Chapter 10
Radiation by a harmonic source: some applications

June 21, 2021

The radiation formulas derived in the last Chapter are applied to a variety of physical problems: antennas, atoms, and the analogous problem of gravitational quadrupole radiation.

Contents

1 Antennas

1.1 Short dipole antenna

In a short, center-fed[1] linear antenna (**Figure 1**) of length $2h$, the current density is (apart from the usual time-varying factor)

$$\mathbf{J}(\mathbf{r}) \quad = \quad I f(z)\, \delta^2(\mathbf{r}_\perp)\, \hat{\mathbf{e}}_z, \quad \text{for } z > 0 \qquad (1)$$

and antisymmetric for $z < 0$. The prefactor I is the current fed into the antenna at the center, so f has to interpolate between $f(0) = 1$ and $f(h) = 0$ (since no current can flow beyond the tip of the antenna). For a *short* antenna, i.e., $h/\lambda \ll 1$, linear interpolation is a good approximation:

$$f(z) \quad = \quad 1 - \frac{z}{h}$$

Then, for the upper half

$$i\omega\rho \quad = \quad \boldsymbol{\nabla} \cdot \mathbf{J} = I\, f'\, \delta^2(\mathbf{r}_\perp) = -\,\frac{I}{h}\delta^2(\mathbf{r}_\perp)$$

from which

$$p \quad = \quad \int z\rho\, d^3r = i\frac{Ih}{2\omega}$$

which is doubled when the lower half is included (opposite J_z and opposite z-variation). Thus the power radiated is

$$P \quad = \quad \frac{1}{3}\frac{1}{4\pi\epsilon_0}\frac{\omega^4}{c^3}\left(\frac{Ih}{\omega}\right)^2 \equiv \frac{1}{2}I^2R \qquad (2)$$

where R is called the *radiation resistance* of the antenna. From (2),

$$\begin{aligned} R \quad &= \quad R_0 \cdot \left(\frac{\omega h}{c}\right)^2 \\ R_0 \quad &= \quad \frac{2}{3}\frac{1}{4\pi\epsilon_0 c} \qquad (3) \end{aligned}$$

Problem 1
Evaluate the universal constant R_0 in ohms. Answer: $R_0 = 20$ ohms. §

Problem 2
A center-fed linear antenna with half-length $h = 1$ cm is driven at 1 MHz (a typical frequency for AM

[1] The antenna should be fed by a coaxial cable; otherwise the cable will also radiate.

radio). If the current amplitude is 0.1 A, find
(a) the power radiated; and
(b) the electric field amplitude (in V m^{-1}) at a distance of 3 km in the plane perpendicular to the antenna. §

1.2 Short solenoid antenna

A short solenoid with circular cross-section of radius a is wound with N turns (**Figure 2**) and driven with a sinusoidal current $I(t) = I \cos\omega t$. The amplitude of the magnetic moment is

$$\mu = NI\,(\pi a^2)$$

and the radiated power is

$$P = \frac{1}{3}\frac{1}{4\pi\epsilon_0}\frac{\omega^4}{c^5}\,\mu^2$$

If a radiation resistance R is similarly defined, then

$$R = R_0 \cdot (\pi N)^2 \left(\frac{\omega a}{c}\right)^4 \tag{4}$$

The suppression factor is even more serious (except for N^2).

1.3 Antenna array

Start with one small current source $\mathbf{J}_0(\mathbf{s})$, for which the multipole expansion applies. But now imagine N such sources arranged in a straight line, centered at positions $n\mathbf{b}$, $n = 0, \ldots, N{-}1$ (**Figure 3a**). The size of the entire array, Nb, is *not* small compared to λ. Now the current distribution is

$$\begin{aligned}
\mathbf{J}(\mathbf{s}) &= \sum_{n=0}^{N-1} \mathbf{J}_0(\mathbf{s}{-}n\mathbf{b}) \\
\tilde{\mathbf{J}}(\mathbf{k}) &= \sum_{n=0}^{N-1} \int \mathbf{J}_0(\mathbf{s}{-}n\mathbf{b}) \exp(-i\mathbf{k}\cdot\mathbf{s})\, d^3s \\
&= \sum_{n=0}^{N-1} \int \mathbf{J}_0(\mathbf{s}) \exp[-i\mathbf{k}\cdot(\mathbf{s}{+}n\mathbf{b})]\, d^3s \\
&= \tilde{\mathbf{J}}_0(\mathbf{k})\, F
\end{aligned} \tag{5}$$

where, compared to a single source, there is an enhancement factor of

$$\begin{aligned}
F &= \sum_{n=0}^{N-1} \exp(-in\beta) = \frac{1 - e^{-iN\beta}}{1 - e^{-i\beta}} \\
&= \exp[-i(N{-}1)\beta/2] \frac{\sin(N\beta/2)}{\sin(\beta/2)}
\end{aligned}$$

in which

$$\beta = \mathbf{k}\cdot\mathbf{b} = (\omega b/c)\sin\theta \tag{6}$$

and θ is the angle between $\mathbf{k}$ and the plane normal to the line of sources (**Figure 3b**).

The intensity will be enhanced by

$$|F|^2 = \left[\frac{\sin(N\beta/2)}{\sin(\beta/2)}\right]^2 \approx N^2 \left[\frac{\sin(N\beta/2)}{N\beta/2}\right]^2 \tag{7}$$

in the domain where $N\beta/2 = O(1)$ and $N \gg 1$.

In the direction $\theta = 0$, $\beta = 0$, the final square bracket is unity, and $|F|^2 = N^2$ — the N sources add coherently, and the intensity goes up not by a factor of N, but by a factor N^2. The first minimum occurs at

$$\begin{aligned}
N\beta/2 &= \pi \\
\sin\theta &= \frac{1}{N}\frac{\lambda}{b}
\end{aligned} \tag{8}$$

If the array spacing b is chosen to be about λ, the first minimum occurs at a very small angle of

$$\theta = O(1/N)$$

In short, the radiated power is squeezed into a small angular range of size $\sim 1/N$, and in this region the power is a factor $\sim N$ larger than the incoherent sum of N sources.

Problem 3
Take $b = \lambda$ and $N = 20$. Plot $|F|^2$ versus θ. §

Problem 4
Suppose the different sources have a phase difference:

$$\mathbf{J}(\mathbf{s}) = \sum_{n=0}^{N-1} \mathbf{J}_0(\mathbf{s}{-}n\mathbf{b})\, e^{in\alpha} \tag{9}$$

Discuss the difference from the case $\alpha = 0$. §

The calculation is similar to the analysis of diffraction. The main lessons are that (a) sharp angular focus is possible, and (b) the direction of the peak can be steered by controlling the phases.

2 Atoms

2.1 Collapse of classical atom

The classical model of a hydrogen (H) atom consists of an electron of mass m and charge[2] $-e$ circulating around a (nearly) stationary nucleus of charge $+e$. For simplicity in the present analysis, assume the orbit to be always nearly circular, with radius r. The circulating electron constitutes an oscillating dipole, which radiates energy; as a result, the electron motion loses energy and the orbital radius

[2] We use q for an arbitrary charge, and e for the fundamental charge.

must decrease gradually.[3] The question is: if the electron starts at some radius r, how long does it take for it to collapse all the way to the nucleus? It will be found below that this time is short, so the classical model is untenable; indeed the model would be untenable unless the time turns out to be cosmological or larger.

The basic equation of energy balance is

$$\boxed{\frac{d}{dt}\left(-\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}\right) = -\frac{2}{3}\frac{e^2}{4\pi\epsilon_0}\frac{\omega^4 r^2}{c^3}} \tag{10}$$

The bracket on the LHS is the energy of the electron, which the virial theorem states is half the potential energy. On the RHS, we have applied the power formula for E1, with an extra factor of 2 because the circulating electron in the x–y plane can be thought of as two linear dipoles, respectively oscillating in the x and y directions.[4] Some simplification leads to

$$\frac{dr}{dt} = -\frac{4}{3}\frac{\omega^4 r^4}{c^3} = -\frac{4}{3}\left(\frac{\omega r}{c}\right)^4 \cdot c \tag{11}$$

But ω and r are dependent, and the former should be eliminated in terms of the latter. The balance of Coulomb and centrifugal forces gives

$$\begin{aligned}\frac{e^2}{4\pi\epsilon_0 r^2} &= m\omega^2 r \\ \omega^2 &= \frac{e^2}{4\pi\epsilon_0 m}\frac{1}{r^3}\end{aligned}$$

The fact that $\omega^2 \propto r^{-3}$ is essentially Kepler's third law transcribed to the electrostatic case.

Putting this into (11) then gives

$$\boxed{\frac{dr}{dt} = -\frac{4}{3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{m^2c^3r^2}}$$

Integrating from $r = r$ to $r = 0$ gives the collapse time t in terms of the formula

$$\begin{aligned}r^3 &= 4\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{t}{m^2c^3} \\ &= 4\left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 (ct) \\ &= 4r_e^2\,(ct)\end{aligned}$$

where

$$r_e = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8 \times 10^{-15}\,\mathrm{m}$$

is the *classical radius of the electron.*

Then we can write

$$\boxed{ct = \frac{r^3}{4r_e^2}} \tag{12}$$

which conveniently relates ct, r and r_e, in a way that is obviously dimensionally consistent.

Problem 5
Fill in all the missing steps in the above derivation, and also find t if $r = 0.05$ nm (the Bohr radius). Answer: $t \approx 10^{-11}$ s. §

2.2 Bohr model

Next take the problem one step further, and calculate, within the Bohr model, the time $\Delta t(n)$ needed to go from the orbit n to the orbit $n-1$, where n is the principal quantum number. Mixing classical and quantum physics is allowed for $n \gg 1$, by the correspondence principle.

Consistency condition
The variable ω has two meanings: (a) the frequency of motion of the source, and (b) the frequency of the emitted radiation. Within classical physics the two are automatically the same, but we need to check that the two meanings agree within the Bohr model.

First, from the Bohr quantization condition, we know the angular momentum is $n\hbar$, hence

$$m\omega r^2 = n\hbar$$

and we know that the radius scales as n^2:

$$r = n^2 r_0$$

where r_0 is the (first) Bohr radius. Thus

$$\omega = \frac{\hbar}{mr_0^2}\cdot\frac{1}{n^3} \tag{13}$$

On the other hand, the energy of state n is

$$E(n) = -\frac{E_0}{n^2}$$

where E_0 is the binding energy. So the energy of the photon emitted is

$$E_\gamma = E(n) - E(n-1) \approx \frac{2E_0}{n^3}$$

and the frequency of the photon is

$$\omega = \frac{E_\gamma}{\hbar} = \frac{2E_0}{\hbar}\cdot\frac{1}{n^3} \tag{14}$$

So we have two expressions for ω: (13) coming from the motion of the source and (14) coming from

[3] In the sense that the change per cycle is small.

[4] The two oscillations are 90 degrees out of phase, so do not interfere.

the frequency of the radiation; they have to agree. We see immediately that they have the same n-dependence.

Problem 6
Check that the prefactors in (13) and (14) also agree:

$$\frac{\hbar}{mr_0^2} \quad = \quad \frac{2E_0}{\hbar}$$

Hint: Consider the centrifugal barrier, which for circular motion is simply related to the KE. Then by the virial theorem for an inverse square force, the KE is equal to the binding energy. §

Transition time
To go from n to $n-1$, the energy loss ΔE is given by E_γ in (14), while the power is

$$\begin{aligned} P &= \frac{2}{3}\frac{e^2}{4\pi\epsilon_0}\frac{\omega^4 r^2}{c^3} \\ &= \frac{2}{3}\frac{e^2}{4\pi\epsilon_0}\cdot\left(\frac{2E_0}{\hbar n^3}\right)^4\cdot\frac{(n^2 r_0)^2}{c^3} \end{aligned} \qquad (15)$$

where (14) has been used for ω. Hence we have[5]

$$\begin{aligned} \frac{P}{\Delta E} &= \frac{2}{3}\frac{e^2}{4\pi\epsilon_0}(2E_0)^3\hbar^{-4}r_0^2\cdot n^{-5} \\ &= \frac{2}{3}\alpha^4\frac{c}{r_0}\cdot n^{-5} \end{aligned}$$

The time taken is $\Delta E/P$:

$$\Delta t(n) \quad = \quad \frac{3}{2\alpha^4}\frac{r_0}{c}\cdot n^5 \qquad (16)$$

which increases very rapidly with n: high-level transitions are slow.

The time to go from state n all the way to the ground state is obtained by summing the above, i.e.,

$$n^5 \quad \mapsto \quad \sum_{k=0}^{n} k^5 \approx \int_0^n k^5\,dk = \frac{1}{6}n^6$$

The sum is dominated by large k, hence it does not matter that (a) the above analysis is only valid for the higher orbits; (b) there is a difference between the sum and the integral; and (c) whether we take the lower limit to be $n=0$ (which really does not exist in the Bohr model) or $n=1$. Thus the time $t(n)$ to collapse *completely* from a state n is given by

$$\boxed{t(n) = \frac{1}{4\alpha^4}\frac{r_0}{c}\cdot n^6} \qquad (17)$$

[5]Note how we manipulate the factors to arrive at a convenient expression, again (a) without electrical units, and (b) manifestly correct dimensionally.

Problem 7
Show that (17) is consistent with (12), both as to (a) the n^6 dependence compared with the r^3 dependence, and (b) the prefactors. §

3 Electric quadrupole radiation

Electric quadrupole (E2) radiation is more complicated, especially in the angular dependence. This Section only deals with some aspects and some special cases. The treatment of the total power also illustrates computational techniques that are useful in other situations.

3.1 Total power

The total power is given by (21). Although the derivation is somewhat lengthy, the result could have been guessed on general grounds, except for the numerical prefactor.

Recall that the general formula for the power radiated into any direction is given by

$$\frac{dP}{d\Omega} \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{1}{8\pi}\frac{\omega^2}{c^3}\left[\tilde{J}_i^* T_{ij}\tilde{J}_j\right] \qquad (18)$$

where for E2

$$\tilde{J}_i \quad = \quad -\frac{1}{6}\omega k_j q_{ij} = -\frac{\omega^2}{6c}\,n_j q_{ij}$$

When this is put into (18) we find

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{4\pi\epsilon_0}\frac{1}{8\pi}\frac{\omega^6}{36c^5}\times \\ &\quad (n_m q_{im}^*)\,(\delta_{ij}-n_i n_j)\,(n_\ell q_{j\ell}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{8\pi}\frac{\omega^6}{36c^5}C_{ijm\ell}\,(q_{im}^* q_{j\ell}) \\ C_{ijm\ell} &\equiv n_m\,(\delta_{ij}-n_i n_j)\,n_\ell \end{aligned} \qquad (19)$$

Integrating over solid angles is the same as multiplying by 4π and taking angular average (denoted by $\langle\ldots\rangle$). Thus

$$P \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{\omega^6}{72c^5}D_{ijm\ell}\,(q_{im}^* q_{j\ell}) \qquad (20)$$

where $D = \langle C\rangle$. Purely as a matter of geometry (Appendix A)

$$D_{ijm\ell} \quad = \quad \frac{1}{15}\,(4\delta_{ij}\delta_{m\ell} - \delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm})$$

When this is put into (20), the contributions of the first and third terms are of the same type, whereas the second term does not contribute because $q_{ii} = 0$. Thus

$$D_{ijm\ell}\left(q^*_{im}q_{j\ell}\right) = \frac{1}{5}q^*_{ij}q_{ij}$$

Therefore

$$\boxed{P = \frac{1}{360}\,\frac{1}{4\pi\epsilon_0}\,\frac{\omega^6}{c^5}\left(q^*_{ij}q_{ij}\right)} \tag{21}$$

3.2 Angular distribution

The angular distribution depends on the relative magnitudes of the elements of q_{ij}. We here consider the case where the system is axially symmetric, so that the quadrupole moment takes the form

$$[q] = \mathrm{diag}\,(-Q/2, -Q/2, Q)$$

All the angular dependence is carried by the factor

$$\begin{aligned}\frac{dP}{d\Omega} &\propto \sigma^*_i T_{ij}\sigma_j \\ \sigma_i &= n_j q_{ij}\end{aligned}$$

Because of cylindrical symmetry, without loss of generality we can take the point of observation to be on the x–z plane, with

$$\begin{aligned}\hat{\mathbf{n}} &= S\hat{\mathbf{e}}_x + C\hat{\mathbf{e}}_z \\ \boldsymbol{\sigma} &= -(Q/2)(S\hat{\mathbf{e}}_x - 2C\hat{\mathbf{e}}_z) \\ \boldsymbol{\sigma}\cdot\hat{\mathbf{n}} &= -(Q/2)(S^2 - 2C^2)\end{aligned}$$

with the shorthand $C = \cos\theta$, $S = \sin\theta$. Then

$$\begin{aligned}&\sigma^*_i T_{ij}\sigma_j = (\boldsymbol{\sigma}^*\cdot\boldsymbol{\sigma}) - |\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}|^2 \\ =\;& (|Q|^2/4)\left[(S^2+4C^2) - (S^2-2C^2)^2\right] \\ =\;& \frac{9}{4}|Q|^2\,S^2C^2\end{aligned}$$

Hence the angular distribution goes as

$$\frac{dP}{d\Omega} \propto \sin^2\theta\,\cos^2\theta \tag{22}$$

- This simple form applies only to the axially symmetric case.
- This distribution vanishes at $\theta = 0$ and $\theta = 90$ deg. The shape is shown qualitatively in **Figure 4**.
- In general, because $\hat{\mathbf{n}}$ appears to the fourth power, we have fourth powers of sin and cos.

4 Gravitational quadrupole radiation

Gravitational waves are much in the news, with LIGO announcing in February 2016 their first direct observation [1], with a second event reported a few months later [2]. However, indirect evidence has been around for some 40 years: the orbit of a binary star system decays slowly with time — because energy is lost through gravitational waves, principally through quadrupole radiation [3, 4, 5, 6, 7]. We take this opportunity to discuss the physics associated with the earlier observation, using a simplified model, and also some qualitative aspects of the LIGO results. The only input we need is that the EM formula can be mapped in the naive way to the gravitational case, with an extra factor of 4 in the power radiated.

4.1 Hulse–Taylor results

In 1974, Hulse and Taylor discovered two neutron stars rotating about each other in a binary system [3, 7]. In fact, only *one* neutron star was observed, as a pulsar (PSR B1913+16). The pulses are of course extremely regular, except for an anomaly in the time of arrival (**Figure 5**): it shifts up and down by about 3.7 seconds, with a period of 7.75 hours.[6] This means that the pulsar is moving towards or away from us in an orbit of period 7.75 hours, and diameter projected along the line of sight of 3.7 light seconds ($\approx 1.1\times10^9$ m).

Thus there must be another unseen star, and the two stars are orbiting around their common center of mass. The parameters of the binary system can be extracted from the anomaly in the time of arrival.

Two masses circulating around the center of mass would constitute a time-varying system that radiates gravitational waves, similar to the case of EM waves.

- The total mass (like the total charge) cannot vary with time, so there is no contribution from this term.
- The analog of E1 is gravitational dipole moment. But there is no gravitational dipole moment about the center of mass. Or, more generally, if the origin is taken at a different point, there would be a dipole moment ($= M\mathbf{R}$) where M is the total mass and $\mathbf{R}$ is the coordinate of the center of mass; the latter cannot change for an isolated system.
- There is also the analog (though in a more complicated form) corresponding to M1. But the gravitational analog of the magnetic moment is just the angular momentum,[7] which again cannot change for an isolated system.

[6] A precise version of Figure 5 with detailed discussions can be found in Ref. [4] and is also reproduced in Ref. [6].

[7] Review the discussion on g-factor.

- Thus the leading term is the analog of E2, i.e., quadrupole radiation.

As energy is lost (very slowly), the orbit shrinks. Hulse and Taylor observed that the period $T \approx 7.75$ hour decreases at a rate

$$\alpha \quad = \quad -\dot{T} = 76 \ \mu\text{s per year}$$

We analyze the orbital decay in a simple model, and try to obtain an estimate[8] for this value of α.

Incidentally, the decay rate is typically expressed in another (equivalent) way. Nearly all accounts of this system show the famous graph of the *cumulative period shift* Δt_c versus time t since the initial measurement. This is a negative quadratic[9]

$$\Delta t_c \quad = \quad -\beta t^2$$

Problem 8
Express β in terms of $\alpha = -\dot{T}$, and give the value of β in units of seconds per decade^2 if $\alpha = 76\ \mu$s per year. This problem requires no knowledge of mechanics or gravity. §

4.2 Formula for radiated power

To map to the gravitational case, simply change

$$\frac{1}{4\pi\epsilon_0} \quad \mapsto \quad G$$

and understand q_{ij} as the analogous quantity in terms of the mass density. The correct answer turns out to be 4 times larger than this naive analogy[10] — this factor of 4 has to be taken on faith unless you know general relativity. Thus the power radiated by a gravitational quadrupole would be[11]

$$\boxed{P = \frac{1}{90}\frac{G\omega^6}{c^5}\left(q^*_{ij}q_{ij}\right)} \tag{23}$$

4.3 The model

Consider two equal masses, each m, a distance $2r$ apart, each moving in the circular orbit of radius r around the center of mass, with an angular frequency Ω. It is observed (through the anomaly in the time of arrival of the pulses) that the diameter of the orbit of one mass is $2r = 2 \times 10^9$ m, and that the period of orbital motion is $T = 7.75$ hours.

Remark on the diameter
The time of 3.7 s translates into a distance $d = 1.1 \times 10^9$ m. However, we assume a circular orbit with a diameter about twice as large; $2r = 2 \times 10^9$ m. Why?

The actual orbit is elliptic. First assume the line of sight is in the orbital plane. When relating orbital size to period of motion, we should use the semi-major axis in the equivalent circular orbit.[12] But the major axis is in general larger than the distance d (**Figure 6**). Secondly, the line of sight is inclined at an angle (in fact about $\chi = 45$ degrees) to the orbital plane, and the actual size of the orbit is larger by a factor $1/\cos\chi$ than what is inferred from the time delays.

Problem 9
Using Newtonian mechanics, find the mass m. Answer: $m \approx 1.6$ solar masses. §

Problem 10
Show that the energy of the system is

$$E \quad = \quad -\frac{Gm^2}{4r}$$

Also verify the virial theorem: the kinetic energy, potential energy and total energy are in the ratio $+1$, -2 and -1. §

$q_{xy} = \sum m(3xy - r^2)$

$= 2mr^2(3\cos\Omega t \sin\Omega t - 1)$

$= 2mr^2(3\tfrac{1}{2}(\sin 2\Omega t) - 1)$

4.4 Decay of orbit

Quadrupole moment
The elements of the quadrupole moment are, e.g.,

$$\begin{aligned} q_{xx}(t) &= \sum m(3x^2 - r^2) \\ &= 2 \times m\left(3r^2\cos^2\Omega t - r^2\right) \\ &= 3mr^2\cos\omega t + \text{const} \end{aligned}$$

$= 3mr^2(\sin\omega t - 1)$

where $\omega = 2\Omega$. Note that the situation repeats itself every half cycle, and $\cos^2\Omega t$ has a time-varying piece at the second harmonic $\cos\omega t$, whose amplitude is

$$q_{xx} = 3\,mr^2 \qquad , \quad q^2_{xx} = 9\,m^2r^4$$

Each of q^2_{yy}, q^2_{xy} and q^2_{yx} makes the same contribution. So

$$\left(q^*_{ij}q_{ij}\right) \quad = \quad 36\,m^2r^4$$

[8] It is only an estimate because, as discussed below, we deal with two equal masses in a circular orbit; in fact, the two masses are unequal, and move in elliptic orbits.

[9] This quadratic dependence can be found in most accounts of this phenomenon, e.g., in Ref. [5].

[10] It is not surprising that the analogy is not perfect. While Newtonian gravity and Coulomb's law are strictly analogous, the same is not true of the time-dependent theories, even when gravity is linearized. Gravitational waves are spin 2, whereas EM waves are spin 1.

[11] Formulas seen in the literature often carry a prefactor of 1/5 instead of 1/90, for two reasons. (a) In the literature, the gravitational quadrupole moment is often defined as the integral of $s_is_j\rho$, without a factor of 3; (b) the answer is often expressed in terms of the average rather than the peak value of q^2_{ij}. These account for factors of $3^2 \times 2 = 18$.

[12] Recall the precise statement of Kepler's third law.

Rate of energy loss
When this is put into (23), we find

$$\boxed{P = \frac{1}{90}\frac{G\omega^6}{c^5}\cdot 36m^2r^4}$$

It is convenient to write this in a manner that is manifestly correct dimensionally:

$$\boxed{P = \frac{2}{5}\frac{Gm^2}{r}\cdot\frac{c}{r}\cdot\left(\frac{\omega r}{c}\right)^6} \tag{24}$$

Thus,

$$\left|\frac{\dot{E}}{E}\right| = \frac{P}{(Gm^2)/(4r)}$$

or

$$\boxed{\left|\frac{\dot{E}}{E}\right| = \frac{8}{5}\frac{c}{r}\cdot\left(\frac{\omega r}{c}\right)^6} \tag{25}$$

written without G or m.

Problem 11
Find the fractional loss of energy per cycle. §

Drift in period
Since $\Omega^2 \propto r^{-3}$, $E \propto r^{-1}$, the period T satisfies

$$\left|\frac{\dot{T}}{T}\right| = \frac{3}{2}\left|\frac{\dot{E}}{E}\right| = \frac{12}{5}\frac{c}{r}\cdot\left(\frac{\omega r}{c}\right)^6 \tag{26}$$

giving

$$\boxed{\dot{T} = \left|\frac{\dot{T}}{T}\right|\frac{4\pi}{\omega} = \frac{48\pi}{5}\left(\frac{\omega r}{c}\right)^5} \tag{27}$$

Using the result from Problem 10, this is evaluated to be 4.6×10^{-12}. The drift in the period over one year would be this number multiplied by 1 year = 3.15×10^7 s, namely about 140 μs.

The actual drift measured per year was about half this number, namely 76 μs per year. The difference should be accounted for by several effects: (a) the geometry including for example the inclination to the line of sight needs to be represented more realistically; (b) the two stars execute highly elliptic orbits ($e = 0.67$); and (c) the two masses and hence the two semi-major axes are unequal. But at least we get the right order of magnitude. A detailed account of these effects is given by Lee [8], and in particular it turns out that (normalized to the same orbital period and total mass), the eccentricity effect is quite substantial.

Shrinkage of radius
In much the same way, we find for the radius r

$$\begin{aligned}\left|\frac{\dot{r}}{r}\right| &= \frac{8}{5}\frac{c}{r}\cdot\left(\frac{\omega r}{c}\right)^6 \\ \frac{dr}{dt} &= -\frac{8}{5}\left(\frac{\omega r}{c}\right)^6 c\end{aligned} \tag{28}$$

To solve this equation, ω should first be eliminated by $Gm = \omega^2r^3$, giving

$$\frac{dr}{dt} = -\frac{8}{5}\frac{G^3m^3}{c^5}\cdot\frac{1}{r^3} \equiv -\frac{C}{r^3} \tag{29}$$

We note that the collapse happens faster and faster.

The time t_c to collapse to a point is thus given by

$$Ct_c = \frac{r^4}{4}$$

Some arithmetic then leads to

$$\boxed{t_c = \frac{5}{64\pi}\left(\frac{\omega r}{c}\right)^{-5} T} \tag{30}$$

where T is the period, and all quantities are evaluated at the starting time. The answer is again expressed without G and m.

For something like the Hulse–Taylor binary system, the collapse time is very long.

4.5 Coalescence of black holes

The event reported by LIGO in February 2016 [1] (observed in September 2015) is the inward spiral and eventual coalescence of two black holes — a much more violent event involving much larger masses (each back hole ~ 30 solar masses) at much closer distances when first detected ($\sim 10^6$ m) and moving at much shorter time scales (period ~ 30 ms). In this case, because the energy involved is much larger, and more importantly because LIGO was designed to directly measure tiny strains, gravitational waves were observed directly.

In this domain, linearized gravity and multipole expansion do not really work, but the model above still provides a good qualitative picture and an order-of-magnitude estimate. Before going into the arithmetic, the point is simply this: If the motion is nearly relativistic, i.e., $\omega r/c$ not very small compared to unity, then the collapse time is only a small multiple of the initial period.

Problem 12
Take (30) at face value even for such violent events, and consider $m = 30$ solar masses, initial period $T = 30$ ms. Find t_c/T, the number of (initial) periods to complete the collapse. §

Chirping
The relation (30) means $r \propto t^{1/4}$, where t now refers to the time before the end-point of collapse. Using Kepler's third law: $T^2 \propto r^3$, we get $T \propto t^{3/8}$, and the frequency then diverges as the end-point is approached, as

$$\boxed{\omega \propto t^{-3/8}} \tag{31}$$

The rising frequency, known as a chirp, is one significant characteristic that allows such an event to be identified; several websites provide sound files that reproduce the chirp.[13]

The intensity of the signal should go as $I \propto P \propto \omega^6 r^4 \propto r^{-5} \propto t^{-5/4}$. Hence the amplitude should increase as

$$\boxed{\text{amp} \propto t^{-5/8}} \tag{32}$$

A rising amplitude is indeed observed in the data.

To conclude, the application to coalescing black holes is only heuristic and order-of-magnitude, and some aspects are qualitatively different from the EM case.[14]

- The event is much more violent; the waves do not satisfy *linear* equations near the source, though of course they do in the radiation zone.
- Unlike the EM case, there is an event horizon, and energy is radiated *inwards* into the horizon as well.
- Quite apart from their orbital motion, the black holes are expected to be spinning rapidly.
- The emitted waves do not travel on a flat background spacetime and will be modified by the intervening medium. (Analogy: if an atom emits broadband radiation in an optical cavity, what one observes outside the cavity depends both on the source, namely the atom, and the medium, namely the cavity. There will be characteristic frequencies which are a property of the cavity.) These signals could in principle reveal properties of the intervening medium, i.e., the curved spacetime outside black holes; see Ref [9] and literature cited therein. In fact, the signal observed by LIGO probably contains a hint of the first of the characteristic frequencies.

[13] It happens that the frequencies are in the audio range, so the chirp can be "heard".
[14] I am grateful to WM Suen who reminded me of the second and third points.

A Some angular averages

Two indices
Start with

$$\langle n_i n_j \rangle = \alpha \, \delta_{ij}$$

the form of which is dictated by symmetry. Set $i = j$ and sum. On the LHS, $n_i n_j \mapsto 1$; and on the RHS $\delta_{ij} \mapsto 3$. So $\alpha = 1/3$, and we get the formula

$$\langle n_i n_j \rangle = \frac{1}{3} \, \delta_{ij} \tag{33}$$

Four indices
Next consider

$$\langle n_i n_j n_m n_\ell \rangle = \beta \, (\delta_{ij}\delta_{m\ell} + \delta_{im}\delta_{j\ell} + \delta_{i\ell}\delta_{jm})$$

Set $i = j$, $\ell = m$ and sum:

$$1 = \beta \, (9 + 3 + 3)$$

giving

$$\begin{aligned} & \langle n_i n_j n_m n_\ell \rangle \\ = & \frac{1}{15} \, (\delta_{ij}\delta_{m\ell} + \delta_{im}\delta_{j\ell} + \delta_{i\ell}\delta_{jm}) \end{aligned} \tag{34}$$

The tensor in question
Since

$$C_{ijm\ell} = \delta_{ij} n_m n_\ell - n_i n_j n_m n_\ell$$

we get

$$\begin{aligned} & D_{ijm\ell} \\ = & \frac{1}{3} \delta_{ij}\delta_{m\ell} - \frac{1}{15} \, (\delta_{ij}\delta_{m\ell} + \delta_{im}\delta_{j\ell} + \delta_{i\ell}\delta_{jm}) \\ = & \frac{1}{15} \, (4\delta_{ij}\delta_{m\ell} - \delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm}) \end{aligned} \tag{35}$$

B Supplement: Virial theorem

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

Reference has been made to the virial theorem. Its restricted form states that for bound orbits (not necessarily periodic, much less circular) in an attractive inverse-square force field, the average values of the kinetic energy K, the potential energy U and the total energy E are in the ratio[15]

$$\boxed{\langle K \rangle : \langle U \rangle : E = 1 : -2 : -1} \tag{36}$$

[15] The total energy does not change, so the average sign is omitted.

To prove this statement, consider

$$\begin{aligned}
& \frac{d}{dt}(\mathbf{p}\cdot\mathbf{r}) \\
= \; & \frac{d\mathbf{p}}{dt}\cdot\mathbf{r}+\mathbf{p}\cdot\frac{d\mathbf{r}}{dt}=\mathbf{F}\cdot\mathbf{r}+\mathbf{p}\cdot\mathbf{v} \qquad (37)
\end{aligned}$$

For an attractive inverse-square force

$$\begin{aligned}
\mathbf{F} &= -\frac{C}{r^2}\,\hat{\mathbf{e}}_r \\
U &= -\frac{C}{r}
\end{aligned}$$

so

$$\mathbf{F}\cdot\mathbf{r} = -\frac{C}{r} = U$$

The second term in (37) is $2K$. Hence

$$\frac{d}{dt}(\mathbf{p}\cdot\mathbf{r}) = U + 2K$$

Average this over a long time T:

$$\frac{1}{T}\int_0^T \frac{d}{dt}(\mathbf{p}\cdot\mathbf{r})\,dt = \langle U\rangle + 2\langle K\rangle$$

But

$$\text{LHS} = \frac{1}{T}\,\mathbf{p}\cdot\mathbf{r}\Big|_{t=0}^{t=T}$$

If the orbit is bounded (without having to assume that it is periodic), the values at the two times are also bounded, and as $T\to\infty$, this quantity vanishes because of the prefactor $1/T$. So for long time averages

$$\langle U\rangle + 2\langle K\rangle = 0 \qquad (38)$$

or in other words

$$\langle K\rangle : \langle U\rangle = 1 : -2 \qquad (39)$$

The relationship to E follows trivially.

Students should generalize this result to an attractive force that goes as $F\propto r^{-n}$ for any n.

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Suen, CY Tam and K Young, "Quasinormal modes of dirty black holes", Phys. Rev. Lett. **78**, 2894 (1997); and many primary references therein.

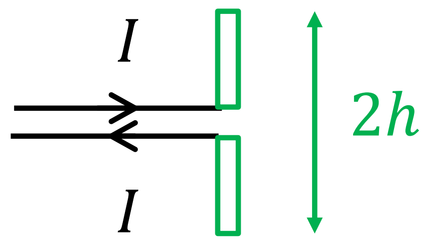

Figure 1 A center-fed antenna

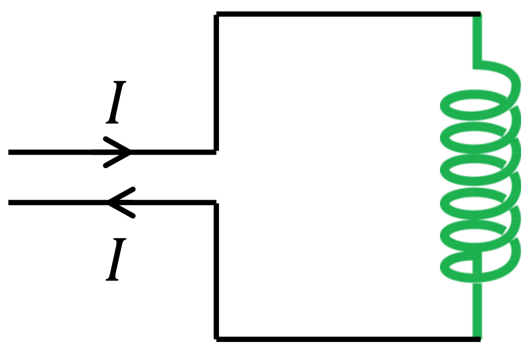

Figure 2 A solenoid as a radiating source

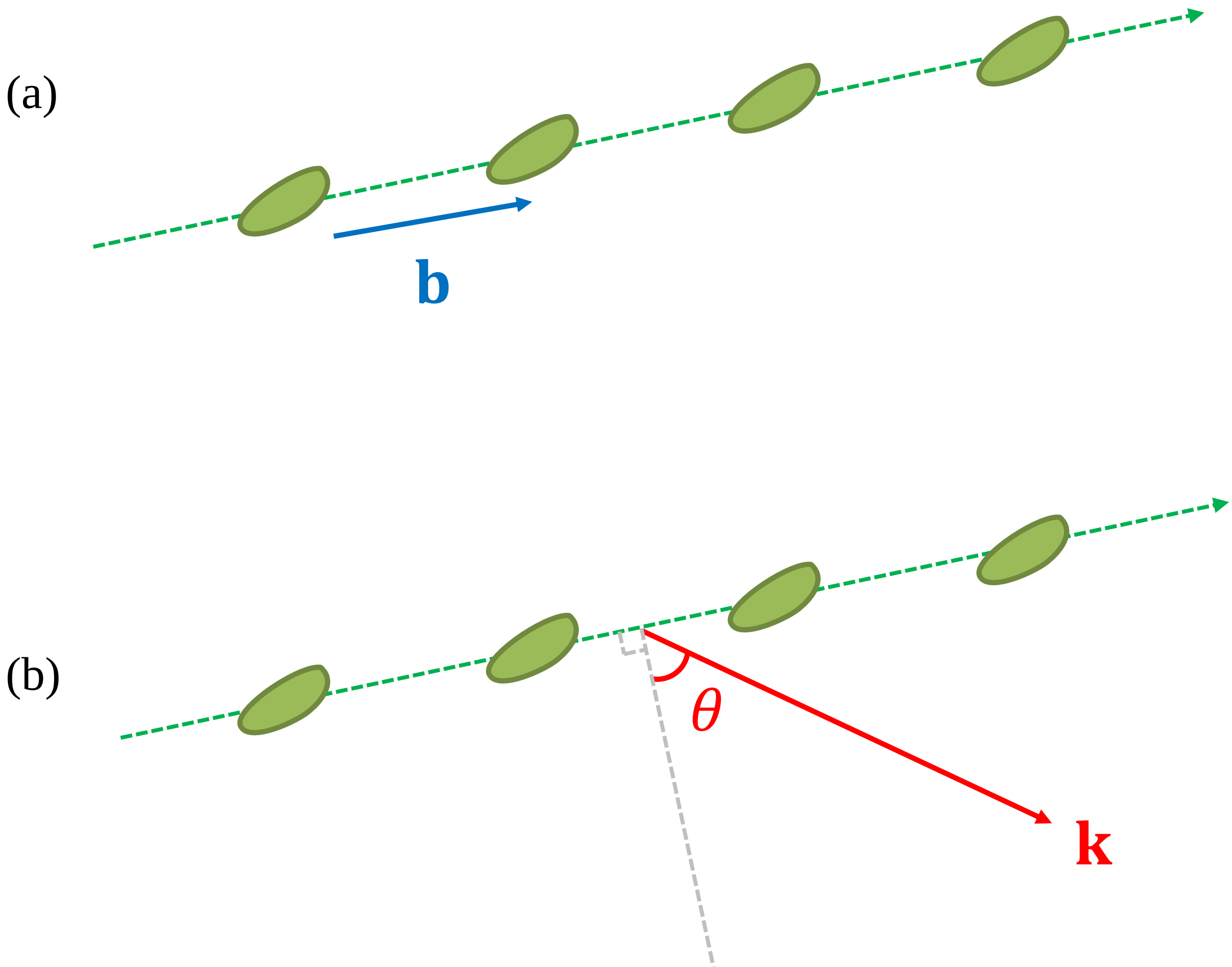

Figure 3 (a) A line of N sources separated by $\mathbf{b}$
(b) The angle θ between the normal to the line of sources and the observation direction

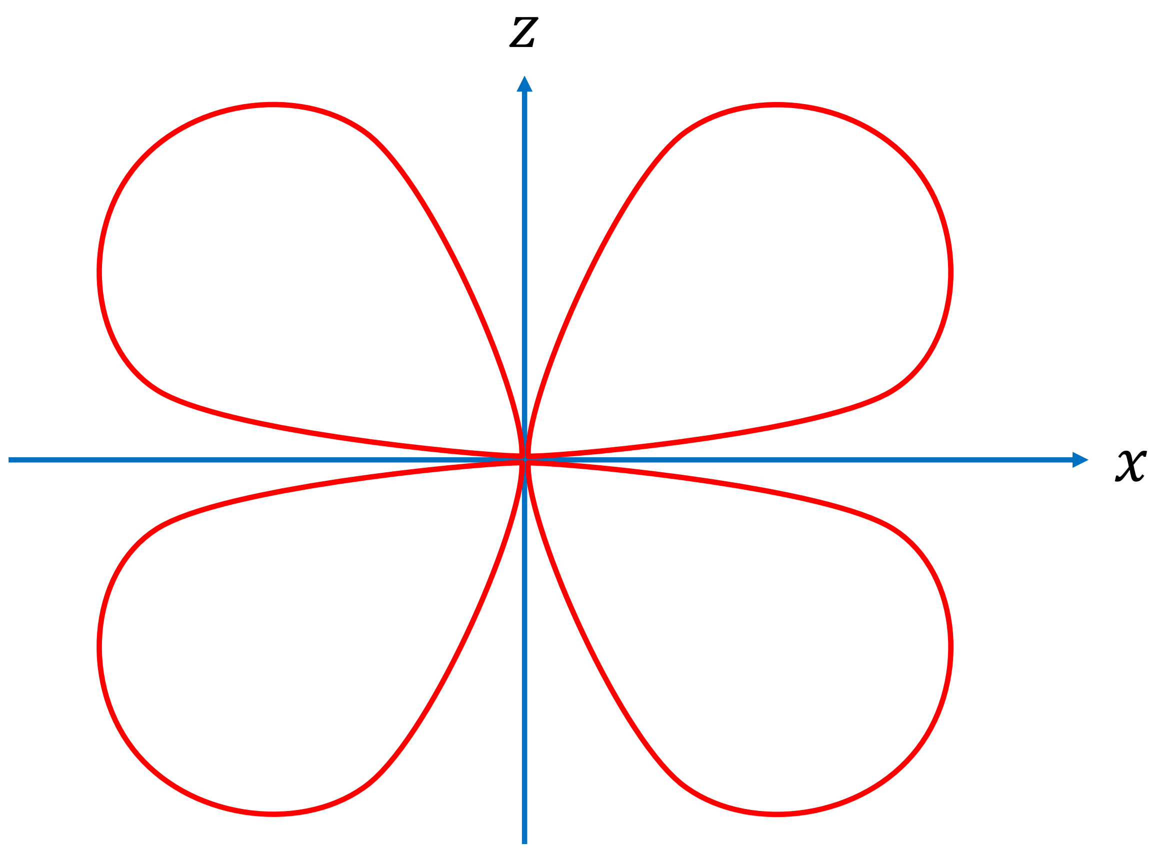

Figure 4 Radiation pattern $\sin^2\theta\cos^2\theta$

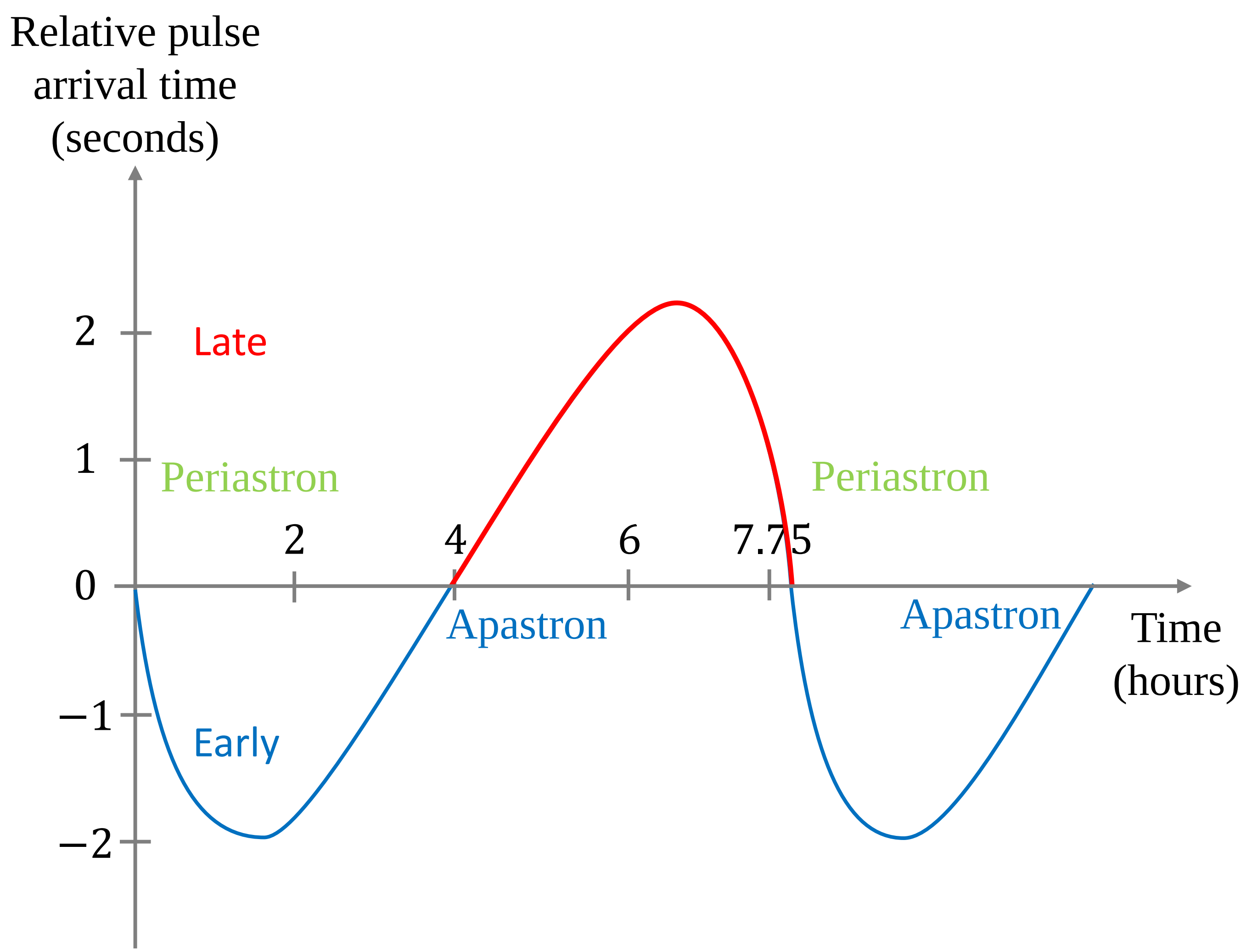

Figure 5 Anomaly in the time of arrival of pulses

Source:
http://www.astro.cornell.edu/academics/courses/astro201/images/bin_puls_arrival.gif

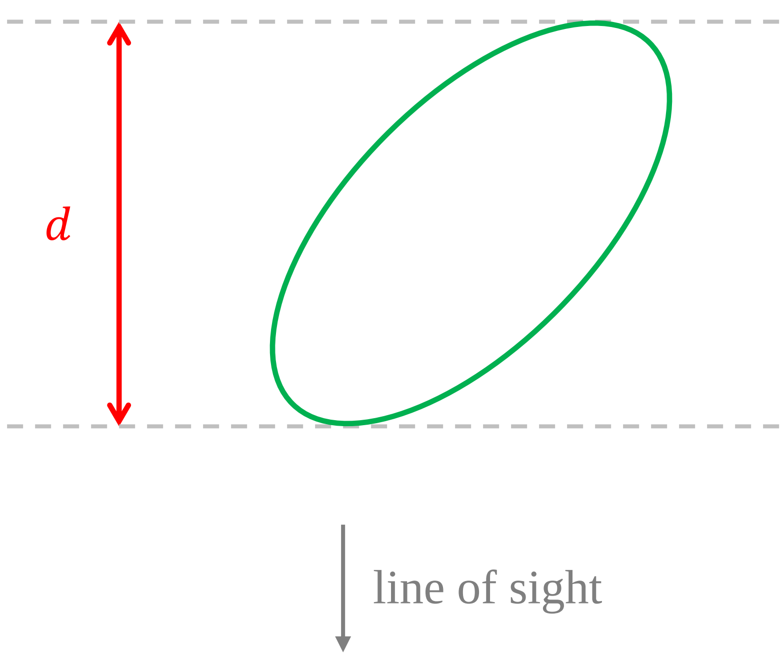

Figure 6 The amplitude in the anomaly of the time of arrival is related to the distance d, which is in general smaller than the major axis if the latter is not along the line of sight.