Chapter 5
Magnetostatics II

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The multipole expansion for magnetism developed up to dipole term. The force and torque on a magnetic dipole are derived. Magnetic material is touched upon, to emphasize that the fields are also produced by currents. A number of further topics are introduced, raising questions such as the physical role of the vector potential and the possibility of monopoles — to be answered later in the course.

Contents

1 Multipole expansion

1.1 The potential

General formula
Recall that (in a suitable gauge), the vector potential in magnetostatics is given as an integral over the current density:

$$\boxed{A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{|\mathbf{r}-\mathbf{s}|}\, J_i(\mathbf{s})\, d^3s} \tag{1}$$

Suppose $\mathbf{J}$ is confined to a region of linear dimension a, i.e., $|\mathbf{s}| \le a$ in the integral, and the field is observed at a large distance, i.e., $r \gg a$. The above formula can then be expanded in powers of the small quantity $s/r \sim a/r$.

Make use of the same expansion as in electrostatics, but here keep only two terms:

$$\frac{1}{|\mathbf{r}-\mathbf{s}|} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{s}}{r^3} + \dots \tag{2}$$

Leading term vanishes
Put the leading term in (2) into (1); this would be the analog of total charge in electrostatics:

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int J_i(\mathbf{s})\, d^3s \tag{3}$$

We claim that the integral is zero. Heuristically, if the current is confined in a finite volume, then any current flowing to the right must eventually turn back and flow to the left, so there is as much positive contribution as negative contribution.

A formal proof proceeds as follows. Start with the identity

$$\frac{\partial}{\partial s_j}(s_i\, J_j) = \delta_{ij} J_j + s_i\,(\boldsymbol{\nabla}\cdot\mathbf{J})$$

On the RHS, the first term is J_i and the second term vanishes by charge conservation, giving

$$J_i = \frac{\partial}{\partial s_j}(s_i\, J_j)$$

In (3), the integral of a total derivative can be converted to a surface integral at infinity (or indeed on any surface beyond a). But $\mathbf{J} = 0$ at spatial infinity, so the integral is zero.

So there is no term in $\mathbf{A}$ that goes as r^{-1}. This is expected, since the formalism started with the assumption that there are no magnetic monopoles.

Dipole

The next term gives

$$A_i(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\frac{r_j}{r^3}\int s_j J_i(\mathbf{s})\, d^3s \tag{4}$$

Now we claim that the integral is antisymmetric in its two indices, i.e.,

$$\int (s_j J_i + s_i J_j)\; d^3s \quad = \quad 0 \tag{5}$$

To prove this, start with the identity

$$\begin{aligned}
&\frac{\partial}{\partial s_k}(s_i s_j J_k) \\
= \quad & \delta_{ki} s_j J_k + \delta_{kj} s_i J_k + s_i s_j\,(\boldsymbol{\nabla}\cdot\mathbf{J}) \\
= \quad & s_j J_i + s_i J_j
\end{aligned}$$

in which we have first differentiated one factor at a time, and then used the conservation of charge to eliminate the last term in the second line. Thus the integrand in (5) is a total derivative, and can be converted to a surface integral at infinity, where $\mathbf{J}$ vanishes.

With this result, we can antisymmetrize the integrand in (4):

$$\begin{aligned}
&\int s_j J_i\, d^3s \\
= \quad & \frac{1}{2}\int (s_j J_i - s_i J_j)\; d^3s \\
= \quad & \frac{1}{2}\,\epsilon_{kji}\int (\mathbf{s}\times\mathbf{J})_k\; d^3s \\
\equiv \quad & \epsilon_{kji}\,\mu_k
\end{aligned} \tag{6}$$

where we have defined the *magnetic moment*

$$\boxed{\boldsymbol{\mu} = \frac{1}{2}\int \mathbf{s}\times\mathbf{J}(\mathbf{s})\, d^3s} \tag{7}$$

as a property of the current distribution. Like the electric dipole moment, it is one power of the displacement $\mathbf{s}$ multiplied by the primary quantity, in this case $\mathbf{J}$. Heuristically, the prefactor $1/2$ is inserted because the cross product implicitly contains two terms. Note however that the magnetic dipole is here defined in terms of the current $\mathbf{J}$, due to the movement of structureless point charges. Strictly speaking, intrinsic magnetic moments require a separate discussion; see Supplement in the previous Chapter and this Chapter.

When this is put into (4), we get

$$A_i(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\,\epsilon_{kji}\,\frac{r_j}{r^3}\,\mu_k \tag{8}$$

This can evidently be written in vector form, which is shown together with the analogous formula for an electric dipole:

$$\begin{aligned}
&\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} \\
&\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\frac{\boldsymbol{\mu}\times\mathbf{r}}{r^3}}
\end{aligned} \tag{9}$$

Problem 1

The definition of $\boldsymbol{\mu}$ involves the displacement $\mathbf{s}$ of the source point from the origin. Therefore it might appear that if the origin is shifted (or equivalently a current distribution is displaced) the magnetic moment would change. Show that this does not happen. §

1.2 The field

To calculate the magnetic field $\mathbf{B}$, take the curl of (9).

$$\begin{aligned}
B_m \quad &= \quad \frac{\mu_0}{4\pi}\,\epsilon_{mnk}\,\epsilon_{kji}\,\mu_j\,\partial_n\frac{r_i}{r^3} \\
&= \quad \frac{\mu_0}{4\pi}\,(\delta_{mj}\delta_{ni} - \delta_{mi}\delta_{nj})\;\mu_j\,\partial_n\frac{r_i}{r^3} \\
&= \quad \frac{\mu_0}{4\pi}\left(\mu_m\,\partial_i\frac{r_i}{r^3} - \mu_j\,\partial_j\frac{r_m}{r^3}\right)
\end{aligned}$$

The first term is zero,[1] since we are concerned with $r > 0$. The second term can be written in a form easier to remember by recalling[2]

$$-\frac{r_m}{r^3} \quad = \quad \partial_m\frac{1}{r}$$

This allows the field to be expressed in the following form, in which the analogous formula for an electric dipole is repeated

$$\begin{aligned}
&E_m = \frac{1}{4\pi\epsilon_0}\,p_j\partial_m\partial_j\frac{1}{r} \\
&\boxed{B_m = \frac{\mu_0}{4\pi}\,\mu_j\,\partial_m\partial_j\frac{1}{r}}
\end{aligned} \tag{10}$$

The field patterns implied by the two formulas in (10) would appear to be identical (say for a unit

[1] Up to a multiplicative constant, this is like the electric field due to a point charge at the origin, so the divergence is a δ-function at the origin.

[2] This is essentially the relationship between electric field and potential for a point charge.

electric dipole and a unit magnetic dipole both in the z-direction). This is indeed true away from the dipole, i.e., in the region $r > a$ in which these formulas are valid. But there is a significant difference, not shown by these formulas, when we come right into the region of the dipole itself. What happens right at the dipole is contained in extra terms proportional to $\delta^3(\mathbf{r})$ and its spatial derivatives; these singular functions are discussed in e.g. Ref. [1] and in Sect. 2.4.

Problem 2
Prove the identity

$$\partial_i \partial_j \frac{1}{r} \quad = \quad \frac{3r_i r_j - r^2 \delta_{ij}}{r^5}$$

and hence write out the Cartesian components of $\mathbf{B}$ implied by (10), in terms of the usual polar coordinates (r, θ, ϕ). §

2 Magnetic dipole moment

The magnetic moment is defined in term of an integral over $\mathbf{s} \times \mathbf{J}$. This Section develops a better intuitive picture.

2.1 Current loop

Consider a wire loop in the x–y plane, in which a current I flows. (**Figure 1a**). Decompose d^3s as

$$d^3s \quad = \quad da\, d\ell$$

where da is the element of area in the cross-section of the wire, and $d\ell$ is a line integral along the wire. The former leads to

$$\mathbf{J}\, da \quad = \quad I\, \hat{\mathbf{e}}_\ell$$

where $\hat{\mathbf{e}}_\ell$ is a unit vector along the wire. Next combine

$$\hat{\mathbf{e}}_\ell\, d\ell \quad = \quad d\boldsymbol{\ell} = d\mathbf{s}$$

where the wire segment $d\boldsymbol{\ell}$ can be identified with the shift $d\mathbf{s}$ of the point $\mathbf{s}$ on the wire. Therefore

$$\boxed{\boldsymbol{\mu} = I\boldsymbol{\mathcal{A}}}$$
$$\boldsymbol{\mathcal{A}} = \frac{1}{2} \oint \mathbf{s} \times d\mathbf{s} \tag{11}$$

Consider an element of the integral shown in **Figure 1b**; it is evidently the triangular area swept out by the movement of the point ds, and $\boldsymbol{\mathcal{A}}$ can be identified as the area of the wire loop, regarded as a vector in the usual way. This agrees with the definition of μ for the specific case of a circular loop in the last Chapter.

2.2 Short solenoid

A solenoid of length L and circular cross-sectional radius a (say with cylinder axis along z) is observed at a distance point $\mathbf{r}$ (**Figure 2**). Assume $a \ll L \ll r$; the latter inequality is what we mean by "short". Given this condition, the solenoid can be regarded as N circular current-carrying loops stacked together, and the dipole formula applies, with $\boldsymbol{\mu} = NI\boldsymbol{\mathcal{A}}$.

2.3 Relationship to angular momentum

Regard the current source as a fluid, with charge density $\rho(\mathbf{s})$, moving with a local fluid velocity $\mathbf{v}(\mathbf{s})$, where $\mathbf{s}$ is an arbitrary source point in the fluid. Then

$$\mathbf{J}(\mathbf{s}) \quad = \quad \rho(\mathbf{s})\mathbf{v}(\mathbf{s}) \tag{12}$$

and the magnetic moment is

$$\boldsymbol{\mu} \quad = \quad \frac{1}{2} \int \mathbf{s} \times \rho(\mathbf{s})\mathbf{v}(\mathbf{s})\, d^3s \tag{13}$$

On the other hand, let the mass density be $\rho_m(\mathbf{s})$; then the total angular momentum is given by a similar expression[3]

$$\mathbf{L} \quad = \quad \int \mathbf{s} \times \rho_m(\mathbf{s})\mathbf{v}(\mathbf{s})\, d^3s \tag{14}$$

If the charge density and the mass density are proportional:

$$\rho(\mathbf{s}) \quad = \quad \kappa\, \rho_m(\mathbf{s})$$

then putting this into (13) and taking out the constant factor κ, we find

$$\boldsymbol{\mu} \quad = \quad \frac{\kappa}{2} \mathbf{L}$$

Moreover, the total charge q and the total mass m must be related as $q = \kappa m$, so we have the important relationship

$$\boxed{\boldsymbol{\mu} = \frac{q}{2m} \mathbf{L}} \tag{15}$$

This applies for example to a collection of electrons (ignoring their spin) in orbital motion, since all electrons regarded as point particles without internal structure have exactly the same charge-to-mass ratio. However, if the charge-to-mass ratio

[3] This is total rather than orbital angular momentum, so normally the symbol $\mathbf{J}$ would be used instead of $\mathbf{L}$. But here we want to avoid confusion with the current density.

of a system (e.g., a collection of protons and electrons)[4] is not uniform, the relation (15) must be modified by a dimensionless parameter g

$$\boxed{\boldsymbol{\mu} = g\,\frac{q}{2m}\,\mathbf{L}} \tag{16}$$

The magnetic moment of elementary particles are specified by their g-factors.

Problem 3
Calculate g for a rotating solid sphere of radius R, in which the mass density is uniform, but whose charge is distributed
(a) also uniformly in the sphere,
(b) uniformly on the *surface* of the sphere, and
(c) uniformly in the region $r \le \alpha R$ for some α in the range $0 < \alpha \le 1$. §

The g-factor of an elementary spin 1/2 particle (such as an electron, muon or quark) turns out to be

$$g \quad = \quad 2 + \text{ quantum corrections}$$

The derivation of the factor 2 was a triumph of the Dirac equation in relativistic quantum mechanics, while the correct computation of the quantum corrections was a triumph of renormalization in QED. At the level CED, we can simply adopt a naive model: in a spinning electron (regarded as made up of even smaller constituent parts), the charge is distributed farther away from the center compared to the mass.

Incidentally, recently (in 2021) there is some excitement about the value of $g-2$ for muons [2], which seems to exhibit a tiny discrepancy between theory and experiment.

2.4 Difference between electric dipole and magnetic dipole

Figure 3a shows an electric dipole: point charges $\pm q$ at $z = \pm a/2$, so that $\mathbf{p}$ points towards $+z$. **Figure 3b** shows a magnetic dipole formed by a short solenoid, with $\boldsymbol{\mu}$ pointing towards $+z$. Far from these objects, the fields are given by similar formulas as in (10). Therefore there is a temptation to say that the magnetic case also consists of two magnetic "poles": an N pole near the top and an S pole near the bottom of the solenoid (**Figure 3c**). Each "pole" supposedly causes a magnetic field that is radial and inverse-square (in analogy with Coulomb's law for a point charge), and that would surely lead to the magnetic formula in (10). That may be a convenient shorthand for describing the situation far from the magnetic dipole, but *it is not correct* at short distances.

At short distances there is a major difference, which can be stated in several equivalent ways.

- Along the z axis (and in a small region around it), the electric field in **Figure 3a** points down, whereas the magnetic field in **Figure 3b** points up.
- More precisely, across the x–y plane, there is a net downward electric flux (in fact equal to the total flux emanating from the positive charge), but the magnetic case shows zero net flux — all the downward flux at large $\rho = \sqrt{x^2+y^2}$ is compensated by the upward flux inside the solenoid.
- The electric lines of force have beginnings (at the positive charge) and ends (at the negative charge); the magnetic flux lines have no beginning or end because $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. If we expand the drawing and focus on a tiny region around the positive charge (**Figure 4a**) and a corresponding region near the end of the solenoid (**Figure 4b**), the flux lines are as shown. Despite the name *dipole*, there is no pole at the end of the solenoid.

To put the matter more graphically, if an electric dipole is broken up, we get two opposite point charges (**Figure 5a**). If a magnetic dipole, e.g., a short solenoid, is broken up, we get two solenoids, each still a dipole, and each still having an N pole and a S pole (**Figure 5b**). You cannot get an isolated N or S pole — which is after all what $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ tells us.

These differences can be phrased mathematically as follows. We note that (10) is not correct right at the dipole, and that there is a strong field in the direction of the dipole. One way to correct for this is to add a term concentrated at the origin. The vector index can only be carried by the respective dipole moment, and we put

$$\begin{aligned} E_m &= \frac{1}{4\pi\epsilon_0}\,p_j\left[\partial_m\partial_j\frac{1}{r} + C\delta_{mj}\delta^3(\boldsymbol{r})\right] \\ B_m &= \frac{\mu_0}{4\pi}\,\mu_j\left[\partial_m\partial_j\frac{1}{r} + C'\delta_{mj}\delta^3(\boldsymbol{r})\right] \end{aligned} \tag{17}$$

for some numerical constants C and C'.

Problem 4
We now determine the constants C and C'.
(a) Represent an electric dipole as two point charges $\pm q$ and positions $\pm\mathbf{a}/2$. Then the charge density is

$$\rho(\mathbf{r}) \quad = \quad q\delta^3(\mathbf{r}-\mathbf{a}/2) - q\delta^3(\mathbf{r}+\mathbf{a}/2)$$

[4] But even a "point" particle can be regarded as made up of even smaller constituent parts, which could be heterogeneous.

Show that for $|\mathbf{a}| \to 0$ this can be written as

$$\rho(\mathbf{r}) = -\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r})$$

Hint: Formal Taylor series in $\mathbf{a}$.
(b) By requiring the divergence of (17) to be correct (i.e., zero in the magnetic case and related to ρ in the electric case), determine the two constants. §

The existence of these δ-function terms, and the fact that they are different in the electric and magnetic cases, constitute a precise statement of the features discussed here. The fact that a bar magnet, regarded as a dipole, is described by (17) with C' rather than C is equivalent to the statement that it is *not* made up of two opposite monopoles.

3 Force and torque on a magnetic dipole

3.1 Force on a current distribution

A current element $\mathbf{J}(\mathbf{s})\, d^3s$ is subject to a magnetic force

$$\mathbf{F} = \mathbf{J}(\mathbf{s})\, d^3s \times \mathbf{B}(\mathbf{s})$$

So for an entire current distribution, the force is, in terms of components

$$F_i = \epsilon_{ijk} \int J_j(\mathbf{s})\, B_k(\mathbf{s})\, d^3s \tag{18}$$

Suppose the current distribution is confined to a small region near the origin,[5] and $\mathbf{B}$ is slowly varying. Then we may expand

$$B_k(\mathbf{s}) \approx B_k + s_m\, \partial_m B_k + \ldots$$

where on the RHS B_k and its derivatives are understood to be evaluated at the origin. The first term when inserted into (18) gives

$$F_i = \epsilon_{ijk} \left[\int J_j(\mathbf{s})\, d^3s \right] B_k \tag{19}$$

and the square bracket vanishes by a previous argument. The second term gives

$$F_i = \epsilon_{ijk} \left[\int J_j(\mathbf{s}) s_m\, d^3s \right] \partial_m B_k \tag{20}$$

Using (6), the square bracket can be expressed as

$$[\ldots] = \epsilon_{\ell m j} \mu_\ell$$

so that

[5]It is easy to generalize to the case where the current distribution is confined to a small region near some point $\mathbf{r}$.

$$\begin{aligned} F_i &= \epsilon_{ijk} \epsilon_{\ell m j} \mu_\ell\, \partial_m B_k \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{i\ell})\, \mu_\ell\, \partial_m B_k \\ &= \mu_k \partial_i B_k - \mu_i \partial_k B_k \end{aligned} \tag{21}$$

The second term vanishes by $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and we finally get

$$F_i = \mu_k \partial_i B_k \tag{22}$$

Assuming $\mathbf{B}$ is an external field, so that $\boldsymbol{\nabla} \times \mathbf{B} = 0$ at the position of the dipole, we can also write the above as $F_i = \mu_k \partial_k B_i$, or in vector notation as

$$\boxed{\mathbf{F} = (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\mathbf{B}} \tag{23}$$

Since we are assuming that $\boldsymbol{\mu}$ is constant (i.e., not induced by the field itself), (22) can also be written as

$$\boxed{\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B})} \tag{24}$$

which implies a potential energy

$$\boxed{U = -\, \boldsymbol{\mu} \cdot \mathbf{B}} \tag{25}$$

Note the analogy with the corresponding formula for an electric dipole in the presence of an electric field.

3.2 Torque on a current distribution

In a similar way, the torque $\boldsymbol{\tau}$ acting on a static current distribution is given by

$$\boldsymbol{\tau} = \int \mathbf{s} \times \left[\mathbf{J}(\mathbf{s})\, d^3s \times \mathbf{B}(\mathbf{s}) \right]$$

Using the triple product formula, and approximating $\mathbf{B}(\mathbf{s}) = \mathbf{B}$ (the value at the origin), we get

$$\boldsymbol{\tau} = \int \left[\mathbf{J}(\mathbf{s} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{s} \cdot \mathbf{J}) \right] d^3s$$

in which $\mathbf{B}$ is a constant independent of $\mathbf{s}$.

Problem 5
Show that

$$\int (\mathbf{s} \cdot \mathbf{J})\, d^3s = 0$$

Hint: Consider the integral of $\boldsymbol{\nabla} \cdot (s^2 \mathbf{J})$. §

Thus we are left with the first term:

$$\tau_i = \int J_i s_j\, d^3s \cdot B_j$$

It is then straightforward to show that

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}} \tag{26}$$

Problem 6
Prove (26). Hint: use (6). §

4 Material medium

This course will not deal with dielectric or magnetic materials, especially the formalism for linear materials so similar to that for EM in vacuum — and because of the similarity so often a cause of conceptual confusion among students. This Section just gives a qualitative account of permanent magnets, simply to point out the difference between total current and free current.

A schematic diagram of a permanent bar magnet is shown in **Figure 6a**, say with an N "pole" at one end and a S "pole" at the other. That means the flux lines are as shown in **Figure 6b**. As emphasized earlier, the flux lines are continuous. So, outside the bar, the lines go from N to S — that is the convention for defining N and S, and is parallel to the case of electric dipoles in defining + and − charges. But inside the bar, the flux lines go from S to N — opposite to the case of electric dipoles.

A *permanent* magnet maintains its field *apparently* without the need for any currents. But according to Ampere's law, that cannot be. A cross-section of the bar is shown in **Figure 6c** and we consider the loop Γ as shown. Outside the bar, the field is very weak, approximately zero, so

$$\oint_\Gamma \mathbf{B} \cdot d\boldsymbol{\ell} \quad = \quad B\ell$$

where B is the magnitude of the field inside the bar and ℓ is the length of the side of the rectangle Γ. So there must be a current passing through the area enclosed by Γ (**Figure 6d**), i.e., on the surface. Where does the current come from if there are no wires?

In a permanent magnet, there is a net electron spin in one direction. Each spinning electron can be thought of as a small current loop, and represented schematically in the sectional view in **Figure 7a**. In the interior of the bar, the currents cancel; however there is a net effective current on the surface, as shown in **Figure 7b**. This effective current acts in the same way as the current in a solenoid.

However, this current is *bound* to the material, and unlike *free* currents does not flow into any wires. Thus the current I and the current density $\mathbf{J}$ can be split into the bound and free components:

$$\begin{aligned} I &= I_b + I_f \\ \mathbf{J} &= \mathbf{J}_b + \mathbf{J}_f \end{aligned} \qquad (27)$$

- The current that should appear in Ampere's law is the total current.
- The current that should appear in relation to any external circuit is the free current.

The handling of this complication, including the introduction of a quantity $\mathbf{H}$ in these circumstances, will not be discussed here. Suffice it to say that there is much confusion on this topic, including not infrequent errors in the literature and in textbooks. Some incorrect statements include: (a) using $\mathbf{H}$ in the Lorentz force law, and (b) referring to $\mathbf{H}$ as the field caused by the free currents.[6] We here only point to the need for caution. Accurate accounts are given by Jackson [3] and by Landau and Lifschitz [4].

We emphasize again that strictly speaking, individual particles with intrinsic magnetic moments have to be dealt with separately, within what we call Model 2 in the Supplement to the last Chapter and this Chapter.

5 Further topics

Here we point to several questions of a conceptual nature that are already foreshadowed, in order to plant these tantalizing questions in the minds of readers, with detailed analysis to be presented separately later.

5.1 Is the vector potential necessary?

The vector potential $\mathbf{A}$ is convenient in decoupling the equations. But is it *necessary*? Fields act on the charges and currents according to the Lorentz force law, which cites $\mathbf{B}$ but not $\mathbf{A}$. Currents produce fields according to the Biot–Savart law (and analogous laws in time-dependent situations), which in its final form does not mention $\mathbf{A}$.

A naive answer is that $\mathbf{A}$ is only an intermediate mathematical tool and not logically necessary. This answer is satisfying since $\mathbf{A}$ can be changed by a gauge transformation

$$\mathbf{A}(\mathbf{r}) \quad \mapsto \quad \mathbf{A}(\mathbf{r}) + \boldsymbol{\nabla}\Lambda(\mathbf{r}) \qquad (28)$$

and is not unique; it is partly a matter of convention, depending on what Λ we choose.

However, this is not entirely true. It turns out that in quantum mechanics (QM), $\mathbf{A}$ is necessary. We shall get a taste of this when we deal (in the classical domain) with the Hamiltonian or Lagrangian formulation of EM. So it seems that, at least in QM, $\mathbf{A}$ is necessary.

But how can a necessary concept be not unique? To put this precisely, suppose we describe the same

[6] The latter is an error since in general $\boldsymbol{\nabla} \cdot \mathbf{H} \neq 0$, and $\mathbf{H}$ is not given by the Bio–Savart law using just the free currents, even though $\boldsymbol{\nabla} \times \mathbf{H}$ is the free current.

physical situation in QM, say involving the energy spectrum of an atom, using the Schrödinger equation with two different sets of $\mathbf{A}$, related by (28). It would seem that we would get two different energy spectra — which would be nonsense.

5.2 Is the vector potential physical?

Obviously it cannot be *completely* physical,[7] since it can be changed as a matter of convention. But it must be *partly* physical, since its curl can be measured. More precisely, suppose a point charge q travels in a region where $\mathbf{B} = 0$ but $\mathbf{A} \neq 0$. Are there circumstances in which a nontrivial effect can be detected? The answer turns out to be yes, and hinges on certain *topological* ideas.

Incidentally students of physics would have encountered questions in a similar vein in other branches of physics.

- Is the QM wavefunction Ψ physical? It must be partly physical since $|\Psi|^2$ can be measured. It cannot be completely physical since we can always change the phase
$$\Psi(\mathbf{r}) \quad \mapsto \quad \Psi(\mathbf{r}) \cdot \exp i\Theta \tag{29}$$
In other words, the phase is only a matter of convention and not physical. One of the most intriguing ideas we shall come to in this course is the deep relationship between (28) and (29).
- Are the components F_x, F_y, F_z of a force $\mathbf{F}$ physical? Again, they must be partly physical, since we can measure the magnitude. But the individual components are a matter of convention, depending on the axes we choose.
- On a curved surface (and spacetime is a curved surface), one can choose many different coordinate systems. Any quantity expressed in a particular coordinate system must again be partly but not completely physical — the components would change when the coordinate system is changed.

Behind all these is a deep worry: If there is a part of the formalism that is not physical and can be changed according to convention, how can we be sure that physics (namely anything that can be measured) is independent of such conventions? Physics being independent of the gauge function Λ is exactly one such example.

One major advance of 20th century physics is to turn this worry around to advantage: We *demand* that the laws of physics must give exactly the same physics even when such conventions change. This concept of *covariance* now permeates modern physics — whether covariance under rotation of Cartesian coordinates, Lorentz transformations, general coordinate transformation, gauge transformations, or transformation between what we call the different colors in QCD. We shall illustrate some of these ideas in the context of CED.

5.3 Can there be monopoles?

For the bar magnets encountered in daily life, there are certainly no individual N and S poles or *monopoles*; the magnetic field is produced by currents. But we can ask a different set of questions:

- Are monopoles theoretically possible?
- If so, could they exist (rarely) in nature?

There are several twists and turns in trying to answer these questions.

Possible?
At the classical level, the answer would seem to be trivially yes. All we need to do is to modify the law to
$$\boldsymbol{\nabla} \cdot \mathbf{B} \quad = \quad \rho_m \tag{30}$$
where ρ_m is the density of magnetic charge.[8] There could be a proportionality constant, depending on the unit adopted for magnetic charge. Additional but straightforward changes would have to be made to the time-dependent equations.

Not possible?
If $\boldsymbol{\nabla} \cdot \mathbf{B} \neq 0$, we can no longer define $\mathbf{A}$. But $\mathbf{A}$ is necessary in QM, so this would seem to rule out magnetic monopoles. That seems like a nice argument as to why monopoles have never been found.

Possible but quantized?
In the 1930s, Dirac found a way out [5]: It is possible to have (30) with $\rho_m \neq 0$ and still be able to define $\mathbf{A}$ well enough[9] to have QM, provided magnetic charges are quantized (see below). The gauge degree of freedom is essential in this argument: Dirac's solution is singular, but this singularity (along a line) can be arbitrarily moved away by a gauge transformation and is therefore unphysical. In the 1970s, Wu and Yang gave an improved argument [6, 7] using the idea of fiber bundles, avoiding all singularities.

Both the Dirac and the Wu–Yang derivations allow monopoles of strength g only under one condition, that

[7] Sometimes people phrase this question as whether the potential is "real" — "real" in the sense of "actual" or "part of reality". But since (especially in QM), there is the danger that "real" is understood in contrast to "imaginary", we shall avoid this terminology.

[8] Here the subscript m refers to magnetic, and ρ_m here should not be confused with the mass density denoted by the same symbol earlier.

[9] The tricky point is, of course, what is "well enough".

$$eg = nK \tag{31}$$

where e is the charge of any fundamental particle (i.e., the electronic charge), K is a universal constant[10] proportional to $\hbar$ and n is an integer. In other words, monopoles, if they exist, must have strengths only in quantized units. Turning this argument around, if a single monopole g exists, then all charges are integer multiples of a fundamental unit, namely $e = ne_0$ where $e_0 = (K/g)$ — resolving the mystery of charge being quantized in *integer* units that has no easy explanation otherwise.

Experimental search
But so far, no monopoles have been definitively found, though there has been at least one possible candidate event [8]. A stringent upper limit has been set experimentally.

Students will understand these remarks better towards the end of this course.

A Supplement: Spin-orbit and hyperfine interaction

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

This Supplement gives a semi-quantitative estimate of the magnitude of the spin-orbit and hyperfine interactions in atomic physics. We focus only on the case of hydrogen (H).

Basic formulas
Elementary accounts of H give for the energies $E_n = -E_B/n^2$. The binding energy and the Bohr radius a are, in terms of the *fine structure constant* α,

$$\begin{aligned} E_B &= \frac{1}{2}mc^2\,\alpha^2 \\ a &= \frac{\hbar}{mc\,\alpha} \\ \alpha &= \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137} \end{aligned} \tag{32}$$

Here m is the electron mass.

Magnetic moment due to orbit
For a general state with orbital angular momentum $\mathbf{L}$, there would be an associated magnetic moment

$$\boldsymbol{\mu} = \frac{-e}{2m}\mathbf{L}$$

giving rise to a magnetic field of strength

$$B \sim \frac{\mu_0}{4\pi}\,\mu\,\frac{1}{r^3} \tag{33}$$

at a distance r. Directional dependence is glossed over, and we only consider the orders of magnitude.

Magnetic moment due to spin
At the same time, associated with the spin $\mathbf{S}$ of the electron is a spin magnetic moment

$$\boldsymbol{\mu}' = \frac{-e}{m}\mathbf{S}$$

Interaction between two magnetic moments
The spin magnetic moment interacts with the field (33), with an energy

$$\begin{aligned} \mathcal{E} &\sim \boldsymbol{\mu}'\cdot\mathbf{B} \\ &\sim \left(\frac{e}{m}\mathbf{S}\right)\cdot\left(\frac{\mu_0}{4\pi}\right)\left(\frac{e}{2m}\mathbf{L}\right)\frac{1}{r^3} \\ &\sim \frac{1}{4\pi\epsilon_0}\frac{e^2}{2m^2c^2}\frac{1}{a^3}\mathbf{L}\cdot\mathbf{S} \end{aligned} \tag{34}$$

in which we have replaced r by the typical magnitude a and assumed that the vector directions of $\mathbf{r}$ do not appear.

Using

$$E_B = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 a}$$

and the fact that $|\mathbf{L}|$ and $|\mathbf{S}|$ are both of order $\hbar$, we can write (34) as

$$\mathcal{E} \sim E_B\frac{\hbar^2}{m^2c^2a^2}$$

Finally, using the expression for a, this reduces to

$$\boxed{\mathcal{E}_{\rm SO} \sim E_B\alpha^2} \tag{35}$$

giving the order-of-magnitude of spin-orbit (SO) interaction energy as two powers of α smaller than the binding energy. With $E_B \sim 10$ eV and $\alpha \sim 10^{-2}$, spin-orbit energy differences are on the order of 10^{-3} eV. Precise expressions would involve first-order perturbation theory on the quantum states.

Hyperfine interaction
The spin-orbit interaction vanishes for the orbital ground state, since $\mathbf{L} = 0$. Instead, there is a smaller *hyperfine interaction*, due to the spin of the proton. That is, we replace the orbital magnetic moment $\sim (e/2m)\,\mathbf{L}$ by the nuclear magnetic moment $\sim (e/M)\,\mathbf{S}'$, where M is the proton mass and S' is the nuclear spin.[11] This leads to an interaction term of the type

$$\mathbf{S}\cdot\mathbf{S}'$$

[10] We do not yet specify it since we have not yet defined the unit in which g is to be measured.

[11] In this order-of-magnitude estimate, we ignore the fact that the proton has a g factor of about 3.

The typical energy for the hyperfine (HF) interaction has a factor m/M compared to the spin-orbit interaction

$$\boxed{\mathcal{E}_{\rm HF} \sim E_B \alpha^2 (m/M)} \tag{36}$$

The factor $m/M \sim 1/2000$ leads to a typical order-of-magnitude 10^{-6} eV. Precise calculations give 5.8×10^{-6} eV, and the wavelength of the associated transition[12] is $\lambda = 21$ cm. The 21 cm line is extremely important in revealing the presence of hydrogen gas in astronomy.

B Supplement: Models for magnetism — Part 2

Each supplement will consist of one or more problems, usually more challenging and of an open nature, to stretch the better students.

This supplement continues the discussion of Model 2 in the Supplement in the last Chapter: a model in which charged particles have intrinsic magnetic moments whose magnitude cannot be changed.

The particles
Consider a dilute[13] collection of point particles labelled by n, each with mass m, and charge q. The positions are $\mathbf{r}_n$ and the spins are $\mathbf{S}_n$ with fixed magnitude: $\mathbf{S}_n^2 = S^2$. Thus there are five (not six) degrees of freedom for each particle. The magnetic moments are

$$\begin{aligned} \boldsymbol{\mu}_n &= \gamma \mathbf{S}_n \\ \gamma &= g\,\frac{q}{2m} \end{aligned} \tag{37}$$

Generalization to particles with difference masses, charges, total spins and different g-factors is straightforward.

Equations of motion for the particles
Now the equations of motion take the form (for simplicity restricting to the non-relativistic case):

$$m\frac{d^2\mathbf{r}_n}{dt^2} = q\,(\mathbf{E} + \mathbf{v}_n \times \mathbf{B}) + \ldots \tag{38}$$

$$\frac{d\mathbf{S}_n}{dt} = \ldots \tag{39}$$

Here $\mathbf{v}_n = d\mathbf{r}_n/dt$ is the velocity of the particle, and the fields are understood to be evaluated at $\mathbf{r}_n$.

[12]Between the states with $\mathbf{S}$ and $\mathbf{S}'$ being parallel and anti-parallel.
[13]This assumption is to avoid any complications arising from collisions.

The first task is to write out all the new terms due to the magnetic dipole moment. The key is to recognize that the force and torque on an *intrinsic* magnetic moment should take the same form as those on a magnetic moment due to a current loop. Be careful with time-dependent terms.

Sources for the fields
The sources for the fields are the charge density ρ, the current density $\mathbf{J}$ and the magnetic moment density $\mathbf{M}$, defined as follows:

$$\begin{aligned} \rho(\mathbf{r}) &= \sum_n q\,\delta^3(\mathbf{r}-\mathbf{r}_n(t)) \\ \mathbf{J}(\mathbf{r}) &= \sum_n q\mathbf{v}_n\,\delta^3(\mathbf{r}-\mathbf{r}_n(t)) \\ \mathbf{M}(\mathbf{r}) &= \sum_n \boldsymbol{\mu}_n\,\delta^3(\mathbf{r}-\mathbf{r}_n(t)) \end{aligned} \tag{40}$$

Note that $\mathbf{J}$ is the free current density, since it is related exclusively to the motion of the point charges.

Modified Maxwell equations
The next task is to add appropriate terms to the RHS of the inhomogeneous Maxwell equations, to account for the effect of the magnetic moments. Again, the key is to recognize that the fields produced by an *intrinsic* magnetic moment should be the same as that produced by a magnetic moment due to a current loop.

Consistency
The modified force law and the modified Maxwell equations must be consistent in the action-reaction sense. This is deferred till we have learnt about the energy and momentum in the fields.

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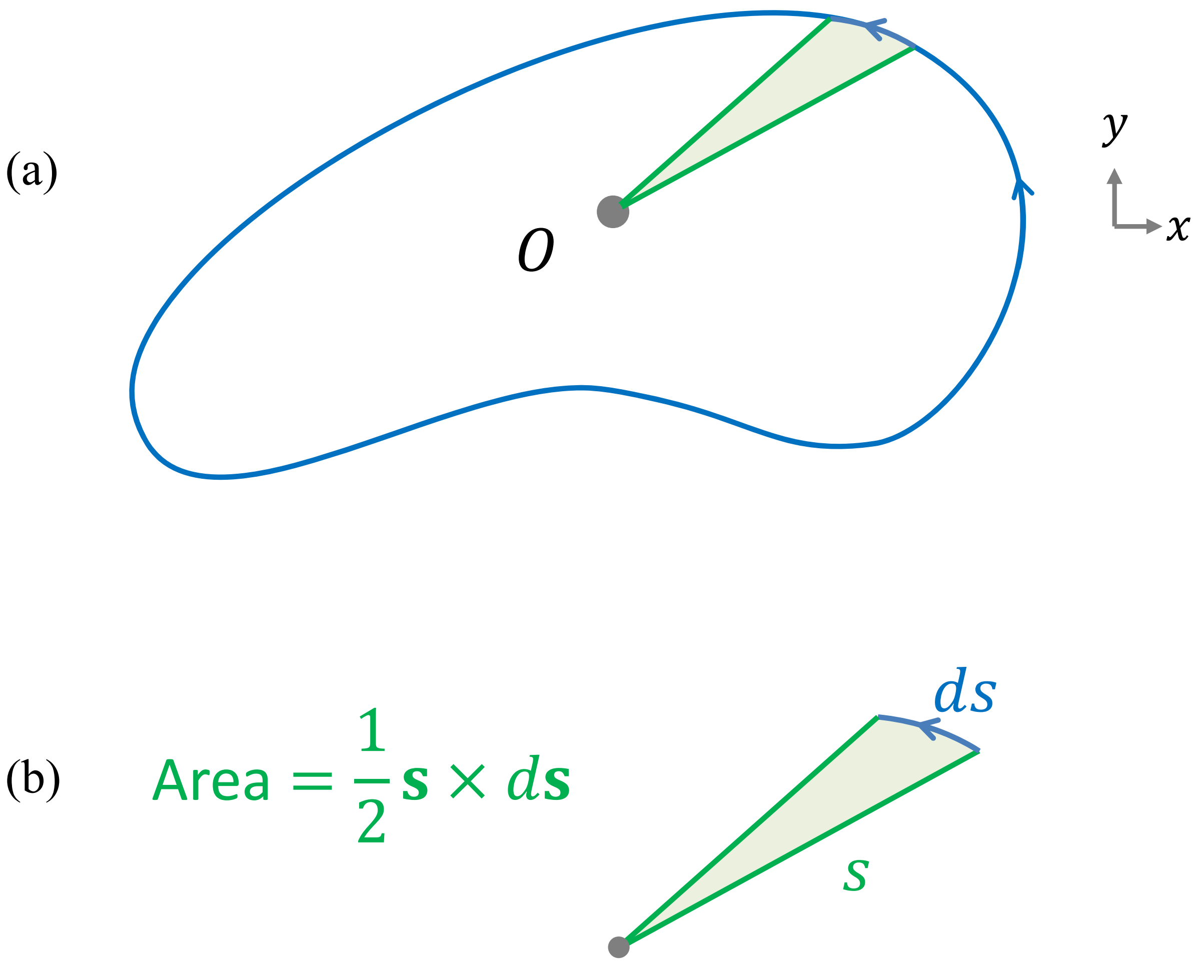


Figure 1

(a) Current loop
(b) An element of area

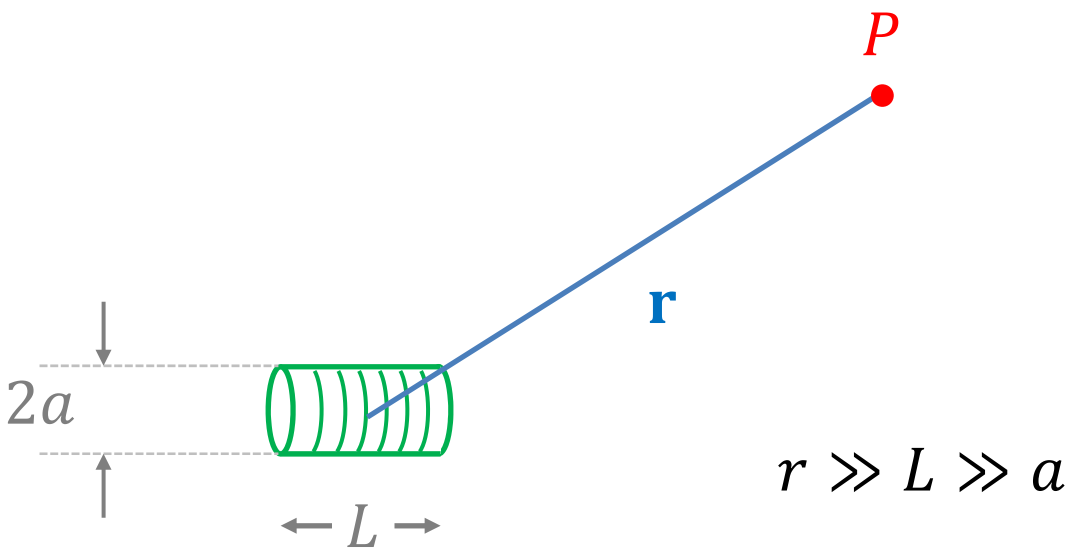


Figure 2

A short solenoid

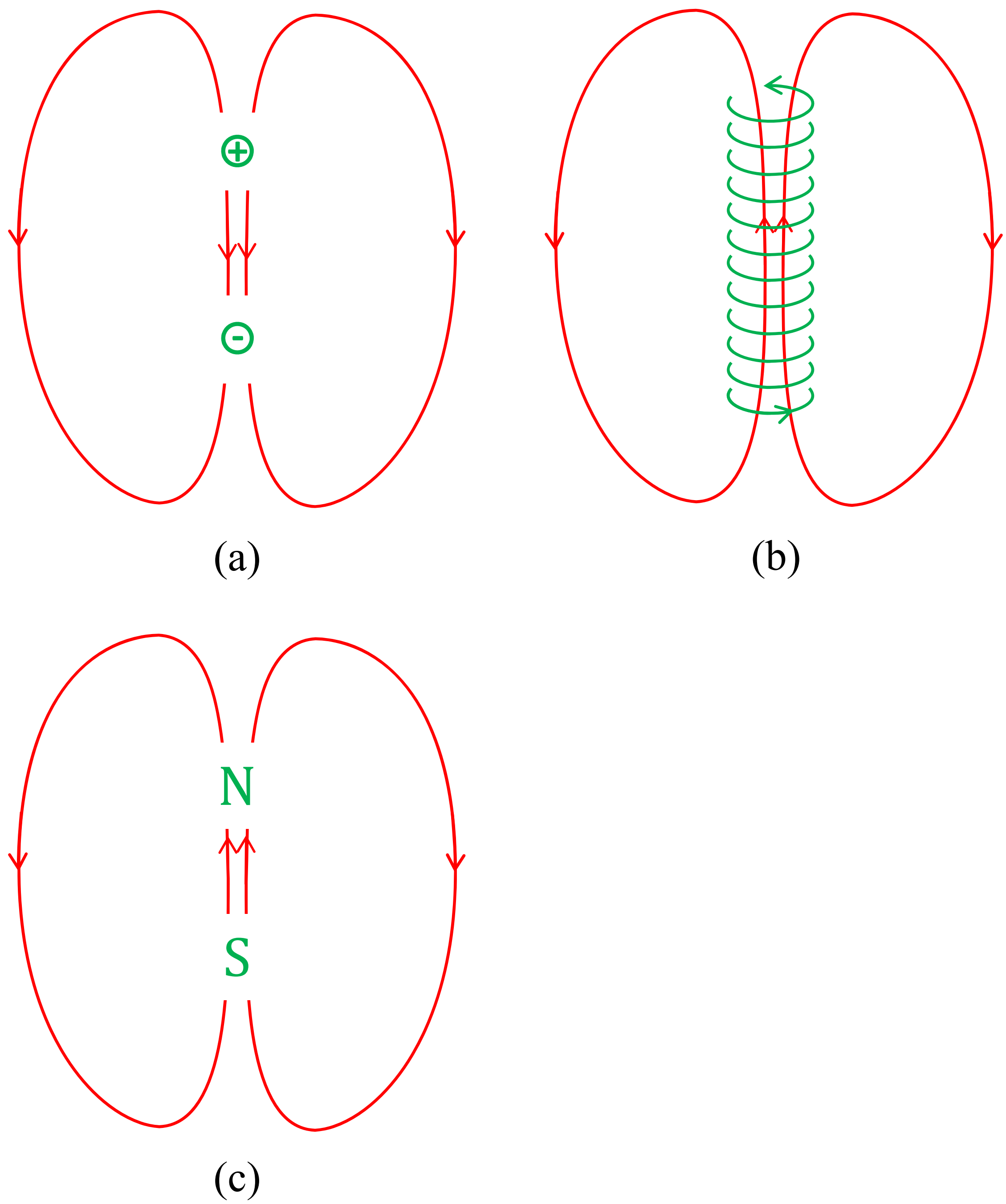
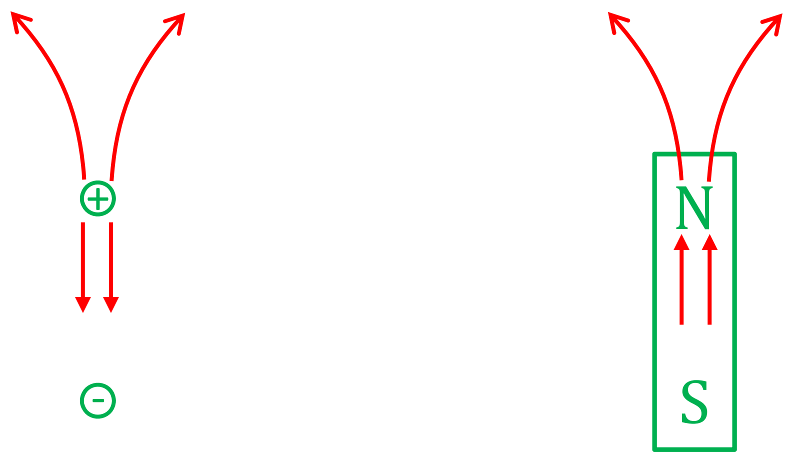


Figure 3

(a) Electric dipole
(b) Magnetic dipole caused by solenoid
(c) Magnetic dipole represented as N and S



(a) (b)

Figure 4

(a) The lines of force near the + charge
(b) The lines of force near the N pole

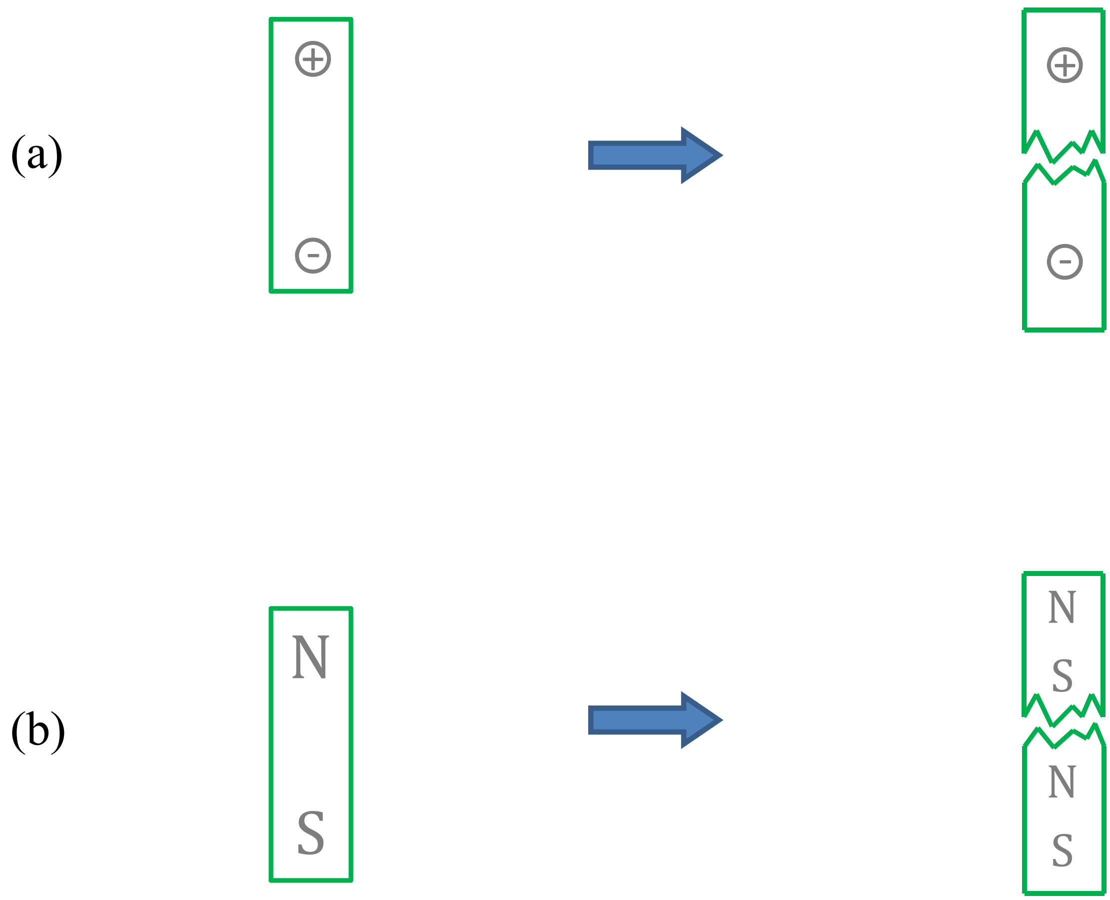


Figure 5

(a) Breaking up an electric dipole
(b) Breaking up a magnetic dipole

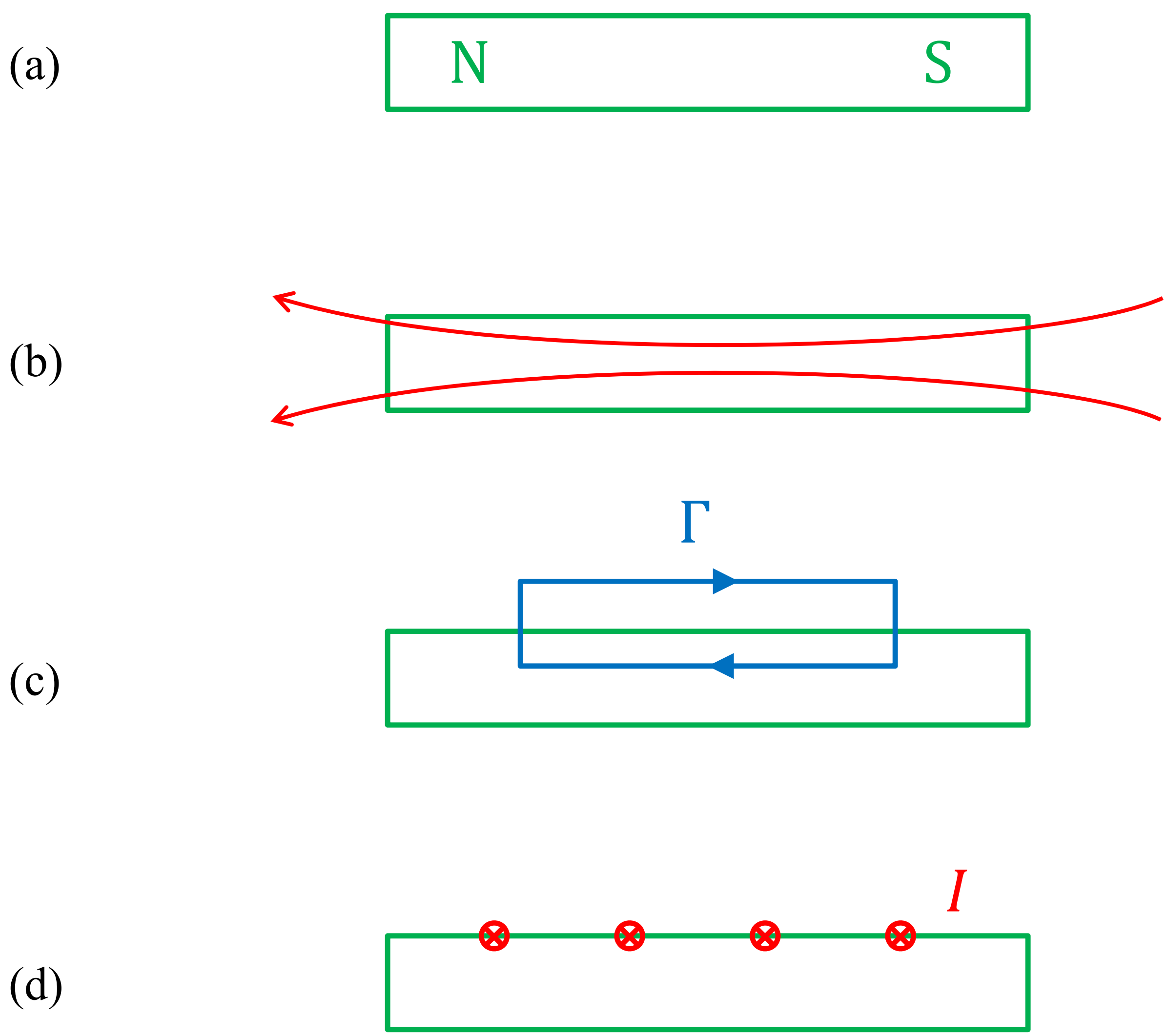


Figure 6
A bar magnet

(a) The poles
(b) The flux lines
(c) A loop for Ampere’s law
(d) Currents on the surface

(a) 

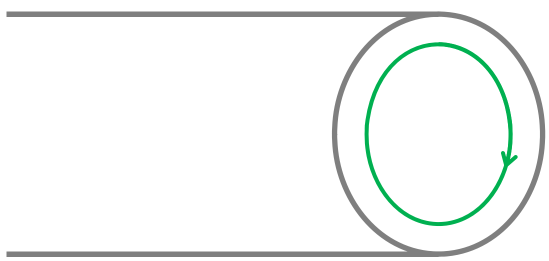
(b) 

Figure 7

(a) Small current loops
(b) Net effect is surface current