Chapter 15
Covariant formulation of electrodynamics

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Electrodynamics is rewritten in a covariant formalism, in which a 4-vector source gives rise to a 4-vector potential.

Contents

Notation

- In this Chapter all factors of c have been kept. Once students become familiar with the relativistic formalism, it is much better to drop all such factors, i.e., use units in which $c = 1$.
- A 4-vector will be denoted with an arrow, e.g., $\vec{u}$. It has components u^0, u^1, u^2, u^3, collectively denoted as u^μ, for example. We shall also say "the vector u^μ" as a shorthand for "the vector $\vec{u}$ whose components are u^μ".
- The index on 3-vectors (e.g., the electric field $\mathbf{E}$) or the spatial components of 4-vectors (e.g., the current density $\mathbf{J}$) can be freely denoted as a superscript or a subscript. Thus $E^1 = E_1$, $J^1 = J_1$ etc.

1 Particle dynamics

Before coming to EM, we first put the dynamics of a free particle into covariant form.

1.1 Four-velocity

Consider a short segment of motion Δx^μ. The usual definition of velocity $v^i = \Delta x^i/\Delta t$ is not a 4-vector (not even the spatial components of a 4-vector), because the denominator transforms in a complicated way. The way out is to divide by an *invariant* that is essentially the time interval.

Consider the *proper time* interval $\Delta\tau$:

$$\begin{aligned}
(c\Delta\tau)^2 &= -\Delta x^\mu \, \Delta x_\mu \\
&= (c\Delta t)^2 - (\Delta\mathbf{x})^2 \\
(\Delta\tau)^2 &= (\Delta t)^2 \, (1 - v^2/c^2) \\
\Delta\tau &= \sqrt{1-\beta^2}\,\Delta t = \gamma^{-1}\Delta t
\end{aligned} \tag{1}$$

which is by definition a 4-scalar and which reduces to Δt for small velocities.

Therefore define *4-velocity* (also taking the limit),

$$\boxed{u^\mu = \frac{dx^\mu}{d\tau}} \tag{2}$$

This is guaranteed to be a 4-vector.

1.2 Four-momentum

The *4-momentum* is defined by

$$\boxed{p^\mu = m u^\mu = m\,\frac{dx^\mu}{d\tau}} \tag{3}$$

in which m is the mass — mass will always refer to rest mass, by definition independent of motion and a 4-scalar. The 4-momentum $\vec{p}$ is by construction a 4-vector.

Spatial components
The spatial components are

$$\boxed{p^i = m u^i = m\,\frac{dx^i}{d\tau} = m\,\frac{dx^i}{dt}\cdot\frac{dt}{d\tau}}$$

From (1), the last factor is $dt/d\tau = \gamma$,

$$p^i = m\gamma v^i = \frac{m v^i}{\sqrt{1 - v^2/c^2}} \tag{4}$$

which recovers the Newtonian expression in the limit $v \ll c$.

Time component
The time component is (multiplying by c for convenience)

$$\begin{aligned} cp^0 &= mcu^0 = mc\,\frac{dx^0}{d\tau} \\ &= mc\cdot\frac{d(ct)}{d\tau} = mc^2\gamma \end{aligned}$$

For small velocities

$$\begin{aligned} mc^2\gamma &= mc^2\left(1 + \beta^2/2 + \ldots\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \ldots \end{aligned}$$

The first term is a constant, interpreted as the rest energy, and the next term is the Newtonian kinetic energy. Thus we interpret cp^0 as the energy E of the particle:

$$\boxed{cp^0 = E}$$

or

$$\boxed{\vec{p} = (E/c, \mathbf{p})} \tag{5}$$

with the spatial components given by (4)

The invariant constructed out of the 4-momentum
Consider a particle with 4-momentum (5), and the invariant

$$\vec{p}\cdot\vec{p} = \mathbf{p}^2 - (E/c)^2 \tag{6}$$

Now go to the frame comoving with the particle, in which the 4-momentum is

$$\vec{p}\,' = (mc, \mathbf{0})$$

and consequently

$$\vec{p}\,'\cdot\vec{p}\,' = -(mc)^2 \tag{7}$$

But since this dot product is an invariant, (6) and (7) must be the same, leading to

$$\boxed{E^2 = p^2c^2 + m^2c^4} \tag{8}$$

This identity can also be obtained by eliminating v from the expressions for E and $\mathbf{p}$.

The advantage of this formula is that it does not involve a factor such as

$$\frac{m}{\sqrt{1 - v^2/c^2}}$$

which would give 0/0 for a photon.

1.3 Conservation of four-momentum

Consider a scattering such as $a + b \to c + d$. The Newtonian (N) and the relativistic (R) laws for conservation of momentum (in the latter case also energy) take the form

$$\begin{aligned} &\text{N:} \qquad \mathbf{P} \equiv \sum \pm m\mathbf{v} = 0 \\ &\text{R:} \qquad \vec{P} \equiv \sum \pm \vec{p} = 0 \end{aligned}$$

where each initial (final) particle contributes to one term with a $+$ $(-)$ sign. Which version is correct? Of course, in the end, experiment tells us that R is correct but N is not. However, there is much we can say *without* doing any experiments.

- N cannot be correct, because it is not covariant. If it holds in one frame, it may not (indeed would not) hold in another frame.
- R is *possibly* correct, because it is covariant. If it holds in one frame, it must hold in another frame:

$$\begin{aligned} P^\mu &= 0 \\ L^\nu{}_\mu P^\mu &= 0 \\ P'^\nu &= 0 \end{aligned}$$

- In fact, we cannot find another candidate "law" which (a) is invariant and (b) reduces to the Newtonian form for small velocities.[1]

Covariance is necessary but not sufficient: a candidate "law" that is covariant is only *possibly* correct, while any candidate "law" that is not covariant is *definitely* wrong.

[1] I do not know of a formal proof that it cannot be done; I do know that nobody has ever been able to do it.

$$\boxed{\text{The principle of relativity is powerful: very few candidates pass the test of covariance; the answer is basically determined before doing any experiments.}}$$

This principle will later be applied to EM, revealing that the laws are heavily constrained — one cannot even contemplate many possible changes.

The relativistic version includes energy in the same statement, and it shows that the rest energy mc^2 is needed. That is how the famous equation $E = mc^2$ comes about for the energy of a particle at rest. The loss of rest mass in nuclear fission and fusion is the source of energy, with conversion at a huge ratio in MKS units — 1 kg converts into 9×10^{16} J.

1.4 Second Law

The conservation of momentum for an isolated system is essentially Newton's First Law; we have just seen how it is to be generalized to relativistic situations. How about Newton's Second Law? Its conventional form is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{9}$$

The obvious way to generalize this is

$$\boxed{\frac{dp^\mu}{d\tau} = K^\mu} \tag{10}$$

By using τ, the LHS is a 4-vector, hence K^μ so defined must also be a 4-vector, and is the interesting thing to study.

The *logical* way to study EM (or any other kind of dynamics that is relativistically covariant) is to write Newton's Second Law in the form (10), and then find a force law that gives K^μ.

But the *historical* and *conventional* way is to consider the force $\mathbf{F}$ instead, as given by (9), which is found experimentally to be

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{11}$$

The next Section analyzes some examples of motion using this conventional formalism.

2 Relativistic motion under EM forces

In a sense the Lorentz force law is a messy way of formulating EM. The force $\mathbf{F} = d\mathbf{p}/dt$ is not (the spatial components of) a 4-vector; it transforms in a complicated way. If (11) is to hold in every frame (and indeed it does), then the RHS must transform in an equally complicated way. Thus, this conventional formalism is not used when we want to talk about covariance; however, it is useful for physics in *one* given frame.

2.1 Motion in a magnetic field

Consider a charged particle q passing through a magnetic field $\mathbf{B}$, say out of the page (**Figure 1**). Since the magnetic field does no work, the magnitude of the momentum does not change; only its direction changes. Therefore the trajectory is an arc of radius R.[2] What does R depend on?

Compare the momenta at t and $t + \Delta t$. In this time, the momentum vector has changed direction by $\Delta\phi$, so

$$\Delta p \approx p\Delta\phi$$

But since the angle changes by 2π in a period T,

$$\begin{aligned} \Delta\phi &= 2\pi\frac{\Delta t}{T} = \omega\Delta t = \frac{v}{R}\Delta t \\ \frac{\Delta p}{\Delta t} &\approx \frac{p\Delta\phi}{\Delta t} = \frac{vp}{R} \\ qvB &= \frac{\Delta p}{\Delta t} = \frac{vp}{R} \end{aligned}$$

so that finally

$$\boxed{R = \frac{p}{qB} \propto p} \tag{12}$$

Note that R is *not* proportional to mv.

Problem 1
Compare the radii of curvature for particles travelling at (a) $0.99\,c$ and (b) $0.9999\,c$. §

This kind of experimental observation verifies that it is correct to use the Lorentz force law with the relativistic momentum $p = m\gamma v$. In fact, the radius of curvature in a known magnetic field is a standard way to measure the momentum and hence the velocity, useful especially if v is close to c.

2.2 Motion under an electric field: parallel case

Consider a charged particle q travelling along a constant electric field $\mathbf{E}$.

$$\frac{dp}{dt} = qE \tag{13}$$

$$\frac{mv}{\sqrt{1 - v^2/c^2}} = p = qEt$$

$$\frac{v}{c} = \beta = \frac{t/\tau}{\sqrt{1 + (t/\tau)^2}} \tag{14}$$

[2] If the magnetic field is uniform, the trajectory is indeed a circle. If the magnetic field is not uniform, the argument below can still be applied to one point of the motion, and R is then the radius of curvature of that part of the trajectory.

assuming that the motion starts from rest, and

$$\frac{1}{\tau} = \frac{qE}{mc}$$

Note that τ can be interpreted as the time taken, according to Newtonian physics, for the particle to attain velocity c.

The solution (14) is sketched in **Figure 2**. For small t, the denominator ~ 1, and the situation is given by line 1, which is the Newtonian result. For large t, the velocity saturates at $v/c \sim 1$. Thus, the limit $v < c$ is built in. Also, qE can be replaced by any force F.

The motion can be analyzed in another way. Start with an identity.

Problem 2
Show that $d(\gamma v)/dt = \gamma^3(dv/dt)$. §

Put this result into (13) and replace qE by any force $F_{||}$ that is in the direction of motion:

$$\begin{aligned} m\gamma^3 \frac{dv}{dt} &= F_{||} \\ M_{||} a &= F_{||} \\ M_{||} &= m\gamma^3 \end{aligned}$$

Thus the effective mass is increased by γ^3.

2.3 Motion under an electric field: perpendicular case

Consider a particle moving originally along x, but subject to an electric field along y. The following applies only to one moment, say $t = 0$, when the velocity is still along x.

$$\begin{aligned} \frac{d}{dt}(m\gamma v_x) &= 0 \\ \frac{d}{dt}(m\gamma v_y) &= F_\perp \end{aligned}$$

where we have generalized to any force $F_\perp$ perpendicular to the direction of motion. The first equation can be analyzed simply: because the force is perpendicular to the motion, the energy does not increase, and γ is instantaneously constant; thus v_x is constant as well. From the second equation.

$$m\frac{d\gamma}{dt} v_y + m\gamma \frac{dv_y}{dt} = F_\perp$$

The first term vanishes because the instantaneous v_y is zero and also because γ is constant. Thus

$$\begin{aligned} M_\perp a &= F_\perp \\ M_\perp &= m\gamma \end{aligned}$$

Magnetic deflection is also a case of force being perpendicular to the motion, and from (12), it is seen that the radius of curvature is modified by a factor of γ (not γ^3).

No "relativistic mass"
There is an important lesson. It is *not* true that all Newtonian formulas can be made correct by replacing the mass m by a unique "relativistic mass". For this reason, the idea of a "relativistic mass" should be avoided.

3 Potential, field and force law

3.1 Four-vector potential

Start with the *postulate* that the scalar potential Φ and the vector potential $\mathbf{A}$ together form a 4-vector field

$$\boxed{\vec{A} = (\Phi/c, \mathbf{A})} \tag{15}$$

The factor of c makes all components have the same units. The proof of this postulate will come from the transformation of the fields and the covariance of the theory.

3.2 Field tensor

Construct the antisymmetric field tensor

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} \tag{16}$$

which must be a $(2,0)$ tensor. The components are as follows

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 \\ &= -\frac{\partial}{\partial(ct)} A^1 - \frac{\partial}{\partial x}(\Phi/c) \\ &= c^{-1}\left(-\frac{\partial \Phi}{\partial x^1} - \frac{\partial A^1}{\partial t}\right)_1 \\ &= E_1/c \\ F^{12} &= \partial^1 A^2 - \partial^2 A^1 \\ &= \frac{\partial}{\partial x^1} A^2 - \frac{\partial}{\partial x^2} A^1 = B_3 \end{aligned}$$

and in general

$$\begin{aligned} F^{0i} = -F^{i0} &= E_i/c \\ F^{ij} = -F^{ji} &= \epsilon_{ijk} B_k \end{aligned}$$

We now gain some appreciation. The formula

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial_t \mathbf{A}$$

is like a "curl" in the 01 directions, similar to

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

In explicit matrix form

$$[F] = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{bmatrix}$$

The first index is the row, so F^{01} is row 0, column 1, i.e., the entry E_1/c.

3.3 Covariant form of the Lorentz force

We claim that K^μ defined by (10) can be written as

$$\boxed{K^\mu = qF^{\mu\nu} u_\nu} \tag{17}$$

Combining it with (10), we need to check

$$\frac{dp^\mu}{d\tau} \stackrel{?}{=} qF^{\mu\nu} u_\nu \tag{18}$$

First of all, note that

$$\begin{aligned} u^\mu &= (\gamma, \gamma \mathbf{v}) \\ u_\mu &= (-\gamma, \gamma \mathbf{v}) \end{aligned}$$

Check spatial components
Take $\mu = 1$ in (18). Then

$$\text{LHS} = \frac{dp^1}{d\tau} = \frac{dp^1}{dt} \cdot \frac{dt}{d\tau} = \gamma \frac{dp^1}{dt} \tag{19}$$

while

$$\begin{aligned} & \text{RHS} = K^1 = qF^{1\nu} u_\nu \\ &= q(F^{10} u_0 + F^{12} u_2 + F^{13} u_3) \\ &= q[(-E_1/c)(-\gamma c) + B_3(\gamma v_2) - B_2(\gamma v_3)] \\ &= \gamma q(E_1 + v_2 B_3 - v_3 B_2) \\ &= \gamma q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})]_1 = \gamma F^1 \end{aligned} \tag{20}$$

Compare (19), (20) and cancel a common factor of γ; we see that this is equivalent to the usual form of the Lorentz force law in the 1-direction. The other spatial components work out in the same way.

Check time components
Take $\mu = 0$ in (18). Then[3]

$$\text{LHS} = \frac{dp^0}{d\tau} = \frac{dp^0}{dt} \cdot \frac{dt}{d\tau} = \gamma \frac{d(E/c)}{dt} \tag{21}$$

while

$$\begin{aligned} & \text{RHS} = K^0 = qF^{0\nu} u_\nu \\ &= qF^{0i} u_i = q(E_i/c)(\gamma v_i) = (\gamma/c)(qE_i)v_i \\ &= (\gamma/c)F_i v_i = (\gamma/c)(\mathbf{F} \cdot \mathbf{v}) \end{aligned} \tag{22}$$

[3] It should be clear that E here, without subscripts, denotes the energy and not (a component of) the electric field.

We have used the fact that the magnetic force does not contribute to $\mathbf{F} \cdot \mathbf{v}$. Compare (21), (22) and cancel a common factor of γ/c; we get

$$\frac{dE}{dt} \stackrel{?}{=} \mathbf{F} \cdot \mathbf{v} \tag{23}$$

which is of course the correct equation for energy conservation.

Assuming that A^μ does indeed transform like a 4-vector, and hence that $F^{\mu\nu}$ does indeed transform like a $(2,0)$ tensor, then it is guaranteed that (18) leads to the same physical consequence in every reference frame.

3.4 Transformation of fields

Since $F^{\mu\nu}$ is a tensor, its tranformation law is

$$\boxed{F'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma}} \tag{24}$$

It suffices to consider a special case. Consider a frame S' moving at speed $V = \beta c$ along the x-direction, relative to a frame S. All other cases of Lorentz transformation can be made up by combining this special case with rotations.

The Lorentz transformation in this special case is given by

$$\begin{aligned} x'^\mu &= L^\mu{}_\nu x^\nu \\ L^\mu{}_\nu &= \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \end{aligned}$$

or, writing out the components explicitly,

$$\begin{aligned} L^0{}_0 = L^1{}_1 &= \gamma \\ L^0{}_1 = L^1{}_0 &= -\beta\gamma \\ L^2{}_2 = L^3{}_3 &= 1 \end{aligned}$$

with all other entries 0.

It is then straightforward to work out the transformation of the fields.

Electric field
First take the component along the direction of relative motion.

$$E_1'/c = F'^{01} = L^0{}_\mu L^1{}_\nu F^{\mu\nu}$$

There are only 4 choices of $(\mu\nu)$, namely $(\mu\nu) = (00), (01), (10), (11)$. Moreover, cases with $\mu = \nu$ vanish because $F^{\mu\nu}$ is antisymmetric; thus we are left with $(\mu\nu) = (01), (10)$:

$$\begin{aligned} & E_1'/c \\ &= L^0{}_0 L^1{}_1 F^{01} + L^0{}_1 L^1{}_0 F^{10} \\ &= (L^0{}_0 L^1{}_1 - L^0{}_1 L^1{}_0)\, F^{01} \\ &= [\gamma \cdot \gamma - (-\gamma\beta)(-\gamma\beta)]\, E_1/c \\ &= \gamma^2(1-\beta^2) E_1/c = E_1/c \end{aligned}$$

Next take a component perpendicular to the direction of motion.

$$E'_2/c = F'^{02} = L^0{}_\mu L^2{}_\nu F^{\mu\nu}$$

The only allowed ν is $\nu = 2$.

$$\begin{aligned} E'_2/c &= L^0{}_\mu F^{\mu 2} \\ &= L^0{}_0 F^{02} + L^0{}_1 F^{12} \\ &= \gamma E_2/c + (-\beta\gamma) B_3 \\ &= \gamma(E_2/c - \beta B_3) \end{aligned}$$

Likewise

$$E'_3/c = \gamma(E_3/c + \beta B_2)$$

Since $\mathbf{V}$ (with magnitude βc) can be regarded as a vector in the $+x$ direction, all these can be summarized as

$$\boxed{\begin{aligned} E'_\parallel &= E_\parallel \\ \mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{V} \times \mathbf{B}_\perp) \end{aligned}} \tag{25}$$

Here $\parallel$ and $\perp$ refer to the direction of $\mathbf{V}$.

If we were to check directly that the Lorentz force law (and later also Maxwell's equations) remains covariant under this transformation, the algebra would be very messy. The covariant formalism bypasses these messy calculations.

Magnetic field
The case of magnetic field is left as an exercise.

Problem 3
Show that

$$\boxed{\begin{aligned} B'_\parallel &= B_\parallel \\ \mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \mathbf{V} \times \mathbf{E}_\perp/c^2) \end{aligned}} \tag{26}$$

Note that these equations can be written more "naturally" in terms of B and E/c, which have the same units. Also note the signs of the cross product terms.

Problem 4
Show that under a Lorentz transformation, the following quantities are unchanged: $\mathbf{B}^2 - (\mathbf{E}/c)^2$, and $\mathbf{E} \cdot \mathbf{B}$. Thus we expect these quantities should be expressible in terms of vectors or tensors, with all indices paired and contracted away. See next two Problems. §

Example of transformation
We consider how the same phenomenon appears to two different observers, in order to illustrate what happens under field transformations. Consider two capacitor plates which create an electric field $\mathbf{E} = E\,\hat{\mathbf{j}}$. Let a particle of mass m and charge q traverse this space at speed v (**Figure 3**). In the lab frame S, this particle experiences a force in the y direction, and accelerates at

$$a_y = \frac{d^2y}{dt^2} = \frac{qE}{M_\perp} = \frac{1}{\gamma}\frac{qE}{m} \tag{27}$$

In the co-moving frame S',

$$\begin{aligned} E'_y &= \gamma(E_y + vB_x) = \gamma E \\ B'_z &= \gamma(B_z - vE_y/c^2) = -\gamma v E/c^2 \end{aligned}$$

Note that a magnetic field appears! In the co-moving frame, the particle is non-relativistic, so Newtonian mechanics apply:

$$\begin{aligned} m\frac{d^2y'}{dt'^2} &= q(E'_y - v'_x B'_z) = \gamma q E \\ a'_y &= \frac{d^2y'}{dt'^2} = \gamma\frac{qE}{m} \end{aligned} \tag{28}$$

The magnetic field does not matter, because the particle has zero velocity.

Are (27) and (28) consistent? To check this, we see that $y' = y$, $t' = t/\gamma$ (because the former is the proper time), $(d/dt') = \gamma(d/dt)$, so (28) becomes

$$\left(\gamma\frac{d}{dt}\right)^2 y = \gamma\frac{qE}{m}$$

agreeing with (27).

An important aspect of the field transformation is that $\mathbf{E}$ and $\mathbf{B}$ are mixed. They are really different aspects of the same thing, i.e., different components of the field tensor $F^{\mu\nu}$.

3.5 Relativistic invariants

Out of the tensor $F^{\mu\nu}$, two quadratic invariants can be constructed.

Square of field tensor
The first is

$$I_1 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} \tag{29}$$

Problem 5
Show that

$$I_1 = \mathbf{B}^2 - (\mathbf{E}/c)^2$$

Because this is defined by (29), it is guaranteed to transform like a 4-scalar, i.e., it is the same in every frame. §

If there is a pure $\mathbf{B}$ field ($\mathbf{E} = 0$) in one frame, then $I_1 > 0$, and it is impossible to transform to another frame so that it becomes a pure $\mathbf{E}$ field ($\mathbf{B} = 0$), since the sign of I_1 would be changed. The reverse is also true.

In a plane wave, $|\mathbf{B}| = |\mathbf{E}|/c$, so $I_1 = 0$. This is consistent since a plane wave ($I_1 = 0$) stays as a plane wave ($I_1' = 0$) in every frame.

Note that I_1 is not proportional to the energy: $\mathbf{B}^2$ and $\mathbf{E}^2$ enter the invariant with *opposite* signs. In fact, it is proportional to the Lagrangian density, a concept to be introduced later.

Field tensor and dual

To introduce the second invariant, start with the *dual tensor*:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{30}$$

Since $\epsilon^{\mu\nu\alpha\beta}$ is a tensor, by the contraction theorem, $\tilde{F}$ transforms as a tensor.

Let us see explicitly what $\tilde{F}^{\mu\nu}$ is.

$$\begin{aligned}\tilde{F}^{01} &= \epsilon^{0123}F_{23} = (+1)B_1 \\ \tilde{F}^{23} &= \epsilon^{2301}F_{01} = (+1)(-E_1/c)\end{aligned}$$

Thus

$$\begin{aligned}F &\mapsto \tilde{F} \\ \mathbf{E}/c &\mapsto \mathbf{B} \\ \mathbf{B} &\mapsto -\mathbf{E}/c\end{aligned} \tag{31}$$

Referring to (17), we see

$$\left[\tilde{F}\right] = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3/c & E_2/c \\ -B_2 & E_3/c & 0 & -E_1/c \\ -B_3 & -E_2/c & E_1/c & 0 \end{bmatrix}$$

With $\tilde{F}$, we can construct another invariant

$$I_2 = \frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} \tag{32}$$

Problem 6

Show that

$$I_2 = -\mathbf{B}\cdot\mathbf{E}/c$$

which must then be the same in every frame. §

For example, a plane wave is characterized by $\mathbf{B}\cdot\mathbf{E} = 0$, and we now see that this condition is the same in every frame.

Duality

Incidentally, the mapping (31) is called a duality transformation; roughly speaking, it interchanges the electric and magnetic fields. EM in vacuum is invariant under duality. Although it has little to do with relativity, this property is further explored in Appendix A, and is important when we come to the possibility of magnetic monopoles.

4 Maxwell's equations

The theory of EM consists of two parts: (a) How do the fields affect the charges and currents — Lorentz force law; (b) How do the charges and currents generate the fields — Maxwell's equations. We have dealt with the first part covariantly. Now we deal with the second part.

4.1 Four Current

Charge

First, charge is a 4-scalar: in a moving frame, the charge measured is the same, $q' = q$. Equivalently, q does not change when the charge goes into motion. The latter statement is implicitly verified to high precision when we think about atoms. The following Problem is just one example of the phenomena from which such a deduction can be made.

Problem 7

Imagine an alternate theory in which q changes by order $(v/c)^2$; to be specific, let us say that for a charge with velocity v

$$q' = [1 + (v/c)^2]\, q$$

Assume the charges of the proton and the electron are equal in magnitude when they are at rest.

(a) Find the fractional difference between the charge of the electron and the charge of the proton in a hydrogen atom.

(b) Find the net charge in 1 kg of hydrogen.

(c) Hence find the force between two such samples of hydrogen 10 m apart. This is so large that such effects can be immediately ruled out. §

Incidentally tight limits have been placed on the neutrality of bulk matter (made up of stationary nuclei and rapidly moving electrons). We cite only two works,[4] from which other references can be traced.

- In 1959, Lyttleton and Bondi [1] argued that a small residual net charge would drive an expansion of the universe. They thereby placed a limit of $\sim 10^{-18}$ on the fractional charge asymmetry. (However, to the best of our knowledge this argument has not been re-examined after the discovery of the acceleration of Hubble expansion in the late 1990s.)
- In a more recent experiment [2], a gas is placed in the space between two spherical shells acting as electrodes, on which an oscillating voltage is applied. If there is a net residual charge, the electrical forces would set up an acoustic

[4] I am grateful to Pui Tak Leung for drawing my attention to these two papers.

wave, which can be large if the frequency of the voltage matches an acoustical resonance. The absence of such an acoustical signal then places a limit on the fractional charge asymmetry; given the experimental set-up, a limit of $\sim 10^{-21}$ was reported. Many earlier experiments are also reviewed in this paper.

Charge density and current

Next, the charge density ρ and the current density $\mathbf{J}$ together form a 4-vector in the form

$$\vec{J} \quad = \quad (\rho c, \mathbf{J}) \tag{33}$$

To see this, we have to write down the expressions for ρ and $\mathbf{J}$. Suppose there is a single charge[5] q at position $\mathbf{X} = \mathbf{X}(t)$. Then

$$\rho(t, \mathbf{x}) \quad = \quad q\, \delta^3(\mathbf{x} - \mathbf{X}(t))$$

Integrating this over space gives the correct charge. The current density is

$$\mathbf{J}(t, \mathbf{x}) \quad = \quad q\mathbf{V}\, \delta^3(\mathbf{x} - \mathbf{X}(t))$$

where $\mathbf{V} = d\mathbf{X}/dt$. Insert the factor

$$1 \quad = \quad \delta(t - T)\, dt$$

where T is an arbitrary constant time. Then

$$\begin{aligned} \rho(t, \mathbf{x}) &= q\, \delta^3(\mathbf{x} - \mathbf{X})\, \delta(t - T)\, dt \\ \mathbf{J}(t, \mathbf{x}) &= q\mathbf{V}\, \delta^3(\mathbf{x} - \mathbf{X})\, \delta(t - T)\, dt \end{aligned}$$

Now

$$\delta^3(\mathbf{x} - \mathbf{X})\, \delta(t - T) \quad = \quad c^{-1} \delta^4(x^\mu - X^\mu)$$

is a scalar, where we have introduced $X^\mu = (cT, \mathbf{X})$. Hence

$$\begin{aligned} & (\rho c, \mathbf{J}) \propto (cdt, \mathbf{V}dt) \\ = \quad & (cdt, d\mathbf{X}) = (cdt, d\mathbf{x}) = dx^\mu \end{aligned}$$

We have changed $\mathbf{x}$ to $\mathbf{X}$ above because all this multiplies a spatial δ-function that ensures the equality of the two variables.

Hence J^μ transforms like a 4-vector. Incidently, since charge conservation can be written as $\partial_\mu J^\mu = 0$, this already *suggests* (but does not prove) that J^μ is a 4-vector. Also, the density and flux of any other quantity will transform like the 4 components of some object.

The formal proof for the case of charge density and current density can be supplemented by the following problem.

Problem 8

(a) There are N charges, each of magnitude q, in a rectangular volume of area A and length L. The charges are not moving. Find J^μ.

(b) Now go to a frame which is moving at a speed $V = \beta c$ along the length direction. In this frame, the charges are (i) moving at a speed $-\beta c$, and contained in a volume $A' \times L'$, where $A' = A$ and $L' = L/\gamma$. Find J'^μ.

(c) Show that J^μ and J'^μ are related exactly by the Lorentz transformation. §

4.2 Homogeneous equations

We simply write down the covariant form of Maxwell's equations, and check that they give the correct result. This is done in two groups, starting with the homogeneous equations.

$$\boxed{\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu} = 0} \tag{34}$$

First, if any 2 indices are the same, this equation is vacuous. For example, let the indices be $\mu = 1$, $\nu = 2$, $\rho = 2$. Then (34) gives

$$\partial^1 F^{22} + \partial^2 F^{21} + \partial^2 F^{12} \quad = \quad 0$$

which is a trivial identity. So we only have to consider the case of all 3 indices being distinct, and these can be identified by the missing index.

Missing index is 0

$$\begin{aligned} \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} &= 0 \\ \frac{\partial}{\partial x} B_1 + \frac{\partial}{\partial y} B_2 + \frac{\partial}{\partial z} B_3 &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{35}$$

Missing index is 1, 2 or 3

Take the missing index to be 1.

$$\begin{aligned} \partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} &= 0 \\ - \frac{\partial}{\partial (ct)} B_1 & \\ + \frac{\partial}{\partial y}(-E_3/c) + \frac{\partial}{\partial z}(E_2/c) &= 0 \\ \frac{\partial}{\partial y} E_z - \frac{\partial}{\partial z} E_y + \frac{\partial}{\partial t} B_x &= 0 \end{aligned}$$

which is one component of

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \quad = \quad 0 \tag{36}$$

Thus (35) for no magnetic charge and (36) for Faraday's law are components of the same covariant equation (34). They are deeply related.

We can also write (34) in terms of the dual tensor as follows:

$$\boxed{\partial^\mu \tilde{F}_{\mu\nu} = 0} \tag{37}$$

[5] Generalization to many charges is straightforward.

To see this, recall that

$$\begin{aligned}\tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \\ \partial^\mu \tilde{F}_{\mu\nu} &= \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\mu F^{\alpha\beta}\end{aligned}$$

Since μ, ν, α, β have to be all different, once we choose ν, it is just the missing index for $\partial^\mu F^{\alpha\beta}$. So (37) recovers (34).

4.3 Inhomogeneous equations

We claim that the inhomogeneous equations can be written as

$$\boxed{\partial^\mu F_{\mu\nu} = -\mu_0 J_\nu} \tag{38}$$

or as

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \tag{39}$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Equivalently one can interchange the two indices in $F_{\mu\nu}$ and have a positive constant $+\mu_0$ on the RHS.

Only μ_0 appears; the other constant ϵ_0 will emerge in terms of μ_0 and c^2.

Index is 0
Take $\nu = 0$.

$$\begin{aligned}\partial_\mu F^{\mu 0} &= -\mu_0 J^0 \\ \frac{\partial}{\partial x}F^{10} + \frac{\partial}{\partial y}F^{20} + \frac{\partial}{\partial z}F^{30} &= -\mu_0 \rho c \\ \frac{\partial}{\partial x}(-E_1/c) & \\ + \frac{\partial}{\partial y}(-E_2/c) + \frac{\partial}{\partial z}(-E_3/c) &= -\mu_0 \rho c \\ \boldsymbol{\nabla}\cdot\mathbf{E} &= \frac{1}{\epsilon_0}\rho \qquad (40)\end{aligned}$$

where in the last step we have used $\mu_0 c^2 = 1/\epsilon_0$. So Gauss' law is recovered.

Index is 1, 2 or 3
For example, take $\nu = 1$.

$$\begin{aligned}\partial_\mu F^{\mu 1} &= -\mu_0 J^1 \\ \frac{\partial}{\partial t}F^{01} & \\ + \frac{\partial}{\partial x}F^{11} + \frac{\partial}{\partial y}F^{21} + \frac{\partial}{\partial z}F^{31} &= -\mu_0 J^1 \\ \frac{\partial}{\partial (ct)}(E_1/c) + \frac{\partial}{\partial y}(-B_3) + \frac{\partial}{\partial z}(B_2) &= -\mu_0 J^1\end{aligned}$$

which is one component of

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\partial_t\mathbf{E} = \mu_0 \mathbf{J} \tag{41}$$

Thus Gauss' law (40) and Ampere's law (41) with displacement current automatically incorporated are components of the same 4-vector equation.

To summarize, Maxwell's equations are

$$\boxed{\begin{aligned}\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\ \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu\end{aligned}} \tag{42}$$

4.4 In terms of potential

If we use

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

then the homogeneous Maxwell equations are automatic. For example

$$\begin{aligned}& \partial^\rho F^{\mu\nu} + \partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} \\ = \; & \underline{\partial^\rho(\partial^\mu A^\nu} - \partial^\nu A^\mu) + \underline{\partial^\mu}(\partial^\nu A^\rho - \underline{\partial^\rho A^\nu}) \\ & + \partial^\nu(\partial^\rho A^\mu - \partial^\mu A^\rho) \\ = \; & 0\end{aligned}$$

because the six terms cancel in pairs, e.g., the two terms underlined.

The inhomogeneous equation becomes

$$\boxed{\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\mu_0 J^\nu} \tag{43}$$

4.5 Gauge transformation

The potential is convenient, but contains "too much" information. In other words, we can make a change — called a gauge transformation — on A^μ and not affect the physics. Let $\Lambda(x)$ be a 4-scalar field, and let

$$\begin{aligned}A^\mu &\mapsto A^\mu + \partial^\mu\Lambda \\ F^{\mu\nu} &\mapsto \partial^\mu(A^\nu + \partial^\nu\Lambda) - \partial^\nu(A^\mu + \partial^\mu\Lambda) \\ &= (\partial^\mu A^\nu - \partial^\nu A^\mu) + (\partial^\mu\partial^\nu - \partial^\nu\partial^\mu)\Lambda \\ &= F^{\mu\nu}\end{aligned}$$

since the order of differentiation in a mixed derivative does not matter. Since classical electromagnetism depends only on $F^{\mu\nu}$, it is invariant under the gauge transformation.

We can make use of gauge transformations to choose any value of $\partial\cdot A = \partial^\mu A_\mu$.

Note that (43) can be written as

$$(\partial\cdot\partial)A^\nu - \partial^\nu(\partial\cdot A) = -\mu_0 J^\nu$$

By using a gauge transformation to choose

$$\mathcal{G} \quad \equiv \quad \partial \cdot A = 0$$

this then becomes decoupled:

$$\boxed{(\partial \cdot \partial) A^\nu = -\mu_0 J^\nu}$$

Check that $\mathcal{G}$ is exactly the same as the gauge function used in the analogous discussion in an earlier Chapter.

5 Summary

All of EM[6] is contained in

$$\begin{aligned} \frac{d}{d\tau}\left(m\frac{dx^\mu}{d\tau}\right) &= \frac{d}{d\tau}p^\mu = qF^{\mu\nu}u_\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \\ \partial_\mu F^{\mu\nu} &= -\mu_0 J^\nu \end{aligned}$$

or with the last two replaced by

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= -\mu_0 J^\nu \end{aligned}$$

If we "hide" indices in the obvious way (just as we write an abstract v or v^i instead of v^1, v^2, v^3 for a 3-vector), all the laws are summarized as follows, with contraction over 4-vector indices indicated by a dot:

$$\boxed{\begin{aligned} \frac{dp}{d\tau} &= q\,F \cdot u \\ (\partial \cdot \partial)A - \partial(\partial \cdot A) &= -\mu_0 J \end{aligned}} \tag{44}$$

Or, introducing the notation

$$(a \wedge b)^{\mu\nu} \quad = \quad a^\mu b^\nu - a^\nu b^\mu$$

then

$$\boxed{\begin{aligned} \frac{dp}{d\tau} &= q\,(\partial \wedge A) \cdot u \\ \partial(\partial \wedge A) &= -\mu_0 J \end{aligned}} \tag{45}$$

Of course the whole theory looks simpler and "prettier". But the following facts are even more important.

- The theory is now obviously consistent with relativity.
- We now have a deeper understanding of the different laws, for example: (a) the electric and magnetic forces are different terms in a "dot product" or contraction; (b) Faraday's law and the absence of magnetic charge are different components of the same 4-vector condition; (c) Ampere's law (with displacement current) and Gauss' law are also different components of the same 4-vector condition.
- Moreover, because of these deep connections, the theory of EM is heavily constrained — one cannot arbitrarily make changes in just *one* of the laws. This we know even *without* reference to experiments.

[6] Many textbooks present EM in material media as a different formalism, which is of course correct. But it is also possible to consider a material medium as a collection of atoms or molecules (sources) in otherwise empty space, to which the so-called vacuum equations apply.

A Duality

You may be intrigued that Maxwell's equations (especially in vacuum) and many of the equations developed in this Chapter seem to display a symmetry between $\mathbf{E}/c$ and $\mathbf{B}$; the factor of c just makes sure that we are talking about quantities with the same units. The concept of *duality* can be expressed more systematically. We do so through several problems.

Problem 9
Define the *complex* vector

$$\mathbf{Z} \quad = \quad \mathbf{E}/c + i\mathbf{B} \tag{46}$$

(a) Show that Maxwell's equations in vacuum can be combined into two:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{Z} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{Z} - ic^{-1}\partial_t \mathbf{Z} &= 0 \end{aligned} \tag{47}$$

(b) Note that (47) is linear. Adding solutions and multiplying by a *real* multiple correspond to the usual notion of superposition. But now there is an additional possibility: multiplying by a *complex* constant, i.e.,

$$\mathbf{Z} \quad \mapsto \quad e^{i\theta}\,\mathbf{Z} \tag{48}$$

What happens to $\mathbf{E}$ and $\mathbf{B}$ under this transformation, for small θ?
(c) Also determine what happens to (48) for $\theta = \pm\pi/2$. Show that it is related to the mapping $F \mapsto \tilde{F}$. §

Problem 10
Show that the Lorentz transformations for $\mathbf{E}$ and $\mathbf{B}$, i.e., (25) and (26), can be summarized into *one* set of transformation laws in terms of $\mathbf{Z}$. Since all contributions to $\mathbf{Z}$ have the same dimensions, the

relative velocity will only appear through $\boldsymbol{\beta} = \mathbf{V}/c$. §

Problem 11
Show that in such a transformation, the quantity $\mathbf{Z}^2 = \mathbf{Z} \cdot \mathbf{Z}$ (note: not $|\mathbf{Z}|^2 = \mathbf{Z}^* \cdot \mathbf{Z}$) is invariant. Relate this to I_1 and I_2 defined in Section 3.5. §

However, duality does not extend to the case where there are charges — unless magnetic monopoles also exist and we also map charges into magnetic monopoles. This possibility will be explored towards the end of the course. It remains a mystery why the formulas shown in this Appendix should work — is it an accident or is there some deep physics?

References

[1] RA Lyttleton and H Bondi, "On the physical consequences of a general excess of charge", Proc. Rpy. Soc. A **252**, 313 (1959). DOI: 10.1098/rspa.1959.0155

[2] G Bressi, et al., "Testing the neutrality of matter by acoustic means in a spherical resonator", Phys. Rev. A **83**, 052101 (2011). DOI: 10.1103/PhysRevA.83.052101

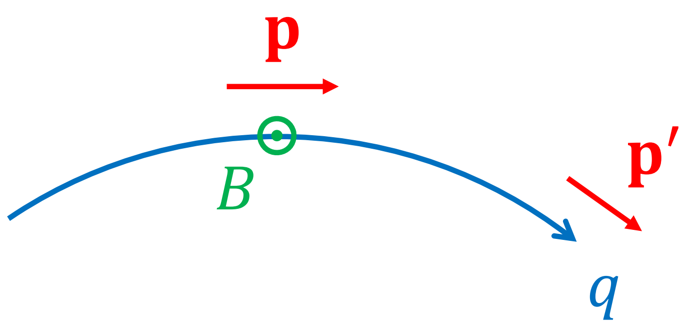

Figure 1 A charged particle moves in a circular path in a constant magnetic field

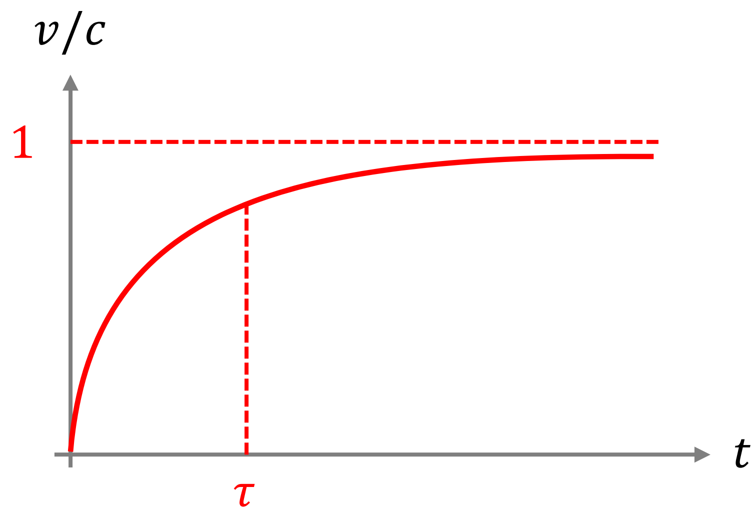

Figure 2 The velocity versus time, for 1D motion along a constant electric field

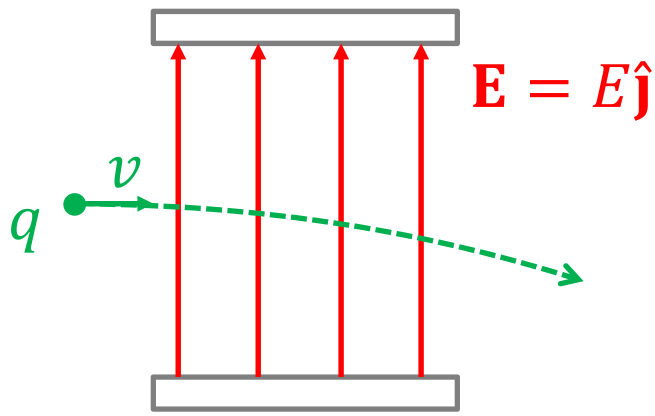

Figure 3 A charged particle moves through the gap between the plates of a capacitor. What does this look like in the frame commoving with the charge?