Chapter 6
Time-dependent electrodynamics

June 12, 2021

The generalization of the laws of EM to time-dependent situations is reviewed.

Contents

1 Introduction

1.1 Static equations

Our earlier account for time-independent phenomena had ended up with two *separate* theories of electrostatics and magnetostatics, governed by the following pairs of laws:

Electrostatics

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \times \mathbf{E} &= 0
\end{aligned}
\tag{1}
$$

Magnetostatics

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
\tag{2}
$$

1.2 Time dependence

The present Chapter explains how these equations are modified in time-dependent situations, i.e., *electrodynamics.*

- Since a current is just a collection of moving charges, it is obvious that electricity and magnetism will be intimately related; we henceforth talk about electromagnetic (EM) phenomena in a single theory.
- It turns out (1) and (2) do not represent the best way to split up the four equations into two pairs; it is more fruitful to split them into two homogeneous equations and two inhomogeneous equations.

Homogeneous equations
The pair

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\
\boldsymbol{\nabla} \times \mathbf{E} &= 0
\end{aligned}
\tag{3}
$$

are homogeneous: they do not depend on the sources. Their generalization is discussed in Section 2. In the covariant formalism (to be presented later), these $1 + 3 = 4$ equations, when suitably generalized, correspond to one single 4-vector equation.

Inhomogeneous equations
The pair

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_0 \mathbf{J}
\end{aligned}
\tag{4}
$$

are inhomogeneous: the sources appear on the RHS. Their generalization is discussed in Sections 3 and 4. These $1 + 3 = 4$ equations, when suitably generalized, again correspond to one single 4-vector equation.

2 Induction

A changing magnetic field will cause an electric field. This conclusion is in fact dictated by the need for covariance, as illustrated by the following thought experiments.

2.1 Thought experiments

First experiment
Imagine permanent magnets causing a magnetic field $\mathbf{B}$ in some region (say vertically downwards on the page) and a charge q passing this region at some speed $\mathbf{v}$ (say horizontally across the page); see **Figure 1a**. The magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ causes the charge to accelerate perpendicular to the page.

Now view the situation in an inertial reference frame moving at the velocity $\mathbf{v}$. The charge is initially at rest, and when the magnetic region passes, it accelerates perpendicular to the page; see **Figure 1b** — after all, it is the same experiment as before. But if the charge is originally at rest, the force can only be electric. So we conclude that when the magnetic region passes, i.e., when at a fixed point the magnetic field changes, somehow an electric field must be induced.

Second experiment
A circuit consists of conducting rails together with a moveable conducting bar; a current I flows in this circuit (**Figure 2**). Suppose there is a uniform magnetic field B inside the loop enclosed by the conductors, perpendicular to the page. Imagine that the bar is moved in the direction shown with velocity v. A charge carrier q in the conducting bar will experience a force qvB along the bar, which will do work $qvB\ell$ when the charge moves from one end of the bar to the other, where ℓ is the length of the bar. There is no external work done in the other parts of the circuit. Hence in one complete circuit, the work done is $qvB\ell$, and the EMF, namely the work done for the whole loop per unit charge, is

$$\mathcal{E} = vB\ell$$

On the other hand the flux in the area enclosed by the loop is

$$\varphi_B = Bx\ell$$

where x is the dimension shown in **Figure 2**. As far as magnitudes are concerned we see that

$$\frac{d\varphi_B}{dt} = B\frac{dx}{dt}\ell = Bv\ell$$

which is the same as $\mathcal{E}$.

Take the sign of $\mathcal{E}$ and the sign of the flux to be related by the usual right-hand rule; then a decreasing flux causes a positive EMF, or

$$\boxed{\mathcal{E} = -\frac{d\varphi_B}{dt}} \tag{5}$$

In this case there is no electric field, but still an EMF.

2.2 Faraday's law

Experiments by Faraday led to the conclusion that (5) is generally correct, not just for the situation described by the above thought experiments (where $\mathbf{B}$ is not changing but the loop is changing), but also for the more common case of a fixed loop with a changing $\mathbf{B}$.

If the loop is not moving, then the EMF can only come from work done by an electric field:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell}$$

whereas the magnetic flux is given by

$$\varphi_B = \int \mathbf{B} \cdot d\mathbf{S}$$

Putting all these together, and denoting the loop as Γ and the corresponding area as S, we get Faraday's law in integral form:

$$\boxed{\oint_\Gamma \mathbf{E} \cdot d\ell = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \tag{6}$$

Convert the line integral on the LHS into a surface integral by Stokes' theorem:

$$\oint_\Gamma \mathbf{E} \cdot d\ell = \int_S (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} \tag{7}$$

When this is combined with (6), we find two surface integrals which are always equal; hence the integrands must be equal:

$$\boxed{\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{8}$$

Since the quantities on both sides are functions of both time and space, and the time derivative is performed at a fixed spatial position, it is indicated as a partial derivative. The above is then the differential form of Faraday's law. It gives precise expression to the idea that a changing magnetic field gives rise to some non-zero electric field — in particular, an electric field with a non-zero curl.

2.3 Consistency condition

Integral formalism
The condition (6) relates a line integral over a closed loop Γ to an integral over a surface S bounded by Γ. All such conditions have to be checked for consistency:

> The same Γ is the boundary to different surfaces, so the RHS must be the same when evaluated for two such surfaces sharing the same boundary.

For example, **Figure 3a** shows the equator of a sphere as the loop Γ, the equatorial plane as the surface S_1, and say the southern hemisphere as the surface S_2. The positive senses are as shown. Schematically we need to check

$$\mathrm{RHS}(S_1) \quad = \quad \mathrm{RHS}(S_2)$$

or equivalently, considering the closed surface S formed by the union of S_1 and S_2,

$$\mathrm{RHS}(S) \quad = \quad 0$$

where now the sign convention is indicated in **Figure 3b**. Thus we need to check

$$\frac{d}{dt} \oint_S \mathbf{B} \cdot d\mathbf{S} \quad = \quad 0 \tag{9}$$

which is of course correct, since, without the time derivative, this is just one of the laws of magnetism, stating that there are no magnetic monopoles.[1]

An equivalent way of arriving at the same consistency condition is to apply (6) to a tiny loop Γ, e.g., the small opening to a large "bag" regarded as the surface S (**Figure 4**). As the opening is shrunk to zero (and assuming that $\mathbf{E}$ is non-singular), the line integral must vanish. The RHS becomes the integral over a closed surface, which must likewise be zero.

Differential formalism
Consider (8) and take the divergence. The LHS must be zero, so the consistency condition is that the RHS must also have zero divergence, i.e.,

$$\boldsymbol{\nabla} \cdot \frac{\partial \mathbf{B}}{\partial t} = \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \mathbf{B}) \quad = \quad 0 \tag{10}$$

This derivation using the differential form is mathematically simpler, but the integral formalism provides a more vivid physical picture.

[1] Incidentally, this means that if there are monopoles, Faraday's law cannot be correct as it stands.

2.4 Lenz's law

Lenz's law is just a physical way of stating the minus sign in (6) or (8). Consider a loop Γ such as that in **Figure 5** and assume that the enclosed magnetic flux (say out of the page) increases. Applying (6) with attention to the right-hand rule convention, we see that the EMF must be in the clockwise sense, and likewise any resultant current — which would (through Ampere's law, and again applying the right-hand rule) produce a magnetic field that would point downwards inside the loop.

Lenz's law states:

> Any magnetic field produced by the induced current (through the EMF) would *oppose* the original change of flux.

If the sign on the RHS of (6) or (8) had been positive, the induced magnetic field would be in the same direction as the original change — and thus would magnify it, leading to an instability, which is impossible.

2.5 Homogeneous equations

Thus the amended homogeneous equations are coupled:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{11}$$

and (3) is just the special case for static situations.

3 Charge conservation

We next come to the inhomogeneous equations (4); on the RHS are respectively ρ and $\mathbf{J}$, which are related in two ways.

- If we go to a moving frame, stationary charge becomes moving charge, and charge density ρ leads to current $\mathbf{J}$.
- Charge is conserved, which means the ρ and $\mathbf{J}$ are related.

Either of these considerations suggests that the two equations — Gauss' law and Ampere's law — must somehow be related through some additional elements that couple the two equations. The covariant point of view will be presented later, but historically the connection between Gauss' law and (the properly modified form of) Ampere's law was discovered through an analysis of their consistency with charge conservation.

3.1 Integral formalism

A closed surface S enclosing a volume V is shown in **Figure 6**. The amount of charge flowing out of an element of area $d\mathbf{S}$ in a unit time is $\mathbf{J} \cdot d\mathbf{S}$, so the total rate of outflow is

$$\oint_S \mathbf{J} \cdot d\mathbf{S}$$

Since charge is conserved, this must lead to a rate of decrease of the total charge in V, i.e.,

$$-\frac{dq}{dt} = -\frac{d}{dt}\int_V \rho\, dV$$

where $dV = d^3r$ represents an element of volume. Equating these gives the law of conservation of charge:

$$\boxed{\frac{d}{dt}\int_V \rho\, dV + \oint_S \mathbf{J} \cdot d\mathbf{S} = 0} \qquad (12)$$

3.2 Differential formalism

Use Gauss' theorem to convert the second term in (12) to a volume integral:

$$\oint_S \mathbf{J} \cdot d\mathbf{S} \;=\; \int_V \boldsymbol{\nabla} \cdot \mathbf{J}\, dV$$

Put this back into (12) and combine into one single volume integral; since the integral must be zero for any volume V, the integrand must vanish, giving

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0} \qquad (13)$$

4 Displacement current

Maxwell realized that the inhomogeneous equations as written in (4) are inconsistent with charge conservation. A physical way to understand this is through a thought experiment.

4.1 Thought experiment

A parallel-plate capacitor is being charged, with current I flowing into the lower plate and out of the upper plate (**Figure 7**). Take a closed loop Γ to be a circle in the plane mid-way between the two plates, and two surfaces both with Γ as the boundary: S_1 is the circular disk in the same plane ("the equatorial plane") and S_2 is a hemisphere that envelopes the lower plate ("the southern hemisphere"). From Ampere's law,

$$\oint_\Gamma \mathbf{B} \cdot d\ell \;=\; \mu_0 \int_{S_i} \mathbf{J} \cdot d\mathbf{S} = \mu_0 I_i \qquad (14)$$

where the RHS can be evaluated for either S_1 or S_2, and the results I_1, I_2 are the currents through S_1, S_2. But $I_1 = 0$, $I_2 = I$. So this equation is inconsistent. It will be recalled that a similar consistency condition was also checked in the case of Faraday's law, but in that case there was no problem.

4.2 Extra term

We guess that we must add an extra term to the RHS of Ampere's law, schematically

$$I \;\mapsto\; I + I_d \qquad (15)$$

where I_d, with the dimensions of current but not really a current, is something that exists in the gap between the two plates, with the property that the sum $I + I_d$ has the same value for the two surfaces — that the sum is continuous. The term I_d is called the *displacement current*, though it has to be emphasized that it is not a current.

What is there in the gap? There is only an electric field, in fact an *increasing* electric field because the plate is being charged. Assume

$$I_d \;=\; k \int_{S_i} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S} \qquad (16)$$

which is of course zero for S_2 outside the capacitor. Can we choose the constant k such that I_d evaluated for S_1 is equal to I?

Assume the gap between the capacitor plates is small. Then the field is uniform, and the RHS of (16) is readily evaluated, giving

$$I_d \;=\; kA\frac{\partial E}{\partial t} \qquad (17)$$

where A is the area of each plate. But in a parallel-plate capacitor,

$$E \;=\; \frac{\sigma}{\epsilon_0} = \frac{q/A}{\epsilon_0}$$

where σ is the charge per unit area and q is the charge on that plate. When this is put into (17), the factor A cancels, and we find

$$I_d \;=\; \frac{k}{\epsilon_0}\frac{dq}{dt} = \frac{k}{\epsilon_0} I \qquad (18)$$

since the charge on the plate increases at the rate I. For the expression (18) to equal I, we should choose $k = \epsilon_0$, and

$$\boxed{I_d = \epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}} \qquad (19)$$

through any surface S.

Adding this to the RHS of Ampere's law, we get

$$\oint_\Gamma \mathbf{B}\cdot d\boldsymbol{\ell} = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{S} + \mu_0\epsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S} \qquad (20)$$

4.3 Differential form

As usual, use Stokes' theorem to convert the LHS of (20) to the surface integral of $\boldsymbol{\nabla}\times\mathbf{B}$. Then, peeling off the integral sign gives the differential form of the modified Ampere's law.

$$\boxed{\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \qquad (21)$$

It is now easy to check consistency with the conservation of charge. Take the divergence of (21) and drop an overall factor of μ_0: we get

$$\begin{aligned} 0 &= \boldsymbol{\nabla}\cdot\mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{E}) \\ &= \boldsymbol{\nabla}\cdot\mathbf{J} + \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) \\ &= \boldsymbol{\nabla}\cdot\mathbf{J} + \frac{\partial \rho}{\partial t} \end{aligned} \qquad (22)$$

where in going to the second line we have used Gauss' law. The final result is consistent because of the conservation of charge.

In other words, the RHS of (21) has zero divergence because of the conservation of charge, and that allows it to be equal to the curl on the LHS. This derivation is in fact equivalent to the integral version,

4.4 Orders of magnitude

In MKSA units

$$\mu_0 = 4\pi\times 10^{-7} \qquad , \qquad \frac{1}{4\pi\epsilon_0} = 9\times 10^9$$

giving

$$\mu_0\epsilon_0 \approx 10^{-17}$$

making the new term insignificant in most cases where the typical length and time scales are not very small in units of metres and seconds. This term becomes important when the fields are changing rapidly, i.e., for high frequencies, as we shall see below.

Problem 1
What is the MKSA unit for the product $\mu_0\epsilon_0$? Determine the precise value (with units) of $(\mu_0\epsilon_0)^{-1/2}$. §

5 Maxwell's equations

5.1 Summary of the equations

To summarize, we now have the four Maxwell equations in two pairs.

Homogeneous equations

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \qquad (23)$$

Inhomogeneous equations

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \boldsymbol{\nabla}\times\mathbf{B} &= \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t} \end{aligned} \qquad (24)$$

Notice the symmetry: changing $\mathbf{B}$ causes $\mathbf{E}$ (Faraday's law), and changing $\mathbf{E}$ causes $\mathbf{B}$ (the new displacement current term).

The rest of this course studies the consequences of these equations together with the Lorentz force law, which constitute the entirety of classical electrodynamics.

5.2 Symmetries

Several symmetries embedded in these equations (and also the Lorentz force law) will be further discussed later in the course.

Lorentz invariance
Historically, Maxwell's equations (and more particularly the behavior of light that is implied and which can also be regarded directly as an observable fact) led to Lorentz invariance. But the modern view is that Lorentz invariance is more general, and logically prior.

But whichever you regard as primary, Maxwell's equations and Lorentz invariance are consistent. It is not obvious from the way the equations are written above, and one of the themes of this course is to make this property obvious.

Gauge invariance
Classical EM depends only on $\mathbf{E}$ and $\mathbf{B}$, so that $\mathbf{A}$ can be changed by $\mathbf{A}\mapsto\mathbf{A}+\boldsymbol{\nabla}\Lambda$. Later we shall learn that this idea should be extended to 4D spacetime, and the scalar potential Φ (which is the time component corresponding to $\mathbf{A}$) can also be transformed. This idea, called gauge invariance, turns out to be both more subtle and more important than might first appear. This will be another theme of the course.

Duality
Let us define

$$\mu_0\epsilon_0 = c^{-2}$$

and also

$$\mathbf{E}' = \mathbf{E}/c$$

These are nothing but changes of notation.

Then in vacuum ($\rho = 0$, $\mathbf{J} = 0$), Maxwell's equations reduce to

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{B} &= 0 \\ \boldsymbol{\nabla}\times\mathbf{E}' &= -\frac{\partial \mathbf{B}}{\partial(ct)} \end{aligned} \tag{25}$$

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E}' &= 0 \\ \boldsymbol{\nabla}\times\mathbf{B} &= +\frac{\partial \mathbf{E}}{\partial(ct)} \end{aligned} \tag{26}$$

which shows an intriguing symmetry between $\mathbf{B}$ and $\mathbf{E}'$. We shall come back to this later — and also ask whether this symmetry can extend to the inhomogeneous equations, which will lead to the idea of magnetic monopoles.

A Supplement: Work done by magnetic field?

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

This Supplement deals with an apparent paradox about work done by the magnetic field. The discussion is within the context of electromagnetism being due to structureless point charges, i.e., Model 1 in the Supplements to Chapters 4 and 5.

Work done on moving bar

Consider the experiment illustrated in **Figure 2**, and for convenience assume the charge carriers are positive, with charges q moving in the direction of the current shown. Let the x-axis point to the right, and the y-axis point upwards on the page.

The current is along $+y$ and the magnetic field is along $+z$, so there is a force on the bar, along $+x$ and with magnitude

$$F_x = I\ell B \tag{27}$$

Since the bar is moving along $+x$, there is work done at a rate of

$$P = F_x\, v = I\ell B v \tag{28}$$

This represents work done by the magnetic force, which seems to contradict the theorem that, within Model 1, the magnetic field does no work.

Analyze in terms of charge carriers

The charge carriers have velocity

$$\mathbf{V} = u\,\hat{\mathbf{e}}_y + v\,\hat{\mathbf{e}}_x \tag{29}$$

where the first term represents the (drift) velocity of the charge carriers along the bar, responsible for the current I.[2]

Now, given the magnetic field $\mathbf{B} = B\,\hat{e}_z$, the force acting on each particle is

$$\begin{aligned} \mathbf{F} &= q\,\mathbf{V}\times\mathbf{B} \\ &= q\,(u\,\hat{\mathbf{e}}_y + v\,\hat{\mathbf{e}}_x)\times B\,\hat{e}_z \\ &= qB\,(u\,\hat{\mathbf{e}}_x - v\,\hat{\mathbf{e}}_y) \end{aligned} \tag{30}$$

It is obvious that $\mathbf{F}$ is perpendicular to $\mathbf{V}$, and therefore the rate of total work done is $\mathbf{F}\cdot\mathbf{V} = 0$.

Let us consider each component of the force. The rate of work done by the x-component is

$$P_x = F_x\, V_x = (qBu)v \neq 0$$

When this is summed over all the charge carriers, it evidently leads to the expression in (28). But there is also work done by the y-component

$$P_y = F_y\, V_y = (-qBv)u \tag{31}$$

which exactly cancels (31).

In other words, the magnetic field

- does positive work in moving the bar perpendicular to the bar itself; and
- does negative work in moving the charge carrier along the bar (i.e., draw energy from the current flow).

and the sum of the two is zero.

Let us further examine the second component of work done more carefully. There are two further possibilities:

- In some versions of this thought experiment, there is a battery that maintains the current going. The battery supplies the energy that is removed by the term P_y.
- If we consider a version of the thought experiment without batteries, then the negative work will slow down the carriers and cause the current to decrease. If we think of the current loop as a magnetic moment, its moment then decreases.[3]

[2] The precise relationship is obvious, and will not be displayed.

[3] A little care is needed here. The area of the loop increase, but the current decreases, so what can we say about their product? The key point to realize is that the latter effect is proportional to $1/R$, where R is the resistance in the circuit, which we can imagine to be very small.

Now we see that to maintain consistency, the magnitude of the magnetic moment of the loop must change in an appropriate manner. Thus, if we repeat this whole story for a tiny loop which is supposed to represent an *intrinsic* dipole moment with *fixed* magnitude, i.e., we ban the second part of the work done, the argument will no longer carry through. That is why we need Model 2: to allow separately for intrinsic dipole moments.

(a)

(b)

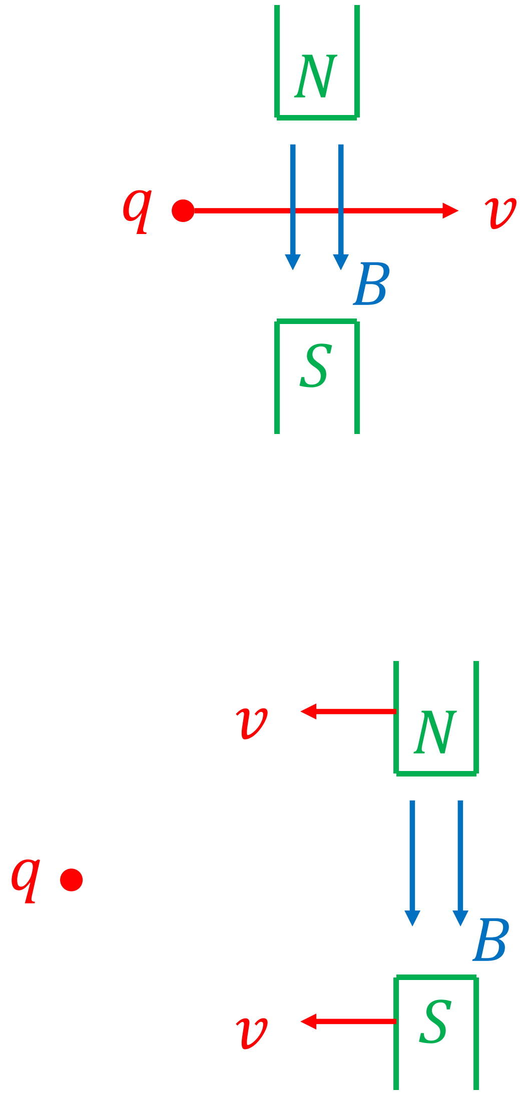


Figure 1 Charge is accelerated perpendicular to the page, as seen in two reference frame
(a) static magnetic field, charge moving
(b) changing magnetic field, charge at rest

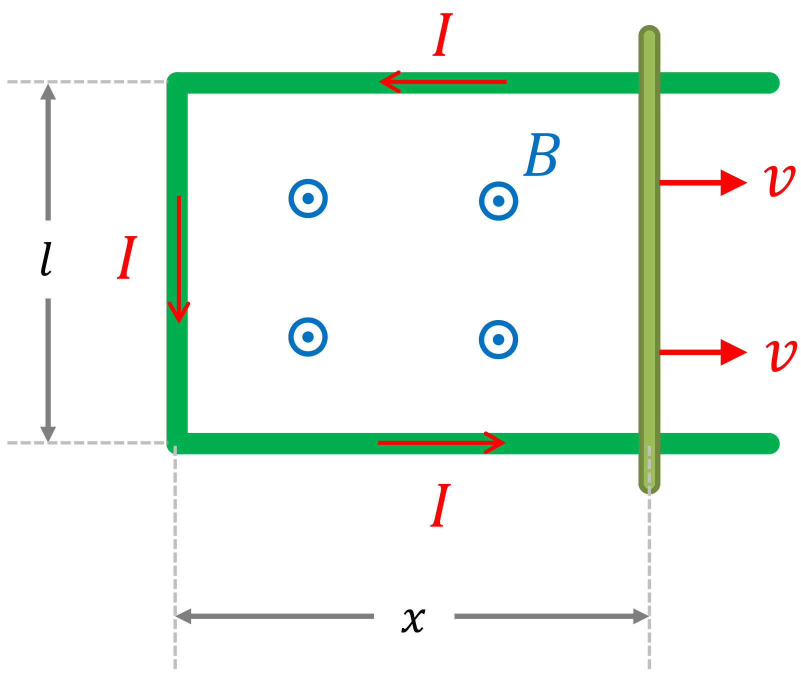
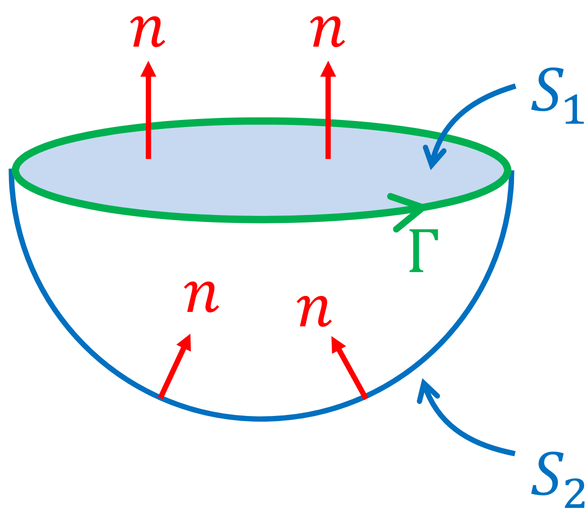


Figure 2 There is an EMF around the loop

(a)



(b)

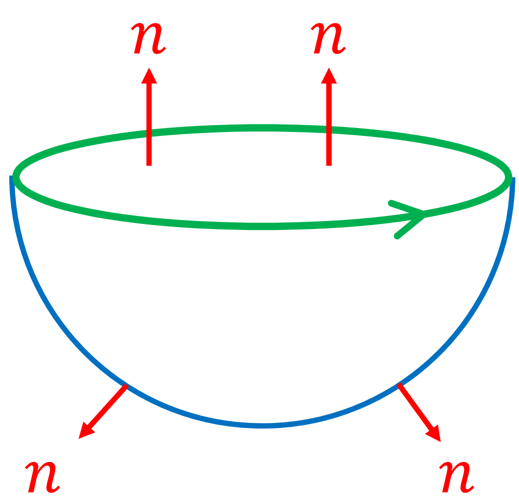


Figure 3 (a) The same loop Γ is the boundary for two surfaces S_1, S_2. The direction of normal follows right hand rule

(b) $S = S_1 + S_2$ is a closed surface. The normal is defined to be outwards

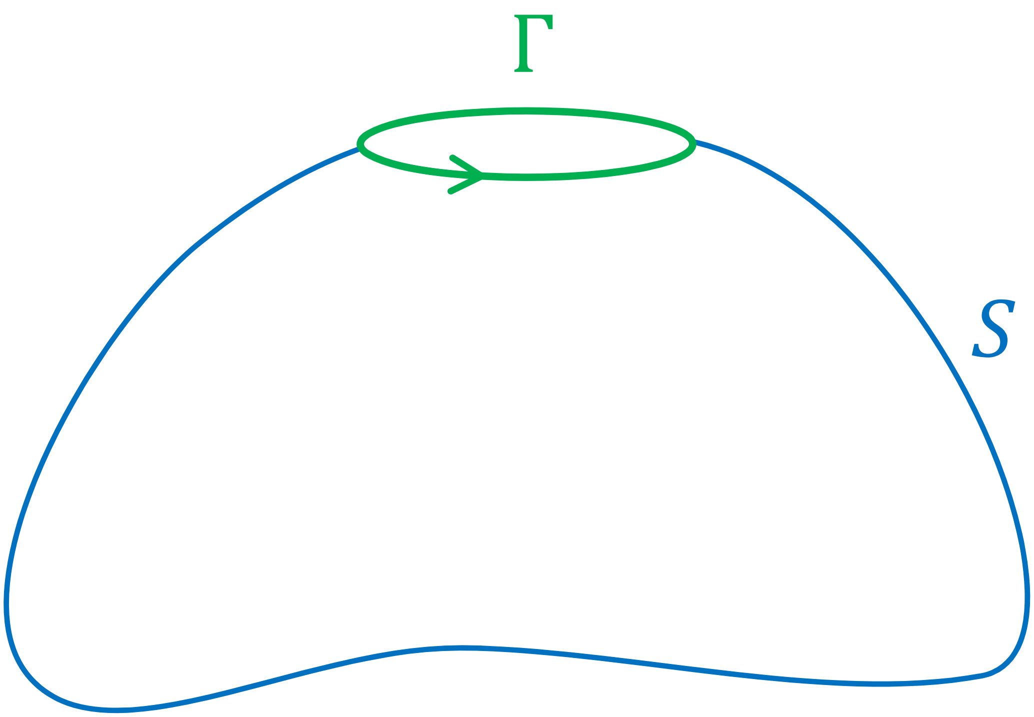


Figure 4 The surface S is a "bag" with a small opening. The edge of the opening is Γ. As Γ $\rightarrow$ 0, the bag becomes a closed surface

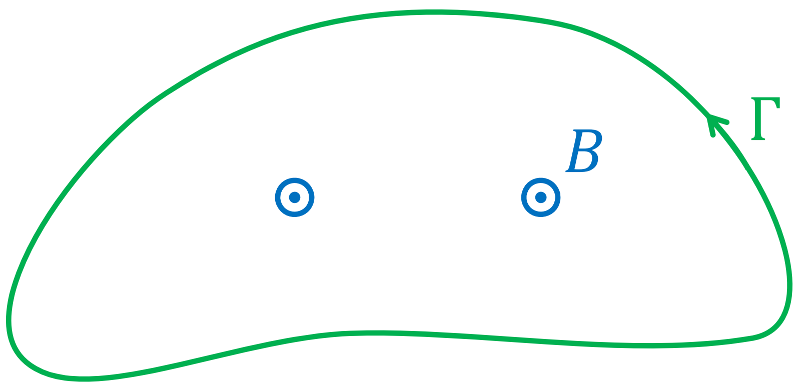


Figure 5 If magnetic field increases, the induced current would be opposite to the sense of Γ

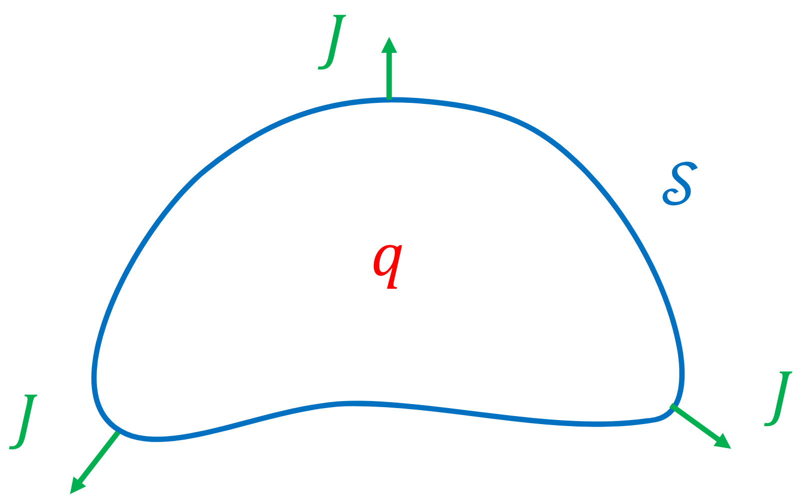


Figure 6 If there is a current J flowing out of the closed surface, then the charge q inside decreases

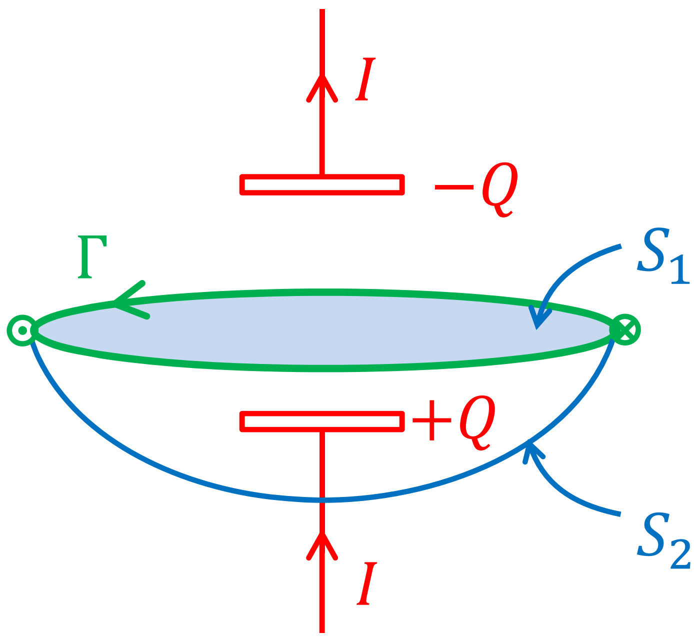


Figure 7 A parallel plate capacitor being charged. The two surfaces S_1, S_2 share the same boundary Γ. There is a current through S_2 but not through S_1