Chapter 21
Aharonov–Bohm effect and other topological issues

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The Aharonov–Bohm effect demonstrates that there are physically relevant aspects of the vector potential going beyond the fields, manifested when there are non-trivial topologies. Flux quantization in superconductors shows similar features, but in a way that depends on a macroscopic wavefunction describing a collective state.

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1 Introduction: topology

In the Chapter on monopoles, it was argued that the relevant degree of freedoms are not the fields, say **B**, but the loop integrals

$$A[\gamma] \quad = \quad \oint_\gamma A_\mu dx^\mu$$

or more precisely $A[\gamma]$ (mod h/q) for a charge q. But is there any difference between these loop integrals and the fields? For simplicity, we examine only loops in space, and with reference to time-independent situations, so the above reduces to

$$A[\gamma] \quad = \quad \oint_\gamma \mathbf{A}\cdot d\mathbf{x}$$

By familiar arguments:
- Any loop γ can be broken up into the sum of small loops γ_j (**Figure 1**).
- For any small loop, the line integral can be converted to the value of $\mathbf{B}\cdot\Delta\mathbf{S}$.

So it would seem that $A[\gamma]$ conveys neither more nor less information than all the fields. That is indeed the case "usually", but not if the domain has a nontrivial topology — and we shall concentrate on such cases in this Chapter.

Suppose a charged particle q moves in a 2D plane, but is *excluded* from a "hole", say a disk of radius R (shaded region in **Figure 2a**). The allowed domain $\mathcal{D}$ (the plane minus the "hole") has a nontrivial topology: a closed loop γ cannot be continuously shrunk to a point (**Figure 2b**). It cannot be reduced to a sum of small loops. In this case, the value of $A[\gamma]$ is *not* completely determined by the values of **B** in $\mathcal{D}$ — the loop integrals convey slightly *more* information. In particular, $A[\gamma]$ depends on the magnetic flux φ in the hole.

To be precise, assume the situation to be cylindrically symmetric, and assume there is no magnetic field outside the "hole". Choose a gauge such that $\mathbf{A} = A(r)\,\hat{\mathbf{e}}_\phi$. Then for γ being a circle of radius r,

$$A(r)\cdot 2\pi r = A[\gamma] \quad = \quad \varphi \tag{1}$$

In particular, $\mathbf{A} \neq 0$ in $\mathcal{D}$.

This can be put more physically. Suppose the charge particle is confined to $\mathcal{D}$, where $\mathbf{B} = 0$. Thus, this particle never experiences any **B**, but it does experience a non-zero **A** — which cannot be completely removed by a gauge transformation. Is there any effect? In other words, does any observable result depend on φ — which is a flux that the charge never "touches"? If there is an effect, then it would really be true that

- there is something that goes beyond the local values of $\mathbf{B}$;[1]
- there is something that goes qualitatively beyond classical EM, which refers only to the local values of $\mathbf{E}$ and $\mathbf{B}$; or equivalently;
- the action formalism (which depends on the potentials and not just the fields) is more comprehensive than just the Lorentz force law.

Aharonov and Bohm (AB) first pointed out that there is indeed such an effect in cases of non-trivial topology [1]. The claim was experimentally verified by Tonomura et al. [2, 3]. This idea is explained in the next Section.

2 The AB experiment

AB proposed a way to test the idea that physics depends on $A[\gamma]$ rather than merely $\mathbf{B}$. **Figure 3a** shows a standard double-slit interference experiment for electrons. On a distant screen, there will be alternate bright and dark fringes depending on the difference in phase between electron waves propagating on the two paths. Now add a magnetic flux φ confined to the shaded region shown in **Figure 3b**, and somehow exclude electrons from this region. AB claimed [1] that the interference fringes will shift when φ is changed — a real and measurable effect even if the electrons never touch any $\mathbf{B}$ field. This is the physical situation to be analyzed in this Section.

The allowed domain $\mathcal{D}$ is all of space minus the shaded region, as discussed in the last Section. Of course, **Figure 3** shows only a cross-section (say the x–y plane) and the excluded region is a cylinder.

2.1 Equation for electron waves

Consider a charge q ($q = -e$ for an electron) moving in the domain $\mathcal{D}$, and assume there is no electric field, so the scalar potential can be ignored. Assume that the energy E of the electron is known. Then, in general, the wavefunction Ψ satisfies

$$\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \Psi \quad = \quad E\,\Psi \tag{2}$$

Solution without vector potential
First review the case where the vector potential is zero. Then (2) reduces to

$$\left(-i\boldsymbol{\nabla}\right)^2 \Psi \quad = \quad k^2\,\Psi$$

where

$$\begin{aligned} k &= \frac{p}{\hbar} \\ \frac{p^2}{2m} &= E \end{aligned} \tag{3}$$

The solution is (up to a constant multiple)

$$\Psi(\mathbf{r}) \quad = \quad \exp[i\xi(\mathbf{r})]$$

and the phase satisfies

$$\boldsymbol{\nabla}\xi(\mathbf{r}) \quad = \quad \mathbf{k} \tag{4}$$

with solution

$$\xi(\mathbf{r}) \quad = \quad \mathbf{k}\cdot\mathbf{r}$$

where $\mathbf{k}$ is any vector with magnitude given by (3). The physical interpretation is obvious.

The phase difference for the two paths (**Figure 3**) is given by

$$\Delta\xi = k\,(r_2 - r_1) \tag{5}$$

and this determines the positions of the maxima and minima.

Problem 1
Suppose the separation between the two slits is a, and the electrons are detected on a screen a large distance L away (**Figure 4**). Show that the vertical position y of the maxima are given by

$$y \quad = \quad n\cdot\frac{L}{a}\,\frac{2\pi}{k} \tag{6}$$

where n is an integer. The minima occur if n is replaced by $n+1/2$. Note that the separation between maxima is essentially the wavelength λ amplified by the geometric ratio L/a. §

Solution with vector potential
In the presence of a vector potential, (4) becomes

$$\boldsymbol{\nabla}\xi(\mathbf{r}) - (q/\hbar)\mathbf{A} \quad = \quad \mathbf{k} \tag{7}$$

with solution, again up to a multiplicative constant

$$\xi(\mathbf{r}) \quad = \quad \mathbf{k}\cdot\mathbf{r} + \frac{q}{\hbar}\int^{\mathbf{r}}\mathbf{A}\cdot d\mathbf{r} \tag{8}$$

Because of the second term in (8), the difference in phase becomes

$$\begin{aligned} \Delta\xi &= k(r_2 - r_1) + \frac{q}{\hbar}\oint\mathbf{A}\cdot d\mathbf{r} \\ &= k(r_2 - r_1) + \frac{q\varphi}{\hbar} \end{aligned} \tag{9}$$

where the closed loop integral is around the path $\gamma_2 - \gamma_1$ (**Figure 5**), and in the last step we have used Stoke's theorem. It is clear that the position

[1] The *local* values mean the values at the position of the charged particle. In classical EM, the values elsewhere do not matter.

y of the maxima (given by $\Delta\xi = n \cdot 2\pi$) will be shifted if φ is changed — even if the electron never "touches" any $\mathbf{B}$.

In principle, if one can continuously increase φ, one should observe the maxima shifting continuously. However, φ is relevant only to the extent of

$$\frac{q\varphi}{\hbar} \pmod{2\pi}$$

or in other words, the only relevant variable is

$$\boxed{\mathrm{frac}\left[\frac{q\varphi}{h}\right]}$$

where frac denotes the fractional part of a real number.

Problem 2
Work out how y in (6) is changed when there is φ. §

Shifting by half a period
A convenient way to express the dependence on φ is to ask how much φ has to change to shift the pattern by half a period, i.e., interchange maxima and minima, as schematically illustrated in **Figure 6**. Obviously we need to make $q\varphi/h$ change by 1/2, i.e.,

$$\Delta\varphi = \frac{h}{2e} \tag{10}$$

where we have specialized to the case where the test charge is an electron, with $|q| = e$.

2.2 Experimental verification

Simple design
Conceptually, the experiment goes as follows. Construct a long thin solenoid with N turns of wire in a length ℓ, carrying a current I. The flux is (mostly) confined in the solenoidal tube, with cross section area A. The flux is

$$\varphi = BA = \mu_0(N/\ell)I \cdot A \tag{11}$$

So theoretically, by tuning I, the interference pattern can be shifted continuously.

Problem 3
Imagine such a solenoid with $N = 10^3$ turns in a length $\ell = 0.1$ m, and a cross-section area $A = 10^{-5}$ m^2. By how much must I change to cause the interference pattern to shift by half a period, i.e., as in (10)? §

It is seen that we are talking about tiny (changes of) current, which makes the practical design and operation almost impossible. There is another difficulty. Even for a long, thin and tightly-wound solenoid, there is inevitably some leakage of flux outside. So there is a small fringe magnetic field $\mathbf{B}$ in the domain $\mathcal{D}$ in which the test charge moves. Therefore even if this effect is observed, one can argue that it may be due to the field $\mathbf{B}$ that the test charge experiences, and not really due to the AB effect caused only by $\mathbf{A}$. This simple design does not work.

Using superconductors
It turns out that in superconductors (SC), especially Type II SC, magnetic fields have the following properties:

- The fields are completely expelled from the bulk of the SC (i.e., $\mathbf{B} = 0$ exactly), either completely (**Figure 7a**) or except in tiny localized vortices (**Figure 7b**).
- The flux in a vortex is quantized as

$$\begin{aligned} \varphi &= n\varphi_0 \\ \varphi_0 &= \frac{h}{2e} \end{aligned} \tag{12}$$

where n is an integer. The factor of 2 is important.

Thus if this experiment is performed using electrons in a toroidal SC, then (a) the problem of fringe fields is avoided, and (b) one unit increase in the quantized flux exactly causes an interchange of maxima and minima as illustrated in **Figure 6**. This was the way the experiment was eventually carried out by Tonomura et al. [2, 3].

But a full understanding requires a brief account of flux quantization in SC.

3 Flux quantization in SC

Superconductivity is a large subject by itself, and there is much recent interest in high-temperature superconductivity in a new class of materials, especially cuprates. The term "high temperature" originally refers to the breakthrough in 1986 to about 30 K [4], which quickly won the Nobel Prize in 1987 [5]. Later work has pushed the superconducting temperature to over 100 K. Here we give an extremely brief account, focusing on flux quantization.

3.1 Properties of SC

Zero resistance
The most familiar property is that below the transition temperature T_c, the electrical resistance R drops abruptly to zero. It is then possible to drive a current through a SC without an electric voltage. More importantly, the current I would not involve any dissipation of energy I^2R. This is the application of superconductivity that the public most

immediately thinks of. This property makes the material a *perfect conductor*.

Problem 4
This problem seeks to understand, in a heuristic and order-of-magnitude fashion, the challenge of electric energy transmission. The description below is naive, and students are advised to think through what is done in practice.
(a) In a typical modern city, the per capita consumption of electrical energy is an average rate of 1 kW. (This is an easy number to remember.) Estimate the total electrical power consumption in Hong Kong per year, in J.
(b) This energy is delivered as VIt, namely, a voltage V driving a current I over a time t (one year).[2] If the voltage V is say 200 V, what is the effective total current I involved?
(c) If all this current is to be delivered over a distance of 10 km through a collection of copper cables with total cross-section 1 m^2 (this is not really the case), how much energy (in J) is dissipated per year? Compare with the answer in (a).
(d) In practice, without SC, how is electrical energy transmitted? Of course, if room-temperature SC is available, and perhaps they can carry large currents, and that would then be another solution. §

Meissner effect
Actually zero resistance is *not* the defining feature of superconductivity. Rather, it is the Meissner effect: the expulsion of all magnetic fields from the SC (or to be very precise, from all regions of the material which is superconducting). This can happen in two ways:

- The **B**-field is completely expelled (**Figure 7a**).
- In Type II SC, it is possible that (in a certain intermediate phase) superconductivity is destroyed in regions called *vortices*, through which the **B**-field can penetrate (**Figure 7b**).

It is beyond this set of lectures to explain the reason why there is no resistance and why there is the Meissner effect. We take these as given.

No current in the bulk
Now we come to a surprising conclusion: there is no current inside the bulk of a SC.

Superconductors do not conduct!

This is not only surprising, but paradoxical, since the naive expectation is that for zero resistance, it should be easy to generate a current.

[2]This is still true for AC, since V and I change signs together.

To prove this point, suppose there is a current density **J** inside the bulk of a SC, as shown in **Figure 8a** by the arrow heads pointing out of the page. Draw a hypothetical loop in the material, enclosing some current. By Ampere's law, there would have to be a magnetic field around that loop — contradicting the Meissner effect.

Current on the surface
However, if we consider a loop which straddles the surface between the superconducting material (or phase) and a region outside it (**Figure 8b**), it is possible to have **B** on part of this loop. So this loop can enclose some currents. In other words, the current **J** must be confined to a thin layer near the surface; the thickness of this thin layer is called the *penetration length* δ.

It may be useful to recall a parallel argument in electrostatics: in the bulk of a conductor, there is no electric field, and no charge; rather, any net charge must reside on a thin layer on the surface. This conclusion is obtained by applying Gauss' law to closed surfaces (a) entirely within the conductor, and (b) straddling the inside and outside of the conductor.

Difficulty in carrying large currents
The potential use of SC as resistanceless cables is actually limited by two considerations. Suppose we try to pass a current I_0 through a cable of radius r.

- The current density is not uniform, and not equal to $I_0/(\pi r^2)$. Rather, it is zero in the bulk, and has a large value $J \sim I_0/(2\pi r\delta)$ near the surface. In other words, only a fraction $\sim \delta/r$ of the cross-section is utilized.
- In addition, superconductivity is destroyed when the current density exceeds a critical value J_c. We shall not go into the reason for this here, but just stress that it is the large surface value of J (large in the sense that it is enhanced by a factor r/δ from the average value) that must be kept below J_c.

3.2 Microscopic view

Charged fluid and macroscopic wavefunction
Resistance disappears because the charges have *condensed* into a charged fluid; all the charge carriers are in the same quantum state, described by the same wavefunction Ψ. Thus $\Psi^*\Psi$ can be thought of not as the probability of finding *one* charge carrier, but (when normalized to the total number of charged carriers N) as the *density* of charge carriers ρ. Because N is large (indeed macroscopic), the fluctuations in ρ is negligible, and ρ can be regarded as nearly constant, mirroring the nearly

constant positive charge density of the ions.

Thus we can write

$$\Psi(\mathbf{r},t) \quad = \quad \sqrt{\rho}\, \exp[i\xi(\mathbf{r},t)] \tag{13}$$

In other words, all the variation appears only in the phase.

Zero velocity
In the bulk of the SC (i.e., excluding a thin layer $\sim \delta$ near the surface), there is no current $\mathbf{J}$. So the velocity $\mathbf{v}$ of the fluid is exactly zero. But in QM,

$$m\mathbf{v} = \mathbf{p} - q\mathbf{A} \quad \mapsto \quad -i\hbar\boldsymbol{\nabla} - q\mathbf{A} \tag{14}$$

and the condition becomes the operator equation

$$(-i\hbar\boldsymbol{\nabla} - q\mathbf{A})\;\Psi \quad = \quad 0 \tag{15}$$

This condition implies that the velocity is exactly zero, and not just the *expectation value* is zero — the latter can be achieved by cancelling positive and negative momenta.

When (13) is put into (15), we find

$$\hbar\boldsymbol{\nabla}\xi - q\mathbf{A} \quad = \quad 0$$

so that

$$\boxed{\xi(\mathbf{r}) = \frac{q}{\hbar}\int^{\mathbf{r}} \mathbf{A}\cdot d\mathbf{r}} \tag{16}$$

In the above, we have specialized to time-independent situations and q is the magnitude of the charge on one particle in the macroscopic wavefunction Ψ.

3.3 Quantization condition

Condition in general
In order for (16) to define a function ξ that depends only on the position $\mathbf{r}$ but not on the path taken to reach $\mathbf{r}$, normally and naively this would require the line integral around a closed loop to be zero. But since ξ is required only in $\exp i\xi$, we can allow the closed-loop integral to be a multiple of 2π. Thus a consistency condition is

$$\boxed{\frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{r} = 2n\pi} \tag{17}$$

where n is an integer. The possibility $n \neq 0$ is an apparent relaxation of the condition.

Simply connected region
Suppose the SC occupies a simply connected region (**Figure 9a**). Consider a closed loop γ and gradually shrink it.

- In any gradual change, the RHS of (17) cannot change, so n stays constant.
- But for a simply connected region, the loop can eventually be shrunk to a point, in which case (provided, of course, that there are no singularities) the LHS must be zero.

Thus in this case, $n = 0$ for all loops, and the relaxation of the condition actually means nothing.

Non-trivial topology
Suppose the SC occupies a region $\mathcal{D}$ that is not simply connected, but contains a "hole" (**Figure 9b**). Then a closed loop γ cannot be continuously shrunk to a point, and we cannot conclude that $n = 0$.

Note that the closed loop integral around γ is the flux through the areas enclosed by γ; since there is no $\mathbf{B}$ in the SC, this is the flux φ through the "hole". Hence we find

$$\begin{aligned} \frac{q}{\hbar}\varphi \quad &= \quad 2n\pi \\ \varphi \quad &= \quad n\cdot\frac{h}{q} \end{aligned} \tag{18}$$

namely as multiples of the fundamental flux unit

$$\boxed{\varphi_0 = \frac{h}{|q|}} \tag{19}$$

Cooper pairs
When these ideas first surfaced, it was thought initially that the charged carriers are electrons, with $q = -e$, and the unit of flux is $\varphi_0 = h/e$.

But it was later realized that this cannot be the case. Single electrons are fermions, and they cannot occupy the same quantum state. If they do not condense into one single state, resistance would not disappear. To cut a long story short:

- Through an attraction mediated by lattice vibrations, electrons attract each other and form *Cooper pairs*. As a system of two fermions, a Cooper pair behaves like a boson.
- The wavefunction Ψ should be thought of as describing the pairs. Since these are bosons, they can condense into one state, and Ψ is a *macroscopic wavefunction.*
- The pairs have a certain binding energy. Above the critical temperature T_c, the pairs dissociate and superconductivity disappears.
- The involvement of the lattice is confirmed by the observation that T_c is changed if the positive ions are replaced by another isotope.

If the basic unit is a pair, then $q = -2e$, and the quantum of flux is

$$\boxed{\varphi_0 = \frac{h}{2e}} \tag{20}$$

The observation of this smaller quantum (by a factor of 2) confirms that the charges are involved in pairs. The factor of 2 is also exactly what is needed in the AB experiment.

3.4 Physical realizations

There are two situations in which such quantization is realized.

Ring geometry
One can have a SC in the form of a ring (**Figure 10a**). When the SC is cooled below T_c, the flux trapped inside the ring must be an integral multiple of φ_0. Above T_c, the flux φ can be any value.

You may ask: How does the flux φ change to an integral multiple? The answer is that some current must be generated on the inner surface of the ring.

Vortices
In Type II SC, it is also possible that superconductivity is destroyed along certain lines, through which magnetic field penetrate (**Figure 10b**). The remaining SC is like a ring around these lines. The flux lines are called *vortices* and typically each one carries one unit of flux.

3.5 Incorrect derivations

Incorrect account are sometimes found in the literature. For example, a web-based course on quantum mechanics [6] says the following:[3]

> If a charged particle travels in a field free region that surrounds another region, in which there is trapped magnetic flux φ, then upon completing a closed loop the particle's wave function will acquire an additional phase factor $\exp(ie\varphi/\hbar c)$. But the wave function must be single valued at any point in space. This can be accomplished if the magnetic flux φ is quantized. We need
>
> $$\frac{e\varphi}{\hbar c} = 2\pi n, \quad n = 0, \pm 1, \pm 2$$

The point is not the factor of 2.[4] Rather, any purported derivation that does not refer to superconductivity or a macroscopic wavefunction is invalid. This is made more explicit in the next Section.

[3]Some symbols have been changed to conform with the notation in this Chapter.

[4]The above formula also differs from those in this Chapter by a factor of c, but that is because of different units being used.

4 Charged particle in a ring

4.1 Conducting ring

Consider a conductor (not SC) in the form of a ring, in which a charge q is allowed to move. The ring encloses a magnetic flux φ, imposed externally. For simplicity, reduce the ring to a circle of radius r, and consider only the degree of freedom ϕ. Then

$$\begin{aligned}
\boldsymbol{\nabla} &\mapsto \frac{1}{r}\frac{d}{d\phi} \\
\mathbf{A} &\mapsto A = \frac{\varphi}{2\pi r}
\end{aligned}$$

where in the second expression the constant value A is obtained by considering the line integral around the circle and converting it to a surface integral. Schrödinger's equation then reads

$$\begin{aligned}
\frac{1}{2m}\left(-i\hbar\boldsymbol{\nabla} - q\mathbf{A}\right)^2 \Psi &= E\,\psi \\
\left(\frac{1}{r}\frac{d}{d\phi} - i\frac{q}{\hbar}A\right)^2 \Psi &= -k^2\,\Psi
\end{aligned}$$

where

$$E = \frac{1}{2m}\hbar^2 k^2$$

Now the wavefunction must be single-valued, so

$$\Psi(\phi) = e^{in\phi}$$

where n is an integer. This then gives

$$\left(n - \frac{q}{h}\varphi\right)^2 = k^2 r^2$$

Introduce

$$\varphi_0 = \frac{h}{q}$$

It is *not* claimed that this is a quantum of flux, i.e., there is no claim that φ/φ_0 is an integer. The energy, which now we label with n, is given by

$$E_n = \frac{\hbar^2}{2mr^2}\left(n - \varphi/\varphi_0\right)^2 \qquad (21)$$

The values of E_n are plotted against φ/φ_0 in **Figure 11a** and the ground state (i.e., the lowest energy for any given value of φ/φ_0, not necessarily $n = 0$) is given in **Figure 11b**. This is a periodic function with period unity in the variable φ/φ_0. Even for fairly small rings, the energy involved per electron is quite small.

Problem 5
What is the maximum value of E_n in **Figure 11b**, if an electron is confined to a ring of radius 1

μm? Only order-of-magnitude is required. (Answer: $E \sim 10^{-8}$ eV.) §

It is tempting to say that the system would settle into the minima of the energy in **Figure 10b**, which would then lead to flux quantization in units of φ_0. But such an argument would be *incorrect* — first because the energy differences are quite small, as shown in the problem above, but much more importantly because in this situation, φ is an *independent* variable imposed externally, and not a dependent variable which adjusts itself according to the laws of dynamics.

4.2 Superconducting ring

However, if we are talking about a SC with a charged fluid with a macroscopic number of charged particles N, all in the same quantum state, then the situation becomes different in two important ways.

- With N particles acting together in the same quantum state, the energy scale involved would be much larger. Say if $N \sim 10^{20}$, then the energy scale involved would be as high as say 10^{12} eV, and there is a large driving force to go to the minimum.
- Much more importantly, φ is now a dynamic variable. In other words, it is not just imposed externally, but can be changed as the system reacts. For a SC, it costs nothing to have a small surface current, to produce an additional φ in the loop, bringing the total φ to an integral multiple of φ_0.

Because of these reasons, a SC will be driven to one of the minima — and that is exactly the phenomenon of flux quantization. This line of reasoning exposes very clearly the difference between a conventional conductor and a SC, and why the essential properties of the latter are necessary for flux quantization.

A Penetration length

The estimation of penetration length is also a topic in electrodynamics (even though it does not involve topological issues that are the theme of the present Chapter). This Appendix gives a simple account that allows the penetration length to be estimated, and in the course of the discussion also introduces (one of) the London equations.

Suppose there are n charge carriers per unit volume, each of mass m, charge q and moving with a fluid velocity $\mathbf{v}$. Then

$$\begin{aligned} \mathbf{J} &= nq\mathbf{v} \\ \frac{\partial \mathbf{J}}{\partial t} &= nq\frac{\partial \mathbf{v}}{\partial t} = nq\frac{q\mathbf{E}}{m} \end{aligned}$$

$$\begin{aligned} \nabla \times \frac{\partial \mathbf{J}}{\partial t} &= \frac{nq^2}{m}\nabla \times \mathbf{E} \\ &= -\frac{nq^2}{m}\frac{\partial \mathbf{B}}{\partial t} \end{aligned}$$

Use Ampere's law, and write

$$\mathbf{J} = \mu_0^{-1}\boldsymbol{\nabla} \times \mathbf{B}$$

Then we get

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\mu_0^{-1}\boldsymbol{\nabla} \times (\nabla \times \mathbf{B})\right] \\ =\ &\frac{\partial}{\partial t}\left[-\frac{nq^2}{m}\mathbf{B}\right] \end{aligned}$$

So far, there is no assumption that this refers to a SC. London proposes that for a SC, the above equation holds even if we peel off the time derivative:

$$\mu_0^{-1}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = -\frac{nq^2}{m}\mathbf{B}$$

Since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = -\nabla^2\mathbf{B}$$

and we get

$$\boxed{\nabla^2\mathbf{B} = \delta^{-2}\,\mathbf{B}} \tag{22}$$

where

$$\delta^{-2} = \frac{\mu_0 nq^2}{m} = \frac{nq^2}{\epsilon_0 mc^2} \tag{23}$$

Take a situation where the SC conductor occupies say the half space $x > 0$ (so $x = 0$ is the surface). Then the appropriate solution to (23) is obviously

$$\mathbf{B}(x) = \mathbf{B}_0 \exp(-x/\delta) \tag{24}$$

giving the physical interpretation of δ as the penetration length into the SC.

To evaluate δ in a convenient way, it is useful to define a length scale a by

$$n = a^{-3}$$

In other words, there is on average one charge per volume a^3. Then (23) can be written as

$$\delta^{-2} = 4\pi\left[\left(\frac{q^2}{4\pi\epsilon_0 a_0}\right)/mc^2\right] a_0^{-2} \cdot \left(\frac{a_0}{a}\right)^3$$

in which we have inserted factors of the Bohr radius a_0. Recall that the square bracket is exactly

α^2, where $\alpha \approx 1/137$ is the fine-structure constant. Then we have

$$\delta = 2 \text{ nm} \times \left(\frac{a}{a_0}\right)^{3/2} \tag{25}$$

Since one expects the charge density to be on the order of one charge per lattice, and the lattice constant is a few times the Bohr radius, we find that in all situations the penetration depth is on the scale of say tens or hundreds of nm.

Incidentally, the expression for δ^{-2} involves the ratio

$$\frac{nq^2}{m} = (nq) \cdot \frac{q}{m}$$

This combination is unchanged if we consider *pairs* as one unit, since

- $q \mapsto 2q$, $n \mapsto n/2$
- $q \mapsto 2q$, $m \mapsto 2m$

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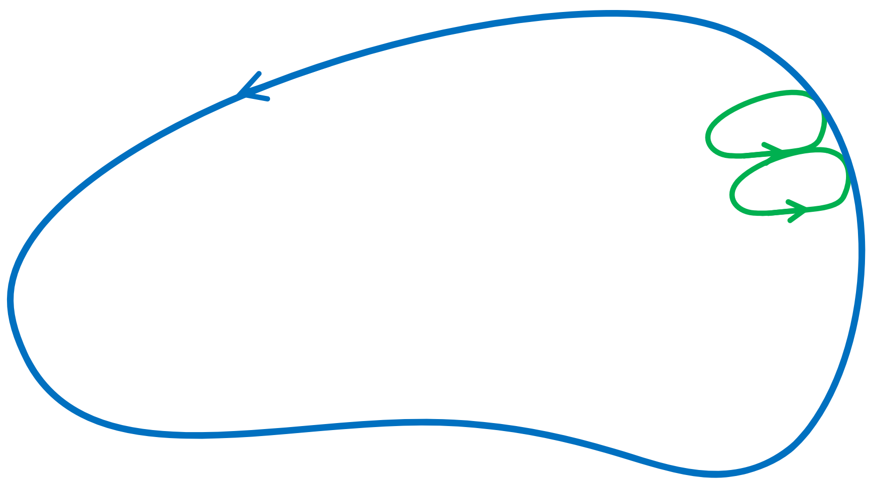


Figure 1 A loop is made up of many small loops.

(a)

(b)

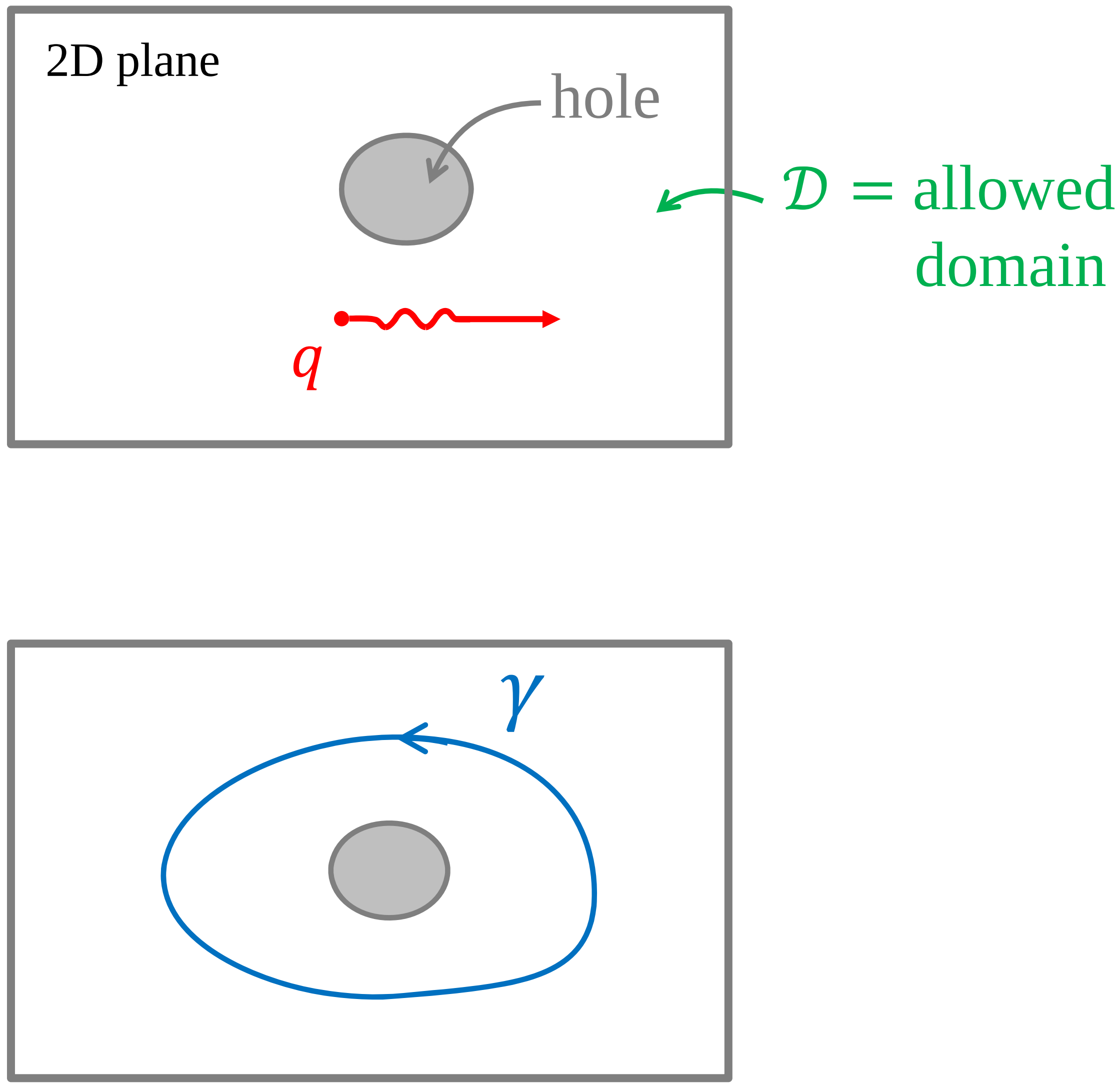


Figure 2 (a) The domain $\mathcal{D}$ is a plane minus a "hole", and has a nontrivial topology.
(b) A loop cannot be continuously shrunk to a point.

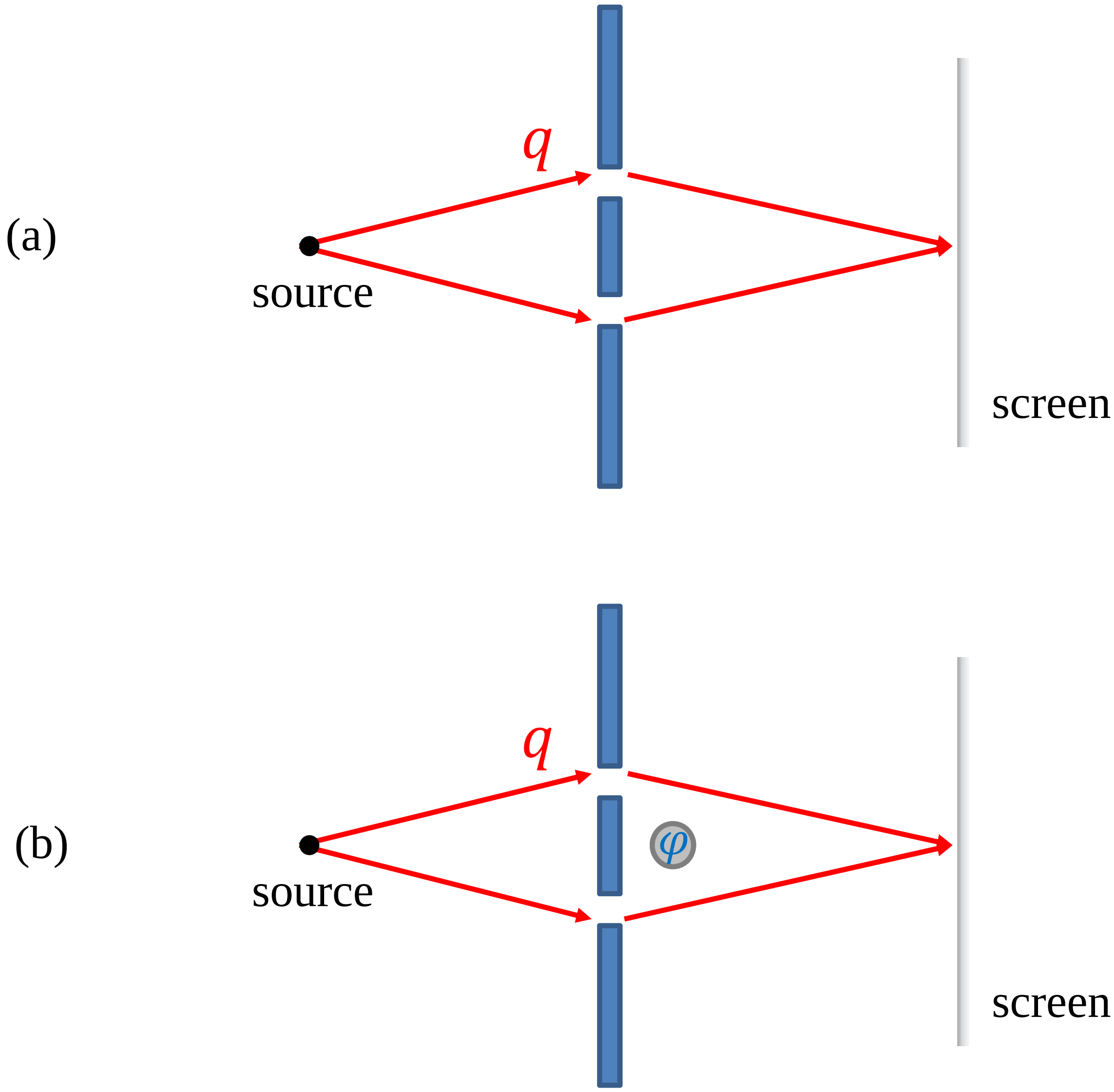


Figure 3 (a) Standard double-slit experiment with charged particles
(b) Adding a flux φ in a region from which the particle is excluded

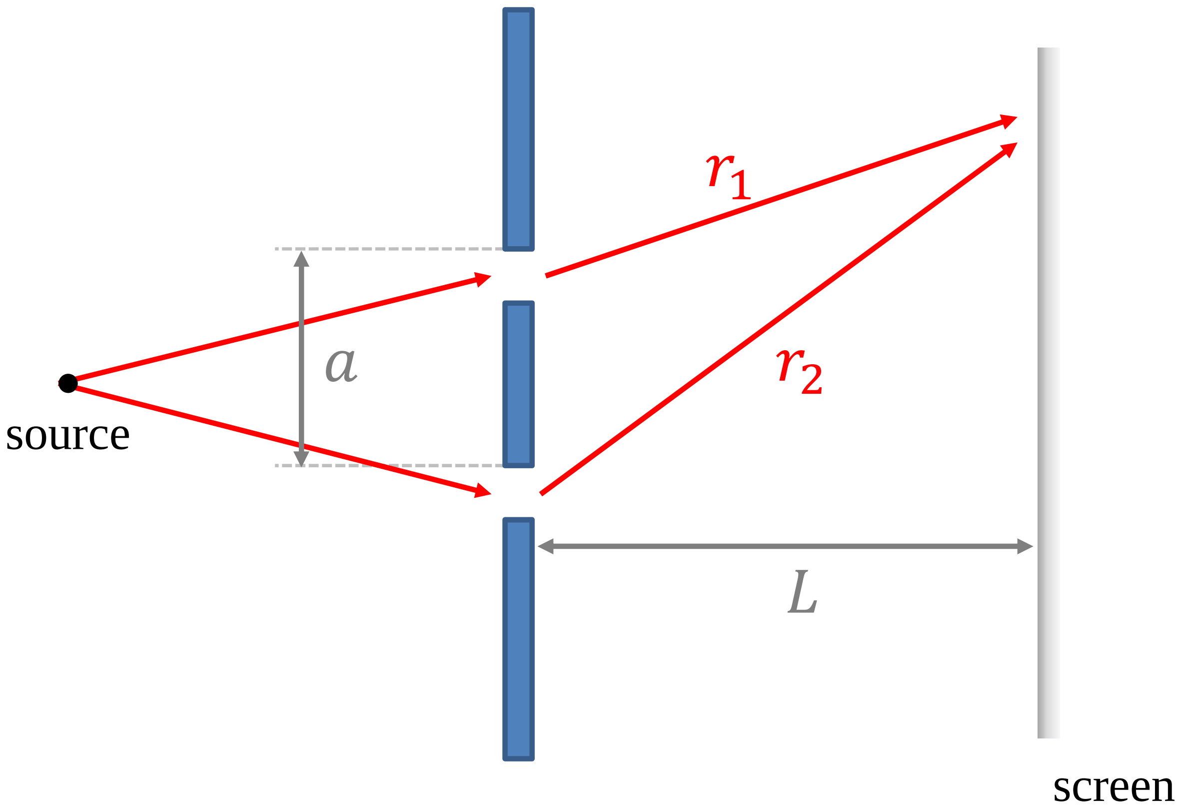


Figure 4 In standard double-slit experiment, interference depends on the difference between path lengths (not to scale)

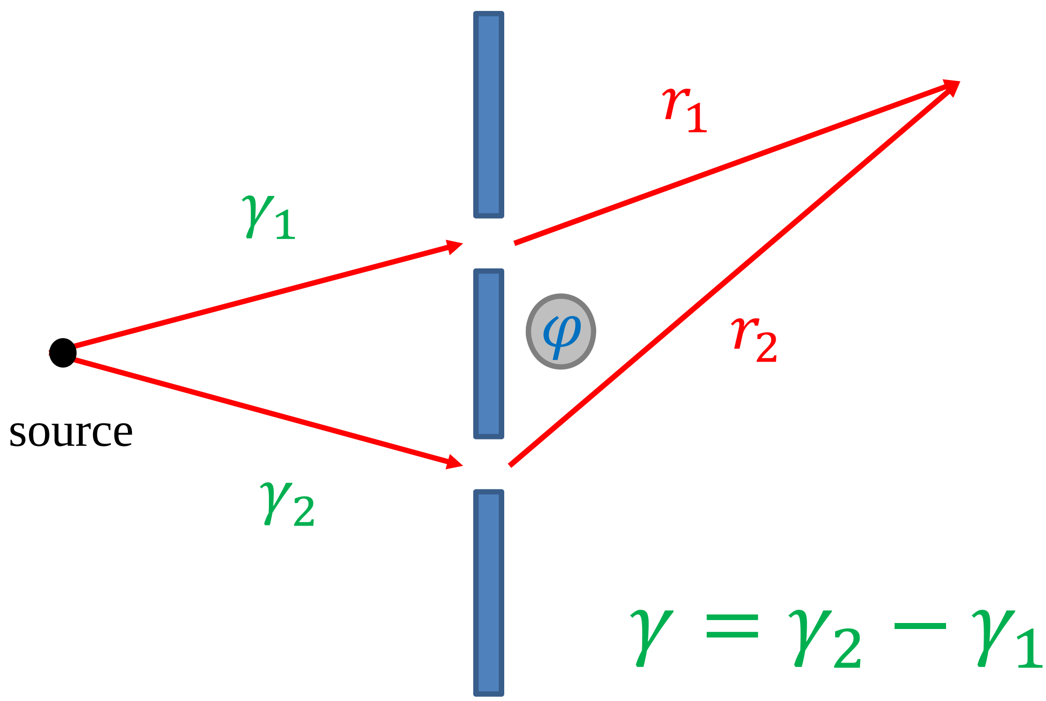
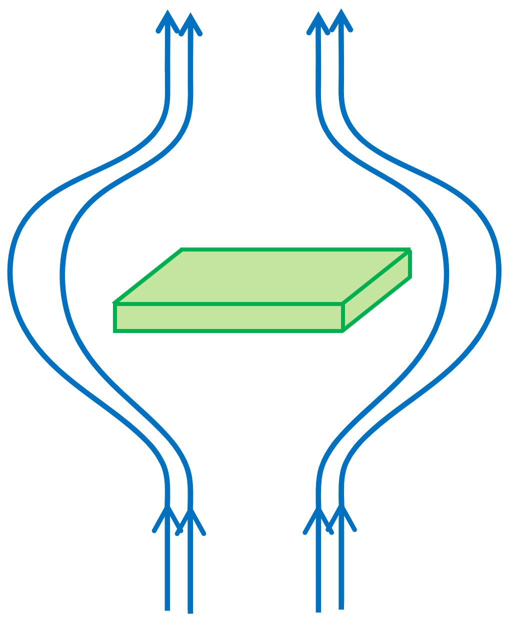
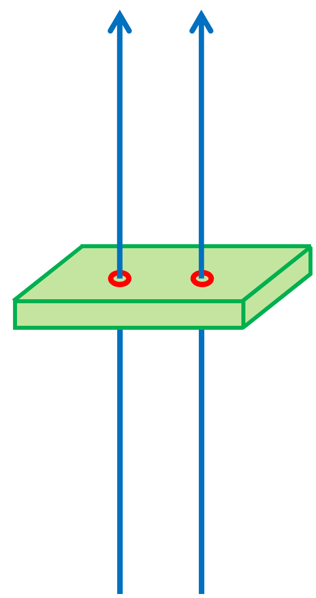


Figure 5 The difference in phase along γ_1 and γ_2 is the same as the closed-loop integral along $\gamma = \gamma_2 - \gamma_1$

Figure 6 Shift of interference pattern by half a fringe



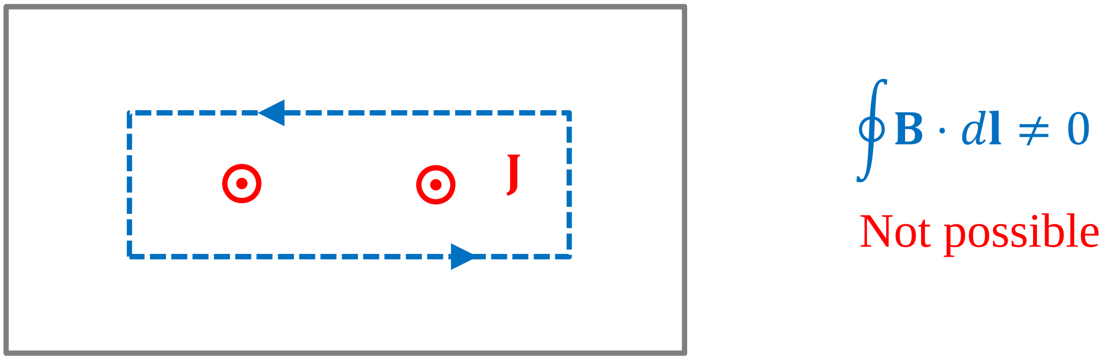
(a)



(b)

Figure 7 Flux is expelled.
(a) Completely
(b) Except in vortices where SC in broken

(a)



(b)

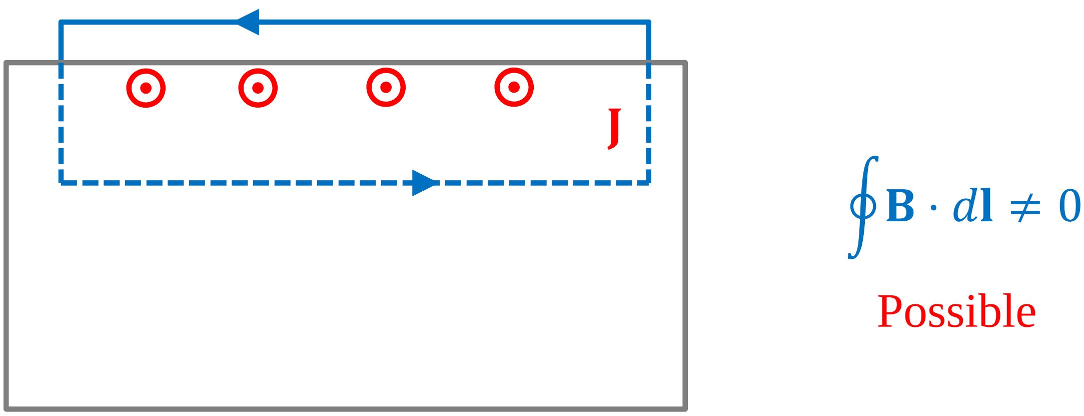
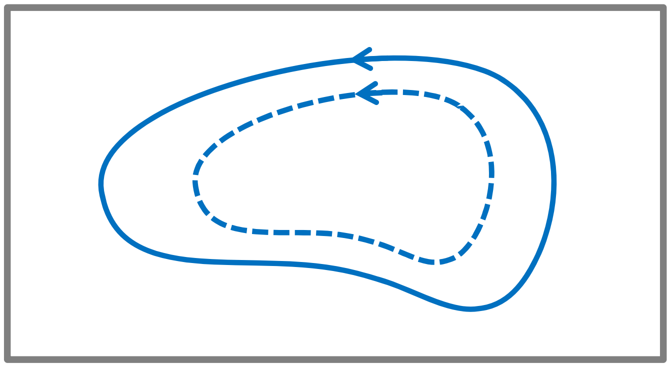


Figure 8 (a) The loop integral of **B** is zero, hence there can be no **J**.

(b) The loop integral of **B** is not zero (because a part of the loop is outside the SC), so there can be a non-zero **J** on the surface.

(a)



(b)

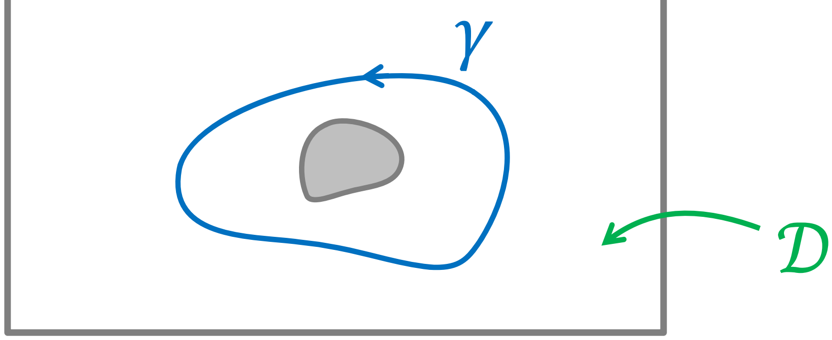
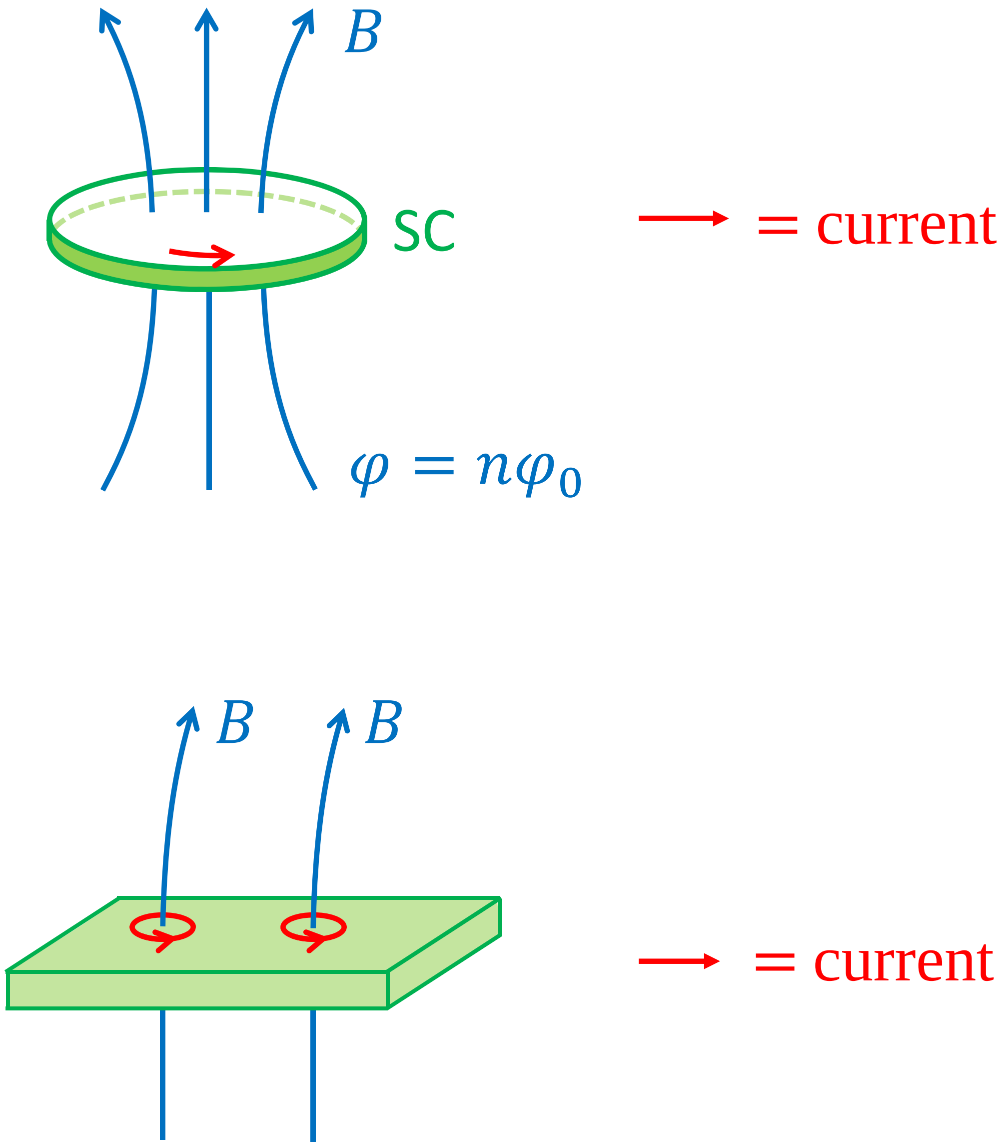


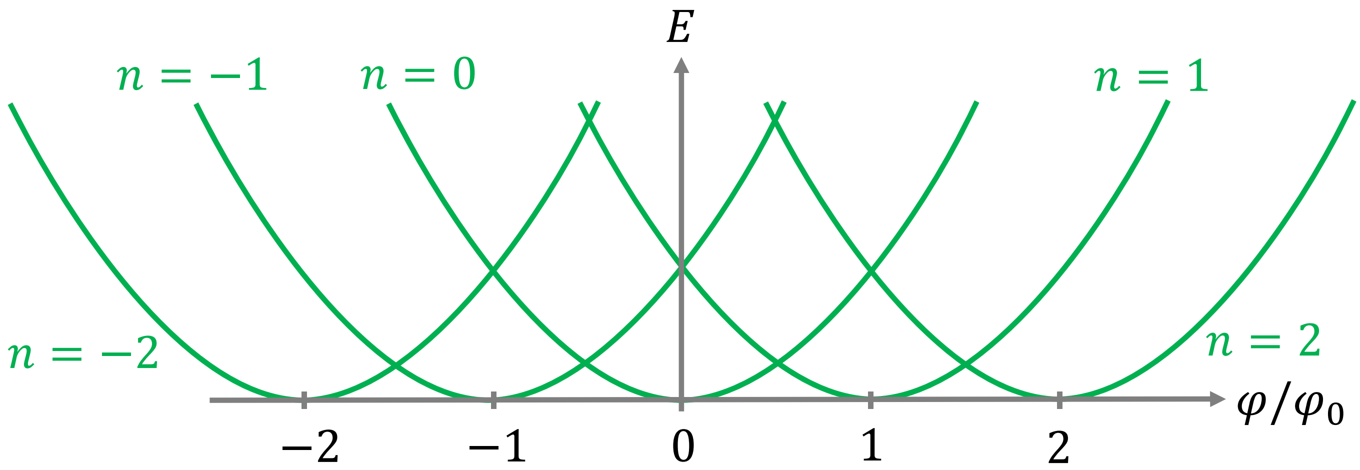
Figure 9 (a) The loop can be continuously shrunk to zero. Hence n in loop integral must be zero.
(b) The loop cannot be continuously shrunk to zero. Hence n in loop integral need not be zero.



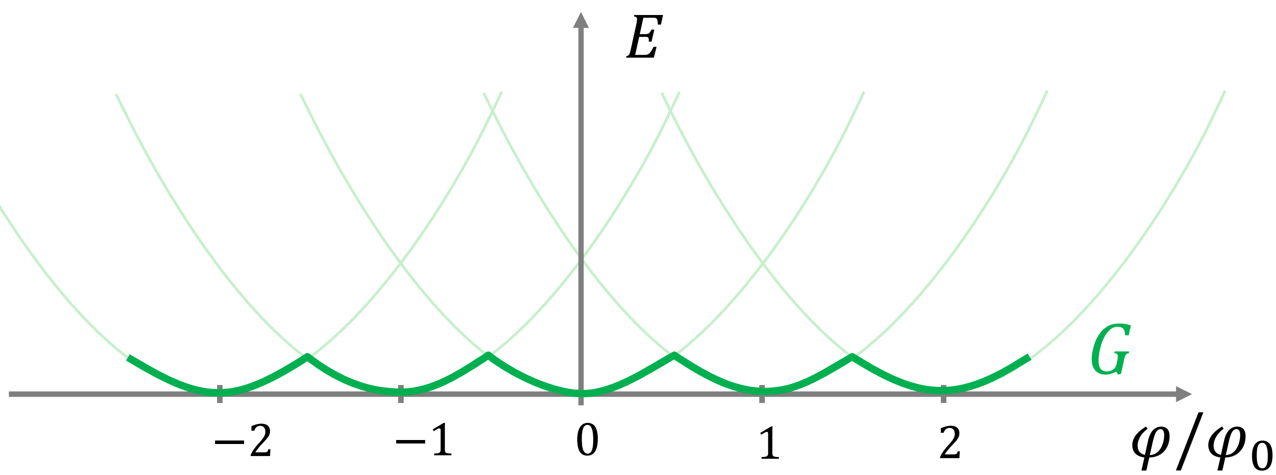
(a)

(b)

Figure 10 (a) Magnetic flux through the area enclosed by a SC ring.
(b) Magnetic flux in vortices.



(a)



(b)

Figure 11 (a) The energy E versus the flux. Different curves are for different values of n.
(b) The lowest energy state.