Chapter 18
Propagator, photon mass, gauge invariance

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An account is given of several related topics: the photon propagator, the possibility of photon mass and the intricacies of gauge invariance. These concepts turn out to be important if ideas from electrodynamics are to be extended to the other interactions, and a brief introduction is given to some aspects of massive vector bosons, especially their propagators.

Contents

1 Propagator

1.1 Simple treatment

This Section first of all repeats, in a covariant notation, some of the results already encountered in solving Maxwell's equations. Recall from the last Chapter that the equations are

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \\ \partial^2 A^\nu - \partial^\nu(\vec{\partial}\cdot\vec{A}) &= -\mu_0 J^\nu \end{aligned} \qquad (1)$$

where $\partial^2 = \vec{\partial}\cdot\vec{\partial}$.

Decoupling

The equations in (1) are coupled: e.g., for $\nu = 0$, the unknowns A^μ with $\mu = 1,2,3$ also appear through the term $(\vec{\partial}\cdot\vec{A})$. Thus, a gauge transformation is used

$$A^\mu \quad \mapsto \quad A^\mu + \partial^\mu\Lambda$$

under which $F^{\mu\nu}$ is unchanged. Then

$$\vec{\partial}\cdot\vec{A} \quad \mapsto \quad \vec{\partial}\cdot\vec{A} + \partial^2\Lambda$$

which can be set to zero by a choice of Λ. Then

$$\boxed{D\,A^\nu = \mu_0 J^\nu} \qquad (2)$$

where

$$D \quad = \quad -\,\partial^2$$

The decoupled equations are easier to solve.

Define the scalar Green's function G satisfying

$$\boxed{D\,G(x) = \delta^4(x)} \qquad (3)$$

in terms of which

$$\boxed{A^\nu(x) = \mu_0 \int G(x-y) J^\nu(y)\, d^4y} \qquad (4)$$

Each component A^ν depends only on the corresponding component J^ν. The Green's function G was solved when we discussed radiation by a moving charge.

Recall that our notation is $d^4x = d^3x\,dt$, $\delta^4(x) = \delta^3(x)\,\delta(t)$.

1.2 Tensor Green's function

Consider a different point of view, *without* first making the gauge transformation to decouple the components. Write (1) as

$$\boxed{D^\mu{}_\nu\, A^\nu = \mu_0 J^\mu} \qquad (5)$$

with

$$D^{\mu}{}_{\nu} = -\left(\partial^2 \delta^{\mu}{}_{\nu} - \partial^{\mu}\partial_{v}\right) \tag{6}$$

Define a tensor Green's function

$$\boxed{D^{\mu}{}_{\nu}\, G^{\nu}{}_{\rho}(x) = \delta^{\mu}{}_{\rho}\, \delta^4(x)} \tag{7}$$

in terms of which

$$\boxed{A^{\mu}(x) = \mu_0 \int G^{\mu}{}_{\nu}(x-y) J^{\nu}(y)\, d^4y} \tag{8}$$

Now J^{ν} can give rise to A^{μ} for $\mu \neq \nu$.

Go to momentum space, i.e.,

$$\begin{aligned} G^{\nu}{}_{\rho}(x) &= \int \tilde{G}^{\nu}{}_{\rho}(k)\, \exp(i\vec{k}\cdot\vec{x})\, \frac{d^4k}{(2\pi)^4} \\ \partial^{\mu} &\mapsto ik^{\mu} \end{aligned}$$

which turns (7) into an *algebraic* equation for each k:

$$\boxed{\tilde{D}^{\mu}{}_{\nu}(k)\, \tilde{G}^{\nu}{}_{\rho}(k) = \delta^{\mu}{}_{\rho}} \tag{9}$$

where

$$\tilde{D}^{\mu}{}_{\nu}(k) = k^2 \delta^{\mu}{}_{\nu} - k^{\mu}k_{\nu} \tag{10}$$

The algebraic equation (9) can be represented in matrix form

$$[\,\tilde{D}\,]\,[\,\tilde{G}\,] = [\,I\,]$$

and immediately solved, at least formally

$$\boxed{[\,\tilde{G}\,] = [\,\tilde{D}\,]^{-1}} \tag{11}$$

This is a general notion:

The Green's function is the inverse of the relevant differential operator.

Singularity

The trouble is that *this does not work*. The reason is that $[\tilde{D}]$ is singular and there is no unique inverse. To prove that a matrix is singular, it suffices to show that it has one zero eigenvector.

Problem 1

Show that $[\tilde{D}(k)]\,[k] = 0$, i.e., $\tilde{D}^{\mu}{}_{\nu}(k)\, k^{\nu} = 0$. In other words, $\vec{k}$ is a zero eigenvector. §

1.3 Gauge fixing

To deal with the singular matrix $[\tilde{D}(k)]$, minimize $\mathcal{S}$ subject to a constraint that

$$\int (\vec{\partial}\cdot\vec{A})^2\, d^4x$$

is some prescribed value (say zero). To impose such a constraint, add this expression to $\mathcal{S}$ with a Lagrange multiplier $\propto \lambda$. Thus the modified Lagrangian density becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4\mu_0}\left[F_{\mu\nu}F^{\mu\nu} + 2\lambda(\vec{\partial}\cdot\vec{A})^2\right] \\ &\quad + \vec{J}\cdot\vec{A} \end{aligned} \tag{12}$$

Note that $\mathcal{L}$ and hence $\mathcal{S}$ are no longer gauge invariant: the added term changes upon a gauge transformation.

Upon a variation, and freely discarding total derivatives (which convert to surface terms at infinity when placed under the integral),

$$\begin{aligned} &\delta\mathcal{L} \\ &= -\frac{1}{\mu_0}\left[F_{\mu\nu}(\partial^{\mu}\delta A^{\nu}) + \lambda(\vec{\partial}\cdot\vec{A})(\partial_{\nu}\delta A^{\nu})\right] \\ &\quad + J_{\nu}\,\delta A^{\nu} \\ &= \frac{1}{\mu_0}\left[\partial^{\mu}F_{\mu\nu}(\delta A^{\nu}) + \lambda(\partial^{\mu}\partial_{\nu}A_{\mu})(\delta A^{\nu})\right] \\ &\quad + J_{\nu}\,(\delta A^{\nu}) \end{aligned}$$

Setting the integral of this to zero and peeling off the factor of δA^{ν}, we find

$$\frac{1}{\mu_0}\left(\partial^{\mu}F_{\mu\nu} + \lambda\partial^{\mu}\partial_{\nu}A_{\mu}\right) + J_{\nu} = 0$$

or

$$\partial^2 A^{\nu} - (1-\lambda)\partial^{\mu}\partial^{\nu}A_{\nu} = -\mu_0 J_{\nu} \tag{13}$$

This equation reduces to (1) if $\lambda = 0$.

The corresponding tensor Green's function should satisfy

$$\begin{aligned} -\left[\partial^2\delta^{\mu}{}_{\nu} - (1-\lambda)\partial^{\mu}\partial_{\nu}\right] G^{\nu}{}_{\rho}(x) &= \delta^{\mu}{}_{\rho}\delta^4(x) \\ \left[k^2\delta^{\mu}{}_{\nu} - (1-\lambda)k^{\mu}k_{\nu}\right] \tilde{G}^{\nu}{}_{\rho}(k) &= \delta^{\mu}{}_{\rho} \end{aligned} \tag{14}$$

Transverse and longitudinal projectors

The equations can be written more compactly if we define the following matrices

$$\begin{aligned} L^{\mu}{}_{\nu} &= \frac{k^{\mu}k_{\nu}}{k^2} \\ T^{\mu}{}_{\nu} &= \delta^{\mu}{}_{\nu} - \frac{k^{\mu}k_{\nu}}{k^2} \end{aligned}$$

They satisfy the properties in the Problem below.

Problem 2

Prove the following identities:

$$\begin{aligned} [\,T\,]^2 &= [\,T\,] \\ [\,L\,]^2 &= [\,L\,] \\ [\,L\,]\,[\,T\,] = [\,T\,]\,[\,L\,] &= [\,0\,] \\ [\,L\,] + [\,T\,] &= [\,I\,] \end{aligned}$$

These show that $[L]$ and $[T]$ are projection operators. §

Problem 3
Show that the inverse of the matrix

$$a\,[\,L\,] + b\,[\,T\,]$$

is

$$a^{-1}\,[\,L\,] + b^{-1}\,[\,T\,]$$

provided $a, b \neq 0$. §

Then (14) becomes

$$\begin{aligned} k^2 \left\{ T^\mu{}_\nu + \lambda L^\mu{}_\nu \right\} \tilde{G}^\nu{}_\rho &= \delta^\mu{}_\rho \\ k^2 \left\{ [\,T\,] + \lambda\,[\,L\,] \right\} [\,\tilde{G}\,] &= [\,I\,] \end{aligned}$$

An inverse exists provided $\lambda \neq 0$, and is readily shown to be

$$\begin{aligned} [\,\tilde{G}(k)\,] &= \frac{1}{k^2} \left\{ [\,T\,] + \lambda^{-1}\,[\,L\,] \right\} \\ \tilde{G}^\mu{}_\nu(k) &= \frac{1}{k^2} \left(T^\mu{}_\nu + \lambda^{-1} L^\mu{}_\nu \right) \end{aligned}$$

Upon defining

$$\xi = \lambda^{-1}$$

we can now write this in the standard form

$$\tilde{G}^{\mu\nu}(k) = \frac{1}{k^2} \left[\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) + \xi \frac{k^\mu k^\nu}{k^2} \right]$$

or

$$\boxed{\tilde{G}^{\mu\nu}(k) = \frac{1}{k^2} \left[\eta^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right]} \qquad (15)$$

To adopt the terminology in particle physics, we call $G^{\mu\nu}$ or $\tilde{G}^{\mu\nu}$ the photon *propagator*.

Some special gauges
The choices of propagator shown in (15) are called R_ξ gauges. Two choices are used commonly:

$$\begin{aligned} \text{Feynman gauge:} \quad & \xi = 1 \\ \text{Lorenz gauge:} \quad & \xi \to 0 \end{aligned}$$

In the present approach, the case $\xi = \lambda^{-1}$ being zero should be understood as a limit.

These gauge choices actually go under a variety of names in the literature. In particular, the Lorenz gauge is now typically (and erroneously) called the Lorentz gauge (although it is indeed Lorentz invariant); it is also called the Landau gauge.

Why gauge fixing term
The method of adding an auxiliary term to $\mathcal{S}$ seems more cumbersome. What is the advantage?

The simple approach in Section 1.1 is at the level of equation of motion, whereas that in Section 1.3 is at the level of action $\mathcal{S}$ (and hence also Lagrangian L and Hamiltonian H). In quantum mechanics, the equation of motion is not enough; we need either $\mathcal{S}$, L or H. For example, in the path integral method, we integrate over all paths (whether or not they satisfy the equation of motion) using a weight

$$\exp(i\mathcal{S}/\hbar) \qquad (16)$$

The path with extremum (or stationary) $\mathcal{S}$ has the largest contributions if $\hbar \to 0$; but we need to know about $\mathcal{S}$ for all other paths (especially if $\hbar$ cannot be regarded as small). So the problem has to be addressed at the level of the action, not just at the level of the equation of motion.

This whole set of issues, and the use of R_ξ gauges, is now a key computational (indeed conceptual) ingredient not only in quantum electrodynamics (QED) but also in non-abelian gauge theories underpinning the Standard Model of elementary particles. There is a further problem when the particle involved (the photon or its analog) has mass.

1.4 Why can we use different parameters?

Go back to the following question: Physics surely cannot depend on the gauge parameter ξ in (15); so how come we are allowed to use different values of ξ in calculations?

Classical level
At the classical level, the answer is straightforward. The ξ-dependent part is proportional to $k^\mu k^\nu$, which upon going back to position space turns into $\partial^\mu \partial^\nu$; through an integration by parts, this acts on J_ν, giving $\partial^\nu J_\nu = 0$ by charge conservation.

Charge conservation is intimately related to gauge invariance, as we can see from Maxwell's equation

$$\partial^\mu F_{\mu\nu} = -\,\mu_0 J_\nu$$

Gauge transformations is rooted in the fact that $F_{\mu\nu}$ is an antisymmetric derivative, which then leads to $\partial^\nu J_\nu = 0$.

Quantum level
At the quantum level, we sample all paths with the weight (16). Paths with different values of $(\vec{\partial} \cdot \vec{A})$ are sampled with a weight determined by λ. Because all these paths are physically equivalent, using different weights (i.e., different λ or ξ) would yield end results that are physically equivalent.

2 Photon mass?

Is it possible that the photon mass is small but non-zero?

2.1 Proca equation

Is it possible that the laws of EM (as we have learnt so far) are not exactly correct, but there are minor corrections? So long as we believe in relativity and superposition, there are very few possible changes to the action: any extra term must be a 4-scalar quadratic in the field. Let us now take the action to be the integral of

$$\begin{aligned}\mathcal{L} \quad = \quad & \frac{1}{4\mu_0}\left(-F^{\mu\nu}F_{\mu\nu} - 2a^{-2}A^\mu A_\mu\right) \\ & + J^\mu A_\mu \end{aligned} \tag{17}$$

where the second term is the new feature. The parameter a has dimension of length. What happens if this extra term is not zero? A gauge-fixing term proportional to $(\vec{\partial}\cdot\vec{A})^2$ is *not* included, for reasons discussed below.

In elementary accounts, possible deviations from the standard theory are usually expressed as a dependence $1/r^{2+\epsilon}$ in Coulomb's law. However, within a relativistic framework, the more natural deviations to consider is an additional term in the action.

Problem 4
By a variation, derive the equation of motion

$$\begin{aligned} & \partial^2 A_\nu - \partial_\nu(\vec{\partial}\cdot\vec{A}) - a^{-2}A_\nu \\ = \quad & -\mu_0 J_\nu \end{aligned} \tag{18}$$

This differs from the usual Maxwell equation only by one extra term. §

In fact, the above equation can be further simplified. If we operate with ∂^ν, the first two terms cancel, and on the RHS, $\partial^\nu J_\nu = 0$ because of charge conservation, so

$$a^{-2}(\vec{\partial}\cdot\vec{A}) \quad = \quad 0$$

So long as $a^{-2} \neq 0$, $\vec{\partial}\cdot\vec{A}$ is fixed, and there is no gauge degree of freedom, and we are not allowed to change $\vec{A} \mapsto \vec{A} + \vec{\partial}\Lambda$.

Putting this back in (18), the second term on the LHS can be dropped, and we find that the equation of motion simplifies to

$$\boxed{\left(\partial^2 - a^{-2}\right)A_\nu = -\mu_0 J_\nu} \tag{19}$$

which is known as the *Proca equation* [1].

2.2 Vacuum solution

Vacuum solutions are plane waves that go as $A_\nu \propto \exp i\vec{k}\cdot\vec{x}$, so $\partial^\mu \mapsto ik^\mu$; so (19), with the RHS set to zero for vacuum, then gives the algebraic equation

$$\vec{k}\cdot\vec{k} + a^{-2} \quad = \quad 0$$

and remembering that $\vec{k} = (\omega/c, \mathbf{k})$, this becomes

$$(\omega/c)^2 = \mathbf{k}^2 + a^{-2}$$

Multiplying by $\hbar$ and invoking wave-particle duality: $E = \hbar\omega$, $\mathbf{p} = \hbar\mathbf{k}$ for the energy and momentum of the corresponding quantas (i.e., photons), we find

$$(E/c)^2 \quad = \quad \mathbf{p}^2 + (\hbar/a)^2$$

But the last term must be m^2c^2 where m is the mass of the particle, so we find

$$\boxed{a = \frac{\hbar}{mc}} \tag{20}$$

i.e., the Compton wavelength of the particle.

In short, allowing for such an additional term is the same as allowing for a photon mass.

2.3 Yukawa potential

Next, consider how Coulomb's law is altered. For this purpose take (19) with $\nu = 0$, restrict to statics, and let the source be a point charge:

$$\begin{aligned} \partial^2 &\mapsto \boldsymbol{\nabla}^2 \\ A^\nu &\mapsto A^0 = \Phi/c \\ J^\nu &\mapsto J^0 = \rho\, c = c\,\delta^3(\mathbf{x}) \end{aligned}$$

and (19) becomes

$$\left(\boldsymbol{\nabla}^2 - a^{-2}\right)\Phi \quad = \quad -\frac{1}{\epsilon_0}\delta^3(\mathbf{x}) \tag{21}$$

First solve this away from the origin, in which case the RHS is zero. Using spherical symmetry

$$\boldsymbol{\nabla}^2 \quad = \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$$

Problem 5
By using the substitution $\Phi(r) = \Psi(r)/r$, show that the spherically symmetric solution to the homogeneous version of (21) is

$$\Phi(r) \quad = \quad A\,\frac{\exp\left(-r/a\right)}{r}$$

where A is some constant. §

The constant A is determined by matching the solution to the δ-function on the RHS. The evaluation can be finessed by recognizing that as $r \to 0$, the exponential factor is unity, and the calculation is exactly the same as the case without the extra term a^{-2} in the equation. Thus

$$\boxed{\Phi(r) = \frac{1}{4\pi\epsilon_0} \frac{\exp(-r/a)}{r}} \tag{22}$$

Compared to the conventional case, the potential (and hence the field) is rapidly cut off beyond $r \sim a$. We refer to a as the *range* of the potential. This type of potential with an exponential cut-off is called a *Yukawa potential*, first introduced for scalar fields. A qualitative discussion was given in Chapter 1.

The same applies to the other components, so the Biot–Savart law for magnetism will have a similar cut-off for $r \gg a$.

2.4 Experimental limits

What experimental limits can be set on the mass of the photon?

Direct method

The direct method is to measure the velocity for different energies. The velocity in units of c is

$$\begin{aligned} \beta &= \frac{pc}{E} = \frac{\sqrt{E^2 - m^2c^4}}{E} \\ &\approx 1 - \frac{m^2c^4}{2E^2} \end{aligned}$$

So if β is measured for two different energies (and ignoring signs),

$$\Delta\beta \approx m^2c^4 \frac{\Delta E}{E^3}$$

or

$$\left(\frac{mc^2}{E}\right)^2 \approx \frac{\Delta\beta}{\Delta E/E}$$

As an example, if photons of energy around $E \sim 1$ eV are measured, and it is found that for a 10 percent change in energy ($\Delta E/E = 0.1$), β changes by no more than one part in 10^7 (which would require extremely accurate timing measurements), then the RHS is bounded by 10^{-6} and we would get the limit $mc^2 < 10^{-3}E \approx 10^{-3}$ eV, a rather weak limit.

Using astronomical magnetic fields

Magnetic fields would be cut off at distances larger than a. So observations of terrestrial magnetic fields (with sources certainly inside the earth) easily set a limit $a > 10^7$ m. Analysis of the magnetic field observed above Mars [2] gave a better limit of $a > 5 \times 10^8$ m, and later astronomical arguments [3] led to improvements by two more orders of magnitude. If we take $a > 10^9$ m, that sets a limit of about $mc^2 < 10^{-16}$ eV, or a ratio of 10^{-25} compared to the proton.

This limit is so tight that it is believed that the photon mass is *exactly zero*.

2.5 Photon mass and gauge invariance

We need a reason for the photon mass to be exactly zero.[1] The argument can be presented at three levels.

- Heuristically, zero is a special number. If we choose the mass randomly, it is most "unlikely" that it is exactly zero.
- A more sophisticated argument is that there are perturbations. **Figure 1** shows a second-order process in which a photon in state 1 turns into an intermediate state j consisting of an electron-positron pair, which then turns back into state 1. Even in non-relativistic quantum mechanics, there is a second-order shift in the energy involving something like

  $$\Delta E \sim \sum_j \langle 1|V|j\rangle \langle j|V|1\rangle \, (E_1 - E_j)^{-1}$$

  where V is the perturbation potential. In relativistic quantum field theory, the formulas are technically more complicated, but the idea is similar. So even if we choose the mass to be zero "to start with", these quantum corrections will cause the energy — and hence the mass — to change. This makes it all the more "unlikely" that the mass is zero, unless there is a "reason" for it.
- Even more seriously, when we actually do the sum, the answer[2] can be expressed in terms of a function $\Pi_{\mu\nu}(k)$ which is formally infinite, at least in the first few terms in the power series expansion in $|k|$. We are stuck unless there is some principle that tells us how to handle the infinities.

It is now believed that the answer is *gauge invariance*.

At the classical level, the Proca equation shows that a photon mass (i.e., a non-zero term $\propto A^\mu A_\mu$

[1] Just as we need a "reason", namely the geometric theory of gravity, to explain why the inertial mass is exactly equal to the gravitational mass.

[2] Not just for the one-loop diagram in Figure 1 but to all orders in perturbation theory.

in $\mathcal{L}$) violates gauge invariance: physics would not be described only by **E** and **B**. Turning it around, if the fields are the only physical variables, there should be gauge invariance, and hence the photon mass has to be zero.[3]

At the quantum level, gauge invariance leads to Ward's identity, which implies, among other things, that

$$k^\mu \, \Pi_{\mu\nu}(k) \quad = \quad 0$$

so that $\Pi_{\mu\nu}$ must have the tensor structure

$$\Pi_{\mu\nu}(k) \quad = \quad (\eta_{\mu\nu} k^2 - k_\mu k_\nu) \, \Pi(k)$$

The two powers of k in the prefactor mean that there is no contribution at $k = 0$. Thus the pole of the propagator at $k^2 = 0$ is unaltered, and the photon mass remains zero to all orders in perturbation theory.[4] Simple accounts of these issues in QED can be found in Ref. [5, 6].

3 Massive vector bosons

3.1 Propagator

The photon does not have mass, but other spin-1 particles (vector bosons) do have mass, for example the $W^\pm$ and Z^0 particles that mediate the weak interactions. Here we take the opportunity to outline a few important issues that arise in these cases.

For this purpose go back to (18) and do not assume that the current is conserved. Since this no longer refers to EM, we may as well absorb the coefficient μ_0 into J_ν. Moreover, in this Section adopt units where $\hbar = c = 1$; then $a^{-2} = m^2$, where m is the boson mass. Going to momentum space and replacing the RHS by a δ-function in order to obtain the Green's function, we have

$$\begin{aligned} \tilde{D}^\mu{}_\nu \, \tilde{G}^\nu{}_\rho &= \delta^\mu{}_\rho \\ \tilde{D}^\mu{}_\nu &= \left(k^2 \delta^\mu{}_\nu - k^\mu k_\nu + m^2 \delta^\mu_\nu \right) \end{aligned} \tag{23}$$

In matrix form

$$\begin{aligned} [\, \tilde{D} \,] &= k^2 \, [\, T \,] + m^2 \, [\, I \,] \\ &= (k^2 + m^2) \, [\, T \,] + m^2 \, [\, L \,] \end{aligned}$$

[3]However, it has been found [4] that for the Proca equation, one can construct a generalized gauge transformation that ensures renormalizability without the need for the Higgs mechanism. This is theoretically interesting, but probably not physically relevant, since QED is now believed to be part of a unified electroweak theory which definitely does need the Higgs mechanism. I thank Pui Tak Leung for drawing my attention to Ref. [4].

[4]Gauge invariance in effect fixes the $|k|^0$ and $|k|^1$ terms in $\Pi_{\mu\nu}$. The $|k|^2$ term, i.e., $\Pi(0)$, is still infinite, but that can be "hidden" in a renormalization constant Z_3.

so its inverse is

$$\begin{aligned} & [\, \tilde{G} \,] \\ = \; & \frac{1}{k^2 + m^2} \, [\, T \,] + \frac{1}{m^2} \, [\, L \,] \\ = \; & \frac{1}{k^2 + m^2} \, \{ [\, I \,] - [\, L \,] \} + \frac{1}{m^2} \, [\, L \,] \\ = \; & \frac{1}{k^2 + m^2} \, [\, I \,] + \left(\frac{-1}{k^2 + m^2} + \frac{1}{m^2} \right) [\, L \,] \end{aligned}$$

The last bracket simplifies to

$$\begin{aligned} & \frac{-1}{k^2 + m^2} + \frac{1}{m^2} \\ = \; & \frac{-\, m^2 + (k^2 + m^2)}{(k^2 + m^2) \, m^2} = \frac{k^2}{(k^2 + m^2) \, m^2} \end{aligned}$$

thus giving

$$[\, \tilde{G} \,] \quad = \quad \frac{1}{k^2 + m^2} \left\{ [\, I \,] + \frac{k^2}{m^2} \, [\, L \,] \right\}$$

or, in explicit component form

$$\boxed{\tilde{G}^\mu{}_\nu = \frac{1}{k^2 + m^2} \left(\delta^\mu{}_\nu + \frac{k^\mu k_\nu}{m^2} \right)} \tag{24}$$

3.2 Asymptotic behavior

There is a fundamental difference between the propagator $\tilde{G}^\mu{}_\nu(k)$ for the photon (15) and for a massive vector boson (24). At large k, the former goes as $|k|^{-2}$, whereas the latter goes as $|k|^0$ due to the longitudinal term. In higher-order perturbation theory, one encounters integrals of the form

$$\int d^4 k \, G^\mu{}_\nu(k) \, \ldots$$

Integrals involving the photon propagator are divergent, but the divergence can be handled.[5] But the corresponding integrals for massive vector bosons would be two powers worse[6] and the problem cannot be cured; it does not lead to a finite theory — unless there is a clever way to cure the asymptotic behavior of the propagator. We say the theory of photons is renormalizable, but the theory for massive vector bosons is (apparently) not renormalizable.

The difference in asymptotic behavior is also related to the fact that a massive vector particle has 3 polarizations whereas a massless photon has only two.

To summarize, we have discovered a tight relationship among the following cluster of concepts.

[5]In effect, "hidden" in a small number of renormalization constants — this achievement, by Schwinger, Tomonaga and Feynman, is by itself highly nontrivial and the subject of a Nobel Prize.

[6]For each loop.

- Gauge invariance.
- Charge conservation.
- The photon mass being zero. (And hence the photon having only two polarizations.)
- The interaction being long range, described by a power law without exponential cut-off.
- The theory being renormalizable, namely that it makes sense when higher-order processes are included.

This leads us to a conundrum with the weak interaction: it seems to be described by vector bosons, but has short range. To allow a mass for the vector bosons would make the theory not renormalizable. This is the puzzle to which the famous Higgs boson [7, 8] provides the answer.

3.3 Gauge fixing for hidden symmetry

But suppose there is a hidden gauge symmetry that allows us to add a gauge-fixing term as before. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\vec{\partial}\cdot\vec{A})^2 \\ &\quad - \frac{m^2}{2}A^\mu A_\mu + J^\mu A_\mu \end{aligned} \tag{25}$$

Variation leads to the equation of motion

$$\left[\delta^\mu{}_\nu \partial^2 - (1{-}\lambda)\partial^\mu\partial_\nu - m^2\delta^\mu{}_\nu\right] A^\nu \quad = \quad -J^\mu$$

The Green's function in momentum space satisfies

$$\left[\delta^\mu{}_\nu k^2 - (1{-}\lambda)k^\mu k_\nu + m^2\delta^\mu{}_\nu\right] \tilde{G}^\nu{}_\rho \quad = \quad \delta^\mu{}_\rho$$

or in matrix form

$$\begin{aligned}\left\{(k^2+m^2)\,[\,I\,] - (1{-}\lambda)k^2\,[\,L\,]\right\}[\,\tilde{G}\,] &= [\,I\,] \\ \left\{(k^2+m^2)\,[\,T\,] + (\lambda k^2+m^2)\,[\,L\,]\right\}[\,\tilde{G}\,] &= [\,I\,]\end{aligned}$$

which gives the inverse

$$\begin{aligned}[\,\tilde{G}\,] &= \frac{1}{k^2+m^2}\,[\,T\,] + \frac{1}{\lambda k^2+m^2}\,[\,L\,] \\ &= \frac{1}{k^2+m^2}\,[\,T\,] + \frac{\xi}{k^2+\xi m^2}\,[\,L\,]\end{aligned}$$

It is conventional to put $[T] = [I] - [L]$ above, in which case the coefficient of $[L]$ becomes

$$\frac{-1}{k^2+m^2} + \frac{\xi}{k^2+\xi m^2} \quad = \quad \frac{-(1{-}\xi)k^2}{(k^2+m^2)(k^2+\xi m^2)}$$

so that we finally have

$$\begin{aligned}&\tilde{G}_{\mu\nu} \\ &= \frac{1}{k^2+m^2}\left[\eta_{\mu\nu} - (1{-}\xi)\frac{k_\mu k_\nu}{k^2+\xi m^2}\right]\end{aligned} \tag{26}$$

For $\lambda = 0$, i.e., $\xi \to \infty$, the second term is $O(|k|^2)$, with the problem as discussed earlier. But for any finite ξ, $[\tilde{G}]$ is $O(|k|^{-2})$ and solves the renormalization problem. Of course, there is a pole at an apparent mass-squared ξm^2, and it turns out (nontrivially) that this does not lead to any physical particle.

Important works

Obviously many subtleties are only hinted at and not explained above. These remaining subtleties are highly nontrivial, and took many years and many physicists to develop and resolve. We simply cite a number of Nobel Prizes related to these issues. Nearly all these works are related to the propagator (26).

- The key idea is the Higgs mechanism (and the Higgs boson that it implies). The idea was invented in 1964 by Englert and Brout, Higgs, and independently but slightly later, also Guralnik, Hagen and Kibble. After the Higgs boson was discovered in 2012, the 2013 Nobel Prize in Physics was awarded to Englert and Higgs. (Brout had died before 2013.)
- The mechanism was used by Weinberg and Salam to develop what has now become the Standard Model of the electroweak interactions.[7] The Higgs mechanism is key to this theory being finite. Weinberg and Salam were awarded the Nobel Prize in Physics 1979.[8]
- The Nobel Prize in Physics 1984 was awarded to Rubbia and van der Meer for the discovery of the massive vector bosons $W^\pm$ and Z^0, which are the carriers of the weak interaction in the Weinberg–Salam model.
- The Higgs mechanism relies on an idea of spontaneous symmetry breaking, for which Nambu was awarded the Nobel Prize in Physics 2008 (half share).
- The detailed and formal proof that all the high-order processes are finite, including elucidation of the various complications, was largely due to 't Hooft and Veltman, who were awarded the Nobel Prize in Physics 1999.

References

[1] See e.g., DN Poenaru and A Calboreanu, "Alexandru Proca (1897-1955) and his equation of the massive vector boson field", Europhysics

[7] In other words, a single theory unifying EM and the weak interactions.

[8] A third share went to Glashow for a slightly different but related part of the problem.

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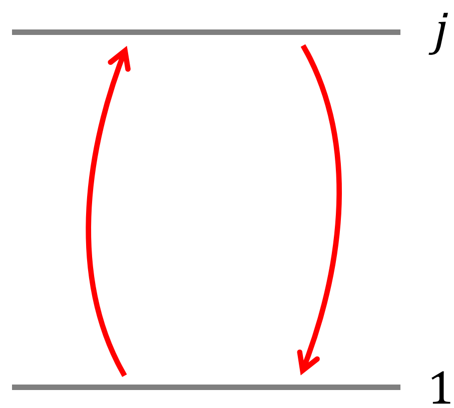

Figure 1 Second-order process in which the system in original state 1 jumps to state j and then returns back to state 1.