Chapter 9
Radiation by a harmonic source

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Radiation by a source varying harmonically with time is analyzed, with attention to the limit where the source is small compared to the wavelength.

Contents

1 Introduction

1.1 Definition of problem

This Chapter considers radiation by a *small* source varying in time harmonically, say as $\cos\omega t$. The source with linear dimension a is said to be small if $a \ll \lambda = 2\pi c/\omega$. The quantity to be evaluated is the power dP radiated into a solid angle $d\Omega$ at spatial infinity (**Figure 1**):

$$\frac{dP}{d\Omega} = ???$$

In the limit that $a/\lambda \ll 1$, the problem is solved in a power series in $(a/\lambda) \propto \omega$. In practice, the most important term is due to the electric dipole (E1), whose contribution goes as ω^4.

The derivation of the formulas is somewhat lengthy. In order not to obscure the physical ideas and the simple results, some qualitative features are first presented heuristically. Precise formulas are presented in the later Sections and physical applications are given in the next Chapter.

1.2 Qualitative features for E1

Order of magnitude

Imagine a point charge q at position[1]

$$z(t) = a\exp(-i\omega t)$$

so that there is a dipole moment[2]

$$p(t) = p\exp(-i\omega t)$$

where $p = qa$.

How much power is radiated? The answer cannot depend on r, the radial distance at which the observation is made, since we implicitly take $r \to \infty$.

The electric field must go as

$$E \propto \frac{q}{4\pi\epsilon_0}$$

So the energy flux goes as

$$S \sim \epsilon_0 E^2 \propto \epsilon_0 \left(\frac{q}{4\pi\epsilon_0}\right)^2 \sim \frac{q^2}{4\pi\epsilon_0}$$

(This is only intended to extract the dependence on q and ϵ_0.)

[1] It is understood that the real part is to be taken for all complex expressions.

[2] For oscillating quantities, the same symbol, without the argument t, shall denote the amplitude.

Using this, and the parameters of the problem, quantities with units of energy $\mathcal{E}$, time t and power P can be constructed:

$$\begin{aligned} \mathcal{E} &\sim \frac{q^2}{4\pi\epsilon_0 a} \\ t &\sim \frac{a}{c} \\ P \sim \frac{\mathcal{E}}{t} &\sim \frac{q^2}{4\pi\epsilon_0 a} \cdot \frac{c}{a} \end{aligned} \tag{1}$$

While this has the right units, it cannot be correct. If the charge is at rest or is in uniform motion, there can be no radiation, so the radiation field should be proportional to the acceleration:

$$E \quad \propto \quad \partial_t^2 z \propto \omega^2 \propto \lambda^{-2}$$

Hence in the intensity there should be a factor λ^{-4}. To convert this into a dimensionless parameter with which to multiply (1), we should have a factor $(a/\lambda)^4$; it is conventional to insert factors of 2π. Thus we arrive at the following estimate:

$$\boxed{P(\mathrm{E1}) \sim \frac{q^2}{4\pi\epsilon_0 a} \cdot \frac{c}{a} \cdot \left(\frac{2\pi a}{\lambda}\right)^4} \tag{2}$$

This can also be written as

$$\boxed{P(\mathrm{E1}) \sim \frac{q^2}{4\pi\epsilon_0} \frac{\omega^4 a^2}{c^3} = \frac{1}{4\pi\epsilon_0} \frac{\omega^4 p^2}{c^3}} \tag{3}$$

Our detailed analysis will provide the exact numerical prefactors of 1/3.

Angular distribution
Suppose a dipole is oscillating in the z direction and the radiation is observed at an angle θ (**Figure 2a**). We know $\mathbf{A} \propto \mathbf{J}$ is along z, but the longitudinal component, i.e., the component along the direction of propagation, should be removed. This introduces a factor $\sin\theta$ into E and hence a factor $\sin^2\theta$ into the intensity or energy flux. Thus we expect

$$\frac{dP}{d\Omega} \quad \propto \quad \sin^2\theta \tag{4}$$

as indicated schematically in **Figure 2b**.

The dependence can be understood heuristically as follows. The lines of force emanate from the charge q, so as the end of the each line of force (imagined to be a "string") is shaken up and down, there will be a transverse wave in the equatorial plane (**Figure 3a**), but not in the direction of shaking (**Figure 3b**).

1.3 Qualitative features for M1

For magnetic dipole (M1) radiation, the qualitative features are similar, with only the following changes.

- The electric dipole p is replaced by the magnetic dipole μ, which dimensionally has an extra power of velocity v. (Magnetic dipole is caused by currents, which are proportional to the velocity of the charges.)
- Thus there is an overall extra factor of c^{-1} in the amplitude or c^{-2} in the intensity.

Taking account of these feature,

$$\boxed{P(\mathrm{M1}) \sim \frac{1}{4\pi\epsilon_0} \frac{\omega^4 \mu^2}{c^5}} \tag{5}$$

Again, detailed calculation will show that there is a prefactor of 1/3.

The angular distribution is again given by (4), where now θ is the angle between $\boldsymbol{\mu}$ and the direction of observation $\hat{\mathbf{n}}$.

By comparing μ and p, it turns out that the order-of-magnitude of the power radiated is reduced compared to E1 by a factor of $(v/c)^2$:

$$\boxed{P(\mathrm{M1}) \sim \frac{q^2}{4\pi\epsilon_0 a} \cdot \frac{c}{a} \cdot \left(\frac{2\pi a}{\lambda}\right)^4 \cdot \left(\frac{v}{c}\right)^2} \tag{6}$$

where v is the typical velocity of the charged particles in the system.

1.4 Qualitative features for E2

The formulas for electric quadrupole (E2) radiation are more complicated, since the quadrupole moment q_{ij} carries two indices. The angular distribution and precise numerical factors will be analyzed for specific examples in the next Chapter. Here we only note that the quadrupole moment has one extra power of a compared to the dipole moment, so the power should have an extra power of a^2, and for dimensional reasons this must then become $(2\pi a/\lambda)^2$. Thus

$$\boxed{P(\mathrm{E2}) \sim \frac{q^2}{4\pi\epsilon_0 a} \cdot \frac{c}{a} \cdot \left(\frac{2\pi a}{\lambda}\right)^6} \tag{7}$$

In practice, the suppression factors for M1 and E2 (relative to E1) are of the same order, by the following argument. In any harmonic motion with amplitude a, $v \sim \omega a$; thus

$$\frac{v}{c} \quad \sim \quad \frac{\omega a}{c} = \frac{2\pi a}{\lambda}$$

Thus for most situations with a source much smaller than the wavelength

$$\boxed{\text{E1} \gg \text{M1} \sim \text{E2}}$$

unless symmetry suppresses or eliminates any of them.

1.5 Schematics of calculation

The key equation to be solved is

$$\left(-\nabla^2 + c^{-2}\partial_t^2\right)\mathbf{A} \quad = \quad \mu_0 \mathbf{J} \tag{8}$$

A similar equation for Φ in terms of ρ can be bypassed, as we shall see.

- In the far zone, we only need to keep terms in $\mathbf{A}$ and in the fields $\mathbf{E}$, $\mathbf{B}$ that go as r^{-1}; higher powers of r^{-1} would not contribute to energy flux at infinity.
- At the distant observation point $\mathbf{r}$, the outgoing radiation is approximately a plane wave propagating along $\hat{\mathbf{n}} = \mathbf{r}/r$.
- Refer to the last Chapter. If the longitudinal component of $\mathbf{A}$ is projected out, i.e.,

$$\mathbf{A} \quad \mapsto \quad \mathbf{A} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{A}) \tag{9}$$

then Φ can be ignored and

$$\mathbf{E} \quad = \quad -\partial_t \mathbf{A} = i\omega\,\mathbf{A} \tag{10}$$

where in the last step the harmonic time dependence $\exp(-i\omega t)$ is used.

- Likewise

$$\mathbf{B} \quad = \quad \boldsymbol{\nabla} \times \mathbf{A} = ik\,\hat{\mathbf{n}} \times \mathbf{A} \tag{11}$$

where $k = \omega/c$ is the magnitude of the wave vector.

- Thus Φ is not needed at all.

All quantities have the same time dependence:

$$\begin{aligned} \mathbf{J}(\mathbf{r}, t) &= \mathbf{J}(\mathbf{r}) \exp(-i\omega t) \\ \mathbf{A}(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}) \exp(-i\omega t) \end{aligned} \tag{12}$$

For convenience, the same symbol is used for the quantities with the time-dependence factored out. Putting this into (8) gives a PDE in the three spatial dimensions only:

$$\boxed{-\left[\nabla^2 + (\omega/c)^2\right]\mathbf{A}(\boldsymbol{r}) = \mu_0 \mathbf{J}(\boldsymbol{r})} \tag{13}$$

So we need to solve (13) and calculate $\mathbf{A}$ to $O(r^{-1})$.

The rest of this Chapter is organized as follows.

- Section 2 solves the PDE for a δ-function source to obtain the Green's function.
- Section 3 then gives the potentials as an integral, which is evaluated in the radiation zone. The vector potential is found to be $\mathbf{A} \sim \tilde{\mathbf{J}}$, where the latter is the spatial Fourier transform of the current density.
- Section 4 develops $\tilde{\mathbf{J}}$ in power of the wave number $\mathbf{k}$, in terms of multipole moments.
- Section 5 shows the explicit formulas for the radiated power, especially for E1 and M1.
- Applications are given in the next Chapter.

2 Green's function

2.1 Definition of Green's function

Define a Green's function G satisfying

$$\boxed{-\left[\nabla^2 + (\omega/c)^2\right] G(\boldsymbol{r}) = \delta^3(\boldsymbol{r})} \tag{14}$$

in terms of which the vector potential is

$$\mathbf{A}(\mathbf{r}) = \mu_0 \int G(\mathbf{r} - \mathbf{s})\mathbf{J}(\mathbf{s})\, d^3 s \tag{15}$$

The formalism is parallel to electrostatics and magnetostatics, which correspond to the special case for $\omega \to 0$.

For simplicity, the factor c^2 in (14) will be dropped, with the understanding that $\omega \mapsto \omega/c$ in the spatial function at the end.

2.2 Position representation

First consider (14) for $r > \delta > 0$ and note that G can only depend on $r = |\mathbf{r}|$, so

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG}{dr}\right) + \omega^2 G \quad = \quad 0 \tag{16}$$

Problem 1
Make the substitution

$$G(r) \quad = \quad H(r)/r$$

(a) Show that (16) reduces to

$$\frac{d^2 H}{dr^2} + \omega^2 H \quad = \quad 0 \tag{17}$$

(b) Hence show that the solution is

$$H(r) \quad = \quad A e^{i\omega r} + B e^{-i\omega r}$$

(c) Restore the time-dependent factor and require the solution to be an outgoing wave. Thus show that $B = 0$.
(d) Then integrate the original equation (14) over

a small sphere of radius r (with $r \to 0$ at the end) and thus show that $A = (1/4\pi)$. §

In short, we have the solution

$$G(\mathbf{r}) \quad = \quad \frac{1}{4\pi}\frac{\exp(i\omega r/c)}{r} \tag{18}$$

in which the factor of c has been restored.

The $\omega = 0$ case can be checked against the solution in statics.

2.3 Momentum representation

An alternate derivation can be given in the momentum representation.

Solution in momentum space
In the momentum representation, (14) becomes

$$\left(k^2 - \omega^2\right)\tilde{G}(k) \quad = \quad 1 \tag{19}$$

The factor of c^2 is again temporarily dropped, and we have used the fact that $\tilde{G}$ can only depend on $k = |\mathbf{k}|$. The solution to (19) is trivial.

Inverse transform to coordinate space
Next transform back to coordinate space:

$$\begin{aligned} G(\mathbf{r}) &= \int \frac{1}{k^2-\omega^2}\exp(i\mathbf{k}\cdot\mathbf{r})\,\frac{d^3k}{(2\pi)^3} \\ &= \frac{1}{8\pi^3}\int \frac{1}{k^2-\omega^2}e^{ikru}\,2\pi\,du\,k^2dk \\ &= \frac{1}{4\pi^2 ir}\int_0^\infty \frac{1}{k^2-\omega^2}\left(e^{ikr}-e^{-ikr}\right)\,k\,dk \end{aligned}$$

In the above, $u = \cos\theta$ (where θ is the angle between $\mathbf{r}$ and $\mathbf{k}$) has been integrated between -1 and $+1$, giving a factor $1/(ikr)$. By changing variable $k \mapsto -k$ in the second term, we then find

$$\begin{aligned} G(\boldsymbol{r}) \quad &= \quad \frac{1}{4\pi^2 ir}\int_{-\infty}^{\infty}\frac{e^{ikr}}{k^2-\omega^2}k\,dk \\ &\equiv \quad \frac{1}{4\pi^2 ir}\,I \end{aligned} \tag{20}$$

Handling the singularities
The integral encounters two singularities, at $k = \pm\omega$, and how they are handled depends on the boundary condition. We choose to take the contour in **Figure 4a**, going over $k = -\omega$ and under $k = +\omega$. Then close the contour in the upper half k-plane; the result is $2\pi i$ times the residue. The integrand is

$$\frac{1}{k-\omega}\cdot\frac{k}{k+\omega}\,e^{ikr}$$

where the first factor is the pole with residue unity, and the other regular factors are to be evaluated at $k = \omega$. Thus

$$I \quad = \quad 2\pi i\cdot\frac{1}{2}\,e^{i\omega r}$$

and when this is put into (20), we recover (18).

Problem 2
Explain why the contour should be closed in the upper half plane and not in the lower half plane. §

Problem 3
Calculate G by taking the integral along the contour in **Figure 4b**. Explain why this solution is rejected. §

3 Potential and field

3.1 Using the Green's function

Henceforth adopt the notation

$$k \quad = \quad \frac{\omega}{c}$$

which is the wave number corresponding to the frequency ω (not to be confused with the dummy variable in Section 2.3). Thus the Green's function is

$$\boxed{G(\mathbf{r}) = \frac{1}{4\pi}\frac{\exp(ikr)}{r}} \tag{21}$$

and the potentials are

$$\begin{aligned} \Phi(\mathbf{r}) \quad &= \quad \frac{1}{4\pi\epsilon_0}\int \frac{\exp(ik|\mathbf{r}-\mathbf{s}|)}{|\mathbf{r}-\mathbf{s}|}\,\rho(\mathbf{s})\,d^3s \\ A_i(\mathbf{r}) \quad &= \quad \frac{\mu_0}{4\pi}\int \frac{\exp(ik|\mathbf{r}-\mathbf{s}|)}{|\mathbf{r}-\mathbf{s}|}\,J_i(\mathbf{s})\,d^3s \end{aligned} \tag{22}$$

This is the exact solution for all problems with a harmonic time dependence. For $\omega = 0$ the solution to the static problem is recovered.

3.2 Fields in radiation zone

Solve for potential
For the problem of radiation, some simplifications are possible. First of all, we only need $\mathbf{A}$. Second, we only keep terms that go as r^{-1}. Thus

$$\frac{1}{|\mathbf{r}-\mathbf{s}|} \quad \mapsto \quad \frac{1}{r}$$

and we have

$$\boxed{A_i(\mathbf{r}) = \frac{\mu_0}{4\pi r}\int \exp(ik|\mathbf{r}-\mathbf{s}|)\,J_i(\mathbf{s})\,d^3s} \tag{23}$$

Moreover, for the argument of the exponential, we have

$$\boxed{k|\mathbf{r}-\mathbf{s}| \approx kr - \mathbf{k}\cdot\mathbf{s}} \tag{24}$$

which can be proved as follows:

$$\begin{aligned} & k|\mathbf{r}-\mathbf{s}| \\ = \;& k(r^2 - 2\mathbf{r}\cdot\mathbf{s} + \ldots)^{1/2} \\ = \;& kr\left(1 - 2\frac{\mathbf{r}\cdot\mathbf{s}}{r^2} + \ldots\right)^{1/2} \\ = \;& kr\left(1 - \frac{\mathbf{r}\cdot\mathbf{s}}{r^2} + \ldots\right) \\ = \;& kr - k\hat{\mathbf{n}}\cdot\mathbf{s} = kr - \mathbf{k}\cdot\mathbf{s} \end{aligned}$$

Terms that go as an inverse power of r have been dropped, and we have introduced

$$\hat{\mathbf{n}} = \frac{\mathbf{r}}{r}$$

as the unit radial vector, as well as

$$\mathbf{k} = k\hat{\mathbf{n}}$$

as the wave vector of the effective plane wave observed at $\mathbf{r}$.

Put these into (23) and taking out all the factors independent of $\mathbf{s}$, we find

$$\begin{aligned} A_i(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \exp(-i\mathbf{k}\cdot\mathbf{s})\, J_i(\mathbf{s})\, d^3s \\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\tilde{J}_i(\mathbf{k}) \end{aligned} \tag{25}$$

where $\tilde{\mathbf{J}}$ is the spatial Fourier transform of the current density.

Restoring the time-dependent factor:

$$\boxed{A_i(\mathbf{r},t) = \frac{e^{i(kr-\omega t)}}{r}\left[\frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{k})\right]} \tag{26}$$

Thus, except for the prefactors describing a spherical outgoing wave with amplitude decreasing as r^{-1}, the rest is easy to remember:

$$\frac{\mu_0}{4\pi}\tilde{J}_i(\mathbf{k})$$

Solve for fields
The fields can also be recorded.

$$\begin{aligned} E_i &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(i\omega)[\tilde{J}_i - n_i(n_j\tilde{J}_j)] \\ B_i &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}(i\omega/c)[\epsilon_{ij\ell}n_j\tilde{J}_\ell] \end{aligned} \tag{27}$$

where $\tilde{J}_i = \tilde{J}_i(\mathbf{k})$. In the above we have used the following prescriptions:

- For $\mathbf{E}$, we remove the longitudinal component of $\mathbf{A}$ and take $-\partial_t = i\omega$.
- For $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, in taking the spatial derivative (a) we make use of the fact that the wave must go as $\exp(i\mathbf{k}\cdot\mathbf{r})$, and (b) there is no need to differentiate the pre-factor $1/r$, since that leads to terms which are negligible.

3.3 Energy flux

Averaging a quadratic quantity
Some care is needed in calculating the energy flux (or any other quadratic quantity). The proper procedure is to *first* take the real parts in each factor, then multiply, then average over time. The following problem provides a useful identity that simplifies the calculation.

Problem 4
Suppose two quantities are given by

$$f(t) = \Re\tilde{f}(t) \quad , \quad g(t) = \Re\tilde{g}(t)$$

where the complex quantities $\tilde{f}$ and $\tilde{g}$ have harmonic time dependence $\exp(-i\omega t)$:

$$\begin{aligned} \tilde{f}(t) &= ae^{i\alpha}\, e^{-i\omega t} \\ \tilde{g}(t) &= be^{i\beta}\, e^{-i\omega t} \end{aligned}$$

Show that

$$\langle f(t)g(t)\rangle = \frac{1}{2}\Re\tilde{f}^*\tilde{g} \tag{28}$$

where $\langle\ldots\rangle$ denotes time average. §

Poynting vector and power
Using this result, the time-averaged Poynting vector is easily found to be

$$\mathbf{S} = \frac{1}{2\mu_0}\mathbf{E}^*\times\mathbf{B} \tag{29}$$

It is a little messy but straightforward to sort out the vector indices and the dependence on the direction $\hat{\mathbf{n}}$ (Appendix A). Of course the direction of $\mathbf{S}$ is along $\hat{\mathbf{n}}$ and its magnitude turns out to be

$$\begin{aligned} S &= \frac{\mu_0}{4\pi}\frac{1}{8\pi r^2}\frac{\omega^2}{c}\left[\tilde{J}_i(\mathbf{k})^*\, T_{ij}\, \tilde{J}_j(\mathbf{k})\right] \\ \frac{dP}{d\Omega} &= \frac{1}{4\pi\epsilon_0}\frac{1}{8\pi}\frac{\omega^2}{c^3}\left[\tilde{J}_i(\mathbf{k})^*\, T_{ij}\, \tilde{J}_j(\mathbf{k})\right] \\ T_{ij} &= \delta_{ij} - n_i n_j \end{aligned} \tag{30}$$

It is conventional to express the result using ϵ_0 rather than μ_0. The matrix T is the transverse projection operator.

Problem 5
Fill in the missing steps above. §

Fourier transform of current density
It remains to evaluate $\tilde{J}_i$. This will be carried out in powers of $k \propto \omega$, or effectively in the small parameter

$$ks \sim a/\lambda$$

Such a power-series expansion is called the multipole expansion; it has many similarities with the multipole expansion in statics, again involving terms of the form

$$\int (ss \ldots s)\,\rho\,d^3 s \quad , \quad \int (ss \ldots s)\,J\,d^3 s$$

4 Multipole expansion

We need to evaluate $\tilde{J}_i$ in powers of k. To start with

$$\tilde{J}_i = \int [(1 - ik_\ell s_\ell + \ldots)J_i] \qquad (31)$$

In this Section, the symbol $d^3 s$ will be understood. The omitted higher-order terms will no longer be displayed. Use the identity

$$\begin{aligned} & \partial_j(s_i J_j) = \delta_{ij} J_j + s_i(\partial_j J_j) \\ = \;& J_i + s_i(-\partial_t \rho) = J_i + i\omega s_i \rho \end{aligned}$$

to obtain

$$J_i = \partial_j(s_i J_j) - i\omega s_i \rho \qquad (32)$$

Put this into (31) to get

$$\begin{aligned} \tilde{J}_i &= \int (1 - ik_\ell s_\ell)\,[\partial_j(s_i J_j) - i\omega s_i \rho] \\ &= ik_j \int s_i J_j - i\omega \int s_i \rho - \omega k_j \int s_i s_j \rho \qquad (33) \end{aligned}$$

where the first term comes from integration by parts.

Next, take the first term and write $s_i J_j$ as the sum of symmetrized and antisymmetrized parts

$$s_i J_j = \frac{1}{2}(s_i J_j + s_j J_i) + \frac{1}{2}(s_i J_j - s_j J_i)$$

Use the identity

$$\begin{aligned} & \partial_k(s_i s_j J_k) \\ =\;& \delta_{ki} s_j J_k + \delta_{kj} s_i J_k + s_i s_j \partial_k J_k \\ =\;& s_j J_i + s_i J_j + s_i s_j(-\partial_t \rho) \\ =\;& s_j J_i + s_i J_j + i\omega s_i s_j \rho \end{aligned}$$

to write the symmetric term as

$$s_j J_i + s_i J_j = \partial_k(s_i s_j J_k) - i\omega s_i s_j \rho$$

The first term does not contribute to the integral, since it can be converted to a surface integral at spatial infinity. Thus we have

$$\begin{aligned} \tilde{J}_i &= \frac{\omega k_j}{2} \int s_i s_j \rho + \frac{ik_j}{2} \int (s_i J_j - s_j J_i) \\ & \quad - i\omega \int s_i \rho - \omega k_j \int s_i s_j \rho \qquad (34) \end{aligned}$$

The first and last terms can be combined, and we have

$$\begin{aligned} \tilde{J}_i = & - i\omega \int s_i \rho + \frac{ik_j}{2} \int (s_i J_j - s_j J_i) \\ & - \frac{\omega k_j}{6} \int (3 s_i s_j - s^2 \delta_{ij}) \rho \end{aligned}$$

In the last term we have added a piece proportional to δ_{ij}, since that leads to a term that goes as k_i, which is longitudinal and therefore does not contribute to the fields. Recalling the definitions of the electric dipole moment, magnetic dipole moment and the electric quadrupole moment, we finally get the compact answer

$$\boxed{\tilde{J}_i = -i\omega p_i + i\epsilon_{ijm} k_j \mu_m - (1/6)\omega k_j q_{ij}} \qquad (35)$$

where p_i, μ_i, q_{ij} are respectively the amplitudes of the corresponding oscillating moments, e.g.,

$$p_i(t) = p_i\, e^{-i\omega t}$$

It is then merely a matter of algebra to put this into (27) to find **E** and **B**.

5 Formulas for the power

This Section analyzes the power radiated by electric dipoles (E1), magnetic dipoles (M1) and electric quadrupoles (E2). Strictly speaking one should add the three amplitudes and then square, so there are cross terms. In most practical applications, only one term dominates, and the three cases can be dealt with separately.

5.1 Electric dipole

For an electric dipole (E1), the first term of (35) give $\tilde{J}_i = -i\omega p_i$, independent of $\hat{\mathbf{n}}$. Put this into the general power formula and note that

$$p_i^* T_{ij} p_j = |p|^2 \sin^2\theta \qquad (36)$$

Integrate over solid angles, noting that the average of $\sin^2\theta$ is $2/3$, and we find

$$\boxed{P = \frac{1}{3}\frac{1}{4\pi\epsilon_0}\frac{\omega^4}{c^3}|p|^2} \qquad (37)$$

For a single charge q oscillating with amplitude a, this can also be written as

$$\boxed{P = \frac{1}{3}\frac{1}{4\pi\epsilon_0}\frac{\omega^4}{c^3}(qa)^2} \tag{38}$$

To compare with the estimate in Section 1, we can write the above as (and indicating explicitly that this refers to electric dipole radiation, denoted as E1)

$$\boxed{P(\mathrm{E1}) = \frac{1}{3}\cdot\frac{q^2}{4\pi\epsilon_0 a}\cdot\frac{c}{a}\cdot\left(\frac{2\pi a}{\lambda}\right)^4} \tag{39}$$

confirming (2), with the exact numerical factor 1/3 now determined. This is by far the most important case for applications.

5.2 Magnetic dipole

For a magnetic dipole (M1), we note from the second term of (35) that compared with E1 we need to replace

$$\mathbf{p} \quad \mapsto \quad -\frac{1}{c}\hat{\mathbf{n}}\times\boldsymbol{\mu}$$

When this replacement is made in (36),

$$p_i^* T_{ij} p_j \quad \mapsto \quad \frac{1}{c^2}\mu_i^* T_{ij}\mu_j$$

To arrive at this result, one can work out the components explicitly, or simply note that $\hat{\mathbf{n}}\times\ldots$ kills the longitudinal component and rotates the two transverse components. Therefore the formula for the power is

$$\boxed{P = \frac{1}{3}\frac{1}{4\pi\epsilon_0}\frac{\omega^4}{c^5}\mu^2} \tag{40}$$

To appreciate the order-of-magnitude estimate, go back to (35) and compare the first and second terms; for the amplitudes there is a ratio

$$\begin{aligned}\frac{\mathrm{M1}}{\mathrm{E1}} &\sim \frac{k}{\omega}\frac{\mu}{p} \sim \frac{1}{c}\frac{(q/m)L}{qa} \\ &\sim \frac{1}{c}\frac{(q/m)mva}{qa} = \frac{v}{c}\end{aligned}$$

in which we have used the fact that

$$\mu \quad = \quad g\frac{q}{m}L$$

for a system of charge q, mass m and angular momentum L, with $g = O(1)$. Thus we have the estimate

$$P(\mathrm{M1}) \quad \sim \quad \frac{q^2}{4\pi\epsilon_0 a}\cdot\frac{c}{a}\cdot\left(\frac{2\pi a}{\lambda}\right)^4\cdot\left(\frac{v}{c}\right)^2 \tag{41}$$

where v is the typical velocity of the charges in the radiating source.

Problem 6
Estimate the suppression ratio for M1/E1 for the intensity in atomic transitions, say for hydrogen. §

5.3 Electric quadrupole

The detailed results for electric quadrupole (E2) will not be dealt with in this Chapter, and only an order-of-magnitude estimate is given. Comparing the first and third terms of (35), we see that in the amplitude there is a ratio of

$$\frac{\mathrm{E2}}{\mathrm{E1}} \quad \sim \quad \frac{k_j q_{ij}}{p_i} \sim ka = \frac{2\pi a}{\lambda}$$

where q is the typical charge and a is the typical linear dimension of the source. This factor appears squared in the intensity, and we obtain the estimate

$$P(\mathrm{E2}) \quad \sim \quad \frac{q^2}{4\pi\epsilon_0 a}\cdot\frac{c}{a}\cdot\left(\frac{2\pi a}{\lambda}\right)^6 \tag{42}$$

Problem 7
Consider the transition from $2p$ to $1s$ in atomic hydrogen.
(a) Find the energy of the photon emitted. (The binding energy of hydrogen is 13.6 eV.)
(b) Find the wavelength λ of the optical transition.
(c) Estimate the suppression ratio for E2/E1 for the intensity, using for a the Bohr radius, $a = 0.05$ nm. §

As explained in Section 1.4, in atomic physics the suppression ratio for M1 and E2 are of the same order.

A Details for Poynting vector

Using (29), we have, for the average

$$\begin{aligned} S_i &= \frac{1}{2\mu_0}\epsilon_{ijk}E_j^* B_k \\ &= \frac{1}{2\mu_0}\left(\frac{\mu_0}{4\pi}\right)^2\frac{1}{r^2}\frac{\omega^2}{c}\times \\ &\quad \epsilon_{ijk}\left[(\delta_{jm}-n_j n_m)\tilde{J}_m^*\right]\left[\epsilon_{kpq}n_p\tilde{J}_q\right] \end{aligned} \tag{43}$$

The last line above is the following expression multplying $\tilde{J}_m^*\tilde{J}_q$:

$$\begin{aligned} &\epsilon_{ijk}\epsilon_{kpq}\left(\delta_{jm}-n_j n_m\right)n_p \\ =\ &(\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})(\delta_{jm}-n_j n_m)\,n_p \\ =\ &(n_i\delta_{jq}-n_j\delta_{iq})(\delta_{jm}-n_j n_m) \\ =\ &(n_i\delta_{mq}-n_m\delta_{iq})-(n_i n_m n_q - n_m\delta_{iq}) \\ =\ &n_i\left(\delta_{mq}-n_m n_q\right) = n_i T_{mq} \end{aligned}$$

It is obvious that S_i should be proportional to n_i. The magnitude of $\mathbf{S}$ is then

$$S = \frac{1}{2\mu_0}\left(\frac{\mu_0}{4\pi}\right)^2 \frac{1}{r^2}\frac{\omega^2}{c}\left[\tilde{J}_m^* T_{mq}\tilde{J}_q\right] \quad (44)$$

which is the result we want.

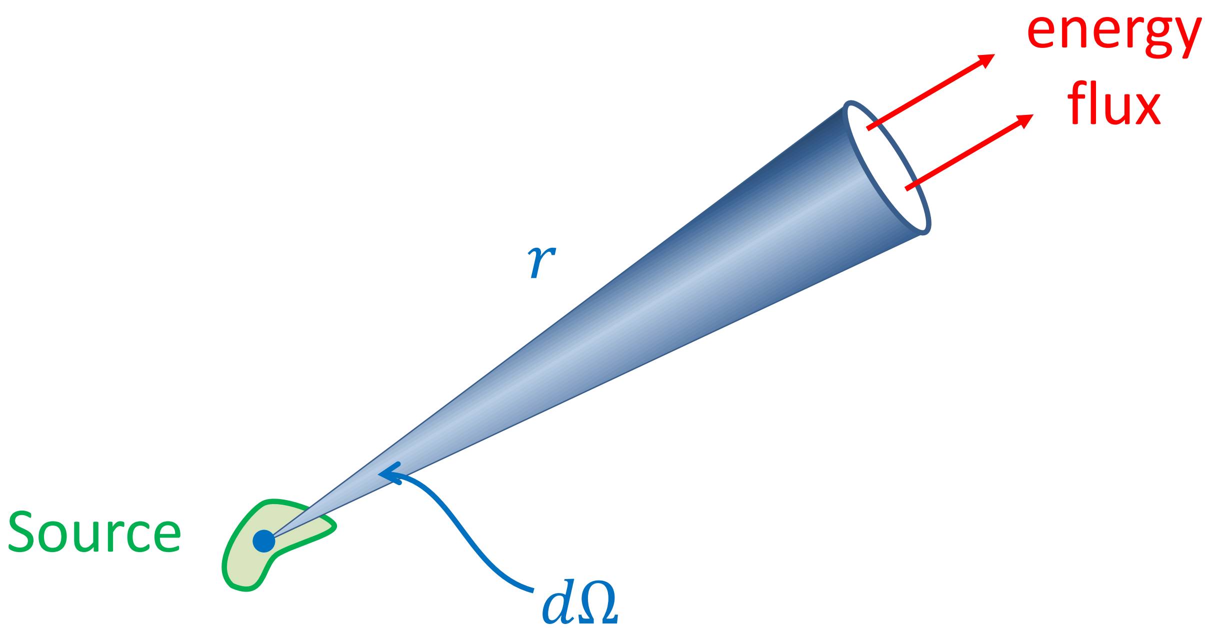
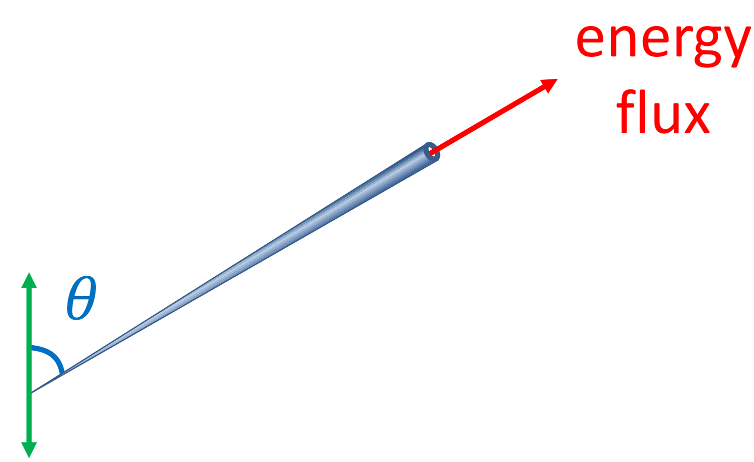


Figure 1 $dP/d\Omega$

(a)



(b)

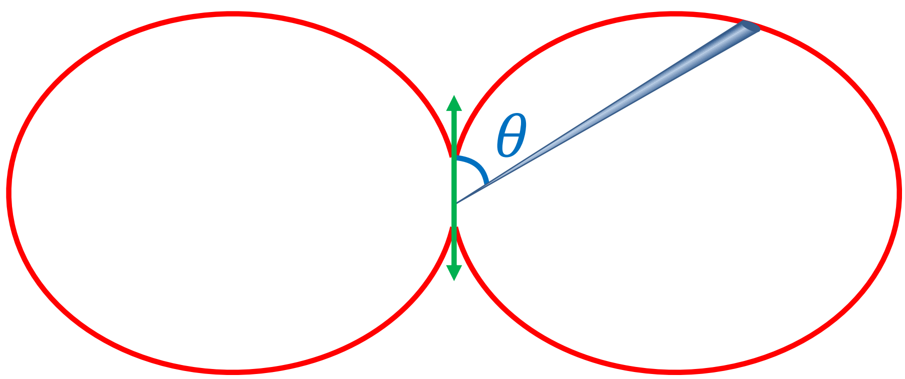


Figure 2 (a) An oscillating dipole radiating into the direction θ
(b) The power pattern

(a)

(b)

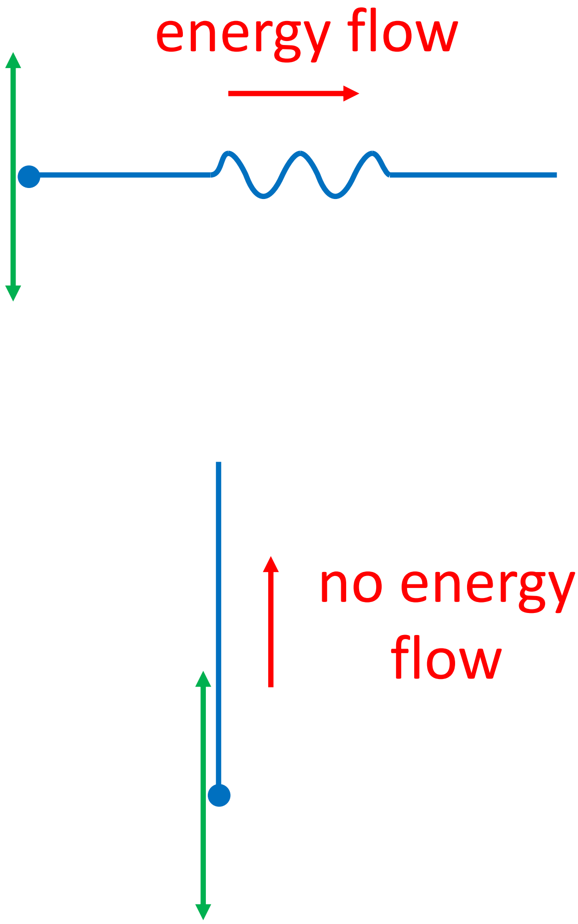


Figure 3 Analogy of a string

(a) Transverse waves are generated if the direction of the string is perpendicular to the shaking

(b) No energy flows if the direction of the string is along the direction of shaking

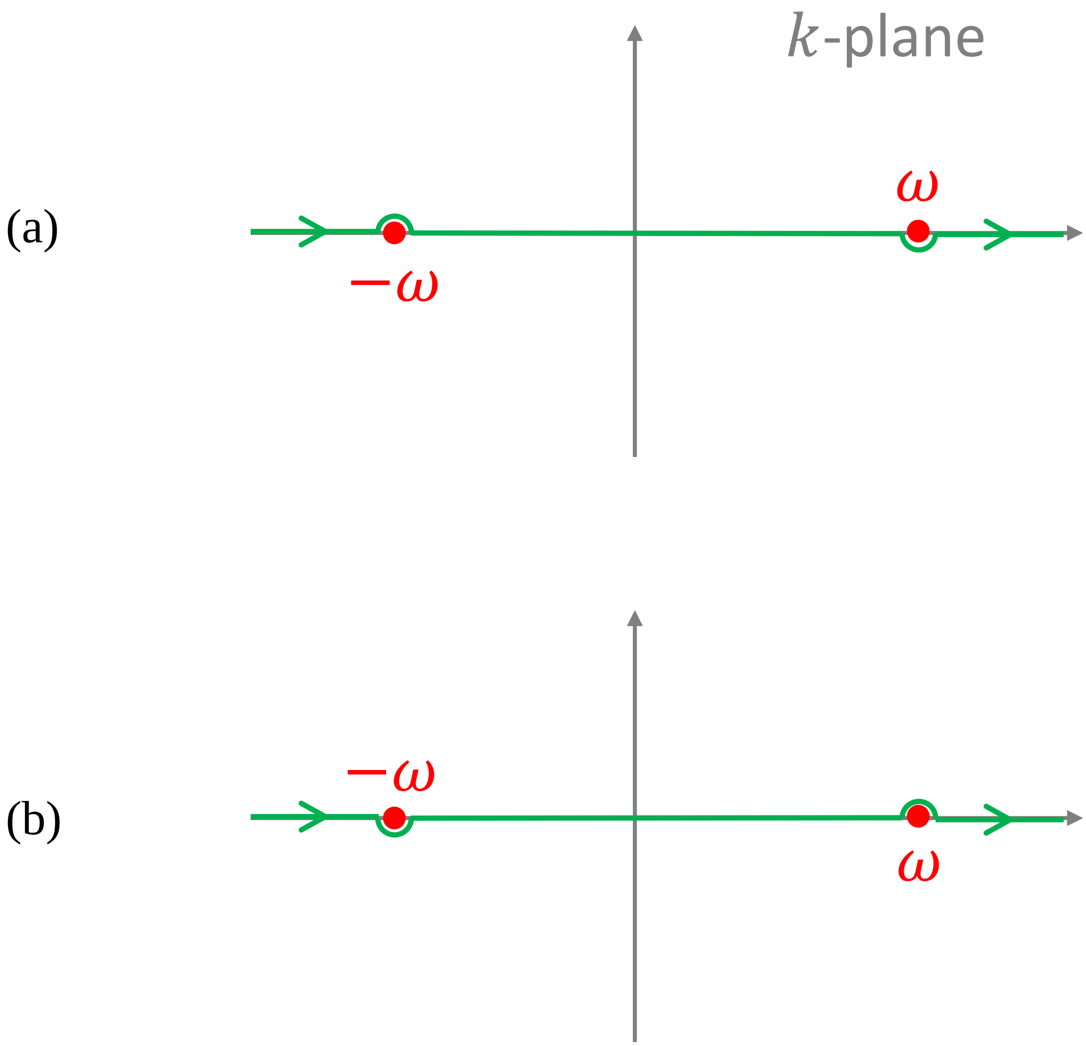


Figure 4 Contours for the integral
(a) The correct choice
(b) An incorrect choice