{r} \tag{34}$$

Problem 6

For $\mathbf{p}$ along $+z$, write out the components of $\mathbf{E}$ in rectangular coordinates, and verify **Figure 1**.

3.5 Charge distribution in slowly changing potential

Consider a charge distribution ρ extending over a region of size $\sim a$, placed in an external potential Φ which is spatially slowly varying. The potential is external in the sense that it is produced by charges elsewhere, so $\nabla^2\Phi = 0$ in the domain of interest. The potential is slowly varying in the sense that the change over the distance a is small.

Energy

The energy of the system is

$$U \quad = \quad \int \rho(\mathbf{r})\,\Phi(\mathbf{r})\,d^3r \tag{35}$$

Assume that the origin is chosen near the center of the charge distribution. Because Φ is slowly varying, it can be expanded in a Taylor series:

$$\Phi(\mathbf{r}) \quad = \quad \Phi + r_i\,\partial_i\Phi + \frac{1}{2}r_ir_j\,\partial_i\partial_j\Phi + \dots$$

where on the RHS, Φ and its derivatives are understood to be evaluated at the origin, and the convention of summing over repeated indices is adopted. In the quadratic term, we can replace

$$r_ir_j \mapsto r_ir_j - (1/3)r^2\delta_{ij}$$

(see also (29)) since that causes an extra term proportional to

$$\delta_{ij}\,\partial_i\partial_j\Phi \quad = \quad \nabla^2\Phi = 0$$

because Φ is external. Put all this into (35) and pull the constant factors Φ, $\partial_i\Phi$ and $\partial_i\partial_j\Phi$ outside the integral. The three remaining integrals are

$$\int \rho\, d^3r = q$$
$$\int \rho r_i\, d^3r = p_i$$
$$\int \rho[r_ir_j - (1/3)r^2\delta_{ij}]\, d^3r = \frac{1}{3}q_{ij}$$

leading to the expression

$$\boxed{U = q\,\Phi + p_i\,\partial_i\Phi + \frac{1}{6}q_{ij}\,\partial_i\partial_j\Phi + \dots} \tag{36}$$

Thus the n-th multipole interacts with the n-th spatial derivative of the external potential. In particular, for a dipole,

$$\boxed{U = -\mathbf{p}\cdot\mathbf{E}} \tag{37}$$

Incidentally, if Φ in (36) is an external potential, we can change

$$\partial_i\partial_j \quad \mapsto \quad \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$$

Force on a dipole
The force on a permanent dipole $\mathbf{p}$ can be evaluated from the energy in (37):

$$\begin{aligned} U &= -p_jE_j \\ F_i &= -\partial_iU = p_j(\partial_iE_j) = p_j(\partial_jE_i) \end{aligned}$$

The last step assumes a static field: $\boldsymbol{\nabla}\times\mathbf{E} = 0$. Thus

$$\boxed{\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}} \tag{38}$$

The force can also be evaluated directly from

$$\mathbf{F} = \int \rho(\mathbf{s})\mathbf{E}(\mathbf{s})\,d^3s$$

Problem 7
Start from the above formula and expand $\mathbf{E}(\mathbf{s})$ about the origin. Pick out the term corresponding to the dipole moment and derive (38). §

Torque on a dipole
The torque on a charge distribution is

$$\boldsymbol{\tau} = \int \mathbf{s}\times\rho(\mathbf{s})\mathbf{E}(\mathbf{s})\,d^3s$$

Approximate $\mathbf{E}(\mathbf{s})$ by the constant value $\mathbf{E}$ at the center of the charge distribution and take this factor out of the integral. Then

$$\boldsymbol{\tau} = \left[\int \mathbf{s}\,\rho(\mathbf{s})d^3s\right]\times\mathbf{E}$$

or simply

$$\boxed{\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}} \tag{39}$$

Van der Waals force
Imagine two neutral atoms (or molecules) that do not have permanent dipole moments, e.g., atoms of He, a distance r apart, with $r \gg$ the atomic radii a. Suppose one of them has an instantaneous dipole moment p. It then produces a field at the position of the other atom of magnitude

$$E \propto \frac{p}{r^3}$$

which, acting on the second atom, polarizes it and produces a dipole moment

$$p' \propto E \propto \frac{p}{r^3}$$

This in turn produces a field back at the position of the first atom, of magnitude

$$E' \propto \frac{p'}{r^3} \propto \frac{p}{r^6}$$

Acting on the first atom, this causes an interaction energy (see (37))

$$U \propto pE' \propto \frac{p^2}{r^6} \tag{40}$$

The factor p^2 should be understood as the time-averaged value.

The upshot is that there will be an interaction energy that goes as r^{-6}. The corresponding force is called the van der Waals force. It is attractive as shown in the following Problem.

Problem 8
Sketch diagrams and determine the sign of the van der Waals potential energy. §

Force on an induced dipole
A neutral particle (e.g., atom, molecule, dust or colloidal particle) would exhibit a dipole moment $\mathbf{p}$ induced by and proportional to the electric field $\mathbf{E}$ acting on the particle:

$$\mathbf{p} = \alpha\mathbf{E}$$

where α is called the *polarizability*. If we naively put this into (37), we find that the potential energy is

$$U = -\alpha E_iE_i = -\alpha E^2$$

But this is wrong by a factor of 2, and the correct potential energy is

$$\boxed{U = -(1/2)\alpha E^2} \tag{41}$$

The reason is that (36) and (37) are derived for a constant $\mathbf{p}$, whereas if we imagine $\mathbf{p}$ increasing as $\mathbf{E}$ is increased from 0, then the total work done will have this factor of 1/2, analogous to the same factor in the potential energy due to a Hookean spring: $U = (1/2)kx^2$ in obvious notation. The force associated with U is also called a *ponderomotive force*.

This result means that a polarizable object is attracted to regions of high E^2. For application to optics, E^2 should be replaced by its time-averaged value.

Problem 9
A thin laser beam (say beam radius of a few μm) is shined vertically onto a liquid surface; thus a column of the liquid is illuminated and becomes a region of high E^2. Sketch the direction of the extra (i.e., E^2-dependent) pressure forces in the liquid. Which way would the liquid surface be displaced? §

This effect, i.e., a vertical movement of the surface, was first observed in detail by Ashkin and Dziedzic [7], but was incorrectly attributed to a radiation force (vertical) at the liquid surface, whereas in fact it is due to the ponderomotive force (horizontal) [8]. The above analysis is only correct to $O((\kappa-1)^1)$, where κ is the dielectric constant: $\kappa \propto N\alpha$, where N is the number of particles per unit volume. The ponderomotive force to higher orders is subtle, and a careful account can be found in Landau and Lifshitz [9] for static fields and the extension to time-varying fields was given by us [10, 11].

4 Macroscopic fields

This Section deals with several related issues all having to do with the presence of matter and the practical need to perform averages in space and time. The purpose is to point out the issues and give some initial (and incomplete) answers, and to highlight some of the subtleties. It is not the intention in this course to go into these issues in detail.

4.1 Microscopic and macroscopic fields

Microscopic fields
The fields discussed so far are the *actual* fields at each position and time, to be called the *microscopic fields*; the same is true of the charge and current densities. For this Section, denote these as $\mathbf{E}'$, $\mathbf{B}'$, ρ', $\mathbf{J}'$, and they satisfy equation such as

$$\boldsymbol{\nabla} \cdot \mathbf{E}' \quad = \quad \frac{\rho'}{\epsilon_0} \tag{42}$$

Macroscopic fields
Consider an EM wave passing through a material medium such as water or a piece of plastic. In these situations, $\mathbf{E}'$, ρ' etc. all show rapid variations on the intermolecular scale, say $a \sim 0.1$ nm (not to mention even smaller scales such as the size of each nucleus). Yet we often describe such a wave with formulas such as

$$\mathbf{E}(\mathbf{r}, t) \quad = \quad \mathbf{E}_0 \cos(\omega t - kz) \tag{43}$$

where $k = 2\pi/\lambda$ is the wave number. The $\mathbf{E}$ field in (43) varies only on the scale λ, say ~ 500 nm, but has negligible variation on the scale a; it is called the *macroscopic field*.

Relationship
The questions are the following.

- How are the macroscopic quantities related to the microscopic quantities?
- What laws do the macroscopic quantities satisfy, and how are these to be derived from the analogous microscopic equations such as (42)?

The answer to the first question is *averaging* in space and time, e.g.,

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \langle \mathbf{E}'(\mathbf{r}, t) \rangle \\ &\equiv \int S(\mathbf{s}, \tau) \mathbf{E}'(\mathbf{r}+\mathbf{s}, t+\tau)\, d^3 s\, d\tau \end{aligned} \tag{44}$$

where the symbol $\langle \ldots \rangle$ denotes an average, and the function S has the following properties

- It is everywhere non-negative.[9]
- It is negligible for $|\mathbf{s}| > R$ or $|\tau| > T$, for some R and T which are (a) much larger than the microscopic scale (say a for the spatial variable) and (b) much smaller than the macroscopic scale (say λ for the spatial variable).
- It has unit weight in the sense

$$\int S(\mathbf{s}, \tau)\, d^3 s\, d\tau \quad = \quad 1$$

For the second question, the Maxwell equations retain the same form, e.g.,

$$\boldsymbol{\nabla} \cdot \mathbf{E} \quad = \quad \frac{\rho}{\epsilon_0} \tag{45}$$

because a linear equation survives averaging.

But for the Lorentz force law, which is *not* linear in these quantities, the problem is more complicated, and will be briefly discussed below.

Problem 10
Formally derive (45) from (42). §

These steps are implicitly invoked whenever we apply Maxwell's equations to a situation such as (43), and was in fact already used in say Problem 9.

4.2 Total charge and free charge

Free charge
Now imagine a situation where say a piece of plastic is placed between the plates of a capacitor. After averaging, we can describe the situation as

$$\boldsymbol{\nabla} \cdot (\epsilon_0 \mathbf{E}) \quad = \quad \rho \tag{46}$$

[9] This condition can be slightly relaxed.

where ρ is the total charge density. But there are two types of charge densities: the free charges such as those on the capacitor plates, and the bound charges arising from the medium being polarized; in obvious notation

$$\rho \quad = \quad \rho_f + \rho_b \tag{47}$$

There is a good reason to treat these two terms separately: usually we only control and specify the free charge. For example, if a current I flows into one of the capacitor plates, then

$$I \quad = \quad \frac{dQ_f}{dt}$$

where Q_f is the free charge on the plate. It is important to realize that (a) Q_f is not related to the total charge Q in any simple manner (except possibly through an approximate and empirical constitutive equation; see below), and (b) the capacitance is the ratio of Q_f (not Q) to voltage.

Bound charge

In order to put the equations into a form that shows the free charges alone as the source, we express ρ_b in terms of the polarization density and move it to the other side. To do so, begin with the formula for the potential due to a single dipole $\mathbf{p}$ at the origin; for simplicity consider a time-independent situation and omit the time variable:

$$\Phi(\mathbf{r}) \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{p_j r_j}{r^3} = -\frac{1}{4\pi\epsilon_0} p_j \frac{\partial}{\partial r_j}\frac{1}{r}$$

More generally, for a dipole at position $\mathbf{s}$:

$$\Phi(\mathbf{r}) \quad = \quad -\frac{1}{4\pi\epsilon_0} p_j \frac{\partial}{\partial r_j}\frac{1}{|\mathbf{r}-\mathbf{s}|}$$

Superposition then gives the potential for a continuous distribution of dipoles with density $\mathbf{P}$:

$$\Phi(\mathbf{r}) \quad = \quad -\frac{1}{4\pi\epsilon_0}\int P_j(\mathbf{s})\,\frac{\partial}{\partial r_j}\frac{1}{|\mathbf{r}-\mathbf{s}|}\,d^3s$$

Change $\partial/\partial r_j$ to $\partial/\partial s_j$ (which costs a minus sign since it acts on a function of $r_j - s_j$) and then integrate by parts for the variable s_j (which costs another minus sign):

$$\Phi(\mathbf{r}) \quad = \quad -\frac{1}{4\pi\epsilon_0}\int \frac{\partial}{\partial s_j} P_j(\mathbf{s})\,\frac{1}{|\mathbf{r}-\mathbf{s}|}\,d^3s$$

The surface term has been discarded assuming that the polarization density exists only in a finite volume. From the last formula, we recognize the corresponding charge density as

$$\boxed{\rho_b = -\,\boldsymbol{\nabla}\cdot\mathbf{P}} \tag{48}$$

A heuristic understanding of this formula is provided by **Figure 3**, which shows the polarization density being along say x and increasing with x, which therefore leads to a negative charge density.

Electric displacement

Now put (48) into (46) and move this term to the other side:

$$\boldsymbol{\nabla}\cdot(\epsilon_0\mathbf{E}+\mathbf{P}) \quad = \quad \rho_f \tag{49}$$

We are thus led to define the *electric displacement*

$$\boxed{\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}} \tag{50}$$

so that

$$\boxed{\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_f} \tag{51}$$

looking very much like the original form of Gauss' law.

4.3 Constitutive equation

Introducing the new vector $\mathbf{D}$ does not by itself help to solve these problems. If we regard $\mathbf{P}$ as something not directly controlled, we need a constitutive equation to express it in terms of variables that we *do* control. As an approximate and empirical law, the dipole moment is caused by and therefore proportional to the imposed electric field:

$$p_i \quad = \quad \alpha E_i$$

where α is called the *polarizability*. Thus the polarization density is

$$\mathbf{P} \quad = \quad \epsilon_0\chi\mathbf{E} \tag{52}$$

Where $\chi = N\alpha/\epsilon_0$ and N is the number density of these particles. We believe Maxwell's equations to be exact, but (52) is empirical and approximate; for example, for strong fields there will be a nonlinear response — the basis of the whole field of *nonlinear optics*.

In any event, this gives

$$\mathbf{D} = \epsilon_0(1+\chi)\mathbf{E} \quad \equiv \quad \epsilon\mathbf{E} \tag{53}$$

and (51) can be written as

$$\boldsymbol{\nabla}\cdot(\epsilon\mathbf{E}) \quad = \quad \rho_f \tag{54}$$

which compared to (12) would suggest that all we have to do is replace

$$\epsilon_0 \quad \mapsto \quad \epsilon \tag{55}$$

It is therefore tempting (and a common mistake[10]) to say (a) $\mathbf{D}$ is the field caused by free charges, and (b) the evaluation of the field is the same as in vacuum, except for the replacement (55). Unfortunately, this is *not* true in general, since

$$\boldsymbol{\nabla} \times \mathbf{D} \quad = \quad \epsilon \, \boldsymbol{\nabla} \times \mathbf{E} + (\boldsymbol{\nabla} \epsilon) \times \mathbf{E} \qquad (56)$$

The first term vanishes in electrostatics, but the second term vanishes *only* if the medium is uniform.

In summary, for *uniform* media much of the formalism from vacuum survives, with only factors of

$$n^2 = \kappa \quad = \quad \frac{\epsilon}{\epsilon_o} \qquad (57)$$

appearing. The dimensionless parameter κ is called the *dielectric constant* and (for optical frequencies) n is the refractive index. (There is some subtlety in the above statement. Students are invited to explain why the dielectric constant of water is measured to be $\kappa \sim 80$, while its refractive index is about $n \sim 1.33$.)

The above is only a sketch for the simplest situations. A more complete discussion, which will not be attempted here, must include the following.

- Extend to time-dependent situations. For example, Ampere's law becomes (say in the absence of magnetic materials)

$$\boldsymbol{\nabla} \times \mathbf{B} \quad = \quad \mu_0 \mathbf{J}_f + \mu_0 \frac{\partial \mathbf{D}}{\partial t} \qquad (58)$$

- Extend to magnetic materials and introduce the field $\mathbf{H}$.

Much of this should be familiar from a course of intermediate EM. We just want to emphasize several conceptual points.

- These formulas make EM in material media deceptively similar to EM in vacuum. But they are based on approximate and empirical constitutive equations, and are not at the same fundamental level as the equations in vacuum.
- Fields such as $\mathbf{D}$ and $\mathbf{H}$ should not be assigned naive physical meanings. For example, $\mathbf{D}$ is the linear combination of two quantities ($\mathbf{E}$ and $\mathbf{P}$) that have quite different physical meanings, and it should not be regarded as the field due to free charges alone.
- As a consequence, one should not naively replace say $\epsilon_0 \mathbf{E} \mapsto \mathbf{D}$ and/or $\mu_0^{-1} \mathbf{B} \mapsto \mathbf{H}$ in various vacuum formulas and then claim that the result applies for a linear material medium. This was essentially the errors behind the so-called Minkowski and Abraham formulations for the momentum of light; see references cited in [10, 11, 13].

[10] For example, *Encyclopedia Britannica* [12] says "Electric displacement ... represents that aspect of an electric field associated solely with the presence of separated free electric charges, purposely excluding the contribution of any electric charges bound together in neutral atoms or molecules."

4.4 Exclusion of self field

There is another subtle point, which does not have an entirely satisfactory solution in classical physics. The electric force acting on a "point" charge q should be related to the electric field $\mathbf{E}$ at its position, which is the sum of the field due to other charges and the field due to itself. The latter, which leads to what we may call the *self force*, has an infinite magnitude and indefinite direction if the charge is regarded as a point.

A partial solution is to regard even a "point" charge (such as an electron) as a little spherical distribution, and apply the laws of EM to each infinitesimal portion — whose self force is likewise infinitesimal (in fact quadratically so) and therefore negligible. The force due to the other parts of the charge distribution must, by symmetry and also by Newton's third law, lead to zero net force on the charge as a whole.

This line of argument is however not entirely satisfactory.

- Although there is no net force, there are forces due to one part of the distribution on another, in other words, internal stresses. Unless there are other internal forces to balance these electromagnetic stresses, the charge distribution would collapse or blow apart.
- One might be tempted to simply postulate the charge distribution as a rigid body. But rigid bodies contradict special relativity. To understand this point, imagine suddenly pushing one end of such a body, say of length L. If the body is rigid, then the other end must start to move immediately. But by special relativity, no signal or effect can travel faster than c, so the other end cannot "know" about the push until a time L/c later. The two perspectives are contradictory.
- The above argument is formulated for a static distribution. By going to another inertial frame, the same result (i.e., the exclusion of self force) must also apply to a uniformly moving charge q. But an accelerating charge is another story, and leads to difficulties.

These difficulties can only be resolved in the quantum domain, and even there, the resolution is by no means simple. The problem of self force gets translated into the problem of self energy, which has a serious divergence problem, solved only by renormalization — which in turn relies on concepts

of gauge invariance that we shall develop towards the end of the course.

So the rest of this course will exclude the self force whenever it is necessary.

4.5 Force law

Not the average field

Maxwell's equations, which are linear, retain the same form upon spatial and temporal averaging. But the Lorentz force law is not linear. Consider a charge distribution, take just the electric term and perform an average, denoted by $\langle \ldots \rangle$. The force density is

$$\langle \rho\, \mathbf{E} \rangle \quad \neq \quad \langle \rho \rangle \, \langle \mathbf{E} \rangle$$

We can put this problem in physical terms. Take a point charge or a polarizable particle X which is a part of the charge distribution. It does not experience the average field, but the field at the *specific* position that it occupies, which is in general not "typical". For example, consider a liquid with X being one molecule. Because we exclude the self force, we want to evaluate the field due to all the *other* charges. But the position of X is a very special one — namely one at which there are no other molecules. In fact, the other molecules can be approximated as uniformly distributed *except* for a spherical cavity of some radius R around X. Accounting for the difference between the average value of $\mathbf{E}$ and the value at the specific position of X (namely at the center of the cavity) leads to the Clausius–Mossotti equation for the dependence of the dielectric constant on the number density:

$$\frac{\kappa - 1}{\kappa + 2} \quad \propto \quad N \tag{59}$$

This is the simplest example of a situation where one cannot simply replace by the average field. We here only emphasize the concepts; the detailed calculations should be familiar from an undergraduate course in intermediate EM.

Net force and internal stress

Nevertheless, in many circumstances the above complication can be ignored. To appreciate this point, divide $\mathbf{E}$ at any point into two components: (a) the field due to external sources (e.g., the charges on a capacitor plate several cm away), and (b) the field due to charges in the medium (e.g., the other molecules in a piece of plastic).

- The former, due to sources far away (compared to the intermolecular distance), has a smooth spatial dependence. It can be freely averaged without changing anything.
- The latter is an internal force. For the net force on the system as a whole (e.g., the piece of plastic), the internal forces must add up to zero by Newton's third law. So averaging it also does not matter.

But what if we go beyond the net force? If we ask about the internal stress, e.g., whether there is an increased pressure, then the complexities do matter, and the result depends on the distribution of other molecules. The internal stress in a dielectric system, e.g., a liquid, is a subject of considerable subtlety, fully resolved only in the last few decades [10, 11, 13].

A Evaluation of an integral

Here we consider the integral

$$I \quad = \quad \int_0^\infty \frac{\sin kr}{k}\, dk = \frac{1}{2} \int_{-\infty}^\infty \frac{\sin kr}{k}\, dk$$

(Incidentally, one can change variable to $\xi = kr$, which immediately shows that I does not depend on r. But we keep the variable of integration as k, for similarity with other problems that will be encountered later in the course.) The integrand is analytic at $k = 0$, so we can distort the contour to go slightly above the origin (**Figure 2**). Next, write the integral as

$$\begin{aligned} I \quad &= \quad \frac{1}{4i} \int_{-\infty}^\infty \frac{e^{ikr} - e^{-ikr}}{k}\, dk \\ &= \quad \frac{1}{4i} \left(I_1 - I_2 \right) \end{aligned}$$

where

$$I_{1,2} \quad = \quad \int_{-\infty}^\infty \frac{e^{\pm ikr}}{k}\, dk \tag{60}$$

Since I_1 and I_2 are not individually analytic at $k = 0$, we are not free to further distort the contour around the origin.

For I_1, close the contour in the upper half plane, enclosing no pole and giving $I_1 = 0$. For I_2, close the contour in the lower half plane, enclosing a pole at the origin with residue 1, and hence getting

$$I_2 \quad = \quad -\, 2\pi i$$

where the minus sign appears because the contour is traversed in a negative sense. This then gives

$$I \quad = \quad \frac{1}{4i} \cdot (2\pi i) = \frac{\pi}{2}$$

B Supplement: Green's function in arbitrary dimensions

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

Consider the Green's function for Poisson's equation in n dimensions

$$-\nabla^2 G(\mathbf{r}) \quad = \quad \delta^n(\mathbf{r}) \tag{61}$$

On dimensional grounds, we must have

$$G(\mathbf{r}) \quad = \quad C(n)\, r^{2-n} \tag{62}$$

Find the numerical coefficient $C(n)$. Also, extend the result to non-integer n (whatever that means), and to the case $n \to 2$. This problem also illustrates the heuristic idea that

$$\ln r \quad \sim \quad \frac{r^0}{0}$$

C Supplement: Oblateness of the sun

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

Consider a body of fluid that is approximately spherical because of its own gravity. If the fluid is rotating at an angular velocity ω, can you estimate the oblateness, in terms of the parameters introduced in this Chapter for discussing the (gravitational) quadrupole moment? Give numerical estimates for the sun and the earth (at one time molten).

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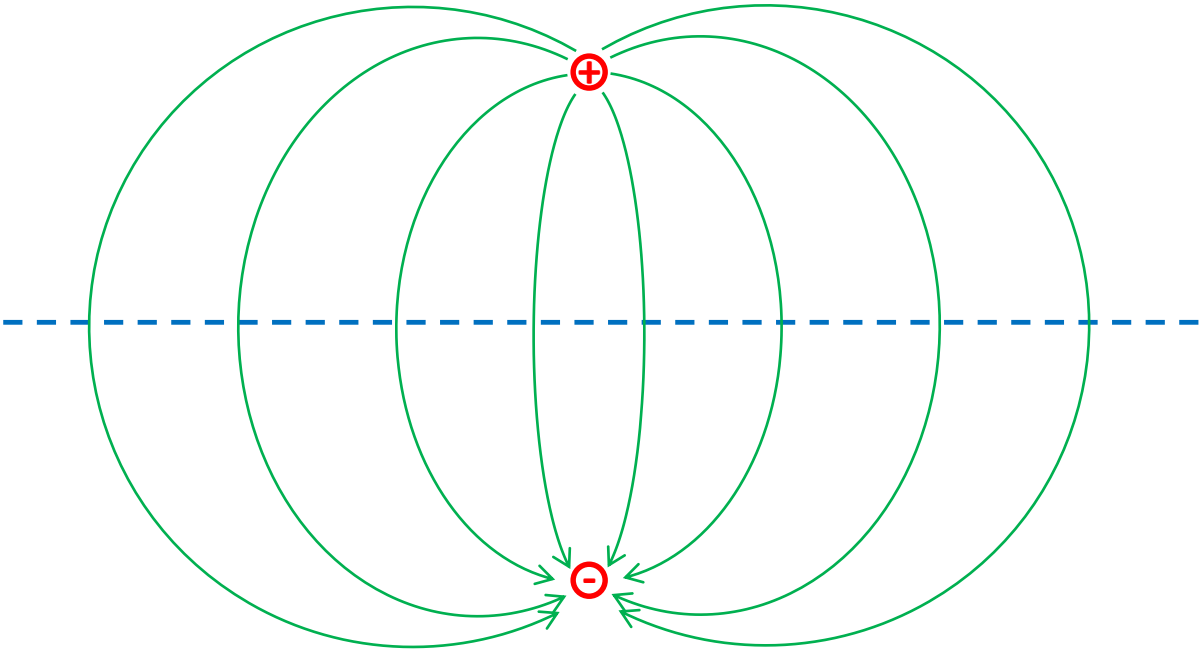

Figure 1

Field pattern for an electric dipole

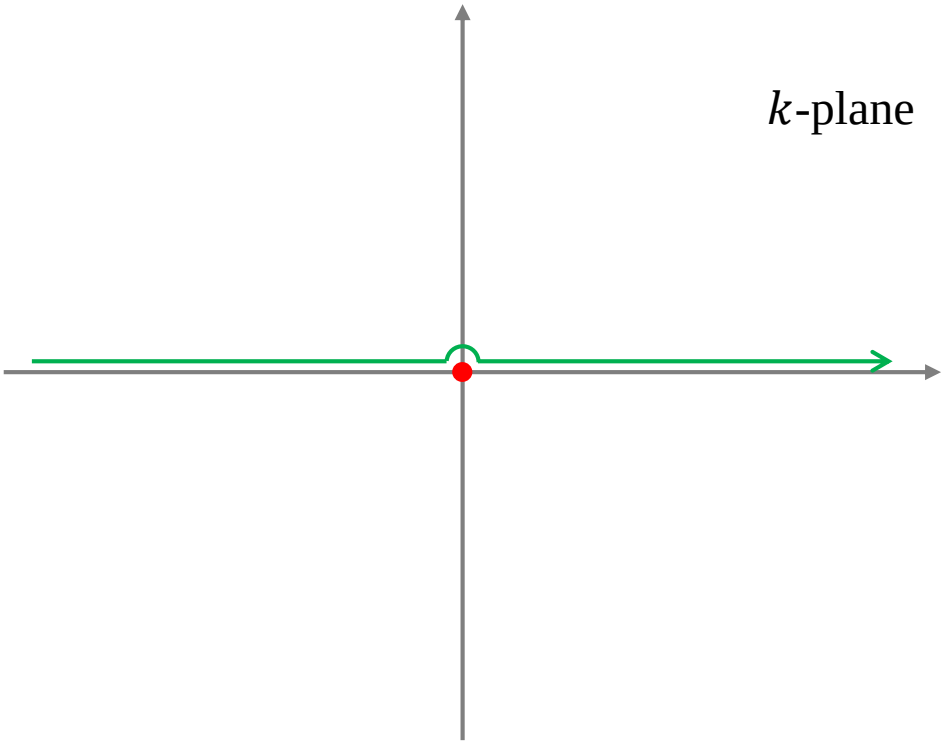

Figure 2

Contour in k-plane

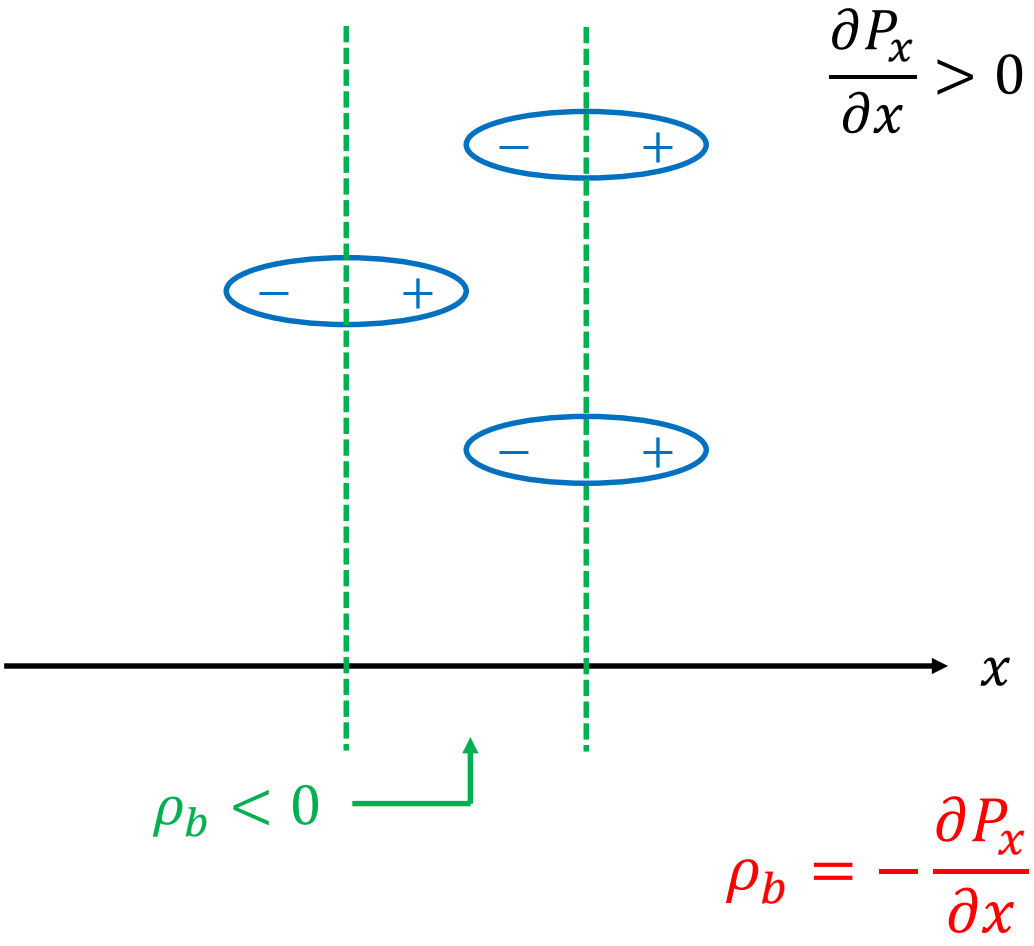

Figure 3

Polarization density along x